Andy's science scratch pad

J. Andrew Fingerhut (andy.fingerhut@gmail.com)

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1 Introduction

This document is a place to write up little bits on science.

Some notation:

$\mathbf{i}$ is the unit vector from left to right. $\mathbf{j}$ is the unit vector upwards. $\mathbf{k}$ is the unit vector pointed out of the page toward the reader.

$\gamma = 1/\sqrt{1 - v^2/c^2}$ is the Lorentz factor.

2 Electromagnetic force between two point charges at rest relative to each other

Scenario 1: There are two point charges a and b both with charge q at rest relative to each other at a distance r apart (see Figure 1). They are at rest relative to us. In this case they both experience a force directly away from the other due to electric repulsion. There is no magnetic force, as both charges are at rest so there are no magnetic fields.

Scenario 2: The same as scenario 1, but both charges are moving with constant velocity v in the upwards direction (see Figure 2). Since they are moving they create magnetic fields.

Questions:

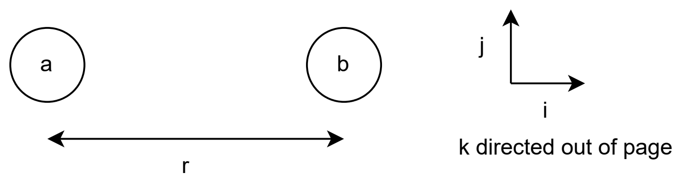

Figure 1: Two point charges at rest

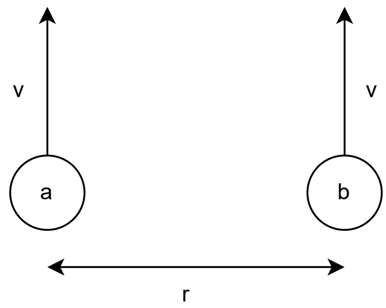

Figure 2: Two point charges moving at same constant velocity

- What is the net force on charge b in each scenario?
- Is it the same in both scenarios, or different?
- Why?

2.1 Scenario 1: Both charges at rest

As mentioned before, there is no current or motion of any charges in this scenario, so no magnetic fields. The electric repulsion force on charge b is easily calculated from Coulomb's Law [4]. Charge b is to the right of charge a, so the direction of the force is $\mathbf{i}$, away from charge a.

$$\mathbf{E}_1 = \frac{1}{4\pi\epsilon_0}\frac{q}{r^2}\mathbf{i} \tag{1}$$

$$\mathbf{B}_1 = 0 \tag{2}$$

$$\mathbf{F}_1 = q(\mathbf{E}_1 + \mathbf{v} \times \mathbf{B}_1) = q\mathbf{E}_1 \tag{3}$$

2.2 Scenario 2: Both charges with equal and constant velocity upwards

2.2.1 Scenario 2 calculated by electromagnetic field equations from Griffiths

The Wikipedia page on the Biot-Savart Law [5] has a subsection titled "Point charge at constant velocity" that says:

> the Biot–Savart law applies only to steady currents and a point charge moving in space does not constitute a steady current

I will thus use the equations in that section to calculate the electric and magnetic fields here. The relevant parts of the Wikipedia page are copied below.

> In the case of a point charged particle q moving at a constant veclocity $\mathbf{v}$, Maxwell's equations give the following expression for the electric field and magnetic field:

$$\mathbf{E} = \frac{q}{4\pi\epsilon_0} \frac{1-\beta^2}{(1-\beta^2\sin^2\theta)^{3/2}} \frac{\hat{\mathbf{r}}'}{|r'|^2} \tag{4}$$

$$\mathbf{B} = \frac{1}{c^2}\mathbf{v} \times \mathbf{E} \tag{5}$$

where:

- $\hat{\mathbf{r}}'$ is the unit vector pointing from the current (non-retarded) position of the particle to the point at which the field is being measured,
- $\beta = v/c$ is the speed in units of c, and
- θ is the angle between $\mathbf{v}$ and $\hat{\mathbf{r}}'$.

The equations above appear to be identical to equations (10.75) and (10.76) in Griffiths [2]. Griffiths comments on the formula for the electric field:

> Notice that $\mathbf{E}$ points along the line from the *present* position of the particle. This is an extraordinary coincidence, since the "message" came from the retarded position. Because of the $\sin^2\theta$ in the denominator, the field of a fast-moving charge is flattened out like a pancake in the direction perpendicular to the motion (Fig. 10.10). In the forward and backward directions $\mathbf{E}$ is reduced by a factor $(1 - v^2/c^2)$ relative to the field of a charge at rest; in the perpendicular direction it is *enhanced* by a factor $1/\sqrt{1 - v^2/c^2}$.

Calculation: To get the force on charge b, we first calculate the $\mathbf{E}$ and $\mathbf{B}$ fields at the position of charge b.

Charge b is directly to the right of charge a, so $\hat{\mathbf{r}}' = \mathbf{i}$ and $\theta = 90°$.

$$\begin{aligned}
\mathbf{E}_2 &= \frac{q}{4\pi\epsilon_0} \frac{1-\beta^2}{(1-\beta^2\sin^2\theta)^{3/2}} \frac{\hat{\mathbf{r}}'}{|r'|^2} && \hat{\mathbf{r}}' = \mathbf{i},\ |r'| = r,\ \theta = 90°, \text{ simplify fraction} \\
&= \frac{q}{4\pi\epsilon_0} \frac{1}{(1-\beta^2)^{1/2}} \frac{\mathbf{i}}{r^2} && \text{part of this is } \gamma, \text{ by (1) the rest is } \mathbf{E}_1 \\
&= \gamma \mathbf{E}_1
\end{aligned} \tag{6}$$

Note that $\mathbf{E}_2$ being γ times larger than $\mathbf{E}_1$ is consistent with the comment from Griffiths above: "in the perpendicular direction it ($\mathbf{E}$) is *enhanced* by a factor $1/\sqrt{1 - v^2/c^2}$".

$$\begin{aligned}
\mathbf{F}_2 &= q(\mathbf{E}_2 + \mathbf{v} \times \mathbf{B}_2) && \text{replace } \mathbf{B}_2 \text{ with (5)} \\
&= q(\mathbf{E}_2 + \mathbf{v} \times (\frac{1}{c^2}\mathbf{v} \times \mathbf{E}_2)) && \mathbf{v} \times \mathbf{E}_2 = -vE_2\mathbf{k} \\
&= q(\mathbf{E}_2 - \frac{vE_2}{c^2}\mathbf{v} \times \mathbf{k}) && \mathbf{v} \times \mathbf{k} = v\mathbf{i} \\
&= q(\mathbf{E}_2 - \frac{v^2E_2}{c^2}\mathbf{i}) \\
&= q(1 - \frac{v^2}{c^2})\mathbf{E}_2 \\
&= \frac{q\mathbf{E}_2}{\gamma^2} && \text{by (6)} \quad \mathbf{E}_2 = \gamma\mathbf{E}_1 \\
&= \frac{q\mathbf{E}_1}{\gamma} && \text{by (3)} \quad \mathbf{F}_1 = q\mathbf{E}_1 \\
&= \frac{\mathbf{F}_1}{\gamma}
\end{aligned}$$

Thus $\mathbf{F}_2$ differs from $\mathbf{F}_1$ by a factor of γ.

TODO: Why?

I do not know how to check the answer below, but it appears that three of the answers to an on-line question similar to mine [3] say that the Lorentz force formula $\mathbf{F} = q(\mathbf{E} + \mathbf{v} \times \mathbf{B})$ is *not* invariant in all inertial frames, but perhaps a slightly modified version of that formula is invariant between different inertial frames. I quote one such answer below:

Just for completeness if permitted: Following Section 3.1 from the book “Gravitation” of Misner, Thorne, and Wheeler the truly (at all speeds) frame independent force is $\frac{dP}{d\tau} = \gamma(E + v \times B)$ (in fact this is only the spacial component of the four force). τ is proper time and γ the well-known Lorentz Factor. – Kurt G. Aug 28, 2021

2.2.2 Scenario 2 calculated by Heaviside-Feynman formula

The Wikipedia page on Jefimenko's Equations [6] has a subsection titled “Heaviside-Feynman formula” that gives equations for the electric and magnetic field at a point due to a single moving point charge.

$$\mathbf{E} = \frac{-q}{4\pi\epsilon_0}\left[\frac{\mathbf{e}_{r'}}{r'^2} + \frac{r'}{c}\frac{d}{dt}\left(\frac{\mathbf{e}_{r'}}{r'^2}\right) + \frac{1}{c^2}\frac{d^2}{dt^2}\mathbf{e}_{r'}\right] \tag{7}$$

$$\mathbf{B} = -\mathbf{e}_{r'} \times \frac{\mathbf{E}}{c} \tag{8}$$

Here $\mathbf{e}_{r'}$ is a unit vector pointing from the observer to the charge and r' is the distance between observer and charge. Since the electromagnetic field propagates at the speed of light, both of these quantities are evaluated at the retarted time $t - r'/c$.

I believe “observer” above means “the position for which we are calculating E and B fields”.

Assume here that the point charges are kept at distance r apart from each other, always horizontally, e.g. because they are connected by a stiff insulating rod. This simplifies our job of calculating E, because then $\mathbf{e}_{r'}$ and r' are unchanging over time, and their derivates are thus 0.

We want to calculate r' as the vector from the position of charge b to the position where charge a was when it emitted an electric field propagated at speed c to b. See Figure 3.

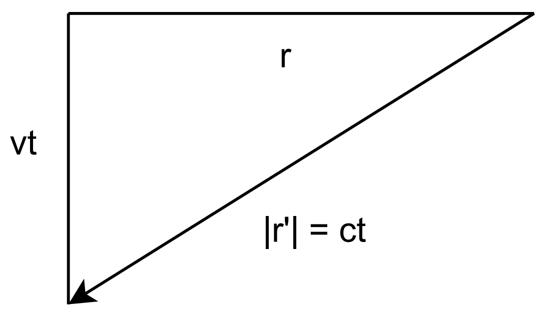

Figure 3: The retarded position of charge a from charge b

Solve for t using Pythagorean theorem since r and v are known constants:

$$\begin{aligned} r^2 + (vt)^2 &= (ct)^2 \\ t^2(c^2 - v^2) &= r^2 \\ t^2 &= \frac{r^2}{c^2 - v^2} \\ t &= \frac{r}{\sqrt{c^2 - v^2}} \\ &= \frac{r}{c\sqrt{1 - v^2/c^2}} \\ &= \gamma r/c \end{aligned}$$

This gives us $r' = ct = \gamma r$, and $\mathbf{e}_{r'}$ is:

$$\begin{aligned} \mathbf{e}_{r'} &= \frac{-r\mathbf{i} - (\gamma r v/c)\mathbf{j}}{\gamma r} \\ &= -\frac{1}{\gamma}\mathbf{i} - \frac{v}{c}\mathbf{j} \end{aligned}$$

Plugging in this value for $\mathbf{e}_{r'}$ into Equation (7) gives:

$$\mathbf{E}_3 = \frac{q}{4\pi\epsilon_0}\left[\frac{\frac{1}{\gamma}\mathbf{i} + \frac{v}{c}\mathbf{j}}{\gamma^2 r^2}\right]$$

Note that $\mathbf{E}_3$ is parallel to $\mathbf{e}_{r'}$, thus $\mathbf{B}_3$ from Equation (8) is 0.
This gives the force on charge b as:

$$\begin{aligned}\mathbf{F}_3 &= q(\mathbf{E}_3 + \mathbf{v} \times \mathbf{B}_3) \\ &= q\mathbf{E}_3\end{aligned}$$

The direction of $\mathbf{F}_3$ is different than $\mathbf{F}_1$ and $\mathbf{F}_2$. Below is the relative magnitude of $\mathbf{E}_3$ to $\mathbf{E}_1$:

$$\begin{aligned}E_3 &= \frac{1}{\gamma^2}E_1 \\ F_3 &= \frac{1}{\gamma^2}F_1\end{aligned}$$

TODO: It seems *very* odd to me that $\mathbf{B}_3 = 0$.
After Feynman explains what the retarded direction and distance **r'** is, he says [1]:

> That would be easy enough to understand, too, but it is also wrong. The whole thing is much more complicated.

Unfortunately there are no footnotes or citation to explain what he meant by this.

3 Simple scenarios in special relativity and Lorentz Ether Theory

Definitions of some terms:

$$\beta = v/c \qquad \text{the } \textit{relativistic velocity}\text{, or } \textit{velocity ratio} \tag{9}$$

$$\gamma = \frac{1}{\sqrt{1-\beta^2}} \qquad \text{the } \textit{Lorentz factor} \tag{10}$$

$$D = \sqrt{\frac{1+\beta}{1-\beta}} \qquad \text{the } \textit{Doppler factor} \tag{11}$$

Later in this article we typically write a subscript after v, β, γ, and D to indicate the entity that has the relevant velocity v. The context should make it clear in what inertial frame that velocity v applies.

We will use these definitions in scenarios where $-c < v < c$. Thus:

$$\begin{aligned}-1 \leq \beta &< 1 \\ \gamma &\geq 1 && \gamma \text{ increases as } |\beta| \text{ does} \\ \lim_{\beta\to 1}\gamma &= +\infty \\ \lim_{\beta\to -1}\gamma &= +\infty \\ D = \gamma(1+\beta) &> 0 && D \text{ increases as } \beta \text{ does} \qquad (12)\end{aligned}$$

3.1 Special Relativity Scenario 1: Two entities moving relative to each other at constant velocity

A is at rest. B is moving at constant velocity v_B relative to A, either directly away from A, or directly towards A. $v_B > 0$ means B moves away from A. $v_B < 0$ means B moves towards A.

3.1.1 B sends periodic pulses to A

B uses his local clock to time the sending of radio pulses to A, sending pulses once every time interval T. At what period does A receive the pulses?

Define $q_A(n)$ to be the time on A's clock when the n-th pulse is transmitted by B, and $q_B(n)$ to be the time on B's clock when it transmits the n-th pulse.

$q_B(n) = nT$ by the setup of the experiment.

By the assumptions of special relativity, A deduces that B's clock is running γ_B times slower than A's clock. Note: A cannot directly observe B's clock, as it is too far away. From this A also deduces:

$$q_A(n) = \gamma_B nT + \Delta \tag{13}$$

Δ is the time that A reads on his clock when B reads 0 on his clock. The value of Δ is irrelevant to the final result we seek.

B's distance from A at A's time t_A is $x_B(t_A) = I_B + v_B t_A$. $I_B > 0$ is B's initial distance from A at $t_A = 0$. The value of I_B is also irrelevant to the final result we seek, as long as it is large enough that $x_B(t_A) > 0$ remains true while B emits all of the pulses we consider (two pulses is enough).

Also by the assumptions of special relativity, A deduces that B's pulse signal will propagate at the one-way speed c. The pulse will thus take time $x_B(q_A(n))/c$ to propagate to A.

A's clock thus shows time $r_A(n) = q_A(n) + x_B(q_A(n))/c$ when the n-th pulse arrives at A. With a little algebra:

$$\begin{aligned}
r_A(n) &= q_A(n) + x_B(q_A(n))/c && \text{definition of } r_A(n)\\
&= q_A(n) + (I_B + v_B q_A(n))/c && \text{substitute equation for } x_B\\
&= (1 + v_B/c) q_A(n) + I_B/c && \text{rearrangement by algebra}\\
&= (1 + \beta_B)(\gamma_B nT + \Delta) + I_B/c && \text{Defn. (9) and Eqn. (13)}\\
&= (1 + \beta_B)\gamma_B T n + ((1 + \beta_B)\Delta + I_B/c) && \text{algebra to collect all terms independent of } n \text{ at the end}
\end{aligned}$$

The time that A measures on his clock between two consecutive received pulses is:

$$\begin{aligned}
r_A(n+1) - r_A(n) &= (1 + \beta_B)\gamma_B T && \text{by calculation of } r_A(n) \text{ above}\\
&= D_B T && \text{Eqn. (12)}
\end{aligned}$$

The pulses arrive at A with a period of $D_B T$ measured on A's clock.

Suppose that A knows somehow the period T that B uses as its period for transmitting pulses, e.g. because they agreed and arranged this beforehand. Then A can measure the period between received pulses on his clock and divide it by T to calculate D_B. By Corollary 1 in Appendix A.2, A can calculate from D_B the one value of β_B in the range $(-1, 1)$ that corresponds to it, and v_B in the range $(-c, +c)$.

Note 1: There is *nothing* in this derivation that relies upon prior knowledge of the Doppler factor, or under what conditions it is applicable. That expression arose in the process of calculating the answer, using only the assumptions that light moves isotropically at speed c in A's frame, and that A can deduce that B's clock is running γ_B times slower than A's clock.

3.1.2 A sends periodic pulses to B

Now A uses his local clock to time the sending of radio pulses to B, sending pulses once every time interval T. At what period does B receive the pulses?

While we could switch perspectives to B's inertial frame, we will not. Instead, we are going to do all of the calculations in A's inertial frame.

The definitions of Section 3 remain the same here, but note that the values of $q_A(n)$ and $q_B(n)$ here are *different* than those in Section 3.1.1.

Define $q_A(n) = nT$ to be the time on A's clock when it transmits its n-th pulse.

At all times t_A, B is a distance $x_B(t_A) = I_B + v_B t_A$ away from A. As in the previous section, $I_B > 0$ is any value large enough that $x_B(t_A)$ remains positive while A emits all of the pulses we consider.

A assumes by special relativity that its pulse propagates with one-way speed c to B. The pulse's position at time $t_A \geq q_A(n)$ is $c(t_A - q_A(n))$.

On A's clock, A deduces that its n-th pulse catches up to B at a time $r_A(n)$ that satisfies the equation:

$$\begin{aligned}
x_B(r_A(n)) &= c(r_A(n) - q_A(n)) && \text{B's position equals light pulse's position} \\
I_B + v_B r_A(n) &= c(r_A(n) - nT) && \text{substitute for defns. of x_B and $q_A(n)$} \qquad (14) \\
I_B/c + (v_B/c) r_A(n) &= r_A(n) - nT && \text{divide by c} \\
nT + I_B/c &= (1 - v_B/c) r_A(n) && \text{a little more algebra} \\
r_A(n) &= \frac{nT}{1 - v_B/c} + \frac{I_B}{c - v_B} && \text{divide by $(1 - v_B/c)$} \\
r_A(n) &= \frac{nT}{1 - \beta_B} + \frac{I_B}{c - v_B} && \text{Defn.(9)} \qquad (15)
\end{aligned}$$

By the assumptions of special relativity, A deduces that B's clock is running γ_B times slower than A's clock. Thus B's time when receiving the n-th pulse is the following, where Δ is the time that B reads on his clock when A reads 0 on his clock (as in the previous section, the value of Δ is irrelevant in our final answer):

$$\begin{aligned}
r_B(n) &= r_A(n)/\gamma_B + \Delta && \text{B's clock slower by factor γ_B} \\
&= \frac{nT}{\gamma_B(1 - \beta_B)} + \left(\frac{I_B}{\gamma_B(c - v_B)} + \Delta\right) && \text{Eqn. (15), move things independent of n to end}
\end{aligned}$$

The time that B measures on his clock between two consecutive received pulses is:

$$\begin{aligned}
r_B(n+1) - r_B(n) &= \frac{T}{\gamma_B(1 - \beta_B)} && \text{by calculation of $r_B(n)$ above} \\
&= \frac{\sqrt{1 - \beta_B^2}}{1 - \beta_B} T && \text{Defn. (10)} \\
&= \sqrt{\frac{1 + \beta_B}{1 - \beta_B}} T && \text{a little algebra} \\
&= D_B T && \text{Defn. (11)}
\end{aligned}$$

B will observe pulses arriving with period $D_B T$ according to B's clock.

While this formula for the period between received pulses is very similar to the one in the previous section, note that both the sending and receiving period are being measured on different clocks than there.

As in the previous section, if B somehow knows T, he can measure the interal between receive pulses, divide it by T to calculate D_B, then use that to calculate β_B and v_B.

3.1.3 Relationship to Lorentz Ether Theory

Note in Sections 3.1.1 and 3.1.2, that except for algebra and the definitions of symbols from Section 3, the only assumptions we used from special relativity were:

- The one-way speed of light is c in all directions in A's inertial frame.
- In A's frame, B's clock runs at a slower rate, by a factor of $1/\gamma_B$, relative to A's clock.

By Lorentz Ether Theory, suppose that somehow we know that A is at rest relative to the ether, then:

- The one-way speed of light is c in all directions relative to the ether, and thus also relative to A.
- B is moving at velocity v_B relative to the ether, and thus B's clock physically runs at a slower rate, $1/\gamma_B$ times as fast as the true time. A's clock runs at the full rate of true time, the same as any other clocks at rest relative to the ether.

TODO: Find a way to explain the following better.

I had heard from a not-yet-in-depth learning about special relativity that when A and B are moving at constant velocity towards or away from each other, that A observed that B's clock ran slower by a factor of γ, and B observed that A's clock ran slower by a factor of γ. (Note: I do not claim that those are fully precise statements, but there is definitely a sense in which special relativity does say something similar to this.)

In A's frame observing B's clock run slower, that seems perfectly consistent with Lorentz Ether Theory's statement that if A is at rest relative to the ether, and B moves at constant velocity v_B relative to the ether, that B experiences duration dilation, i.e. its clock physically runs slower than A's by a factor of γ_B. In this situation A's clock runs at full speed, i.e. γ_B times *faster* than B's.

But what about Lorentz Ether Theory's position on the converse statement? That is, from B's point of view, does B observe A's clock running γ_B times slower? If so, how can that possibly make sense?

I now believe that the answer is that the statements in special relativity can be made a bit more precise by saying something like this: Because A is following special relativity's assumptions, i.e. in A's frame the speed of light propagates isotropically at constant speed c, therefore A can deduce that any clocks moving at constant speed v directly towards or away from A run γ times slower, and make further calculations from that deduction.

A does not actually *observe* such clocks directly over any appreciable interval of time, so they are always, or almost always, so far away that A cannot make *any* direct observations of how fast such clocks are running. By "direct" observations I mean "with light propagation delay very close to 0 between A and the entity being observed".

Suppose in some future context of knowledge that not only is Lorentz Ether Theory proven, but in such a way that we know how to measure our speed relative to the ether.

Then, in the scanario described, we would know that A's clock is running at full speed, and light propagates isotropically at constant speed c relative to the ether, and thus also relative to A.

Everyone with this knowledge would be able to deduce that B's clock is running γ times slower than full speed. Also, that A's clock is running γ times *faster* than B's clock (and that all of A's local physical processes are proceeding γ times faster than similar local physical processes of B).

Further, light does *not* propagate at the same speed in all directions relative to B. It does so only with respect to the ether.

We could also prove that if one chose to make calculations using special relativity's assumptions in B's frame, one would get the same answers to these calculations that you do when using Lorentz Ether Theory.

A hint of corroboration can be seen in the Wikipedia page on time dilation, which says in the introduction [7]:

> The dilation compares "wristwatch" clock readings between events measured in different inertial frames and is not observed by visual comparison of clocks across moving frames.

It seems that any statement similar to:

- A observes B's clock running slower than their own.

could be said in much more detail as either of the following:

- In accordance with special relativity's time synchronization convention that light propagates in A's inertial frame isotropically with speed c, A deduces that B's clock runs slower than A's.
- A deduces, using the postulate that light moves isotropically at speed c, that B's clock runs slower, and can then make further consistent calculations based on this deduction.

And you can swap A and B in that statement. Without defining new precise terminology, a shorter precise statement would perhaps be:

- According to SR, in A's inertial frame we deduce that B's clock runs slower.

3.2 Special Relativity Scenario 2: Three entities, two of them moving at constant velocity

A is at rest. B is moving at constant velocity v_B relative to A, which is away from A if $v_B > 0$, or towards A if $v_B < 0$. C is moving at constant velocity v_C relative to A, with same sign conventions as B's velocity.

For B sending periodic pulses to A or vice versa, everything in Sections 3.1.1 and 3.1.2 applies without change. For C sending periodic pulses to A or vice versa, everything in Section 3.1.1 and 3.1.2 applies, except replace B subscripts with C subscripts, i.e. use v_C, γ_C, β_C, and D_C.

So it is only pulses between B and C that might present something new here.

Our calculations in the following sections for ping signals between B and C assume that throughout the entire duration of sending and receiving the pulses of interest, B's position according to its x coordinate is always less than C's x coordinate. If that condition is ever violated because B crosses paths with C, the results presented here are no longer applicable after that occurs. We use I_B and I_C to represent their initial distances from A, and require that these positions, combined with their velocities, will ensure this.

3.2.1 B sends periodic pulses to C

B uses his local clock to time the sending of radio pulses to C, sending pulses once every time interval T, according to B's clock. At what period does C receive the pulses?

To make the calculations as easily applicable to Lorentz Ether Theory as possible, all calculations will be done in A's inertial frame.

Define $q_A(n)$ and $q_B(n)$ the same way as they were in Section 3.1.1. As explained there, when A makes the assumptions according to special relativity theory:

$$\begin{aligned}
q_B(n) &= nT && \text{by the setup of the experiment} \\
t_A &= \gamma_B t_B + \Delta_B && B\text{'s clock runs } \gamma_B \text{ times slower than } A\text{'s} \\
q_A(n) &= \gamma_B nT + \Delta_B && && (16) \\
t_A &= \gamma_C t_C + \Delta_C && C\text{'s clock runs } \gamma_C \text{ times slower than } A\text{'s} \\
t_C &= t_A/\gamma_C - \Delta_C/\gamma_C && \text{equivalent to previous equation, but solved for } t_C && (17) \\
x_B(t_A) &= I_B + v_B t_A && \text{relationship of } B\text{'s position and time, in } A\text{'s frame} && (18) \\
x_C(t_A) &= I_C + v_C t_A && \text{relationship of } C\text{'s position and time, in } A\text{'s frame} && (19)
\end{aligned}$$

Also by special relativity assumptions, A considers the pulse to travel from B to C at constant speed c. The n-th pulse is emitted at time $q_A(n)$ in A's frame, so its position as a function of A's time t_A is:

$$\begin{aligned}
l_A(n, t_A) &= \text{position of } B \text{ when emitted} \\
&\quad + \text{distance traveled after emission} \\
&= x_B(q_A(n)) + (t_A - q_A(n))c && \text{for any time } t_A \geq q_A(n) \\
&= I_B + v_B q_A(n) + (t_A - q_A(n))c && \text{substitute Eqn. (18)} \\
&= (v_B - c)q_A(n) + t_A c + I_B && \text{algebra} \\
&= (v_B - c)\gamma_B nT + t_A c + ((v_B - c)\Delta_B + I_B) && \text{substitute Eqn. (16)} \\
&= (v_B - c)\gamma_B nT + t_A c + Z && Z \text{ is constants, independent of } n \text{ and } t_A
\end{aligned}$$

To find A's time $r_A(n)$ when C receives the pulse, solve for the time that makes the pulse position the same as C's position:

$$\begin{aligned}
x_C(r_A(n)) &= l_A(n, r_A(n)) \\
I_C + v_C r_A(n) &= (v_B - c)\gamma_B nT + r_A(n)c + Z && \text{substitute Eqn. (19)} \\
(v_C - c)r_A(n) &= (v_B - c)\gamma_B nT + (Z - I_C) && \text{algebra} \\
r_A(n) &= \frac{v_B - c}{v_C - c}\gamma_B nT + \frac{Z - I_C}{v_C - c} && \text{divide by } v_C - c \\
r_A(n) &= \frac{1 - \beta_B}{1 - \beta_C}\gamma_B nT + \frac{Z - I_C}{v_C - c} && \text{defn. of } \beta_B, \beta_C \\
r_A(n) &= \frac{1 - \beta_B}{1 - \beta_C}\gamma_B nT + Y && Y \text{ is a constant, independent of } n
\end{aligned}$$

According to A and its special relativity assumptions, C's clock runs slower, at a rate $1/\gamma_C$ times that of A's clock. So C's time $r_C(n)$ to receive the n-th pulse sent by B is:

$$\begin{aligned}
r_C(n) &= \frac{1}{\gamma_C} r_A(n) - \Delta_C/\gamma_C && \text{Eqn. (17)} \\
&= \frac{\gamma_B}{\gamma_C}\left(\frac{1 - \beta_B}{1 - \beta_C}\right) nT + \frac{Y - \Delta_C}{\gamma_C}
\end{aligned}$$

The time that B measures on his clock between two consecutive received pulses is:

$$
\begin{aligned}
r_C(n+1) - r_C(n) &= \frac{\gamma_B}{\gamma_C}\left(\frac{1-\beta_B}{1-\beta_C}\right)T \\
&= \sqrt{\frac{1-\beta_C^2}{1-\beta_B^2}}\left(\frac{1-\beta_B}{1-\beta_C}\right)T && \text{Defn. (10)} \\
&= \sqrt{\frac{1+\beta_C}{1-\beta_C}}\sqrt{\frac{1-\beta_B}{1+\beta_B}}T && \text{algebra} \\
&= (D_C/D_B)T && \text{defn. of } D_B, D_C
\end{aligned}
$$

So when B sends pulses with period T according to B's clock, C receives from B pulses with period $(D_C/D_B)T$ on C's clock.

Since $D > 0$ and it increases with β (see Appendix A.2), and thus also with v:

- If $v_c > v_B$, then C and B are moving away relative to each other, and $D_C > D_B$, thus $(D_C/D_B) > 1$.
- If $v_c < v_B$, then C and B are getting closer over time, and $D_C < D_B$, thus $(D_C/D_B) < 1$.

Aside: If you are curious, the answer provided by ChatGPT for solving this problem can be found in Appendix B.1.

Near the end of Sections 3.1.1 and 3.1.2, we noted that if the receiver knew somehow the period T that the sender is sending pulses, the receiver could calculate a D value that enabled the receiver to determine what the velocity of B is.

In the current scenario, the receiver C, if it knows T, can measure the time interval between pulses it receives, divide by T, and calculate (D_C/D_B).

Thus, with the measurement of the interval between received pulses, given any two of T, v_A, and v_B, C can calculate the other one of those (and given values for two of those quantities, there is only one value possible for the remaining one). But from the measurement of the interval between received pulses and T, that is not enough information for C to calculate either one of v_B or v_C.

However, it *can* do the following. Define $D_r = (D_C/D_B)$. From Corollary 3 in Appendix A.2, we know that if $\beta_r = \beta_C \ominus \beta_B$ for some β_C, β_B values in the range $(-1, +1)$, then $D_r = D_C/D_B$.

TODO: I am fairly certain it is straightforward to prove the converse: If in this scenario C calculates the value value of D_r, then calculates $\beta_r = (D_r^2 - 1)/(D_r^2 + 1)$, then the only possible pairs of values β_C, β_B that could have given this measurement are those satisfying $\beta_r = \beta_C \ominus \beta_B$. There are an unlimited number of such pairs of values.

This strongly suggests that the receiver can calculate the *relative* velocity between B and C, at least in some sense. I am not sure yet how to explain that further.

Also note that if we follow Lorentz Ether Theory, but we have no knowledge of how to determine our motion relative to the ether, these results show that the receiver can, with knowledge of the sender's period T, still calculate this kind of relative velocity described, which is interesting.

3.2.2 C sends periodic pulses to B

The derivation is nearly identical to that in Section 3.2.1. Here we only mention a few equations along the way that have noticeable differences.

The formula $l_A(n, t_A)$ for the position of the n-th pulse emitted by C at A's time t_A is:

$$l_A(n, t_A) = (v_C + c)\gamma_C nT - t_A c + Z' \tag{20}$$

A's time when B receives the n-th pulse $r_A(n)$ is:

$$r_A(n) = \frac{1+\beta_C}{1+\beta_B}\gamma_C nT + Y' \tag{21}$$

and B's time when it receives the n-th pulse from C is:

$$r_B(n) = \frac{1}{\gamma_B} r_A(n) + (\text{constants independent of } n)$$

and finally:

$$r_B(n+1) - r_B(n) = (D_C/D_B)T$$

Thus B, after measuring the interval between received pulses, if it knows T somehow and can calculate (D_C/D_B), can also calculate the same kind of relative velocity between B and C as described at the end of the previous section.

3.3 Summary of results in this section

Here is a summary of what we have shown so far.

- In the scenario of Section 3.1 where A is at rest, and B is moving at velocity v_B relative to A in A's frame (negative for velocity towards A, positive for velocity away from A):
  - When B sends pulse signals every time period T according to B's clock, A measures received pulses every time period D_BT according to A's clock.
  - When A sends pulse signals every time period T according to A's clock, B measures received pulses every time period D_BT according to B's clock.
- In the scenario of Section 3.2 where A is at rest, B and C are moving at velocity v_B and v_C relative to A in A's frame (same sign conventions as above), and the position of C is always "to the right" of B during the scenario:
  - When B sends pulse signals every time period T according to B's clock, C measures received pulses every time period $(D_C/D_B)T$ according to C's clock.
  - When C sends pulse signals every time period T according to C's clock, B measures received pulses every time period $(D_C/D_B)T$ according to B's clock.

In every scenarios, all calculations of movement of B, C, and pulse signals were done in A's frame. Only in the first or last steps did we do any time conversions between clocks running at different rates. Thus both special relativity and Lorentz Ether Theory predict the same measurements.

The equations for the received time intervals all contain one or more factors of D that are relativistic Doppler factors. These arose naturally out of the calculations, not from any prior knowledge of Doppler factors. The velocities involved in these Doppler factors are strongly related to the relative velocity of the sender and the receiver.

If we adopt Lorentz Ether Theory, but remain ignorant of how to measure our velocity relative to the ether, Scenario 2's results strongly suggest a proper understanding of relative velocity.

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A Miscellaneous math facts

A.1 Math facts about relativistic velocity addition and subtraction in one dimension

All of the facts here are quite simple to see. I write them out primarily as an aid to thinking about and remembering them, and they might also be useful to refer to from elsewhere in this document.

Note that this appendix is restricted to proofs of simple mathematical relationships about the definitions of one-dimensional relativistic velocity addition and subtraction formulas. This appendix makes no claims about the physical meaning of these operations.

I have read that relativistic velocity addition in 3 dimensions is not associative. TODO: If I add any discussion of 3-dimensional relativity examples to this document, it would be nice to give an example, perhaps in a separate appendix.

In one dimension, though, all velocities of subliminal entities can be represented by v such that $-c < v < c$, where negative velocities are in the opposite direction along the line than positive velocities.

The can also be represented as relativistic velocity $\beta = v/c$, i.e. a fraction of c. These are in the range $-1 < \beta < 1$.

I will use the notation $v \oplus w$ for relativistic velocity addition. While I will use the same symbol $\beta_v \oplus \beta_w$ for relativistic velocity addition of relativistic velocity values, one should be careful to note that the definition of the operator $\oplus$ is slightly different for these two cases.

$$v \oplus w = \frac{v + w}{1 + \frac{vw}{c^2}} \tag{22}$$

$$v \ominus w = \frac{v - w}{1 - \frac{vw}{c^2}} \tag{23}$$

$$\beta_v \oplus \beta_w = \frac{\beta_v + \beta_w}{1 + \beta_v \beta_w} \tag{24}$$

$$\beta_v \ominus \beta_w = \frac{\beta_v - \beta_w}{1 - \beta_v \beta_w} \tag{25}$$

The proofs are only given for the relativistic velocity formulas (24) and (25). The proofs for equations (22) and (23) are nearly identical.

We will use the symbols a, b, c to represent arbitrary real values between -1 and 1, to avoid writing β with subscripts all over the place.

$$a \oplus b = b \oplus a \quad \text{addition is commutative} \tag{26}$$

$$(a \oplus b) \oplus c = a \oplus (b \oplus c) \quad \text{addition is associative (1-d only, not 3-d!)} \tag{27}$$

$$a \oplus 0 = 0 \oplus a = a \tag{28}$$

$$a \oplus (-b) = a \ominus b \tag{29}$$

$$a \ominus (-b) = a \oplus b \tag{30}$$

$$0 \ominus b = -b \tag{31}$$

$$a \ominus b = -(b \ominus a) \tag{32}$$

The following inequalities only hold if $0 < a < 1$ and $0 < b < 1$:

$$a \oplus b > a \tag{33}$$

$$a \oplus b > b \tag{34}$$

$$a \oplus b < a + b \tag{35}$$

The following inequality holds for all $-1 < a < 1$ and $-1 < b < 1$:

$$-1 < a \oplus b < 1 \tag{36}$$

The last one is only slightly different if we use velocity addition on normal velocities, i.e. velocities that are not relativistic velocities, where $-c < v < c$ and $-c < w < c$:

$$-c < v \oplus w < c \tag{37}$$

The inequalities all have examples where the two sides can be very nearly equal, such as these:

$$\begin{aligned} 0.1 \oplus 0.001 &\approx 0.100989901 > 0.1 \\ 0.1 \oplus 0.1 &\approx 0.198019802 < 0.1 + 0.1 \\ 0.99 \oplus 0.99 &\approx 0.999949498 < 1 \end{aligned}$$

Proving most of these is as simple as substituting the definition of $a \oplus b$ and $a \ominus b$, and a tiny amount of algebra.

Proving associativity (27) is only a little bit more algebra, but we will do the steps here:

$$\begin{aligned} (a \oplus b) \oplus c &= \frac{\frac{a+b}{1+ab} + c}{1 + \left(\frac{a+b}{1+ab}\right) c} && \text{use Defn. (24) twice} \\ &= \frac{(a+b) + c(1+ab)}{(1+ab) + (a+b)c} && \text{multiply numerator and denominator by } (1+ab) \\ &= \frac{a+b+c+abc}{1+ab+ac+bc} && \text{multiply out the terms} \end{aligned}$$

Similarly:

$$\begin{aligned} a \oplus (b \oplus c) &= \frac{a + \frac{b+c}{1+bc}}{1 + a\left(\frac{b+c}{1+bc}\right)} && \text{use Defn. (24) twice} \\ &= \frac{a(1+bc) + (b+c)}{(1+bc) + a(b+c)} && \text{multiply numerator and denominator by } (1+bc) \\ &= \frac{a+abc+b+c}{1+bc+ab+ac} && \text{multiply out the terms} \end{aligned}$$

The above two final results are easily seen to be equal.

The inequalities are also not difficult to prove, but we will write out their short proofs. Recall that these inequalities are true only for $0 < a < 1$ and $0 < b < 1$. Similar inequality hold if both a and b are negative.

For the proof of inequality (33), recall that we can multiply or divide both sides of an inequality by the same positive number, and the resulting inequality is true if and only if the original one was. The symbol $\Leftrightarrow$ below means "if and only if", i.e. the expression before is true if and only if the expression after is true.

$$\begin{aligned} a < \frac{a+b}{1+ab} &\Leftrightarrow a + a^2 b < a + b && \text{multiply both sides by } 1+ab\text{, which is positive} \\ &\Leftrightarrow a^2 b < b && \text{subtract } a \text{ from both sides} \\ &\Leftrightarrow a^2 < 1 && \text{divide both sides by } b\text{, which is positive} \end{aligned}$$

The last inequality is true because $a < 1$. The proof of inequality (34) is the same as above.

To prove $a \oplus b < a + b$ (inequality (35)):

$$\begin{aligned} \frac{a+b}{1+ab} < a + b &\Leftrightarrow a + b < (a+b)(1+ab) && \text{multiply both sides by } 1+ab\text{, which is positive} \\ &\Leftrightarrow 1 < (1+ab) && \text{divide both sides by } a+b\text{, which is positive} \\ &\Leftrightarrow 0 < ab && \text{subtract 1 from both sides} \end{aligned}$$

The final inequality is true because both a and b are positive.

Now to prove $a \oplus b < 1$ (part of inequality (36)), but recall now we are doing so for the more general case of all values $-1 < a < 1$ and $-1 < b < 1$:

$$\begin{aligned} \frac{a+b}{1+ab} < 1 &\Leftrightarrow a + b < 1 + ab && \text{multiply both sides by } 1+ab\text{, which is positive} \\ &\Leftrightarrow b - ab < 1 - a && \text{subtract } a+ab \text{ from both sides} \\ &\Leftrightarrow b(1-a) < 1 - a && \text{algebra} \\ &\Leftrightarrow b < 1 && \text{divide both sides by } 1-a\text{, which is positive} \end{aligned}$$

And the last inequality is true. Proving the part that

$$
\begin{aligned}
-1 < \frac{a+b}{1+ab} &\Leftrightarrow -(1+ab) < a+b && \text{multiply both sides by } 1+ab \text{, which is positive} \\
&\Leftrightarrow -(1+a) < b+ab && \text{add } ab-a \text{ to both sides} \\
&\Leftrightarrow -(1+a) < b(1+a) && \text{algebra} \\
&\Leftrightarrow -1 < b && \text{divide both sides by } 1+a \text{, which is positive}
\end{aligned}
$$

And the last inequality is true.

A.2 Math facts about the Doppler factor

This appendix is restricted to proofs of mathematical relationships. It makes no claims about the physical meaning of the equations involved.

Define D as given by Definition (11), repeated here:

$$D = \sqrt{\frac{1+\beta}{1-\beta}}$$

Define it over the entire domain $-1 < \beta < 1$.

- D approaches 0 as β approaches -1.
- D approaches $+\infty$ as β approaches 1.
- D is an increasing function of β.

The last fact is straightforward to confirm by taking the derivative of D with respect to β, and confirming that it is positive everwhere for β in the range $(-1, +1)$.

Corollary 1. *For any value of $D > 0$ that satisfies Definition (11), there is exactly one value of β, $-1 < \beta < 1$, that corresponds to that value of D. That value is $\beta = \frac{D^2-1}{D^2+1}$.*

Plugging the value $\frac{D^2-1}{D^2+1}$ into Definition (11) for β takes only a little algebra to simplify and show it is equal to D.

Theorem 2. *Let β_u, β_v be any two values in the range $(-1, +1)$. Let $\beta_w = \beta_u \oplus \beta_v$, using the one-dimensional relativistic velocity addition formula for β values (Defn. (24)). Let D_u, D_v, D_w be calculated from $\beta_u, \beta_v, \beta_w$, respectively. Then $D_w = D_u D_v$.*

Proof.

$$
\begin{aligned}
D_w &= \sqrt{\frac{1+\beta_w}{1-\beta_w}} && \text{defn. of } D_w \\
&= \sqrt{\frac{1+(\beta_v \oplus \beta_w)}{1-(\beta_v \oplus \beta_w)}} && \text{Defn. of } \beta_w \\
&= \sqrt{\frac{1+\frac{\beta_v+\beta_w}{1+\beta_v\beta_w}}{1-\frac{\beta_v+\beta_w}{1+\beta_v\beta_w}}} && \text{Defn. (24) of Appendix A.1} \\
&= \sqrt{\frac{(1+\beta_v\beta_w)+(\beta_v+\beta_w)}{(1+\beta_v\beta_w)-(\beta_v+\beta_w)}} && \text{multiply numerator and denominator by } (1+\beta_v\beta_w) \\
&= \sqrt{\frac{(1+\beta_v)(1+\beta_w)}{(1-\beta_v)(1-\beta_w)}} && \text{algebra (factoring)} \\
&= \sqrt{\frac{1+\beta_v}{1-\beta_v}}\sqrt{\frac{1+\beta_w}{1-\beta_w}} && \text{algebra} \\
&= D_v D_w && \text{defn. of } D_v, D_w
\end{aligned}
$$

□

Corollary 3. *Under the same conditions as Theorem 2, except $\beta_w = \beta_u \ominus \beta_v$, using the one-dimensional relativistic velocity subtraction formula for β values (Defn. (25)), $D_w = D_u/D_v$.*

Proof. $\beta_w = \beta_u \ominus \beta_v = \beta_u \oplus (-\beta_v)$. Thus by the previous theorem, $D_w = D_u D_{-v}$, where D_{-v} is calculated from the definition for D with $-\beta_v$. It is straightforward to see from the definition of D that $D_{-v} = 1/D_v$, so $D_w = D_u/D_v$. □

Apparently these facts are well-known among those working with relativistic Doppler factors.

B Double-check results from ChatGPT

B.1 Scenario 2, B sends periodic pulses to C, double-check by ChatGPT

Since at the time of first performing the calculations in Section 3.2.1 I was still fairly new to such things, I wanted a way to double-check the results. I asked ChatGPT what the period would be that C would receive pulses from B and it gave an answer close to the following.

Note: I only asked it about the case where $v_C > v_B > 0$.

Calculate the velocity of B in C's frame using relativistic velocity subtraction:

$$u' = \frac{v_B - v_C}{1 - \frac{v_B v_C}{c^2}} \tag{38}$$

$$\beta' = \frac{\beta_B - \beta_C}{1 - \beta_B \beta_C} \tag{39}$$

Since $v_B < v_C$, the numerator is negative. The denominator is positive. Thus $u' < 0$ and $\beta' < 0$. Thus in C's frame, B is moving in the negative x direction, and C is at rest.

For receding motion:

$$T_{\text{observed}} = T_{\text{emitted}} \sqrt{\frac{1+\beta}{1-\beta}} \tag{40}$$

where:

$$\beta = \frac{|u'|}{c} = |\beta'| \tag{41}$$

Since $\beta' < 0$:

$$|\beta'| = -\beta' = \frac{\beta_C - \beta_B}{1 - \beta_B \beta_C} \tag{42}$$

From here on this is mostly just algebra:

$$\begin{aligned}
\frac{T_{\text{observed}}}{T_{\text{emitted}}} &= \sqrt{\frac{1+\beta}{1-\beta}} \\
&= \sqrt{\frac{1 + \frac{\beta_C - \beta_B}{1-\beta_B\beta_C}}{1 - \frac{\beta_C - \beta_B}{1-\beta_B\beta_C}}} \\
&= \sqrt{\frac{1 - \beta_C\beta_B + (\beta_C - \beta_B)}{1 - \beta_C\beta_B - (\beta_C - \beta_B)}} \\
&= \sqrt{\frac{(1+\beta_C)(1-\beta_B)}{(1-\beta_C)(1+\beta_B)}} \\
&= D_C/D_B && \text{defn. of } D_B, D_C
\end{aligned}$$

This is the same result, calculated in a fairly different way, using the assumptions of special relativity, which ChatGPT is much better at answering questions about than it is any alternatives that are not special relativity.