Andy's science scratch pad

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May 31, 2025

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1 Introduction

This document is a place to write up little bits on science.

Some notation:

$\mathbf{i}$ is the unit vector from left to right. $\mathbf{j}$ is the unit vector upwards. $\mathbf{k}$ is the unit vector pointed out of the page toward the reader.

$\gamma = 1/\sqrt{1 - v^2/c^2}$ is the Lorentz factor.

2 Electromagnetic force between two point charges at rest relative to each other

Scenario 1: There are two point charges a and b both with charge q at rest relative to each other at a distance r apart (see Figure 1). They are at rest relative to us. In this case they both experience a force directly away from the other due to electric repulsion. There is no magnetic force, as both charges are at rest so there are no magnetic fields.

Scenario 2: The same as scenario 1, but both charges are moving with constant velocity v in the upwards direction (see Figure 2). Since they are moving they create magnetic fields.

Questions:

- What is the net force on charge b in each scenario?

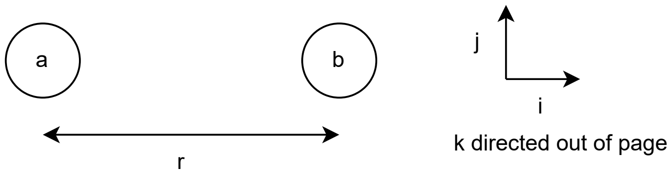

Figure 1: Two point charges at rest

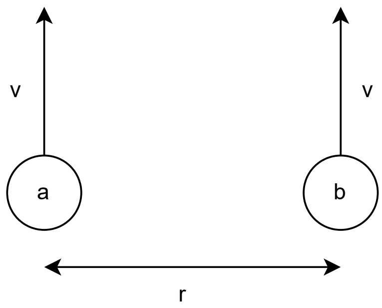

Figure 2: Two point charges moving at same constant velocity

- Is it the same in both scenarios, or different?
- Why?

2.1 Scenario 1: Both charges at rest

As mentioned before, there is no current or motion of any charges in this scenario, so no magnetic fields. The electric repulsion force on charge b is easily calculated from Coulomb's Law [4]. Charge b is to the right of charge a, so the direction of the force is $\mathbf{i}$, away from charge a.

$$\mathbf{E}_1 = \frac{1}{4\pi\epsilon_0}\frac{q}{r^2}\mathbf{i} \tag{1}$$

$$\mathbf{B}_1 = 0 \tag{2}$$

$$\mathbf{F}_1 = q(\mathbf{E}_1 + \mathbf{v} \times \mathbf{B}_1) = q\mathbf{E}_1 \tag{3}$$

2.2 Scenario 2: Both charges with equal and constant velocity upwards

2.2.1 Scenario 2 calculated by electromagnetic field equations from Griffiths

The Wikipedia page on the Biot-Savart Law [5] has a subsection titled "Point charge at constant velocity" that says:

> the Biot–Savart law applies only to steady currents and a point charge moving in space does not constitute a steady current

I will thus use the equations in that section to calculate the electric and magnetic fields here. The relevant parts of the Wikipedia page are copied below.

> In the case of a point charged particle q moving at a constant veclocity $\mathbf{v}$, Maxwell's equations give the following expression for the electric field and magnetic field:

$$
\mathbf{E} = \frac{q}{4\pi\epsilon_0} \frac{1-\beta^2}{(1-\beta^2 \sin^2\theta)^{3/2}} \frac{\hat{\mathbf{r}}'}{|r'|^2} \tag{4}
$$

$$
\mathbf{B} = \frac{1}{c^2}\mathbf{v} \times \mathbf{E} \tag{5}
$$

where:

- $\hat{\mathbf{r}}'$ is the unit vector pointing from the current (non-retarded) position of the particle to the point at which the field is being measured,
- $\beta = v/c$ is the speed in units of c, and
- θ is the angle between $\mathbf{v}$ and $\hat{\mathbf{r}}'$.

The equations above appear to be identical to equations (10.75) and (10.76) in Griffiths [2]. Griffiths comments on the formula for the electric field:

> Notice that $\mathbf{E}$ points along the line from the *present* position of the particle. This is an extraordinary coincidence, since the "message" came from the retarded position. Because of the $\sin^2\theta$ in the denominator, the field of a fast-moving charge is flattened out like a pancake in the direction perpendicular to the motion (Fig. 10.10). In the forward and backward directions $\mathbf{E}$ is reduced by a factor $(1 - v^2/c^2)$ relative to the field of a charge at rest; in the perpendicular direction it is *enhanced* by a factor $1/\sqrt{1 - v^2/c^2}$.

Calculation: To get the force on charge b, we first calculate the $\mathbf{E}$ and $\mathbf{B}$ fields at the position of charge b.

Charge b is directly to the right of charge a, so $\hat{\mathbf{r}}' = \mathbf{i}$ and $\theta = 90°$.

$$
\begin{aligned}
\mathbf{E}_2 &= \frac{q}{4\pi\epsilon_0} \frac{1-\beta^2}{(1-\beta^2 \sin^2\theta)^{3/2}} \frac{\hat{\mathbf{r}}'}{|r'|^2} && \hat{\mathbf{r}}' = \mathbf{i},\ |r'| = r,\ \theta = 90°, \text{ simplify fraction} \\
&= \frac{q}{4\pi\epsilon_0} \frac{1}{(1-\beta^2)^{1/2}} \frac{\mathbf{i}}{r^2} && \text{part of this is } \gamma, \text{ by (1) the rest is } \mathbf{E}_1 \\
&= \gamma \mathbf{E}_1
\end{aligned} \tag{6}
$$

Note that $\mathbf{E}_2$ being γ times larger than $\mathbf{E}_1$ is consistent with the comment from Griffiths above: "in the perpendicular direction it ($\mathbf{E}$) is *enhanced* by a factor $1/\sqrt{1 - v^2/c^2}$".

$$
\begin{aligned}
\mathbf{F}_2 &= q(\mathbf{E}_2 + \mathbf{v} \times \mathbf{B}_2) && \text{replace } \mathbf{B}_2 \text{ with (5)} \\
&= q(\mathbf{E}_2 + \mathbf{v} \times (\frac{1}{c^2}\mathbf{v} \times \mathbf{E}_2)) && \mathbf{v} \times \mathbf{E}_2 = -vE_2\mathbf{k} \\
&= q(\mathbf{E}_2 - \frac{vE_2}{c^2}\mathbf{v} \times \mathbf{k}) && \mathbf{v} \times \mathbf{k} = v\mathbf{i} \\
&= q(\mathbf{E}_2 - \frac{v^2 E_2}{c^2}\mathbf{i}) \\
&= q(1 - \frac{v^2}{c^2})\mathbf{E}_2 \\
&= \frac{q\mathbf{E}_2}{\gamma^2} && \text{by (6)} \quad \mathbf{E}_2 = \gamma\mathbf{E}_1 \\
&= \frac{q\mathbf{E}_1}{\gamma} && \text{by (3)} \quad \mathbf{F}_1 = q\mathbf{E}_1 \\
&= \frac{\mathbf{F}_1}{\gamma}
\end{aligned}
$$

Thus $\mathbf{F}_2$ differs from $\mathbf{F}_1$ by a factor of γ.

TODO: Why?

I do not know how to check the answer below, but it appears that three of the answers to an on-line question similar to mine [3] say that the Lorentz force formula $\mathbf{F} = q(\mathbf{E} + \mathbf{v} \times \mathbf{B})$ is *not* invariant in all inertial frames, but perhaps a slightly modified version of that formula is invariant between different inertial frames. I quote one such answer below:

Just for completeness if permitted: Following Section 3.1 from the book "Gravitation" of Misner, Thorne, and Wheeler the truly (at all speeds) frame independent force is $\frac{dP}{d\tau} = \gamma(E + v \times B)$ (in fact this is only the spacial component of the four force). τ is proper time and γ the well-known Lorentz Factor. – Kurt G. Aug 28, 2021

2.2.2 Scenario 2 calculated by Heaviside-Feynman formula

The Wikipedia page on Jefimenko's Equations [6] has a subsection titled "Heaviside-Feynman formula" that gives equations for the electric and magnetic field at a point due to a single moving point charge.

$$\mathbf{E} = \frac{-q}{4\pi\epsilon_0}\left[\frac{\mathbf{e}_{r'}}{r'^2} + \frac{r'}{c}\frac{d}{dt}\left(\frac{\mathbf{e}_{r'}}{r'^2}\right) + \frac{1}{c^2}\frac{d^2}{dt^2}\mathbf{e}_{r'}\right] \tag{7}$$

$$\mathbf{B} = -\mathbf{e}_{r'} \times \frac{\mathbf{E}}{c} \tag{8}$$

Here $\mathbf{e}_{r'}$ is a unit vector pointing from the observer to the charge and r' is the distance between observer and charge. Since the electromagnetic field propagates at the speed of light, both of these quantities are evaluated at the retarted time $t - r'/c$.

I believe "observer" above means "the position for which we are calculating E and B fields".

Assume here that the point charges are kept at distance r apart from each other, always horizontally, e.g. because they are connected by a stiff insulating rod. This simplifies our job of calculating E, because then $\mathbf{e}_{r'}$ and r' are unchanging over time, and their derivates are thus 0.

We want to calculate r' as the vector from the position of charge b to the position where charge a was when it emitted an electric field propagated at speed c to b. See Figure 3.

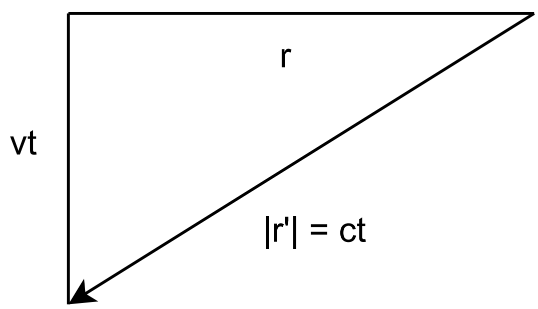

Figure 3: The retarded position of charge a from charge b

Solve for t using Pythagorean theorem since r and v are known constants:

$$\begin{aligned} r^2 + (vt)^2 &= (ct)^2 \\ t^2(c^2 - v^2) &= r^2 \\ t^2 &= \frac{r^2}{c^2 - v^2} \\ t &= \frac{r}{\sqrt{c^2 - v^2}} \\ &= \frac{r}{c\sqrt{1 - v^2/c^2}} \\ &= \gamma r/c \end{aligned}$$

This gives us $r' = ct = \gamma r$, and $\mathbf{e}_{r'}$ is:

$$\begin{aligned} \mathbf{e}_{r'} &= \frac{-r\mathbf{i} - (\gamma r v/c)\mathbf{j}}{\gamma r} \\ &= -\frac{1}{\gamma}\mathbf{i} - \frac{v}{c}\mathbf{j} \end{aligned}$$

Plugging in this value for $\mathbf{e}_{r'}$ into Equation (7) gives:

$$\mathbf{E}_3 = \frac{q}{4\pi\epsilon_0}\left[\frac{\frac{1}{\gamma}\mathbf{i} + \frac{v}{c}\mathbf{j}}{\gamma^2 r^2}\right]$$

Note that $\mathbf{E}_3$ is parallel to $\mathbf{e}_{r'}$, thus $\mathbf{B}_3$ from Equation (8) is 0.

This gives the force on charge b as:

$$\begin{aligned}\mathbf{F}_3 &= q(\mathbf{E}_3 + \mathbf{v} \times \mathbf{B}_3)\\ &= q\mathbf{E}_3\end{aligned}$$

The direction of $\mathbf{F}_3$ is different than $\mathbf{F}_1$ and $\mathbf{F}_2$. Below is the relative magnitude of $\mathbf{E}_3$ to $\mathbf{E}_1$:

$$\begin{aligned}E_3 &= \frac{1}{\gamma^2}E_1\\ F_3 &= \frac{1}{\gamma^2}F_1\end{aligned}$$

TODO: It seems *very* odd to me that $\mathbf{B}_3 = 0$.

After Feynman explains what the retarded direction and distance **r'** is, he says [1]:

> That would be easy enough to understand, too, but it is also wrong. The whole thing is much more complicated.

Unfortunately there are no footnotes or citation to explain what he meant by this.

3 Simple scenarios in special relativity and Lorentz Ether Theory

Definitions of some terms:

$$\beta = v/c \qquad \text{the } \textit{relativistic velocity}\text{, or } \textit{velocity ratio} \tag{9}$$

$$\gamma = \frac{1}{\sqrt{1-\beta^2}} \qquad \text{the } \textit{Lorentz factor} \tag{10}$$

$$D = \sqrt{\frac{1+\beta}{1-\beta}} \qquad \text{the } \textit{Doppler factor} \tag{11}$$

Later in this article we typically write a subscript after v, β, γ, and D to indicate the entity that has the relevant velocity v. The context should make it clear in what inertial frame that velocity v applies.

We only use these definitions in scenarios where $0 \leq v < c$. Thus:

$$\begin{aligned}0 \leq \beta &< 1 && \\ \gamma &\geq 1 && \gamma \text{ increases as } \beta \text{ does}\\ \lim_{\beta\to 1}\gamma &= +\infty && \\ D = \gamma(1+\beta) &\geq \gamma && D \text{ increases as } \beta \text{ does, faster than } \gamma\end{aligned}$$

3.1 Special Relativity Scenario 1: Two entities moving away from each other at constant velocity

A is at rest. B is moving at constant velocity v_B away from A.

3.1.1 B sends periodic pulses to A

B uses his local clock to time the sending of radio pulses to A, sending pulses once every time interval T. At what period does A receive the pulses?

For simplicity of calculations, we will assume that B passed A's position at time 0, and A and B synchronized their clocks at that time. However, note that A will receive the pulses at the same period in this scenario, regardless of whether they ever synchronized their clocks.

Define $t_A(n)$ to be the time on A's clock when the n-th pulse is transmitted by B, and $t_B(n)$ to be the time on B's clock when it transmits the n-th pulse.

$t_B(n) = nT$ by the setup of the experiment.

By the assumptions of special relativity, A deduces that B's clock is running γ_B times slower than A's clock. Note: A cannot directly observe B's clock, as it is too far away. From this A also assumes:

$$t_A(n) = \gamma_B nT \tag{12}$$

B is a distance $v_B t_A(n)$ away from A at that time.

Also by the assumptions of special relativity, A deduces that B's pulse signal will propagate at the one-way speed c. The pulse will thus take time $v_B t_A(n)/c$ to propagate to A.

A's clock thus shows time $t_A(n) + v_B t_A(n)/c$ when the n-th pulse arrives at A. With a little algebra:

$$\begin{aligned}
t_A(n) + v_B t_A(n)/c &= (1 + v_B/c)t_A(n) && \text{rearrangement by algebra} \\
&= nT(1+\beta_B)\gamma_B && \text{Defn. (9) and Eqn. (12)} \\
&= nT\frac{1+\beta_B}{\sqrt{1-\beta_B^2}} && \text{Defn. (10)} \\
&= nT\sqrt{\frac{1+\beta_B}{1-\beta_B}} && \text{algebra} \\
&= nTD_B && \text{algebra} \qquad \text{Defn. (11)}
\end{aligned}$$

The pulses arrive at A with a period of $D_B T$.

Note: There is *nothing* in this derivation that relies upon prior knowledge of the Doppler factor, or under what conditions it is applicable. The expression $\sqrt{(1+\beta_B)/(1-\beta_B)}$ arose in the process of calculating the answer, using only the assumptions that light moves isotropically at speed c in A's frame, and that A can deduce that B's clock is running γ_B times slower than A's clock.

3.1.2 A sends periodic pulses to B

Now A uses his local clock to time the sending of radio pulses to B, sending pulses once every time interval T. At what period does B receive the pulses?

While we could switch perspectives to B's inertial frame, we will not. Instead, we are going to do all of the calculations in A's inertial frame.

The definitions of Section 3 remain the same here, but note that the values of $t_A(n)$ and $t_B(n)$ here are *different* than those in Section 3.1.1.

Define $t_A(n) = nT$ to be the time on A's clock when it transmits its n-th pulse.

At all times t_A, B is a distance $v_B t_A$ away from A. A assumes by special relativity that its pulse propagates with one-way speed c to B. The pulse's position at time $t_A \geq nT$ is $c(t_A - nT)$.

On A's clock, A by assumption and inference calculates that its n-th pulse catches up to B at a time $r_A(n)$ that satisfies the equation:

$$\begin{aligned}
v_B r_A(n) &= c(r_A(n) - nT) && B\text{'s position equals light pulse's position} \\
(v_B/c) r_A(n) &= r_A(n) - nT && \text{divide by } c \\
nT &= (1 - v_B/c) r_A(n) && \text{a little more algebra} \\
r_A(n) &= \frac{nT}{1 - v_B/c} \\
r_A(n) &= \frac{nT}{1-\beta_B} && \text{Defn.(9)} \qquad (13)
\end{aligned}$$

By the assumptions of special relativity, A deduces that B's clock is running γ_B times slower than A's clock. Thus B's time when receiving the n-th pulse is:

$$\begin{aligned}
r_B(n) &= r_A(n)/\gamma_B && B\text{'s clock slower by factor } \gamma_B \\
r_B(n) &= \frac{nT}{\gamma_B(1-\beta_B)} && \text{Eqn. (13)} \\
r_B(n) &= nT\frac{\sqrt{1-\beta_B^2}}{1-\beta_B} && \text{Defn. (10)} \\
r_B(n) &= nT\sqrt{\frac{1+\beta_B}{1-\beta_B}} && \text{a little algebra} \\
r_B(n) &= nTD_B && \text{Defn. (11)}
\end{aligned}$$

B will observe pulses arriving with period $D_B T$ according to B's clock.

Note from the previous section that A sees this same period, on A's clock, for the period of pulses that A receives from B.

3.1.3 Relationship to Lorentz Ether Theory

Note in Sections 3.1.1 and 3.1.2, that except for algebra and the definitions of symbols from Section 3, the only assumptions we used from special relativity were:

- The one-way speed of light is c in all directions, as measured by A.
- B's clock runs at a slower rate, by a factor of $1/\gamma_B$, relative to A's clock.

By Lorentz Ether Theory, if A is at rest relative to the ether, then:

- The one-way speed of light is c in all directions relative to the ether, and thus A, at rest relative to the ether, will measure that speed for light in all directions.
- B is moving at velocity v_B relative to the ether, and thus B's clock physically runs at a slower rate, by a factor of $1/\gamma_B$. A's clock runs at the full rate of all clocks at that are at rest relative to the ether.

TODO: Find a way to explain the following better.

I had heard from a not-yet-in-depth learning about special relativity that when A and B are moving at constant velocity towards or away from each other, that A observed that B's clock ran slower by a factor of γ, and B observed that A's clock ran slower by a factor of γ. (Note: I do not claim that those are fully precise statements, but there is definitely a sense in which special relativity does say something similar to this.)

In A's frame observing B's clock run slower, that seems perfectly consistent with Lorentz Ether Theory's statement that if A is at rest relative to the ether, and B moves at constant velocity v_B relative to the ether, that B experiences duration dilation, i.e. its clock physically runs slower than A's by a factor of γ_B. In this situation A's clock runs at full speed, i.e. γ_B times *faster* than B's.

But what about Lorentz Ether Theory's position on the converse statement? That is, from B's point of view, does B observe A's clock running γ_B times slower? If so, how can that possibly make sense?

I now believe that the answer is that the statements in special relativity can be made a bit more precise by saying something like this: Because A is following special relativity's assumptions, i.e. in A's frame the speed of light propagates isotropically at constant speed c, therefore A can deduce that any clocks moving at constant speed v directly towards or away from A run γ times slower, and make further calculations from that deduction.

A does not actually *observe* such clocks directly over any appreciable interval of time, so they are always, or almost always, so far away that A cannot make *any* direct observations of how fast such clocks are running. By "direct" observations I mean "with light propagation delay very close to 0 between A and the entity being observed".

Suppose in some future context of knowledge that not only is Lorentz Ether Theory proven, but in such a way that we know how to measure our speed relative to the ether.

Then, in the scenario described, we would know that A's clock is running at full speed, and light propagates isotropically at constant speed c relative to the ether, and thus also relative to A.

Everyone with this knowledge would be able to deduce that B's clock is running γ times slower than full speed. Also, that A's clock is running γ times *faster* than B's clock (and that all of A's local physical processes are proceeding γ times faster than similar local physical processes of B).

Further, light does *not* propagate at the same speed in all directions relative to B. It does so only with respect to the ether.

We could also prove that if one chose to make calculations using special relativity's assumptions in B's frame, one would get the same answers to these calculations that you do when using Lorentz Ether Theory.

A hint of corroboration can be seen in the Wikipedia page on time dilation, which says in the introduction [7]:

> The dilation compares "wristwatch" clock readings between events measured in different inertial frames and is not observed by visual comparison of clocks across moving frames.

It seems that any statement similar to:

- A observes B's clock running slower than their own.

could be said in much more detail as either of the following:

- In accordance with special relativity's time synchronization convention that light propagates in A's inertial frame isotropically with speed c, A deduces that B's clock runs slower than A's.
- A deduces, using the postulate that light moves isotropically at speed c, that B's clock runs slower, and can then make further consistent calculations based on this deduction.

And you can swap A and B in that statement. Without defining new precise terminology, a shorter precise statement would perhaps be:

- According to SR, in A's inertial frame we deduce that B's clock runs slower.

3.2 Special Relativity Scenario 2: Three entities, two of them moving away from the first at constant velocity

A is at rest. B is moving at constant velocity v_B away from A. C is moving at constant velocity $v_C > v_B$ away from A, in same direction that B is moving.

For B sending periodic pulses to A or vice versa, everything in Sections 3.1.1 and 3.1.2 applies without change. For C sending periodic pulses to A or vice versa, everything in Section 3.1.1 and 3.1.2 applies, except replace B subscripts with C subscripts, i.e. use v_C, γ_C, β_C, and D_C.

So it is only pulses between B and C that might present something new here.

3.2.1 B sends periodic pulses to C

B uses his local clock to time the sending of radio pulses to C, sending pulses once every time interval T, according to B's clock. At what period does C receive the pulses?

For simplicity of calculations, we will assume that both B and C pass A's position at time 0, and A, B, and C all synchronize their clocks at that time. As before, note that the calculation of the period for someone receiving pulses is unaffected by whether such synchronization is done.

In order to make the calculations as applicable to Lorentz Ether Theory as possible, all calculations will be done in A's inertial frame.

Define $t_A(n)$ and $t_B(n)$ the same way as they were in Section 3.1.1. As explained there, when A makes the assumptions according to special relativity theory:

$$t_B(n) = nT \qquad \text{by the setup of the experiment}$$

$$t_A(n) = \gamma_B nT \qquad B\text{'s clock runs } \gamma_B \text{ times slower than } A\text{'s} \tag{14}$$

$$x_B(t_A) = v_B t_A \qquad \text{relationship of } B\text{'s position and time, in } A\text{'s frame} \tag{15}$$

$$x_C(t_A) = v_C t_A \qquad \text{relationship of } C\text{'s position and time, in } A\text{'s frame} \tag{16}$$

Also by special relativity assumptions, A considers the pulse to travel from B to C at constant speed c. The n-th pulse is emitted at time $t_A(n)$ in A's frame, so its position as a function of A's time t_A is:

$$\begin{aligned}
l_A(n, t_A) &= \text{position of } B \text{ when emitted} \\
&\quad + \text{distance traveled after emission} \\
&= x_B(t_A(n)) + (t_A - t_A(n))c && \text{for any time } t_A \geq t_A(n) \\
&= v_B t_A(n) + (t_A - t_A(n))c && \text{substitute Eqn. (15)} \\
&= -(c - v_B)t_A(n) + t_A c && \text{algebra} \\
&= -(c - v_B)\gamma_B nT + t_A c && \text{substitute Eqn. (14)}
\end{aligned}$$

To find A's time when C receives the pulse, solve for t_A that makes the pulse position the same as C's position:

$$\begin{aligned}
x_C(t_A) &= l_A(n, t_A) \\
v_C t_A &= -(c - v_B)\gamma_B nT + t_A c \\
(c - v_B)\gamma_B nT &= (c - v_C)t_A && \text{algebra} \\
t_A &= \frac{c - v_B}{c - v_C}\gamma_B nT && \text{algebra} \\
t_A &= \frac{1 - \beta_B}{1 - \beta_C}\gamma_B nT && \text{defn. of } \beta_B, \beta_C
\end{aligned}$$

According to A and its special relativity assumptions, C's clock runs slower, at a rate $1/\gamma_C$ times that of A's clock. So C's time $r_C(n)$ to receive the n-th pulse sent by B is:

$$\begin{aligned}
r_C(n) &= \frac{1}{\gamma_C} t_A && (17) \\
&= \frac{\gamma_B}{\gamma_C}\left(\frac{1 - \beta_B}{1 - \beta_C}\right) nT && (18) \\
&= \sqrt{\frac{1 - \beta_C^2}{1 - \beta_B^2}}\left(\frac{1 - \beta_B}{1 - \beta_C}\right) nT && (19) \\
&= \sqrt{\frac{1 + \beta_C}{1 - \beta_C}}\sqrt{\frac{1 - \beta_B}{1 + \beta_B}} nT && \text{algebra} \quad (20) \\
&= (D_C/D_B)nT && \text{defn. of } D_B, D_C \quad (21)
\end{aligned}$$

So when B sends pulses with period T according to B's clock, C receives from B pulses with period $(D_C/D_B)T$ on C's clock.

Since D increases with β, and thus also with v, and since $v_C > v_B$, $D_C > D_B$, and $(D_C/D_B) > 1$. Thus C's measured period of receiving pulses is larger than B's measured period of sending pulses.

Note: If you perform the calculations in a similar scenario where $v_C < v_B$, they end up nearly identical to what is shown above. The final answer is still $(D_C/D_B)T$ for the period at which C receives pulses, but in this case $(D_C/D_B) < 1$ because $v_C < v_B$. Thus C's measured period of receiving pulses is shorter than B's measured period of sending pulses.

3.2.2 B sends periodic pulses to C, double-check by ChatGPT

Since at the time of performing calculations in the previous section I was still fairly new to such things, I wanted a way to double-check the results. I asked ChatGPT what the period would be that C would receive pulses from B and it gave an answer close to the following.

Calculate the velocity of B in C's frame using relativistic velocity subtraction:

$$u' = \frac{v_B - v_C}{1 - \frac{v_B v_C}{c^2}} \tag{22}$$

$$\beta' = \frac{\beta_B - \beta_C}{1 - \beta_B \beta_C} \tag{23}$$

Since $v_B < v_C$, the numerator is negative. The denominator is positive. Thus $u' < 0$ and $\beta' < 0$. Thus in C's frame, B is moving in the negative x direction, and C is at rest.

For receding motion:

$$T_{\text{observed}} = T_{\text{emitted}} \sqrt{\frac{1+\beta}{1-\beta}} \tag{24}$$

where:

$$\beta = \frac{|u'|}{c} = |\beta'| \tag{25}$$

Since $\beta' < 0$:

$$|\beta'| = -\beta' = \frac{\beta_C - \beta_B}{1 - \beta_B \beta_C} \tag{26}$$

From here on this is mostly just algebra:

$$\begin{aligned}
\frac{T_{\text{observed}}}{T_{\text{emitted}}} &= \sqrt{\frac{1+\beta}{1-\beta}} \\
&= \sqrt{\frac{1 + \frac{\beta_C - \beta_B}{1-\beta_B\beta_C}}{1 - \frac{\beta_C - \beta_B}{1-\beta_B\beta_C}}} \\
&= \sqrt{\frac{1 - \beta_C\beta_B + (\beta_C - \beta_B)}{1 - \beta_C\beta_B - (\beta_C - \beta_B)}} \\
&= \sqrt{\frac{(1+\beta_C)(1-\beta_B)}{(1-\beta_C)(1+\beta_B)}} \\
&= D_C/D_B && \text{defn. of } D_B, D_C
\end{aligned}$$

This is the same result, calculated in a fairly different way, using the assumptions of special relativity, which ChatGPT is much better at answering questions about than it is any alternatives that are not special relativity.

3.2.3 C sends periodic pulses to B

The derivation is nearly identical to that in Section 3.2.1. Here we only mention a few equations along the way that have noticeable differences.

The formula $l_A(n, t_A)$ for the position of the n-th pulse emitted by C at A's time t_A is:

$$l_A(n, t_A) = (c + v_C)\gamma_C nT - t_A c \tag{27}$$

A's time when B receives the n-th pulse t_A is:

$$t_A = \frac{1+\beta_C}{1+\beta_B}\gamma_C nT \tag{28}$$

and B's time when it receives the n-th pulse from C is:

$$\begin{aligned}
r_B(n) &= \frac{1}{\gamma_B} t_A \\
&= (D_C/D_B) nT
\end{aligned}$$

That is exactly the same local time period that C measures for pulses sent from B.

Note: As noted at the end of Section 3.2.1, if you perform the calculations for the same scenario, except $v_C < v_B$, the final answer is the same period $(D_C/D_B)T$. As noted there, this is less than T when $v_C < v_B$.

3.2.4 A note on Doppler factors and relativistic velocity addition and subtraction

Note that a consequence of the derivation in Section 3.2.2 is the following.

If you do relativistic velocity subtraction of u minus v, resulting in w, then the Doppler factor of w is $D_w = D_u/D_v$.

Although not shown in that section, it is pretty much the same algebra to show that when you do relativistic velocity addition of u plus v resulting in w, the Doppler factor of w is $D_w = D_u D_v$.

This is apparently a well-known result among those working with relativistic Doppler factors.

TODO: Consider adding a short section showing this more directly with a bit of algebra, perhaps?

3.3 Summary of results in this section

What have we actually proved here?

For a scenario where two or more entities are moving relative to each other in a straight line, then according to special relativity, or Lorentz Ether Theory:

- Both theories predict the same measurements in a simple experiment where one entity sends pings with period T (according to the sender's local clock), and the receiver measures the period of the received pings, using its local clock, as period DT where D always looks like a Doppler factor for some speed v related to the relative speed of the sender and receiver.

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A Miscellaneous math facts about relativistic velocity addition and subtraction in one dimension

All of the facts here are quite simple to see. I write them out primarily as an aid to thinking about and remembering them, and they might also be useful to refer to from elsewhere in this document.

Note that this appendix is restricted to proofs of simple mathematical relationships about the definitions of one-dimensional relativistic velocity addition and subtraction formulas. This appendix makes no claims about the physical meaning of these operations.

I have read that relativistic velocity addition in 3 dimensions is not associative. TODO: If I add any discussion of 3-dimensional relativity examples to this document, it would be nice to give an example, perhaps in a separate appendix.

In one dimension, though, all velocities of subliminal entities can be represented by v such that $-c < v < c$, where negative velocities are in the opposite direction along the line than positive velocities.

The can also be represented as relativistic velocity $\beta = v/c$, i.e. a fraction of c. These are in the range $-1 < \beta < 1$.

I will use the notation $v \oplus w$ for relativistic velocity addition. While I will use the same symbol $\beta_v \oplus \beta_w$ for relativistic velocity addition of relativistic velocity values, one should be careful to note that the definition of the operator $\oplus$ is slightly different for these two cases.

$$v \oplus w = \frac{v + w}{1 + \frac{vw}{c^2}} \tag{29}$$

$$v \ominus w = \frac{v - w}{1 - \frac{vw}{c^2}} \tag{30}$$

$$\beta_v \oplus \beta_w = \frac{\beta_v + \beta_w}{1 + \beta_v \beta_w} \tag{31}$$

$$\beta_v \ominus \beta_w = \frac{\beta_v - \beta_w}{1 - \beta_v \beta_w} \tag{32}$$

The proofs are only given for the relativistic velocity formulas (31) and (32). The proofs for equations (29) and (30) are nearly identical.

We will use the symbols a, b, c to represent arbitrary real values between -1 and 1, to avoid writing β with subscripts all over the place.

$$a \oplus b = b \oplus a \qquad \text{addition is commutative} \tag{33}$$

$$(a \oplus b) \oplus c = a \oplus (b \oplus c) \qquad \text{addition is associative (1-d only, not 3-d!)} \tag{34}$$

$$a \oplus 0 = 0 \oplus a = a \tag{35}$$

$$a \oplus (-b) = a \ominus b \tag{36}$$

$$a \ominus (-b) = a \oplus b \tag{37}$$

$$0 \ominus b = -b \tag{38}$$

$$a \ominus b = -(b \ominus a) \tag{39}$$

The following inequalities only hold if $0 < a < 1$ and $0 < b < 1$:

$$a \oplus b > a \tag{40}$$

$$a \oplus b > b \tag{41}$$

$$a \oplus b < a + b \tag{42}$$

The following inequality holds for all $-1 < a < 1$ and $-1 < b < 1$:

$$-1 < a \oplus b < 1 \tag{43}$$

The last one is only slightly different if we use velocity addition on normal velocities, i.e. velocities that are not relativistic velocities, where $-c < v < c$ and $-c < w < c$:

$$-c < v \oplus w < c \tag{44}$$

The inequalities all have examples where the two sides can be very nearly equal, such as these:

$$0.1 \oplus 0.001 \approx 0.100989901 > 0.1$$
$$0.1 \oplus 0.1 \approx 0.198019802 < 0.1 + 0.1$$
$$0.99 \oplus 0.99 \approx 0.999949498 < 1$$

Proving most of these is as simple as substituting the definition of $a \oplus b$ and $a \ominus b$, and a tiny amount of algebra.

Proving associativity (34) is only a little bit more algebra, but we will do the steps here:

$$\begin{aligned}
(a \oplus b) \oplus c &= \frac{\frac{a+b}{1+ab} + c}{1 + \left(\frac{a+b}{1+ab}\right) c} && \text{use Defn. (31) twice} \\
&= \frac{(a+b) + c(1+ab)}{(1+ab) + (a+b)c} && \text{multiply numerator and denominator by } (1+ab) \\
&= \frac{a+b+c+abc}{1+ab+ac+bc} && \text{multiply out the terms}
\end{aligned}$$

Similarly:

$$\begin{aligned}
a \oplus (b \oplus c) &= \frac{a + \frac{b+c}{1+bc}}{1 + a\left(\frac{b+c}{1+bc}\right)} && \text{use Defn. (31) twice} \\
&= \frac{a(1+bc) + (b+c)}{(1+bc) + a(b+c)} && \text{multiply numerator and denominator by } (1+bc) \\
&= \frac{a + abc + b + c}{1 + bc + ab + ac} && \text{multiply out the terms}
\end{aligned}$$

The above two final results are easily seen to be equal.

The inequalities are also not difficult to prove, but we will write out their short proofs. Recall that these inequalities are true only for $0 < a < 1$ and $0 < b < 1$. Similar inequality hold if both a and b are negative.

For the proof of inequality (40), recall that we can multiply or divide both sides of an inequality by the same positive number, and the resulting inequality is true if and only if the original one was. The symbol $\Leftrightarrow$ below means "if and only if", i.e. the expression before is true if and only if the expression after is true.

$$\begin{aligned}
a < \frac{a+b}{1+ab} &\Leftrightarrow a + a^2 b < a + b && \text{multiply both sides by } 1+ab\text{, which is positive} \\
&\Leftrightarrow a^2 b < b && \text{subtract } a \text{ from both sides} \\
&\Leftrightarrow a^2 < 1 && \text{divide both sides by } b\text{, which is positive}
\end{aligned}$$

The last inequality is true because $a < 1$. The proof of inequality (41) is the same as above.

To prove $a \oplus b < a + b$ (inequality (42)):

$$\begin{aligned}
\frac{a+b}{1+ab} < a + b &\Leftrightarrow a + b < (a+b)(1+ab) && \text{multiply both sides by } 1+ab\text{, which is positive} \\
&\Leftrightarrow 1 < (1+ab) && \text{divide both sides by } a+b\text{, which is positive} \\
&\Leftrightarrow 0 < ab && \text{subtract 1 from both sides}
\end{aligned}$$

The final inequality is true because both a and b are positive.

Now to prove $a \oplus b < 1$ (part of inequality (43)), but recall now we are doing so for the more general case of all values $-1 < a < 1$ and $-1 < b < 1$:

$$\begin{aligned}
\frac{a+b}{1+ab} < 1 &\Leftrightarrow a + b < 1 + ab && \text{multiply both sides by } 1+ab\text{, which is positive} \\
&\Leftrightarrow b - ab < 1 - a && \text{subtract } a+ab \text{ from both sides} \\
&\Leftrightarrow b(1-a) < 1 - a && \text{algebra} \\
&\Leftrightarrow b < 1 && \text{divide both sides by } 1-a\text{, which is positive}
\end{aligned}$$

And the last inequality is true. Proving the part that

$$\begin{aligned}
-1 < \frac{a+b}{1+ab} &\Leftrightarrow -(1+ab) < a + b && \text{multiply both sides by } 1+ab\text{, which is positive} \\
&\Leftrightarrow -(1+a) < b + ab && \text{add } ab - a \text{ to both sides} \\
&\Leftrightarrow -(1+a) < b(1+a) && \text{algebra} \\
&\Leftrightarrow -1 < b && \text{divide both sides by } 1+a\text{, which is positive}
\end{aligned}$$

And the last inequality is true.