Andy's science scratch pad

J. Andrew Fingerhut[1]

[1]andy.fingerhut@gmail.com

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1 Introduction

This document is a place to write up little bits on science.

Some notation:

$\mathbf{i}$ is the unit vector from left to right. $\mathbf{j}$ is the unit vector upwards. $\mathbf{k}$ is the unit vector pointed out of the page toward the reader.

$\gamma = 1/\sqrt{1 - v^2/c^2}$ is the Lorentz factor.

2 Electromagnetic force between two point charges at rest relative to each other

Scenario 1: There are two point charges a and b both with charge q at rest relative to each other at a distance r apart (see Figure 1). They are at rest relative to us. In this case they both experience a force directly away from the other due to electric repulsion. There is no magnetic force, as both charges are at rest so there are no magnetic fields.

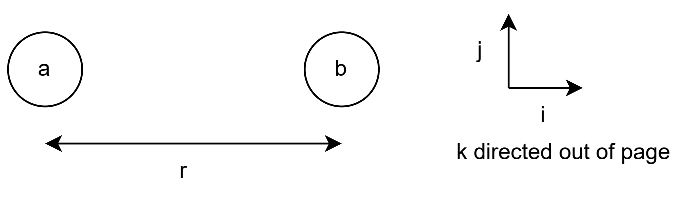

Figure 1: Two point charges at rest

Scenario 2: The same as scenario 1, but both charges are moving with constant velocity v in the upwards direction (see Figure 2). Since they are moving they create magnetic fields.

Questions:

- What is the net force on charge b in each scenario?
- Is it the same in both scenarios, or different?
- Why?

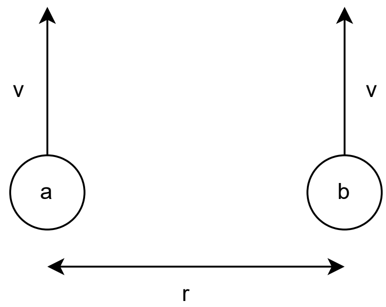

Figure 2: Two point charges moving at same constant velocity

2.1 Scenario 1: Both charges at rest

As mentioned before, there is no current or motion of any charges in this scenario, so no magnetic fields. The electric repulsion force on charge b is easily calculated from Coulomb's Law [**?**]. Charge b is to the right of charge a, so the direction of the force is $\mathbf{i}$, away from charge a.

$$\mathbf{E}_1 = \frac{1}{4\pi\epsilon_0}\frac{q}{r^2}\mathbf{i} \tag{1}$$

$$\mathbf{B}_1 = 0 \tag{2}$$

$$\mathbf{F}_1 = q(\mathbf{E}_1 + \mathbf{v} \times \mathbf{B}_1) = q\mathbf{E}_1 \tag{3}$$

2.2 Scenario 2: Both charges with equal and constant velocity upwards

2.2.1 Scenario 2 calculated by electromagnetic field equation set 1

The Wikipedia page on the Biot-Savart Law [**?**] has a subsection titled "Point charge at constant velocity" that says:

> the Biot–Savart law applies only to steady currents and a point charge moving in space does not constitute a steady current

I will thus use the equations in that section to calculate the electric and magnetic fields here. The relevant parts of the Wikipedia page are copied below.

> In the case of a point charged particle q moving at a constant veclocity v, Maxwell's equations give the following expression for the electric field and magnetic field:

$$\mathbf{E} = \frac{q}{4\pi\epsilon_0}\frac{1-\beta^2}{(1-\beta^2\sin^2\theta)^{3/2}}\frac{\hat{\mathbf{r}}'}{|r'|^2} \tag{4}$$

$$\mathbf{B} = \frac{1}{c^2}\mathbf{v} \times \mathbf{E} \tag{5}$$

where:

- $\hat{\mathbf{r}}'$ is the unit vector pointing from the current (non-retarded) position of the particle to the point at which the field is being measured,
- $\beta = v/c$ is the speed in units of c, and
- θ is the angle between $\mathbf{v}$ and $\hat{\mathbf{r}}'$. Alternatively, these can be derived by considering the Lorentz tranfromation of the Coulomb's force (in four-force form) in the source charge's inertial frame.

Calculation: To get the force on charge b, we first calculate the $\mathbf{E}$ and $\mathbf{B}$ fields at the position of charge b.

Charge b is directly to the right of charge a, so $\hat{\mathbf{r}}' = \mathbf{i}$ and $\theta = 90°$.

$$
\begin{aligned}
\mathbf{E}_2 &= \frac{q}{4\pi\epsilon_0} \frac{1-\beta^2}{(1-\beta^2\sin^2\theta)^{3/2}} \frac{\hat{\mathbf{r}}'}{|r'|^2} && \hat{\mathbf{r}}' = \mathbf{i},\ |r'| = r,\ \theta = 90°, \text{ simplify fraction} \\
&= \frac{q}{4\pi\epsilon_0} \frac{1}{(1-\beta^2)^{1/2}} \frac{\mathbf{i}}{r^2} && \text{part of this is } \gamma, \text{ by (1) the rest is } \mathbf{E}_1 \\
&= \gamma \mathbf{E}_1
\end{aligned} \tag{6}
$$

$$
\begin{aligned}
\mathbf{F}_2 &= q(\mathbf{E}_2 + \mathbf{v} \times \mathbf{B}_2) && \text{replace } \mathbf{B}_2 \text{ with (5)} \\
&= q(\mathbf{E}_2 + \mathbf{v} \times (\frac{1}{c^2}\mathbf{v} \times \mathbf{E}_2)) && \mathbf{v} \times \mathbf{E}_2 = -vE_2\mathbf{k} \\
&= q(\mathbf{E}_2 - \frac{vE_2}{c^2}\mathbf{v} \times \mathbf{k}) && \mathbf{v} \times \mathbf{k} = v\mathbf{i} \\
&= q(\mathbf{E}_2 - \frac{v^2E_2}{c^2}\mathbf{i}) \\
&= q(1 - \frac{v^2}{c^2})\mathbf{E}_2 \\
&= \frac{q\mathbf{E}_2}{\gamma^2} && \text{by (6)}\quad \mathbf{E}_2 = \gamma\mathbf{E}_1 \\
&= \frac{q\mathbf{E}_1}{\gamma} && \text{by (3)}\quad \mathbf{F}_1 = q\mathbf{E}_1 \\
&= \frac{\mathbf{F}_1}{\gamma}
\end{aligned}
$$

Thus $\mathbf{F}_2$ differs from $\mathbf{F}_1$ by a factor of γ.

TODO: Why?

I do not know how to check the answer below, but it appears that three of the answers to an on-line question similar to mine [**?**] say that the Lorentz force formula $\mathbf{F} = q(\mathbf{E} + \mathbf{v} \times \mathbf{B})$ is *not* invariant in all inertial frames, but perhaps a variant of it is. I quote one such answer below:

> Just for completeness if permitted: Following Section 3.1 from the book "Gravitation" of Misner, Thorne, and Wheeler the truly (at all speeds) frame independent force is $\frac{dP}{d\tau} = \gamma(E + v \times B)$ (in fact this is only the spacial component of the four force). τ is proper time and γ the well-known Lorentz Factor. – Kurt G. Aug 28, 2021

2.2.2 Scenario 2 calculated by Heaviside-Feynman formula

The Wikipedia page on Jefimenko's Equations [**?**] has a subsection titled "Heaviside-Feynman formula" that gives equations for the electric and magnetic field at a point due to a single moving point charge.

$$
\mathbf{E} = \frac{-q}{4\pi\epsilon_0} \left[\frac{\mathbf{e}_{r'}}{r'^2} + \frac{r'}{c}\frac{d}{dt}\left(\frac{\mathbf{e}_{r'}}{r'^2}\right) + \frac{1}{c^2}\frac{d^2}{dt^2}\mathbf{e}_{r'} \right] \tag{7}
$$

$$
\mathbf{B} = -\mathbf{e}_{r'} \times \frac{\mathbf{E}}{c} \tag{8}
$$

Here $\mathbf{e}_{r'}$ is a unit vector pointing from the observer to the charge and r' is the distance between observer and charge. Since the electromagnetic field propagates at the speed of light, both of these quantities are evaluated at the retarted time $t - r'/c$.

I believe "observer" above means "the position for which we are calculating E and B fields".

Assume here that the point charges are kept at distance r apart from each other, always horizontally, e.g. because they are connected by a stiff insulating rod. This simplifies our job of calculating E, because then $\mathbf{e}_{r'}$ and r' are unchanging over time, and their derivates are thus 0.

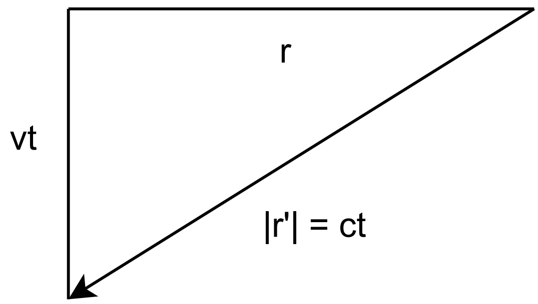

Figure 3: The retarded position of charge a from charge b

We want to calculate r' as the vector from the position of charge b to the position where charge a was when it emitted an electric field propagated at speed c to b. See Figure 3.

Solve for t using Pythagorean theorem since r and v are known constants:

$$\begin{aligned} r^2 + (vt)^2 &= (ct)^2 \\ t^2(c^2 - v^2) &= r^2 \\ t^2 &= \frac{r^2}{c^2 - v^2} \\ t &= \frac{r}{\sqrt{c^2 - v^2}} \\ &= \frac{r}{c\sqrt{1 - v^2/c^2}} \\ &= \gamma r/c \end{aligned}$$

This gives us $r' = ct = \gamma r$, and $\mathbf{e}_{r'}$ is:

$$\begin{aligned} \mathbf{e}_{r'} &= \frac{-r\mathbf{i} - (\gamma r v/c)\mathbf{j}}{\gamma r} \\ &= -\frac{1}{\gamma}\mathbf{i} - \frac{v}{c}\mathbf{j} \end{aligned}$$

Plugging in this value for $\mathbf{e}_{r'}$ into Equation (7) gives:

$$\mathbf{E}_3 = \frac{q}{4\pi\epsilon_0}\left[\frac{\frac{1}{\gamma}\mathbf{i} + \frac{v}{c}\mathbf{j}}{\gamma^2 r^2}\right]$$

Note that $\mathbf{E}_3$ is parallel to $\mathbf{e}_{r'}$, thus $\mathbf{B}_3$ from Equation (8) is 0.

This gives the force on charge b as:

$$\begin{aligned} \mathbf{F}_3 &= q(\mathbf{E}_3 + \mathbf{v} \times \mathbf{B}_3) \\ &= q\mathbf{E}_3 \end{aligned}$$

The direction of $\mathbf{F}_3$ is different than $\mathbf{F}_1$ and $\mathbf{F}_2$. Below is the relative magnitude of $\mathbf{E}_3$ to $\mathbf{E}_1$:

$$\begin{aligned} E_3 &= \frac{1}{\gamma^2}E_1 \\ F_3 &= \frac{1}{\gamma^2}F_1 \end{aligned}$$

TODO: It seems *very* odd to me that $\mathbf{B}_3 = 0$.

After Feynman explains what the retarded direction and distance **r'** is, he says [**?**]:

> That would be easy enough to understand, too, but it is also wrong. The whole thing is much more complicated.

Unfortunately there are no footnotes or citation to explain what he meant by this.

References

[] Feynman. Feynman's Lectures on Physics Volume I Chapter 28: Electromagnetic Radiation, 2025. URL https://www.feynmanlectures.caltech.edu/I_28.html#Ch28-S1-p10.

[] Physics Stack Exchange. How is Lorentz force frame-independent?, 2022. URL https://physics.stackexchange.com/questions/661883/how-is-lorentz-force-frame-independent.

[] Wikipedia. Coulomb's Law, 2025. URL https://en.wikipedia.org/wiki/Coulomb%27s_law.

[] Wikipedia. Biot-Savart Law, 2025. URL https://en.wikipedia.org/wiki/Biot%E2%80%93Savart_law.

[] Wikipedia. Jefimenko's Equations, 2025. URL https://en.wikipedia.org/wiki/Jefimenko%27s_equations.