Andy's science scratch pad

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1 Introduction

This document is a place to write up little bits on science.

Some notation:

$\mathbf{i}$ is the unit vector from left to right. $\mathbf{j}$ is the unit vector upwards. $\mathbf{k}$ is the unit vector pointed out of the page toward the reader.

$\gamma = 1/\sqrt{1 - v^2/c^2}$ is the Lorentz factor.

2 Electromagnetic force between two point charges at rest relative to each other

Scenario 1: There are two point charges a and b both with charge q at rest relative to each other at a distance r apart (see Figure 1). They are at rest relative to us. In this case they both experience a force directly away from the other due to electric repulsion. There is no magnetic force, as both charges are at rest so there are no magnetic fields.
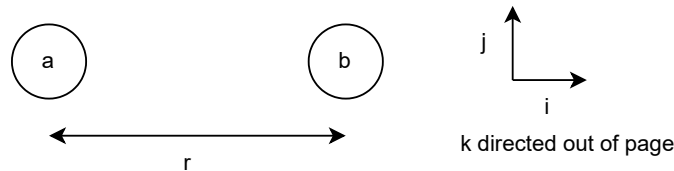

Figure 1: Two point charges at rest

Scenario 2: The same as scenario 1, but both charges are moving with constant velocity v in the upwards direction (see Figure 2). Since they are moving they create magnetic fields.
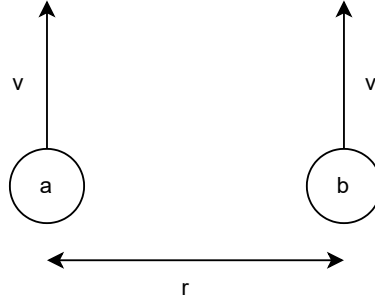
Questions:

Figure 2: Two point charges moving at same constant velocity

- What is the net force on charge b in each scenario?

- Is it the same in both scenarios, or different?

- Why?

2.1 Scenario 1: Both charges at rest

As mentioned before, there is no current or motion of any charges in this scenario, so no magnetic fields. The electric repulsion force on charge b is easily calculated from Coulomb's Law [4]. Charge b is to the right of charge a, so the direction of the force is $\mathbf{i}$, away from charge a.

$$\mathbf{E}_1 = \frac{1}{4\pi\epsilon_0}\frac{q}{r^2}\mathbf{i} \tag{1}$$

$$\mathbf{B}_1 = 0 \tag{2}$$

$$\mathbf{F}_1 = q(\mathbf{E}_1 + \mathbf{v} \times \mathbf{B}_1) = q\mathbf{E}_1 \tag{3}$$

2.2 Scenario 2: Both charges with equal and constant velocity upwards

2.2.1 Scenario 2 calculated by electromagnetic field equations from Griffiths

The Wikipedia page on the Biot-Savart Law [5] has a subsection titled "Point charge at constant velocity" that says:

> the Biot–Savart law applies only to steady currents and a point charge moving in space does not constitute a steady current

I will thus use the equations in that section to calculate the electric and magnetic fields here. The relevant parts of the Wikipedia page are copied below.

> In the case of a point charged particle q moving at a constant veclocity $\mathbf{v}$, Maxwell's equations give the following expression for the electric field and magnetic field:
>
> $$\mathbf{E} = \frac{q}{4\pi\epsilon_0}\frac{1-\beta^2}{(1-\beta^2\sin^2\theta)^{3/2}}\frac{\hat{\mathbf{r}}'}{|r'|^2} \tag{4}$$
>
> $$\mathbf{B} = \frac{1}{c^2}\mathbf{v} \times \mathbf{E} \tag{5}$$

where:

- $\hat{\mathbf{r}}'$ is the unit vector pointing from the current (non-retarded) position of the particle to the point at which the field is being measured,

- $\beta = v/c$ is the speed in units of c, and

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- θ is the angle between $\mathbf{v}$ and $\hat{\mathbf{r}}'$.

The equations above appear to be identical to equations (10.75) and (10.76) in Griffiths [2]. Griffiths comments on the formula for the electric field:

> Notice that $\mathbf{E}$ points along the line from the *present* position of the particle. This is an extraordinary coincidence, since the "message" came from the retarded position. Because of the $\sin^2 \theta$ in the denominator, the field of a fast-moving charge is flattened out like a pancake in the direction perpendicular to the motion (Fig. 10.10). In the forward and backward directions $\mathbf{E}$ is reduced by a factor $(1 - v^2/c^2)$ relative to the field of a charge at rest; in the perpendicular direction it is *enhanced* by a factor $1/\sqrt{1 - v^2/c^2}$.

Calculation: To get the force on charge b, we first calculate the $\mathbf{E}$ and $\mathbf{B}$ fields at the position of charge b.

Charge b is directly to the right of charge a, so $\hat{\mathbf{r}}' = \mathbf{i}$ and $\theta = 90°$.

$$
\begin{aligned}
\mathbf{E}_2 &= \frac{q}{4\pi\epsilon_0} \frac{1 - \beta^2}{(1 - \beta^2 \sin^2 \theta)^{3/2}} \frac{\hat{\mathbf{r}}'}{|r'|^2} && \hat{\mathbf{r}}' = \mathbf{i}, \ |r'| = r, \ \theta = 90°, \text{ simplify fraction} \\
&= \frac{q}{4\pi\epsilon_0} \frac{1}{(1 - \beta^2)^{1/2}} \frac{\mathbf{i}}{r^2} && \text{part of this is } \gamma, \text{ by (1) the rest is } \mathbf{E}_1 \\
&= \gamma \mathbf{E}_1 && (6)
\end{aligned}
$$

Note that $\mathbf{E}_2$ being γ times larger than $\mathbf{E}_1$ is consistent with the comment from Griffiths above: "in the perpendicular direction it ($\mathbf{E}$) is *enhanced* by a factor $1/\sqrt{1 - v^2/c^2}$".

$$
\begin{aligned}
\mathbf{F}_2 &= q(\mathbf{E}_2 + \mathbf{v} \times \mathbf{B}_2) && \text{replace } \mathbf{B}_2 \text{ with (5)} \\
&= q(\mathbf{E}_2 + \mathbf{v} \times (\frac{1}{c^2} \mathbf{v} \times \mathbf{E}_2)) && \mathbf{v} \times \mathbf{E}_2 = -vE_2\mathbf{k} \\
&= q(\mathbf{E}_2 - \frac{vE_2}{c^2} \mathbf{v} \times \mathbf{k}) && \mathbf{v} \times \mathbf{k} = v\mathbf{i} \\
&= q(\mathbf{E}_2 - \frac{v^2 E_2}{c^2} \mathbf{i}) && \\
&= q(1 - \frac{v^2}{c^2}) \mathbf{E}_2 && \\
&= \frac{q\mathbf{E}_2}{\gamma^2} && \text{by (6)} \ \ \mathbf{E}_2 = \gamma \mathbf{E}_1 \\
&= \frac{q\mathbf{E}_1}{\gamma} && \text{by (3)} \ \ \mathbf{F}_1 = q\mathbf{E}_1 \\
&= \frac{\mathbf{F}_1}{\gamma} &&
\end{aligned}
$$

Thus $\mathbf{F}_2$ differs from $\mathbf{F}_1$ by a factor of γ.

TODO: Why?

I do not know how to check the answer below, but it appears that three of the answers to an on-line question similar to mine [3] say that the Lorentz force formula $\mathbf{F} = q(\mathbf{E} + \mathbf{v} \times \mathbf{B})$ is *not* invariant in all inertial frames, but perhaps a slightly modified version of that formula is invariant between different inertial frames. I quote one such answer below:

> Just for completeness if permitted: Following Section 3.1 from the book "Gravitation" of Misner, Thorne, and Wheeler the truly (at all speeds) frame independent force is $\frac{dP}{d\tau} = \gamma(E + v \times B)$ (in fact this is only the spacial component of the four force). τ is proper time and γ the well-known Lorentz Factor. – Kurt G. Aug 28, 2021

2.2.2 Scenario 2 calculated by Heaviside-Feynman formula

The Wikipedia page on Jefimenko's Equations [6] has a subsection titled "Heaviside-Feynman formula" that gives equations for the electric and magnetic field at a point due to a single moving point charge.

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$$\mathbf{E} = \frac{-q}{4\pi\epsilon_0}\left[\frac{\mathbf{e}_{r'}}{r'^2} + \frac{r'}{c}\frac{d}{dt}\left(\frac{\mathbf{e}_{r'}}{r'^2}\right) + \frac{1}{c^2}\frac{d^2}{dt^2}\mathbf{e}_{r'}\right] \tag{7}$$

$$\mathbf{B} = -\mathbf{e}_{r'} \times \frac{\mathbf{E}}{c} \tag{8}$$

Here $\mathbf{e}_{r'}$ is a unit vector pointing from the observer to the charge and r' is the distance between observer and charge. Since the electromagnetic field propagates at the speed of light, both of these quantities are evaluated at the retarted time $t - r'/c$.

I believe "observer" above means "the position for which we are calculating E and B fields".

Assume here that the point charges are kept at distance r apart from each other, always horizontally, e.g. because they are connected by a stiff insulating rod. This simplifies our job of calculating E, because then $\mathbf{e}_{r'}$ and r' are unchanging over time, and their derivates are thus 0.

We want to calculate r' as the vector from the position of charge b to the position where charge a was when it emitted an electric field propagated at speed c to b. See Figure 3.
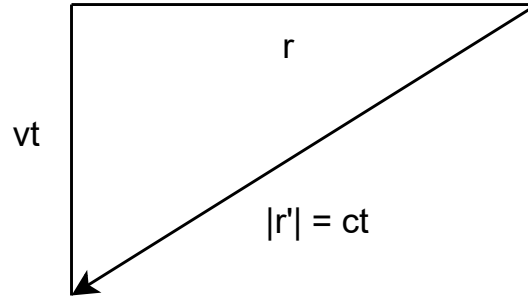

Figure 3: The retarded position of charge a from charge b

Solve for t using Pythagorean theorem since r and v are known constants:

$$r^2 + (vt)^2 = (ct)^2$$
$$t^2(c^2 - v^2) = r^2$$
$$t^2 = \frac{r^2}{c^2 - v^2}$$
$$t = \frac{r}{\sqrt{c^2 - v^2}}$$
$$= \frac{r}{c\sqrt{1 - v^2/c^2}}$$
$$= \gamma r/c$$

This gives us $r' = ct = \gamma r$, and $\mathbf{e}_{r'}$ is:

$$\mathbf{e}_{r'} = \frac{-r\mathbf{i} - (\gamma r v/c)\mathbf{j}}{\gamma r}$$
$$= -\frac{1}{\gamma}\mathbf{i} - \frac{v}{c}\mathbf{j}$$

Plugging in this value for $\mathbf{e}_{r'}$ into Equation (7) gives:

$$\mathbf{E}_3 = \frac{q}{4\pi\epsilon_0}\left[\frac{\frac{1}{\gamma}\mathbf{i} + \frac{v}{c}\mathbf{j}}{\gamma^2 r^2}\right]$$

Note that $\mathbf{E}_3$ is parallel to $\mathbf{e}_{r'}$, thus $\mathbf{B}_3$ from Equation (8) is 0.

4

This gives the force on charge b as:

$$\mathbf{F}_3 = q(\mathbf{E}_3 + \mathbf{v} \times \mathbf{B}_3)$$
$$= q\mathbf{E}_3$$

The direction of $\mathbf{F}_3$ is different than $\mathbf{F}_1$ and $\mathbf{F}_2$. Below is the relative magnitude of $\mathbf{E}_3$ to $\mathbf{E}_1$:

$$E_3 = \frac{1}{\gamma^2} E_1$$
$$F_3 = \frac{1}{\gamma^2} F_1$$

TODO: It seems *very* odd to me that $\mathbf{B}_3 = 0$.

After Feynman explains what the retarded direction and distance **r'** is, he says [1]:

> That would be easy enough to understand, too, but it is also wrong. The whole thing is much more complicated.

Unfortunately there are no footnotes or citation to explain what he meant by this.

3 Simple scenarios in special relativity and Lorentz Ether Theory

Definitions of some terms:

$$\beta_B = v_B/c \tag{9}$$

$$\gamma_B = \frac{1}{\sqrt{1 - \beta_B^2}} \tag{10}$$

$$D_B = \sqrt{\frac{1 + \beta_B}{1 - \beta_b}} \tag{11}$$

3.1 Special Relativity Scenario 1: Two entities moving away from each other at constant velocity

A is at rest. B is moving at constant velocity v_B away from A.

3.1.1 B sends periodic pulses to A

B uses his local clock to time the sending of radio pulses to A, sending pulses once every time interval T. At what period does A receive the pulses?

For simplicity of calculations, we will assume that B passed A's position at time 0, and A and B synchronized their clocks at that time. However, note that A will receive the pulses at the same period in this scenario, regardless of whether they ever synchronized their clocks.

Define $t_A(n)$ to be the time on A's clock when the n-th pulse is transmitted by B, and $t_B(n)$ to be the time on B's clock when it transmits the n-th pulse.

$t_B(n) = nT$ by the setup of the experiment.

By the assumptions of special relativity, A deduces that B's clock is running γ_B times slower than A's clock. Note: A cannot directly observe B's clock, as it is too far away. From this A also assumes:

$$t_A(n) = \gamma_B nT \tag{12}$$

B is a distance $v_B t_A(n)$ away from A at that time.

Also by the assumptions of special relativity, A deduces that B's pulse signal will propagate at the one-way speed c. The pulse will thus take time $v_B t_A(n)/c$ to propagate to A.

A's clock thus shows time $t_A(n) + v_B t_A(n)/c$ when the n-th pulse arrives at A. With a little algebra:

$$t_A(n) + v_B t_A(n)/c = (1 + v_B/c)t_A(n) \qquad \text{rearrangement by algebra}$$
$$= nT(1 + \beta_B)\gamma_B \qquad \text{Defn. (9) and Eqn. (12)}$$
$$= nT \frac{1 + \beta_B}{\sqrt{1 - \beta_B^2}} \qquad \text{Defn. (10)}$$
$$= nT \sqrt{\frac{1 + \beta_B}{1 - \beta_B}} \qquad \text{algebra}$$
$$= nT D_B \qquad \text{algebra} \qquad\qquad \text{Defn. (11)}$$

The pulses arrive at A with a period of $D_B T$.

3.1.2 A sends periodic pulses to B

Now A uses his local clock to time the sending of radio pulses to B, sending pulses once every time interval T. At what period does B receive the pulses?

While we could switch perspectives to B's inertial frame, we will not. Instead, we are going to do all of the calculations in A's inertial frame.

The definitions of Section 3 remain the same here, but note that the values of $t_A(n)$ and $t_B(n)$ here are *different* than those in Section 3.1.1.

Define $t_A(n) = nT$ to be the time on A's clock when it transmits its n-th pulse.

At all times t_A, B is a distance $v_B t_A$ away from A. A assumes by special relativity that its pulse propagates with one-way speed c to B. The pulse's position at time $t_A \geq nT$ is $c(t_A - nT)$.

On A's clock, A by assumption and inference calculates that its n-th pulse catches up to B at a time $r_A(n)$ that satisfies the equation:

$$v_B r_A(n) = c(r_A(n) - nT) \qquad \text{B's position equals light pulse's position}$$
$$(v_B/c)r_A(n) = r_A(n) - nT \qquad \text{divide by } c$$
$$nT = (1 - v_B/c)r_A(n) \qquad \text{a little more algebra}$$
$$r_A(n) = \frac{nT}{1 - v_B/c}$$
$$r_A(n) = \frac{nT}{1 - \beta_B} \qquad \text{Defn.(9)} \qquad\qquad (13)$$

By the assumptions of special relativity, A deduces that B's clock is running γ_B times slower than A's clock. Thus B's time when receiving the n-th pulse is:

$$r_B(n) = r_A(n)/\gamma_B \qquad \text{B's clock slower by factor } \gamma_B$$
$$r_B(n) = \frac{nT}{\gamma_B(1 - \beta_B)} \qquad \text{Eqn. (13)}$$
$$r_B(n) = nT \frac{\sqrt{1 - \beta_B^2}}{1 - \beta_B} \qquad \text{Defn. (10)}$$
$$r_B(n) = nT \sqrt{\frac{1 + \beta_B}{1 - \beta_B}} \qquad \text{a little algebra}$$
$$r_B(n) = nT D_B \qquad \text{Defn. (11)}$$

B will observe pulses arriving with period $D_B T$ according to B's clock.

Note from the previous section that A sees this same period, on A's clock, for the period of pulses that A receives from B.

3.1.3 Relationship to Lorentz Ether Theory

Note in Sections 3.1.1 and 3.1.2, that except for algebra and the definitions of symbols from Section 3, the only assumptions we used from special relativity were:

- The one-way speed of light is c in all directions, as measured by A.

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- B's clock runs at a slower rate, by a factor of $1/\gamma_B$, relative to A's clock.

By Lorentz Ether Theory, if A is at rest relative to the ether, then:

- The one-way speed of light is c in all directions relative to the ether, and thus A, at rest relative to the ether, will measure that speed for light in all directions.

- B is moving at velocity v_B relative to the ether, and thus B's clock physically runs at a slower rate, by a factor of $1/\gamma_B$. A's clock runs at the full rate of all clocks at that are at rest relative to the ether.

References

[1] Feynman. Feynman's Lectures on Physics Volume I Chapter 28: Electromagnetic Radiation, 2025. URL `https://www.feynmanlectures.caltech.edu/I_28.html#Ch28-S1-p10`.

[2] David J. Griffiths. *Introduction to Electrodynamics, Section 10.3.2 "The fields of a moving point charge"*, pages 456–462. Pearson Education, 4th edition, 2013.

[3] Physics Stack Exchange. How is Lorentz force frame-independent?, 2022. URL `https://physics.stackexchange.com/questions/661883/how-is-lorentz-force-frame-independent`.

[4] Wikipedia. Coulomb's Law, 2025. URL `https://en.wikipedia.org/wiki/Coulomb%27s_law`.

[5] Wikipedia. Biot-Savart Law page, section titled "Point charge at constant velocity", 2025. URL `https://en.wikipedia.org/wiki/Biot%E2%80%93Savart_law#Point_charge_at_constant_velocity`.

[6] Wikipedia. Jefimenko's Equations page, section title "Heaviside-Feynman formula", 2025. URL `https://en.wikipedia.org/wiki/Jefimenko%27s_equations#Heaviside%E2%80%93Feynman_formula`.

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