Andy's math/science background information

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Contents

1 Introduction

This document is a place to write up notes on background in math that help in learning science.

2 Notation

$\mathbb{R}^3$ - The set of all points in 3-dimensional space, i.e. where each point is specified by a sequence of 3 real-valued coordinates.

∇ - Called "nabla", and often called "del". Some history of it can be found on the Nabla Symbol page [12].

∇f - Gradient of a scalar function $f : \mathbb{R}^3 \to \mathbb{R}$.

grad$f$ - another way to write the gradient of f, ∇f

TODO: Any 3Blue1Brown-quality YouTube videos showing examples and definition of gradient, divergence, and curl? If so, give links to them below in appropriate sections.

2.1 Maxwell's Equations

In partial differentiatal form:

$$\nabla \cdot \mathbf{E} = \frac{\rho}{\epsilon_0} \qquad \text{Gauss's Law} \tag{1}$$

$$\nabla \cdot \mathbf{B} = 0 \qquad \text{Gauss's Law for magnetism} \tag{2}$$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \qquad \text{Maxwell-Faraday Equation (Faraday's law of induction)} \tag{3}$$

$$\nabla \times \mathbf{B} = \mu_0(\mathbf{J} + \epsilon_0 \frac{\partial \mathbf{E}}{\partial t}) \qquad \text{Ampère-Maxwell law} \tag{4}$$

where:

- $\mathbf{E}$ is the electric field
- $\mathbf{B}$ is the magnetic field
- ρ is the electric charge density
- $\mathbf{J}$ is the current density
- ϵ_0 is the vacuum permittivity
- μ_0 is the vacuum permeability

Statement of the equations in English:

Gauss's Law (equation (1)) says that electric fields are "produced" radially outward from volumes with positive electric charge, and "consumed" radially into volumes with negative electric charge.

Gauss's Law for Magnetism (equation (**??**)) says that there are no magnetic monopoles, i.e. no sources or sinks for net magnetic field. All magnetic flux "swirls around" in loops.

2.1.1 Questions about Maxwell's Equations

Questions about Gauss's Law TODO: Is Gauss's Law a strict generalization of Coulomb's Law? In what ways? One way is that Gauss's Law gives the electric field generated by a distribution of charge, whereas Coulomb's Law gives the force between two point charges. So applying to a continuous distribution of charge is one generalization, and what electric field is created by that distribution of charge is another (as long as you then combine Gauss's Law with the Lorentz force law in order to calculate the forces).

TODO: It seems that Gauss's Law, if you take ρ and $\mathbf{E}$ as functions of time, gives an electric field that changes instantaneously everywhere when the charge distribution ρ changes. There does not appear to be anything related to retarded positions of charge in that equation anywhere. How does the finite speed of electric field propagation enter into Maxwell's equations?

Doing a Google search on the question "is Gauss's law consistent with relativistic effects" gives answers that it *is* consistent with special relativity, including the following paragraphs:

- Maxwell's equations are relativistic: Gauss's law is one of Maxwell's equations. These equations, which describe classical electromagnetism, are inherently Lorentz invariant. This means they hold true in all inertial reference frames, consistent with the principle of relativity, which states that the laws of physics are the same for all observers in uniform motion relative to one another.

- Lorentz covariance: Maxwell's equations, and thus Gauss's law, can be written in a covariant form that is explicitly consistent with Lorentz transformations. These transformations describe how physical quantities change when moving from one inertial frame to another, incorporating relativistic effects like time dilation and length contraction.

Questions about Gauss's Law for Magnetism Figures for electromagnetic waves that are the propagation of light in a vacuum show E and B fields as perpendicular to each other, and each a sinusoid with the same phase as each other. These figures must be dramatic simplifications of the full E and B fields, e.g. perhaps they show at one instant in time all E and B fields only along the line that is the "center" of the ray of light? In particular, I am wondering how they look in all of space, such that it is clear that their divergence is 0 at all times, at all positions in space. It might be instructive to have a visual representation, perhaps an animation, somehow showing what the E field is at all points in space at one particular instant of time. The B field has a very similar shape. And if you know what the shape of these fields are at one instant of time, then you know that they merely "shift" in the direction of light propagation over time at speed c, without changing shape.

Is there a way to write them that has only ρ without J? If one emphasizes the view that all current is due to moving charge, is there a way to write Maxwell's equations in a way that makes this clear? For example, is there a way to write it with some single function x that somehow represents both ρ and J?

I realize that it is possible to have non-zero J even when ρ is a constant over time, e.g. a wire carrying a constant current in a loop. Here charge is moving, but the density of the charge at any one point in space remains the same over time, because in any given sub-volume within the wire, as much charge is entering that volume per unit time as is leaving per unit time.

Answer from Google to this search phrase: "are there versions of Maxwell's equations that combine charge density and current density"

Yes, Maxwell's equations can be formulated in a way that combines charge density and current density into a single entity called the four-current. This formulation is part of the covariant formulation of electromagnetism, which is a more elegant and relativistic way to express Maxwell's equations.

More details available on the Wikipedia page "Covariant formulation of classical electromagnetism" [9].

Since this is a four-vector, it has just as much "information" as the scalar field ρ plus the vector field J. It makes sense that if someone had found a way to combine ρ and J into a single time-varying scalar field, they would have done so long ago.

3 Gradient

Gradient has been generalized to many coordinate systems other than the 3-dimensional Cartesian coordinate system $\mathbb{R}^3$, but I will focus on $\mathbb{R}^3$. The Wikipedia page on Gradient [11] is fairly clear for me, as long as I skim over the parts that generalize it to other coordinate systems.

Griffiths [4] Section 1.2.2 "Gradient" is good at giving the definition and some useful examples and properties of the gradient. He defines $\boldsymbol{\nabla} T$ this way, where $\hat{\mathbf{x}}$, $\hat{\mathbf{y}}$, and $\hat{\mathbf{z}}$ are the unit vectors in the direction of the three coordinate axes:

$$\boldsymbol{\nabla} T \equiv \frac{\partial T}{\partial x}\hat{\mathbf{x}} + \frac{\partial T}{\partial y}\hat{\mathbf{y}} + \frac{\partial T}{\partial z}\hat{\mathbf{z}}$$

where:

- $T : \mathbb{R}^3 \to \mathbb{R}$ is a scalar function that is continuous and differentiable

Some properties of the gradient:

- $\boldsymbol{\nabla} T(x, y, z)$ is the gradient evaluated at a position given by (x, y, z). The vector points in a direction that function T increases most quickly.
- The function T can be approximated at points $\mathbf{r}$ near $\mathbf{r}_0 = (x, y, z)$ by the linear function $T(\mathbf{r}_0) + (\mathbf{r} - \mathbf{r}_0) \cdot \boldsymbol{\nabla} T(\mathbf{r}_0)$
- The instantaneous rate of change of T in direction $\mathbf{u}$ (a unit vector) from $\mathbf{r}_0$ is $\mathbf{u} \cdot \boldsymbol{\nabla} T(\mathbf{r}_0)$. Note that it is always 0 in a direction perpendicular to $\boldsymbol{\nabla} T(\mathbf{r}_0)$, and the negative of the magnitude of $\boldsymbol{\nabla} T(\mathbf{r}_0)$ in the opposite direction.

- Consider a "level set", i.e. a surface defined by all of the points $\mathbf{r}$ where $T(\mathbf{r}) = c$ for some constant c. Then $\boldsymbol{\nabla} T$ evaluated at any point on that surface, is normal to the surface.

If T is a function of other parameters, e.g. of time, then $\boldsymbol{\nabla} T$ is also a function of those parameters.

4 Fundamental Theorems of Calculus

See Section 1.3.2 "Fundamental Theorem of Calculus" through Section 1.3.5 "The Fundamental Theorem for Curls" of Griffiths [4] for some more discussion.

4.1 The Fundamental Theorem of Calculus

$$\int_a^b F(x)\,dx = f(b) - f(a) \tag{5}$$

where $F(x) = df/dx$, and a, b are any real numbers.

4.2 The Fundamental Theorem for Gradients

For any scalar function $T(x, y, z)$ and any two points $\mathbf{a}$ and $\mathbf{b}$:

$$\int_{\mathbf{a}}^{\mathbf{b}} (\boldsymbol{\nabla} T) \cdot d\mathbf{l} = T(\mathbf{b}) - T(\mathbf{a}) \tag{6}$$

Corollary 1: $\int_{\mathbf{a}}^{\mathbf{b}} (\boldsymbol{\nabla} T) \cdot d\mathbf{l}$ is independent of the path taken from $\mathbf{a}$ to $\mathbf{b}$.

Corollary 2: $\oint (\boldsymbol{\nabla} T) \cdot d\mathbf{l} = 0$, since the beginning and end points are identical, and hence $T((b) - T(\mathbf{a}) = 0$.

4.3 The Fundamental Theorem for Divergences

Also known as Gauss's theorem, Green's theorem, and the divergence theorem.

For any vector field $\mathbf{F}$, and any volume $\mathcal{V}$ with boundary surface $\mathcal{S}$:

$$\int_{\mathcal{V}} (\boldsymbol{\nabla} \cdot \mathbf{F}) d\tau = \oint_{\mathcal{S}} \mathbf{F} \cdot d\mathbf{a} \tag{7}$$

4.4 The Fundamental Theorem for Curls

Also known as Stokes' theorem.

For any vector field $\mathbf{F}$, and surface $\mathcal{S}$ with boundary / perimeter path $\mathcal{P}$:

$$\int_{\mathcal{S}} (\boldsymbol{\nabla} \times \mathbf{F}) \cdot d\mathbf{a} = \oint_{\mathcal{P}} \mathbf{F} \cdot d\mathbf{l} \tag{8}$$

The right-hand side is sometimes called the *circulation* of $\mathbf{F}$.

Corollary 1: $\int (\boldsymbol{\nabla} \times \mathbf{F}) \cdot d\mathbf{a}$ depends only on the boundary line, not on the particular surface used.

Corollary 2: $\oint (\boldsymbol{\nabla} \times \mathbf{F}) \cdot d\mathbf{a} = 0$ for any closed surface, since the boundary line, like the mouth of balloon, shrinks down to a point, and hence the right hand side of the equation above vanishes.

5 Potentials

See Section 1.6.2 "Potentials" of Griffiths [4] for some more discussion.

A *curl-less* or *irrotational* vector field $\mathbf{F}$ is one where $\boldsymbol{\nabla} \times \mathbf{F} = 0$. Examples include:

- The electric field in electrostatic scenarios (Equation (3) when $\mathbf{B}$ is unchanging over time).
- Gravitational fields in classical mechanics.
- Constant vector field.

Theorem 1. *The following conditions are equivalent, i.e. vector field* $\boldsymbol{F}$ *satisfies one if and only if it satsfies all the others:*
(a) $\boldsymbol{\nabla} \times \boldsymbol{F} = 0$
(b) $\int_{\boldsymbol{a}}^{\boldsymbol{b}} \boldsymbol{F} \cdot d\boldsymbol{l}$ *is independent of path, for any given endpoints* $\boldsymbol{a}$ *and* $\boldsymbol{b}$*.*
(c) $\oint \boldsymbol{F} \cdot d\boldsymbol{l} = 0$ *for any closed loop.*
(d) $\boldsymbol{F}$ *is the gradient of some scalar function:* $\boldsymbol{F} = -\boldsymbol{\nabla} V$*.*

Note that V is not unique. Any constant can be added to V without affecting the value of its gradient. See Appendix C for a proof of Theorem 1.

A *divergence-less* vector field $\mathbf{F}$ is one where $\boldsymbol{\nabla} \cdot \mathbf{F} = 0$. Such a field is also called a *solenoidal* field, or an *incompressible* field.

The name incompressible comes from fluid mechanics, when studying fluids that have a constant density. The origin of the word "solenoid" is from Greek root words meaning "resembling a pipe". A wire coiled many times into the shape of the curved part of a cylinder's surface is shaped like a section of pipe. The magnetic field created when current flows through a wire in this shape is divergence-less. (It turns out that all magnetic fields in all known scenarios are solenoidal, so this property of magnetic fields is not restricted to those created by solenoid-shaped current carrying wires. Somehow this name became associated with such vector fields).

Examples of solenoidal fields include:

- A magnetic field (Equation (2)).
- The velocity field of an incompressible fluid.
- Current density, in a scenario where the charge density is not changing.

Theorem 2. *The following conditions are equivalent:*
(a) $\boldsymbol{\nabla} \cdot F = 0$
(b) $\int \boldsymbol{F} \cdot d\boldsymbol{a}$ *is independent of surface, for any given boundary line.*
(c) $\oint \boldsymbol{F} \cdot d\boldsymbol{a} = 0$ *for any closed surface.*
(d) $\boldsymbol{F}$ *is the curl of some scalar function:* $\boldsymbol{F} = \boldsymbol{\nabla} \times A$*.*

A is not unique. The gradient of any scalar function can be added to A without affecting the curl, since the curl of a gradient is 0. See Appendix D for a proof of Theorem 2.

Theorem 3 (Helmholtz decomposition)**.** *All vector fields* $\boldsymbol{F}$ *can be written as the gradient of a scalar plus the curl of a vector:*

$$\boldsymbol{F} = -\boldsymbol{\nabla} V + \boldsymbol{\nabla} \times A \tag{9}$$

TODO: What is the proof of this? It might be on this page [14], or some page linked from it.

TODO: Add "The Helmholtz Theorem" here (the one that given a scalar field that is the divergence of F, and a vector field C that is the curl of F, and suitable boundary conditions, we can uniquely determine F.

6 History of vectors and operations on them

Before vectors were invented, quaternions were invented by William Rowan Hamilton.

Some articles and videos related to this topic:

- James Propp's "Hamilton's Quaternions, or, The Trouble with Triples" [5]
- James Propp's "Twisty Numbers for a Screwy Universe" [6] (perhaps not as relevant directly to quaternions and vectors, but complex numbers are a fairly direct precursor to quaternions).
- 3Blue1Brown video "Visualizing quaternions (4d numbers) with stereographic projection [2]. This is not so much about the history of quaternions, as it is understanding them.

7 Galilean relativity

In classical Newtonian mechanics, physical laws are invariant under Galilean relativity, i.e. any laws that hold in one inertial frame also hold in all other inertial frames, i.e. those that move with a constant velocity relative to the first frame.

This applies to Newton's three laws of motion, for example.

It does not apply to measurements of quantities that depend upon velocity, e.g. an object's momentum or kinetic energy.

However, the properties that momentum is conserved, and energy is conserved, is preserved across inertial frames, even if the absolute values of the kinetic energy and momentum differ between frames.

If there are forces that are velocity-dependent, e.g. a frictional force resisting an object traveling through a medium like air or water that is a function of velocity relative to that medium, then as long as that force is always written as a function of the relative velocity between the object and the medium, then note that the measurements of such relative velocities are the same in different inertial frames.

For other examples and some mathematical derivations, see [3].

See this video [1] that gives a similar mathematical demonstration that Newton's second law is invariant under Galilean relativity, but Maxwell's equations are *not*.

8 Experiments

- 1848-49: Fizeau's measurement of the speed of light in air [7]
- 1851: Fizeau's experiment to measure the relative speed of light in a moving medium (water) [8]

9 Questions

- Historically, how was aberration of light distinguished/separated from other effects like parallax?

10 Index of A History of the Theories of Aether and Electricity

All page numbers are of the form "I(number)" for a page number in Volume I, or "II(number)" for a page in Volume II.

Done for I, pp. 1-102

10.1 Definitions of some key terms

aberration of light

diffraction of light Diffraction of light is the bending of light waves as they pass around the edges of an obstacle or through a narrow opening. This bending causes the light to spread out and illuminate areas where a shadow would otherwise be expected. Diffraction is a wave phenomenon, meaning it’s a characteristic behavior of waves, including light, sound, and water waves.

Bending Around Obstacles: When light encounters an obstacle, like the edge of a barrier or a small opening, it doesn’t just travel in a straight line. Instead, the light waves spread out and bend around the obstacle, a phenomenon known as diffraction.

Interference: Diffraction often occurs alongside interference, where waves combine, either constructively (creating brighter areas) or destructively (creating darker areas). This combination of diffraction and interference is responsible for the characteristic patterns observed in diffraction experiments.

Wavelength Dependence: The extent of diffraction depends on the wavelength of the light and the size of the opening or obstacle. When the wavelength of the light is comparable to the size of the opening or obstacle, diffraction is more pronounced.

Examples: A classic example is the single-slit experiment where light passes through a narrow opening and creates a diffraction pattern on a screen behind it. Another example is the silver lining seen around clouds in the sky, caused by sunlight diffracting through water droplets or ice crystals.

diffusion of light Light diffusion is the process where light scatters or spreads out in various directions, rather than traveling in a straight line. This happens when light encounters a medium or surface with irregularities, causing it to scatter at different angles. This scattering results in a softer, more even light, reducing harsh shadows and glare.

Scattering: When light hits a rough or uneven surface (like a matte surface or a translucent material), it scatters in multiple directions instead of reflecting at a single angle.

Softening: This scattering effect diffuses the light, making it appear softer and more gentle.

Reduced glare: By scattering the light, diffusion reduces the intensity of the light source, minimizing glare and harsh shadows.

Examples:

- Transmission diffusion: Light passing through a frosted glass or plastic panel.
- Reflection diffusion: Light bouncing off a whiteboard or textured wall.
- Softboxes and diffusers: Professional tools like softboxes and diffusion panels are used in photography and filmmaking to soften and spread out light.
- Overcast days: The clouds act as a natural diffuser, scattering sunlight and creating soft, even light.

dispersion of light Dispersion of light is the phenomenon where white light, upon passing through a medium like a prism, separates into its constituent colors (like in a rainbow). This happens because different wavelengths (colors) of light bend at slightly different angles when they enter and exit the medium. When light passes from one medium to another (like from air to glass), it bends (refracts). The amount of bending depends on the wavelength of the light. Shorter wavelengths (like violet and blue) bend more than longer wavelengths (like red and orange).

induction of electric charge

induction of electric current

interference

polarization of light

reflection

refraction Refraction of light is the bending of light as it passes from one medium to another, caused by a change in the speed of light. This bending occurs because light travels at different speeds in different materials. For example, light travels slower in water than in air, causing it to bend when entering or exiting water.

10.2 Noteworthy events in aether vs. action at a distance

Vol. I, p. 28:

> Newton claimed nothing more for his discovery than that it provided the necessary instrument for methematical prediction, and he pointed out that it did not touch on the question of the mechanism of gravity. As to this, he conjectured that the density of the aether might vary from place to place, and that bodies might tend to move from the denser parts of the medium toward the rarer; but whether this were the true explanation or not, at any rate, he said, to suppose 'that one body may act upon another at a distance through a vacuum, without the mediation of anything else, ... is to me so great an absurdity, that I believe no man, who has in philosophical matters a competent faculty for thinking, can ever fall into.'

According to Wikipedia [15], Newton wrote this in 1692 in a letter to Richard Bentley, 5 years after Principia was first published in 1687. I do not know if Newton said anything similar in Principia itself. Very likely he did not, or else it would be more widely quoted on this topic.

Vol. I, p. 30:

> The rejection of the inverse-square law of gravitation by the French Cartesians antagonised the younger disciples of Newton to such an extent that the latter hardened into opposition not only to the vortices but to the whole body of Cartesian notions, including the aether. In the second edition (1713) of the *Principia*, there is a preface written by Roger Cotes (1682-1716), in which the Newtonian law of action at a distance is championed as being the only formulation of the facts of experience which does not introduce unverifiable and useless suppositions.

Vol. I, p. 31:

> When the eighteenth-century natural philosophers found by experience that the Newtonian law was marvellously powerful, yielding formulae by which practically every observable motion in the solar system could be predicted, while on the other hand the search for an explanation of inter-phenomena led to no practical results, opinion set in favour of Cotes's attitude, which came to prevail widely; and in the middle of the century R. G. Boscovich (1711-87) ... attempted to account for all known physical effects in terms of action at a distance between point particles.

References

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[3] Galilean Transformation, 2025. URL `https://qiweb.tudelft.nl/mecharela/galilean_transformation`.

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A Proof that curl of a gradient is 0

A.1 Proof from Griffith's definition of curl and gradient

Definition of a gradient for any scalar field T:

$$\boldsymbol{\nabla} T \equiv \frac{\partial T}{\partial x}\hat{\mathbf{x}} + \frac{\partial T}{\partial y}\hat{\mathbf{y}} + \frac{\partial T}{\partial z}\hat{\mathbf{z}}$$

Definition of a curl for any vector field $\mathbf{v}$:

$$\begin{aligned}\boldsymbol{\nabla} \times \mathbf{v} &\equiv \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ v_x & v_y & v_z \end{vmatrix} \\ &= \hat{\mathbf{x}}(\frac{\partial v_z}{\partial y} - \frac{\partial v_y}{\partial z}) + \hat{\mathbf{y}}(\frac{\partial v_x}{\partial z} - \frac{\partial v_z}{\partial x}) + \hat{\mathbf{z}}(\frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y})\end{aligned}$$

First, simply apply the definitions to $\boldsymbol{\nabla} \times (\boldsymbol{\nabla} T)$. Then do a little bit of algebra:

$$\boldsymbol{\nabla} \times (\boldsymbol{\nabla} T) = \hat{\mathbf{x}}(\frac{\partial}{\partial y}(\frac{\partial T}{\partial z}) - \frac{\partial}{\partial z}(\frac{\partial T}{\partial y})) + \hat{\mathbf{y}}(\frac{\partial}{\partial z}(\frac{\partial T}{\partial x}) - \frac{\partial}{\partial x}(\frac{\partial T}{\partial z})) + \hat{\mathbf{z}}(\frac{\partial}{\partial x}(\frac{\partial T}{\partial y}) - \frac{\partial}{\partial y}(\frac{\partial T}{\partial x})) \tag{10}$$

As long as T satisfies appropriate conditions, then Clairaut's theorem (aka Schwartz's theorem) says that regardless of which order you take partial derivatives, the final result is equal. I believe a stronger condition than necessary is that T has continuous second partial derivatives everywhere. See [13] for more precise conditions.

As long as taking second partial derivatives of T by variables in different orders gives the same result, we can see that the right hand side of (10) must be 0.

$$\boldsymbol{\nabla} \times (\boldsymbol{\nabla} T) = 0 \tag{11}$$

A.2 A proof based on Stokes' Theorem

This alternate is intended to perhaps give a little more insight into *why* the curl of the gradient is always 0.

Let T be any scalar field, and $\mathbf{F} = \boldsymbol{\nabla} T$ be its gradient. By the fundamental theorem for gradients (Section 4.2) Corollary 2, $\oint \mathbf{F} \cdot d\mathbf{l} = 0$ for all closed paths.

By Stokes' theorem (Section 4.4):

$$\begin{aligned}\int_{\mathcal{S}} (\boldsymbol{\nabla} \times \mathbf{F}) \cdot d\mathbf{a} &= \oint_{\mathcal{P}} \mathbf{F} \cdot d\mathbf{l} \\ &= 0\end{aligned}$$

By Lemma 4 (see Section A.3), The only way that the flux of $\boldsymbol{\nabla} \times \mathbf{F}$ can be 0 for all surfaces $\mathcal{S}$ is if $\boldsymbol{\nabla} \times \mathbf{F} = 0$ everywhere.

A.3 Technical lemma

Lemma 4. *Let $\boldsymbol{F}$ be a continuous vector field where*

$$\int_{\mathcal{S}} \boldsymbol{F} \cdot d\boldsymbol{a} = 0 \tag{12}$$

for all surfaces $\mathcal{S}$. Then $\boldsymbol{F} = 0$.

Proof. Performing a proof by contradiction, we suppose that there is a point $\mathbf{p}$ where $\mathbf{F}(p) \neq 0$, and deduce that there must be a surface where $\int_{\mathcal{S}} \mathbf{F} \cdot d\mathbf{a} \neq 0$.

Let $\mathbf{u}$ be the unit vector $\mathbf{F}(\mathbf{p})/|\mathbf{F}(\mathbf{p})|$, and define the scalar field $G(\mathbf{x}) = \mathbf{F}(\mathbf{x}) \cdot \mathbf{u}$. G is continuous, and $G(\mathbf{p}) = |\mathbf{F}(\mathbf{p})| = c > 0$.

By continuity of G, there is an open neighborhood V of $\mathbf{p}$ such that $G(\mathbf{x}) \geq c/2$ for all points $\mathbf{x}$ in that neighborhood. Let surface $\mathcal{S}$ be a circular disc contained entirely inside of V, centered at $\mathbf{p}$, with normal $\mathbf{u}$. Then:

$$\begin{aligned}\int_{\mathcal{S}} \mathbf{F} \cdot d\mathbf{a} = & \int_{\mathcal{S}} G(\mathbf{x}) ds \\ \geq & \int_{\mathcal{S}} (c/2) ds \\ = & (c/2) Area(\mathcal{S}) \\ > & 0\end{aligned}$$

□

B Proof that divergence of a curl is 0

B.1 Proof from Griffith's definition of divergence and curl

There is a straightforward proof of $\boldsymbol{\nabla} \cdot (\boldsymbol{\nabla} \times \mathbf{F}) = 0$ for all vector fields that follows from the definitions, with only a little bit of algebra and a use of Clairaut's/Schwartz's theorem, very similar to the proof given in Appendix A.1. I won't write it out explicitly here unless it becomes very useful to have as a reference, because it is very straightforward, and does not provide much insight into why it is true.

B.2 A proof based on Stokes' and Gauss's Theorems

Let $\mathbf{F}$ be any vector field, and $\mathbf{G} = \boldsymbol{\nabla} \times \mathbf{F}$ its curl. By Stokes' theorem (Section 4.4) Corollary 2:

$$\oint_{\mathcal{S}} \mathbf{G} \cdot d\mathbf{a} = 0 \tag{13}$$

for any closed surface $\mathcal{S}$. By Gauss's theorem (Section 4.3):

$$\begin{aligned}\int_{\mathcal{V}} (\boldsymbol{\nabla} \cdot \mathbf{G}) d\tau &= \oint_{\mathcal{S}} \mathbf{G} \cdot d\mathbf{a} \\ &= 0\end{aligned}$$

It is a fundamental result in calculus that if the integral of a continuous function over every volume is zero, then the function itself must be identically zero. This is often referred to as a consequence of the Fundamental Lemma of the Calculus of Variations [10]. Thus $\boldsymbol{\nabla} \cdot \mathbf{G} = \boldsymbol{\nabla} \cdot (\boldsymbol{\nabla} \times F)$ is identically 0.

C Proof of theorem about irrotational fields

That is, a proof of Theorem 1 from Section 5. This proof is essentially the same as one given in Section 2.3.1 "Introduction to Potential" of [4].

Let $\mathbf{F}$ be a vector field such that $\boldsymbol{\nabla} \times \mathbf{F} = 0$. Stokes' theorem says, for all vector fields $\mathbf{F}$:

$$\int_{\mathcal{S}} (\boldsymbol{\nabla} \times \mathbf{F}) \cdot d\mathbf{a} = \oint_{\mathcal{P}} \mathbf{F} \cdot d\mathbf{l}$$

Because $\boldsymbol{\nabla} \times \mathbf{F} = 0$, the left-hand side is 0 for all surfaces $\mathcal{S}$. Thus for all paths $\mathcal{P}$ that are boundaries of appropriate surfaces $\mathcal{S}$, the right-hand side is also 0.

$$\oint_{\mathcal{P}} \mathbf{F} \cdot d\mathbf{l} = 0 \tag{14}$$

I will call such paths "Stokes paths" here, for brevity (TODO: It might be nice to give the full set of restrictions that a Stokes path must satisfy). This proves that (a) implies (c) for Theorem 1, at least for closed loops that are Stokes paths.

Consider an arbitrary distinct pair of points $\mathbf{a}$ and $\mathbf{b}$, and any closed loop Stokes path $\mathcal{P}$ that includes both of those points. Let $\mathcal{P}_1$ be the part of $\mathcal{P}$ from $\mathbf{a}$ to $\mathbf{b}$, and $\mathcal{P}_2$ be the remaining part of the path from $\mathbf{b}$ back to $\mathbf{a}$. Let $\bar{\mathcal{P}_2}$ be the same as $\mathcal{P}_2$, but in the reverse direction. Then:

$$\begin{aligned}
0 &= \oint_{\mathcal{P}} \mathbf{F} \cdot d\mathbf{l} && \text{by (14)} \\
&= \int_{\mathcal{P}_1} \mathbf{F} \cdot d\mathbf{l} + \int_{\mathcal{P}_2} \mathbf{F} \cdot d\mathbf{l} && \text{break } \mathcal{P} \text{ into two subpaths} \\
&= \int_{\mathcal{P}_1} \mathbf{F} \cdot d\mathbf{l} - \int_{\bar{\mathcal{P}_2}} \mathbf{F} \cdot d\mathbf{l} && \text{line integral of reverse path is negative of forward path}
\end{aligned}$$

Therefore:

$$\int_{\mathcal{P}_1} \mathbf{F} \cdot d\mathbf{l} = \int_{\bar{\mathcal{P}_2}} \mathbf{F} \cdot d\mathbf{l} \tag{15}$$

Thus for any pair of Stokes paths $\mathcal{P}_1$ and $\bar{\mathcal{P}_2}$ from $\mathbf{a}$ to $\mathbf{b}$, their path integrals are equal. This proves that (c) implies (b) for Theorem 1, for Stokes paths.

Now consider (b) as our starting point. Pick some arbitrary point $\mathbf{d}$. Define the scalar function $V(\mathbf{r})$ as:

$$V(\mathbf{r}) = -\int_{\mathbf{d}}^{\mathbf{r}} \mathbf{F} \cdot d\mathbf{l} \tag{16}$$

Because of (b), the particular path over which these line integrals are performed does not affect the result.

$$\begin{aligned}
V(\mathbf{b}) - V(\mathbf{a}) &= -\int_{\mathbf{d}}^{\mathbf{b}} \mathbf{F} \cdot d\mathbf{l} + \int_{\mathbf{d}}^{\mathbf{a}} \mathbf{F} \cdot d\mathbf{l} \\
&= -\int_{\mathbf{a}}^{\mathbf{d}} \mathbf{F} \cdot d\mathbf{l} - \int_{\mathbf{d}}^{\mathbf{b}} \mathbf{F} \cdot d\mathbf{l} \\
&= -\int_{\mathbf{a}}^{\mathbf{b}} \mathbf{F} \cdot d\mathbf{l}
\end{aligned}$$

By the fundamental theorem for gradients (Section 4.2):

$$\int_{\mathbf{a}}^{\mathbf{b}} (\boldsymbol{\nabla} V) \cdot d\mathbf{l} = V(\mathbf{b}) - V(\mathbf{a}) \tag{17}$$

Thus:

$$\int_{\mathbf{a}}^{\mathbf{b}} (\boldsymbol{\nabla} V) \cdot d\mathbf{l} = -\int_{\mathbf{a}}^{\mathbf{b}} \mathbf{F} \cdot d\mathbf{l} \tag{18}$$

Since this is true for all pairs of points $\mathbf{a}$ and $\mathbf{b}$, the functions being integrated must be equal:

$$\mathbf{F} = -\boldsymbol{\nabla} V \tag{19}$$

This proves that (b) implies (d) for Theorem 1, for Stokes paths.

Section A contains the proof that (d) implies (a).

D Proof of theorem about solenoidal fields

That is, a proof of Theorem 2 from Section 5. This proof is strongly hinted at by Problem 5.31 in Section 5.4.1 "The Vector Potential" of [4].

TODO: Proof that (a) implies (c) is straightforward from Gauss's theorem.

TODO: Proof that (c) implies (b). Basic idea: Given a boundary line, pick an arbitrary closed surface that completely contains that boundary line. By (b) we know that the total flux through the entire closed surface is 0. Assuming the boundary line divides that closed surface into 2 non-closed surfaces (TODO: I assume that some conditions must hold for the boundary line in order for that to be true, at least that the boundary line is simple?) S_1 and S_2, the flux through S_1 must be the negative of the flux through S_2. We can replace S_2 with any other surface S_3 such that $S_1 \cup S_3$ is a closed surface, and the flux through that new closed surface is also 0, so the flux through S_3 must be the same as that through S_2, and S_3 has the same boundary line as S_2.

TODO: Proof that (b) implies (d) is (I hope) what Griffiths Problem 5.31 helps me to flesh out in more detail.

Section B contains the proof that (d) implies (a).