Example 2. Show that the inclusion relation $\subseteq$ is a partial ordering on the power set of a set S.

Solution : Since $A \subseteq A$ whenever A is a subset of S, $\subseteq$ is reflexive. It is antisymmetric since $A \subseteq B$ and $B \subseteq A$ imply that $A = B$. Finally $\subseteq$ is transitive, since $A \subseteq B$ and $B \subseteq C$ imply that $A \subseteq C$. Hence, $\subseteq$ is a partial ordering on P(S), and $(P(S), \subseteq)$ is a poset.

Example 3. Let R be a binary relation on the set of all positive integers such that R = {(a, b)/a = b²)}. Is R reflexive ? Symmetric ? Antisymmetric ? Transitive ? An equivalence relation ? A partial ordering relation ? [MCA, MU, Nov. 1990, Dec. 1992]

Solution : $R = \{(a, b)/a, b$ are positive integers and $a = b^2\}$. For R to be reflexive, we should have a R a for all positive integers a. But aRa holds only when $a = a^2$ by hypothesis. Now $a = a^2$ is not true for all positive integers. Infact only for the positive integer $a = 1$, we have $a = a^2$. Hence R is not reflexive.

For R to be symmetric, if aRb then we should have bRa. But aRb implies $a = b^2$. But $a = b^2$ does not imply $b = a^2$ always for positive integers. For instance $16 = 4^2$ but $4 \neq 16^2$. Hence aRb does not imply bRa. Hence R is not symmetric.

For R to be anti-symmetric, for positive integers a, b if a R, b and $b\,R\,a$ hold, then $a = b$. aRb implies $a = b^2$ and bRa implies $b = a^2$, So if $a = b^2$ and $b = a^2$, then $a = b^2 = (a^2)^2 = a^4$ i.e, $a^4 - a = 0$, i.e., $a\,(a^3 - 1) = 0$. Since a is not a positive integer, $a \neq 0$ so that $a^3 - 1 = 0$ i.e., $a^3 = 1$ i.e., $a = 1$. This means $b = a^2 = 1$. Thus aRb and bRa imply $a = b = 1$. Hence R is anti-symmetric.

For R to be transitive, if aRb holds and bRc holds, the aRc should hold.

i.e., aRb implies $a = b^2$ and bRc implies $b = c^2$.

So that $a = b^2 = c^4$. Hence aRc does not hold.

For example, $256 = 16^2$ and $16 = 4^2$ but $256 \neq 4^2$ (in fact $256 = 4^4$). Thus R is not transitive.

Also, R is not an equivalence relation as an equivalence relation is reflexive, symmetric and transitive. R is also not a partial ordering relation, as a partial ordering relation is reflexive, anti-symmetric and transitive.

Example 4. Let X = {2, 3, 6, 12, 24, 36} and the relation $\leq$ be such that x $\leq$ y if x divides y. Draw the Hasse diagram of (x, $\leq$)

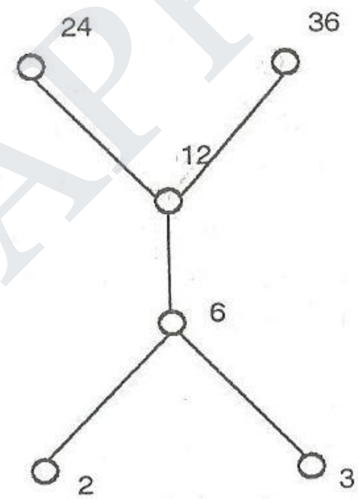
Solution :

The relation

$R = \{(x, y) / x \mid y\}, x \leq y$

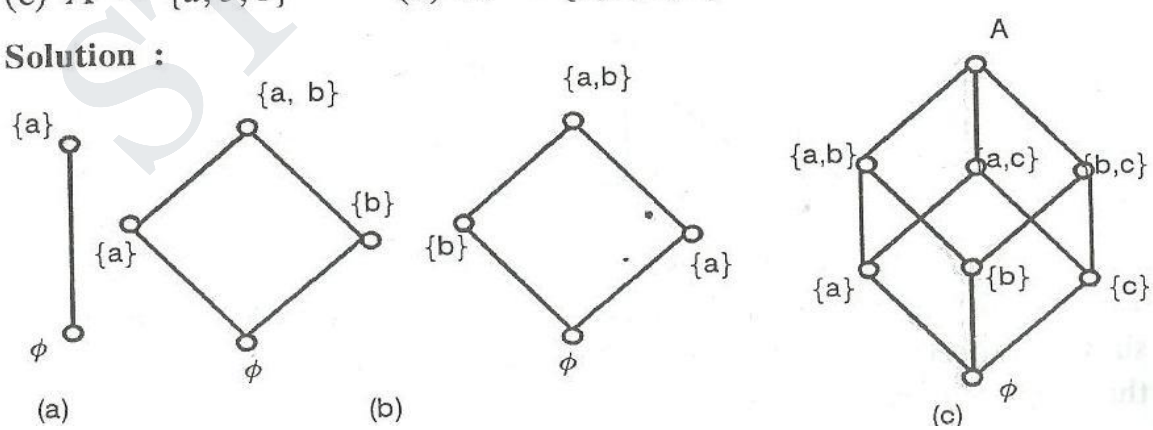
$= \{(2, 6), (2, 12), (2, 24), (2, 36),$
$(3, 6), (3, 12), (3, 24), (3, 36),$
$(6, 12), (6, 24), (6, 36)$
$(12, 24), (12, 36)\}$

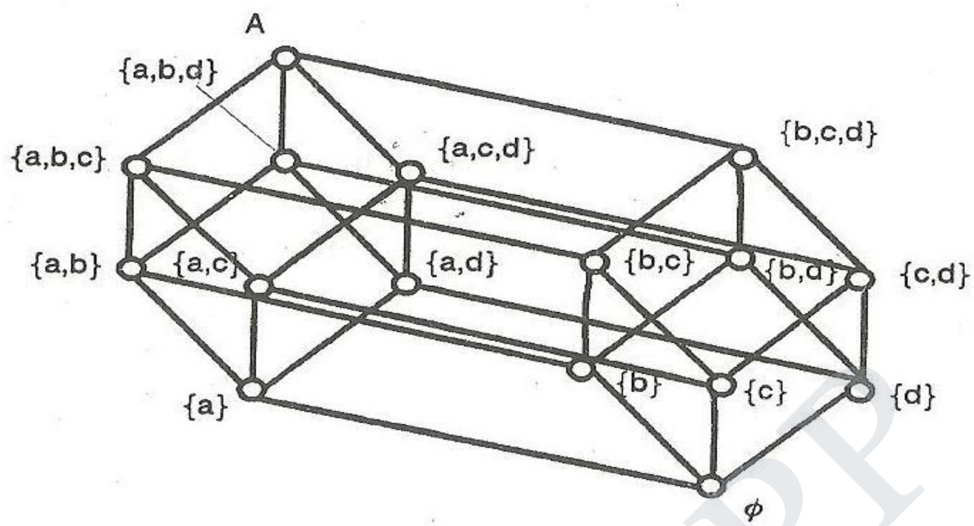
The Hasse diagram is



Example 5. Let A be a given finite set and P(A) its power set. Let $\subseteq$ be the inclusion relation on the elements of $\rho(A)$. Draw Hasse diagram of $P(A), \subseteq)$ for (a) A = $\{a\}$ (b) A = $\{a, b\}$
(c) A = $\{a, b, c\}$ (d) A = $\{a, b, c, d\}$

Solution :



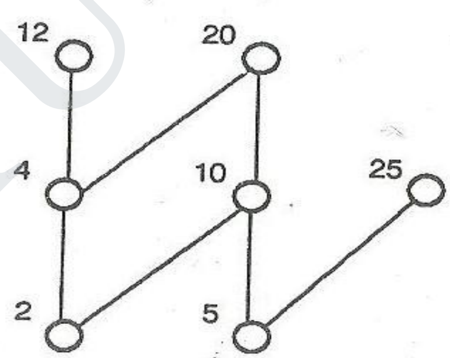


Example 6. Give a relation which is both a partially ordering relation and an equivalence relation on a set.

Solution : Equality, similarity of triangles are the examples of relation which are both a partial ordering relation and an equivalence relation.

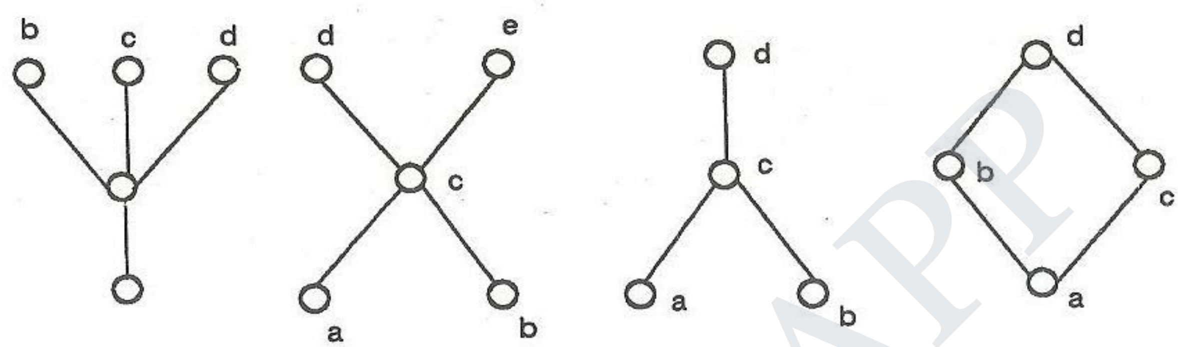
Example 7. Which elements of the poset {{2, 4, 5, 10, 12, 20, 25}, |} are maximal, and which are minimal?

Solution : Draw Hasse diagram



From the figure this poset shows that the maximal elements are 12, 20 and 25 and the minimal elements are 2 and 5. As this example shows, a poset can have more than one maximal element and more than one minimal element.

Example 8. Determine whether the posets represents by each of the Hasse diagrams in the following figure, have a greatest element and a least element.



(a) (b) (c) (d)

Solution : The least element of the poset with Hasse diagram (a) is a. This poset has no greatest element. The poset with Hasse diagram (b) has neither a least not a greatest element. The poset with Hasse diagram (c) has no least element. Its greatest element is *d*. The poset with Hasse diagram (d) has least element *a* and greatest element *d*.

Example 9. Let S be a set. Determine whether there is a greatest element and a least element in the poset (P(S), $\subseteq$).

Solution : The least element is the empty set since $\phi \subseteq T$ for any subset T of S. The set S is the greatest element in this poset. Since $T \subseteq S$ whenever T is a subset of S.

Example 10. Is there a greatest element and a least element in the poset $(Z^+, 1)$?

Solution : The integer 1 is the least element since 1/n whenever n is a positive integer. Since then is no integer that is divisible by all positive integers, there is no greatest element.

5.2 Properties of Lattices - Lattices as Algebraic Systems - Sublattices - Direct Product and Homomorphism - Some Special Lattices

In order to emphasize the role of an ordering relation, a lattice is first introduced as a partially ordered set. Both lattices and Boolean algebra have important applications in the theory and design of computers. There are many other areas such as engineering and science to which Boolean algebra is applied.

Def. Lattice

A lattice in a partially ordered set (L, $\leq$) in which every pair of elements $a, b \in$ L has a greatest lower bound and a least upper bound.

Def. Greatest Lower Bound (GLB) and Least Upper Bound (LUB)

The GLB of a subset $\{a, b\} \subseteq$ L will be denoted by $a * b$ and the least upper bound by $a \oplus b$

i.e., GLB $\{a, b\} = a * b$ (meet or product of a and b)

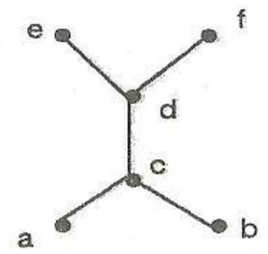
LUB $\{a, b\} = a \oplus b$ (join or sum of a and b)

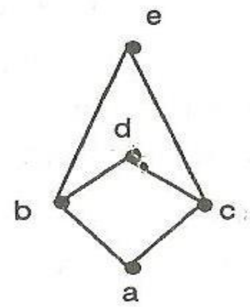
Note : 1. From the definition of a lattice that both $*$ and $\oplus$ are binary operations on L because of the uniqueness of the LUB and GLB of any subset of a poset.

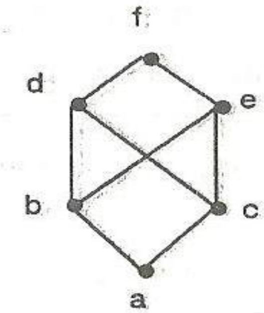
2. If is obvious that, a totally ordered set is trivially a lattice, but not all partially ordered sets are lattices, can be concluded from Hasse diagrams of posets.

Remark : GLB, LUB may or may-not exist for a subset.

Solution : Given :







(i) Doesnot represent a lattice, because $e \oplus f$ does not exists.

(ii) Does not represent a lattice, because $b \oplus c$ does not exits.

(iii) Does not represent a lattice, because neither $d \oplus c$ not $b * c$ exists.

Example 7.

Let the sets $S_0, S_1, ..., S_7$ be given by

$S_0 = \{a, b, c, d, e, f\}$, $S_1 = \{a, b, c, d, e\}$

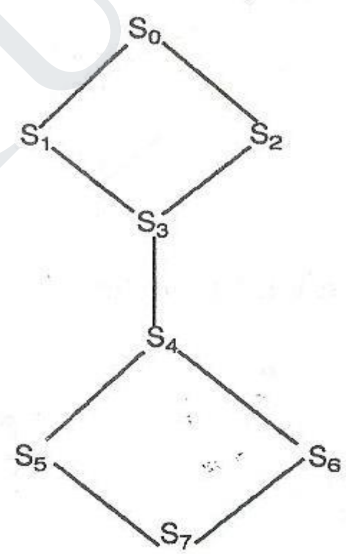
$S_2 = \{a, b, c, d, e, f\}$, $S_3 = \{a, b, c, e\}$

$S_4 = \{a, b, c\}$ $S_5 = \{a, b\}$, $S_6 = \{a, c\}$

$S_7 = \{a\}$

Draw the diagram of $(L, \subseteq)$ where $L = \{S_0, S_1, S_2, ..., S_7\}$

Solution :



Some properties of Lattices

PROPERTY 1 : Let (L, $\leq$) be a lattice. For any $a, b, c \in$ L we have $a * a = a$ and $a \oplus a = a$ [Idempotent law]

Proof: Let $a, b, c \in$ L, by the definition of GLB of a and b we have

$$a * b \leq a \qquad \text{... (i)}$$

and if $a \leq a$ and $a \leq b$, then

$$a \leq a * b \qquad \text{... (ii)}$$

As $a \leq a$ from (i) and (ii) we have

$$a * a \leq a \text{ and } a \leq a * a$$

By the antisymmetric property if follows that $a = a * a$

Similarly we can prove that $a \oplus a = a$

PROPERTY 2. Show that the operation of meet are join on a lattice are associative.

Solution : To prove : $(a * b) * c = a * (b * c)$

Let $a, b, c \in$ L by the definition we have

$$(a * b) * c \leq a * b$$

and $\quad (a * b) * c \leq c$

By the definition of GLB of a and b, we have $a * b \leq a$ and $a * b \leq b$, so by transitive property of $\leq$ we have

$$(a * b) * c \leq a$$

and $\quad (a * b) * c \leq b$

As $\quad (a * b) * c \leq b$ and $(a * b) * c \leq c$

We see that $(a * b) * c$ is lower bound for b and c. From the definition of $b * c$ it follows that $(a * b) * c \leq b * c$

As $\quad (a * b) * c \leq a$ and $(a * b) * c \leq b * c$

From the definition of $a * (b * c)$, we have

$$(a * b) * c \leq a * (b * c) \qquad \text{... (i)}$$

Now $\quad a * (b * c) \leq a$ and $a * (b * c) \leq b * c$

As $\quad b * c \leq b$, by tansitivity $a * (b * c) \leq b$

since $\quad a * (b * c) \leq a$, and $a * (b * c) \leq b$

We have $a * (b * c) \leq (a * b)$

As $\quad a * (b * c) \leq b * c \leq c$

$$a * (b * c) \leq (a * b) * c \qquad \text{(ii)}$$

From (i) and (ii), by antisymmetric property, if follows that

$$a * (b * c) = (a * b) * c$$

Similarly, we can prove that $a \oplus (b \oplus c) = (a \oplus b) \oplus c$

PROPERTY 3. Show that the operation of meet and join on a lattice are commutative law. i.e., $a * b = b * a$ and $a \oplus b = b \oplus a$

Solution : Given $a, b \in$ L both $a * b$ and $b * a$ are GLB of a and b. By the uniqueness of GLB of a and b, we have $a * b = b * a$. Similarly $a \oplus b = b \oplus a$ holds good.

PROPERTY 4. Absorption law $a * (a \oplus b) = a$ and $a \oplus (a * b) = a$

Solution : Let $a, b \in$ L. Then $a \leq a$ and $a \leq a \oplus b$. So $a \leq a * (a \oplus b)$. On the other hand $a * (a \oplus b) \leq a$. By annisymmetric property of $\leq$ we have $a = a * (a \oplus b)$

Similarly we have $a = a \oplus (a * b)\ \forall\ a, b \in$ L

THEOREMS

Theorem 1.

Let (L, $\leq$) be a lattice in which $*$ and $\oplus$ denotes the operations of meet and join respectively. For any $a, b \in$ L.

$$\mathbf{a \leq b \Leftrightarrow a * b = a \Leftrightarrow a \oplus b = b}$$

Proof : First let us prove that $a \leq b \Leftrightarrow a * b = a$

Let us assume that $a \leq b$ and also we know that $a \leq a$.

$$\therefore\ a \leq a * b \qquad \dots (1)$$

But, from definition of $a * b$, we have

$$a * b \leq a \qquad \dots (2)$$

Hence $a \leq b \Rightarrow a * b = a$ [From (1) and (2)]

Next, assume that $a * b = a$, but it is only possible if $a \leq b$.

That is $a * b = a \Rightarrow a \leq b$

Combining these two results, we get

$$a \leq b \Leftrightarrow a * b = a$$

To show that $a \leq b \Leftrightarrow a \oplus b = b$ in a similar way.

From $a * b = a$, We have

$$b \oplus (a + b) = b \oplus a = a \oplus b$$

But $\quad b \oplus (a * b) = b$

Hence $\quad a \oplus b = b$ follows that $a * b = a$

Theorem 2.

Let (L, $\leq$) be a lattice. For any $a, b \in$ L the following are equivalent.
(i) a $\leq$ b, (ii) a $*$ b = a, (iii) a $\oplus$ b = b

Proof : At first, consider (i) $\Leftrightarrow$ (ii)

We have $a \leq a$, assume $a \leq b$. Therefore $a \leq a * b$. By the definition of GLB, we have

$$a * b \leq a$$

Hence by antisymmetric property, $a * b = a$

Assume that $a * b = a$, but is only possible if

$$a \leq b \Rightarrow a * b = a \Rightarrow a \leq b.$$

Combining these two results, we have $a \leq b \Leftrightarrow a * b = a$

Similarly, $a \leq b \Leftrightarrow a \oplus b = b$

Alternatively, (ii) $\Leftrightarrow$ (iii) as follows :

Assume $a * b = a$, we have $b \oplus (a * b) = b \oplus a = a \oplus b$, but by absorption $b \oplus (a * b) = b$. Hence $a \oplus b = b$.

But, from definition of $a * b$, we have

$$a * b \leq a \qquad \dots (2)$$

Hence $a \leq b \Rightarrow a * b = a$ [From (1) and (2)]

Next, assume that $a * b = a$, but it is only possible if $a \leq b$.

That is $a * b = a \Rightarrow a \leq b$

Combining these two results, we get

$$a \leq b \Leftrightarrow a * b = a$$

To show that $a \leq b \Leftrightarrow a \oplus b = b$ in a similar way.

From $a * b = a$, We have

$$b \oplus (a + b) = b \oplus a = a \oplus b$$

But $\quad b \oplus (a * b) = b$

Hence $\quad a \oplus b = b$ follows that $a * b = a$

Theorem 2.

Let (L, $\leq$) be a lattice. For any $a, b \in$ L the following are equivalent.
(i) a $\leq$ b, (ii) a $*$ b = a, (iii) a $\oplus$ b = b

Proof : At first, consider (i) $\Leftrightarrow$ (ii)

We have $a \leq a$, assume $a \leq b$. Therefore $a \leq a * b$. By the definition of GLB, we have

$$a * b \leq a$$

Hence by antisymmetric property, $a * b = a$

Assume that $a * b = a$, but is only possible if

$$a \leq b \Rightarrow a * b = a \Rightarrow a \leq b.$$

Combining these two results, we have $a \leq b \Leftrightarrow a * b = a$

Similarly, $a \leq b \Leftrightarrow a \oplus b = b$

Alternatively, (ii) $\Leftrightarrow$ (iii) as follows :

Assume $a * b = a$, we have $b \oplus (a * b) = b \oplus a = a \oplus b$, but by absorption $b \oplus (a * b) = b$. Hence $a \oplus b = b$.

which is inequality (i) in 2.

Hence the theorem.

Theorem 4.

In a lattice (L, $\leq$), show that (i) $(a * b) \oplus (c * d) \leq (a \oplus c) * (b \oplus d)$
(ii) $(a * b) \oplus (b * c) \oplus (c * a) \leq (a \oplus b) * (b \oplus c) * (c \oplus a)$, $\forall a, b, c \in L$

Proof : Let $a, b, c \in L$

Then $\quad a * b \leq a$ (or) $b \leq a \; a \oplus b \qquad \dots (1)$

$a * b \leq a \leq c \oplus a \qquad \dots (2)$

$a * b \leq b \leq b \oplus c \qquad \dots (3)$

Using (1), (2) and (3), we get

$$a * b \leq (a \oplus b) * (b \oplus c) * (c \oplus a)$$

Similarly $b * c \leq (a \oplus b) * (b \oplus c) * (c \oplus a)$

$$c * a \leq (a \oplus b) * (b \oplus c) * (c \oplus a)$$

This proves (ii)

We have $a \leq a \oplus c$

$$b \leq b \oplus d$$

$$\therefore \quad (a * b) \leq (a \oplus c) * (b \oplus d)$$

We know that $\quad c \leq a \oplus c \qquad \dots (4)$

$d \leq b \oplus d \qquad \dots (5)$

$$\therefore \quad c * d \leq (a \oplus c) * (b \oplus d)$$

By (4) and (5), $(a * b) \oplus (c * d) \leq (a \oplus c) * (b \oplus d)$. This proves (i)

Theorem 5.

In a lattice (L, $\leq$), prove that for $a, b, c \in L$
(i) $(a * b) \oplus (a * c) \leq a * (b \oplus (a * c))$
(ii) $(a \oplus b) * (a \oplus c) \geq a \oplus (b * (a \oplus c))$

Proof : We know that $a * b \leq a, a * c \leq a$

$\therefore\ (a * b) \oplus (a * c) \leq a \oplus a \ = \ a$ ... (1)

Also $a * b \leq b,\ a * c \leq a * c$

$\Rightarrow\ (a * b) \oplus (a * c) \leq b \oplus (a * c)$ ... (2)

From (1) and (2), $(a * b) \oplus (a * c) \leq a * (b \oplus (a * c))$

This proves (i)

We know that $a \leq a \oplus b\ ;\ a \leq a \oplus c$

$$\Rightarrow a = a * a \leq (a \oplus b) * (a \oplus c)$$

Further $b \leq a \oplus b;\ a \oplus c \leq a \oplus c$

$$\Rightarrow b * (a \oplus c) \leq (a \oplus b) * (a \oplus c)$$

By (3) & (4), $a \oplus (b * (a \oplus c)) \leq (a \oplus b) * (a \oplus c)$

This proves (ii)

Theorem 6.

In a lattice if $a \leq b \leq c$, show that
(i) $a \oplus b = b * c$ (ii) $(a * b) \oplus (b * c) = (a \oplus b) * (a \oplus c) = b$

Proof : Let $a \leq b \leq c$

$$a \leq b \Rightarrow a \oplus b = b\ ,\ a * b = a$$

$$b \leq c \Rightarrow b \oplus d = c\ ,\ b * c = b$$

$$a \leq c \Rightarrow a \oplus c = c\ ,\ a * c = a$$

$\therefore\ a \oplus b = b = b * c$ (i) follows

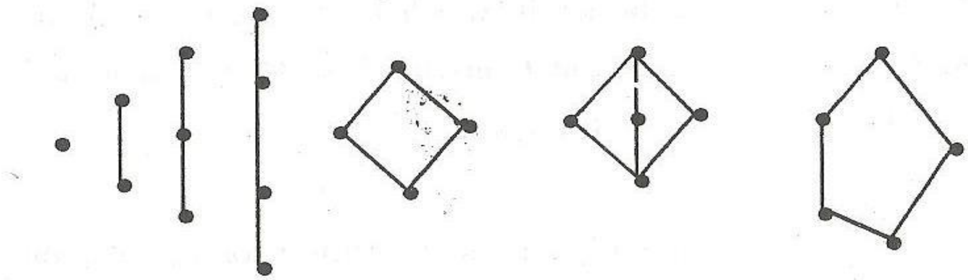
Now $(a * b) \oplus (b * c) \ = \ a \oplus b = b$

$(a \oplus b) * (a \oplus c) = b * c = b$ (ii) follows

Theorem 7.

Draw Hasse diagram of all lattices with upto five elements.

Solution : The following Hasse diagrams are the lattice with 1, 2, 3, 4 and 5 elements.



Note :

Terminology

In logic notation	In set theory notation	Computer Designer's notation	Read as
$\vee$	$\cup$	$\oplus$	joint 'or' sum
$\wedge$	$\cap$	$*$	'Meet' and product
$\neg$	$\subset$	$-$, $'$	Complement
$\leq$	$\subseteq$	$\leq$	partially ordered set.

Def. A lattice is an algebraic system $(L, *, \oplus)$ with two binary operations $*$ and $\oplus$ on L which are both (1) commutative and (2) associative and (3) satisfy the absorption laws.

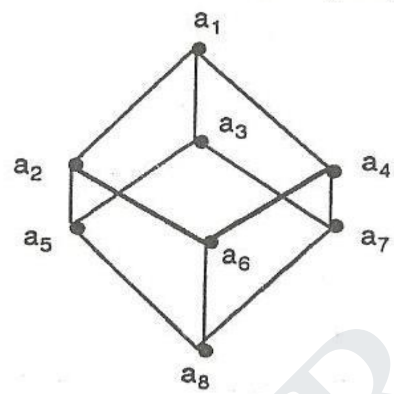
Def. Sublattice

Let $(L, *, \oplus)$ be a lattice and let $S \leq L$ be a subset of L. The algebra $(S, *, \oplus)$ is a sublattice of $(L, *, \oplus)$ iff S is closed under both operations $*$ and $\oplus$

Example 1. Let (L, $\leq$) be a lattice in which L = $\{a_1, a_2, \ldots a_8\}$ and S_1, S_2 and S_3 be the sublattices of L given by $S_1 = \{a_1, a_2, a_4, a_6\}$, $S_2 = \{a_3, a_5, a_7, a_8\}$ and $S_3 = \{a_1, a_2, a_4, a_8\}$

The diagram of (L, $\leq$) is

Observe that (S_1, $\leq$) and (S_2 $\leq$) are sublattices of (L, $\leq$), but (S_3, $\leq$) is not a sublattices because $a_2, a_4 \in S_3$ but $a_2 * a_4 = a_6 \notin S_3$.



Note that (S_3, $\leq$) is a lattice.

Def.: Let (L, $*$, $\oplus$) and (S, $\wedge$, $\vee$) be two lattices. The algebraic system (L $\times$ S, $\cdot$, $+$) in which the binary operations + and $\bullet$ on L $\times$ S are such that for any (a_1, b_1) and (a_2, b_2) in L $\times$ S

$$(a_1, b_1) \cdot (a_2, b_2) = (a_1 * a_2, b_1 \wedge b_2)$$

$$(a_1, b_1) + (a_2, b_2) = (a_1 \oplus a_2, b_1 \vee b_2)$$

is called the direct product of the lattices (L, $*$, $\oplus$) and (S, $\wedge$, $\vee$)

Def. Lattice homomorphism

Let (L, $*$, $\oplus$) and (S, $\wedge$, $\vee$) be two lattices. A mapping $g : L \to S$ is called a lattice homomorphism from the lattice (L, $*$, $\oplus$) to (S, $\wedge$, $\vee$) if for any $a, b \in L$

$$g(a * b) = g(a) \wedge g(b) \text{ and } g(a \oplus b) = g(a) \vee g(b).$$

Note : Observe that both the operations of meet and join are preserved. These may be mappings which preserve only one of the two operations. Such mappings are not lattice homomorphisms.

Def. A lattice is called complete if each of its non empty subsets has a least upper bound and a greatest lower bound.

Def. In a bounded lattice $(L, *, \oplus, 0, 1)$ an element $b \in L$ is called a complement of an element $a \in L$ if

$$a * b = 0 \text{ and } a \oplus b = 1$$

Def. A lattice $(L, *, \oplus, 0, 1)$ is said to be a complemented lattice if every element of L has atleast one complement.

Def. A lattice $(L, *, \oplus)$ is called a distributive lattice if for any $a, b, c \in L$

$$a * (b \oplus c) = (a * b) \oplus (a * c)$$

$$a \oplus (b * c) = (a \oplus b) * (a \oplus c)$$

In other words, in a distributive lattice the operations $*$ and $\oplus$ distribute over each other.

Def. Modular

A lattice $(L, \wedge, \vee)$ is called modular if for all $x, y, z \in L$, $x \leq z$ $\Rightarrow x \vee (y \wedge z) = (x \vee y) \wedge z$ (modular equations).

Note : We have (by modular inequality) if $x \leq z \Rightarrow x \vee (y \wedge z) \leq (x \vee y) \wedge z$ holds in any lattice. Therefore to show that a lattice L is modular it is enough to show if,

$$x \leq z \Rightarrow x \vee (y \wedge z) \geq (x \vee y) \wedge z \text{ holds in L.}$$

THEOREMS

Theorem 1.

Every chain is a distributive lattice. **[A.U M/J 2013]**

Proof : Let $(L, \leq)$ be a chain

Let $a, b, c \in L$

Consider the following possible cases

(i) $a \leq b$ or $a \leq c$ and

(ii) $a \geq b$ and $a \geq c$

2. If a homomorphism $g : L \rightarrow S$ of two lattices $(L, *, \oplus)$ and $(S, \wedge, \vee)$ is bijective i.e., one-to-one onto, then g is called an isomorphism. If there exists an isomorphism between two lattices then the lattices are called isomorphic.

3. A homomorphism $g : L \rightarrow L$ where $(L, *, \oplus)$ is a lattice is called an endomorphism.

If $g : L \rightarrow L$ is an isomorphism, then g is called an automorphism.

4. If $g : L \rightarrow L$ is an endomorphism, then the image set of g is a sublattice of L.

Def. Let $(P, \leq)$ and $(Q, \leq')$ be two partially ordered sets. A mapping $f : P \rightarrow Q$ is said to be order-preserving relative to the ordering $\leq$ in P and $\leq'$ in Q iff for any $a, b \in P$ such that $a \leq b$, $f(a) \leq' f(b)$ in Q.

Note : If $(P, \leq)$ and $(Q, \leq')$ are lattices and $g : P \rightarrow Q$ is a lattice homomorphism, then g is order-preserving.

Def. Two partially ordered sets $(P, \leq)$ and $(Q, \leq')$ are called order-isomorphic if there exists a mapping $f : P \rightarrow Q$ which is bijective and if both f and f^{-1} are order-preserving.

Def. Let $(L, *, \oplus)$ be a lattice and $S \subseteq L$ be a finite subset of L where $S = \{a_1, a_2, \dots a_n\}$. The greatest lower bound and the least upper bound of S can be expressed as

$$\text{GLB } S = \mathop{*}_{i=1}^{n} a_i \quad \text{and} \quad \text{LUB } S = \bigoplus_{i=1}^{n} a_i \quad \dots (1)$$

where $\mathop{*}_{i=1}^{2} a_i = a_1 * a_2$ and $\mathop{*}_{i=1}^{k} a_i = \mathop{*}_{i=1}^{k-1} a_i * a_k$, $k = 3,4,\dots$

A similar representation can be given for $\bigoplus_{i=1}^{n} a_i$. Because of the associativity of the operations $*$ and $\oplus$, we can write

$$\mathop{*}_{i=1}^{n} a_i = a_1 * a_2 * \dots * a_n \quad \text{and}$$

$$\bigoplus_{i=1}^{n} a_i = a_1 \oplus a_2 \oplus \dots \oplus a_n$$

We shall now show that the distributive law

$$a * (b \oplus c) = (a * b) \oplus (a * c)$$

In case (i) $a \leq b$ or $a \leq c$ then we have $a * b = a$

$a \oplus a = a$, $a * c = a$ and

$$\Rightarrow a \leq b \oplus c$$

So $a * (b \oplus c) = a$ ... (1)

and $(a * b) \oplus (a * c) = a \oplus a = a$ ... (2)

(1) + (2) we get

$$a * (b \oplus c) = (a * b) \oplus (a * c)$$

In case (ii)

If $a \geq b$ and $a \geq c$ then we have $a * b = b$, $a * c = c$ and $b \oplus c \leq a$

So that $a * (b \oplus c) = b \oplus c$ ... (3) and

$(a * b) \oplus (a * c) = b \oplus c$ ... (4)

From (3) and (4) we get

$$a * (b \oplus c) = (a * b) \oplus (a * c)$$

Theorem 2.

Let (L, *, ⊕) be a distributive lattice. **[A.U. N/D 2004]**

For any a, b, c ∈ L

$$(\mathbf{a * b = a * c}) \wedge (\mathbf{a \oplus b = a \oplus c}) \Rightarrow \mathbf{b = c}$$

Proof : $(a * b) \oplus c = (a * c) \oplus c = c$ ... (1)

$$\begin{aligned}(a * b) \oplus c &= (a \oplus c) * (b \oplus c)\\ &= (a \oplus b) * (b \oplus c)\\ &= b \oplus (a * c)\\ &= b \oplus (a * b)\\ &= b \quad \text{... (2)}\end{aligned}$$

From (1) and (2) $b = c$

Theorem 3.

Every distributive lattice is modular. **[A.U. N/D 2004]**

Proof : Let (L, $\leq$) be a distributive lattice

For all $a, b, c \in$ L, we have

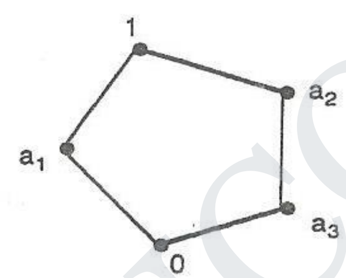
$$a \oplus (b * c) = (a \oplus b) * (a \oplus c)$$

Thus if $a \leq c$, then $a \oplus c = c$ and

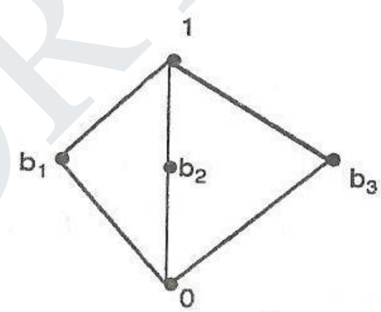
$$a \oplus (b * c) = (a \oplus b) * c$$

So if $a \leq c$, the modular equation is satisfied and L is modular.

Example 1. Show that the Lattices given by the diagrams are not distributive.



(A)



(B)

Solution : In lattice (A),

$$a_3 * (a_1 \oplus a_2) = a_3 * 1 = a_3 = (a_3 * a_1) \oplus (a_3 * a_2)$$

$$a_1 * (a_2 \oplus a_3) = 0 = (a_1 * a_2) \oplus (a_1 * a_3)$$

but $$a_2 * (a_1 \oplus a_3) = a_2 * 1 = a_2$$

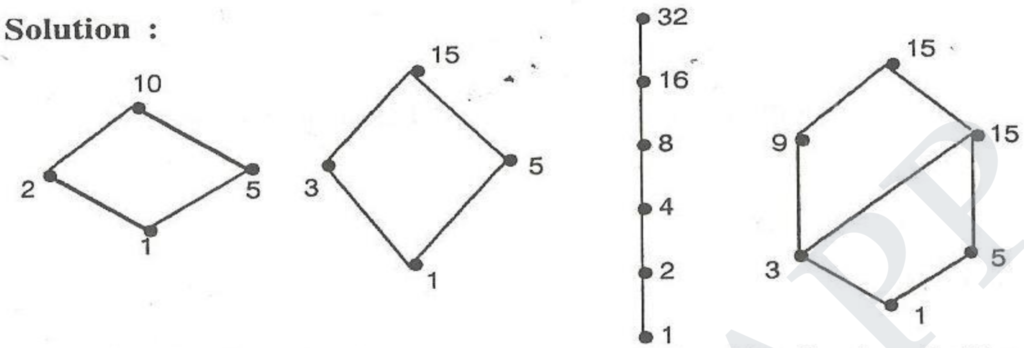
$$(a_2 * a_1) \oplus (a_2 * a_3) = 0 \oplus a_3 = a_3$$

Hence the Lattice (A) is not distributive.

In (B), $b_1 * (b_2 \oplus b_3) = b_1$ while $(b_1 * b_2) \oplus (b_1 * b_3) = 0$ which shows that the Lattice is not distributive.

Example 2. If D(n) denotes the lattice of all the divisors of the integer n draw the Hasse diagrams of D(10), D(15), D(32) and D(45)

Solution :



Example 3. Let L be a complemented, distributive lattice. For a, b $\in$ L the following are equivalent.

(i) a $\leq$ b (ii) a $*$ b$'$ = 0 (iii) a$'$ $\oplus$ b = 1
(iv) b$'$ $\leq$ a$'$ where '1' denotes corresponding complement.

(OR)

Show that in a distributive and complemented lattice.

$$a \leq b \Leftrightarrow a * b' = 0 \Leftrightarrow a' \oplus b = 1 \Leftrightarrow b' \leq a'$$

Solution : $a \leq b \Rightarrow a \oplus b = b$ [A.U. N/D 2004] [A.U M/J 2013]

$\Rightarrow (a \oplus b) * b' = 0$ as $b * b' = 0$

$\Rightarrow (a * b') \vee (b * b') = 0$

$\Rightarrow a * b' = 0$ as $b * b' = 0$

Hence (i) $\Rightarrow$ (ii)

$a * b' = 0$

$\Rightarrow (a * b') = 1$

$\Rightarrow a' \oplus (b') = 1$

$\Rightarrow a' \oplus b = 1$

Hence (ii) $\Rightarrow$ (iii)

$a' \oplus b = 1 \Rightarrow (a' \oplus b) * b' = b'$

$\Rightarrow (a' * b') \oplus (b * b') = b'$ (distributive law)

$\Rightarrow a' * b' = b'$ as $b * b' = 0$

$\Rightarrow b' \leq a'$

Hence (iii) $\Rightarrow$ (iv)

$$b' \leq a' \Rightarrow a' * b' = b'$$

$$\Rightarrow a \oplus b = b$$

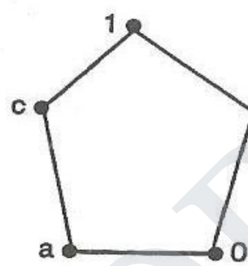
(taking complement on both sides by Demorgan's law)

$$\Rightarrow a \leq b$$

Hence (iv) $\Rightarrow$ (i)

Thus (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) $\Rightarrow$ (iv) $\Rightarrow$ (i)

Example 4. Prove that the following lattice is not modular.



Solution : For this lattice when $a \leq c$

$$a \oplus (b * c) \neq (a \oplus b) * c$$

Since $\quad a \oplus (b * c) = a \oplus 0 = a$

but $\quad (a \oplus b) * c = 1 * c = c$

$\therefore$ it is not a modular lattice.

Def. Enumeration : A one-to-one correspondence with the elements of a set is called an enumeration.

Example 5. Theorem : State and prove Isotonicity property in a lattice.

Proof : Let $(L, \leq)$ be a lattice. For $a, b, c \in L$, the following properties called isotonicity laws.

$$b \leq c \Rightarrow a * b \leq a * c \; ; \; a \oplus b \leq a \oplus c$$

(i.e.,) $\quad b \leq c \Rightarrow a \wedge b \leq a \wedge c \; ; \; a \vee b \leq a \vee c$

Let us assume that $b \leq c$