Andy's math/science background information

J. Andrew Fingerhut (andy.fingerhut@gmail.com)

June 1, 2025

Contents

1 Introduction

This document is a place to write up notes on background in math that help in learning science.

2 Notation

$\mathbb{R}^3$ - The set of all points in 3-dimensional space, i.e. where each point is specified by a sequence of 3 real-valued coordinates.

∇ - Called "nabla", and often called "del". Some history of it can be found on the Nabla Symbol page [3].

$\boldsymbol{\nabla} f$ - Gradient of a scalar function $f : \mathbb{R}^3 \to \mathbb{R}$.

grad$f$ - another way to write the gradient of f, $\boldsymbol{\nabla} f$

TODO: Any 3Blue1Brown-quality YouTube videos showing examples and definition of gradient, divergence, and curl? If so, give links to them below in appropriate sections.

2.1 Maxwell's Equations

In partial differentiatal form:

$$\boldsymbol{\nabla} \cdot \mathbf{E} = \frac{\rho}{\epsilon_0} \qquad \text{Gauss's Law} \tag{1}$$

$$\boldsymbol{\nabla} \cdot \mathbf{B} = 0 \qquad \text{Gauss's Law for magnetism} \tag{2}$$

$$\boldsymbol{\nabla} \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \qquad \text{Maxwell-Faraday Equation (Faraday's law of induction)} \tag{3}$$

$$\boldsymbol{\nabla} \times \mathbf{B} = \mu_0(\mathbf{J} + \epsilon_0 \frac{\partial \mathbf{E}}{\partial t}) \qquad \text{Ampère-Maxwell law} \tag{4}$$

where:

- $\mathbf{E}$ is the electric field
- $\mathbf{B}$ is the magnetic field
- ρ is the electric charge density
- $\mathbf{J}$ is the current density

- ϵ_0 is the vacuum permittivity
- μ_0 is the vacuum permeability

Statement of the equations in English:

Gauss's Law (equation (1)) says that electric fields are "produced" radially outward from volumes with positive electric charge, and "consumed" radially into volumes with negative electric charge.

Gauss's Law for Magnetism (equation (**??**)) says that there are no magnetic monopoles, i.e. no sources or sinks for net magnetic field. All magnetic flux "swirls around" in loops.

2.1.1 Questions about Maxwell's Equations

Questions about Gauss's Law TODO: Is Gauss's Law a strict generalization of Coulomb's Law? In what ways? One way is that Gauss's Law gives the electric field generated by a distribution of charge, whereas Coulomb's Law gives the force between two point charges. So applying to a continuous distribution of charge is one generalization, and what electric field is created by that distribution of charge is another (as long as you then combine Gauss's Law with the Lorentz force law in order to calculate the forces).

TODO: It seems that Gauss's Law, if you take ρ and $\mathbf{E}$ as functions of time, gives an electric field that changes instantaneously everywhere when the charge distribution ρ changes. There does not appear to be anything related to retarded positions of charge in that equation anywhere. How does the finite speed of electric field propagation enter into Maxwell's equations?

Doing a Google search on the question "is Gauss's law consistent with relativistic effects" gives answers that it *is* consistent with special relativity, including the following paragraphs:

- Maxwell's equations are relativistic: Gauss's law is one of Maxwell's equations. These equations, which describe classical electromagnetism, are inherently Lorentz invariant. This means they hold true in all inertial reference frames, consistent with the principle of relativity, which states that the laws of physics are the same for all observers in uniform motion relative to one another.
- Lorentz covariance: Maxwell's equations, and thus Gauss's law, can be written in a covariant form that is explicitly consistent with Lorentz transformations. These transformations describe how physical quantities change when moving from one inertial frame to another, incorporating relativistic effects like time dilation and length contraction.

Questions about Gauss's Law for Magnetism Figures for electromagnetic waves that are the propagation of light in a vacuum show E and B fields as perpendicular to each other, and each a sinusoid with the same phase as each other. These figures must be dramatic simplifications of the full E and B fields, e.g. perhaps they show at one instant in time all E and B fields only along the line that is the "center" of the rawy of light? In particular, I am wondering how they look in all of space, such that it is clear that their divergence is 0 at all times, at all positions in space. It might be instructive to have a visual representation, perhaps an animation, somehow showing what the E field is at all points in space at one particular instant of time. The B field has a very similar shape. And if you know what the shape of these fields are at one instant of time, then you know that they merely "shift" in the direction of light propagation over time at speed c, without changing shape.

Is there a way to write them that has only ρ without J? If one emphasizes the view that all current is due to moving charge, is there a way to write Maxwell's equations in a way that makes this clear? For example, is there a way to write it with some single function x that somehow represents both ρ and J?

I realize that it is possible to have non-zero J even when ρ is a constant over time, e.g. a wire carrying a constant current in a loop. Here charge is moving, but the density of the charge at any one point in space remains the same over time, because in any given sub-volume within the wire, as much charge is entering that volume per unit time as is leaving per unit time.

Answer from Google to this search phrase: "are there versions of Maxwell's equations that combine charge density and current density"

Yes, Maxwell's equations can be formulated in a way that combines charge density and current density into a single entity called the four-current. This formulation is part of the covariant formulation of electromagnetism, which is a more elegant and relativistic way to express Maxwell's equations.

More details available on the Wikipedia page "Covariant formulation of classical electromagnetism" [**?**].

3 Gradient

Gradient has been generalized to many coordinate systems other than the 3-dimensional Cartesian coordinate system $\mathbb{R}^3$, but I will focus on $\mathbb{R}^3$. The Wikipedia page on Gradient [2] is not too bad for me, as long as I skim over the parts that generalize it to other coordinate systems.

Griffiths [1] Section 1.2.2 "Gradient" is good at giving the definition and some useful examples and properties of the gradient. He defines $\boldsymbol{\nabla} T$ this way, where $\hat{\mathbf{x}}$, $\hat{\mathbf{y}}$, and $\hat{\mathbf{z}}$ are the unit vectors in the direction of the three coordinate axes:

$$\boldsymbol{\nabla} T \equiv \frac{\partial T}{\partial x}\hat{\mathbf{x}} + \frac{\partial T}{\partial y}\hat{\mathbf{y}} + \frac{\partial T}{\partial z}\hat{\mathbf{z}}$$

where:

- $T : \mathbb{R}^3 \to \mathbb{R}$ is a scalar function that is continuous and differentiable

Some properties of the gradient:

- $\boldsymbol{\nabla} T(x, y, z)$ is the gradient evaluated at a position given by (x, y, z). The vector points in a direction that function T increases most quickly.
- The function T can be approximated at points $\mathbf{r}$ near $\mathbf{r}_0 = (x, y, z)$ by the linear function $T(\mathbf{r}_0) + (\mathbf{r} - \mathbf{r}_0) \cdot \boldsymbol{\nabla} T(\mathbf{r}_0)$
- The instantaneous rate of change of T in direction $\mathbf{u}$ (a unit vector) from $\mathbf{r}_0$ is $\mathbf{u} \cdot \boldsymbol{\nabla} T(\mathbf{r}_0)$. Note that it is always 0 in a direction perpendicular to $\boldsymbol{\nabla} T(\mathbf{r}_0)$, and the negative of the magnitude of $\boldsymbol{\nabla} T(\mathbf{r}_0)$ in the opposite direction.
- Consider a "level set", i.e. a surface defined by all of the points $\mathbf{r}$ where $T(\mathbf{r}) = c$ for some constant c. Then $\boldsymbol{\nabla} T$ evaluated at any point on that surface, is normal to the surface.

References

[1] David J. Griffiths. *Introduction to Electrodynamics.* Pearson Education, 4th edition, 2013.

[] Wikipedia. Covariant formulation of classical electromagnetism, 2025. URL `https://en.wikipedia.org/wiki/Covariant_formulation_of_classical_electromagnetism`.

[2] Wikipedia. Gradient, 2025. URL `https://en.wikipedia.org/wiki/Gradient`.

[3] Wikipedia. Nabla Symbol, 2025. URL `https://en.wikipedia.org/wiki/Nabla_symbol`.