Andy's science scratch pad

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Contents

1 Introduction

This document is a place to write up little bits on science.

Some notation:

$\mathbf{i}$ is the unit vector from left to right. $\mathbf{j}$ is the unit vector upwards. $\mathbf{k}$ is the unit vector pointed out of the page toward the reader.

$\gamma = 1/\sqrt{1 - v^2/c^2}$ is the Lorentz factor.

2 Electromagnetic force between two point charges at rest relative to each other

Scenario 1: There are two point charges a and b both with charge q at rest relative to each other at a distance r apart (see Figure 1). They are at rest relative to us. In this case they both experience a force directly away from the other due to electric repulsion. There is no magnetic force, as both charges are at rest so there are no magnetic fields.

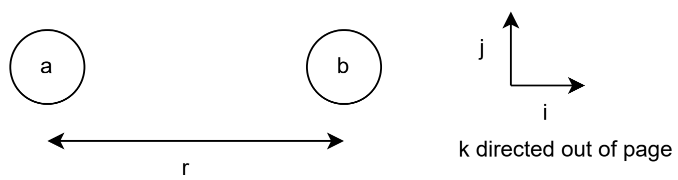

Figure 1: Two point charges at rest

Scenario 2: The same as scenario 1, but both charges are moving with constant velocity v in the upwards direction (see Figure 2). Since they are moving they create magnetic fields.

Questions:

- What is the net force on charge b in each scenario?
- Is it the same in both scenarios, or different?
- Why?

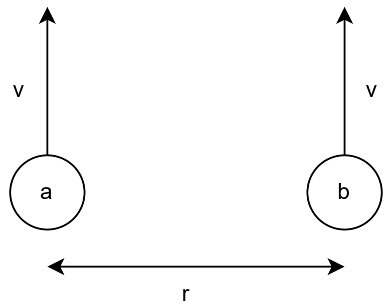

Figure 2: Two point charges moving at same constant velocity

2.1 Scenario 1: Both charges at rest

As mentioned before, there is no current or motion of any charges in this scenario, so no magnetic fields. The electric repulsion force on charge b is easily calculated from Coulomb's Law [9]. Charge b is to the right of charge a, so the direction of the force is $\mathbf{i}$, away from charge a.

$$\mathbf{E}_1 = \frac{1}{4\pi\epsilon_0}\frac{q}{r^2}\mathbf{i} \tag{1}$$

$$\mathbf{B}_1 = 0 \tag{2}$$

$$\mathbf{F}_1 = q(\mathbf{E}_1 + \mathbf{v} \times \mathbf{B}_1) = q\mathbf{E}_1 \tag{3}$$

2.2 Scenario 2: Both charges with equal and constant velocity upwards

2.2.1 Scenario 2 calculated by electromagnetic field equations from Griffiths

The Wikipedia page on the Biot-Savart Law [10] has a subsection titled "Point charge at constant velocity" that says:

> the Biot–Savart law applies only to steady currents and a point charge moving in space does not constitute a steady current

I will thus use the equations in that section to calculate the electric and magnetic fields here. The relevant parts of the Wikipedia page are copied below.

> In the case of a point charged particle q moving at a constant veclocity $\mathbf{v}$, Maxwell's equations give the following expression for the electric field and magnetic field:

$$\mathbf{E} = \frac{q}{4\pi\epsilon_0}\frac{1-\beta^2}{(1-\beta^2\sin^2\theta)^{3/2}}\frac{\hat{\mathbf{r}}'}{|r'|^2} \tag{4}$$

$$\mathbf{B} = \frac{1}{c^2}\mathbf{v} \times \mathbf{E} \tag{5}$$

where:

- $\hat{\mathbf{r}}'$ is the unit vector pointing from the current (non-retarded) position of the particle to the point at which the field is being measured,
- $\beta = v/c$ is the speed in units of c, and
- θ is the angle between $\mathbf{v}$ and $\hat{\mathbf{r}}'$.

The equations above appear to be identical to equations (10.75) and (10.76) in Griffiths [4]. Griffiths comments on the formula for the electric field:

> Notice that $\mathbf{E}$ points along the line from the *present* position of the particle. This is an extraordinary coincidence, since the "message" came from the retarded position. Because of the $\sin^2\theta$ in the denominator, the field of a fast-moving charge is flattened out like a pancake in the direction perpendicular to the motion (Fig. 10.10). In the forward and backward directions $\mathbf{E}$ is reduced by a factor $(1-v^2/c^2)$ relative to the field of a charge at rest; in the perpendicular direction it is *enhanced* by a factor $1/\sqrt{1-v^2/c^2}$.

Calculation: To get the force on charge b, we first calculate the $\mathbf{E}$ and $\mathbf{B}$ fields at the position of charge b.

Charge b is directly to the right of charge a, so $\hat{\mathbf{r}}' = \mathbf{i}$ and $\theta = 90°$.

$$\begin{aligned}
\mathbf{E}_2 &= \frac{q}{4\pi\epsilon_0}\frac{1-\beta^2}{(1-\beta^2\sin^2\theta)^{3/2}}\frac{\hat{\mathbf{r}}'}{|r'|^2} && \hat{\mathbf{r}}' = \mathbf{i},\ |r'| = r,\ \theta = 90°, \text{ simplify fraction} \\
&= \frac{q}{4\pi\epsilon_0}\frac{1}{(1-\beta^2)^{1/2}}\frac{\mathbf{i}}{r^2} && \text{part of this is } \gamma, \text{ by (1) the rest is } \mathbf{E}_1 \\
&= \gamma\mathbf{E}_1
\end{aligned} \tag{6}$$

Note that $\mathbf{E}_2$ being γ times larger than $\mathbf{E}_1$ is consistent with the comment from Griffiths above: "in the perpendicular direction it ($\mathbf{E}$) is *enhanced* by a factor $1/\sqrt{1-v^2/c^2}$".

$$\begin{aligned}
\mathbf{F}_2 &= q(\mathbf{E}_2 + \mathbf{v}\times\mathbf{B}_2) && \text{replace } \mathbf{B}_2 \text{ with (5)} \\
&= q(\mathbf{E}_2 + \mathbf{v}\times(\frac{1}{c^2}\mathbf{v}\times\mathbf{E}_2)) && \mathbf{v}\times\mathbf{E}_2 = -vE_2\mathbf{k} \\
&= q(\mathbf{E}_2 - \frac{vE_2}{c^2}\mathbf{v}\times\mathbf{k}) && \mathbf{v}\times\mathbf{k} = v\mathbf{i} \\
&= q(\mathbf{E}_2 - \frac{v^2E_2}{c^2}\mathbf{i}) \\
&= q(1-\frac{v^2}{c^2})\mathbf{E}_2 \\
&= \frac{q\mathbf{E}_2}{\gamma^2} && \text{by (6)}\quad \mathbf{E}_2 = \gamma\mathbf{E}_1 \\
&= \frac{q\mathbf{E}_1}{\gamma} && \text{by (3)}\quad \mathbf{F}_1 = q\mathbf{E}_1 \\
&= \frac{\mathbf{F}_1}{\gamma}
\end{aligned}$$

Thus $\mathbf{F}_2$ differs from $\mathbf{F}_1$ by a factor of γ.

TODO: Why?

I do not know how to check the answer below, but it appears that three of the answers to an on-line question similar to mine [7] say that the Lorentz force formula $\mathbf{F} = q(\mathbf{E} + \mathbf{v}\times\mathbf{B})$ is *not* invariant in all inertial frames, but perhaps a slightly modified version of that formula is invariant between different inertial frames. I quote one such answer below:

> Just for completeness if permitted: Following Section 3.1 from the book "Gravitation" of Misner, Thorne, and Wheeler the truly (at all speeds) frame independent force is $\frac{dP}{d\tau} = \gamma(E + v\times B)$ (in fact this is only the spacial component of the four force). τ is proper time and γ the well-known Lorentz Factor. – Kurt G. Aug 28, 2021

2.2.2 Scenario 2 calculated by Heaviside-Feynman formula

The Wikipedia page on Jefimenko's Equations [11] has a subsection titled "Heaviside-Feynman formula" that gives equations for the electric and magnetic field at a point due to a single moving point charge.

$$\mathbf{E} = \frac{-q}{4\pi\epsilon_0}\left[\frac{\mathbf{e}_{r'}}{r'^2} + \frac{r'}{c}\frac{d}{dt}\left(\frac{\mathbf{e}_{r'}}{r'^2}\right) + \frac{1}{c^2}\frac{d^2}{dt^2}\mathbf{e}_{r'}\right] \tag{7}$$

$$\mathbf{B} = -\mathbf{e}_{r'}\times\frac{\mathbf{E}}{c} \tag{8}$$

Here $\mathbf{e}_{r'}$ is a unit vector pointing from the observer to the charge and r' is the distance between observer and charge. Since the electromagnetic field propagates at the speed of light, both of these quantities are evaluated at the retarted time $t - r'/c$.

I believe "observer" above means "the position for which we are calculating E and B fields".

Assume here that the point charges are kept at distance r apart from each other, always horizontally, e.g. because they are connected by a stiff insulating rod. This simplifies our job of calculating E, because then $\mathbf{e}_{r'}$ and r' are unchanging over time, and their derivates are thus 0.

We want to calculate r' as the vector from the position of charge b to the position where charge a was when it emitted an electric field propagated at speed c to b. See Figure 3.

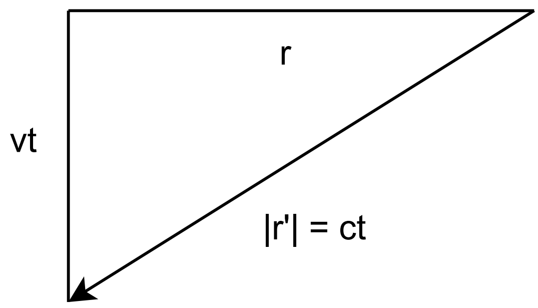

Figure 3: The retarded position of charge a from charge b

Solve for t using Pythagorean theorem since r and v are known constants:

$$\begin{aligned} r^2 + (vt)^2 &= (ct)^2 \\ t^2(c^2 - v^2) &= r^2 \\ t^2 &= \frac{r^2}{c^2 - v^2} \\ t &= \frac{r}{\sqrt{c^2 - v^2}} \\ &= \frac{r}{c\sqrt{1 - v^2/c^2}} \\ &= \gamma r/c \end{aligned}$$

This gives us $r' = ct = \gamma r$, and $\mathbf{e}_{r'}$ is:

$$\begin{aligned} \mathbf{e}_{r'} &= \frac{-r\mathbf{i} - (\gamma r v/c)\mathbf{j}}{\gamma r} \\ &= -\frac{1}{\gamma}\mathbf{i} - \frac{v}{c}\mathbf{j} \end{aligned}$$

Plugging in this value for $\mathbf{e}_{r'}$ into Equation (7) gives:

$$\mathbf{E}_3 = \frac{q}{4\pi\epsilon_0}\left[\frac{\frac{1}{\gamma}\mathbf{i} + \frac{v}{c}\mathbf{j}}{\gamma^2 r^2}\right]$$

Note that $\mathbf{E}_3$ is parallel to $\mathbf{e}_{r'}$, thus $\mathbf{B}_3$ from Equation (8) is 0.

This gives the force on charge b as:

$$\begin{aligned} \mathbf{F}_3 &= q(\mathbf{E}_3 + \mathbf{v} \times \mathbf{B}_3) \\ &= q\mathbf{E}_3 \end{aligned}$$

The direction of $\mathbf{F}_3$ is different than $\mathbf{F}_1$ and $\mathbf{F}_2$. Below is the relative magnitude of $\mathbf{E}_3$ to $\mathbf{E}_1$:

$$E_3 = \frac{1}{\gamma^2}E_1$$

$$F_3 = \frac{1}{\gamma^2}F_1$$

TODO: It seems *very* odd to me that $\mathbf{B}_3 = 0$.

After Feynman explains what the retarded direction and distance $\mathbf{r}'$ is, he says [2]:

> That would be easy enough to understand, too, but it is also wrong. The whole thing is much more complicated.

Unfortunately there are no footnotes or citation to explain what he meant by this.

3 Simple time scenarios in special relativity and Lorentz Ether Theory

Definitions of some terms:

$$\beta = v/c \quad \text{the } \textit{relativistic velocity}\text{, or } \textit{velocity ratio} \tag{9}$$

$$\gamma = \frac{1}{\sqrt{1-\beta^2}} \quad \text{the } \textit{Lorentz factor} \tag{10}$$

$$D = \sqrt{\frac{1+\beta}{1-\beta}} \quad \text{the } \textit{Doppler factor} \tag{11}$$

Later in this article we typically write a subscript after v, β, γ, and D to indicate the entity that has the relevant velocity v. The context should make it clear in what inertial frame that velocity v applies.

We will use these definitions in scenarios where $-c < v < c$. Thus:

$$\begin{aligned}
-1 \le \beta &< 1 \\
\gamma &\ge 1 && \gamma \text{ increases as } |\beta| \text{ does} \\
\lim_{\beta\to 1} \gamma &= +\infty \\
\lim_{\beta\to -1} \gamma &= +\infty \\
D = \gamma(1+\beta) &> 0 && D \text{ increases as } \beta \text{ does}
\end{aligned} \tag{12}$$

3.1 Special Relativity Scenario 1a: Two entities moving relative to each other at constant velocity

A is at rest. B is moving at constant velocity v_B relative to A, either directly away from A, or directly towards A. $v_B > 0$ means B moves away from A. $v_B < 0$ means B moves towards A.

3.1.1 B sends periodic pulses to A

B uses his local clock to time the sending of light pulses to A, sending pulses once every time interval T. At what period does A receive the pulses?

Define $q_A(n)$ to be the time on A's clock when the n-th pulse is transmitted by B, and $q_B(n)$ to be the time on B's clock when it transmits the n-th pulse.

$q_B(n) = nT$ by the setup of the experiment.

By the assumptions of special relativity, A deduces that B's clock is running γ_B times slower than A's clock. Note: A cannot directly observe B's clock, as it is too far away. From this A also deduces:

$$q_A(n) = \gamma_B nT + \Delta \tag{13}$$

Δ is the time that A reads on his clock when B reads 0 on his clock. The value of Δ is irrelevant to the final result we seek.

B's distance from A at A's time t_A is $x_B(t_A) = I_B + v_B t_A$. $I_B > 0$ is B's initial distance from A at $t_A = 0$. The value of I_B is also irrelevant to the final result we seek, as long as it is large enough that $x_B(t_A) > 0$ remains true while B emits all of the pulses we consider (two pulses is enough).

Also by the assumptions of special relativity, A deduces that B's pulse signal will propagate at the one-way speed c. The pulse will thus take time $x_B(q_A(n))/c$ to propagate to A.

A's clock thus shows time $r_A(n) = q_A(n) + x_B(q_A(n))/c$ when the n-th pulse arrives at A. With a little algebra:

$$\begin{aligned}
r_A(n) &= q_A(n) + x_B(q_A(n))/c && \text{definition of } r_A(n) \\
&= q_A(n) + (I_B + v_B q_A(n))/c && \text{substitute equation for } x_B \\
&= (1 + v_B/c)q_A(n) + I_B/c && \text{rearrangement by algebra} \\
&= (1 + \beta_B)(\gamma_B nT + \Delta) + I_B/c && \text{Defn. (9) and Eqn. (13)} \\
&= (1 + \beta_B)\gamma_B Tn + ((1 + \beta_B)\Delta + I_B/c) && \text{algebra to collect all terms independent of } n \text{ at the end}
\end{aligned}$$

The time that A measures on his clock between two consecutive received pulses is:

$$\begin{aligned}
r_A(n+1) - r_A(n) &= (1 + \beta_B)\gamma_B T && \text{by calculation of } r_A(n) \text{ above} \\
&= D_B T && \text{Eqn. (12)}
\end{aligned}$$

The pulses arrive at A with a period of $D_B T$ measured on A's clock.

Suppose that A knows somehow the period T that B uses as its period for transmitting pulses, e.g. because they agreed and arranged this beforehand. Then A can measure the period between received pulses on his clock and divide it by T to calculate D_B. By Corollary 3 in Appendix A.4, A can calculate from D_B the one value of β_B in the range $(-1, 1)$ that corresponds to it, and v_B in the range $(-c, +c)$.

Note 1: There is *nothing* in this derivation that relies upon prior knowledge of the Doppler factor, or under what conditions it is applicable. That expression arose in the process of calculating the answer, using only the assumptions that light moves isotropically at speed c in A's frame, and that A can deduce that B's clock is running γ_B times slower than A's clock.

3.1.2 A sends periodic pulses to B

Now A uses his local clock to time the sending of light pulses to B, sending pulses once every time interval T. At what period does B receive the pulses?

While we could switch perspectives to B's inertial frame, we will not. Instead, we are going to do all of the calculations in A's inertial frame.

The definitions of Section 3 remain the same here, but note that the values of $q_A(n)$ and $q_B(n)$ here are *different* than those in Section 3.1.1.

Define $q_A(n) = nT$ to be the time on A's clock when it transmits its n-th pulse.

At all times t_A, B is a distance $x_B(t_A) = I_B + v_B t_A$ away from A. As in the previous section, $I_B > 0$ is any value large enough that $x_B(t_A)$ remains positive while A emits all of the pulses we consider.

A assumes by special relativity that its pulse propagates with one-way speed c to B. The pulse's position at time $t_A \geq q_A(n)$ is $c(t_A - q_A(n))$.

On A's clock, A deduces that its n-th pulse catches up to B at a time $r_A(n)$ that satisfies the equation:

$$\begin{aligned}
x_B(r_A(n)) &= c(r_A(n) - q_A(n)) && B\text{'s position equals light pulse's position} \\
I_B + v_B r_A(n) &= c(r_A(n) - nT) && \text{substitute for defns. of } x_B \text{ and } q_A(n) \qquad (14) \\
I_B/c + (v_B/c) r_A(n) &= r_A(n) - nT && \text{divide by } c \\
nT + I_B/c &= (1 - v_B/c) r_A(n) && \text{a little more algebra} \\
r_A(n) &= \frac{nT}{1 - v_B/c} + \frac{I_B}{c - v_B} && \text{divide by } (1 - v_B/c) \\
r_A(n) &= \frac{nT}{1 - \beta_B} + \frac{I_B}{c - v_B} && \text{Defn.(9)} \qquad (15)
\end{aligned}$$

By the assumptions of special relativity, A deduces that B's clock is running γ_B times slower than A's clock. Thus B's time when receiving the n-th pulse is the following, where Δ is the time that B reads on his clock when A reads 0 on his clock (as in the previous section, the value of Δ is irrelevant in our final answer):

$$\begin{aligned}
r_B(n) &= r_A(n)/\gamma_B + \Delta && B\text{'s clock slower by factor } \gamma_B \\
&= \frac{nT}{\gamma_B(1 - \beta_B)} + (\frac{I_B}{\gamma_B(c - v_B)} + \Delta) && \text{Eqn. (15), move things independent of } n \text{ to end}
\end{aligned}$$

The time that B measures on his clock between two consecutive received pulses is:

$$\begin{aligned}
r_B(n+1) - r_B(n) &= \frac{T}{\gamma_B(1-\beta_B)} && \text{by calculation of } r_B(n) \text{ above} \\
&= \frac{\sqrt{1-\beta_B^2}}{1-\beta_B} T && \text{Defn. (10)} \\
&= \sqrt{\frac{1+\beta_B}{1-\beta_B}} T && \text{a little algebra} \\
&= D_B T && \text{Defn. (11)}
\end{aligned}$$

B will observe pulses arriving with period $D_B T$ according to B's clock.

While this formula for the period between received pulses is very similar to the one in the previous section, note that both the sending and receiving period are being measured on different clocks than there.

As in the previous section, if B somehow knows T, he can measure the interal between receive pulses, divide it by T to calculate D_B, then use that to calculate β_B and v_B.

3.1.3 Relationship to Lorentz Ether Theory

Note in Sections 3.1.1 and 3.1.2, that except for algebra and the definitions of symbols from Section 3, the only assumptions we used from special relativity were:

- The one-way speed of light is c in all directions in A's inertial frame.
- In A's frame, B's clock runs at a slower rate, by a factor of $1/\gamma_B$, relative to A's clock.

By Lorentz Ether Theory, suppose that somehow we know that A is at rest relative to the ether, then:

- The one-way speed of light is c in all directions relative to the ether, and thus also relative to A.
- B is moving at velocity v_B relative to the ether, and thus B's clock physically runs at a slower rate, $1/\gamma_B$ times as fast as the true time. A's clock runs at the full rate of true time, the same as any other clocks at rest relative to the ether.

TODO: Find a way to explain the following better.

I had heard from a not-yet-in-depth learning about special relativity that when A and B are moving at constant velocity towards or away from each other, that A observed that B's clock ran slower by a factor of γ, and B observed that A's clock ran slower by a factor of γ. (Note: I do not claim that those are fully precise statements, but there is definitely a sense in which special relativity does say something similar to this.)

In A's frame observing B's clock run slower, that seems perfectly consistent with Lorentz Ether Theory's statement that if A is at rest relative to the ether, and B moves at constant velocity v_B relative to the ether, that B experiences duration dilation, i.e. its clock physically runs slower than A's by a factor of γ_B. In this situation A's clock runs at full speed, i.e. γ_B times *faster* than B's.

But what about Lorentz Ether Theory's position on the converse statement? That is, from B's point of view, does B observe A's clock running γ_B times slower? If so, how can that possibly make sense?

I now believe that the answer is that the statements in special relativity can be made a bit more precise by saying something like this: Because A is following special relativity's assumptions, i.e. in A's frame the speed of light propagates isotropically at constant speed c, therefore A can deduce that any clocks moving at constant speed v directly towards or away from A run γ times slower, and make further calculations from that deduction.

A does not actually *observe* such clocks directly over any appreciable interval of time, so they are always, or almost always, so far away that A cannot make *any* direct observations of how fast such clocks are running. By "direct" observations I mean "with light propagation delay very close to 0 between A and the entity being observed".

Suppose in some future context of knowledge that not only is Lorentz Ether Theory proven, but in such a way that we know how to measure our speed relative to the ether.

Then, in the scenario described, we would know that A's clock is running at full speed, and light propagates isotropically at constant speed c relative to the ether, and thus also relative to A.

Everyone with this knowledge would be able to deduce that B's clock is running γ times slower than full speed. Also, that A's clock is running γ times *faster* than B's clock (and that all of A's local physical processes are proceeding γ times faster than similar local physical processes of B).

Further, light does *not* propagate at the same speed in all directions relative to B. It does so only with respect to the ether.

We could also prove that if one chose to make calculations using special relativity's assumptions in B's frame, one would get the same answers to these calculations that you do when using Lorentz Ether Theory.

A hint of corroboration can be seen in the Wikipedia page on time dilation, which says in the introduction [15]:

> The dilation compares "wristwatch" clock readings between events measured in different inertial frames and is not observed by visual comparison of clocks across moving frames.

It seems that any statement similar to:

- A observes B's clock running slower than their own.

could be said in much more detail as either of the following:

- In accordance with special relativity's time synchronization convention that light propagates in A's inertial frame isotropically with speed c, A deduces that B's clock runs slower than A's.
- A deduces, using the postulate that light moves isotropically at speed c, that B's clock runs slower, and can then make further consistent calculations based on this deduction.

And you can swap A and B in that statement. Without defining new precise terminology, a shorter precise statement would perhaps be:

- According to SR, in A's inertial frame we deduce that B's clock runs slower.

3.2 Special Relativity Scenario 1b: Three entities, two of them moving at constant velocity along x axis

A is at rest. B is moving at constant velocity v_B relative to A, which is away from A if $v_B > 0$, or towards A if $v_B < 0$. C is moving at constant velocity v_C relative to A, with same sign conventions as B's velocity.

For B sending periodic pulses to A or vice versa, everything in Sections 3.1.1 and 3.1.2 applies without change. For C sending periodic pulses to A or vice versa, everything in Section 3.1.1 and 3.1.2 applies, except replace B subscripts with C subscripts, i.e. use v_C, γ_C, β_C, and D_C.

So it is only pulses between B and C that might present something new here.

Our calculations in the following sections for pulse signals between B and C assume that throughout the entire duration of sending and receiving the pulses of interest, either:

- B's position according to its x coordinate is always less than C's x coordinate, or
- B's position according to its x coordinate is always greater than C's x coordinate, or

If that condition is ever violated because B crosses paths with C, the results presented here are no longer applicable after that occurs. We use I_B and I_C to represent their initial distances from A, and require that these positions, combined with their velocities, will ensure this.

3.2.1 B sends periodic pulses to C

B uses his local clock to time the sending of light pulses to C, sending pulses once every time interval T, according to B's clock. At what period does C receive the pulses?

To make the calculations as easily applicable to Lorentz Ether Theory as possible, all calculations will be done in A's inertial frame.

Define $q_A(n)$ and $q_B(n)$ the same way as they were in Section 3.1.1. As explained there, when A makes the assumptions according to special relativity theory:

$$\begin{aligned}
q_B(n) &= nT && \text{by the setup of the experiment} \\
t_A &= \gamma_B t_B + \Delta_B && B\text{'s clock runs } \gamma_B \text{ times slower than } A\text{'s} \\
q_A(n) &= \gamma_B nT + \Delta_B && && (16) \\
t_A &= \gamma_C t_C + \Delta_C && C\text{'s clock runs } \gamma_C \text{ times slower than } A\text{'s} \\
t_C &= t_A/\gamma_C - \Delta_C/\gamma_C && \text{equivalent to previous equation, but solved for } t_C && (17) \\
x_B(t_A) &= I_B + v_B t_A && \text{relationship of } B\text{'s position and time, in } A\text{'s frame} && (18) \\
x_C(t_A) &= I_C + v_C t_A && \text{relationship of } C\text{'s position and time, in } A\text{'s frame} && (19)
\end{aligned}$$

Also by special relativity assumptions, A considers the pulse to travel from B to C at constant speed c. The n-th pulse is emitted at time $q_A(n)$ in A's frame, so its position as a function of A's time t_A is given by $l_A(n, t_A)$ below. Note that d is 1 if C's x coordinate is greater than B's, and thus the pulse of interest is moving in the increasing x direction. Otherwise d is -1.

$$\begin{aligned}
l_A(n, t_A) &= \text{position of } B \text{ when emitted} \\
&\quad + \text{distance traveled after emission} \\
&= x_B(q_A(n)) + d(t_A - q_A(n))c && \text{for any time } t_A \geq q_A(n) \\
&= I_B + v_B q_A(n) + d(t_A - q_A(n))c && \text{substitute Eqn. (18)} \\
&= (v_B - dc)q_A(n) + dct_A + I_B && \text{algebra} \\
&= (v_B - dc)\gamma_B nT + dct_A + ((v_B - c)\Delta_B + I_B) && \text{substitute Eqn. (16)} \\
&= (v_B - dc)\gamma_B nT + dct_A + Z && Z \text{ is constants, independent of } n \text{ and } t_A
\end{aligned}$$

To find A's time $r_A(n)$ when C receives the pulse, solve for the time that makes the pulse position the same as C's position:

$$\begin{aligned}
x_C(r_A(n)) &= l_A(n, r_A(n)) \\
I_C + v_C r_A(n) &= (v_B - dc)\gamma_B nT + dcr_A(n) + Z && \text{substitute Eqn. (19)} \\
(v_C - dc)r_A(n) &= (v_B - dc)\gamma_B nT + (Z - I_C) && \text{algebra} \\
r_A(n) &= \frac{v_B - dc}{v_C - dc}\gamma_B nT + \frac{Z - I_C}{v_C - c} && \text{divide by } v_C - dc \\
r_A(n) &= \frac{d - \beta_B}{d - \beta_C}\gamma_B nT + \frac{Z - I_C}{v_C - c} && \text{defn. of } \beta_B, \beta_C \\
r_A(n) &= \frac{d - \beta_B}{d - \beta_C}\gamma_B nT + Y && Y \text{ is a constant, independent of } n
\end{aligned}$$

According to A and its special relativity assumptions, C's clock runs slower, at a rate $1/\gamma_C$ times that of A's clock. So C's time $r_C(n)$ to receive the n-th pulse sent by B is:

$$\begin{aligned}
r_C(n) &= \frac{1}{\gamma_C} r_A(n) - \Delta_C/\gamma_C && \text{Eqn. (17)} \\
&= \frac{\gamma_B}{\gamma_C}\left(\frac{d - \beta_B}{d - \beta_C}\right) nT + \frac{Y - \Delta_C}{\gamma_C}
\end{aligned}$$

The time that B measures on his clock between two consecutive received pulses is:

$$\begin{aligned}
r_C(n+1) - r_C(n) &= \frac{\gamma_B}{\gamma_C}\left(\frac{d - \beta_B}{d - \beta_C}\right) T \\
&= \sqrt{\frac{1 - \beta_C^2}{1 - \beta_B^2}}\left(\frac{d - \beta_B}{d - \beta_C}\right) T && \text{Defn. (10)}
\end{aligned}$$

From here there are two slightly different cases, depending on the value of d. First let us finish the problem for the case of $d = 1$, when C is at a greater x coordinate than B:

$$\begin{aligned}
r_C(n+1) - r_C(n) &= \sqrt{\frac{1 + \beta_C}{1 - \beta_C}}\sqrt{\frac{1 - \beta_B}{1 + \beta_B}} T && \text{algebra} \\
&= (D_C/D_B)T && \text{defn. of } D_B, D_C
\end{aligned}$$

Now for the case $d = -1$, when C is at a lesser x coordinate than B:

$$\begin{aligned}
r_C(n+1) - r_C(n) &= \sqrt{\frac{1-\beta_C^2}{1-\beta_B^2}}\left(\frac{-1-\beta_B}{-1-\beta_C}\right)T \\
&= \sqrt{\frac{1-\beta_C^2}{1-\beta_B^2}}\left(\frac{1+\beta_B}{1+\beta_C}\right)T && \text{algebra} \\
&= \sqrt{\frac{1-\beta_C}{1+\beta_C}}\sqrt{\frac{1+\beta_B}{1-\beta_B}}T && \text{algebra} \\
&= (D_B/D_C)T && \text{defn. of } D_B, D_C
\end{aligned}$$

So when B sends pulses with period T according to B's clock, and C is at a greater x coordinate than B, C receives from B pulses with period $(D_C/D_B)T$ on C's clock. If C is at a lesser x coordinate than B, C receives pulses with period $(D_B/D_C)T$ on C's clock.

Since $D > 0$ and it increases with β (see Appendix A.4), and thus also with v, (D_C/D_B) is greater than 1 if $v_C > v_B$, otherwise less than 1, and (D_B/D_C) is greater than 1 if $v_C < v_B$.

	$v_c < v_B$	$v_c > v_B$
C is at greater x coordinate than B	C and B getting closer. $(D_C/D_B) < 1$	C and B getting further. $(D_C/D_B) > 1$
C is at lesser x coordinate than B	C and B getting further. $(D_B/D_C) > 1$	C and B getting closer. $(D_B/D_C) < 1$

All of these cases are a bit annoying. It is good to note that as long as you use the one of D_C/D_B and D_B/D_C that is greater than 1 if B and C are moving away relative to each other, and the one that is less than 1 if B and C are getting closer over time, you will get the right answer.

Aside: If you are curious, the answer provided by ChatGPT for solving this problem can be found in Appendix D.1.

Near the end of Sections 3.1.1 and 3.1.2, we noted that if the receiver knew somehow the period T that the sender is sending pulses, the receiver could calculate a D value that enabled the receiver to determine what the velocity of B is.

In the current scenario, the receiver C, if it knows T, can measure the time interval between pulses it receives, divide by T, and calculate (D_C/D_B).

Thus, with the measurement of the interval between received pulses, given any two of T, v_A, and v_B, C can calculate the other one of those (and given values for two of those quantities, there is only one value possible for the remaining one). But from the measurement of the interval between received pulses and T, that is not enough information for C to calculate either one of v_B or v_C.

However, it *can* do the following. Define $D_r = (D_C/D_B)$. From Corollary 5 in Appendix A.4, we know that if $\beta_r = \beta_C \ominus \beta_B$ for some β_C, β_B values in the range $(-1, +1)$, then $D_r = D_C/D_B$.

TODO: I am fairly certain it is straightforward to prove the converse: If in this scenario C calculates the value value of D_r, then calculates $\beta_r = (D_r^2 - 1)/(D_r^2 + 1)$, then the only possible pairs of values β_C, β_B that could have given this measurement are those satisfying $\beta_r = \beta_C \ominus \beta_B$. There are an unlimited number of such pairs of values.

This strongly suggests that the receiver can calculate the *relative* velocity between B and C, at least in some sense. I am not sure yet how to explain that further.

Also note that if we follow Lorentz Ether Theory, but we have no knowledge of how to determine our motion relative to the ether, these results show that the receiver can, with knowledge of the sender's period T, still calculate this kind of relative velocity described, which is interesting.

3.2.2 C sends periodic pulses to B

The derivation is nearly identical to that in Section 3.2.1. Here we only mention a few equations along the way that have noticeable differences.

The formula $l_A(n, t_A)$ for the position of the n-th pulse emitted by C at A's time t_A is given below. Here we use the same choice of $d = 1$ if C's x coordinate is greater than B, otherwise $d = -1$.

$$l_A(n, t_A) = (v_C + dc)\gamma_C nT - dct_A + Z' \tag{20}$$

A's time when B receives the n-th pulse $r_A(n)$ is:

$$r_A(n) = \frac{d + \beta_C}{d + \beta_B} \gamma_C n T + Y' \tag{21}$$

and B's time when it receives the n-th pulse from C is:

$$r_B(n) = \frac{1}{\gamma_B} r_A(n) + \text{(constants independent of } n\text{)}$$

and finally:

$$\begin{aligned} r_B(n+1) - r_B(n) &= (D_C/D_B)T && \text{for case of } d = 1 \\ r_B(n+1) - r_B(n) &= (D_B/D_C)T && \text{for case of } d = -1 \end{aligned}$$

Thus B, after measuring the interval between received pulses, if it knows T somehow and can calculate the correct one of (D_C/D_B) and (D_B/D_C), can also calculate the same kind of relative velocity between B and C as described at the end of the previous section.

3.3 Special Relativity Scenario 1c: Three entities, two of them moving at constant velocity towards or away from each other, but otherwise arbitrary 3-d directions

This is a generalized version of Special Relativity Scenario 1b in Section 3.2.

A is at rest at the origin, as before.

B is moving at constant velocity $\mathbf{v}_B$ relative to A, but here it is allowed to be any general 3-d direction and magnitude, not restricted to lie along the x axis.

Similarly for C, moving at constant velocity $\mathbf{v}_C$ relative to A.

B and C are restricted in their movement in the following way: relative to each other, they must be moving directly towards or directly away from the other. If they are moving directly towards each other, they must be moving towards the same position, arriving there at the same time. If they are moving directly away from each other, they must be moving away from a common starting position, having left it at the same time. We define $\mathbf{x}_0$ to be this common position, and the equations of position for B and C are:

$$\mathbf{x}_B(t) = \mathbf{x}_0 + \mathbf{v}_B t_A \tag{22}$$

$$\mathbf{x}_C(t) = \mathbf{x}_0 + \mathbf{v}_C t_A \tag{23}$$

These positions are relative to A, and t_A is the time in A's frame. If $t < 0$, B and C are moving towards $\mathbf{x}_0$, and thus towards each other. If $t > 0$, B and C are moving away from $\mathbf{x}_0$, and thus away from each other.

B's clock runs $1/\gamma_B$ times the speed of A's clock. C's clock runs $1/\gamma_C$ times the speed of A's clock.

The calculations are a bit more involved than those in the previous section, but basically they are straightforward applications of Lemma 8. We placed the details in Appendix E.

When B sends pulses with period T_B measured on B's clock, C receives pulses with period T_C measured on C's clock, where:

$$T_C = \frac{\gamma_B}{\gamma_C} T_B \left[1 + \frac{\gamma_C^2}{c^2} \left((v_C^2 - \mathbf{v}_B \cdot \mathbf{v}_C) + \sqrt{(v_C^2 - \mathbf{v}_B \cdot \mathbf{v}_C)^2 + (1 - \beta_C^2) c^2 |\mathbf{v}_C - \mathbf{v}_B|^2} \right) \right] \tag{24}$$

TODO: Add equation for C sending pulses to B.

3.4 Summary of results in this section

TODO: Add summary of results of Section 3.3 here.

Here is a summary of what we have shown so far.

- In the scenario of Section 3.1 where A is at rest, and B is moving at velocity v_B relative to A in A's frame (negative for velocity towards A, positive for velocity away from A):
  - When B sends pulse signals every time period T according to B's clock, A measures received pulses every time period $D_B T$ according to A's clock.

- When A sends pulse signals every time period T according to A's clock, B measures received pulses every time period $D_B T$ according to B's clock.

- In the scenario of Section 3.2 where A is at rest, B and C are moving at velocity v_B and v_C relative to A in A's frame (same sign conventions as above), and the position of C is always "to the right" of B during the scenario:
  - When B sends pulse signals every time period T according to B's clock, C measures received pulses every time period $(D_C/D_B)T$ according to C's clock.
  - When C sends pulse signals every time period T according to C's clock, B measures received pulses every time period $(D_C/D_B)T$ according to B's clock.

The table below is another way to summarize the results of Scenario 1a. All formulas for periods have been simplified by using $T = 1$, i.e. you can multiply them all by T to get the original answer derived earlier. It also shows how much the previous period is multiplied by to get the next period in the sequence of steps from the sender to the receiver.

	Period measured locally on A's clock	multiply by factor	Period deduced by A, on A's clock, at B's location	multiply by factor	Period measured locally on B's clock
$A \to B$	$1 \longrightarrow$	$\frac{1}{1-\beta_B} \longrightarrow$	$\frac{1}{1-\beta_B} \longrightarrow$	$\frac{1}{\gamma_B} \longrightarrow$	D_B
$A \leftarrow B$	$D_B \longleftarrow$	$(1+\beta_B) \longleftarrow$	$\gamma_B \longleftarrow$	$\gamma_B \longleftarrow$	1

And the table below is another way to summarize the results of Scenario 1b:

	Period measured locally on B's clock	multiply by factor	Period deduced by A, on A's clock, at B's location	multiply by factor	Period deduced by A, on A's clock, at C's location	multiply by factor	Period measured locally on C's clock
$B \to C$	$1 \longrightarrow$	$\gamma_B \longrightarrow$	$\gamma_B \longrightarrow$	$\frac{1-\beta_B}{1-\beta_C} \longrightarrow$	$\frac{1-\beta_B}{1-\beta_C}\gamma_B \longrightarrow$	$\frac{1}{\gamma_C} \longrightarrow$	(D_C/D_B)
$B \leftarrow C$	$(D_C/D_B) \longleftarrow$	$\frac{1}{\gamma_B} \longleftarrow$	$\frac{1+\beta_C}{1+\beta_B}\gamma_C \longleftarrow$	$\frac{1+\beta_C}{1+\beta_B} \longleftarrow$	$\gamma_C \longleftarrow$	$\gamma_C \longleftarrow$	1

In every scenarios, all calculations of movement of B, C, and pulse signals were done in A's frame. Only in the first or last steps did we do any time conversions between clocks running at different rates. Thus both special relativity and Lorentz Ether Theory predict the same measurements.

The equations for the received time intervals all contain one or more factors of D that are relativistic Doppler factors. These arose naturally out of the calculations, not from any prior knowledge of Doppler factors. The velocities involved in these Doppler factors are strongly related to the relative velocity of the sender and the receiver.

If we adopt Lorentz Ether Theory, but remain ignorant of how to measure our velocity relative to the ether, Scenario 1b's results strongly suggest a proper understanding of relative velocity.

4 Simple length contraction scenarios in special relativity and Lorentz Ether Theory

4.1 Special Relativity Scenario 2a: Two rods moving towards each other at constant velocity

In this scenario (see Figure 4), there are two rods R and S with the same rest length L. They are parallel to each other, and both lie along a common straight line, initially at some distance apart. We will say R is to the right of S for the purposes of drawing and discussing events of interest.

There is a person A at rest relative to R, at a distance $D \gg L$ from the R's center, at a direction U perpendicular to R's length. Rod S is moving left to right with velocity v ($0 < v < c$) relative to R.

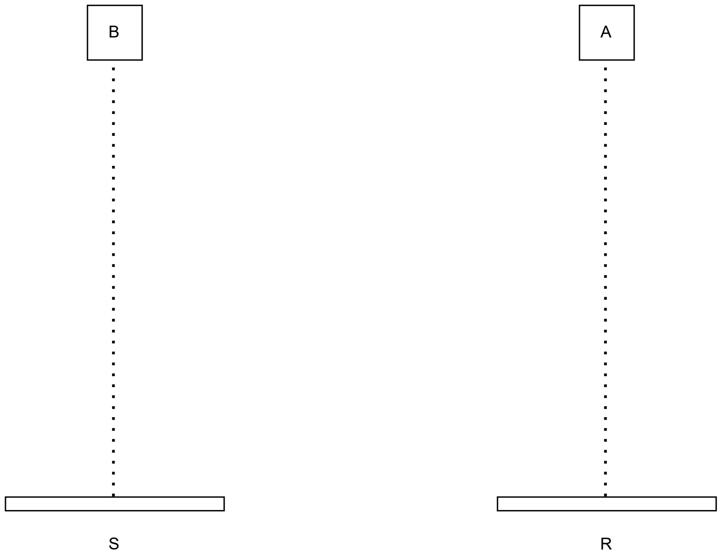

Figure 4: Two collinear rods moving towards each other, observers far away in perpendicular direction

There is another person B at rest relative to S at a distance D from S's center, in the same direction U from S's center.

There are 3 events of interest, shown in Figure 5:

- Event 1: The left end of R meets the right end of S.
- Event 2: The left end of R meets the left end of S.
- Event 3: The right end of R meets the right end of S.

In the figure, the location of each event is shown by a small circle on the appropriate end of R.

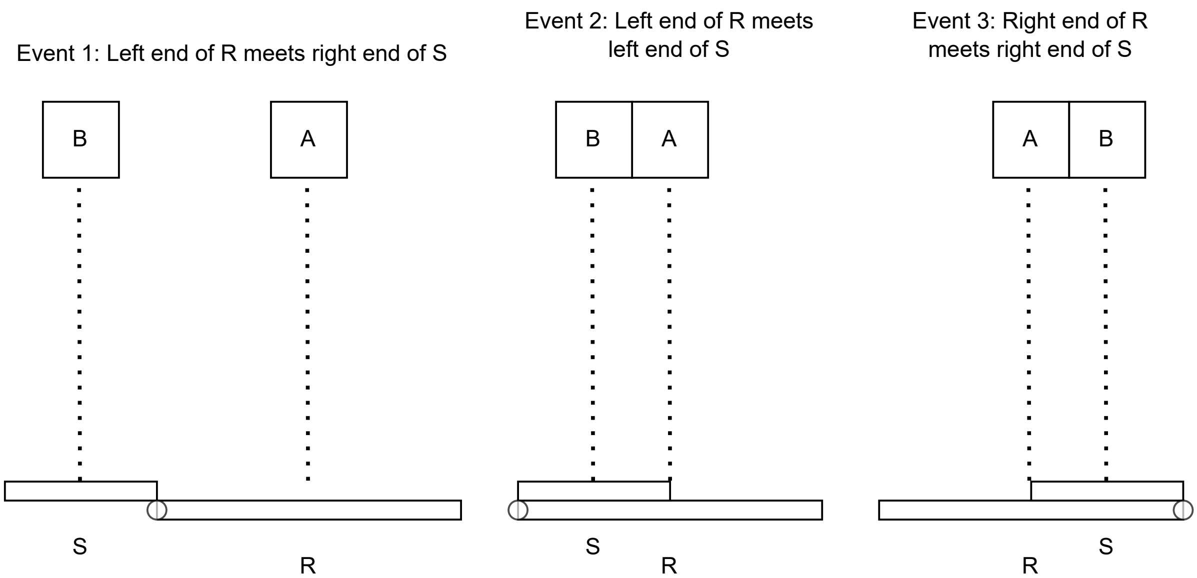

Figure 5: Events of interest for Scenario 2a

Each of these events causes a light pulse to be emitted from the left or right end of rod R, when the event is detected locally by some devices built into the rods at each end.

The figure shows the order of events as observed by A in A's frame. The figure shows the moving rod S contracts in length by a large factor, to emphasize this. We assume that the pulses can be distinguished by A in some way, e.g. by relative intensity, frequency, and/or duration.

I have read that according to special relativity, because $D \gg L$, the propagation time of the pulses from their sending points to A should be effectively equal, and thus the relative order that A receives the pulses is the same order that they are sent, and also the difference in their arrival times should be equal to the difference in their sending times.

In A's frame, the sending times of the pulses are as follows. (Detail: These equations can use v instead of $|v|$ because we are assuming $v > 0$.)

$$
\begin{aligned}
q_{1,A} &= 0 && \text{for convenience as a starting time} \\
q_{2,A} &= \frac{L/\gamma}{v} && \text{distance moved by } S \text{ since Event 1 is its contracted length} \\
q_{3,A} &= \frac{L}{v} && \text{distance moved by } S \text{ since Event 1 is } R\text{'s rest length}
\end{aligned}
$$

Let $r_{1,A}, r_{2,A}, r_{3,A}$ be the times that A measures for receiving the 3 pulses.

$$
\begin{aligned}
r_{2,A} - r_{1,A} &= \frac{L}{\gamma v} && \text{equal to } q_{2,A} - q_{1,A} \\
r_{3,A} - r_{1,A} &= \frac{L}{v} && \text{equal to } q_{3,A} - q_{1,A}
\end{aligned}
$$

According to special relativity, if we calculate in B's frame the interval between B receiving the pulses, we find that Event 3's pulse reaches B *before* Event 2's pulse, because in B's frame, S is its full rest length L, and R is contracted in length. So the sequence of events is the "mirror image" of Figure 5.

$$
\begin{aligned}
q_{1,B} &= 0 && \text{see Note 1 below} \\
q_{2,B} &= \frac{L}{v} && \text{distance moved by } R \text{ since Event 1 is } S\text{'s rest length} \\
q_{3,B} &= \frac{L/\gamma}{v} && \text{distance moved by } R \text{ since Event 1 is its contracted length} \\
r_{2,B} - r_{1,B} &= \frac{L}{v} && \text{equal to } q_{2,B} - q_{1,B} \\
r_{3,B} - r_{1,B} &= \frac{L}{\gamma v} && \text{equal to } q_{3,B} - q_{1,B}
\end{aligned}
$$

Note 1: This is just a convenient starting time, with no attempt to quantify any relationship between this and $q_{1,A}$.

Events 1 and 2 are timelike separated, because they are at the same location, and one occurs earlier than the other.

Events 1 and 3 are timelike separated, because an object with mass traveling at speed faster than v relative to A, but less than c, in the same direction as S's velocity can move from event 1 to the place where event 2 occurs, before event 2 occurs.

Events 2 and 3 are spacelike separated, because it would require faster-than-light travel in A's frame to move from the left end of R starting when event 2 occurs, to the right end of R by the time event 3 occurs. As one example, note that rod S in Figure 5 is drawn half the length of rod R, so $\gamma = 2$ in that figure. This magnitude of length contraction requires $v \approx 0.866c$. Something would need to travel the entire rest length of R in the time it takes S to move a distance $0.5L$, i.e. it would need to move twice as fast as S, or $1.732c$ to reach the right end of R before event 3 occurred.

Both in SR and LET, there are locations where an observer at rest relative to A will receive the pulses from event 2 before receiving the pulse from event 3. There are other locations where an observer receives the pulses in the opposite order. See the next section for many details on this.

4.1.1 Where do observers at rest in A's frame receive event 2 or 3 pulse first?

For this section, assume either Special Relativity where rod R and observer A are motionless with respect to us, or Lorentz Ether Theory where they are motionless with respect to the ether. Thus in this section, light propagates in all directions with speed c. For SR, all times will be measured in R's frame. For LET, all times are true times, as measured by clocks at rest relative to the ether. We will analyze more general situations later, if we can get a good understanding of this one.

In scenario 2a, consider every location in the XY plane. A receiver at rest relative to rod R will receive the pulse from event 2 before the pulse from event 3 in many locations, e.g. at the location of event 2 at the left end of R. Similarly, there are many locations that will receive the event 3 pulse first, e.g. at the location of event 3 at the right end of R. There are also many locations that receive both pulses at the same time.

What is the shape of all such places?

Let R's left end be placed at coordinates $(-L/2, 0)$ and its right end at $(L/2, 0)$. As discussed in the previous section, the sending times of pulses for events 1, 2, and 3 are as follows (we omit the A in the subscripts for brevity here);

$$
\begin{aligned}
q_1 &= 0 && \text{for convenience as a starting time} \\
q_2 &= \frac{L/\gamma}{v} && \text{distance moved by } S \text{ since Event 1 is its contracted length} \\
q_3 &= \frac{L}{v} && \text{distance moved by } S \text{ since Event 1 is } R\text{'s rest length}
\end{aligned}
$$

At an arbitrary location (x, y), it receives the pulse from event 2 at time e_2, and the pulse from event 3 at time e_3, given by:

$$
\begin{aligned}
e_2 &= q_2 + d_l/c && d_l \text{ is distance between left end of } R \text{ and } (x, y) \\
e_3 &= q_3 + d_r/c && d_r \text{ is distance between right end of } R \text{ and } (x, y)
\end{aligned}
$$

For a location that receives pulse 3 time k later than receiving pulse 2 (or before if $k < 0$) the following equation must be true:

$$
\begin{aligned}
e_3 - e_2 &= k \\
(q_3 + \frac{d_r}{c}) - (q_2 + \frac{d_l}{c}) &= k && \text{defns. of } e_2, e_3 \\
d_r - d_l &= c(k - (q_3 - q_2)) && \text{algebra} \qquad (25)
\end{aligned}
$$

Below we derive equivalent formulas for $q_3 - q_2$:

$$
\begin{aligned}
q_3 - q_2 &= \frac{L}{v} - \frac{L}{\gamma v} && \text{defns. of } q_2, q_3 \\
&= \frac{L}{v}(1 - \frac{1}{\gamma}) && \text{algebra} \\
&= \frac{L}{\beta c}(1 - \frac{1}{\gamma}) && \text{replace } v \text{ with } \beta c \qquad (26) \\
&= \frac{L}{c}\frac{\beta\gamma}{\gamma + 1} && \text{Lemma 6} \qquad (27)
\end{aligned}
$$

The equivalent equation below is a bit easier to see the behavior as β approaches 0 or 1:

$$
\frac{\beta\gamma}{\gamma + 1} = \beta(1 - \frac{1}{\gamma + 1})
$$

When β approaches 0, this value approaches $0/2 = 0$. When β approaches 1, this value approaches $1(1 - 0) = 1$. It is straightforward to verify by calculating the derivative that this function is strictly increasing with β over the interval $(0, 1)$.

Substituting Equation (27) into Equation (25) gives:

$$
\begin{aligned}
d_r - d_l &= c\left(k - \left(\frac{L}{c}\right)\frac{\beta\gamma}{\gamma + 1}\right) \\
&= ck - \frac{L\beta\gamma}{\gamma + 1} \qquad (28)
\end{aligned}
$$

If we restrict our attention to situations where L, β, and k are given constants, this equation is of the form $d_r - d_l$ equal to a constant. The shape of all such curves is a hyperbola [12].

Figure 6 shows the XY plane with x and y coordinates in units of light-seconds. $L = 1$ light-second, with rod R parallel to the x axis, centered at the origin. $\beta = 0.5$ for this figure. First consider the line labeled $k = 0.27$ in the figure. This is the line where $k = q_3 - q_2$, so the pulse for event 3 arrives $k = 0.27$ sec after the pulse for event 2. This is exactly the difference in times between when the pulses

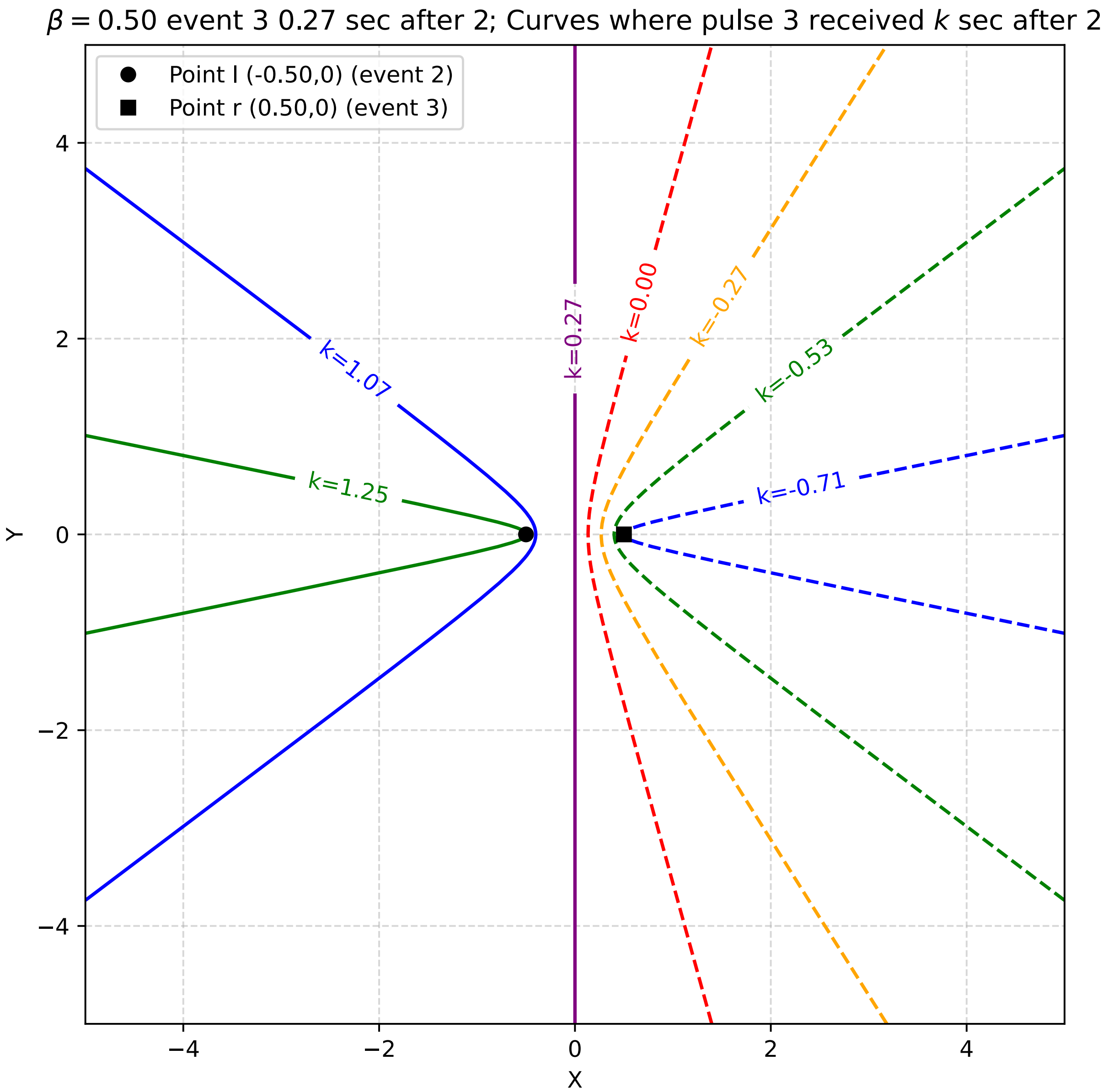

Figure 6: Locations with selected values of k for $\beta = 0.5$, $q_3 - q_2 = 0.27$ sec

are generated, so this line is all of the points equally distant in length from the two ends of rod R; a vertical line along the y axis.

Next consider the curve labeled $k = 0$. These are all locations where the two pulses arrive simultaneously. Since pulse 2 is sent 0.27 sec earlier than pulse 3, all of these locations are 0.27 light-seconds further from the left end of R than from the right end.

The last curve we will give special attention to is the curve labeled $k = -0.27$. These are all locations where pulse 3 (emitted later), arrives 0.27 *earlier* than pulse 2 (emitted earlier), which is the relative order and separation in time that B receives the pulses according to SR.

For the line $y = D$ where A and B are located (far above the bounds of the figure), A is on the line $k = 0.27$ at coordinates $(0, D)$. But B is moving to the right at speed v, so even though it is near A when the pulses are emitted, B is a significant distance to the right of A when B receives the pulses. Is it so far to the right that it is on the $k = -0.27$ line?

As we will show in the next section, B reaches far enough to the right that if you measure the difference in times that it receives the pulses, according to time in A's frame, and then multiply that by $1/\gamma$ to account for B's clock running slower than A's clock, the result is -0.27 sec. And this correspondence is true for all β values, not only $\beta = 0.5$ used in Figure 6.

4.1.2 Difference in time of pulses arriving to B, calculated in A's frame

For calculating when B receives the pulses from events 2 and 3, it is useful to know the location of B at any time where it is convenient to determine. At the time of event 1 ($q_1 = 0$) the right end of rod S is at the same x coordinate as the left end of rod R, $x = -L/2$. The center of rod S is thus half of S's contracted length further to the left, at $x = -(L/2)(1+1/\gamma)$. B's x coordinate is the same as the center of S at all times.

Time when B receives pulse from event 2 Note that this time is calculated in R's frame. In the notation of Lemma 8:

$$\begin{aligned}
\mathbf{s} &= (-L/2, 0) && \text{pulse sent from } \mathbf{s}\text{, location of event 2} \\
t_0 &= \frac{L}{\gamma v} && \text{pulse sent at time } t_0\text{, time of event 2} \\
\mathbf{s} + \mathbf{r}_1 &= ((-L/2)(1+1/\gamma), D) && \mathbf{s} + \mathbf{r}_1 \text{ is location of } B \text{ at time } t_1 = 0 \\
\mathbf{r}_1 &= (-L/(2\gamma), D) && \text{subtract } \mathbf{s} \text{ from previous line to get } \mathbf{r}_1 \\
\mathbf{r}_0 &= \mathbf{r}_1 + \mathbf{v}(t_0 - t_1) \\
&= (-L/(2\gamma) + vL/(\gamma v), D) \\
&= (L/(2\gamma), D)
\end{aligned}$$

The result of Lemma 8 gives us the time (in R's frame) when B receives the event 2 pulse:

$$e_2 = \frac{L}{\gamma v} + \frac{\gamma^2}{c^2}\left[\frac{Lv}{2\gamma} + \sqrt{\frac{L^2 v^2}{4\gamma^2} + (1-\beta^2)c^2(\frac{L^2}{4\gamma^2} + D^2)}\right] \tag{29}$$

Time when B receives pulse from event 3 This is very similar to the calculation for event 2. Again in the notation of Lemma 8:

$$\begin{aligned}
\mathbf{s} &= (L/2, 0) && \text{pulse sent from } \mathbf{s}\text{, location of event 3} \\
t_0 &= \frac{L}{v} && \text{pulse sent at time } t_0\text{, time of event 3} \\
\mathbf{s} + \mathbf{r}_1 &= ((-L/2)(1+1/\gamma), D) && \mathbf{s} + \mathbf{r}_1 \text{ is location of } B \text{ at time } t_1 = 0\text{, same as above} \\
\mathbf{r}_1 &= (-L - L/(2\gamma), D) && \text{subtract } \mathbf{s} \text{ from previous line to get } \mathbf{r}_1 \\
\mathbf{r}_0 &= \mathbf{r}_1 + \mathbf{v}(t_0 - t_1) \\
&= (-L - L/(2\gamma) + vL/v, D) \\
&= (-L/(2\gamma), D)
\end{aligned}$$

The result of Lemma 8 gives us the time (in R's frame) when B receives the event 3 pulse:

$$e_3 = \frac{L}{v} + \frac{\gamma^2}{c^2}\left[-\frac{Lv}{2\gamma} + \sqrt{\frac{L^2 v^2}{4\gamma^2} + (1-\beta^2)c^2(\frac{L^2}{4\gamma^2} + D^2)}\right] \tag{30}$$

Relative time when B receives pulses from events 2 and 3 It is convenient that the subexpressions of Equations (29) and (30) under the radicals are equal, so they cancel out when we determine

$e_3 - e_2$, resulting in:

$$\begin{aligned}
e_3 - e_2 &= \frac{L}{v}(1 - \frac{1}{\gamma}) + \frac{\gamma^2}{c^2}\left[-\frac{Lv}{2\gamma} - \frac{Lv}{2\gamma}\right] \\
&= \frac{L}{v}(1 - \frac{1}{\gamma}) - \frac{\gamma L v}{c^2} && \text{algebra} \\
&= \frac{L}{\beta c}(1 - \frac{1}{\gamma}) - \frac{\gamma L \beta}{c} && \text{replace } v \text{ with } \beta c \text{ and algebra} \\
&= \frac{L}{c}\left[\frac{1}{\beta}(1 - \frac{1}{\gamma}) - \beta\gamma\right] && \text{algebra} \\
&= \frac{L}{c}\left(\frac{\beta\gamma}{\gamma+1} - \beta\gamma\right) && \text{Lemma 6} \\
&= \frac{L\beta\gamma}{c}\left(\frac{1}{\gamma+1} - 1\right) && \text{algebra} \\
&= \frac{L\beta\gamma}{c}\left(\frac{-\gamma}{\gamma+1}\right) && \text{algebra} \\
&= -\frac{L}{c}\left(\frac{\beta\gamma^2}{\gamma+1}\right) && \text{algebra} \qquad (31)
\end{aligned}$$

We want to show that when we convert this difference $e_3 - e_2$ into B's slower-running clock, by multiplying by $1/\gamma$, we get the *negative* of $q_3 - q_2$, where $q_3 - q_2$ is the (positive) time that event 3's pulse is sent after event 2's pulse is sent, in A's frame.

$$\begin{aligned}
\frac{1}{\gamma}(e_3 - e_2) &= -\frac{L}{\gamma c}\left(\frac{\beta\gamma^2}{\gamma+1}\right) && \text{Eqn. (31)} \\
&= -\frac{L}{c}\left(\frac{\beta\gamma}{\gamma+1}\right) && \text{algebra} \\
&= -(q_3 - q_2) && \text{Eqn. (27)}
\end{aligned}$$

4.2 Special Relativity Scenario 2b: Two rods moving towards each other at constant velocity along x axis, and both relative to observer

Scenario 2b is a small generalization of Scenario 2a. The intent in doing this is that in LET, if A and B do not know their motion relative to the ether, the results in this section will show what they will be able to observe and measure, for at least some kinds of motion of both of them relative to the ether, while they only directly observe their uniform constant motion to each other and the rods.

TODO: It would be nice to generalize this further to A and B at uniform speed motion towards each other, but they both have an arbitrary 2-dimensional velocity vector $\mathbf{v} = (v_x, v_y)$ in the XY plane.

The only differences between Scenario 2b and Scenario 2a are that rod R and A are moving parallel to the x direction with velocity v_R. The velocity of S and B is denoted v_S to distinguish it from v_R.

S starts at a lesser x coordinate than R. We want S to catch up to R, so we only consider $v_S > v_R$. We also want S to be length-contracted to a shorter length than R, to make it similar to Scenario 2a and reduce the number of cases to consider, so we only consider $|v_S| > |v_R|$.

We also add another observer E that is at rest relative to S, but unlike B it remains at an x coordinate (as measured in rest frame) that is at an offset Z away from B, i.e. E is at the same position as B if $Z = 0$, it is further to the left of B if $Z < 0$, and it is to the right of B if $Z > 0$.

All calculations are done in the rest frame, except for conversion of time intervals to/from A and B's frame as near to the beginning or end of the calculation as possible. This keeps all calculations consistent with LET in the rest frame, or SR in the frame of a new observer O with velocity 0.

R is contracted to length L/γ_R, and S is contracted to length L/γ_S. Events 1, 2, and 3 are the same as in Scenario 2a.

$$x_R(t) = x_{R,0} + v_R t \qquad x_R(t) \text{ is } x \text{ coordinate of } R\text{'s center, and } A\text{, at time } t \qquad (32)$$

$$x_S(t) = x_{S,0} + v_S t \qquad x_S(t) \text{ is } x \text{ coordinate of } S\text{'s center, and } B\text{, at time } t \qquad (33)$$

$$x_E(t) = x_{S,0} + Z + v_S t \qquad x_E(t) \text{ is } x \text{ coordinate of } E \text{ at time } t \qquad (34)$$

The details of the calculations are in Appendix F. They are very similar to those for Scenario 2a.

The result is that the times $e_{2,A}$ and $e_{3,A}$ when A receives the pulses from event 2 and 3, as measured in the rest frame, differ by the following (a positive value, since A receives the pulse for event 2 before the pulse for event 3):

$$d_A = e_{3,A} - e_{2,A} = \frac{\gamma_R L}{c}\left[\frac{\beta_S\gamma_S - \beta_R\gamma_R}{\gamma_S + \gamma_R}\right] \tag{35}$$

The times $e_{2,B}$ and $e_{3,B}$ when B receives the pulses, as measured in the rest frame, differ by the following (a negative value, since B receives the pulse for event 3 before the pulse for event 2):

$$d_B = e_{3,B} - e_{2,B} = -\frac{\gamma_S L}{c}\left[\frac{\beta_S\gamma_S - \beta_R\gamma_R}{\gamma_S + \gamma_R}\right] \tag{36}$$

The difference between receiving the pulses as measured on A's slower clock is $1/\gamma_R$ times Equation (35). Similarly the difference between receiving the pulses as measured on B's slower clock is $1/\gamma_S$ times Equation (36). It is easily seen from the equations above that:

$$\frac{1}{\gamma_R}(e_{3,A} - e_{2,A}) = -\frac{1}{\gamma_S}(e_{3,B} - e_{2,B}) \tag{37}$$

Similar to what was described at the end of Section 3.2.1 for Scenario 1b, where B and C, both in motion relative to A, get measurements consistent with B being at rest and C having relative velocity equal to $v_C \ominus v_B$ relative to B (or C at rest and B having relative velocity $v_B \ominus v_C$ relative to C), there is a similar result here.

The difference in time measured by A on its local clock of the two pulses is:

$$\begin{aligned}
\frac{1}{\gamma_R}(e_{3,A} - e_{2,A}) &= \frac{L}{c}\left[\frac{\beta_S\gamma_S - \beta_R\gamma_R}{\gamma_S + \gamma_R}\right] && \text{Eqn. (35)}\\
&= \frac{L}{c}\tanh\frac{\eta_S - \eta_R}{2} && \text{Lemma 1 in Appendix A.2}\\
&= \frac{L}{c}\tanh\frac{\eta_\Delta}{2} && \text{define } \eta_\Delta = \eta_S - \eta_R
\end{aligned}$$

where $\eta_S = \tanh^{-1}\beta_S$ and $\eta_R = \tanh^{-1}\beta_R$. The rapidity value η_Δ corresponds to the relativistic velocity $\beta_\Delta = \beta_S \ominus \beta_R$. Thus as long as A knows L, it can calculate a value of β_Δ from its measured difference between the pulse arrival times T:

$$\begin{aligned}
T &= \frac{L}{c}\tanh\frac{\eta_\Delta}{2} && \text{from above}\\
\tanh\frac{\eta_\Delta}{2} &= \frac{cT}{L} && \text{algebra}
\end{aligned} \tag{38}$$

then:

$$\begin{aligned}
\beta_\Delta &= \tanh\eta_\Delta && \text{relationship between } \beta \text{ and } \eta \text{ Eqn. (61)}\\
&= \frac{2\tanh\frac{\eta_\Delta}{2}}{1+\tanh^2\frac{\eta_\Delta}{2}} && \text{Eqn. (78)}\\
&= \frac{2(\frac{cT}{L})}{1+(\frac{cT}{L})^2} && \text{Eqn. (38)}
\end{aligned}$$

B measures exactly the negative of the time difference, and following the same kinds of calculations can calculate the negative of β_Δ that A does.

Also from Appendix F we have this somewhat more complex expression for the difference in time when the two pulses reach observer E, which like B is also at rest relative to rod S, but is always at an x coordinate that is an offset Z away from B, measured in the rest frame.

$$d_E = e_{3,E} - e_{2,E} = (e_{3,B} - e_{2,B}) + \frac{L\gamma_S}{c}\left[\frac{-2Z\gamma_S}{|Z_-|\sqrt{\gamma_S^2 + (D/Z_-)^2} + |Z_+|\sqrt{\gamma_S^2 + (D/Z_+)^2}}\right] \tag{39}$$

where:

$$Z_{-} = Z - \frac{L}{2\gamma_S}$$
$$Z_{+} = Z + \frac{L}{2\gamma_S}$$

Now we will find the answers to these questions, and create some charts with example values:

- Where must E start in order for E to be exactly at A's position at the time that A receives the pulse from Event 3? (Note that since E and A will be at the same position, E will also receive the pulse from Event 3 at the same time.)
- How does d_E compare to d_A when E and A receive Event 3's pulse at the same time and place?

To answer the first question, note that we are asking for the value of Z that makes $x_E(e_{3,A}) = x_R(e_{3,A})$ true.

$$\begin{aligned}
x_E(e_{3,A}) &= x_R(e_{3,A}) \\
x_{S,0} + Z + v_S e_{3,A} &= x_{R,0} + v_R e_{3,A} && \text{Eqns. (32) and (34)} \\
Z &= (x_{R,0} - x_{S,0}) + (v_R - v_S)e_{3,A} && \text{algebra} \\
Z &= \frac{L}{2}(\frac{1}{\gamma_R} + \frac{1}{\gamma_S}) + (v_R - v_S)e_{3,A} && \text{Eqn. (149) from Appendix F} \qquad (40)
\end{aligned}$$

Now to calculate d_E and compare it to d_A, we plug in that value of Z into Equation (39).

Figure 7 shows a series of similar drawings representing 8 different times of interest in a scenario where:

- The rod lengths are $L = 1$ light-second in rest length.
- observers A, B, and E are all $D = 3$ light-seconds above the line of the rods.
- A and rod R at rest so $\beta_R = 0$.
- B, E, and rod S are moving from left to right with speed $\beta_S = 0.866$, so $\gamma_S = 2$, and rod S is length-contracted by a factor of 2.
- The value of $Z = -2.884$ light-seconds was chosen so that E and A receive the pulse for event 3 at the same time and place.

Each sub-figure is labeled with its time in seconds, as measured on A's clock (at rest relative to the ether frame), and has a brief description of the event of interest that occurs at that time.

TODO: How does the aberration of light affect what the observers see in this scenario, if it does? For example, does A see the two light pulses come from the direction shown in A's reference frame, nearly straight down, but B sees them come in from a diagonal direction from down and to the left? If so, then what about SR's perspective of B's inertial frame, where the corresponding series of pictures would indicate that B receives the pulses from nearly straight down, but A would receive them from a diagonal direction down and to the right? Does it make a difference whether the light pulses are emitted from rod R or S, as to which direction A, B see them come from? If so, why? I believe the answer is that A sees the direction of the pulses from nearly straight down. B sees them from the same direction that A does, because of aberration due to B's motion.

Figure 8 compares the values of d_E and d_A when E and A receive pulse 3 at the same time and place, for $\beta_R = 0$, and all values of β_S from 0.001 up to 0.999. $L = 1$ light-second, and $D = 3$ light-seconds. The previous series of 8 figures has the measurements from this figure for $\beta_S = 0.866$.

Let us examine the curves one at a time. The curve labeled Z/D shows the value of Z that leads to E and A being at the same place when they both receive the pulse for Event 3. It appears close to a straight line, suggesting that either $Z = -\beta_S D$, or quite close to it. As can be seen by examining the curve labeled $Z + (beta * D)$, we see that Z is always lower than $-\beta_S D$, but never by more than 0.5.

The top two curves with stars and X's labeling them **d_E_restclock** and **d_A_restclock**. The curve with label beginning **1** shows that d_E is always larger than d_A, but never by more than about 5%.

The curve labeled **d_E_Bclock** shows the value of d_E/γ_S, i.e. the difference between receiving the pulses from Event 3, minus the time that the pulse from Event 2 is received, by E, measured on its own clock that is running slower than the rest clock by a factor of $1/\gamma_S$.

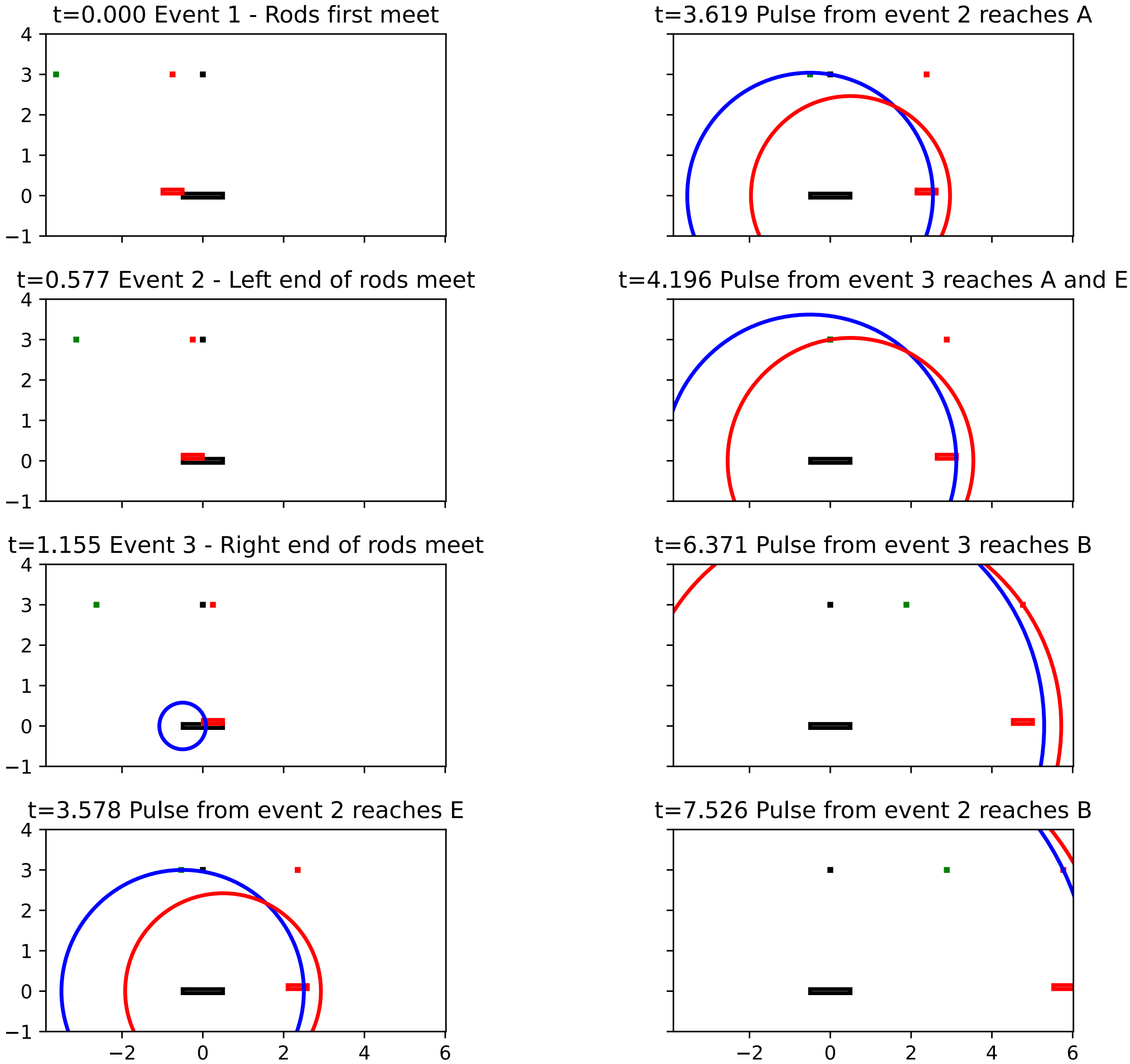

Figure 7: 8 times of interest in Scenario 2b: R, A black, S, B red, E green, circles show expanding wavefronts of pulses from Event 2 and 3

Figure 9 is the same as Figure 8, except it is for $D = 1000$, so all of the observers are much further away from the rods. Here the top two curves with stars and X's labeling them are so close that one overlaps the other at this scale: the curves labeled **d_E_restclock** and **d_A_restclock**. The curve with label beginning **1000** shows that d_E and d_A are not equal, but they never differ by more than about 1 part in 4000.

5 Special Relativity Scenario 3b: Angled rods length-contracted diagonally

A slightly less general scenario is described by Iyer and Prabhu [5], where one rod is at rest and the other is moving towards it with constant velocity. Here I start by analyzing the more general scsenario where both rods are moving relative to an observer, as well as each rod having its own observer at rest relative to it. The more restrictive case of one of the rods at rest is easily derived from the more general result by plugging in its velocity as 0.

See Appendix B for notes on why a rod, where its length is at an angle to its direction of velocity, is at a different angle when it is length-contracted.

Figure 10 shows the initial configuration, and the two events of interest. Both rods R and S have a

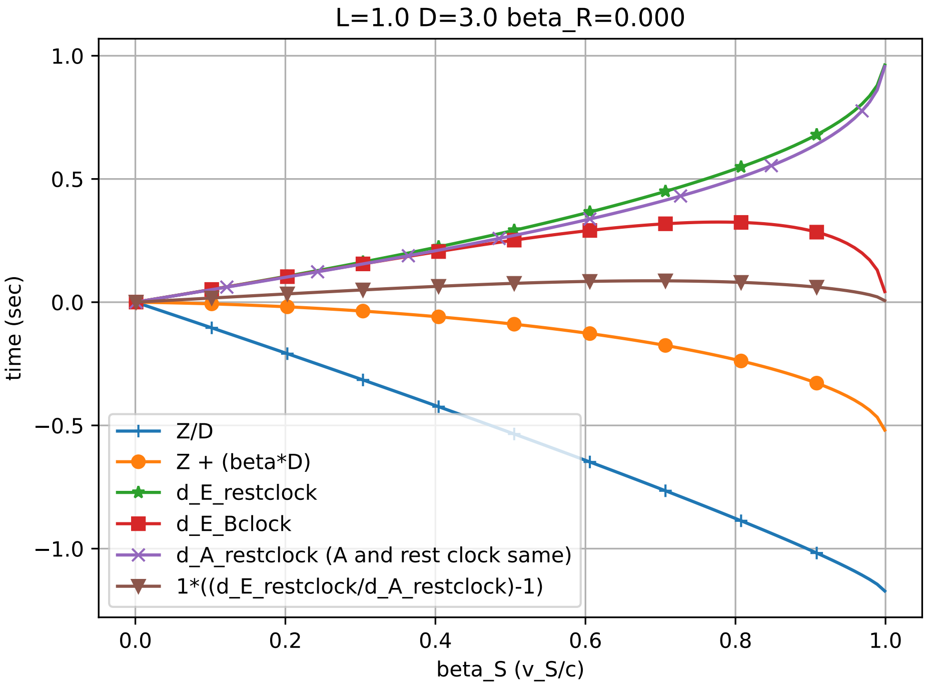

Figure 8: Comparing d_E to d_A when E and A receive pulse 3 at the same time and place, $D = 3$

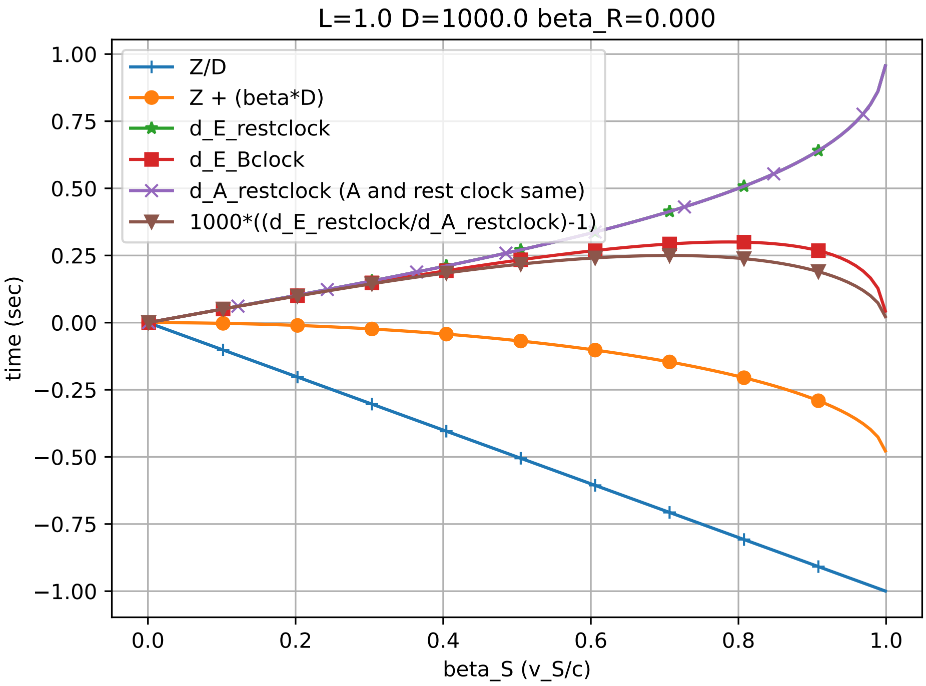

Figure 9: Comparing d_E to d_A when E and A receive pulse 3 at the same time and place, $D = 1000$

rest length and angle such that their projection on the x axis has width $2w$, and their projection on the y axis has height $2h$. Because of length contraction, rod R moving in the positive x direction with speed v_R has width $2w_R = 2w/\gamma_R$, and rod S with speed v_S has width $2w_S = 2w/\gamma_S$. This figure is drawn from the perspective of an observer C at rest relative to the velocities v_R and v_S. $v_S > v_R > 0$, so S catches up to R.

Rod R has radio pulse emitters at both ends, and detectors that trigger the corresponding emitter when the rod S's corresponding end is nearly touching it, such that a light pulse is emitted at the moment when the ends of the rods nearly touch. The rods move in parallel planes that are very close to each other, but without the rods actually colliding. We leave it to other authors to discuss the possible

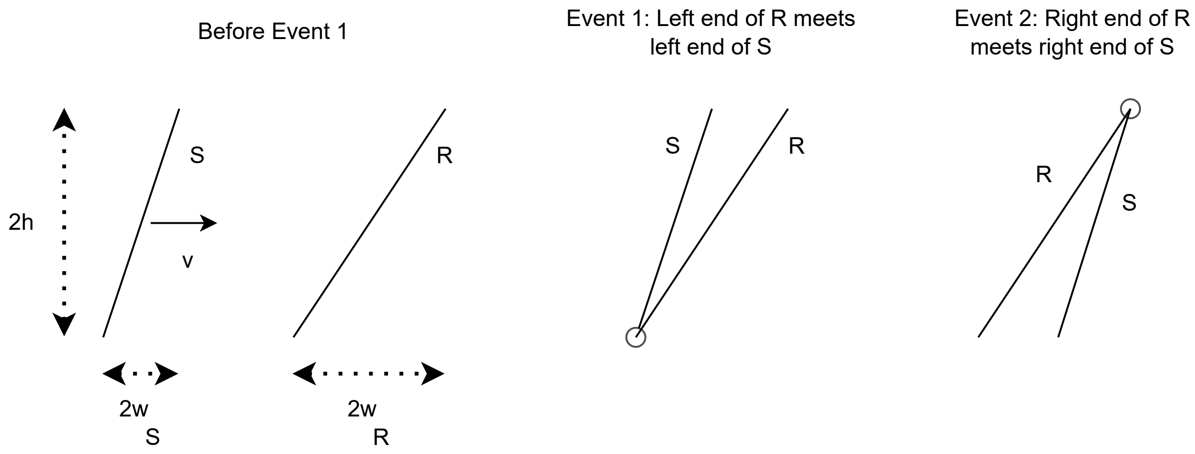

Figure 10: Two angled rods moving towards each other, and 2 events of interest

consequences of the rods colliding at such speeds.

Observer A is at rest relative to R, at a distance D above R's center, in the positive z direction. Similarly B is at rest relative to S, at a distance D above S's center.

Observer D we will consider last. D lies on the line between A and B, at a position to be specified later, and with a velocity v_D whose rapidity is the average of the rapidities of v_R and v_S (see Appendix A.2 for more on rapidity). With velocity v_D, D observes both rods with the same length-contracted width, and thus we will see that with the proper position, D receives the pulses from both events simultaneously.

The details of the calculation can be found in Appendix G. The difference in time of the pulses arriving to A, B, and D are as follows, measured in C's frame:

$$e_{2,A} - e_{1,A} = \frac{2\gamma_R w}{c}\left[\frac{\beta_S\gamma_S - \beta_R\gamma_R}{\gamma_S + \gamma_R}\right] \tag{41}$$

$$e_{2,B} - e_{1,B} = -\frac{2\gamma_S w}{c}\left[\frac{\beta_S\gamma_S - \beta_R\gamma_R}{\gamma_S + \gamma_R}\right] \tag{42}$$

$$e_{2,D} - e_{1,D} = 0$$

Note that these are the same as the results of Scenario 2b, with the only minor difference being that the rod length L in Scenario 2b is replaced here with the width of the angled rod's projection onto the x axis $2w$. The height h does not affect the results here, and thus these results are equivalent to Scenario 2b in the case $h = 0$.

As for Scenario 2b, we see here that if we calculate the difference in time when the pulses are received by A on A's clock (by dividing by γ_R), and the difference in time received by B on B's clock (by dividing by γ_S), B has the negative of A's difference, because B receives the pulses in the opposite order.

$$\frac{1}{\gamma_R}(e_{2,A} - e_{1,A}) = -\frac{1}{\gamma_S}(e_{2,B} - e_{1,B}) \tag{43}$$

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A Miscellaneous math facts

A.1 Math facts about relativistic velocity addition and subtraction in one dimension

All of the facts here are quite simple to validate. I write them out primarily as an aid to thinking about and remembering them, and they might also be useful to refer to from elsewhere in this document.

Note that this appendix is restricted to proofs of simple mathematical relationships about the definitions of one-dimensional relativistic velocity addition and subtraction formulas. This appendix makes no claims about the physical meaning of these operations.

I have read that relativistic velocity addition in 3 dimensions is not associative (see Appendix A.3). In one dimension, though, all velocities of subliminal entities can be represented by v such that $-c < v < c$, where negative velocities are in the opposite direction along the line than positive velocities.

The can also be represented as relativistic velocity $\beta = v/c$, i.e. a fraction of c. These are in the range $-1 < \beta < 1$.

I will use the notation $v \oplus w$ for relativistic velocity addition. While I will use the same symbol $\beta_v \oplus \beta_w$ for relativistic velocity addition of relativistic velocity values, one should be careful to note that the definition of the operator $\oplus$ is slightly different for these two cases.

$$v \oplus w = \frac{v+w}{1+\frac{vw}{c^2}} \tag{44}$$

$$v \ominus w = \frac{v-w}{1-\frac{vw}{c^2}} \tag{45}$$

$$\beta_v \oplus \beta_w = \frac{\beta_v+\beta_w}{1+\beta_v\beta_w} \tag{46}$$

$$\beta_v \ominus \beta_w = \frac{\beta_v-\beta_w}{1-\beta_v\beta_w} \tag{47}$$

The proofs are only given for the relativistic velocity formulas (46) and (47). The proofs for equations (44) and (45) are nearly identical.

We will use the symbols a, b, c to represent arbitrary real values between -1 and 1, to avoid writing β with subscripts all over the place.

$$a \oplus b = b \oplus a \qquad \text{addition is commutative (1-d only, not 3-d!)} \tag{48}$$

$$(a \oplus b) \oplus c = a \oplus (b \oplus c) \qquad \text{addition is associative (1-d only, not 3-d!)} \tag{49}$$

$$a \oplus 0 = 0 \oplus a = a \tag{50}$$

$$a \oplus (-b) = a \ominus b \tag{51}$$

$$a \ominus (-b) = a \oplus b \tag{52}$$

$$0 \ominus b = -b \tag{53}$$

$$a \ominus b = -(b \ominus a) \tag{54}$$

The following inequalities only hold if $0 < a < 1$ and $0 < b < 1$:

$$a \oplus b > a \tag{55}$$

$$a \oplus b > b \tag{56}$$

$$a \oplus b < a + b \tag{57}$$

The following inequality holds for all $-1 < a < 1$ and $-1 < b < 1$:

$$-1 < a \oplus b < 1 \tag{58}$$

The last one is only slightly different if we use velocity addition on normal velocities, i.e. velocities that are not relativistic velocities, where $-c < v < c$ and $-c < w < c$:

$$-c < v \oplus w < c \tag{59}$$

The inequalities all have examples where the two sides can be very nearly equal, such as these:

$$\begin{aligned} 0.1 \oplus 0.001 &\approx 0.100989901 > 0.1 \\ 0.1 \oplus 0.1 &\approx 0.198019802 < 0.1 + 0.1 \\ 0.99 \oplus 0.99 &\approx 0.999949498 < 1 \end{aligned}$$

Proving most of these is as simple as substituting the definition of $a \oplus b$ and $a \ominus b$, and a tiny amount of algebra.

Proving associativity (49) is only a little bit more algebra, but we will do the steps here:

$$\begin{aligned} (a \oplus b) \oplus c &= \frac{\frac{a+b}{1+ab} + c}{1 + \left(\frac{a+b}{1+ab}\right) c} && \text{use Defn. (46) twice} \\ &= \frac{(a+b) + c(1+ab)}{(1+ab) + (a+b)c} && \text{multiply numerator and denominator by } (1+ab) \\ &= \frac{a+b+c+abc}{1+ab+ac+bc} && \text{multiply out the terms} \end{aligned}$$

Similarly:

$$
\begin{aligned}
a \oplus (b \oplus c) &= \frac{a + \frac{b+c}{1+bc}}{1 + a\left(\frac{b+c}{1+bc}\right)} && \text{use Defn. (46) twice} \\
&= \frac{a(1+bc) + (b+c)}{(1+bc) + a(b+c)} && \text{multiply numerator and denominator by } (1+bc) \\
&= \frac{a + abc + b + c}{1 + bc + ab + ac} && \text{multiply out the terms}
\end{aligned}
$$

The above two final results are easily seen to be equal.

The inequalities are also not difficult to prove, but we will write out their short proofs. Recall that these inequalities are true only for $0 < a < 1$ and $0 < b < 1$. Similar inequality hold if both a and b are negative.

For the proof of inequality (55), recall that we can multiply or divide both sides of an inequality by the same positive number, and the resulting inequality is true if and only if the original one was. The symbol $\Leftrightarrow$ below means "if and only if", i.e. the expression before is true if and only if the expression after is true.

$$
\begin{aligned}
a < \frac{a+b}{1+ab} &\Leftrightarrow a + a^2 b < a + b && \text{multiply both sides by } 1+ab\text{, which is positive} \\
&\Leftrightarrow a^2 b < b && \text{subtract } a \text{ from both sides} \\
&\Leftrightarrow a^2 < 1 && \text{divide both sides by } b\text{, which is positive}
\end{aligned}
$$

The last inequality is true because $a < 1$. The proof of inequality (56) is the same as above.

To prove $a \oplus b < a + b$ (inequality (57)):

$$
\begin{aligned}
\frac{a+b}{1+ab} < a + b &\Leftrightarrow a + b < (a+b)(1+ab) && \text{multiply both sides by } 1+ab\text{, which is positive} \\
&\Leftrightarrow 1 < (1+ab) && \text{divide both sides by } a+b\text{, which is positive} \\
&\Leftrightarrow 0 < ab && \text{subtract 1 from both sides}
\end{aligned}
$$

The final inequality is true because both a and b are positive.

Now to prove $a \oplus b < 1$ (part of inequality (58)), but recall now we are doing so for the more general case of all values $-1 < a < 1$ and $-1 < b < 1$:

$$
\begin{aligned}
\frac{a+b}{1+ab} < 1 &\Leftrightarrow a + b < 1 + ab && \text{multiply both sides by } 1+ab\text{, which is positive} \\
&\Leftrightarrow b - ab < 1 - a && \text{subtract } a + ab \text{ from both sides} \\
&\Leftrightarrow b(1-a) < 1 - a && \text{algebra} \\
&\Leftrightarrow b < 1 && \text{divide both sides by } 1-a\text{, which is positive}
\end{aligned}
$$

And the last inequality is true. Proving the part that

$$
\begin{aligned}
-1 < \frac{a+b}{1+ab} &\Leftrightarrow -(1+ab) < a + b && \text{multiply both sides by } 1+ab\text{, which is positive} \\
&\Leftrightarrow -(1+a) < b + ab && \text{add } ab - a \text{ to both sides} \\
&\Leftrightarrow -(1+a) < b(1+a) && \text{algebra} \\
&\Leftrightarrow -1 < b && \text{divide both sides by } 1+a\text{, which is positive}
\end{aligned}
$$

And the last inequality is true.

A.2 Using hyperbolic functions for relativistic velocity arithmetic in one dimension

Logarithms and exponential functions are well known for the following properties:

$$
\begin{aligned}
l &= \log_{10} x \\
x &= 10^l \\
l_3 = l_1 + l_2 &\Leftrightarrow x_3 = x_1 x_2 && + \text{ of logs corresponds to multiplication of numbers} \\
l_3 = l_1 - l_2 &\Leftrightarrow x_3 = x_1/x_2 && - \text{ of logs corresponds to division of numbers} \\
l_3 = c l_1 &\Leftrightarrow x_3 = x_1^c && \text{multiplying log by constant corresponds to exponentiation of numbers} \\
l_3 = l_1/c &\Leftrightarrow x_3 = x_1^{1/c} && \text{dividing log by constant corresponds to taking } c\text{-th root of numbers}
\end{aligned}
$$

Hyperbolic functions [13] have a similar correspondence to relativistic velocities β and related quantities like the Lorentz factor γ.

$$
\begin{aligned}
\eta &= \tanh^{-1} \beta && \text{calculate the } \textit{rapidity } \eta \text{ from relativistic velocity} && (60) \\
\beta &= \tanh \eta && \text{get } \beta \text{ from rapidity} && (61) \\
\gamma &= \cosh \eta && \text{get } \gamma \text{ from rapidity} && (62) \\
D &= \sqrt{\frac{1+\beta}{1-\beta}} = e^\eta && \text{get Doppler factor from rapidity} && (63) \\
\beta\gamma &= \sinh \eta && \text{get } \beta\gamma \text{ from rapidity} && (64) \\
\eta_3 = \eta_1 + \eta_2 &\Leftrightarrow \beta_3 = \beta_1 \oplus \beta_2 && + \text{ of rapidity corresponds to 1-d relativistic velocity } \oplus && (65) \\
&\Leftrightarrow D_3 = D_1 D_2 && \text{and to 1-d Doppler factor multiplication} && (66) \\
\eta_3 = \eta_1 - \eta_2 &\Leftrightarrow \beta_3 = \beta_1 \ominus \beta_2 && - \text{ of rapidity corresponds to 1-d relativistic velocity } \ominus && (67) \\
&\Leftrightarrow D_3 = D_1/D_2 && \text{and to 1-d Doppler factor division} && (68) \\
& && && (69)
\end{aligned}
$$

Also:

- Multiplying a rapidity by a positive number n corresponds to repeatedly adding that relativistic velocity n times.
- Dividing a rapidity by a positive number n corresponds to finding a relativistic velocity β such that repeated velocity addition of β n times will give back the original relativistic velocity.

There are an abundance of identities involving hyperbolic functions, similar to the large variety of identities relating trigonometric functions. These identities can be useful for finding relationships between relativistic β and γ values.

Examples of a few hyperbolic function identities, but there are many more [13]:

$$\sinh(-x) = -\sinh x \tag{70}$$
$$\cosh(-x) = \cosh x \tag{71}$$
$$\tanh(-x) = -\tanh x \tag{72}$$
$$\sinh(x \pm y) = \sinh x \cosh y \pm \cosh x \sinh y \tag{73}$$
$$\cosh(x \pm y) = \cosh x \cosh y \pm \sinh x \sinh y \tag{74}$$
$$\tanh(x \pm y) = \frac{\tanh x \pm \tanh y}{1 \pm \tanh x \tanh y} \tag{75}$$
$$\sinh x + \sinh y = 2 \sinh\left(\frac{x+y}{2}\right) \cosh\left(\frac{x-y}{2}\right) \tag{76}$$
$$\cosh x + \cosh y = 2 \cosh\left(\frac{x+y}{2}\right) \cosh\left(\frac{x-y}{2}\right) \tag{77}$$
$$\tanh 2x = \frac{2 \tanh x}{1 + \tanh^2 x} \tag{78}$$

Below are a couple of examples of the use of hyperbolic functions, relevant elsewhere in this document, that I have personally found much more difficult to validate without using hyperbolic functions.

Lemma 1. *Let β_1, β_1 be any real values in the range $(-1,1)$, γ_1, γ_1 their corresponding Lorentz factors, and $\eta_i = \tanh^{-1}\beta_i$ their rapidities. Then:*

$$\frac{\beta_1\gamma_1 - \beta_2\gamma_2}{\gamma_1 + \gamma_2} = \tanh\frac{\eta_1 - \eta_2}{2} \tag{79}$$

Proof.

$$\begin{aligned}
\frac{\beta_1\gamma_1 - \beta_2\gamma_2}{\gamma_1 + \gamma_2} &= \frac{\sinh\eta_1 - \sinh\eta_2}{\cosh\eta_1 + \cosh\eta_2} && \text{Eqns. (64) and (62)} \\
&= \frac{2\sinh\left(\frac{\eta_1-\eta_2}{2}\right)\cosh\left(\frac{\eta_1+\eta_2}{2}\right)}{2\cosh\left(\frac{\eta_1+\eta_2}{2}\right)\cosh\left(\frac{\eta_1-\eta_2}{2}\right)} && \text{Eqns. (76) and (77)} \\
&= \frac{\sinh\left(\frac{\eta_1-\eta_2}{2}\right)}{\cosh\left(\frac{\eta_1-\eta_2}{2}\right)} && \text{cancel common terms} \\
&= \tanh\frac{\eta_1-\eta_2}{2} && \tanh x = \sinh x/\cosh x
\end{aligned}$$

□

Corollary 2. *Let β be any real value in the range $(-1,1)$, γ its corresponding Lorentz factor, and $\eta = \tanh^{-1}\beta$ its rapidity. Then:*

$$\frac{\beta\gamma}{\gamma + 1} = \tanh\frac{\eta}{2} \tag{80}$$

Proof. Follows directly from Lemma 1 by substituting $\beta_2 = 0$, from which follows $\gamma_2 = 1$ and $\eta_2 = 0$. □

A.3 Relativistic velocity addition and subtraction in 3 dimensions

A.3.1 Decomposing vector u into components parallel and perpendicular to vector v

Sometimes it can be useful to take vectors $\mathbf{u}$ and $\mathbf{v}$, and express $\mathbf{u}$ as a sum of two vectors $\mathbf{u}_{||}$ and $\mathbf{u}_\perp$ that satisfy these conditions:

- $\mathbf{u} = \mathbf{u}_{||} + \mathbf{u}_\perp$
- $\mathbf{u}_{||}$ is parallel to $\mathbf{v}$.
- $\mathbf{u}_\perp$ is perpendicular to $\mathbf{v}$, i.e. $\mathbf{u}_\perp \cdot \mathbf{v} = 0$.

The direction of $\mathbf{u}_{||}$ is parallel to $\mathbf{v}$. The magnitude of $\mathbf{u}_{||}$ is the portion of $\mathbf{u}$'s magnitude that is parallel to $\mathbf{v}$, which is $u\cos\theta$, where θ is the angle between $\mathbf{u}$ and $\mathbf{v}$.

$$\begin{aligned}
\mathbf{u}_{||} &= (\text{unit vector in direction of } \mathbf{v})(\text{magnitude of } \mathbf{u}_{||}) \\
&= \left(\frac{\mathbf{v}}{v}\right)(u\cos\theta) \\
&= \left(\frac{\mathbf{v}}{v}\right)\left(u\frac{\mathbf{u}\cdot\mathbf{v}}{uv}\right) && \mathbf{u}\cdot\mathbf{v} = uv\cos\theta \\
&= \left(\frac{\mathbf{v}}{v^2}\right)(\mathbf{u}\cdot\mathbf{v}) && \text{algebra} \\
\mathbf{u}_{||} &= \frac{\mathbf{u}\cdot\mathbf{v}}{v^2}\mathbf{v} && \text{algebra} \qquad (81)
\end{aligned}$$

This vector equation turns out to be useful in finding an equation for $\mathbf{u}_\perp$ in other forms:

$$(\mathbf{u}\times\mathbf{v})\times\mathbf{v} = (\mathbf{u}\cdot\mathbf{v})\mathbf{v} - v^2\mathbf{u} \tag{82}$$

Divide both sides of that equation by v^2 and rearrange a bit to get:

$$\frac{(\mathbf{u}\cdot\mathbf{v})}{v^2}\mathbf{v} = \frac{1}{v^2}(\mathbf{u}\times\mathbf{v})\times\mathbf{v} + \mathbf{u} \tag{83}$$

$$\begin{aligned}
\mathbf{u} &= \mathbf{u}_{\|} + \mathbf{u}_{\perp} \\
\mathbf{u}_{\perp} &= \mathbf{u} - \mathbf{u}_{\|} && \text{algebra} \\
&= \mathbf{u} - \frac{\mathbf{u}\cdot\mathbf{v}}{v^2}\mathbf{v} && \text{Eqn. (81)} \\
&= \mathbf{u} - (\frac{1}{v^2}(\mathbf{u}\times\mathbf{v})\times\mathbf{v} + \mathbf{u}) && \text{Eqn. (83)} \\
\mathbf{u}_{\perp} &= -\frac{1}{v^2}(\mathbf{u}\times\mathbf{v})\times\mathbf{v} && \text{algebra} \qquad (84)
\end{aligned}$$

A.3.2 Relativistic velocity addition and subtraction in 3 dimensions

Let:

$\mathbf{v}_B$	velocity of frame B relative to frame A
$\mathbf{u}'$	velocity of object in frame B
$\mathbf{v}_A$	velocity of object in frame A

$$\mathbf{v}_A = \mathbf{v}_B \oplus \mathbf{u}' = \frac{\mathbf{u}' + \left[(\gamma_B - 1)\frac{\mathbf{v}_B\cdot\mathbf{u}'}{v_B^2} + \gamma_B\right]\mathbf{v}_B}{\gamma_B(1+\frac{\mathbf{v}_B\cdot\mathbf{u}'}{c^2})} \qquad [16] \qquad (85)$$

$$= \frac{\mathbf{v}_B + \mathbf{u}' + \frac{\gamma_B}{\gamma_B+1}(\mathbf{u}'\times\mathbf{v}_B)\times\mathbf{v}_B/c^2}{1+\frac{\mathbf{v}_B\cdot\mathbf{u}'}{c^2}} \qquad (86)$$

$$= \mathbf{v}_{A\|} + \mathbf{v}_{A\perp} \qquad (87)$$

where:

$$\mathbf{v}_{A\|} = \frac{\mathbf{v}_B + \mathbf{u}'_{\|}}{1+\frac{\mathbf{v}_B\cdot\mathbf{u}'}{c^2}} \qquad (88)$$

$$\mathbf{v}_{A\perp} = \frac{\mathbf{u}'_{\perp}}{\gamma_B(1+\frac{\mathbf{v}_B\cdot\mathbf{u}'}{c^2})} \qquad (89)$$

and formulas for $\mathbf{u}'_{\|}$ and $\mathbf{u}'_{\perp}$ can be found in the previous section. It is only a few steps of algebra to plug Equations (88) and (89) into (87) and show it is equal to (85), so we will not do so here. Similarly, it is not many steps to show that Equations (85) and (86) are equal, using (84).

Relativistic velocity subtraction:

$$\mathbf{u}' = \mathbf{v}_A \ominus \mathbf{v}_B = \frac{\mathbf{v}_A - \mathbf{v}_B + \frac{\gamma_B}{\gamma_B+1}(\mathbf{v}_A\times\mathbf{v}_B)\times\mathbf{v}_B/c^2}{1-\frac{\mathbf{v}_A\cdot\mathbf{v}_B}{c^2}} \qquad [6] \qquad (90)$$

$$= \mathbf{u}'_{\|} + \mathbf{u}'_{\perp} \qquad (91)$$

where:

$$\mathbf{u}'_{\|} = \frac{\mathbf{v}_{A\|} - \mathbf{v}_B}{1-\frac{\mathbf{v}_A\cdot\mathbf{v}_B}{c^2}} \qquad (92)$$

$$\mathbf{u}'_{\perp} = \frac{\mathbf{v}_{A\perp}}{\gamma_B(1-\frac{\mathbf{v}_A\cdot\mathbf{v}_B}{c^2})} \qquad (93)$$

I was able to get Mathematica to confirm that for all vectors $\mathbf{u}$, $\mathbf{v}$:

$$(\mathbf{u}\oplus\mathbf{v})\ominus\mathbf{u} = \mathbf{v} \qquad (94)$$

$$\mathbf{u}\oplus(\mathbf{v}\ominus\mathbf{u}) = \mathbf{v} \qquad (95)$$

We will prove Equation (94) below. We will not prove Equation (95) here – its proof is nearly identical to the one below.

Let $K_- = 1 - \frac{(\mathbf{u}\oplus\mathbf{v})\cdot\mathbf{u}}{c^2}$ and $K_+ = 1 + \frac{\mathbf{u}\cdot\mathbf{v}}{c^2}$. Then:

$$\begin{aligned}
(\mathbf{u}\oplus\mathbf{v})\ominus\mathbf{u} &= \frac{(\mathbf{u}\oplus\mathbf{v})_\parallel - \mathbf{u}}{K_-} + \frac{(\mathbf{u}\oplus\mathbf{v})_\perp}{\gamma_u K_-} && \text{Eqns. (91), (92), and (93)}\\
&= \frac{\frac{\mathbf{u}+\mathbf{v}_\parallel}{K_+} - \mathbf{u}}{K_-} + \frac{\frac{\mathbf{v}_\perp}{\gamma_u K_+}}{\gamma_u K_-} && \text{Eqns. (88) and (89)}\\
&= \frac{1}{K_+K_-}\left[\mathbf{u} + \mathbf{v}_\parallel - K_+\mathbf{u} + \frac{\mathbf{v}_\perp}{\gamma_u^2}\right] && \text{algebra}\\
&= \frac{1}{K_+K_-}\left[\mathbf{u} + \mathbf{v}_\parallel - (1 + \frac{\mathbf{u}\cdot\mathbf{v}}{c^2})\mathbf{u} + \frac{\mathbf{v}_\perp}{\gamma_u^2}\right] && \text{substitute } K_+ \text{ with its value in one place}\\
&= \frac{1}{K_+K_-}\left[\mathbf{v}_\parallel - \frac{\mathbf{u}\cdot\mathbf{v}}{c^2}\mathbf{u} + \frac{\mathbf{v}_\perp}{\gamma_u^2}\right] && \text{algebra}
\end{aligned}$$

Recall that $\mathbf{v}_\parallel = \frac{\mathbf{u}\cdot\mathbf{v}}{u^2}\mathbf{u}$ (Equation (81)), so $(\mathbf{u}\cdot\mathbf{v})\mathbf{u} = u^2\mathbf{v}_\parallel$. Substituting the right hand side for the left where it appears in the equation above:

$$\begin{aligned}
&= \frac{1}{K_+K_-}\left[\mathbf{v}_\parallel - \frac{u^2}{c^2}\mathbf{v}_\parallel + \frac{\mathbf{v}_\perp}{\gamma_u^2}\right] && \text{substitution}\\
&= \frac{1}{K_+K_-}\left[(1-\beta_u^2)\mathbf{v}_\parallel + (1-\beta_u^2)\mathbf{v}_\perp\right] && \text{algebra, } \beta_u = u/c,\ 1/\gamma_u^2 = (1-\beta_u^2)\\
&= \frac{1}{K_+K_-}(1-\beta_u^2)\mathbf{v} && \text{algebra, } \mathbf{v}_\parallel + \mathbf{v}_\perp = \mathbf{v} && (96)
\end{aligned}$$

Now to simplify K_+K_-:

$$\begin{aligned}
K_+K_- &= K_+\left(1 - \frac{(\mathbf{u}\oplus\mathbf{v})\cdot\mathbf{u}}{c^2}\right) && \text{defn of } K_-\\
&= K_+\left(1 - \frac{(\frac{\mathbf{u}+\mathbf{v}_\parallel}{K_+})\cdot\mathbf{u}}{c^2}\right) && \text{only need parallel part of } (\mathbf{u}\oplus\mathbf{v}) \text{ because of dot product with } \mathbf{u}\\
&= K_+\left(1 - \frac{u^2 + \mathbf{u}\cdot\mathbf{v}_\parallel}{c^2K_+})\right) && \text{algebra}\\
&= K_+ - \frac{u^2}{c^2} + \frac{\mathbf{u}\cdot\mathbf{v}_\parallel}{c^2} && \text{algebra}\\
&= 1 + \frac{\mathbf{u}\cdot\mathbf{v}}{c^2} - \beta_u^2 + \frac{\mathbf{u}\cdot\mathbf{v}_\parallel}{c^2} && \text{defn. of } K_+,\ \beta_u = u/c\\
&= 1 - \beta_u^2 && \text{defn. of } \mathbf{v}_\parallel \text{ means } \mathbf{u}\cdot\mathbf{v}_\parallel = \mathbf{u}\cdot\mathbf{v}
\end{aligned}$$

Substituting this result for K_+K_- into Equation (96) we have shown that $(\mathbf{u}\oplus\mathbf{v})\ominus\mathbf{u} = \mathbf{v}$.

A.3.3 Identities about 3-d relativistic velocity addition and subtraction

Identity	True in 1-d?	True in 3-d?
$a \oplus 0 = 0 \oplus a = a$	yes	yes
$0 \ominus b = -b$	yes	yes
$(a \oplus b) \ominus a = b$	yes	yes
$a \oplus (b \ominus a) = b$	yes	yes
$\lvert a \oplus b\rvert = \lvert b \oplus a\rvert$	yes	yes (Lemma 13)
$\lvert a \ominus b\rvert = \lvert b \ominus a\rvert$	yes	yes (Lemma 13)
$\lvert a \oplus b\rvert = c$ if $\lvert a\rvert = c$ or $\lvert b\rvert = c$	yes	yes (Lemma 15)
$a \oplus b = b \oplus a$	yes	no
$(a \oplus b) \oplus d = a \oplus (b \oplus d)$	yes	no
$a \oplus (-b) = a \ominus b$	yes	no
$a \ominus (-b) = a \oplus b$	yes	no
$a \ominus b = -(b \ominus a)$	yes	no

Note that if we restrict our attention to 3-d vectors that are all along a line of some unit vector $\hat{\mathbf{d}}$, then all such 3-d vectors are 1-dimensional along that line, and all 1-d identities above apply to that subset of 3-d velocity vectors.

3-d counterexample where $a \oplus b \neq b \oplus a$

$$\begin{aligned} a &= (0.5c, 0, 0) \\ b &= (0, 0.5c, 0) \qquad \text{Note: } a \cdot b = 0 \text{ simplifies Eqn. (85) a lot} \\ a \oplus b &= (0.5c, 0.433c, 0) \\ b \oplus a &= (0.433c, 0.5c, 0) \end{aligned}$$

3-d counterexample where $(a \oplus b) \oplus d \neq a \oplus (b \oplus d)$

$$\begin{aligned} a &= (0.5c, 0, 0) \\ b &= (0, 0.5c, 0) \\ (a \oplus b) \oplus b &= (0.462c, 0.708c, 0) \\ a \oplus (b \oplus b) &= (0.5c, 0.693c, 0) \end{aligned}$$

3-d counterexample where $a \oplus (-b) \neq a \ominus b$

$$\begin{aligned} a &= (0.4c, 0.1c, 0) \\ b &= (0.2c, 0.3c, 0) \\ a \oplus (-b) &= (0.219c, -0.201c, 0) \\ a \ominus b &= (0.207c, -0.213c, 0) \end{aligned}$$

The same counterexample deomnstrates that $a \ominus (-b) \neq a \oplus b$.

3-d counterexample where $a \ominus b \neq -(b \ominus a)$

$$\begin{aligned} a &= (0.4c, 0.1c, 0) \\ b &= (0.2c, 0.3c, 0) \\ a \ominus b &= (0.207c, -0.213c, 0) \\ -(b \ominus a) &= (0.212c, -0.201c, 0) \end{aligned}$$

A.4 Math facts about the Doppler factor and Lorentz factor

This appendix is restricted to proofs of mathematical relationships. It makes no claims about the physical meaning of the equations involved.

Define D as given by Definition (11), repeated here:

$$D = \sqrt{\frac{1+\beta}{1-\beta}} \qquad \text{Repeat of Defn. (11)}$$

Define it over the entire domain $-1 < \beta < 1$.

- D approaches 0 as β approaches -1.
- D approaches $+\infty$ as β approaches 1.
- D is an increasing function of β.

The last fact is straightforward to confirm by taking the derivative of D with respect to β, and confirming that it is positive for all β in the range $(-1, +1)$.

Corollary 3. *For any value of $D > 0$ that satisfies Definition (11), there is exactly one value of β, $-1 < \beta < 1$, that corresponds to that value of D. That value is $\beta = \frac{D^2-1}{D^2+1}$.*

Plugging the value $\frac{D^2-1}{D^2+1}$ into Definition (11) for β takes only a little algebra to simplify and show it is equal to D.

These two equations pop up only occasionly:

$$D = \gamma(1+\beta) \tag{97}$$

$$D = \frac{1}{\gamma(1-\beta)} \tag{98}$$

Simple algebra to confirm two of the identities above:

$$\begin{aligned}
\gamma(1+\beta) &= \frac{1+\beta}{\sqrt{1-\beta^2}} && \text{Defn. of } \gamma \\
&= \frac{\sqrt{(1+\beta)(1+\beta)}}{\sqrt{(1+\beta)(1-\beta)}} && \text{algebra} \\
&= \sqrt{\frac{1+\beta}{1-\beta}} && \text{algebra, cancelling factor of } \sqrt{1+\beta} \\
&= D && \text{Defn. of } D
\end{aligned}$$

And the second:

$$\begin{aligned}
\frac{1}{\gamma(1-\beta)} &= \frac{\sqrt{1-\beta^2}}{1-\beta} && \text{Defn. of } \gamma \\
&= \frac{\sqrt{(1+\beta)(1-\beta)}}{\sqrt{(1-\beta)(1-\beta)}} && \text{algebra} \\
&= \sqrt{\frac{1+\beta}{1-\beta}} && \text{algebra, cancelling factor of } \sqrt{1-\beta} \\
&= D && \text{Defn. of } D
\end{aligned}$$

Theorem 4. *Let β_u, β_v be any two values in the range $(-1, +1)$. Let $\beta_w = \beta_u \oplus \beta_v$, using the one-dimensional relativistic velocity addition formula for β values (Defn. (46)). Let D_u, D_v, D_w be calculated from $\beta_u, \beta_v, \beta_w$, respectively. Then $D_w = D_u D_v$.*

Proof.

$$\begin{aligned}
D_w &= \sqrt{\frac{1+\beta_w}{1-\beta_w}} && \text{defn. of } D_w \\
&= \sqrt{\frac{1+(\beta_v \oplus \beta_w)}{1-(\beta_v \oplus \beta_w)}} && \text{Defn. of } \beta_w \\
&= \sqrt{\frac{1+\frac{\beta_v+\beta_w}{1+\beta_v\beta_w}}{1-\frac{\beta_v+\beta_w}{1+\beta_v\beta_w}}} && \text{Defn. (46) of Appendix A.1} \\
&= \sqrt{\frac{(1+\beta_v\beta_w)+(\beta_v+\beta_w)}{(1+\beta_v\beta_w)-(\beta_v+\beta_w)}} && \text{multiply numerator and denominator by } (1+\beta_v\beta_w) \\
&= \sqrt{\frac{(1+\beta_v)(1+\beta_w)}{(1-\beta_v)(1-\beta_w)}} && \text{algebra (factoring)} \\
&= \sqrt{\frac{1+\beta_v}{1-\beta_v}}\sqrt{\frac{1+\beta_w}{1-\beta_w}} && \text{algebra} \\
&= D_v D_w && \text{defn. of } D_v, D_w
\end{aligned}$$

□

Corollary 5. *Under the same conditions as Theorem 4, except $\beta_w = \beta_u \ominus \beta_v$, using the one-dimensional relativistic velocity subtraction formula for β values (Defn. (47)), $D_w = D_u / D_v$.*

Proof. $\beta_w = \beta_u \ominus \beta_v = \beta_u \oplus (-\beta_v)$. Thus by the previous theorem, $D_w = D_u D_{-v}$, where D_{-v} is calculated from the definition for D with $-\beta_v$. It is straightforward to see from the definition of D that $D_{-v} = 1/D_v$, so $D_w = D_u/D_v$. □

Apparently these facts are well-known among those working with relativistic Doppler factors.

The following lemmas are occasionally useful.

Lemma 6. *Let β be in range $0 < \beta < 1$, and let $\gamma = 1/\sqrt{1-\beta^2}$ be the Lorentz factor corresponding to it. Then:*

$$\frac{1}{\beta}(1 - \frac{1}{\gamma}) = \frac{\beta\gamma}{\gamma+1} \tag{99}$$

Proof.

$$\begin{aligned}
\frac{1}{\beta}(1 - \frac{1}{\gamma}) &= \frac{1}{\beta}\frac{\gamma-1}{\gamma}\frac{\gamma+1}{\gamma+1} && \text{algebra} \\
&= \frac{1}{\beta}\frac{\gamma^2-1}{\gamma(\gamma+1)} && \text{algebra} \\
&= \frac{1}{\beta}\frac{\gamma^2(1-\frac{1}{\gamma^2})}{\gamma(\gamma+1)} && \text{algebra} \\
&= \frac{1}{\beta}\frac{\gamma(1-(1-\beta^2))}{\gamma+1} && \text{defn. of } \gamma \text{, algebra} \\
&= \frac{\beta\gamma}{\gamma+1} && \text{algebra}
\end{aligned}$$

□

Lemma 7. *Let β_1, β_2 be in range $0 < \beta_1, \beta_2 < 1$, and let $\gamma_i = 1/\sqrt{1-\beta_i^2}$ be the Lorentz factors corresponding to them. Then:*

$$\frac{\frac{1}{\gamma_1} - \frac{1}{\gamma_2}}{\beta_2 - \beta_1} = \frac{\gamma_1\gamma_2(\beta_1+\beta_2)}{\gamma_1+\gamma_2} \tag{100}$$

Proof.

$$\begin{aligned}
\frac{\frac{1}{\gamma_1} - \frac{1}{\gamma_2}}{\beta_2 - \beta_1} &= \frac{\frac{1}{\gamma_1} - \frac{1}{\gamma_2}}{\beta_2 - \beta_1}\left(\frac{\frac{1}{\gamma_1} + \frac{1}{\gamma_2}}{\frac{1}{\gamma_1} + \frac{1}{\gamma_2}}\right) && \text{algebra} \\
&= \frac{\frac{1}{\gamma_1^2} - \frac{1}{\gamma_2^2}}{\beta_2 - \beta_1}\left(\frac{1}{\frac{1}{\gamma_1} + \frac{1}{\gamma_2}}\right) && \text{algebra} \\
&= \frac{(1-\beta_1^2)-(1-\beta_2^2)}{\beta_2 - \beta_1}\left(\frac{1}{\frac{1}{\gamma_1} + \frac{1}{\gamma_2}}\right) && \text{defn. of } \gamma \\
&= \frac{\beta_2^2 - \beta_1^2}{\beta_2 - \beta_1}\left(\frac{1}{\frac{1}{\gamma_1} + \frac{1}{\gamma_2}}\right) && \text{algebra} \\
&= \frac{\beta_2 + \beta_1}{\frac{1}{\gamma_1} + \frac{1}{\gamma_2}} && \text{algebra} \\
&= \frac{\gamma_1\gamma_2(\beta_1+\beta_2)}{\gamma_1+\gamma_2} && \text{multiply num. and denom. by } \gamma_1\gamma_2
\end{aligned}$$

□

A.5 When a moving observer intersects with a spherically spreading pulse

Note: All calculations and results of lemmas in this appendix calculate a pulse receive time under the assumptions given, e.g. that the signal propagates at the same speed c in all directions. Thus the results apply in any inertial frame when assuming the postulates of SR, but note that the receive time is in the frame where the receiver is moving with velocity v, *not* the receiver's time dilated frame. In LET, these lemmas only apply in the frame that is at rest relative to the ether, and the times calculated are in that frame, not the pulse receiver's clock.

Lemma 8 below slightly generalizes Lemma 9, to enable you to choose one time t_0 for the pulse to be emitted, and a different time t_1 to correspond with a starting position for O.

Lemma 8. *A spherically spreading "pulse" (e.g. a light pulse, or a brief sound wave) is emitted from a position* $\mathbf{s}$ *at time* t_0*, propagating in all directions at speed* c*.*

An observer O *is moving at constant velocity* $\mathbf{v}$ *starting at position* $\mathbf{s} + \mathbf{r}_1$ *at time* t_1*, so its position is:*

$$\mathbf{r}(t) = \mathbf{s} + \mathbf{r}_1 + \mathbf{v}(t - t_1) \tag{101}$$

If $|\mathbf{v}| < c$*, then there is exactly one time* $t_r \geq t_0$ *that* O *encounters the pulse signal, given by:*

$$t_r = t_0 + \frac{\gamma^2}{c^2}\left[(\mathbf{r}_0 \cdot \mathbf{v}) + \sqrt{(\mathbf{r}_0 \cdot \mathbf{v})^2 + (1 - \beta^2)c^2 r_0^2}\right] \tag{102}$$

where:

$$\mathbf{r}_0 = \mathbf{r}_1 + \mathbf{v}(t_0 - t_1)$$

It is assumed that O*'s constant velocity motion has been occurring since the earlier of times* t_0 *and* t_1*.*

The proof of the above is straightforward, using time $(t - t_0)$ instead of t. Note that the equation for the position $\mathbf{r}(t)$ in the Lemma becomes as follows when replace $\mathbf{r}_1$ as shown:

$$\begin{aligned}
\mathbf{r}(t) &= \mathbf{s} + \mathbf{r}_1 + \mathbf{v}(t - t_1) \\
&= \mathbf{s} + \mathbf{r}_0 - \mathbf{v}(t_0 - t_1) + \mathbf{v}(t - t_1) && \text{substitute } \mathbf{r}_1 \text{ with } \mathbf{r}_0 - \mathbf{v}(t_0 - t_1) \\
&= \mathbf{s} + \mathbf{r}_0 + \mathbf{v}(t - t_0) && \text{algebra}
\end{aligned}$$

A.5.1 Proof for simplified case

Lemma 9. *A spherically spreading "pulse" (e.g. a light pulse, or a brief sound wave) is emitted from location* $\mathbf{s}$ *at time* $t = 0$*, propagating in all directions at speed* c*.*

An observer O *is moving at constant velocity* $\mathbf{v}$ *starting at position* $\mathbf{s} + \mathbf{r}_0$ *at* $t = 0$*, so its position is:*

$$\mathbf{r}(t) = \mathbf{s} + \mathbf{r}_0 + \mathbf{v}t \tag{103}$$

If $|\mathbf{v}| < c$*, then there is exactly one time* $t_r \geq 0$ *that* O *encounters the pulse signal, given by:*

$$t_r = \frac{\gamma^2}{c^2}\left[(\mathbf{r}_0 \cdot \mathbf{v}) + \sqrt{(\mathbf{r}_0 \cdot \mathbf{v})^2 + (1 - \beta^2)c^2 r_0^2}\right] \tag{104}$$

Proof. Any time t when O is at the wavefront of the pulse must be one where the distance from O to $\mathbf{s}$ is equal to the distance that the pulse has traveled from $\mathbf{s}$:

$$\begin{aligned}
|\mathbf{r}(t) - \mathbf{s}| &= ct \\
|\mathbf{r}_0 + \mathbf{v}t| &= ct && \text{substitute value of } \mathbf{r}(t) \\
(\mathbf{r}_0 + \mathbf{v}t) \cdot (\mathbf{r}_0 + \mathbf{v}t) &= c^2 t^2 && \text{square both sides} \\
r_0^2 + 2(\mathbf{r}_0 \cdot \mathbf{v})t + v^2 t^2 &= c^2 t^2 && \text{expand dot product} \\
(v^2 - c^2)t^2 + 2(\mathbf{r}_0 \cdot \mathbf{v})t + r_0^2 &= 0 && \text{rearrange in preparation for quadratic equation on } t
\end{aligned}$$

The discriminant Δ when applying the quadratic equation is:

$$\begin{aligned}
\Delta &= b^2 - 4ac \\
&= 4(\mathbf{r}_0 \cdot \mathbf{v})^2 - 4(v^2 - c^2)r_0^2 \\
&= 4(\mathbf{r}_0 \cdot \mathbf{v})^2 - 4(\beta^2 c^2 - c^2)r_0^2 && v = \beta c \\
&= 4\left[(\mathbf{r}_0 \cdot \mathbf{v})^2 + (1 - \beta^2)c^2 r_0^2\right] && \text{algebra}
\end{aligned}$$

Plugging Δ into the full quadratic equation to solve for t:

$$
\begin{aligned}
t &= \frac{-2(\mathbf{r}_0 \cdot \mathbf{v}) \pm \sqrt{4\left[(\mathbf{r}_0 \cdot \mathbf{v})^2 + (1-\beta^2)c^2 r_0^2\right]}}{2(v^2 - c^2)} \\
&= -\frac{\gamma^2}{c^2}\left[-(\mathbf{r}_0 \cdot \mathbf{v}) \pm \sqrt{(\mathbf{r}_0 \cdot \mathbf{v})^2 + (1-\beta^2)c^2 r_0^2}\right] && \text{cancel 2, } 1/(v^2-c^2) = -\gamma^2/c^2 \\
&= \frac{\gamma^2}{c^2}\left[(\mathbf{r}_0 \cdot \mathbf{v}) \mp \sqrt{(\mathbf{r}_0 \cdot \mathbf{v})^2 + (1-\beta^2)c^2 r_0^2}\right]
\end{aligned}
$$

We want only a positive solution, since that is after the pulse is emitted. $(\mathbf{r}_0 \cdot \mathbf{v})$ could be positive or negative, so only the + alternative of $\mp$ will be positive in that situation. □

B Length contraction

The left side of Figure 11 shows the top view of a 3-dimensional rectangular prism at rest. The right side shows that object moving with velocity $v = 0.866c$ to the right, and how it would appear length-contracted by a factor of $\gamma = 2$.

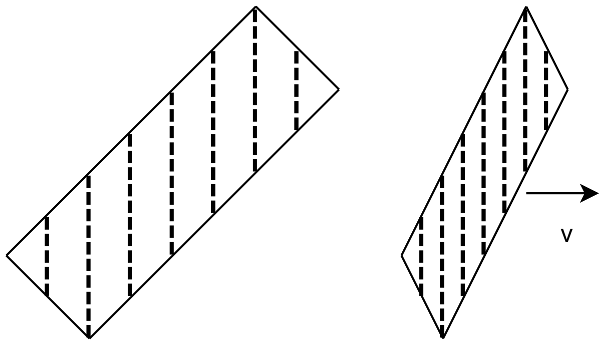

Figure 11: Length contraction of a rectangular prism moving with velocity at an angle to its sides

One way to think about length contraction of an arbitrary shape is to imagine slicing it up by many parallel planes, all planes perpendicular to the direction of the velocity vector. The left side of the figure shows 7 equally spaced planes as dashed lines.

When length contraction occurs, imagine that each pair of these planes gets closer to each other by having their original distance of separation multiplied by $1/\gamma$, which is 1/2 in the example figure on the right.

Note that the internal angles at the corners change, with the ones on top and bottom getting smaller, and the ones on left and right getting larger. The angles of the long sides relative to the velocity vector $\mathbf{v}$ are different for the length-constracted shape than when it was at rest. The height of each "slice" of the object remains the same – only its width contracts.

If we imagine a rod as a rectangular prism where its length is so much larger than its width that the width is nearly 0 by comparison, note that the drawing of the length-contracted version relative to the at-rest original would look like that in Figure 12. Thus the only measurements that we care about, i.e. the length and angle of the rod, change.

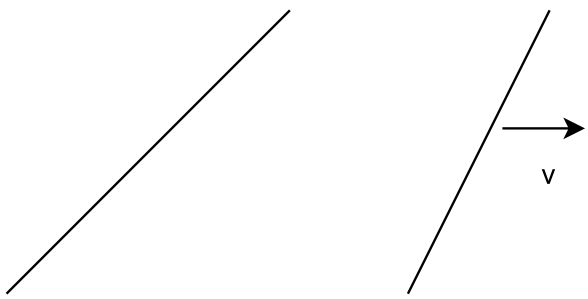

Figure 12: Length contraction of a thin rod moving with velocity at an angle to its sides

C Aberration

The Wikipedia article on aberration in astronomy [8] is quite detailed, and I will not attempt to repeat it here.

TODO: It would be good to add some more details here about the history of the physical theories explaining aberration, or at least the most modern understanding of the physical explanation of aberration according to Special Relativity and Lorentz Ether Theory.

For now, I will simply give the most basic classical explanation of aberration, while warning the reader that this explanation does not help explain aberration for a telescope filled with a material like water with a refractive index significantly different than 1.

See Figure 13 for a picture that helps visualize an early classical explanation of aberration.

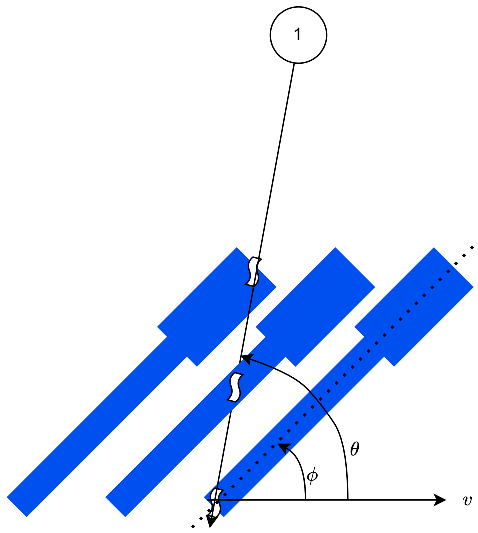

Figure 13: An early classical explanation of aberration

An outline of a telescope is shown moving from left to right with velocity $\mathbf{v}$. A light beam arrives from a source at position 1, from a direction that makes an angle θ with $\mathbf{v}$. In this explanation, we consider that the light is moving with velocity c relative to an ether that is at rest relative to the source, and the telescope is moving with velocity $\mathbf{v}$ relative to the ether.

In the leftmost image of the telescope, earliest in time, the light beam enters the top of the telescope. The middle image of the telescope shows a short time later, after the light beam has traveled about halfway down the length of the telescope, while the telescope is moving. The right image of the telescope shows the light beam leaving the bottom end of the telescope. The angle of the telescope has been chosen such that the light beam travels a path down the length of the telescope. The circle labeled 2 shows the apparent position of the light from the source, at an angle ϕ away from the velocity vector $\mathbf{v}$.

Figure 14 has all of the same angles between $\mathbf{v}$, the light beam, and the telescope. The sides have been labeled with their lengths. The side of the right triangle adjacent to angle θ has not been labeled in the figure. It has length $c\cos\theta$. The side adjacent to angle ϕ thus has a total length equal to both sides of this equation:

$$s\cos\phi = v + c\cos\theta$$

That plus the identity $s\sin\phi = c\sin\theta$ gives:

$$\frac{s\sin\phi}{s\cos\phi} = \frac{c\sin\theta}{v + c\cos\theta}$$

$$\tan\phi = \frac{\sin\theta}{v/c + \cos\theta} \tag{105}$$

The correct equation for Special Relativity and Lorentz Ether Theory is given below. Note: Currently we are giving this with no explanation of how it was derived or validated. It would be best to include proof, or at least some evidence, for why this equation is correct.

$$\tan\phi = \frac{\sin\theta}{\gamma(v/c + \cos\theta)} \tag{106}$$

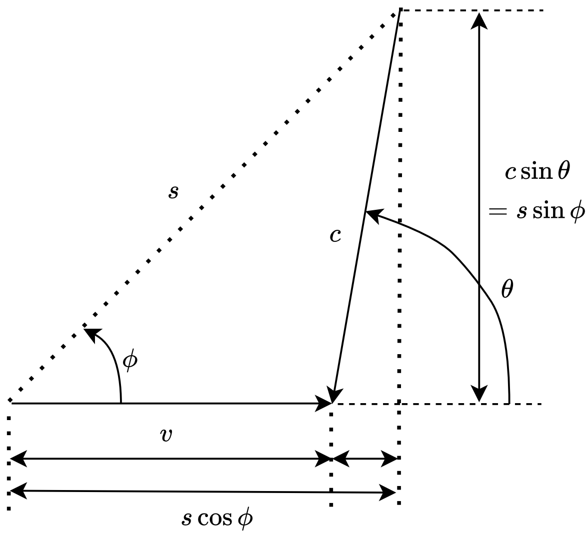

Figure 14: Mathematical analysis of classical explanation of aberration

Lemma 10. *Let* $\mathbf{v}$ *be the velocity of an observer relative to the sender of a light beam. Let the vector* $\mathbf{u}$ *be the direction the light beam travels from the source to the observer, with* $|\mathbf{u}| = c$*. Then after aberration is taken into account, the direction which the light beam appears to the observer to arrive is* $\mathbf{v} \oplus \mathbf{u}$*, and* $|\mathbf{v} \oplus \mathbf{u}| = c$*.*

Proof. According to Lemma 14, the angle ϕ between $\mathbf{v} \oplus \mathbf{u}$ and $\mathbf{v}$ satisfies:

$$\tan\phi = \frac{\sin\theta}{\gamma(v/c + \cos\theta)} \tag{107}$$

where θ is the angle between $\mathbf{v}$ and $\mathbf{u}$. Furthermore, $\mathbf{v} \oplus \mathbf{u}$ lies in the same plane as vectors $\mathbf{v}$ and $\mathbf{u}$, and so does the apparent direction from which the light beam arrives according to earlier in this section.

By Lemma 15, the magnitude of $\mathbf{v} \oplus \mathbf{u}$ is equal to c. □

C.1 Relationship between relativistic velocity addition and aberration

Lemma 11. *Given vectors* $\mathbf{v}$ *and* $\mathbf{u}$ *with angle* θ *between them, the following equations are true:*

$$|\mathbf{v} \oplus \mathbf{u}| = \frac{1}{K}\sqrt{(v + u\cos\theta)^2 + \frac{u^2 \sin^2\theta}{\gamma_v^2}} \tag{108}$$

$$|\mathbf{v} \oplus \mathbf{u}| = \frac{1}{K}\sqrt{|\mathbf{v} + \mathbf{u}|^2 - \frac{v^2 u^2 \sin^2\theta}{c^2}} \tag{109}$$

where:

$$\gamma_v = \frac{1}{\sqrt{1 - v^2/c^2}} \qquad K = 1 + \frac{\mathbf{v} \cdot \mathbf{u}}{c^2} \tag{110}$$

Proof.

$$\begin{aligned}
|\mathbf{v} \oplus \mathbf{u}|^2 &= (\mathbf{v} \oplus \mathbf{u}) \cdot (\mathbf{v} \oplus \mathbf{u}) \\
&= (\frac{\mathbf{v} + \mathbf{u}_\parallel}{K} + \frac{\mathbf{u}_\perp}{\gamma_v K}) \cdot (\frac{\mathbf{v} + \mathbf{u}_\parallel}{K} + \frac{\mathbf{u}_\perp}{\gamma_v K}) && \text{defn. of } \mathbf{v} \oplus \mathbf{u} \\
&= \frac{1}{K^2} \left[(\mathbf{v} + \mathbf{u}_\parallel) \cdot (\mathbf{v} + \mathbf{u}_\parallel) + \frac{1}{\gamma_v^2} (\mathbf{u}_\perp \cdot \mathbf{u}_\perp) \right] && \text{other partial products are 0} \\
&= \frac{1}{K^2} \left[|\mathbf{v} + \mathbf{u}_\parallel|^2 + \frac{u_\perp^2}{\gamma_v^2} \right] && \text{algebra} \\
&= \frac{1}{K^2} \left[(v + u\cos\theta)^2 + \frac{u^2 \sin^2\theta}{\gamma_v^2} \right] && |\mathbf{u}_\parallel| = u\cos\theta,\ |\mathbf{u}_\perp| = u\sin\theta
\end{aligned}$$

Taking the square root of both sides proves Equation (108). Continuing with our derivation we get:

$$\begin{aligned}
&= \frac{1}{K^2} \left[v^2 + 2vu\cos\theta + u^2\cos^2\theta + u^2\sin^2\theta - \beta_v^2 u^2 \sin^2\theta \right] && \text{algebra, } 1/\gamma_v^2 = (1 - \beta_v^2) \\
&= \frac{1}{K^2} \left[v^2 + 2vu\cos\theta + u^2 - \beta_v^2 u^2 \sin^2\theta \right] && \sin^2\theta + \cos^2\theta = 1 \\
&= \frac{1}{K^2} \left[|\mathbf{v} + \mathbf{u}|^2 - \beta_v^2 u^2 \sin^2\theta \right] && \text{algebra} \\
&= \frac{1}{K^2} \left[|\mathbf{v} + \mathbf{u}|^2 - \frac{v^2 u^2 \sin^2\theta}{c^2} \right] && \beta_v = v/c
\end{aligned}$$

Taking the square root of both sides proves Equation (109). □

Lemma 12. *Given vectors* $\mathbf{v}$ *and* $\mathbf{u}$ *with angle* θ *between them, the following equations are true:*

$$|\mathbf{v} \ominus \mathbf{u}| = \frac{1}{K} \sqrt{(v\cos\theta - u)^2 + \frac{v^2 \sin^2\theta}{\gamma_u^2}} \tag{111}$$

$$|\mathbf{v} \ominus \mathbf{u}| = \frac{1}{K} \sqrt{|\mathbf{v} - \mathbf{u}|^2 - \frac{v^2 u^2 \sin^2\theta}{c^2}} \tag{112}$$

where:

$$\gamma_u = \frac{1}{\sqrt{1 - u^2/c^2}} \qquad K = 1 - \frac{\mathbf{v} \cdot \mathbf{u}}{c^2}$$

Proof. We omit the details of the proof. It is nearly identical to the proof of Lemma 11. □

Lemma 13. *Given vectors* $\mathbf{u}$ *and* $\mathbf{v}$*:*

$$\begin{aligned}
|\mathbf{u} \oplus \mathbf{v}| &= |\mathbf{v} \oplus \mathbf{u}| \\
|\mathbf{u} \ominus \mathbf{v}| &= |\mathbf{v} \ominus \mathbf{u}|
\end{aligned}$$

Proof. Examining Equation (109) from Lemma 11, we see that it is symmetric in $\mathbf{u}$ and $\mathbf{v}$, giving the same result when they are swapped. The same goes for Equation (112) from Lemma 12. □

Lemma 14. *Given non-0 vectors* $\mathbf{v}$ *and* $\mathbf{u}$ *with angle* θ *between them, the following equation is true for the angle* ϕ *between* $\mathbf{v} \oplus \mathbf{u}$ *and* $\mathbf{v}$*:*

$$\tan\phi = \frac{\sin\theta}{\gamma(v/u + \cos\theta)} \tag{113}$$

Proof.

$$\mathbf{v} \oplus \mathbf{u} = \frac{\mathbf{v} + \mathbf{u}_\parallel}{K} + \frac{\mathbf{u}_\perp}{\gamma K} \qquad \text{Eqn. (87)}$$

where:

$$\begin{aligned}
\gamma &= \frac{1}{\sqrt{1 - v^2/c^2}} \\
K &= 1 + \frac{\mathbf{v} \cdot \mathbf{u}}{c^2}
\end{aligned}$$

The angle ϕ between $\mathbf{v} \oplus \mathbf{u}$ and $\mathbf{v}$ satisfies the following equation, because this relationship is true for the dot product and angle between any two vectors:

$$\cos\phi = \frac{(\mathbf{v} \oplus \mathbf{u}) \cdot \mathbf{v}}{|\mathbf{v} \oplus \mathbf{u}||\mathbf{v}|} \tag{114}$$

Then an equation equal to $(\mathbf{v} \oplus \mathbf{u}) \cdot \mathbf{v}$:

$$\begin{aligned}
(\mathbf{v} \oplus \mathbf{u}) \cdot \mathbf{v} &= \frac{1}{K}\left[(\mathbf{v} + \mathbf{u}_{\|}) \cdot \mathbf{v} + \frac{1}{\gamma}(\mathbf{u}_{\perp} \cdot \mathbf{v})\right] && \text{defn. of } \mathbf{v} \oplus \mathbf{u}\text{, algebra} \\
&= \frac{1}{K}|\mathbf{v} + \mathbf{u}_{\|}||\mathbf{v}| && \mathbf{v} \text{ and } \mathbf{u}_{\|} \text{ are parallel} \\
&= \frac{v}{K}(v + u\cos\theta) && (115)
\end{aligned}$$

Plugging Equations (108) and (115) into Equation (114) gives:

$$\begin{aligned}
\cos\phi &= \frac{\frac{v}{K}(v + u\cos\theta)}{\frac{v}{K}\sqrt{(v + u\cos\theta)^2 + \frac{u^2\sin^2\theta}{\gamma^2}}} \\
&= \frac{v + u\cos\theta}{\sqrt{(v + u\cos\theta)^2 + \frac{u^2\sin^2\theta}{\gamma^2}}} && \text{algebra}
\end{aligned}$$

Note that $\cos\phi$ is of the form $a/\sqrt{a^2+b^2}$. Thus $\sin\phi$ is equal to $b/\sqrt{a^2+b^2}$, and $\tan\phi = b/a$:

$$\begin{aligned}
\tan\phi &= \frac{\frac{u\sin\theta}{\gamma}}{v + u\cos\theta} \\
&= \frac{\sin\theta}{\gamma(v/u + \cos\theta)}
\end{aligned} \tag{116}$$

□

Lemma 15. *Given vectors* $\mathbf{v}$ *and* $\mathbf{u}$*, if either* $|\mathbf{v}| = c$ *or* $|\mathbf{u}| = c$*, then* $|\mathbf{v} \oplus \mathbf{u}| = c$.

Proof. Start by proving the case when $|\mathbf{u}| = c$. By Equation (108) of Lemma 11

$$\begin{aligned}
|\mathbf{v} \oplus \mathbf{u}| &= \frac{1}{K}\sqrt{(v + u\cos\theta)^2 + \frac{u^2\sin^2\theta}{\gamma_v^2}} \\
&= \frac{1}{1 + \frac{\mathbf{v}\cdot\mathbf{u}}{c^2}}\sqrt{(v + u\cos\theta)^2 + (1 - \beta_v^2)u^2\sin^2\theta} && \text{substitute values of } K \text{ and } 1/\gamma_v^2 \\
&= \frac{1}{1 + \frac{u\cos\theta}{c}}\sqrt{(c + u\cos\theta)^2} && 1 - \beta_v^2 = 0 \text{ when } v = c \\
&= \frac{c}{c + u\cos\theta}(c + u\cos\theta) && \text{algebra} \\
&= c
\end{aligned}$$

By Lemma 13, we must also get the same magnitude if we swap the order of arguments to $\oplus$, so $|\mathbf{v}+\mathbf{u}| = c$ if $|\mathbf{u}| = c$. □

D Double-check results from ChatGPT

D.1 Scenario 1b, B sends periodic pulses to C, double-check by ChatGPT

Since at the time of first performing the calculations in Section 3.2.1 I was still fairly new to such things, I wanted a way to double-check the results. I asked ChatGPT what the period would be that C would receive pulses from B and it gave an answer close to the following.

Note: I only asked it about the case where $v_C > v_B > 0$.

Calculate the velocity of B in C's frame using relativistic velocity subtraction:

$$u' = \frac{v_B - v_C}{1 - \frac{v_B v_C}{c^2}} \tag{117}$$

$$\beta' = \frac{\beta_B - \beta_C}{1 - \beta_B \beta_C} \tag{118}$$

Since $v_B < v_C$, the numerator is negative. The denominator is positive. Thus $u' < 0$ and $\beta' < 0$. Thus in C's frame, B is moving in the negative x direction, and C is at rest.

For receding motion:

$$T_{\text{observed}} = T_{\text{emitted}} \sqrt{\frac{1+\beta}{1-\beta}} \tag{119}$$

where:

$$\beta = \frac{|u'|}{c} = |\beta'| \tag{120}$$

Since $\beta' < 0$:

$$|\beta'| = -\beta' = \frac{\beta_C - \beta_B}{1 - \beta_B \beta_C} \tag{121}$$

From here on this is mostly just algebra:

$$\begin{aligned}
\frac{T_{\text{observed}}}{T_{\text{emitted}}} &= \sqrt{\frac{1+\beta}{1-\beta}} \\
&= \sqrt{\frac{1 + \frac{\beta_C - \beta_B}{1-\beta_B\beta_C}}{1 - \frac{\beta_C - \beta_B}{1-\beta_B\beta_C}}} \\
&= \sqrt{\frac{1 - \beta_C\beta_B + (\beta_C - \beta_B)}{1 - \beta_C\beta_B - (\beta_C - \beta_B)}} \\
&= \sqrt{\frac{(1+\beta_C)(1-\beta_B)}{(1-\beta_C)(1+\beta_B)}} \\
&= D_C/D_B && \text{defn. of } D_B, D_C
\end{aligned}$$

This is the same result, calculated in a fairly different way, using the assumptions of special relativity, which ChatGPT is much better at answering questions about than it is any alternatives that are not special relativity.

D.2 Using the Doppler factor in special relatitivity in longitudinal scenarios

When authors write about relative motion between the source and receiver of a light signal, where the source and receiver are moving in a straight line at constant velocity directly towards or away from each other, some adopt the sign convention that $v > 0$ means they are moving towards each other, whereas others adopt the opposite convention where $v > 0$ means they are moving away from each other.

Regardless of the convention used, physically what happens is that when the sender and receiver are moving away from each other, for periodic pulse signals the period between them is larger at the receiver than the sender, due to a combination of time dilation at the sender as observed by the receiver (according to special relativity's conventions), and because each pulse or wave cycle takes longer to reach the receiver than the previous one. For sinusoidal signals, the frequency is smaller at the receiver and the wavelength is longer.

The opposite of all of these statements is true if the sender is moving towards the receiver. This is the best way to confirm what the correct velocity sign convention is for any formula using the Doppler factor.

Examples of authors using different velocity sign conventions include:

- Feynman Lectures, Volume I, Section 34-6 "The Doppler effect" [3]: $v > 0$ for source moving towards the receiver.
- Wikipedia page "The Relativistic Doppler effect" [14]: $v > 0$ for source moving away from the receiver.

- A conversation I had with ChatGPT in July 2025 (see Appendix D.2.3): $v > 0$ for source moving towards the receiver.

Since there seems to be no consensus on the sign convention used, in the rest of this document I will use the convention of $v > 0$ means the sender and receiver are moving away from each other.

D.2.1 Using the Doppler factor, according to the Feynman Lectures

In Section 34-6 "The Doppler effect" of the Feynman Lectures, Volume I [3], Feynman consistently uses the convention that $v > 0$ means that the sender and receiver are moving towards each other. Quotes include:

> and at the same time the whole atom, the whole oscillator, is moving along in a direction toward the observer at velocity v.

Later:

> Suppose, now, that the source is standing still and is emitting waves at frequency ω_0, while the observer is moving with speed v toward the source.

Feynman derives the frequency observed by the receiver ω is related to the frequency transmitted by the sender ω_0 as:

$$\begin{aligned}\omega &= \frac{\omega_0\sqrt{1-v^2/c^2}}{1-v/c} && \text{Feynman Eqn. (34.12)}\\ &= \omega_0\sqrt{\frac{1+\beta}{1-\beta}}\\ &= \omega_0 D\end{aligned}$$

If we replace ω (frequency of the receiver) with $1/T_r$ and ω_0 (frequency of the sender) with $1/T_s$, we get:

$$\frac{T_r}{T_s} = \frac{1}{D} \tag{122}$$

When the sender and receiver are moving away from each other, $v < 0$ by the velocity sign convention used by Feynman here. Thus $\beta < 0$, and $D < 1$. The ratio $1/D > 1$, indicating the receiver's pulses are further apart than the sender's.

D.2.2 Using the Doppler factor, according to Wikipedia

The Wikipedia page "The Relativistic Doppler effect" [14]. in its section "Summary of major results" gives its velocity sign convention as follows:

> In the following table, it is assumed that for $\beta = v/c > 0$ the receiver r and the source s are moving away from each other

The expression f_s is used for the frequency as transmitted by the sender, and f_r is the frequency as measured by the receiver.

$$\frac{f_s}{f_r} = \sqrt{\frac{1+\beta}{1-\beta}} \tag{123}$$

Period T is $1/f$, so we can write the relationship between the sender and receiver's period as:

$$\frac{T_r}{T_s} = \sqrt{\frac{1+\beta}{1-\beta}} = D \tag{124}$$

When the sender and receiver are moving away from each other, $\beta > 0$ by the sign convention used in that Wikipedia article. Thus $D > 1$, indicating the receiver's pulses are further apart than the sender's.

D.2.3 Using the Doppler factor, according to ChatGPT

In a fairly long interaction with ChatGPT that I had regarding questions related to Special Relativity, I asked:

> What are the rules for the beta value to use in the Doppler factor formula?

This section contains the answer I received, which is the same sign convention as used in the Feynman lectures mentioned earlier.

My goal is to compare these rules against the way I have used the formula for D elsewhere in this document, and do one of the following:

- update the document to make my use of D consistent with widely accepted rules for its use, or
- use a different symbol and name for the formula, perhaps calling mine a "modified Doppler factor", and describe as clearly as possible the difference between them.

Here begins the ChatGPT answer to the question above.

Here is a clear, precise technical guide for choosing the correct β sign in the relativistic Doppler factor:

Doppler factor formula For longitudinal (collinear) motion, the relativistic Doppler factor is:

$$D = \sqrt{\frac{1+\beta}{1-\beta}} \tag{125}$$

where:

$$\beta = \frac{v}{c} \tag{126}$$

The observed period:

$$T_{obs} = \frac{T_{emit}}{D} \tag{127}$$

and the observed frequency:

$$f_{obs} = D f_{emit} \tag{128}$$

Rule for β

$$\beta = \frac{v_{source, relative to observer}}{c} \tag{129}$$

where:

- $v_{source, relative to observer}$ is the velocity of the source measured in the observer's frame,
- Positive if source is moving toward the observer,
- Negative if source is moving away from the observer.

Rule for interpreting the sign: If the source and observer are approacahing each other, use:

$$\beta > 0 \tag{130}$$

leading to:

$$D > 1 \Rightarrow f_{obs} > f_{emit} \tag{131}$$

(blueshift, period decreases).

If the source and observer are receding from each other, use:

$$\beta < 0 \tag{132}$$

leading to:

$$D < 1 \Rightarrow f_{obs} < f_{emit} \tag{133}$$

(redshift, period increases).

How to determine β systematically:

1. Choose observer's frame as the reference.
2. Compute source's velocity in that frame.
3. If source moving toward observer: $\beta > 0$. If source moving away from observer: $\beta < 0$.
4. Plug into:

$$D = \sqrt{\frac{1+\beta}{1-\beta}} \tag{134}$$

It will automatically give you the correct Doppler shift direction (blushift or redshift).

E Details of Scenario 1c calculations

E.1 B sends periodic pulses to C

B sends its n-th pulse at the following times, on B's and A's clock:

$$q_B(n) = t_{0,B} + nT_B \tag{135}$$

$$q_A(n) = t_{0,A} + \gamma_B n T_B \tag{136}$$

What is C's time when it receives the B's n-th pulse? We find it using Lemma 9. Here are the symbols mentioned in that Lemma, with their values relevant for this situation:

$$\begin{aligned}
\mathbf{v} &= \mathbf{v}_C && \text{pulse receiver } C\text{'s velocity}\\
\mathbf{s} &= \mathbf{x}_B(q_A(n)) = \mathbf{x}_0 + \mathbf{v}_B t_{0,A} + \gamma_B \mathbf{v}_B n T_B\\
t_0 &= q_A(n) = t_{0,A} + \gamma_B n T_B\\
\mathbf{s} + \mathbf{r}_1 &= \mathbf{x}_0 && \text{location of } C \text{ at time } 0\\
t_1 &= 0\\
\mathbf{r}_1 &= -\mathbf{v}_B t_{0,A} - \gamma_B \mathbf{v}_B n T_B && \text{subtract } \mathbf{s} \text{ from } \mathbf{s}+\mathbf{r}_1\\
\mathbf{r}_0 &= \mathbf{r}_1 + \mathbf{v}_C(t_0 - t_1)\\
&= -\mathbf{v}_B t_{0,A} - \gamma_B \mathbf{v}_B n T_B + \mathbf{v}_C(t_{0,A} + \gamma_B n T_B) && \text{substitute } \mathbf{r}_1 \text{ and } t_0 \text{ from above}\\
&= (\mathbf{v}_C - \mathbf{v}_B)(\gamma_B n T_B + t_{0,A}) && \text{algebra}
\end{aligned}$$

Plug those into the equation for t_r from Lemma 8 to get $e_C(n)$:

$$e_C(n) = t_{0,A} + \gamma_B n T_B + \frac{\gamma_C^2}{c^2}\left[(\mathbf{r}_0 \cdot \mathbf{v}_C) + \sqrt{(\mathbf{r}_0 \cdot \mathbf{v}_C)^2 + (1-\beta_C^2)c^2 r_0^2}\right] \quad \text{substitute } t_0 \text{ but not yet } \mathbf{r}_0$$

To begin the substitutions and simplification of this, start with the expression $\mathbf{r}_0 \cdot \mathbf{v}_C$:

$$\begin{aligned}
\mathbf{r}_0 \cdot \mathbf{v}_C &= (\mathbf{v}_C - \mathbf{v}_B)(\gamma_B n T_B + t_{0,A}) \cdot \mathbf{v}_C\\
&= (v_C^2 - \mathbf{v}_B \cdot \mathbf{v}_C)(\gamma_B n T_B + t_{0,A})
\end{aligned}$$

Substituting into the formula for $e_C(n)$:

$$e_C(n) = t_{0,A} + \gamma_B n T_B + \frac{\gamma_C^2}{c^2}\left[(v_C^2 - \mathbf{v}_B \cdot \mathbf{v}_C)(\gamma_B n T_B + t_{0,A}) + \sqrt{F}\right]$$

where:

$$\begin{aligned}
F &= (v_C^2 - \mathbf{v}_B \cdot \mathbf{v}_C)^2(\gamma_B n T_B + t_{0,A})^2 + (1-\beta_C^2)c^2|\mathbf{v}_C - \mathbf{v}_B|^2(\gamma_B n T_B + t_{0,A})^2\\
&= (\gamma_B n T_B + t_{0,A})^2\left[(v_C^2 - \mathbf{v}_B \cdot \mathbf{v}_C)^2 + (1-\beta_C^2)c^2|\mathbf{v}_C - \mathbf{v}_B|^2\right]\\
&= (\gamma_B n T_B + t_{0,A})^2 G
\end{aligned}$$

Note that the part given the name G does not depend on n. Plugging that back into the equation for $e_C(n)$ and doing a little bit more algebra we can get:

$$\begin{aligned} e_C(n) &= t_{0,A} + \gamma_B n T_B + \frac{\gamma_C^2}{c^2}\left[(v_C^2 - \mathbf{v}_B \cdot \mathbf{v}_C)(\gamma_B n T_B + t_{0,A}) + (\gamma_B n T_B + t_{0,A})\sqrt{G}\right] \\ &= t_{0,A} + \gamma_B n T_B + \frac{\gamma_C^2}{c^2}(\gamma_B n T_B + t_{0,A})\left[(v_C^2 - \mathbf{v}_B \cdot \mathbf{v}_C) + \sqrt{G}\right] \end{aligned}$$

When calculating the difference between two consecutive pulses arriving to C, all terms independent of n cancel out, and we are left with the following:

$$e_C(n+1) - e_C(n) = \gamma_B T_B + \frac{\gamma_C^2}{c^2}(\gamma_B T_B)\left[(v_C^2 - \mathbf{v}_B \cdot \mathbf{v}_C) + \sqrt{G}\right]$$

On C's clock, the period is $1/\gamma_C$ times the period on A's clock, thus:

$$\frac{T_C}{T_B} = \frac{\gamma_B}{\gamma_C}\left[1 + \frac{\gamma_C^2}{c^2}\left((v_C^2 - \mathbf{v}_B \cdot \mathbf{v}_C) + \sqrt{(v_C^2 - \mathbf{v}_B \cdot \mathbf{v}_C)^2 + (1-\beta_C^2)c^2|\mathbf{v}_C - \mathbf{v}_B|^2}\right)\right] \tag{137}$$

E.2 Relationship with relativistic velocity subtraction

Starting from Equation (137), let us derive another expression that is more similar to Equation (112), to help us see their relationship to one another.

First we will find a different way to write the expression under the radical from Equation (137):

$$\begin{aligned} &(v_C^2 - \mathbf{v}_B \cdot \mathbf{v}_C)^2 + (1-\beta_C^2)c^2|\mathbf{v}_C - \mathbf{v}_B|^2 \\ &= v_C^2(v_C - \frac{\mathbf{v}_B \cdot \mathbf{v}_C}{v_C})^2 - v_C^2|\mathbf{v}_C - \mathbf{v}_B|^2 + c^2|\mathbf{v}_C - \mathbf{v}_B|^2 && \text{algebra, } c\beta_C = v_C \\ &= v_C^2\left[(v_C - \frac{\mathbf{v}_B \cdot \mathbf{v}_C}{v_C})^2 - |\mathbf{v}_C - \mathbf{v}_B|^2\right] + c^2|\mathbf{v}_C - \mathbf{v}_B|^2 && \text{algebra} \\ &= v_C^2\left[(v_C^2 - 2v_B v_C\cos\theta + v_B^2\cos^2\theta) - (v_C^2 - 2v_C v_B\cos\theta + v_B^2)\right] + c^2|\mathbf{v}_C - \mathbf{v}_B|^2 && \text{algebra} \\ &= v_C^2\left[v_B^2\cos^2\theta - v_B^2\right] + c^2|\mathbf{v}_C - \mathbf{v}_B|^2 && \text{cancel like terms} \\ &= c^2|\mathbf{v}_C - \mathbf{v}_B|^2 - v_C^2 v_B^2\sin^2\theta && 1 - \sin^2\theta = \cos^2\theta \\ &= c^2\left[|\mathbf{v}_C - \mathbf{v}_B|^2 - \frac{v_C^2 v_B^2 \sin^2\theta}{c^2}\right] && \text{algebra} \end{aligned}$$

In this section we will define:

$$F = \sqrt{|\mathbf{v}_C - \mathbf{v}_B|^2 - \frac{v_C^2 v_B^2 \sin^2\theta}{c^2}} \tag{138}$$

So the above expression under the radical becomes c^2F^2, and the entire Equation (137) can be written:

$$\begin{aligned} \frac{T_C}{T_B} &= \frac{\gamma_B}{\gamma_C}\left[1 + \frac{\gamma_C^2}{c^2}\left((v_C^2 - \mathbf{v}_B \cdot \mathbf{v}_C) + \sqrt{(v_C^2 - \mathbf{v}_B \cdot \mathbf{v}_C)^2 + (1-\beta_C^2)c^2|\mathbf{v}_C - \mathbf{v}_B|^2}\right)\right] \\ &= \frac{\gamma_B}{\gamma_C}\left[1 + \frac{\gamma_C^2}{c^2}\left((v_C^2 - \mathbf{v}_B \cdot \mathbf{v}_C) + cF\right)\right] && \text{substitute } F \\ &= \frac{\gamma_B}{\gamma_C}\left[1 + \gamma_C^2\left(\frac{v_C^2}{c^2} - \frac{\mathbf{v}_B \cdot \mathbf{v}_C}{c^2} + \frac{F}{c}\right)\right] && \text{algebra} \\ &= \frac{\gamma_B}{\gamma_C}\left[1 + \gamma_C^2\left((\beta_C^2 - 1) + (1 - \frac{\mathbf{v}_B \cdot \mathbf{v}_C}{c^2}) + \frac{F}{c}\right)\right] && \text{algebra, } \beta_C = v_C/c \\ &= \frac{\gamma_B}{\gamma_C}\left[1 + \gamma_C^2(\beta_C^2 - 1) + \gamma_C^2\left((1 - \frac{\mathbf{v}_B \cdot \mathbf{v}_C}{c^2}) + \frac{F}{c}\right)\right] && \text{algebra} \\ &= \frac{\gamma_B}{\gamma_C}\left[\gamma_C^2\left((1 - \frac{\mathbf{v}_B \cdot \mathbf{v}_C}{c^2}) + \frac{F}{c}\right)\right] && \gamma_C^2(1-\beta_C^2) = 1 \\ &= \gamma_B\gamma_C\left[(1 - \frac{\mathbf{v}_B \cdot \mathbf{v}_C}{c^2}) + \frac{F}{c}\right] && \text{algebra} \end{aligned} \tag{139}$$

Define K like that given in Lemma 12:

$$K = 1 - \frac{\mathbf{v}_B \cdot \mathbf{v}_C}{c^2}$$

and we see that we can write Equation (139) as:

$$\frac{T_C}{T_B} = \gamma_B \gamma_C \left(K + \frac{F}{c} \right) \tag{140}$$

Also, we can write Equation (112) as:

$$|\mathbf{v}_{rel}| = |\mathbf{v}_C \ominus \mathbf{v}_B| = \frac{F}{K} \tag{141}$$

Let $\beta_{rel} = v_{rel}/c$. Then the Doppler factor for this relative velocity $\mathbf{v}_{rel}$ is:

$$\begin{aligned}
D_{rel} &= \sqrt{\frac{1+\beta_{rel}}{1-\beta_{rel}}} \\
&= \sqrt{\frac{1+v_{rel}/c}{1-v_{rel}/c}} && \beta_{rel} = v_{rel}/c \\
&= \sqrt{\frac{1+\frac{F}{cK}}{1-\frac{F}{cK}}} && \text{Eqn. (141)} \\
&= \sqrt{\frac{cK+F}{cK-F}} && \text{multiply num. and denom. by } cK \\
&= \frac{cK+F}{\sqrt{(cK)^2 - F^2}} && \text{multiply num. and denom. by } \sqrt{cK+F} \qquad (142)
\end{aligned}$$

Find a way to rewrite $(cK)^2 - F^2$:

$$\begin{aligned}
(cK)^2 - F^2 &= \left(c(1 - \frac{\mathbf{v}_B \cdot \mathbf{v}_C}{c^2}) \right)^2 - \left(|\mathbf{v}_C - \mathbf{v}_B|^2 - \frac{v_C^2 v_B^2 \sin^2\theta}{c^2} \right) && \text{defn. of } K, F \\
&= \left(c^2 - 2v_B v_C \cos\theta + \frac{v_B^2 v_C^2 \cos^2\theta}{c^2} \right) - (v_C^2 - 2v_B v_C \cos\theta + v_B^2) + \frac{v_C^2 v_B^2 \sin^2\theta}{c^2} && \text{algebra} \\
&= c^2 + \frac{v_B^2 v_C^2}{c^2} - (v_B^2 + v_C^2) && \text{algebra} \\
&= c^2 \left(1 - \frac{v_B^2}{c^2} - \frac{v_C^2}{c^2} + \frac{v_B^2}{c^2}\frac{v_C^2}{c^2} \right) && \text{algebra} \\
&= c^2 \left(1 - \frac{v_B^2}{c^2} \right) \left(1 - \frac{v_C^2}{c^2} \right) && \text{algebra} \\
&= \frac{c^2}{\gamma_B^2 \gamma_C^2} && \text{defn. of } \gamma
\end{aligned} \tag{143}$$

Substituting this into Equation (142) we find:

$$\begin{aligned}
D_{rel} &= \frac{cK+F}{\frac{c}{\gamma_B \gamma_C}} \\
&= \gamma_B \gamma_C \left(K + \frac{F}{c} \right)
\end{aligned} \tag{144}$$

Thus we have shown that T_C/T_B from Equation (140), which was derived with no knowledge of the equation for 3-d relativistic velocity difference, nor of the relativistic Doppler factor, is equal to D_{rel} from Equation (147).

In addition to the proof above, I have written a simple program [1] that demonstrates those equations give the same result for many randomly selected pairs of values for $\mathbf{v}_B$, $\mathbf{v}_C$, which is strong evidence that there are no mistakes in the proof.

E.3 C sends periodic pulses to B

The calculations are nearly the same as in Appendix E.1, so we will omit most details and give only the results of a few of the steps.

C sends its n-th pulse at the following times, on C's and A's clock:

$$q_C(n) = t_{0,C} + nT_C \tag{145}$$
$$q_A(n) = t_{0,A} + \gamma_C n T_C \tag{146}$$

B's time when it receives the n-th pulse from C, as measured on A's clock, is:

$$e_B(n) = t_{0,A} + \gamma_C n T_C + \frac{\gamma_B^2}{c^2}(\gamma_C n T_C + t_{0,A}) \left[(v_B^2 - \mathbf{v}_C \cdot \mathbf{v}_B) + \sqrt{(v_B^2 - \mathbf{v}_C \cdot \mathbf{v}_B)^2 + (1 - \beta_B^2)c^2 |\mathbf{v}_B - \mathbf{v}_C|^2} \right]$$

The period (measured on A's clock) between B receiving consecutive pulses is:

$$e_B(n+1) - e_B(n) = \gamma_C T_C \left[1 + \frac{\gamma_B^2}{c^2} \left((v_B^2 - \mathbf{v}_C \cdot \mathbf{v}_B) + \sqrt{(v_B^2 - \mathbf{v}_C \cdot \mathbf{v}_B)^2 + (1 - \beta_B^2)c^2 |\mathbf{v}_B - \mathbf{v}_C|^2} \right) \right]$$

Then we divide by γ_B to calculate the period between B receiving pulses, as measured on B's slower-running clock:

$$\frac{T_B}{T_C} = \frac{\gamma_C}{\gamma_B} \left[1 + \frac{\gamma_B^2}{c^2} \left((v_B^2 - \mathbf{v}_C \cdot \mathbf{v}_B) + \sqrt{(v_B^2 - \mathbf{v}_C \cdot \mathbf{v}_B)^2 + (1 - \beta_B^2)c^2 |\mathbf{v}_B - \mathbf{v}_C|^2} \right) \right]$$

By the same set of steps at the beginning of Appendix E.2 that led to finding F of Equation (138), the expression under the radical above will give exactly the same formula for F. Note that expression is symmetric in $\mathbf{v}_B, \mathbf{v}_C$, i.e. swapping the values of $\mathbf{v}_B$ and $\mathbf{v}_C$ will give the same value for F. Continuing, a similar sequence of steps as in Appendix E.2 leads to:

$$\frac{T_B}{T_C} = \gamma_B \gamma_C \left(K + \frac{F}{c} \right) \tag{147}$$

where K and F are exactly the same as in that appendix. K is also symmetric in $\mathbf{v}_B, \mathbf{v}_C$, as F is.

By Lemma 13, $|\mathbf{v}_B \ominus \mathbf{v}_C|$ is also symmetric in $\mathbf{v}_B, \mathbf{v}_C$. So we get the same result for the period of pulses sent from C to B as we got for pulses from B to C:

$$\frac{T_B}{T_C} = D_{rel} = \sqrt{\frac{1 + \beta_{rel}}{1 - \beta_{rel}}} \tag{148}$$

where $\beta_{rel} = v_{rel}/c = |\mathbf{v}_B \ominus \mathbf{v}_C|/c$.

F Details of Scenario 2b calculations

See Section 4.2 for some preliminaries.

We choose event 1 to occur at time $q_1 = 0$ as before. This equation must hold at event 1:

x coordinate of right end of $S = x$ coordinate of left end of R

$$\begin{aligned}
x_S(q_1) + \frac{L}{2\gamma_S} &= x_R(q_1) - \frac{L}{2\gamma_R} && \\
x_{S,0} + 0 + \frac{L}{2\gamma_S} &= x_{R,0} + 0 - \frac{L}{2\gamma_R} && \text{defns. of } x_S(t), x_R(t) \\
x_{R,0} - x_{S,0} &= \frac{L}{2}\left(\frac{1}{\gamma_R} + \frac{1}{\gamma_S}\right) && \text{algebra} \qquad (149)
\end{aligned}$$

We let $x_{S,0}$ be some arbitrary real number, and then the equation above defines what $x_{R,0}$ must be.

Event 2 occurs at time q_2 determined by:

$$\begin{aligned}
x \text{ coordinate of left end of } S &= x \text{ coordinate of left end of } R \\
x_S(q_2) - \frac{L}{2\gamma_S} &= x_R(q_2) - \frac{L}{2\gamma_R} \\
x_{S,0} + v_S q_2 - \frac{L}{2\gamma_S} &= x_{R,0} + v_R q_2 - \frac{L}{2\gamma_R} && \text{defns. of } x_S(t), x_R(t) \\
(v_S - v_R) q_2 &= (x_{R,0} - x_{S,0}) + \frac{L}{2}\left(\frac{1}{\gamma_S} - \frac{1}{\gamma_R}\right) && \text{algebra} \\
q_2 &= \frac{\frac{L}{2}\left(\frac{1}{\gamma_R} + \frac{1}{\gamma_S} + \frac{1}{\gamma_S} - \frac{1}{\gamma_R}\right)}{v_S - v_R} && \text{Eqn. (149) and algebra} \\
q_2 &= \frac{L}{\gamma_S (v_S - v_R)} && \text{algebra} \qquad (150)
\end{aligned}$$

To find the x coordinate where event 2 occurs, plug time $t = q_2$ into $x_S(t)$ and subtract half the contracted length of rod S:

$$\begin{aligned}
s_2 &= x_S(q_2) - \frac{L}{2\gamma_S} \\
s_2 &= x_{S,0} + v_S \frac{L}{\gamma_S (v_S - v_R)} - \frac{L}{2\gamma_S} && \text{defn. of } x_S(t) \text{ and Eqn. (150)} \\
s_2 &= x_{S,0} + \frac{L}{\gamma_S}\left(\frac{v_S}{v_S - v_R} - \frac{1}{2}\right) && \text{algebra} \qquad (151)
\end{aligned}$$

Event 3 occurs at time q_3 determined by:

$$\begin{aligned}
x \text{ coordinate of right end of } S &= x \text{ coordinate of right end of } R \\
x_S(q_3) + \frac{L}{2\gamma_S} &= x_R(q_3) + \frac{L}{2\gamma_R} \\
x_{S,0} + v_S q_3 + \frac{L}{2\gamma_S} &= x_{R,0} + v_R q_3 + \frac{L}{2\gamma_R} && \text{defns. of } x_S(t), x_R(t) \\
(v_S - v_R) q_3 &= (x_{R,0} - x_{S,0}) + \frac{L}{2}\left(\frac{1}{\gamma_R} - \frac{1}{\gamma_S}\right) && \text{algebra} \\
q_3 &= \frac{\frac{L}{2}\left(\frac{1}{\gamma_R} + \frac{1}{\gamma_S} + \frac{1}{\gamma_R} - \frac{1}{\gamma_S}\right)}{v_S - v_R} && \text{Eqn. (149) and algebra} \\
q_3 &= \frac{L}{\gamma_R (v_S - v_R)} && \text{algebra} \qquad (152)
\end{aligned}$$

To find the x coordinate where event 3 occurs, plug time $t = q_3$ into $x_S(t)$ and add half the contracted length of rod S:

$$\begin{aligned}
s_3 &= x_S(q_3) + \frac{L}{2\gamma_S} \\
s_3 &= x_{S,0} + v_S \frac{L}{\gamma_R (v_S - v_R)} + \frac{L}{2\gamma_S} && \text{defn. of } x_S(t) \text{ and Eqn. (152)} \\
s_3 &= x_{S,0} + \frac{L v_S}{\gamma_R (v_S - v_R)} + \frac{L}{2\gamma_S} && \text{algebra} \qquad (153)
\end{aligned}$$

Find time in rest frame that event 2 pulse reaches A In the symbols used for Lemma 8, here are the relevant details about event 2 and A:

$$\begin{aligned}
\mathbf{v} &= (v_R, 0) && A \text{ velocity same as } R \\
\mathbf{s} &= (s_2, 0) = (x_{S,0} + \frac{L}{\gamma_S}\left(\frac{v_S}{v_S - v_R} - \frac{1}{2}\right), 0) && \text{event 2 pulse is from left end of } R\text{, Eqn. (151)} \\
t_0 &= q_2 = \frac{L}{\gamma_S(v_S - v_R)} && \text{event 2 pulse time from Eqn. (150)} \\
\mathbf{s} + \mathbf{r}_1 &= (x_{R,0}, D) = (x_{S,0} + \frac{L}{2}(\frac{1}{\gamma_R} + \frac{1}{\gamma_S}), D) && \text{location of } A \text{ at time 0} \\
t_1 &= 0 \\
\mathbf{r}_1 &= (\frac{L}{2\gamma_R} + \frac{L}{\gamma_S} - \frac{Lv_S}{\gamma_S(v_S - v_R)}, D) && \text{subtract } \mathbf{s} \text{ from } \mathbf{s} + \mathbf{r}_1 \\
\mathbf{r}_0 &= \mathbf{r}_1 + \mathbf{v}_R(t_0 - t_1) \\
&= (\frac{L}{2\gamma_R} + \frac{L}{\gamma_S} - \frac{Lv_S}{\gamma_S(v_S - v_R)} + \frac{Lv_R}{\gamma_S(v_S - v_R)}, D) && \text{substitute } \mathbf{r}_1 \text{ and } t_0 \text{ from above} \\
&= (\frac{L}{2\gamma_R} + \frac{L}{\gamma_S} - \frac{L}{\gamma_S}, D) && \text{algebra} \\
&= (\frac{L}{2\gamma_R}, D) && \text{algebra}
\end{aligned}$$

Plug those into the equation for t_r from Lemma 8 to get $e_{2,A}$:

$$e_{2,A} = \frac{L}{\gamma_S(v_S - v_R)} + \frac{\gamma_R^2}{c^2}\left[\frac{Lv_R}{2\gamma_R} + \sqrt{\frac{L^2 v_R^2}{4\gamma_R^2} + (1 - \beta_R^2)c^2(\frac{L^2}{4\gamma_R^2} + D^2)}\right]$$

Find time in rest frame that event 3 pulse reaches A In the symbols used for Lemma 8, here are the relevant details about event 3 and A:

$$\begin{aligned}
\mathbf{v} &= (v_R, 0) && A \text{ velocity same as } R \\
\mathbf{s} &= (s_3, 0) = (x_{S,0} + \frac{Lv_S}{\gamma_R(v_S - v_R)} + \frac{L}{2\gamma_S}, 0) && \text{event 3 pulse is from right end of } R\text{, Eqn. (153)} \\
t_0 &= q_3 = \frac{L}{\gamma_R(v_S - v_R)} && \text{event 3 pulse time from Eqn. (152)} \\
\mathbf{s} + \mathbf{r}_1 &= (x_{R,0}, D) = (x_{S,0} + \frac{L}{2}(\frac{1}{\gamma_R} + \frac{1}{\gamma_S}), D) && \text{location of } A \text{ at time 0} \\
t_1 &= 0 \\
\mathbf{r}_1 &= (\frac{L}{2\gamma_R} - \frac{Lv_S}{\gamma_R(v_S - v_R)}, D) && \text{subtract } \mathbf{s} \text{ from } \mathbf{s} + \mathbf{r}_1 \\
\mathbf{r}_0 &= \mathbf{r}_1 + \mathbf{v}_R(t_0 - t_1) \\
&= (\frac{L}{2\gamma_R} - \frac{Lv_S}{\gamma_R(v_S - v_R)} + \frac{Lv_R}{\gamma_R(v_S - v_R)}, D) && \text{substitute } \mathbf{r}_1 \text{ and } t_0 \text{ from above} \\
&= (\frac{L}{2\gamma_R} - \frac{L}{\gamma_R}, D) && \text{algebra} \\
&= (-\frac{L}{2\gamma_R}, D) && \text{algebra}
\end{aligned}$$

Plug those into the equation for t_r from Lemma 8 to get $e_{3,A}$:

$$e_{3,A} = \frac{L}{\gamma_R(v_S - v_R)} + \frac{\gamma_R^2}{c^2}\left[-\frac{Lv_R}{2\gamma_R} + \sqrt{\frac{L^2 v_R^2}{4\gamma_R^2} + (1 - \beta_R^2)c^2(\frac{L^2}{4\gamma_R^2} + D^2)}\right]$$

Find difference in times that event 2 and event 3 pulses reach A, on A's clock The expressions under the radicals are equal, and cancel out:

$$\begin{aligned}
e_{3,A} - e_{2,A} &= \left[\frac{L}{\gamma_R(v_S - v_R)} - \frac{L}{\gamma_S(v_S - v_R)}\right] + \frac{\gamma_R^2}{c^2}\left[-\frac{Lv_R}{2\gamma_R} - \frac{Lv_R}{2\gamma_R}\right] \\
&= \frac{L}{c}\frac{\frac{1}{\gamma_R} - \frac{1}{\gamma_S}}{(\beta_S - \beta_R)} - \frac{L\gamma_R\beta_R}{c} \\
&= \frac{L}{c}\left[\frac{\frac{1}{\gamma_R} - \frac{1}{\gamma_S}}{(\beta_S - \beta_R)} - \gamma_R\beta_R\right] && (154) \\
&= \frac{L}{c}\left[\frac{\gamma_R\gamma_S(\beta_R + \beta_S)}{\gamma_R + \gamma_S} - \gamma_R\beta_R\right] && \text{Lemma 7} \\
&= \frac{\gamma_R L}{c}\left[\frac{\beta_S\gamma_S - \beta_R\gamma_R}{\gamma_S + \gamma_R}\right] && \text{a few algebra steps} \quad (155)
\end{aligned}$$

Since all of these calculations were done with the assumption that $v_S > v_R$ and $v_S > 0$, it follows that $\beta_S > \beta_R$, $\beta_S > 0$, and $\gamma_S > \gamma_R$. Thus $\beta_S\gamma_S > \beta_R\gamma_R$, and $e_{3,A} - e_{2,A}$ is positive.

As a quick double-check, let us see what happens if we plug in $v_R = 0$ and thus also $\beta_R = 0$ and $\gamma_R = 1$:

$$e_{3,A} - e_{2,A} = \frac{L}{c}\left(\frac{\beta_S\gamma_S}{\gamma_S + 1}\right) \qquad \text{plug in values for } R$$

This is the same as Equation (27), as it should be when R and A are at rest.

Find time in rest frame that event 2 pulse reaches B In the symbols used for Lemma 8, here are the relevant details about event 2 and B:

$$\begin{aligned}
\mathbf{v} &= (v_S, 0) && B \text{ velocity same as } S \\
\mathbf{s} &= (s_2, 0) = (x_{S,0} + \frac{L}{\gamma_S}\left(\frac{v_S}{v_S - v_R} - \frac{1}{2}\right), 0) && \text{event 2 pulse is from left end of } R\text{, Eqn. (151)} \\
t_0 &= q_2 = \frac{L}{\gamma_S(v_S - v_R)} && \text{event 2 pulse time from Eqn. (150)} \\
\mathbf{s} + \mathbf{r}_1 &= (x_{S,0}, D) && \text{location of } B \text{ at time 0} \\
t_1 &= 0 \\
\mathbf{r}_1 &= (-\frac{L}{\gamma_S}\left(\frac{v_S}{v_S - v_R} - \frac{1}{2}\right), D) && \text{subtract } \mathbf{s} \text{ from } \mathbf{s} + \mathbf{r}_1 \\
\mathbf{r}_0 &= \mathbf{r}_1 + \mathbf{v}_S(t_0 - t_1) \\
&= (-\frac{L}{\gamma_S}\left(\frac{v_S}{v_S - v_R} - \frac{1}{2}\right) + \frac{Lv_S}{\gamma_S(v_S - v_R)}, D) && \text{substitute } \mathbf{r}_1 \text{ and } t_0 \text{ from above} \\
&= (\frac{L}{2\gamma_S}, D) && \text{algebra}
\end{aligned}$$

Plug those into the equation for t_r from Lemma 8 to get $e_{2,B}$:

$$e_{2,B} = \frac{L}{\gamma_S(v_S - v_R)} + \frac{\gamma_S^2}{c^2}\left[\frac{Lv_S}{2\gamma_S} + \sqrt{\frac{L^2v_S^2}{4\gamma_S^2} + (1 - \beta_S^2)c^2(\frac{L^2}{4\gamma_S^2} + D^2)}\right]$$

Find time in rest frame that event 3 pulse reaches B In the symbols used for Lemma 8, here are the relevant details about event 2 and B:

$$\begin{aligned}
\mathbf{v} &= (v_S, 0) && B \text{ velocity same as } S \\
\mathbf{s} = (s_3, 0) &= (x_{S,0} + \frac{Lv_S}{\gamma_R(v_S - v_R)} + \frac{L}{2\gamma_S}, 0) && \text{event 3 pulse is from right end of } R\text{, Eqn. (153)} \\
t_0 = q_3 &= \frac{L}{\gamma_R(v_S - v_R)} && \text{event 3 pulse time from Eqn. (152)} \\
\mathbf{s} + \mathbf{r}_1 &= (x_{S,0}, D) && \text{location of } B \text{ at time 0} \\
t_1 &= 0 \\
\mathbf{r}_1 &= (-\frac{Lv_S}{\gamma_R(v_S - v_R)} - \frac{L}{2\gamma_S}, D) && \text{subtract } \mathbf{s} \text{ from } \mathbf{s} + \mathbf{r}_1 \\
\mathbf{r}_0 &= \mathbf{r}_1 + \mathbf{v}_S(t_0 - t_1) \\
&= (-\frac{Lv_S}{\gamma_R(v_S - v_R)} - \frac{L}{2\gamma_S} + \frac{Lv_S}{\gamma_R(v_S - v_R)}, D) && \text{substitute } \mathbf{r}_1 \text{ and } t_0 \text{ from above} \\
&= (-\frac{L}{2\gamma_S}, D) && \text{algebra}
\end{aligned}$$

Plug those into the equation for t_r from Lemma 8 to get $e_{3,B}$:

$$e_{3,B} = \frac{L}{\gamma_R(v_S - v_R)} + \frac{\gamma_S^2}{c^2}\left[-\frac{Lv_S}{2\gamma_S} + \sqrt{\frac{L^2v_S^2}{4\gamma_S^2} + (1-\beta_S^2)c^2(\frac{L^2}{4\gamma_S^2} + D^2)}\right]$$

Find difference in times that event 2 and event 3 pulses reach B, on A's clock The expressions under the radicals are equal, and cancel out:

$$\begin{aligned}
e_{3,B} - e_{2,B} &= \left[\frac{L}{\gamma_R(v_S - v_R)} - \frac{L}{\gamma_S(v_S - v_R)}\right] + \frac{\gamma_S^2}{c^2}\left[-\frac{Lv_S}{2\gamma_S} - \frac{Lv_S}{2\gamma_S}\right] \\
&= \frac{L}{c}\frac{\frac{1}{\gamma_R} - \frac{1}{\gamma_S}}{(\beta_S - \beta_R)} - \frac{L\gamma_S\beta_S}{c} \\
&= \frac{L}{c}\left[\frac{\frac{1}{\gamma_R} - \frac{1}{\gamma_S}}{(\beta_S - \beta_R)} - \gamma_S\beta_S\right] && (156) \\
&= \frac{L}{c}\left[\frac{\gamma_R\gamma_S(\beta_R + \beta_S)}{\gamma_R + \gamma_S} - \gamma_S\beta_S\right] && \text{Lemma 7} \\
&= -\frac{\gamma_S L}{c}\left[\frac{\beta_S\gamma_S - \beta_R\gamma_R}{\gamma_S + \gamma_R}\right] && \text{a few algebra steps} \quad (157)
\end{aligned}$$

For the same reasons that were explained above why $e_{3,A} - e_{2,A}$ is positive, we know that $e_{3,B} - e_{2,B}$ must be negative.

Details of observer E Introduce a new observer E that is at rest relative to rod S, and always at y coordinate D, like B is, but E begins at a different x position than B does. E's initial x coordinate is $x_{S,0} + Z$, where Z is any real number.

$$\begin{aligned}
\mathbf{v} &= \mathbf{v}_S && && (158) \\
\mathbf{t_1} &= 0 && \text{lemma lets us pick any time } t_1 \text{ convenient for us} && (159) \\
\mathbf{s} + \mathbf{r}_1 &= (x_{S,0} + Z, D) && E \text{ starts at offset } Z \text{ from } B\text{'s starting position} && (160)
\end{aligned}$$

Event 2, Observer E: The expression $Z + \frac{L}{2\gamma_S}$ appears often enough that we will use the symbol Z_+ to denote it.

$$
\begin{aligned}
\mathbf{r}_1 &= (Z - q_2 v_S + \frac{L}{2\gamma_S}, D) \\
\mathbf{r}_0 &= \mathbf{r}_1 + \mathbf{v}_S(t_0 - t_1) = (Z_+, D) && t_0 = q_2, t_1 = 0 \\
\mathbf{r}_0 \cdot \mathbf{v}_S &= Z_+ v_S \\
r_0^2 &= Z_+^2 + D^2 \\
e_{2,E} &= q_2 + \frac{\gamma_S^2}{c^2}\left[Z_+ v_S + \sqrt{Z_+^2 v_S^2 + (1-\beta_S^2)c^2(Z_+^2 + D^2)} \right]
\end{aligned} \tag{161}
$$

Event 3, Observer E: The expression $Z - \frac{L}{2\gamma_S}$ appears often enough that we will use the symbol Z_- to denote it.

$$
\begin{aligned}
\mathbf{r}_1 &= (Z - q_3 v_S - \frac{L}{2\gamma_S}, D) \\
\mathbf{r}_0 &= \mathbf{r}_1 + \mathbf{v}_S(t_0 - t_1) = (Z_-, D) && t_0 = q_3, t_1 = 0 \\
\mathbf{r}_0 \cdot \mathbf{v}_S &= Z_- v_S \\
r_0^2 &= Z_-^2 + D^2 \\
e_{3,E} &= q_3 + \frac{\gamma_S^2}{c^2}\left[Z_- v_S + \sqrt{Z_-^2 v_S^2 + (1-\beta_S^2)c^2(Z_-^2 + D^2)} \right]
\end{aligned} \tag{162}
$$

Find difference in times that event 2 and event 3 pulses reach E, on A's clock Note that this time the expressions on the radicals are not equal, and do not cancel. We will use F to denote the last term, containing $\frac{\gamma_S^2}{c^2}$ times the difference of those radicals.

$$
\begin{aligned}
e_{3,E} - e_{2,E} &= q_3 - q_2 + \frac{\gamma_S^2}{c^2}(Z_- v_S - Z_+ v_S) + F \\
&= q_3 - q_2 - \frac{\gamma_S^2}{c^2}(\frac{L v_S}{\gamma_S}) + F \\
&= (e_{3,B} - e_{2,B}) + F
\end{aligned} \tag{163}
$$

Notice that except for the term containing F, everything else is equal to $(e_{3,B} - e_{2,B})$, so we will spend the rest of our effort focusing only on F. Let us focus first on only the first radical expression of F:

$$
\begin{aligned}
&\sqrt{Z_-^2 v_S^2 + (1-\beta_S^2)c^2(Z_-^2 + D^2)} \\
&= |Z_-|\sqrt{v_S^2 + (1-\beta_S^2)c^2(1 + (D/Z_-)^2)} && \text{factor out } |Z_-| \\
&= c|Z_-|\sqrt{\beta_S^2 + (1-\beta_S^2)(1 + (D/Z_-)^2)} && v_S = \beta_S c, \text{ factor out } c \\
&= c|Z_-|\sqrt{\beta_S^2 + 1 - \beta_S^2 + (1-\beta_S^2)(D/Z_-)^2)} && \text{algebra} \\
&= c|Z_-|\sqrt{1 + (D/Z_-)^2/\gamma_S^2} && \text{algebra, } (1-\beta_S^2) = 1/\gamma_S^2 \\
&= \frac{c|Z_-|}{\gamma_S}\sqrt{\gamma_S^2 + (D/Z_-)^2} && \text{factor out } 1/\gamma_S^2
\end{aligned}
$$

The exact same sequence of steps shows the following for the other radical expression:

$$
\begin{aligned}
&\sqrt{Z_+^2 v_S^2 + (1-\beta_S^2)c^2(Z_+^2 + D^2)} \\
&= \frac{c|Z_+|}{\gamma_S}\sqrt{\gamma_S^2 + (D/Z_+)^2}
\end{aligned}
$$

Thus F can be written as:

$$
\begin{aligned}
F &= \frac{\gamma_S^2}{c^2}(\frac{c}{\gamma_S})\left[|Z_-|\sqrt{\gamma_S^2 + (D/Z_-)^2} - |Z_+|\sqrt{\gamma_S^2 + (D/Z_+)^2} \right] \\
&= \frac{\gamma_S}{c}\left[|Z_-|\sqrt{\gamma_S^2 + (D/Z_-)^2} - |Z_+|\sqrt{\gamma_S^2 + (D/Z_+)^2} \right]
\end{aligned}
$$

Temporarily using a and b to denote these expressions:

$$a = |Z_-|\sqrt{\gamma_S^2 + (D/Z_-)^2}$$
$$b = |Z_+|\sqrt{\gamma_S^2 + (D/Z_+)^2}$$

then $F = \frac{\gamma_S}{c}(a - b)$. Focusing on the $a - b$ part, we can multiply it by $(a + b)/(a + b)$, resulting in $(a^2 - b^2)/(a + b)$. The numerator of this can be simplified quite a bit:

$$\begin{aligned}
a^2 - b^2 &= Z_-^2(\gamma_S^2 + (D/Z_-)^2) - Z_+^2(\gamma_S^2 + (D/Z_+)^2) \\
&= (Z_-^2\gamma_S^2 + D^2) - (Z_+^2\gamma_S^2 + D^2) && \text{algebra} \\
&= \gamma_S^2(Z_-^2 - Z_+^2) && \text{algebra} \\
&= \gamma_S^2\left[(Z - \frac{L}{2\gamma_S})^2 - (Z + \frac{L}{2\gamma_S})^2\right] && \text{substitute definitinos of } Z_-,\, Z_+ \\
&= \gamma_S^2\left[(Z^2 - 2Z\frac{L}{2\gamma_S} + \frac{L^2}{4\gamma_S^2}) - (Z^2 + 2Z\frac{L}{2\gamma_S} + \frac{L^2}{4\gamma_S^2})\right] && \text{algebra} \\
&= \gamma_S^2\left[-\frac{2LZ}{\gamma_S}\right] && \text{algebra} \\
&= -2LZ\gamma_S && \text{algebra}
\end{aligned}$$

Plugging all of this back into the equation for F:

$$\begin{aligned}
F &= (\frac{\gamma_S}{c})\frac{a^2 - b^2}{a + b} \\
&= (\frac{\gamma_S}{c})\frac{-2LZ\gamma_S}{|Z_-|\sqrt{\gamma_S^2 + (D/Z_-)^2} + |Z_+|\sqrt{\gamma_S^2 + (D/Z_+)^2}} \\
&= \frac{L\gamma_S}{c}\left[\frac{-2Z\gamma_S}{|Z_-|\sqrt{\gamma_S^2 + (D/Z_-)^2} + |Z_+|\sqrt{\gamma_S^2 + (D/Z_+)^2}}\right] \qquad (164)
\end{aligned}$$

When Z approaches $+\infty$ or $-\infty$, both $(D/Z_-)^2$ and $(D/Z_+)^2$ approach 0, and the radical expressions approach γ_S. $|Z_-| + |Z_+| = 2|Z|$ when $|Z| > \frac{L}{2\gamma_S}$. Thus the entire expression in square brackets approaches $+1$ when Z approaches $-\infty$, and approaches -1 when Z approaches $+\infty$.

$$\begin{aligned}
F &\to \frac{L\gamma_S}{c} && \text{as } Z \to -\infty \\
F &\to -\frac{L\gamma_S}{c} && \text{as } Z \to +\infty \\
F &= 0 && \text{when } Z = 0
\end{aligned}$$

Combining Equations (163) and (164), we get:

$$e_{3,E} - e_{2,E} = (e_{3,B} - e_{2,B}) + \frac{L\gamma_S}{c}\left[\frac{-2Z\gamma_S}{|Z_-|\sqrt{\gamma_S^2 + (D/Z_-)^2} + |Z_+|\sqrt{\gamma_S^2 + (D/Z_+)^2}}\right] \qquad (165)$$

G Details of Scenario 3b calculations

See Section 5 for some preliminaries.

Without loss of generality, we choose event 1 to occur at time 0, when rod R's center is at the origin. The equations for the x coordinates of R and S are:

$$x_R(t) = v_R t$$
$$x_S(t) = x_{S,0} + v_S t$$

To find the value of $x_{S,0}$ that is consistent with the choices above, note that event 1 at $t = 0$ occurs when the following equation is true, because $w_R - w_S$ is the difference in the half-widths of R and S, so their centers must be that far apart in order for their left ends to meet.

$$\begin{aligned} x_R(0) - x_S(0) &= w_R - w_S \\ -x_{S,0} &= w_R - w_S \\ x_{S,0} &= w_S - w_R \end{aligned}$$

As for most of these scenarios, we will be applying Lemma 8 to determine the times when the light pulses from both events will reach each observer. It is useful to define these vectors:

$$\mathbf{R} = (w_R, h, 0) \quad \text{offset from center of } R \text{ to its top right end} \tag{166}$$
$$\mathbf{S} = (w_S, h, 0) \quad \text{offset from center of } S \text{ to its top right end} \tag{167}$$
$$\mathbf{D} = (0, 0, D) \quad \text{offset from center of } R\ (S) \text{ to observer } A\ (B) \tag{168}$$

Here are the details of each event, given using the symbols of that lemma.

Event 1:

$$t_0 = 0 \tag{169}$$
$$\mathbf{s} = x_R(t_0) - \mathbf{R} = -\mathbf{R} \quad -\mathbf{R} \text{ is offset from } R\text{'s center to event 1} \tag{170}$$

Event 2: Event 2 occurs at time q_2 when the center of S is to the right of the center of R by $w_R - w_S$:

$$\begin{aligned} x_S(q_2) - x_R(q_2) &= w_R - w_S \\ (w_S - w_R + v_S q_2) - (v_R q_2) &= w_R - w_S && \text{defns. of } x_S(t) \text{ and } x_R(t) \\ (v_S - v_R) q_2 &= 2(w_R - w_S) && \text{algebra} \\ q_2 &= \frac{2(w_R - w_S)}{v_S - v_R} && (171) \end{aligned}$$

$$t_0 = q_2 = \frac{2(w_R - w_S)}{v_S - v_R} \tag{172}$$
$$\mathbf{s} = x_R(q_2) + \mathbf{R} = \mathbf{f} + \mathbf{R} \quad +\mathbf{R} \text{ is offset from } R\text{'s center to event 2} \tag{173}$$

where:

$$\mathbf{f} = q_2 \mathbf{v}_R \tag{174}$$

Here are the details of each observer, given using the symbols of the lemma.

Observer A:

$$\mathbf{v} = \mathbf{v}_R \tag{175}$$
$$\mathbf{t_1} = 0 \quad \text{lemma lets us pick any time } t_1 \text{ convenient for us} \tag{176}$$
$$\mathbf{s} + \mathbf{r}_1 = \mathbf{D} \quad \text{at that time we know } A \text{ is above the origin} \tag{177}$$

Observer B:

$$\mathbf{v} = \mathbf{v}_S \tag{178}$$
$$\mathbf{t_1} = 0 \quad \text{lemma lets us pick any time } t_1 \text{ convenient for us} \tag{179}$$
$$\mathbf{s} + \mathbf{r}_1 = x_{S,0}\hat{\mathbf{i}} + \mathbf{D} = (w_S - w_R)\hat{\mathbf{i}} + \mathbf{D} \quad B \text{ is above } S\text{'s center} \tag{180}$$

Observer D: Note that we want $\mathbf{v}_D$ to be related to $\mathbf{v}_R$ and $\mathbf{v}_S$ such that D's rapidity (see Appendix A.2) is the average of the rapidity of R and S.

$$\mathbf{v} = \mathbf{v}_D = \beta_D c = \frac{\beta_R \gamma_R + \beta_S \gamma_S}{\gamma_R + \gamma_S} c \quad \text{Lemma 1 with } \eta_1 = \tanh^{-1} \beta_R \text{ and } \eta_2 = -\tanh^{-1} \beta_S \tag{181}$$
$$\mathbf{t_1} = 0 \quad \text{lemma lets us pick any time } t_1 \text{ convenient for us} \tag{182}$$
$$\mathbf{s} + \mathbf{r}_1 = -k\hat{\mathbf{i}} + \mathbf{D} \quad k \text{ to be determined later} \tag{183}$$

Now all that is left is applying Lemma 8 for each pair of event and observer of interest to us, and some algebra to simplify.

Event 1, Observer A:

$$\begin{aligned}
\mathbf{r}_1 &= \mathbf{D} - (-\mathbf{R}) = \mathbf{R} + \mathbf{D} \\
\mathbf{r}_0 &= \mathbf{r}_1 + \mathbf{v}_R(t_0 - t_1) = \mathbf{r}_1 \\
\mathbf{r}_0 \cdot \mathbf{v}_R &= w_R v_R \\
r_0^2 &= R^2 + D^2 && \mathbf{R} \cdot \mathbf{D} = 0 \\
e_{1,A} &= \frac{\gamma_R^2}{c^2}\left[w_R v_R + \sqrt{(w_R v_R)^2 + (1 - \beta_R^2)c^2 r_0^2}\right]
\end{aligned}$$

Event 2, Observer A:

$$\begin{aligned}
\mathbf{r}_1 &= \mathbf{D} - (\mathbf{f} + \mathbf{R}) = \mathbf{D} - \mathbf{f} - \mathbf{R} \\
\mathbf{r}_0 &= \mathbf{r}_1 + \mathbf{v}_R(t_0 - t_1) \\
&= (\mathbf{D} - \mathbf{f} - \mathbf{R}) + \mathbf{v}_R q_2 \\
&= \mathbf{D} - \mathbf{R} && \mathbf{f} \text{ and } \mathbf{v}_R q_2 \text{ are equal and cancel} \\
\mathbf{r}_0 \cdot \mathbf{v}_R &= -w_R v_R && \mathbf{D} \cdot \mathbf{v}_R = 0 \\
r_0^2 &= R^2 + D^2 && \mathbf{R} \cdot \mathbf{D} = 0 \\
e_{2,A} &= q_2 + \frac{\gamma_R^2}{c^2}\left[-w_R v_R + \sqrt{(w_R v_R)^2 + (1 - \beta_R^2)c^2 r_0^2}\right]
\end{aligned}$$

Observer A difference in time receiving pulses:

$$\begin{aligned}
e_{2,A} - e_{1,A} &= q_2 - 2\frac{\gamma_R^2}{c^2}(w_R v_R) \\
&= \frac{2(w_R - w_S)}{v_S - v_R} - 2\frac{\gamma_R^2}{c^2}\left(\frac{w}{\gamma_R}\beta_R c\right) && \text{Defn. of } q_2,\ w_R, \text{ and } v_R = \beta_R c \\
&= 2w\left[\frac{\frac{1}{\gamma_R} - \frac{1}{\gamma_S}}{\beta_S c - \beta_R c} - \frac{\gamma_R}{c}\beta_R\right] && \text{algebra, defn. of } w_S,\ w_R \\
&= \frac{2w}{c}\left[\frac{\frac{1}{\gamma_R} - \frac{1}{\gamma_S}}{\beta_S - \beta_R} - \gamma_R \beta_R\right] && \text{algebra. Same as Eqn. (154) except } 2w \text{ instead of } L \\
&= \frac{2\gamma_R w}{c}\left[\frac{\beta_S \gamma_S - \beta_R \gamma_R}{\gamma_S + \gamma_R}\right] && \text{So equal to Eqn. (155) with same replacement}
\end{aligned}$$

Event 1, Observer B:

$$\begin{aligned}
\mathbf{r}_1 &= (w_S - w_R)\hat{\mathbf{i}} + \mathbf{D} - (-\mathbf{R}) = (w_S - w_R)\hat{\mathbf{i}} + \mathbf{R} + \mathbf{D} \\
\mathbf{r}_0 &= \mathbf{r}_1 + \mathbf{v}_R(t_0 - t_1) = \mathbf{r}_1 \\
\mathbf{r}_0 \cdot \mathbf{v}_S &= (w_S - w_R)v_S + w_R v_S = w_S v_S \\
r_0^2 &= ((w_S - w_R)\hat{\mathbf{i}} + \mathbf{R})^2 + D^2 && \mathbf{R} \cdot \mathbf{D} = 0 \\
e_{1,B} &= \frac{\gamma_S^2}{c^2}\left[w_S v_S + \sqrt{(w_S v_S)^2 + (1 - \beta_S^2)c^2 r_0^2}\right]
\end{aligned}$$

Event 2, Observer B:

$$\begin{aligned}
\mathbf{r}_1 &= (w_S - w_R)\hat{\mathbf{i}} + \mathbf{D} - (\mathbf{f} + \mathbf{R}) = (w_S - w_R)\hat{\mathbf{i}} - \mathbf{f} - \mathbf{R} + \mathbf{D} \\
\mathbf{r}_0 &= \mathbf{r}_1 + \mathbf{v}_R(t_0 - t_1) \\
&= (w_S - w_R)\hat{\mathbf{i}} - \mathbf{f} - \mathbf{R} + \mathbf{D} + \mathbf{v}_S q_2 \\
&= (w_S - w_R)\hat{\mathbf{i}} + q_2(\mathbf{v}_S - \mathbf{v}_R) - \mathbf{R} + \mathbf{D} && \mathbf{f} = q_2\mathbf{v}_R \\
&= (w_S - w_R)\hat{\mathbf{i}} + \frac{2(w_R - w_S)}{v_S - v_R}(\mathbf{v}_S - \mathbf{v}_R) - \mathbf{R} + \mathbf{D} && \text{defn. of } q_2 \\
&= (w_S - w_R)\hat{\mathbf{i}} + 2(w_R - w_S)\hat{\mathbf{i}} - \mathbf{R} + \mathbf{D} && \text{algebra} \\
&= (w_R - w_S)\hat{\mathbf{i}} - \mathbf{R} + \mathbf{D} && \text{algebra} \\
\mathbf{r}_0 \cdot \mathbf{v}_S &= (w_R - w_S)v_S - w_R v_S \\
&= -w_S v_S \\
r_0^2 &= (-(w_S - w_R)\hat{\mathbf{i}} - \mathbf{R})^2 + D^2 && \mathbf{R}\cdot\mathbf{D} = 0 \\
&= ((w_S - w_R)\hat{\mathbf{i}} + \mathbf{R})^2 + D^2 && \text{same as for Event 1, Observer } B \text{ above} \\
e_{2,B} &= q_2 + \frac{\gamma_S^2}{c^2}\left[-w_S v_S + \sqrt{(w_S v_S)^2 + (1 - \beta_S^2)c^2 r_0^2}\right]
\end{aligned}$$

Observer B difference in time receiving pulses:

$$\begin{aligned}
e_{2,B} - e_{1,B} &= q_2 - 2\frac{\gamma_S^2}{c^2}(w_S v_S) \\
&= \frac{2(w_R - w_S)}{v_S - v_R} - 2\frac{\gamma_S^2}{c^2}\left(\frac{w}{\gamma_S}\beta_S c\right) && \text{Defn. of } q_2,\ w_S, \text{ and } v_S = \beta_S c \\
&= 2w\left[\frac{\frac{1}{\gamma_R} - \frac{1}{\gamma_S}}{\beta_S c - \beta_R c} - \frac{\gamma_S}{c}\beta_S\right] && \text{algebra, defn. of } w_S,\ w_R \\
&= \frac{2w}{c}\left[\frac{\frac{1}{\gamma_R} - \frac{1}{\gamma_S}}{\beta_S - \beta_R} - \gamma_S\beta_S\right] && \text{algebra. Same as Eqn. (156) except } 2w \text{ instead of } L \\
&= -\frac{2\gamma_S w}{c}\left[\frac{\beta_S\gamma_S - \beta_R\gamma_R}{\gamma_S + \gamma_R}\right] && \text{So equal to Eqn. (157) with same replacement}
\end{aligned}$$

Event 1, Observer D:

$$\begin{aligned}
\mathbf{r}_1 &= -k\hat{\mathbf{i}} + \mathbf{D} - (-\mathbf{R}) = \mathbf{R} - k\hat{\mathbf{i}} + \mathbf{D} \\
\mathbf{r}_0 &= \mathbf{r}_1 + \mathbf{v}_R(t_0 - t_1) = \mathbf{r}_1 && t_0 = t_1 = 0 \\
\mathbf{r}_0 \cdot \mathbf{v}_D &= w_R v_D - k v_D && (184) \\
r_0^2 &= (w_R - k)^2 + h^2 + D^2 && \mathbf{R}\cdot\mathbf{D} = 0 \\
e_{1,D} &= \frac{\gamma_D^2}{c^2}\left[(w_R - k)v_D + \sqrt{((w_R - k)v_D)^2 + (1 - \beta_D^2)c^2 r_0^2}\right]
\end{aligned}$$

Event 2, Observer D:

$$\begin{aligned}
\mathbf{r}_1 &= -k\hat{\mathbf{i}} + \mathbf{D} - (\mathbf{f} + \mathbf{R}) = -\mathbf{R} - k\hat{\mathbf{i}} - \mathbf{f} + \mathbf{D} \\
\mathbf{r}_0 &= \mathbf{r}_1 + \mathbf{v}_D(t_0 - t_1) \\
&= \mathbf{r}_1 + q_2\mathbf{v}_D && t_0 - t_1 = q_2 \\
&= -\mathbf{R} - k\hat{\mathbf{i}} - q_2\mathbf{v}_R + \mathbf{D} + q_2\mathbf{v}_D && \mathbf{f} = q_2\mathbf{v}_R \\
&= -\mathbf{R} - k\hat{\mathbf{i}} + q_2(\mathbf{v}_D - \mathbf{v}_R) + \mathbf{D} && \text{algebra} \\
\mathbf{r}_0 \cdot \mathbf{v}_D &= -w_R v_D - k v_D + q_2(v_D - v_R)v_D && (185) \\
r_0^2 &= (-w_R - k + q_2(v_D - v_R))^2 + h^2 + D^2 && \mathbf{R}\cdot\mathbf{D} = 0
\end{aligned}$$

In order for the terms under the radicals to cancel out when we calculate $e_{2,D} - e_{1,D}$, we want the squares of Equations (184) and (185) to be equal, and also the expressions for r_0^2 to be equal. This is achieved with the following value for k:

$$\begin{aligned}
w_R v_D - k v_D &= -(-w_R v_D - k v_D + q_2(v_D - v_R)v_D) \\
w_R - k &= w_R + k - q_2(v_D - v_R) && \text{divide by } v_D\text{, algebra} \\
-2k &= -q_2(v_D - v_R) && \text{algebra} \\
k &= q_2(v_D - v_R)/2
\end{aligned}$$

Observer D difference in time receiving pulses:

$$\begin{aligned}
e_{2,D} - e_{1,D} &= q_2 + \frac{\gamma_D^2}{c^2}(-2w_R v_D + q_2(v_D - v_R)v_D) \\
&= \gamma_D^2 \left[q_2 \left(\frac{1}{\gamma_D^2} + \frac{1}{c^2}(v_D - v_R)v_D \right) - \frac{2w_R v_D}{c^2} \right] && \text{collect terms with } q_2\text{, factor out } \gamma_D^2 \\
&= \gamma_D^2 \left[\frac{2(w_R - w_S)}{(\beta_S - \beta_R)c} \left((1 - \beta_D^2) + (\beta_D - \beta_R)\beta_D \right) - \frac{2w_R \beta_D}{c} \right] && \text{substitute } q_2 \text{ with value, } v = \beta c\text{, } 1/\gamma^2 = 1 - \\
&= \frac{2\gamma_D^2}{c} \left[\frac{w(\frac{1}{\gamma_R} - \frac{1}{\gamma_S})}{\beta_S - \beta_R} \left(1 - \beta_D^2 + \beta_D^2 - \beta_R \beta_D \right) - \frac{w\beta_D}{\gamma_R} \right] && \text{replace } w_i = w/\gamma_i\text{, factor out } 2/c \\
&= \frac{2w\gamma_D^2}{c\gamma_R\gamma_S(\beta_S - \beta_R)} \left[(\gamma_S - \gamma_R)(1 - \beta_R\beta_D) - (\beta_S - \beta_R)\beta_D\gamma_S \right] && \text{factor out } w/(\gamma_R\gamma_S(\beta_S - \beta_R)) \\
&= C\left[\gamma_S - \gamma_R - \gamma_S\beta_R\beta_D + \gamma_R\beta_R\beta_D - \beta_S\beta_D\gamma_S + \beta_R\beta_D\gamma_S \right] && \text{replace initial factor with } C\text{, algebra} \\
&= C\left[\gamma_S - \gamma_R + (\gamma_R\beta_R - \gamma_S\beta_S)\beta_D \right] && \text{algebra} \\
&= C\left[\gamma_S - \gamma_R + (\gamma_R\beta_R - \gamma_S\beta_S)\frac{\beta_R\gamma_R + \beta_S\gamma_S}{\gamma_R + \gamma_S} \right] && \text{replace } \beta_D \text{ with its value} \\
&= \frac{C}{\gamma_R + \gamma_S}\left[\gamma_S^2 - \gamma_R^2 + \gamma_R^2\beta_R^2 - \gamma_S^2\beta_S^2 \right] && \text{factor out } 1/(\gamma_R + \gamma_S)\text{, algebra} \\
&= \frac{C}{\gamma_R + \gamma_S}\left[\gamma_S^2(1 - \beta_S^2) - \gamma_R^2(1 - \beta_R^2) \right] && \text{algebra} \\
&= 0 && \gamma_i^2(1 - \beta_i^2) = 1
\end{aligned}$$

H ChatGPT conversation about Reichenbach ϵ convention and relationship to one-way speed of light

I asked ChatGPT this question: If the Richenbach epsilon clock synchronization convention is used, would that imply a particular function for the one-way speed of light in all 3-d directions?

ChaptGPT's reponse: Short answer: Yes – choosing a Reichenbach synchronization convention (possibly allowing the ϵ parameter to depend on direction) is exactly equivalent to choosing a function for the one-way speed of light in every direction. The convention determines the one-way speeds; conversely, specifying a physically allowed one-way speed field determines the corresponding ϵ-function.

Below I give the precise relations, simple parametrizations you can use in 3-D, the constraints they must satisfy, and some useful examples (including the Einstein choice as a special case).

1) One-way speed $\leftrightarrow$ Reichenbach ϵ for a given direction

Fix a spatial direction given by a unit vector $\hat{\mathbf{n}}$. Let a light pulse travel a distance L along $\hat{\mathbf{n}}$ from A to B and then back from B to A. Denote the forward one-way speed along $\hat{\mathbf{n}}$ by $c_+(\hat{\mathbf{n}})$ and the backward one by $c_-(\hat{\mathbf{n}})$ The round-trip time for distance L is

$$T = \frac{L}{c_+(\hat{\mathbf{n}})} + \frac{L}{c_-(\hat{\mathbf{n}})} \tag{186}$$

Reichenbach synchronizes the forward arrival event at B at the time

$$t_B = t_A + \epsilon(\hat{\mathbf{n}})T \tag{187}$$

where $0 < \epsilon(\hat{\mathbf{n}}) < 1$. From this one gets the direct algebraic relations

$$c_+(\hat{\mathbf{n}}) = \frac{c}{2\epsilon(\hat{\mathbf{n}})}, c_-(\hat{\mathbf{n}}) = \frac{c}{2(1-\epsilon(\hat{\mathbf{n}}))} \tag{188}$$

provided the two-way (round-trip) speed is defined to be c (i.e. $T = 2L/c$; this is the usual empirical input). These formulae follow by noting $T = 2L/c \to L/(\epsilon T) = c/(2\epsilon)$, etc. (Andy: I have confirmed all of this with only a small amount of algebra required.)

So the one-way speed in direction $\hat{\mathbf{n}}$ is determined by the ϵ assigned to that direction.

2) Consistency / symmetry constraint

If $\epsilon(\hat{\mathbf{n}})$ is taken to parametrize the forward leg in direction $\hat{\mathbf{n}}$, then for the same spatial line but opposite direction one must have

$$\epsilon(-\hat{\mathbf{n}}) = 1 - \epsilon(\hat{\mathbf{n}}) \tag{189}$$

because the forward leg in $-\hat{\mathbf{n}}$ is the backward leg in $\hat{\mathbf{n}}$. Equivalently, one may define an odd function $s(\hat{\mathbf{n}})$ by

$$\epsilon(\hat{\mathbf{n}}) = \frac{1}{2}(1 + s(\hat{\mathbf{n}}), s(-\hat{\mathbf{n}}) = -s(\hat{\mathbf{n}}) \tag{190}$$

and then

$$c_{one-way}(\hat{\mathbf{n}}) = \frac{c}{1 + s(\hat{\mathbf{n}})} \tag{191}$$

The constraint $0 < \epsilon < 1$ becomes $|s(\hat{\mathbf{n}})| < 1$ for all $\hat{\mathbf{n}}$, so the denominator never vanishes and speeds are positive.

3) Useful parametrizations

(a) Isotropic (Einstein) choice. $\epsilon(\hat{\mathbf{n}}) = \frac{1}{2}$ (so $s(\hat{\mathbf{n}}) = 0$ for all $\hat{\mathbf{n}}$). Then

$$c_{one-way}(\hat{\mathbf{n}}) = c \tag{192}$$

i.e. isotropic one-way speed.

(b) Linear/first-order anisotropy. Pick a constant vector $\mathbf{k}$ with $|\mathbf{k}| < 1$ and set

$$s(\hat{\mathbf{n}}) = \mathbf{k} \cdot \hat{\mathbf{n}}, \epsilon(\hat{\mathbf{n}}) = \frac{1}{2}(1 + \mathbf{k} \cdot \hat{\mathbf{n}}) \tag{193}$$

Then

$$c_{one-way}(\hat{\mathbf{n}}) = \frac{c}{1 + \mathbf{k} \cdot \hat{\mathbf{n}}} \tag{194}$$

and automatically $c_{one-way}(-\hat{\mathbf{n}}) = c/(1 - \mathbf{k} \cdot \hat{\mathbf{n}})$. This is a simple, commonly used model for anisotropic synchronization.

(c) Arbitrary direction dependence. Any odd function $s(\hat{\mathbf{n}})$ with $|\mathbf{s}| < 1$ gives a valid synchronization and thus a well-defined one-way speed field $c/(1 + s(\hat{\mathbf{n}}))$.

4) Relation to Lorentz/LET pictures

- Einstein synchronization is the special case $\mathbf{s} \equiv 0$.
- In LET (or any preferred-frame theory) one can choose the synchronization tied to the preferred frame; that choice corresponds to a particular $s(\hat{\mathbf{n}})$ that reproduces the one-way speed field of that ether frame. Conversely, choosing a non-Einstein ϵ in an SR formalism simply amounts to adopting coordinates in which light's coordinate one-way speed is anisotropic – it does not change measurable round-trip results.

5) Constraints, interpretational remarks

- Empirical indistinguishability: All choices of $\epsilon(\hat{\mathbf{n}})$ (equivalently $s(\hat{\mathbf{n}})$) that respect the round-trip speed c are empirically indistinguishable in standard experiments, because those experiments measure round-trip quantities or are otherwise insensitive to the simultaneity convention.
- Mathematical constraint: $|s(\hat{\mathbf{n}})| < 1$ for positivity; also enforce $s(-\hat{\mathbf{n}}) = -s(\hat{\mathbf{n}})$.
- Coordinate vs physical statement: Specifying $\epsilon(\hat{\mathbf{n}})$ is choosing coordinates (a foliation of spacetime into "simultaneity slices"). Different choices change coordinate one-way speeds but not invariant physics (unless you posit a physically detectable preferred frame).

6) Mathematica-ready formulae

Define a unit vector $\mathbf{n}$ and an odd scalar function `s[n_]` satisfying `Abs[s[n]] < 1`. Then one-way speed in Mathematica:

```
cOneWay[n_] := c/(1 + s[n])
(* require s[-n] == -s[n] *)
```

For the linear model with vector k:

```
s[n_] := k . n
cOneWay[n_] := c/(1 + k . n)
```

Einstein choice: `k -> {0,0,0}` gives `cOneWay[n_] == c`.

7) Short worked example

If you pick $\mathbf{k} = \alpha \hat{\mathbf{e}}_x$ with small $|\alpha|$, then

$$c_{one-way}(\hat{\mathbf{n}}) = \frac{c}{1 + \alpha \cos\theta} \approx c(1 - \alpha \cos\theta + ...) \tag{195}$$

where θ is the angle between $\hat{\mathbf{n}}$ and the x-axis. For small α this is a small dipole anisotropy.

Bottom line

- Yes: Reichenbach synchronization determines a (possibly direction-dependent) one-way speed field.
- The precise mapping is
$$c_{one-way}(\hat{\mathbf{n}}) = \frac{c}{2\epsilon(\hat{\mathbf{n}})} = \frac{c}{1 + s(\hat{\mathbf{n}})} \tag{196}$$
with
$$\epsilon(\hat{\mathbf{n}}) = \frac{1}{2}(1 + s(\hat{\mathbf{n}})), s(-\hat{\mathbf{n}}) = -s(\hat{\mathbf{n}}) \tag{197}$$
- Einstein's convention is the special isotropic choice $\epsilon = \frac{1}{2}$ (or $s \equiv 0$); other choices produce anisotropic one-way speeds but leave all empirical two-way measurements unchanged.

I followed up with this question: Are there any constraints on the magnitude of s(n) in different directions, other than that it is an odd function? For example, could there be a 5-degree arc of directions for n where s(n) oscillated between 0 and almost 1 multiple times?

ChaptGPT's reponse: Good question – and you're right to suspect there's more to it than just oddness. Short answer:

No, an arbitrary wild oscillation of $s(\hat{\mathbf{n}})$ (subject only to $s(-\hat{\mathbf{n}}) = -s(\hat{\mathbf{n}})$ and $|s| < 1$ is not generally admissible as a global, physically sensible one-way-speed / synchronization convention. Beyond oddness and positivity, there are additional mathematical and physical constraints (continuity/integrability, and compatibility with a global simultaneity foliation) that rule out pathological direction-dependent oscillations in general.

Below I explain why, give the precise restrictions, and show when simple models (like the linear $s(\hat{\mathbf{n}}) = \mathbf{k} \cdot \hat{\mathbf{n}}$) are the only globally consistent position-independent choices.

1) Minimal obvious requirement

At every direction $\hat{\mathbf{n}}$,

$$|s(\hat{\mathbf{n}})| < 1, s(-\hat{\mathbf{n}}) = -s(\hat{\mathbf{n}}) \tag{198}$$

so one-way speeds $c/(1 + s)$ are positive and finite, and opposite directions transform correctly. That alone does not guarantee global consistency.

2) Why arbitrary angular oscillation can fail: the integrability / global-time condition

Choosing a synchronization convention on all of space is equivalent to choosing a global time function $T(x)$ (a foliation of spacetime by simultaneity slices). The one-way speed in direction $\hat{\mathbf{n}}$ must be consistent with propagation of light between nearby spatial points using that same time function.

Heuristically, for a small spatial step $dx = dl\hat{\mathbf{n}}$ the light travel time (one-way) according to your chosen simultaneity must be

$$dt_{light}(\hat{\mathbf{n}}) = \frac{dl}{c_{one-way}(\hat{\mathbf{n}})} = \frac{dl[1 + s(\hat{\mathbf{n}})]}{c} \tag{199}$$

If your time coordinate is $T(x) = t - \Phi(x)$ (where t is some background inertial time and Φ is the spatial desynchronization function), then the difference in T between the two ends equals the coordinate travel time minus the difference $\Phi(x + dx) - \Phi(x)$. For consistency (i.e. for this differential relation to hold for every infinitesimal displacement and every path), the angular dependence of $s(\hat{\mathbf{n}})$ must be derivable from a scalar field $\Phi(x)$.

Concretely, to first order in small dl one gets a relation of the schematic form

$$\frac{dl[1 + s(\hat{\mathbf{n}})]}{c} = \frac{dl}{c} + \frac{1}{c}\mathbf{n} \cdot (\boldsymbol{\nabla}\Phi(x))dl \tag{200}$$

so (dropping the trivial 1)

$$s(\hat{\mathbf{n}}) \approx \mathbf{n} \cdot (\boldsymbol{\nabla}\Phi(x)) \tag{201}$$

Thus locally the directional dependence of s must equal the directional projection of a spatial gradient. That implies:

- For a position-independent (homogeneous) synchronization (same everywhere), $s(\hat{\mathbf{n}})$ must be the projection of a constant vector $\mathbf{k}$:

$$s(\hat{\mathbf{n}}) = \mathbf{k} \cdot \hat{\mathbf{n}}, |\mathbf{k}| < 1 \tag{202}$$

  This is the familiar linear model giving $c_{one-way}(\hat{\mathbf{n}}) = c/(1 + \mathbf{k} \cdot \hat{\mathbf{n}})$.

- If $s(\hat{\mathbf{n}})$ cannot be written (locally) as $\mathbf{n} \cdot$ (some gradient), then there is no single-valued global scalar $\Phi(x)$ producing that directional dependence. The would-be time coordinate would become *path-dependent*, i.e. the time assigned to a distant event would depend on how you integrate across space – which destroys the notion of a single global simultaneity.

So arbitrary angular oscillations (e.g. a 5° arc in which s flips between 0 and nearly 1 many times) cannot arise from any smooth, single-valued $\Phi(x)$ and therefore do not define a consistent global synchronization.

3) Consequences of violating integrability

If you attempted to impose such a pathological $s(\hat{\mathbf{n}})$:

- The assigned time difference between two spatial points would depend on the path chosen between them (non-integrability). That is not acceptable for a global clock synchronization.
- You could get contradictory statements of simultaneity (A simultaneous with B via one route but not via another).
- Such constructions can even produce coordinate pathologies that make the "time function" multi-valued or discontinuous, and prevent defining a well-behaved inertial coordinate chart.
- Physical invariants (round-trip speed c) might still hold locally, but you lose the global coordinate consistency that underlies conventional descriptions of dynamics.

4) Smoothness and physical reasonableness

Even if you restrict to integrable s coming from $\boldsymbol{\nabla}\Phi$, you usually want Φ (hence s) to be at least continuous (and typically differentiable) to avoid pathological coordinate systems. Physically:

- Discontinuous or wildly oscillatory s would imply highly nonphysical, non-smooth time slices.
- Practical clock synchronization presumes smooth, slowly varying desynchronization fields – abrupt high-frequency angular oscillations make no operational sense.

So besides integrability, continuity / smoothness is a practical constraint.

5) Examples and special cases

- Allowed homogeneous family: $s(\hat{\mathbf{n}}) = \mathbf{k}\cdot\hat{\mathbf{n}}$. This is globally consistent: choose $\Phi(x) = \mathbf{k}\cdot\mathbf{x}$, then $T = t - \Phi(x)$ yields the desired one-way speeds everywhere. This is the standard Reichenbach/Lorentz-style anisotropy with constant vector $\mathbf{k}$ (e.g. an "ether wind" model).
- Position-dependent but integrable: $s(\hat{\mathbf{n}}, \mathbf{x}) = \mathbf{K}(\mathbf{x}) \cdot \hat{\mathbf{n}}$ with $\mathbf{K}(\mathbf{x}) = \boldsymbol{\nabla}\Phi(\mathbf{x})$. Works if $\mathbf{K}$ is curl-free (i.e. a gradient) and smooth.

- Forbidden pathological case: $s(\hat{\mathbf{n}})$ that oscillates rapidly within a small angular patch and cannot be represented as $\mathbf{k}\cdot\hat{\mathbf{n}}$ (or as $\nabla\Phi\cdot\hat{\mathbf{n}}$ locally). This cannot come from any scalar desynchronization and so is not a valid global synchronization.

6) Intuition: why only 'projection-of-a-vector' patterns survive

A synchronization choice is really a choice of how much one shifts clocks as a function of spatial position: $\Phi(\mathbf{x})$. The directional asymmetry seen in one-way light speed must be the directional derivative of that shift – i.e. a projection of $\nabla\Phi$ onto the direction $\hat{\mathbf{n}}$. That's why the simplest and physically meaningful anisotropies are linear in direction (a dot product with a vector field). Any more exotic angular dependence that cannot be represented as such a projection is incompatible with a single-valued Φ.

7) Final compact answer

- Necessary conditions: $s(-\hat{\mathbf{n}}) = -s(\hat{\mathbf{n}})$ and $|s(\hat{\mathbf{n}})| < 1$.
- Additional (crucial) condition for a global, position-independent synchronization: there must exist a constant vector $\mathbf{k}$ with $|\mathbf{k}| < 1$ such that $s(\hat{\mathbf{n}}) = \mathbf{k}\cdot\hat{\mathbf{n}}$. For position-dependent but globally consistent synchrony, $s(\hat{\mathbf{n}},\mathbf{x}) = \mathbf{K}(\mathbf{x})\cdot\hat{\mathbf{n}}$ with $\mathbf{K} = \nabla\Phi$.
- Therefore: a wildly oscillatory $s(\hat{\mathbf{n}})$ on a small angular arc is not allowed unless it can be produced as the directional projection of some scalar field's gradient; generic oscillations cannot.

If you like, I can:

- Show a short derivation relating $s(\hat{\mathbf{n}})$ to $\nabla\Phi$ for the infinitesimal displacement argument,
- Give an explicit counterexample demonstrating path-dependence when $s(\hat{\mathbf{n}})$ is chosen non-integrably, or
- Produce a visualization comparing the allowed linear model $s = \mathbf{k}\cdot\hat{\mathbf{n}}$ with a pathological oscillatory $s(\hat{\mathbf{n}})$.

Which would help you most?