2 . ASSUME p(k) : 1.1! + 2.2! +3.3! +……+k.k!

= (k+1)! – 1 is true

CLAIM : p(k+1) is true.

P(k+1) = 1.1! + 2.2! +3.3! +……+k.k! +(k+1)(k+1)!

= (k+1)!-1 +(k+1)(k+1)!

= (k+1)! [(1+k+1)] – 1

= (k+1)! (k+2) – 1

= (k+2) ! – 1

= [(k+1) + 1]! - 1

P(k+1)is true.

BY THE PRINCIPLE OF MATHEMATICAL INDUCTION,

P(n) : 1.1! + 2.2! +3.3! +……+n.n! = (n+1)! – 1 , n>=1

EXAMPLE 3 : Use mathematical induction , prove that $\sum_{m=0}^{n} 3^m = \frac{(3\wedge n+1)-1}{2}$

SOLUTION:

Let p(n): $3^0 + 3^1 + \ldots\ldots 3^n = \frac{(3\wedge n+1)-1}{2}$

1.p(0) : $3^0 = \frac{(3\wedge 0+1)-1}{2} = \frac{2}{2} = 1$ is true .

2.ASSUME

P(k):): $3^0 + 3^1 + \ldots\ldots 3^n = \frac{(3\wedge k+1)-1}{2}$ is true.

CLAIM : p(k+1)is true.

P (k+1) :): $3^0 + 3^1 + 3^2 + \ldots\ldots + 3^k + 3^{k+1}$

$= \frac{(3\wedge k+1)-1}{2} + 3^{k+1}$ using (1)

$= \frac{(3\wedge k+1)+2.(3\wedge k+1)-1}{2}$

$= \frac{3(3\wedge k+1)-1}{2}$

$= \frac{(3\wedge k+2)-1}{2}$

$= \frac{(3\wedge(k+1)+1)-1}{2}$

P(k+1)is true.

By the principle of mathematical induction.

P(n): $\sum^{n}_{m=0} 3^{m} = \frac{(3\wedge n+1)-1}{2}$ is true for n>=0

EXAMPLE 4 :Use mathematical induction , prove that $\frac{1}{\sqrt{1}}+\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{3}}+\ldots+\frac{1}{\sqrt{n}} > \sqrt{n}$, n>=2

SOLUTION:

Let p(n): $\frac{1}{\sqrt{1}}+\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{3}}+\ldots+\frac{1}{\sqrt{n}} > \sqrt{n}$, n>=2

1.p(2): that $\frac{1}{\sqrt{1}}+\frac{1}{\sqrt{2}}$ = (1.707) > $\sqrt{2}$ + (1.414) is true

2.ASSUME

P(k): that $\frac{1}{\sqrt{1}}+\frac{1}{\sqrt{2}}+\ldots+\frac{1}{\sqrt{k}} > \sqrt{k}$ is true -> (1)

CLAIM : p(k+1) is true.

P(k+1) : $\frac{1}{\sqrt{1}}+\frac{1}{\sqrt{2}}+\ldots+\frac{1}{\sqrt{k}}+\frac{1}{\sqrt{k+1}}$

$\sqrt{k}+\frac{1}{\sqrt{k+1}}$ using (1)

$\frac{\sqrt{k}\sqrt{k+1}+1}{\sqrt{k+1}}$

$\frac{\sqrt{k(k+1)}+1}{\sqrt{k+1}}$

$> \frac{\sqrt{k.k}+1}{\sqrt{k+1}}$

$> \frac{k+1}{\sqrt{k+1}}$

$> \sqrt{k+1}$

P(k+1) $> \sqrt{k+1}$

P(K+1) is true

BY THE PRINCIPLE OF MATHEMATICAL INDUCTION.

that $\frac{1}{\sqrt{1}}+\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{3}}+\ldots+\frac{1}{\sqrt{n}} > \sqrt{n+1}$

EXAMPLE 5: Using mathematical induction ,prove that $1^2 + 3^2 + 5^2 + \ldots\ldots (2n-1)^2 = \frac{n(2n-1)(2n+1)}{3}$

SOLUTION :

Let p(n): $1^2 + 3^2 + 5^2 + \ldots\ldots (2n-1)^2 = \frac{1}{3} n(2n-1)(2n+1)$

1.p(1): $1^2 = \frac{1}{3} 1(2-1)(2+1) = \frac{1}{3} . 3$

=1 is true.

2.ASSUME p(k)is true.

$1^2 + 3^2 + 5^2 + \ldots\ldots (2k-1)^2 = \frac{1}{3} n(2k-1)(2k+1)$ -> (1) Is true.

CLAIM : p(k+1) is true.

$P(k+1) = \frac{1}{3} k(2k-1)(2k+1) + (2k+1)^2$ using (1)

$= \frac{1}{3} (2k+1)[k(2k-1) + 3(2k+1)]$

$= \frac{1}{3} (2k+1)(2k2+5k+3)$

$= \frac{1}{3} (2k+1)(2k+3)(k+1)$

$= \frac{1}{3} (k+1)[2(k+1)-1][2(k+1)+1]$

P(k+1) is true .

BY THE PRINCIPLE OF MATHEMATICAL INDUCTION,

$P(n) = 1^2 + 3^2 + 5^2 + \ldots\ldots (2n-1)^2 = \frac{n(2n-1)(2n+1)}{3}$

EXAMPLE 6:Use mathematical induction to show that $n^3 - n$ is divisible by 3. For n Ɛ Z^+

SOLUTION:

Let p(n): $n^n - n$ is divisible by 3.

1. p(1): $1^3 - 1$ is divisible by 3,is true.
2. ASSUME p(k): $k^3 - k$ is divisible by 3. -> (1)

CLAIM : p(k+1) is true .

P (k+1): $(k+1)^3 - (k+1)$

$= k^3 + 3k^2 + 3k + 1 - k - 1$

$= (k^3-k) + 3(k^2+k)$ ->(2)

(1) => $k^3 - k$ Is divisible by 3 and $3(k^2 + k)$ is divisible by 3 ,we have equation (2) is divisibleby 3

Therefore P(k+1) is true.

By the principle of mathematical induction , $n^3 - n$ is divisible by 3.

2.2 Strong Induction

There is another form of mathematics induction that is often useful in proofs.In this form we use the basis step as before, but we use a different inductive step. We assume that p(j)is true for j=1…,k and show that p(k+1)must also be true based on this assumption . This is called strong Induction (and sometimes also known as the second principles of mathematical induction).

We summarize the two steps used to show that p(n)is true for all positive integers n.

Basis Step : The proposition P(1) is shown to be true

Inductive Step: It is shown that

$$[P(1)\wedge P(2)\wedge\ldots\ldots\wedge P(k)] \rightarrow P(k+1)$$

NOTE:

The two forms of mathematical induction are equivalent that is, each can be shown to be valid proof technique by assuming the other

EXAMPLE 1: Show that if n is an integer greater than 1, then n can be written as the product of primes.

SOLUTION:

Let P(n) be the proportion that n can be written as the product of primes

Basis Step : P(2) is true , since 2 can be written as the product of one prime

Inductive Step: Assume that P(j) is positive for all integer j with j<=k. To complete the Inductive Step, it must be shown that P(k+1) is trueunder the assumption.

There are two cases to consider namely

i) When (k+1) is prime
ii) When (k+1) is composite

Case 1 : If (k+1) is prime, we immediately see that P(k+1) is true.

Case 2: If (k+1) is composite

Then it can be written as the product of two positive integers a and b with 2<=a<b<=k+1. By the Innduction hypothesis, both a and b can be written as the product of primes, namely those primes in the factorization of a and those in the factorization of b .

The Well-Ordering Property:

The validity of mathematical induction follows from the following fundamental axioms about the set of integers.

Every non-empty set of non negative integers has a least element.

The well-ordering property can often be used directly in the proof.

Problem :

What is wrong with this "Proof" by strong induction ?

Theorem :

For every non negative integer n, 5n = 0

Proof:

Basis Step: 5 – 0 = 0

Inductive Step: Suppose that 5j = 0 for all non negative integers j with o<=j<=k. Write k+1 = i+j where I and j are natural numbers less than k+1. By the induction hypothesis

5(k+1) = 5(i+j) = 5i + 5j = 0+ 0 =0

Example 1:

Among any group of 367 people, there must be atleast 2 with same birthday, because there are only 366 possible birthdays.

Example 2:

In any group of 27 English words, there must be at least two, that begins with the same letter, since there are only 26 letters in English alphabet

Example 3:

Show that among 100 people , at least 9 of them were born in the same month

Solution :

Here, No of Pigeon = m = No of People = 100

No of Holes = n = No of Month = 12

Then by generalized pigeon hole principle

{[100-1]/12}+1 = 9, were born in the same month

Combinations:

Each of the difference groups of sections which can be made by taking some or all of a number of things at a time is called a combinations.

The number of combinations of ‘n’ things taken ‘r’ as a time means the number as groups of ‘r’ things which can be formed from the ‘n’ things.

It denoted by nCr.

The value of nCr :

Each combination consists /r/ difference things which can be arranged among themselves in r! Ways. Hence the number of arrangement for all the combination is nCr x r! . This is equal to the permulations of ‘n’ difference things taken ‘r’ as a time.

nCr x r! =n P r

nCr = n P r / r! --------------------→(A)

= n (n-1) , (n-2)........(n-r+1) / 1,2,3,.......r

Cor 1 : nPr = n! / (n-r)! --------→(B)

Substituting (B) in (A) we get

nCr = n! / (n-r)!r!

Cor 2: To prove that nCr = nCn-r

Proof :

nCr = n! / r!(n-r)! --------------------→(1)

nCn-r = n! / (n-r)! [n-(n-r)]!

$= n! / (n-r)!\, r!$ ---------------------------→(2)

From 1 and 2 we get

$nCr = nCn-r$

Example :

$30C_{28} = 30\ C_{30-28}$

$= 30\ C_2$

$30 \times 29 / 1 \times 2$

Example 2:

In how many can 5 persons be selected from amongs 10 persons ?

Sol :

The selection can be done in $10C_5$ ways.

$= 10 \times 9 \times 8 \times 7 \times 6 / 1 \times 2 \times 3 \times 4 \times 5$

$= 9 \times 28$ ways.

Example 5 :

How many ways are there to from a commitiee , if the consists of 3 educanalis and 4 socialist if there are 9 educanalists and 11 socialists.

Sol : The 3 educanalist can be choosen from a educanalist in 9C3 ways. The 4 socialist can be choosen from 11 socialist in 11C4 ways.

∴ By products rule , the number of ways to select the commitiee is

$= 9C_3 . 11C_4$

$= 9! \;/ 3!\, 6! \;\;.\;\; 11! / 4!\, 7!$

= 84 x 330

27720 ways.

Example 6 :

1. A team of 11 players is so be chosen from 15 members. In how ways can this be done if

i. One particular player is always included.
ii. Two such player have always to be included.

Sol : Let one player be fixed the remaining players are 14 . Out of these 14 players we have to select 10 players in $14C_{10}$ ways.

$14C_4$ ways. [.`. $nCr = nC_{n-r}$]

➔ 14x13x12x11 / 1x2x3x4

➔1001 ways.

2. Let 2 players be fixed. The remaining players are 13. Out of these players we have to select a players in $13C_9$ ways.

$13C_4$ ways [.`. $nC_r = nC_{n-r}$]

➔13x12x11x10 / 1x2x3x4 ways

➔715 ways.

Example 9 :

Find the value of 'r' if $20C_r = 20_{Cr-2}$

Sol : Given $20\ C_r = 20C_{20-(r-2)}$ ➔r=20-(r+2) ----------------→(1)

(1) -----→ r=20 - r – 2

2r = 18

$r = 9$

Example 12 :

From a commitiee consisting of 6 men and 7 women in how many ways can be select a committee of

(1) 3men and 4 women.
(2) 4 members which has atleast one women.
(3) 4 persons of both sexes.
(4) 4 person in which Mr. And Mrs kannan is not included.

Sol :

(a) 3 men can be selected from 6 men is $6C_3$ ways. 4 women can be selected from 7 women in $7C_4$ ways.

.˙. By product rule the committee of 3 men and 4 women can be selected in

$$6C_3 \times 7C_4 \text{ ways} = \frac{6x5x4x}{1x2x3} \;X\; \frac{7x6x5x4}{1x2x3x4}$$

=700 ways.

(b) For the committee of atleast one women we have the following possibilities

1. 1 women and 3 men
2. 2 women and 2 men
3. 3 women and 1 men
4. 4 women and 0 men

There fore the selection can be done in

$= 7C_1 \text{ x } 6C_3 + 7C_2 \text{ x } 6C_2 + 7C_3 \text{ x} 6C_1 + 7C_4 \text{ x } 6C_6$ ways

= 7x20+21x15+35x6+35x1

=140x315x210x35

=700 ways.

(d) For the committee of bath sexes we have the following possibilities .

1. 1 men and 3 women
2. 2 men and 2 women
3. 3 men and 1 women

Which can be done in

$=6C_1 x 7C_3 + 6C_2 x 7C_2 + 6C_3 x 7C_1$

=6x35+15x21+20x7

=210+315+140

=665 ways.

Sol : (1) 4 balls of any colour can be chosen from 11 balls (6+5) in $11C_4$ ways.

=330 ways.

(2) The 2 white balls can be chosen in $6C_2$ ways. The 2 red balls can be chosen in $5C_2$ ways. Number of ways selecting 4 balls 2 must be red.

$=6C_2 + 5C_2$

$$= \frac{6 \times 5}{1 \times 2} + \frac{5 \times 4}{1 \times 2}$$

$=15 + 10$

$=25$ ways.

Number of ways selecting 4 balls and all Of same colour is $= 6C_4 + 5C_4$

$=15+5$

$=20$ways.

Definition

.

A Linear homogeneous recurrence relation of degree K with constant coefficients is a recurrence relation of the form

The recurrence relation in the definition is linesr since the right hand side is the sum of multiplies of the previous terms of sequence.

The recurrence relation is homogeneous , since no terms occur that are not multiplies of the aj"s.

The coefficients of the terms of the sequence are all constants ,rather than function that depends on "n".

The degree is k because an is exrressed in terms of the previous k terms of the sequence

Ex:4 The recurrence relation

$H_n = 2H_{n-1}+1$

Is not homogenous

Ex: 5 The recurrence relation

$B_n = nB_{n-1}$

Does not have constant coefficient

Ex:6 The relation $T(k)=2[T(k-1)]^2 KT(K-3)$

Is a third order recurrence relation &

T(0),T(1),T(2) are the initial conditions.

Ex:7 The recurrence relation for the function

f : N->Z defined by

f(x)=2x,Ɐ x € N is given by

f(n+1)=f(n)+2,n>=0 with f(0)=0

f(1)=f(0)+2=0+2=2

f(2)=f(1)+2=2+2=4 and so on.

It is a first order recurrence relation.

2.3 Recurrence relations.

Definition

An equation that expresses a_n, the general term of the sequence $\{a_n\}$ in terms of one or more of the previous terms of the sequence , namely $a_0,a_{1,\ldots\ldots,}a_{n-1}$,for all integers n with n>=0,where n_0 is a non –ve integer is called a recurrence relation for $\{a_n\}$ or a difference equation.

If the terms of a recurrence relation satisfies a recurrence relation , then the sequence is called a solution of the recurrence relation.

For example ,we consider the famous Fibonacci sequence

0,1,1,2,3,5,8,13,21,.....,

which can be represented by the recurrence relation.

$$F_n=F_{n-1}+F_{n-2},n>=2$$

& $F_0=0,F_1=1$. Here $F_0=0$ & $F_1=1$ are called initial conditions.

It is a second order recurrence relation.

2.4 Solving Linear Homogenous Recurrence Relations with Constants Coefficients.

Step 1: Write down the characteristics equation of the given recurrence relation .Here ,the degree of character equation is 1 less than the number of terms in recurrence relations.

Step 2: By solving the characteristics equation first out the characteristics roots.

Step 3: Depends upon the nature of roots ,find out the solution a_n as follows:

Case 1: Let the roots be real and distinct say $r_1, r_2, r_3, \ldots, r_n$ then

$$A_n = \alpha_1 r_1^n + \alpha_2 r_2^n + \alpha_3 r_3^n + \ldots\ldots + \alpha_n r_n^n,$$

Where $\alpha_1, \alpha_2, \ldots, \alpha_n$ are arbitrary constants.

Case 2: Let the roots be real and equal say $r_1 = r_2 = r_3 = r_n$ then

$$A_n = \alpha_1 r_1^n + n\alpha_2 r_2^n + n^2 \alpha_3 r_3^n + \ldots\ldots + n^2 \alpha_n r_n^n,$$

Where $\alpha_1, \alpha_2, \ldots, \alpha_n$ are arbitrary constants.

Case 3: When the roots are complex conjugate, then

$$a_n = r^n(\alpha_1 \cos n\theta + \alpha_2 \sin n\theta)$$

Case 4: Apply initial conditions and find out arbitrary constants.

Note:

There is no single method or technique to solve all recurrence relations. There exist some recurrence relations which cannot be solved. The recurrence relation.

$S(k)=2[S(k-1)]^2-kS(k-3)$ cannot be solved.

Example 1: If sequence $a_n=3.2^n$,n>=1, then find the recurrence relation.

Solution:

For n>=1

$a_n=3.2^n$,

now, $a_{n-1}=3.2^{n-1,}$

$=3.2^n / 2$

$a_{n-1}=a^n/2$

$a_n = 2(a_n-1)$

$a_n = 2a_n-1$, for $n\geq 1$ with $a_n=3$

Example 2 :

Find the recurrence relation for S(n) = 6(-5),$n\geq 0$

<u>Sol :</u>

Given $S(n) = 6(-5)^n$

$S(n-1) = 6(-5)^{n-1}$

$=6(-5)^n / -5$

$S(n-1) = S(n) / -5$

$S_n = -5.5 (n-1)$, $n\geq 0$ with s(0) =6

Example 5: Find the relation from $Y_k = A.2^k + B.3^k$

Sol :

Given $Y_k = A.2^k + B.3^k$ --------------→(1)

$Y_{k+1} = A.2^{k+1} + B.3^{k+1}$

$= A.2^k.2 + B3^k.3$

$Y_{k+1} = 2A.2^k + 3B.3^k$ ------------------------→(2)

$Y_{k+2} = 4A.2^k + 9B.3^k$ ------------------------→(3)

(3) – 5(2) + 6(1)

➔ $y_{k+2} - 5y_{k+1} + 6y_k = 4A.2^k + 9B.3^k - 10A.2^k - 15B.3^k + 6A.2^k + 6B.3^k$

$= 0$

∴ $Y_{k+1} - 5y_{k+1} + 6y_k = 0$ in the required recurrence relation.

Example 9 :

Solve the recurrence relation defind by $S_o = 100$ and S_k (1.08) S_{k-1} for $k \geq 1$

Sol ;

Given $S_0 = 100$

$S_k = (1.08)\ S_{k-1}\ ,\ k \geq 1$

$S_1 = (1.08)\ S_0 = (1.08)100$

$S_2 = (1.08)\ S_1 = (1.08)(1.08)100$

$= (1.08)^2\ 100$

$S_3 = (1.08)\ S_2 = (1.08)(1.08)^2 100$

$= = (1.08)^3\ 100$

$S_k = (1.08)S_{k-1} = (1.08)^k 100$

Example 15 : Find an explicit formula for the Fibonacci sequence .

Sol ;

Fibonacci sequence 0,1,2,3,4........ satisify the recurrence relation

$fn= f_{n-1} + f_{n-2}$

$fn- f_{n-1} - f_{n-2} = 0$

& also satisfies the initial condition $f_0=0,f_1=1$

Now , the characteristic equation is

$r_2-r-1 =0$

Solving we get $r=1\underline{+}\ 1+4 / 2$

$= 1\underline{+}\ 5 / 2$

Sol :

$fn = \alpha_1\ (1+ 5 / 2)^n + \alpha_2\ (1- 5 / 2)^n$ ----→(A)

given f_0 =0 put n=0 in (A) we get

$f0 = \alpha_1\ (1+ 5 / 2)^0 + \alpha_2\ (1- 5 / 2)^0$

(A)➔ α1 +α2 =0 ------------------------→(1)

given f_1 =1 put n=1 in (A) we get

$f_1 = \alpha_1\ (1+ 5 / 2)^1 + \alpha_2\ (1- 5 / 2)^1$

(A)➔$(1+ 5 / 2)^n + \alpha_2\ (1- 5 / 2)^n\ \alpha_2 = 1$ -------→(2)

To solve(1) and (2)

$(1)\ X(1+5/2) \Rightarrow (1+5/2)\alpha_1 + (1+5/2)\alpha_2 = 0 \text{-----}\rightarrow (3)$

$(1+5/2)\alpha_1 + (1+5/2)\alpha_2 = 1 \text{------}\rightarrow (2)$

(-) (-) (-)

$1/2\ \alpha_2 + 5/2\ \alpha_2 - 1/2\ \alpha_2 + 5/2\ \alpha_2 = -1$

$2\ 5\ \ d_2 = -1$

$\alpha_2 = -1/5$

Put $\alpha_2 = -1/5$ in eqn (1) we get α_1 1/5

Substituting these values in (A) we get

Solution $fn = 1/5\ (1+5/2)^n - 1/5\ (1+5/2)^n$

Example 13 ;

Solve the recurrence equation

$a_n = 2a_{n-1} - 2a_{n-2}$, $n \geq 2$ & $a_0 = 1$ & $a_1 = 2$

Sol :

The recurrence relation can be written as

$a_n - 2a_{n-1} + 2a_{n-2} = 0$

The characteristic equation is

$r2 - 2r - 2 = 0$

Roots are $r = 2 \pm 2i / 2$

$= 1 \pm i$

LINEAR NON HOMOGENEOUS RECRRENCE RELATIONS WITH CONSTANT COEFFICIENTS

A recurrence relation of the form

$a_n = c_1\ a_{n-1}+c_2\ a_{n-2}+\ldots\ldots\ldots.c_k\ a_{n-k}$+F(n)……………   (A)

Where c_1 , c_2 ,…. c_k are real numbers and F(n) is a function not identically zero depending only on n,is called a non-homogeneous recurrence relation with constant coefficient.

Here ,the recurrence relation

$a_n = c_1\ a_{n-1}+c_2\ a_{n-2}+\ldots\ldots\ldots.c_k\ a_{n-k}$+F(n)……………   (B)

Is called Associated homogeneous recurrence relation.

NOTE:

(B) is obtained from (A) by omitting F(n) for example ,the recurrence relation

$a_n = 3\ a_{n-1}+2_n$ is an example of non-homogeneous recurrence relation .Its associated

Homogeneous linear equation is

$a_n = 3\ a_{n-1}$ [By omitting F(n) = 2n]

PROCEDURE TO SOLVE NON-HOMOGENEOUS RECURRENCE RELATIONS:

The solution of non-homogeneous recurrence relations is the sum of two solutions.

1.solution of Associated homogeneous recurrence relation (By considering RHS=0).

2.Particular solution depending on the RHS of the given recurrence relation