(i) **Claim :** $a \vee b \le a \vee c$

Let $x = a \vee c$. Then x is lub of a & c

$\Rightarrow x$ is an upper bound of a & c.

$\therefore a \le x,\ c \le x$

But $b \le c,\ c \le x \Rightarrow b \le x$

Also $a \le x$

$\therefore x$ is upper bound of a & b

But $a \vee b$ is lub of a & b

$\therefore a \vee b \le x = a \vee c$

(ii) **Claim :** $a \wedge b \le a \wedge c$

Let $y = a \wedge b \Rightarrow y$ is glb of a & b

$\therefore y$ is a lower bound of a & b

$y \le a,\ y \le b$

Using $b \le c,\ y \le a,\ y \le c$

$\therefore y$ is a lower bound of a & c

But $a \wedge c$ is glb of a and c

$\therefore y \le a \wedge c \Rightarrow a \wedge b \le a \wedge c$

Example 6. If (L, $\wedge$, $\vee$) is a complemented distributive lattice, then the De Morgan's laws are valid. **[A.U M/J 2013]**

i.e., $\overline{a \vee b} = \bar{a} \wedge \bar{b}$ **and** $\overline{a \wedge b} = \bar{a} \vee \bar{b}$, $\forall\ a, b \in L$

Proof : If (L, $\wedge$, $\vee$) is a complemented distributive lattice. We have the complement of any element is unique in the distributive lattice.

Let $a, b \in L$

Let $\bar{a}$ and $\bar{b}$ are complements of a and b respectively.

$\therefore$ We have $a \wedge \bar{a} = 0,\ a \vee \bar{a} = 1,\ b \wedge \bar{b} = 0,\ b \vee \bar{b} = 1$

Now,
$$(a \vee b) \vee (\bar{a} \wedge \bar{b}) = ((a \vee b) \vee \bar{a}) \wedge ((a \vee b) \vee \bar{b})$$
$$= (a \vee (b \vee \bar{a})) \wedge (a \vee (b \vee \bar{b}))$$

$$= (a \vee (\bar{a} \vee b)) \wedge (a \vee 1)$$

$$= ((a \vee \bar{a}) \vee b) \wedge (a \vee 1)$$

$$= 1 \wedge 1 = 1$$

and $$(a \vee b) \wedge (\bar{a} \wedge \bar{b}) = (a \wedge (\bar{a} \wedge \bar{b})) \vee (b \wedge (\bar{a} \wedge \bar{b}))$$

$$= ((a \wedge \bar{a}) \wedge \bar{b}) \vee (b \wedge (\bar{b} \wedge \bar{a}))$$

$$= (0 \wedge \bar{b}) \vee ((b \wedge \bar{b}) \wedge \bar{a})$$

$$= (0 \wedge \bar{b}) \vee (0 \wedge \bar{a})$$

$$= 0 \vee 0 = 0$$

$\therefore$ $\bar{a} \wedge \bar{b}$ is the complement of $a \vee b$ and its unique.

Example 7. Let (L, $\wedge$, $\vee$) be a distributive lattice and a, b, c $\in$ L If $a \wedge b = a \wedge c$ and $a \vee b = a \vee c$, then $b = c$

[Cancellation laws are valid in a Distributive lattice]

Proof : Let (L, $\wedge$, $\vee$) be any distributive lattice and $a, b, c \in$ L, such that

$$a \wedge b = a \wedge c$$

and $$a \vee b = a \vee c$$

Now, $$(a \wedge b) \vee c = (a \vee c) \wedge (b \vee c) \qquad [\because \text{ L is distributive}]$$

$$= (a \vee b) \wedge (b \vee c)$$

$$= (b \vee a) \wedge (b \vee c)$$

$$= b \vee (a \wedge c)$$

$$= b \vee (a \wedge b)$$

$$= b$$

and $(a \wedge b) \vee c = (a \wedge c) \vee c = c$

Thus $\quad b = (a \wedge b) \vee c = c$

So that, $a \wedge b = a \wedge c$

and $\quad a \vee b = a \vee c$

$\Rightarrow b = c$

That is, the cancellation law is valid in a distributive lattice.

Example 8. Theorem : A modular lattice is distributive if and only if none of its sublattice is isomorphic to the diamond lattice M_5.

Proof : We have, the diamond lattice M_5 is not distributive lattice, therefore any lattice having sublattice isomorphic to M_5 cannot be distributive.

Conversely, let $(L, \leq)$ be any modular lattice but not distributive lattice. We show that L has a sublattice isomorphic to M_5. Since $(L, \leq)$ is not distributive lattice, then we an find $x, y, z \in L$ such that

$$(x \wedge y) \vee (y \wedge z) \vee (z \wedge x) < (x \vee y) \wedge (y \vee z) \wedge (z \vee x)$$

Now let, $\quad u = (x \wedge y) \vee (y \wedge z) \vee (z \wedge x)$

$v = (x \vee y) \wedge (y \vee z) \wedge (z \vee x)$

$a = u \vee (x \wedge v)$

$b = u \vee (y \wedge v)$ and

$c = u \vee (z \wedge v)$

Then the elements u, v, a, b, c are distinct and form a sublattice to L, and $S = \{u, a, b, c, v\}$ is isomorphic to the diamond lattice M_5.

5.3 BOOLEAN ALGEBRA

Def. Boolean Algebra

A Boolean algebra is a complemented, distributive lattice.

Note : 1. George boole in 1854 had given a set of basic rules for logic in his book "The laws of thought". Boolean algebra provides the operations and rules working with the binary set $\{0, 1\}$

2. Electronic circuites and switching matchings are working with the rules of Boolean algebra.

Properties

A Boolean algebra will generally be denoted by $(B, *, \oplus, ', 0, 1)$ in which $(B, *, \oplus)$ is a lattice with two binary operations $*$ and $\oplus$ called the meet and join respectively. The corresponding partially ordered set will be denoted by $(B, \leq)$. The bounds of the lattice are denoted by 0 and 1, where 0 is the least element and 1 the greatest element of $(B, \leq)$. Since $(B, *, \oplus)$ is complemented and because of the fact that it is a distributive lattice, each element of B has a unique complement. We shall denote the unary operation of complementation by $'$, so that for any $a \in B$, the complement of a is denoted by $a' \in B$.

Most of the properties of a Boolean algebra have been derived in the previous section. We shall list some of the important properties here. It may be mentioned that the properties listed here are not independent of each other.

A Boolean algebra (B, $*$, $\oplus$, $'$, 0, 1) satisfies the following properties in which a, b and c denote any elements of the set B.

1. (B, $*$, $\oplus$) is a lattice in which the operations $*$ and $\oplus$ satisfy the following identities :

(L-1) $a * a = a$ **(L-1)$'$** $a \oplus a = a$

(L-2) $a * b = b * a$ **(L-2)$'$** $a \oplus b = b \oplus a$ [Commutative law]

(L-3) $(a * b) * c = a * (b * c)$ **(L-3)$'$** $(a \oplus b) \oplus c = a \oplus (b \oplus c)$ [Associative law]

(L-4) $a * (a \oplus b) = a$ **(L-4)$'$** $a \oplus (a * b) = a$

2. (B, $*$, $\oplus$) is a distributive lattice and satisfies these identities :

(D-1) $a * (b \oplus c) = (a * b) \oplus (a * c)$
(D-2) $a \oplus (b * c) = (a \oplus b) * (a \oplus c)$ } [Distributive laws]

(D-3) $(a * b) \oplus (b * c) \oplus (c * a) = (a \oplus b) * (b \oplus c) * (c \oplus a)$

(D-4) $a * b = a * c$, and $a \oplus b = a \oplus c \Rightarrow b = c$

3. (B, $*$, $\oplus$, 0, 1) is a bounded lattice in which for any $a \in$ B, the following hold :

(B-1) $0 \leq a \leq 1$

(B-2) $a * 0 = 0$ **(B-2)$'$** $a \oplus 1 = 1$ [Dominance laws]

(B-3) $a * 1 = a$ **(B-3)$'$** $a \oplus 0 = a$ [Identity laws]

4. (B, $*$, $\oplus$, $'$, 0, 1) is a uniquely complemented lattice in which the complement of any element $a \in$ B is denoted by $a' \in$ B and satisfies the following identities :

(C-1) $a * a' = 0$ **(C-1)$'$** $a \oplus a' = 1$ [Complement laws]

(C-2) $0' = 1$ **(C-2)** $1' = 0$ [Zero and one law]

(C-3) $(a * b)' = a' \oplus b'$ **(C-3)$'$** $(a \oplus b)' = a' * b'$ [De-Morgan's laws]

5. There exists a partial ordering relation $\leq$ on B such that

(P-1) $a * b = \text{GLB}\{a, b\}$ (P-1)′ $a \oplus b = \text{LUB}\{a, b\}$

(P-2) $a \leq b \Leftrightarrow a * b = a \Leftrightarrow a \oplus b = b$

(P-3) $a \leq b \Leftrightarrow a * b' = 0 \Leftrightarrow b' \leq a' \Leftrightarrow a' \oplus b = 1$

Example :

Let $B = \{0, 1\}$ be a set. The operations $*, \oplus, '$ on B are defined by

$*$	0	1
0	0	0
1	0	1

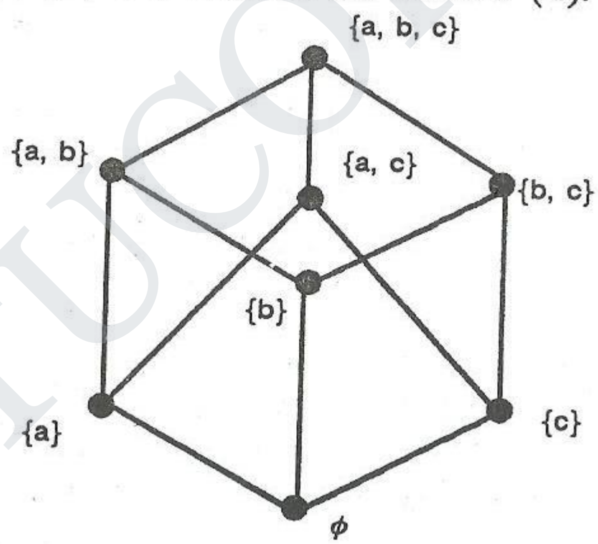
$\oplus$	0	1
0	0	1
1	1	1

x	x'
0	1
1	0

Clearly $< B, *, \oplus, ', 0, 1 >$ is a Boolean algebra.

Example :

Let A $= \{a, b, c\}$ and consider the lattice $P(A)$.



Clearly $< P(A), \cap, \cup >$ is a Boolean Algebra.

Def. Sub-Boolean algebra

Let (B, $*, \oplus, '$, 0, 1) be a Boolean algebra and S $\subseteq$ B. If S contains the elements 0 and 1 and is closed under the operations $*, \oplus$ and $'$, then (S, $*, \oplus, '$, 0, 1) is called a sub-boolean algebra.

Def. Join-irreducible.

Let $(L, *, \oplus)$ be a lattice. An element $a \in L$ is called join-irreducible if it cannot be expressed as the join of two distinct elements of L. In otherwords $a \in L$ is join-irreducible

If for any $a_1, a_2 \in L$

$$a = a_1 \oplus a_2 \Rightarrow (a = a_1) \vee (a = a_2)$$

Def. (Direct product)

Let $(L, \oplus, *)$ and $(S, \vee, \wedge)$ be two lattices. Then the direct product of L, and S is defined by $(L \times S, +, *)$ where $+$ and $*$ are defined by the following manners.

$$(a_1, b_1) + (a_2, b_2) = (a_1 \oplus a_2, b_1 \vee b_2)$$

$$(a_1, b_1) . (a_2, b_2) = (a_1 * a_2, b_1 \wedge b_2), \forall a_1, a_2 \in L, \forall b_1, b_2 \in$$

Def. Lattice homomorphism

Let $(L, \oplus, *)$ and $(S, \vee, \wedge)$ be two lattices. A map $f : L \to S$ is called a homomorphism if

$$g(a \oplus b) = g(a) \vee g(b)$$

$$g(a * b) = g(a) \wedge g(b) \ ; \forall a, b \in L$$

Note: 1. The binary operation $\oplus$ and $*$ are preserved under Lattice Homomorphism.

2. $g : (L, \oplus, *, \leq) \to (S, \vee, \wedge, \leq')$ is called an order homomorphism, then $a \leq b \Rightarrow g(a) \leq' g(b)$, for all $a, b \in L$

Example 1. Theorem : In a Boolean lattice, prove that the De-Morgan's laws. **[A.U. M/J 2006]**

Proof : Let $(L, \oplus, *)$ be Boolean lattice.

(i.e.,) L is a complemented and distributive lattice.

The De-Morgan's laws are

(i) $\overline{a \oplus b} = \bar{a} * \bar{b}$; (ii) $\overline{a * b} = \bar{a} \oplus \bar{b}, \forall \bar{a}, a, b \in L$

Assume that $a, b \in L$. There exists elts

$\bar{a}, \bar{b} \in L$ such that $a \oplus \bar{a} = 1$; $a * \bar{a} = 0$; $b \oplus \bar{b} = 1$; $b * \bar{b} = 0$

(i) Claim : $\overline{a \oplus b} = \bar{a} * \bar{b}$

Now
$$
\begin{aligned}
(a \oplus b) \oplus (\bar{a} * \bar{b}) &= [(a \oplus b) \oplus \bar{a}] * [(a \oplus b) \oplus \bar{b}] \\
&= [a \oplus \bar{a} \oplus b] * [a \oplus b \oplus \bar{b}] \\
&= [1 \oplus b] * [a \oplus 1] \\
&= 1 * 1 = 1
\end{aligned}
$$

$$
\begin{aligned}
(a \oplus b) * (\bar{a} * \bar{b}) &= [(a \oplus b) * \bar{a}] * [(a \oplus b) * \bar{b}) \\
&= [(a * \bar{a}) \oplus (b * \bar{a})] * [(a * \bar{b}) \oplus (b * \bar{b})] \\
&= [0 \oplus (b * \bar{a})] * [(a * \bar{b}) \oplus 0] \\
&= (b * \bar{a} * (a * \bar{b}) \\
&= b * (\bar{a} * a) * b = \bar{b} * 0 * \bar{b} = 0
\end{aligned}
$$

Hence claim (i) is proved.

(ii) Claim : $\overline{a * b} = \bar{a} \oplus \bar{b}$

Now
$$
\begin{aligned}
(a * b) \oplus (\bar{a} \oplus \bar{b}) &= [(a * b) \oplus \bar{a}] \oplus [(a * b) \oplus \bar{b}] \\
&= [(a \oplus \bar{a}) * (b \oplus \bar{a})] \oplus [(a \oplus \bar{b}) * (b \oplus \bar{b})] \\
&= [1 * (b \oplus \bar{a})] \oplus [(a \oplus \bar{b}) * 1] \\
&= (b \oplus \bar{a}) \oplus (a \oplus \bar{b}) \\
&= b \oplus (\bar{a} \oplus a) \oplus \bar{b} \\
&= b \oplus 1 \oplus \bar{b} = b \oplus \bar{b} = 1
\end{aligned}
$$

$$
\begin{aligned}
(a * b) * (\bar{a} \oplus \bar{b}) &= [(a * b) * \bar{a}] \oplus [(a * b) * \bar{b}] \\
&= (a * \bar{a} * b) \oplus [a * b * \bar{b}]
\end{aligned}
$$

$$= (0 * b) \oplus [a * 0]$$

$$= 0 * 0 = 0$$

Therefore claim (ii) is proved

Hence the De-Morgan's laws are proved.

Example 2. Show that $(P(A), \cup, \cap, \subseteq)$ is a Boolean algebra.

Proof : We know that $(P(A)), \subseteq)$ is a lattice.

For any $X, Y, Z \in P(A)$, $X \cap (Y \cup Z) = (X \cap Y) \cup (X \cap Z)$

$$X \cup (Y \cap Z) = (X \cup Y) \cap (X \cup Z)$$

Also $\forall X \in P(A)$, there exists a subset $\overline{X}$ of A such that

$$X \cup \overline{X} = A, \quad X \cap \overline{X} = \{ \} = \phi$$

Zero elt of P(A) is $\{ \}$ = least elt.

The greatest elt of P(A) is A.

$\therefore$ $(P(A), \cup, \cap, \subseteq)$ is a Boolean algebra.

Example 3. Consider the $D_{30} = \{1, 2, 3, 5, 6, 10, 15, 30\}$ is a lattice (infact Boolean algebra) with relation "divides".

Solution : Now

$2 * 1 = 1$	$3 * 1 = 1$	$5 * 1 = 1$
$2 * 2 = 2$	$3 * 2 = 1$	$5 * 2 = 1$
$2 * 3 = 1$	$3 * 3 = 3$	$5 * 3 = 1$
$2 * 5 = 1$	$3 * 5 = 3$	$5 * 5 = 5$
$2 * 6 = 2$	$3 * 6 = 3$	$5 * 6 = 1$
$2 * 10 = 1$	$3 * 10 = 1$	$5 * 10 = 5$
$2 * 15 = 1$	$3 * 15 = 3$	$5 * 15 = 5$
$2 * 30 = 2$	$3 * 30 = 3$	$5 * 30 = 5$

$\therefore$ 2, 3, 5 are atoms in D_{30}.

Unit I

LOGIC AND PROOFS

PART – A

1. Without constructing the truth table show that $p \rightarrow (q \rightarrow p) \equiv \neg p(p \rightarrow q)$

Solution

$p \rightarrow (q \rightarrow p) \equiv p \rightarrow (\neg q \vee p)$

$\equiv \neg p \vee (\neg q \vee p)$

$\equiv \neg p \vee (p \vee \neg q)$

$\equiv (\neg p \vee p) \vee \neg q$

$\equiv T \vee \neg q$

$\equiv T.$

2. Prove that $p \rightarrow q$ is logically prove that $(\neg p \vee q)$

Solution:

p	q	$p \rightarrow q$	$\neg p \vee \vee q$
T	T	T	T
T	F	F	F
F	T	T	T
F	F	T	T

3. Define a tautology. With an example.

A statement that is true for all possible values of its propositional variables is called a tautology universely valid formula or a logical truth.

Example: $p \vee \neg q$ is a tautology.

4. When do you say that two compound statement proposition are equivalent.

Two compound proposition are said to be equivalent if then the have identical truth tables.

5. Give an indirect proof of the theorem if 3n+2 is odd, then n is odd.

Solution:

P: 3n+2 is odd

Q: n is odd

Hypothesis: Assume that $p \rightarrow$ q is false.

Assume that p *is true and* q is false.

i.e, n is not odd $\Rightarrow$ n is even.

Analysis: If n is even then n=2k for some integer k.

$3n+2 = 3(2k)+2.$

$= 6k+2.$

$= 2(3k+1)$

6. Define a universal specification.

$(x)A(x) \Rightarrow A(y)$

If a statement of the form (x)A(x) is assumed to be true , then the universal quantifier can be dropped to obtain A(y) is true for any arbitrary object y in the universe.

7.Show that {∨,∧} is not functionally complete.

Solution: $\neg p$ cannot be expressed using the connectives {∨,∧} .since no sets contribution of the statement exists {∨,∧} as input if T and the output is F.

8. Write the converse, inverse, contra positive of 'If you work hard then you will be rewarded'

Solution:

p: you will be work hard.

q: you will be rewarded.

¬p: You will not be work hard.

¬ q: You will no tbe rewarded.

Converse: q→ p, If you will be rewarded then you will be work hard

Contrapositive: ¬ q→ p,if You will not be rewarded then You will not be work hard.

Inverse: ¬ p→ ¬ q, if You will not be work hard then You will no tbe rewarded.

9. let E={ -1, 0,1,2 denote a universe of discourse . If P(x ,y) : x +y =1 find the truth value of ($\forall x$) ($\exists y$).

Solution: Given E={ -1, 0,1,2} P(x,y): x+y=1

($\forall x$) ($\exists y$). P(x,y) is true since 2+(-1)=1
1+0=1.

10.obtain the disjunctive normal forms of p ∧(p→ q)

Solution: let s⇔ p ∧(p→ q)

⇔ $p \wedge(\neg p \vee \vee\ q)$

⇔ $(p \wedge \neg p) \vee$ (p ∧ q)

11. Show that (p ∧ q) ⇒ (p→ q).

Solution:

To prove: (p ∧ q)→ (p→ q). is a tautology.

p	q	p ∧ q	p→ q	(p ∧ q)→ (p→ q).
T	T	T	T	T
T	F	F	F	T
F	T	F	T	T
F	F	F	T	T

12.Write an equivalent formula for p∧(q ↔r) which contains neither the bi onditional nor conditional.

Solution :

$p \wedge (q \leftrightarrow r) \Leftrightarrow (p \wedge (q \rightarrow r) \wedge (r \rightarrow q)$

$\Leftrightarrow (p \wedge (\neg q \vee r) \wedge (\neg r \vee q).$

13. Show that (x) (H(x) →M(x)) ∧ H(s) ⇒ M(s)

Solution :

Steps	Premises	Rule	Reason
1	(x) (H(x) →M(x))	P	Given premise
2	H(s) →M(s)	US (1)	(Vx) p (x) ⇒ $p(y)$
3	H(s)	P	Given premise
4	M(s)	T	(2) (3) (p → q, p ⇒ q)

14. Show that ¬ p(a,b) follows logically from (x) (y) (p(x,y) →w(x,y)) *and* ¬w(a, b)

Solution :

1. (x) (y) (p(x,y) → w(x,y)) p
2. (y), p(a,y) → w(a,y) US, (1)
3. P(a,b) → w(a,b) US (2)
4. $\neg w(a, b)$ p Given
5. $\neg p(a, b)$ T (3),(4), (p→ Q) ∧ ¬ Q⇒ ¬p

15. Symbolise: For every x, these exixts a y such that $x^2+y^2 \geq 100$

Solution :

$(\forall x)(\exists y)(x^2+y^2 \geq 100)$

PART – B

1. a) Prove that $(P \rightarrow Q) \wedge (Q \rightarrow R) \rightarrow (P \rightarrow R)$

Proof:

Let S: $(P \rightarrow Q) \wedge (Q \rightarrow R) \rightarrow (P \rightarrow R)$

To prove: S is a tautology

P	Q	R	$(P \rightarrow Q)$	$(Q \rightarrow R)$	$(P \rightarrow R)$	$(P \rightarrow Q) \wedge (Q \rightarrow R)$	S
T	T	T	T	T	T	T	T
T	T	F	T	F	F	F	T
T	F	T	F	T	T	F	T
T	F	F	F	T	T	T	T
F	T	T	T	T	T	T	T
F	T	F	T	F	T	F	T
F	F	T	T	T	T	T	T
F	F	F	T	T	T	T	T

The last column shows that S is a tautology

2. Show that ¬ (p↔q) ≡ (p∨q) ⋀ ¬ (p∧q) without constructing the truth table

Solution :

¬ (p↔q) ≡ (p∨q) ⋀ ¬ (p∧q)

¬ (p↔q) ≡ ¬ (p→q) ⋀ (q→p)

≡ ¬ (¬ p∨q) ⋀ (¬ q∨p)

≡ ¬ (¬ p∨q) ⋀ ¬ q) ∨((¬ p∨q) ⋀p)

≡ ¬ (¬ p⋀ ¬ q) ∨(q⋀ ¬ q) ∨ ((¬ p⋀p) ∨(q⋀p)

≡ ¬ (¬ p∨q) ∨F∨F∨(q⋀p)

≡ ¬ (¬ p∨q) ∨(q⋀p)

≡(p∨q) ⋀ (q⋀p).

3. Obtain PCNF of (¬ p→ r) ∧(q↔ p). and hence obtain its PDNF.

Solution:

PCNF:

S⇔ (¬ p → r) ∧ (q ↔ p).

⇔ (¬ p→ r) ∧((q→ p). ∧ (p → q)

⇔ (p∨ r) ∧((¬ q∨ p). ∧ (¬ p ∨ q)

⇔ ((p∨ r) ∨F) ∧((¬ q∨ p).∨ F) ∧ ((¬ p ∨ q) ∨ F)

⇔ ((p ∨ r) ∨ (q ∧ ¬ q))∧((¬ q∨ p).∨ (r ∧ ¬ r)) ∧ ((¬ p ∨ q) ∨ (p ∧ ¬ p)) .

⇔ ((p ∨ r ∨ q)∧(p ∨ r ∨ ¬ q)) ∧ ((¬ q∨ p∨ r) ∧.(¬ q ∨ p ∨ ¬ r) ∧ ((¬ p ∨ q ∨ r) ∨ (¬ p ∨ q ∨ ¬ r)

⇔ ((p ∨ r ∨ q) ∧ ((¬ q∨ p∨ r) ∧.(¬ q ∨ p ∨ ¬ r) ∧ ((¬ p ∨ q ∨ r) ∨ (¬ p ∨ q ∨ ¬ r)

PCNF of S: ((p ∨ r ∨ q) ∧ ((¬ q∨ p∨ r) ∧.(¬ q ∨ p ∨ ¬ r) ∧ ((¬ p ∨ q ∨ r) ∨ (¬ p ∨ q ∨ ¬ r)

PCNF of ¬ S: (p∨q∨r) ∧ (¬ p ∨ ¬ q ∨ r|) ∧ (¬ p ∨ ¬ q ∨ ¬ r)

PDNF of S: (p∧q∧r) ∨ (¬ p ∧ ¬ q ∧ r) ∨ (¬ p ∨∧ ¬ q ∧ ¬ r).

4. Prove that $\sqrt{2}$ is irrational.

Solution :

Suppose $\sqrt{2}$ is irrational.

$\therefore \sqrt{2} = \frac{p}{q}$ for p,q $\in z, q \neq 0, p \& q$ have no common divisor.

$\therefore \frac{p^2}{q^2} = 2 \Rightarrow p^2 = 2q^2$.

Since p^2 is an even integer, p is an even integer.

$\therefore$ p= 2m for some integer m.

$\therefore\ (2m)^2 = 2q^2 \Rightarrow q^2 = 2m^2$

Since q^2 is an even integer, q is an even integer.

$\therefore$ q= 2k f or some integer k.

Thus p & q are even . Hence they have a common factor 2. Which is a contradiction to our assumption.

$\therefore \sqrt{2}$ is irrational.

5. Verify that validating of the following inference. If one person is more successful than another, then he has worked harder to deserve success. Ram has not worked harder than Siva. Therefore, Ram is not more successful than Siva.

Solution:

Let the universe consists of all persons.

Let S(x,y): x is more successful than y.

H(x,y): x has worked harder than y to deserve success.

a: Ram

b: Siva

Then, given set of premises are

1) (x) (y) [S(x,y) $\rightarrow$ H(x,y)]
2) $\neg$ H(a,b)
3) Conslution is $\neg$ S(a,b).

{1}	1) (x) (y) [S(x,y) $\rightarrow$ H(x,y)]	Rule P
{2}	2) (y) [S(a,y) $\rightarrow$ H(a,y)]	Rule US
{3}	3) [S(a,b) $\rightarrow$ H(a,b)]	Rule US
{4}	4) $\neg$ H(a,b)	Rule P
{5}	5) $\neg$ S(a,b)	Rule T ($\neg$ P, P $\rightarrow$ Q $\Rightarrow$ $\neg$ Q)

Unit – II

COMBINATORICS

PART - A

1. Pigeon Hole Principle:

If (n=1) pigeon occupies 'n' holes then atleast one hole has more than 1 pigeon.
Proof:
Assume (n+1) pigeon occupies 'n' holes.
Claim: Atleast one hole has more than one pigeon.
Suppose not, ie. Atleast one hole has not more than one pigeon.
Therefore, each and every hole has exactly one pigeon.
Since, there are 'n' holes, which implies, we have totally 'n' pigeon.
Which is a $\Rightarrow \Leftarrow$ to our assumption that there are (n+1) pigeon.
Therefore, atleast one hole has more than 1 pigeon.

2. Prove that $nP_r = (n-r+1)*nP_{r-1}$

Solution:

We know that $nP_r = \frac{n!}{(n-r)!}$

$$nP_{r-1} = \frac{n!}{[n-(r-1)]!}$$

But n! = n*(n-1)!

(n-r+1)! = (n-r+1) (n-r)!

Now (n-r+1) *nP_{r-1}

$= (n-r+1) * \frac{n!}{[n-(r-1)]!}$

$= \frac{(n-r+1)*n!}{[n-r+1)!} \quad = \frac{n!}{(n-r)!} \quad = nP_r$

3. In how many ways can letters of the word "INDIA" be arranged?

Solution:

The word contains 5 letters of which 2 are I's.

The number of words possible = $\frac{5!}{2!}$ = $\frac{5*4*3*2*1}{2*1}$ = 60

4. Use mathematical induction to show that $n! \geq 2^{n+1}$, n = 5,6,…..

Solution:

Let P(n): $n! \geq 2^{n+1}$, n = 5,6,…..

Assume P(5): $5! \geq 2^{5+1}$ is true

Assume P(k): $k! \geq 2^{k+1}$ is true

Claim: P(k+1) is true.

Using (1), we have, $k! \geq 2^{k+1}$

Multiply both sides by 2, we have $2\,k! \geq 2.\,2^{k+1}$ is true

$(k+1)\,k! \geq 2^{k+2}$

$(k+1)\,! \geq 2^{k+2}$

$\therefore$ P(k+1) is true

Hence by the principle of mathematical induction, $n! \geq 2^{n+1}$, for n = 5,6,…..

5. Find the value of n if $nP_3 = 5nP_2$

Solution:

$nP_3 = 5nP_2$

$n(n-1)(n-2) = 5\,n\,(n-1)$

$n-2 = 5$

$n=7$

6. How many ways are these to select five players from 10 member tennis team to make a trip to match to another school.

Solution:

5 members can be selected from 10 members in $10C_5$ ways.

Now, $10C_5 = \frac{10!}{5!5!}$ = 252 ways.

7. If the sequence $a_n = 3.2^n$, $n \geq 1$, then find the corresponding recurrence relation.

Solution:

For $n \geq 1$ $\quad a_n = 3.2^n$

Now, $\quad a_{n-1} = 3.2^{n-1} \quad = 3.\ 2^{n-1} \quad = 3.\ \dfrac{2^n}{2}$

$$a_{n-1} = \frac{a^n}{2} \quad \Rightarrow a_n = 2\,(a_{n-1})$$

$$\Rightarrow a_n = 2\,a_{n-1}, \text{ for } n \geq 1, \text{ with } a_0 = 3.$$

PART –B

1. Find the number of distinct permutation that can be formed. From all the letter of each word (1) RADAR (2) UNUSAL.

Solution :

(1) The word RADAR contains 5 letters of which 2 A's and 2 R's are there

The number of possible words $= \dfrac{5!}{2!2!}$

$$= \frac{120}{2*2} = 30$$

No. of distinct permutation = 30

The word UNUSAL. contains 7 letters of which 3 U's are there

(2) The number of possible words $= \dfrac{7!}{3!}$

$$=840$$

No. of distinct permutation = 840.

2. **Use mathematical Induction, prove that** $\dfrac{1}{\sqrt{1}} + \dfrac{1}{\sqrt{2}} + \dfrac{1}{\sqrt{3}} + \ldots + \dfrac{1}{\sqrt{n}} > \sqrt{n}$ **for n ≥ 2.**

Solution:

Let P(n): $\frac{1}{\sqrt{1}}+\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{3}}+\ldots.+\frac{1}{\sqrt{n}}>\sqrt{n}$, $n \geq 2$.

i) P(2): $\frac{1}{\sqrt{1}}+\frac{1}{\sqrt{2}}$ = (1.707) > $\sqrt{2}$ = (1.414) is true.

ii) P(k): $\frac{1}{\sqrt{1}}+\frac{1}{\sqrt{2}}+\ldots\ldots+\frac{1}{\sqrt{k}}>\sqrt{k}$ is true. ……….1

Claim: P(k+1): $\frac{1}{\sqrt{1}}+\frac{1}{\sqrt{2}}+\ldots\ldots+\frac{1}{\sqrt{k}}+\frac{1}{\sqrt{k+1}}$

$$= \sqrt{k}+\frac{1}{\sqrt{k+1}} = \frac{\sqrt{k.\sqrt{k+1}}+1}{\sqrt{k+1}}$$

$$= \sqrt{k+1}$$

P(k+1) > $\sqrt{k+1}$ is true.

By the principle of mathematical Induction, $\frac{1}{\sqrt{1}}+\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{3}}+\ldots.+\frac{1}{\sqrt{n}}>\sqrt{n}+1$.

3. Use mathematical Induction, prove that $\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$

Solution:

Let P(n): $1^2+2^2+3^2+\ldots\ldots+n^2 = \frac{n(n+1)(2n+1)}{6}$

i) P(1): $1^2 = \frac{1(1+1)(2.1+1)}{6}$ is true.

ii) Assume P(k): $1^2+2^2+3^2+\ldots\ldots+k^2 = \frac{k(k+1)(2k+1)}{6}$ is true.

Where k is any integer,

iii) Claim: P(k+1) is true.

Now, P(k+1): $1^2+2^2+3^2+\ldots\ldots+k^2+(k+1)^2$

$$= \frac{k(k+1)(2k+1)}{6}+(k+1)^2$$

$$= \frac{k(k+1)(2k+1)+6(k+1)^2}{6}$$

$$= \frac{(k+1)((k+1)+1)(2(k+1)+1)}{6}$$ P(k+1) is true.

By the principle of mathematical Induction, $\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$ is true for all 'n'

4. **Use mathematical Induction, prove that (3^n+7^n-2) is divisible by 8, for $n \geq 1$.**
Solution:

Let P(n): (3^n+7^n-2) is divisible by 8

i) P(1): (3^1+7^1-2) is divisible by 8, is true.
ii) Assume P(k): (3^k+7^k-2) is divisible by 8, is true. …….1

Claim: P(k+1) is true.

$P(k+1) = 3^{k+1}+7^{k+1}-2 \quad = 3.\ 3^k+7.7^k-2 \quad = 3.3^k+3.7^k+4-7^k-6+4$

$= 3\ (3^k+7^k-2) + 4(7^k+1)$ ……..2

$4(7^k+1)$ is divisible by 8. (7^k+1 is an even number, for $k \geq 1$)

$3(3^k+7^k-2)$ is divisible by 8 Using 1
$P(k+1) = 3(3^k+7^k-2) + 4(7^k+1)$ is divisible by 8 is true.
P(k+1) is true.

By the principle of mathematical Induction, P(n): (3^n+7^n-2) is divisible by 8

5. **Solve the recurrence relation $a_n = 2a_{n-1} – 2a_{n-2}$, $n\geq2$ and $a_0 = 1$ & $a_1 = 2$.**
Solution:
The recurrence relation can be rewritten as $a_n-2a_{n-1} + 2a_{n-2} = 0$.
The Characteristic equation is $n^2-2r+2 = 0$

Roots are $r = \frac{2 \pm 2i}{2} = 1 \pm i$

The modulus (amplitude) from of $1 \pm i = \sqrt{2}\,(\cos\frac{\pi}{4} \pm i\sin\frac{\pi}{4})$

Solution: $a_n = (\sqrt{2})^n\,(C_1.\cos\frac{n\pi}{4} + C_2.\sin\frac{n\pi}{4})$

Given $a_o = 1$, put n=0 in (A)

$a_o = (\sqrt{2})^0\,(C_1+0)$

we get $C_1 = 1$
Given $a_1 = 2$, put n=1 in (A)

$a_1 = (\sqrt{2})^1\,(C_1.\cos\frac{n\pi}{4} + C_2.\sin\frac{n\pi}{4})$

$2 = \sqrt{2}\,(C_1\,\frac{1}{\sqrt{2}} + C_2.\,\frac{1}{\sqrt{2}})$