Andy's math/science background information

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1 Introduction

This document is a place to write up notes on background in math that help in learning science.

2 Notation

$\mathbb{R}^3$ - The set of all points in 3-dimensional space, i.e. where each point is specified by a sequence of 3 real-valued coordinates.

∇ - Called "nabla", and often called "del". Some history of it can be found on the Nabla Symbol page [**?**].

$\boldsymbol{\nabla} f$ - Gradient of a scalar function $f : \mathbb{R}^3 \to \mathbb{R}$.

grad$f$ - another way to write the gradient of f, $\boldsymbol{\nabla} f$

TODO: Any 3Blue1Brown-quality YouTube videos showing examples and definition of gradient, divergence, and curl? If so, give links to them below in appropriate sections.

2.1 Maxwell's Equations

In partial differentiatal form:

$$\boldsymbol{\nabla} \cdot \mathbf{E} = \frac{\rho}{\epsilon_0} \qquad \text{Gauss's Law} \tag{1}$$

$$\boldsymbol{\nabla} \cdot \mathbf{B} = 0 \qquad \text{Gauss's Law for magnetism} \tag{2}$$

$$\boldsymbol{\nabla} \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \qquad \text{Maxwell-Faraday Equation (Faraday's law of induction)} \tag{3}$$

$$\boldsymbol{\nabla} \times \mathbf{B} = \mu_0(\mathbf{J} + \epsilon_0 \frac{\partial \mathbf{E}}{\partial t}) \qquad \text{Ampère-Maxwell law} \tag{4}$$

where:

- $\mathbf{E}$ is the electric field
- $\mathbf{B}$ is the magnetic field
- ρ is the electric charge density
- $\mathbf{J}$ is the current density
- ϵ_0 is the vacuum permittivity

- μ_0 is the vacuum permeability

Statement of the equations in English:

Gaus's Law (equation (1)) says that electric fields are "produced" radially outward from volumes with positive electric charge, and "consumed" radially into volumes with negative electric charge.

TODO: Is this a strict generalization of Coulomb's Law? In what ways? One way is that Gauss's Law gives the electric field generated by a distribution of charge, whereas Coulomb's Law gives the force between two point charges. So applying to a continuous distribution of charge is one generalization, and what electric field is created by that distribution of charge is another (as long as you then combine Gauss's Law with the Lorentz force law in order to calculate the forces).

TODO: It seems that Gauss's Law, if you take ρ and $\mathbf{E}$ as functions of time, gives an electric field that changes instantaneously everywhere when the charge distribution ρ changes. There does not appear to be anything related to retarded positions of charge in that equation anywhere. How does the finite speed of electric field propagation enter into Maxwell's equations?

Doing a Google search on the question "is Gauss's law consistent with relativistic effects" gives answers that it *is* consistent with special relativity, including the following paragraphs:

- Maxwell's equations are relativistic: Gauss's law is one of Maxwell's equations. These equations, which describe classical electromagnetism, are inherently Lorentz invariant. This means they hold true in all inertial reference frames, consistent with the principle of relativity, which states that the laws of physics are the same for all observers in uniform motion relative to one another.
- Lorentz covariance: Maxwell's equations, and thus Gauss's law, can be written in a covariant form that is explicitly consistent with Lorentz transformations. These transformations describe how physical quantities change when moving from one inertial frame to another, incorporating relativistic effects like time dilation and length contraction.

3 Gradient

Gradient has been generalized to many coordinate systems other than the 3-dimensional Cartesian coordinate system $\mathbb{R}^3$, but I will focus on $\mathbb{R}^3$. The Wikipedia page on Gradient [?] is not too bad for me, as long as I skim over the parts that generalize it to other coordinate systems.

Griffiths [?] Section 1.2.2 "Gradient" is good at giving the definition and some useful examples and properties of the gradient. He defines $\boldsymbol{\nabla} T$ this way, where $\hat{\mathbf{x}}$, $\hat{\mathbf{y}}$, and $\hat{\mathbf{z}}$ are the unit vectors in the direction of the three coordinate axes:

$$\boldsymbol{\nabla} T \equiv \frac{\partial T}{\partial x}\hat{\mathbf{x}} + \frac{\partial T}{\partial y}\hat{\mathbf{y}} + \frac{\partial T}{\partial z}\hat{\mathbf{z}}$$

where:

- $T : \mathbb{R}^3 \to \mathbb{R}$ is a scalar function that is continuous and differentiable

Some properties of the gradient:

- $\boldsymbol{\nabla} T(x, y, z)$ is the gradient evaluated at a position given by (x, y, z). The vector points in a direction that function T increases most quickly.
- The function T can be approximated at points $\mathbf{r}$ near $\mathbf{r}_0 = (x, y, z)$ by the linear function $T(\mathbf{r}_0) + (\mathbf{r} - \mathbf{r}_0) \cdot \boldsymbol{\nabla} T(\mathbf{r}_0)$
- The instantaneous rate of change of T in direction $\mathbf{u}$ (a unit vector) from $\mathbf{r}_0$ is $\mathbf{u} \cdot \boldsymbol{\nabla} T(\mathbf{r}_0)$. Note that it is always 0 in a direction perpendicular to $\boldsymbol{\nabla} T(\mathbf{r}_0)$, and the negative of the magnitude of $\boldsymbol{\nabla} T(\mathbf{r}_0)$ in the opposite direction.
- Consider a "level set", i.e. a surface defined by all of the points $\mathbf{r}$ where $T(\mathbf{r}) = c$ for some constant c. Then $\boldsymbol{\nabla} T$ evaluated at any point on that surface, is normal to the surface.

References

[] David J. Griffiths. *Introduction to Electrodynamics*. Pearson Education, 4th edition, 2013.

[] Wikipedia. Gradient, 2025. URL `https://en.wikipedia.org/wiki/Gradient`.

[] Wikipedia. Nabla Symbol, 2025. URL `https://en.wikipedia.org/wiki/Nabla_symbol`.