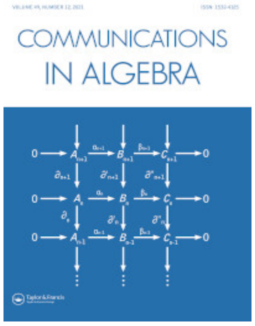

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Characters of the nullcone related to involutions of reductive groups

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ABSTRACT

Let G be a reductive algebraic group over an algebraically closed field F of characteristic other than 2, let θ be an involution of G, let $K = (G^\theta)^\circ$, and let $\mathfrak{p}$ be the -1-eigenspace of $d\theta$ in $\mathrm{Lie}(G)$. Then the adjoint action of G on $\mathrm{Lie}(G)$ restricts to an action of K on $\mathcal{N}$, the variety of nilpotent elements in $\mathfrak{p}$. We describe the structure of $\mathcal{N}$ as a K-module, providing a formula for the multiplicities of the simple highest weight K-modules as composition factors of the homogeneous parts of the coordinate ring $F[\mathcal{N}]$. The multiplicity formula gives complete information about $F[\mathcal{N}]$ when $F = \mathbb{C}$ but only partial information when $\mathrm{char}(F) > 0$. This result is an adaptation of analogous results of Hesselink when $F = \mathbb{C}$ and Friedlander and Parshall when $\mathrm{ch}(F) > 0$.

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1. Introduction

1.1. Notation and background

Let G be a reductive linear algebraic group defined over an algebraically closed field F. Throughout, F will either be $\mathbb{C}$ (or more generally, any algebraically closed field of characteristic 0) or a field k whose characteristic $\mathrm{char}(k)$ is greater than 2. Further restrictions will later be placed on $\mathrm{char}(k)$. Let θ be an involution on G, and let G^θ denote the subgroup $\{g \in G : \theta(g) = g\}$. Then $K := (G^\theta)^\circ$, the connected component of G^θ containing the identity, is also reductive by, for example, [14, Section 1]. (Any results cited here from [14] are valid whenever $\mathrm{char}(F) \neq 2$, despite the assumption there that $\mathrm{char}(F) = 0$.) Let $\mathfrak{g} = \mathrm{Lie}(G)$. The differential $d\theta$ of θ is an involution on $\mathfrak{g}$, and by convention we will denote $d\theta$ by just θ. We thus get a decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$, where $\mathfrak{k}$ (resp., $\mathfrak{p}$) is the $+1$ (resp., -1) -eigenspace of θ in $\mathfrak{g}$. Further, $\mathfrak{k} = \mathrm{Lie}(K)$ and the adjoint action of G on $\mathfrak{g}$ restricts to an action of K on $\mathfrak{p}$. We are interested here in the action of K on the nullcone $\mathcal{N}$ of $\mathfrak{p}$, which is the set of nilpotent elements in $\mathfrak{p}$, and on the algebra $F[\mathfrak{p}]$ of polynomial functions defined on $\mathfrak{p}$.

Kostant and Rallis extensively investigated the action of K on $\mathfrak{p}$, $\mathcal{N}$, and $F[\mathfrak{p}]$ in [9] when $F = \mathbb{C}$ and θ arises from the Cartan decomposition of the real form $\mathfrak{g}_\mathbb{R}$ of $\mathfrak{g}$. Richardson dealt with the global analogue of Kostant and Rallis' work in [11], giving results related to the action of K on G/K. In [10], Levy extended many of the results of [9] to fields of positive characteristic, and these play a key role below.

Throughout [10], it was assumed that G satisfies the "standard hypotheses", and therefore we will assume them as well; in particular, we assume:

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- When $F = k$, $\mathrm{char}(F)$ is a good prime for G. This means $\mathrm{char}(F)$ is greater than any coefficient used to write a root of G as a $\mathbb{Z}$-linear combination of simple roots of G.
- The derived subgroup of G is simply connected.
- There is a nondegenerate, G-equivariant, symmetric, bilinear form defined on $\mathfrak{g}$.

In the rest of this section, we establish the setup to be used throughout the paper and review some standard facts about algebraic groups and their representations. In Section 2, we prove that $F[\mathfrak{p}]$ is free over $F[\mathfrak{p}]^K$, the subalgebra of K-invariant polynomials. Section 3 uses this freeness result to achieve a positive-characteristic version of Kostant and Rallis' separation of variables decomposition found in [9, Theorem 15]. Section 4 contains the main result, a formula for the multiplicities of the simple highest weight modules in the homogeneous parts of the coordinate ring $F[\mathcal{N}]$. A full formula is given in the case when $F = \mathbb{C}$ and a partial one when $\mathrm{char}(F)$ is a good prime for G. These multiplicity formulas are derived using the methods in [6] (in the $F = \mathbb{C}$ case) and [4] (in the $F = k$ case), both of which rely on the decompositions from Section 3.

1.2. Tori, root systems, and Weyl groups

The following setup and notation will be used throughout. Let B_G be a θ-stable Borel subgroup of G and T_G a θ-stable maximal torus of G contained in B_G. (The existence of these subgroups is guaranteed by [13, Theorem 7.5].) Let Φ_G be the root system of G relative to T_G with positive roots Φ_G^+ determined by B_G.

By [11, Lemma 5.1], $B := B_G \cap K$ is a Borel subgroup of K containing the maximal torus $T := T_G \cap K$ of K. Let Φ be the root system of K relative to T with positive roots Φ^+ determined by B. Let $\rho = \frac{1}{2}\sum_{\alpha\in\Phi^+}\alpha$. Additionally, we will let Δ denote the basis of simple roots of Φ.

The character and cocharacter groups of T are $X(T) = \mathrm{Hom}(T, F^\times)$ and $Y(T) = \mathrm{Hom}(F^\times, T)$, respectively. There is a partial order $\leqslant$ defined on $X(T)$ by $\mu \leqslant \lambda$ if and only if $\lambda - \mu = \sum_{\alpha\in\Delta} c_\alpha \alpha$ with $c_\alpha \in \mathbb{N}\cup\{0\}$ for each α. Given $\lambda \in X(T)$ and a T-module M, if $M_\lambda := \{x \in M \ : \ t\cdot x = \lambda(t)x \text{ for all } t \in T\} \neq \{0\}$, then λ is a weight of T on M, and M_λ is the corresponding weight space of λ in M.

We can derive Φ, Φ^+, and the set of weights of T on $\mathfrak{p}$ (which we will denote $\Phi_\mathfrak{p}$) from Φ_G as follows. First consider the sets $\Phi_{G,T} := \{\alpha|_T \ : \ \alpha \in \Phi_G\} \cup \{0\}$ and $\Phi_{G,T}^+ := \{\alpha|_T \ : \ \alpha \in \Phi_G^+\}$. We have that and $\Phi^+ \subseteq \Phi_{G,T}^+$. When considering $\mathfrak{g}$ as a K-module, we will denote it $\mathfrak{g}^{(K)}$. Then $\Phi_{G,T}$ is the set of weights of T on $\mathfrak{g}^{(K)}$, and the corresponding weight space decomposition is

$$\mathfrak{g}^{(K)} = \bigoplus_{\chi\in\Phi_{G,T}} \mathfrak{g}_\chi^{(K)},$$

where

$$\mathfrak{g}_\chi^{(K)} = \bigoplus_{\{\alpha\in\Phi_G:\alpha|_T=\chi\}} \mathfrak{g}_\alpha.$$

Therefore, the set of weights of T on $\mathfrak{k}$ is

$$\Phi_\mathfrak{k} := \{\chi \in \Phi_{G,T} \ : \ \mathfrak{g}_\chi^{(K)} \cap \mathfrak{k} \neq \{0\}\},$$

with $\Phi = \Phi_\mathfrak{k} \setminus \{0\}$ and $\Phi^+ = \Phi_\mathfrak{k} \cap \Phi_{G,T}^+$.

Similarly, we have

$$\Phi_\mathfrak{p} = \{\chi \in \Phi_{G,T} \ : \ \mathfrak{g}_\chi^{(K)} \cap \mathfrak{p} \neq \{0\}\},$$

and the weight space of a weight $\chi \in \Phi_\mathfrak{p}$ is $\mathfrak{p}_\chi := \mathfrak{g}_\chi^{(K)} \cap \mathfrak{p}$.

Lemma 1.2.1.

a. The weight 0 is in $\Phi_\mathfrak{p}$ if and only if $\mathrm{rank}(G) > \mathrm{rank}(K)$. When $0 \in \Phi_\mathfrak{p}$, $\dim(\mathfrak{p}_0) = \mathrm{rank}(G) - \mathrm{rank}(K)$.

b. If $\chi \in \Phi_{\mathfrak{p}} \setminus \{0\}$, then $\dim(\mathfrak{p}_\chi) = 1$.

Proof. By [11, Lemma 5.3], the centralizer $Z_G(T)$ of T in G is a maximal torus of G containing T_G, and so $Z_G(T) = T_G$. Thus

$$\dim(\mathfrak{g}_0) = \dim(\mathrm{Lie}(T_G)) = \dim(\mathrm{Lie}(Z_G(T))).$$

Since $\mathfrak{h} := \mathrm{Lie}(T)$ consists of semisimple elements,

$$\mathrm{Lie}(Z_G(T)) = \mathfrak{z}_{\mathfrak{g}}(\mathfrak{h}) = \mathfrak{g}_0^{(K)},$$

where $\mathfrak{z}_{\mathfrak{g}}(\mathfrak{h})$ is the centralizer of $\mathfrak{h}$ in $\mathfrak{g}$. Thus $\dim(\mathfrak{g}_0) = \dim(\mathfrak{g}_0^{(K)})$, which gives

$$\dim(\mathfrak{g}_0^{(K)}) - \dim(\mathfrak{g}_0^{(K)} \cap \mathfrak{k}) = \mathrm{rank}(G) - \mathrm{rank}(K),$$

and part (a) immediately follows.

For part (b), let χ be a nonzero weight in $\Phi_{\mathfrak{p}}$. Then for $\alpha, \beta \in \Phi_G$, $\alpha|_T = \beta|_T$ if and only if $\theta^*(\alpha) = \beta$, where θ^* is the graph automorphism of Φ_G induced by θ. If $\alpha = \beta$, then $\mathfrak{g}_\chi^{(K)} = \mathfrak{g}_\alpha$, which means $\dim(\mathfrak{p}_\chi) = 1$. If $\alpha \neq \beta$, then $\mathfrak{g}_\chi^{(K)} = \mathfrak{g}_\alpha \oplus \mathfrak{g}_\beta$. Since $\chi \neq 0$, we thus have $\dim(\mathfrak{k}_\chi) = \dim(\mathfrak{p}_\chi) = 1$. □

Remark 1.2.2. The result in [12, 1.8] implies that when G is semisimple, $0 \in \Phi_{\mathfrak{p}}$ if and only if θ is an outer automorphism. Consequently, when θ is inner, $\Phi_{G,T} = \Phi_G$ and $\Phi_{G,T}^+ = \Phi_G^+$.

There is a pairing $\langle \cdot, \cdot \rangle : X(T) \times Y(T) \to \mathbb{Z}$ defined by $(\lambda \circ \psi)(a) = a^{\langle \lambda, \psi \rangle}$ for $a \in F^\times$. This pairing extends to the Euclidean spaces $E_X := X(T) \otimes_{\mathbb{Z}} \mathbb{R}$ and $E_Y := Y(T) \otimes_{\mathbb{Z}} \mathbb{R}$. To each root $\alpha \in \Phi$, we denote the coroot of α in $Y(T)$ relative to $\langle \cdot, \cdot \rangle$ by $\alpha^\vee$. Let $X(T)^+$ denote the set $\{\lambda \in X(T) \ : \ \langle \lambda, \alpha^\vee \rangle \geq 0 \mathrm{for all} \alpha \in \Delta\}$ of dominant weights in $X(T)$.

Let W be the Weyl group of K relative to T. The subset $X(T)^+$ of E_X is a fundamental domain for the natural action of W on E_X (denoted $w(v)$ for $w \in W$ and $v \in E_X$).

The group W also acts on E_X and $X(T)$ by the *dot action*: $w \cdot \lambda = w(\lambda + \rho) - \rho$. The set

$$C_0 := \{\lambda \in X(T) \ : \ \langle \lambda + \rho, \alpha^\vee \rangle \geq 0 \text{ for all } \alpha \in \Phi^+\}$$

is a fundamental domain for the dot action of W on $X(T)$.

Now suppose $F = k$ with $\mathrm{char}(k) = p$. Let $\alpha \in \Phi$ and $n \in \mathbb{N}$. The set

$$L_{\alpha,n} := \{v \in E_X \ : \ \langle v + \rho, \alpha^\vee \rangle = np\}$$

is a hyperplane in E_X. Let $s_{\alpha,n}$ be the reflection of E_X across $L_{\alpha,n}$, and let W_p be the group generated by the set $\{s_{\alpha,n}\}_{\alpha \in \Phi, n \in \mathbb{N}}$. This is called the *affine Weyl group* with respect to p; it extends the dot action of W on E_X and $X(T)$. The set

$$C_p := \{\lambda \in X(T) \ : \ 0 \leq \langle \lambda + \rho, \alpha^\vee \rangle \leq p \text{ for all } \alpha \in \Phi^+\}$$

is called the *standard alcove* for W_p in E_X, and it is a fundamental domain for the dot action of W_p on $X(T)$.

A torus S in G is said to be θ-*split* if $\theta(s) = s^{-1}$ for all $s \in S$. Fix a θ-split torus A which is maximal among all θ-split tori in G, and let $\mathfrak{a} = \mathrm{Lie}(A)$. Then $\mathfrak{a}$ is a maximal torus in $\mathfrak{p}$ in the sense of [9, Section I.1] or [10, Section 2].

Let Φ_A be the set of roots of G relative to A. In [11, Section 4], Richardson constructed a vector space in which Φ_A is a (not necessarily reduced) root system. He called the corresponding Weyl group W_A the *little Weyl group* as opposed to the "big" Weyl group of G relative to T_G.

For the Weyl groups W and W_A, we denote by $l(w)$ the length of the Weyl group element w relative to the basis of simple roots of the corresponding root system, and we let $\det(w) = (-1)^{l(w)}$.

1.3. Simple K-modules

The representation theory of reductive groups (as found in [7, Part II], for example) states that the simple K-modules correspond to the elements of $X(T)^+$ as follows. For $\lambda \in X(T)$, let F_λ be the one-dimensional B-module defined by λ. Let $H^0(\lambda)$ denote the K-module induced from F_λ. Then $H^0(\lambda) \neq 0$ if and only if $\lambda \in X(T)^+$ (see [7, Proposition II.2.6]). When $F = \mathbb{C}$, $H^0(\lambda)$ is a simple K-module, but when $F = k$, it is only the case that $H^0(\lambda)$ contains a unique simple submodule ([7, Corollary II.2.3]), which we denote $L(\lambda)$. If $\mu \in X(T)$ is a weight of T on $L(\lambda)$, then $\mu \leqslant \lambda$ ([7, Proposition II.2.4]), and $L(\lambda)$ is thus called the simple K-module of highest weight λ. The dual K-module $L(\lambda)^*$ is isomorphic to $L(-w_0(\lambda))$ (see [7, Corollary II.2.5]), where w_0 is the longest element in W.

Suppose $F = k$. By [7, Corollary II.5.6], we have that if $\lambda \in C_p \cap X(T)^+$, then $L(\lambda) = H^0(\lambda)$. This leads to the following known result.

Proposition 1.3.1. *If M is a K-module with the property that the highest weights of its composition factors all lie in $C_p \cap X(T)^+$, then M is semisimple.*

Proof. Suppose $L(\lambda)$ and $L(\mu)$ are any two composition factors of M. Then $L(\lambda) = H^0(\lambda)$ and $L(\mu) = H^0(\mu)$, so that by [7, Proposition II.2.14],

$$\mathrm{Ext}^1_K(L(\lambda), L(\mu)) = 0.$$

Therefore, M is the direct sum of its composition factors and thus semisimple. □

2. Invariant theory

Let $F[\mathfrak{p}]$ be the ring of polynomial functions defined on $\mathfrak{p}$ with coefficients in the field F. Formally, $F[\mathfrak{p}]$ is $S(\mathfrak{p}^*)$, the symmetric algebra on $\mathfrak{p}^*$. The action of K on $F[\mathfrak{p}]$ is given by $(h \cdot f)(x) = f(h^{-1} \cdot x)$ for $h \in K, f \in F[\mathfrak{p}]$, and $x \in \mathfrak{p}$. The following subrings of $F[\mathfrak{p}]$ will also play a role: $F[\mathfrak{p}]^K$ denotes the subring of K-invariant polynomials on $\mathfrak{p}$, J the subring of $F[\mathfrak{p}]^K$ consisting of polynomials with 0 constant term, and $\langle J \rangle$ the ideal generated by J in $F[\mathfrak{p}]$. Each of these rings has an $\mathbb{N}$-grading given by homogeneous degree, and the homogeneous parts are denoted $F_n[\mathfrak{p}]$, $F_n[\mathfrak{p}]^K$, J_n, and $\langle J \rangle_n$, respectively. We will also need the algebras $F[\mathfrak{a}] = S(\mathfrak{a}^*)$ and $F[\mathfrak{a}]^{W_A}$, the W_A-invariant polynomials in $F[\mathfrak{a}]$. Finally, if $\mathbf{V}$ is an affine variety in F^m, then $\mathcal{I}(\mathbf{V})$ will denote the ideal of polynomials in $F[x_1, ..., x_m]$ which vanish on $\mathbf{V}$.

2.1. Review of known results

The following results are well-known for both $F = \mathbb{C}$ and $F = k$.

(IT1) The nullcone $\mathcal{N}$ is an affine K-variety which is the zero-set of the ideal $\langle J \rangle$. (This is established in [9] when $F = \mathbb{C}$ and in [10] when $F = k$.)

(IT2) The algebra $F[\mathfrak{p}]^K$ is generated by finitely many algebraically independent homogeneous polynomials $f_1, ..., f_r$, where $r = \dim(\mathfrak{a})$. Further, we have the identity

$$\sum_{w \in W_A} z^{l(w)} = \prod_{i=1}^{r} \frac{1 - z^{d_i}}{1 - z},$$

where d_i is the degree of f_i for $1 \leq i \leq r$. (See [10, Theorem 4.9 and Lemma 4.11] for a proof that holds for both $F = \mathbb{C}$ and $F = k$.)

(IT3) The list of degrees in (IT2) is the same whether $F = \mathbb{C}$ or $F = k$. (See [2, V.5.1, Corollary].)

(IT4) The natural embedding $\mathfrak{a} \to \mathfrak{p}$ induces an isomorphism $F[\mathfrak{p}]^K \to F[\mathfrak{a}]^{W_A}$. (See, for example, [5, Theorem 12.4.5] when $F = \mathbb{C}$ and [10, Theorem 4.9] when $F = k$.)

(IT5) The algebra $F[\mathfrak{a}]$ is a free $F[\mathfrak{a}]^{W_A}$-module. (This follows when $\mathrm{char}(F) \neq 2$ from [3, Theorem 2c and Corollary to Theorem 2], identifying $F[\mathfrak{a}]$ with $S(X(A) \otimes_{\mathbb{Z}} F)$.)

2.2. The module of covariants is free

The algebra $F[\mathfrak{p}]$, when considered as an $F[\mathfrak{p}]^K$-module, is called the *module of covariants*. The following proposition (already known when $F = \mathbb{C}$), is a special case of [1, Proposition 1.1], the proof of which does not depend on $\mathrm{char}(F)$.

Proposition 2.2.1. *If $F = \mathbb{C}$ or $F = k$, then $F[\mathfrak{p}]$ is a free module over $F[\mathfrak{p}/\mathfrak{a}] \otimes F[\mathfrak{p}]^K$. In particular, $F[\mathfrak{p}]$ is free over $F[\mathfrak{p}]^K$.*

Proof. Let V be a vector space over F, and let H be a subspace of V. Let I be a graded subalgebra of $F[V]$. Proposition 1.1 in [1] states that if:

(i) the restriction morphism $\mathrm{res} : I \to F[H]$ is injective, and
(ii) $F[H]$ is a free module over $\mathrm{res}(I)$,

then $F[V]$ is a free module over $F[V/H] \otimes I$. Letting $V = \mathfrak{p}$, $H = \mathfrak{a}$, and $I = F[\mathfrak{p}]^K$, we have that (i) holds by (IT4) and (ii) holds by (IT5). □

3. Separation of variables

In this section, we recall the Kostant and Rallis' Separation of Variables Theorem from [9] and then adapt the analogous decomposition of $F[\mathfrak{g}]$ (for $F = k$) given by Friedlander and Parshall in [4] to $F[\mathfrak{p}]$.

3.1. Decomposition in characteristic 0

Let $F = \mathbb{C}$. Kostant and Rallis' Separation of Variables Theorem ([9, Theorem 15]) states that there is a graded K-submodule $H = \bigoplus H_n$ of $F[\mathfrak{p}]$ such that the map $F[\mathfrak{p}]^K \otimes H \to F[\mathfrak{p}]$ defined by $f \otimes g \mapsto fg$ is an isomorphism of K-modules.

By [9, Theorem 14] (using $\xi = 0$), $\mathcal{I}(\mathcal{N}) = \langle J \rangle$, and by [9, Lemma 18], $F[\mathfrak{p}] = \langle J \rangle \oplus H$. These results allow us to identify H with $F[\mathcal{N}]$ as graded K-modules.

3.2. Decomposition in good characteristic

Now let $F = k$. We would like to recover Kostant and Rallis' separation of variables decomposition, but the methods they used depend on the semisimplicity of $\mathbb{C}[\mathfrak{p}]$ as a K-module and thus do not work in this case. We can, however, obtain a partial version of the decomposition given the following setup.

Recall that T is a fixed maximal torus of K and that C_p denotes the standard alcove for W_p in E_X, where $p = \mathrm{char}(F)$. Let $\mathcal{S}$ be the set of maximal weights relative to $\preccurlyeq$ of T on $\mathfrak{p}$, and let N_p be the largest non-negative integer such that $N_p\delta \in C_p$ for all $\delta \in \mathcal{S}$. The maximal weights of T on the dual $\mathfrak{p}^*$ are precisely those weights of the form $-w_0(\delta)$ for $\delta \in \mathcal{S}$. Then by adapting [4, Proposition 4.4], we have the following.

Proposition 3.2.1. *For $n \leq N_p$, $F_n[\mathfrak{p}]$ is a semisimple* K-*module.*

Proof. The possible composition factors of $F_n[\mathfrak{p}] = S^n(\mathfrak{p}^*)$ are the duals of those of $S^n(\mathfrak{p})$. In other words, the composition factors of $F_n[\mathfrak{p}]$ are the simple K-modules $L(\lambda)^* \cong L(-w_0(\lambda))$, where λ is a dominant weight such that $\lambda \preccurlyeq n\delta$ for some $\delta \in \mathcal{S}$. Since $n\delta \in C_p \cap X(T)^+$, so is any such λ. Because $-w_0(\rho) = \rho$, $-w_0(C_p) = C_p$, which means that if $\lambda \preccurlyeq n\delta$ for $\delta \in \mathcal{S}$, then $-w_0(\lambda) \in C_p \cap X(T)^+$, and the result now follows from Proposition 1.3.1. □

For each $n \geq 0$, let H_n be a subspace of $F_n[\mathfrak{p}]$ such that $F_n[\mathfrak{p}] \cong \langle J \rangle_n \oplus H_n$, and let $H = \bigoplus_{n \geq 0} H_n$. Then

$$F[\mathfrak{p}] \cong \langle J \rangle \oplus H. \tag{1}$$

We are now able to partially recover the decomposition of $F[\mathfrak{p}]$ from the previous subsection. In the following, by $(F[\mathfrak{p}]^K \otimes H)_n$ we mean the sum $\bigoplus_{i=0}^n (F_i[\mathfrak{p}]^K \otimes H_{n-i})$.

Proposition 3.2.2. *For $n \leq N_p$, the map $(F[\mathfrak{p}]^K \otimes H)_n \to F_n[\mathfrak{p}]$ defined by $f \otimes g \mapsto fg$ is a K-module isomorphism.*

Proof. When $n \leq N_p$, we can choose H_n to be a K-submodule of $F_n[\mathfrak{p}]$ by Proposition 3.2.1. Thus $(F[\mathfrak{p}]^K \otimes H)_n$ is a K-module. The proof of [9, Lemma 19] works in our current context, and using (1) also, we can conclude that $F[\mathfrak{p}] = F[\mathfrak{p}]^K H$. Because $F[\mathfrak{p}]$ is a free $F[\mathfrak{p}]^K$-module by Proposition 2.2.1, we therefore have that the map

$$F[\mathfrak{p}]^K \otimes H \to F[\mathfrak{p}]$$

defined by $f \otimes g \mapsto fg$ is an isomorphism of graded vector spaces. Since H_n is K-stable when $n \leq N_p$, the vector space isomorphism $F_n[\mathfrak{p}] \cong \langle J \rangle_n \oplus H_n$ is thus actually a K-module isomorphism, and the map $(F[\mathfrak{p}]^K \otimes H)_n \to F_n[\mathfrak{p}]$ for $n \leq N_p$ is also a K-module isomorphism. □

We can also partially recover Kostant and Rallis' identification of H with $F[\mathcal{N}]$ thanks to the following.

Proposition 3.2.3. *Let $F = k$. The ideal $\langle J \rangle$ is radical in $F[\mathfrak{p}]$. In particular, $\mathcal{I}(\mathcal{N}) = \langle J \rangle$.*

Proof. By [10, Theorem 5.1], each irreducible component of $\mathcal{N}$ has codimension $r = \dim(\mathfrak{a})$ in $\mathfrak{p}$ and contains an open dense K-orbit. This is a version of [9, Theorem 9] that holds when $F = k$. Also, [10, Corollary 6.31] extends [9, Theorem 13] to $F = k$, proving that the differentials of the generating polynomials of $F[\mathfrak{p}]^K$ are linearly independent at each regular element $x \in \mathfrak{p}$. The proof of [9, Theorem 14], which uses a characteristic-independent commutative algebra argument, thus carries over to $F = k$. □

Consequently, H is isomorphic to $F[\mathcal{N}]$ as a graded vector space, and H_n is isomorphic to $F_n[\mathcal{N}]$ as a K-module when $n \leq N_p$.

Example 3.2.4. We calculate the values of N_p here in a few specific cases to give a sense for the range of n-values for which Proposition 3.2.1, Proposition 3.2.2, and, later, Theorem 4.3.1 hold. Below, e_i denotes the ith coordinate function on the standard maximal torus T in K, and J_t is the $t \times t$ matrix with 1's on the skew diagonal and 0's elsewhere.

1. Let $G = \mathrm{GL}_n(F)$, $n \geq 3$, with a Type AI involution θ, which is defined by $\theta(g) = J_n (g^{-1})^\top J_n$. Then $K = (G^\theta)^\circ = \mathrm{SO}_n(F)$. The rank of K is $\lfloor \frac{n}{2} \rfloor$, which we will denote by l. If n is odd, then Φ is of Type B_l, and if n is even, then Φ is of Type D_l. Either way, $\Phi_\mathfrak{p} = \Phi \cup \{\pm 2e_1, \ldots, \pm 2e_l\} \cup \{0\}$, with unique maximal weight $\delta = 2e_1$. When n is odd,

$$\rho = \left(l - \frac{1}{2}\right)e_1 + \left(l - \frac{3}{2}\right)e_2 + \cdots + \frac{3}{2}e_{l-1} + \frac{1}{2}e_l,$$

and when n is even,

$$\rho = (l-1)e_1 + (l-2)e_2 + \cdots + 2e_{l-2} + e_{l-1}.$$

For a non-negative integer m, the largest integer value of $\langle m\delta + \rho, \alpha^\vee\rangle$ for $\alpha \in \Phi^+$ is thus $2m + n - 3$, meaning that

$$N_p = \begin{cases} \lfloor (p-n+3)/2 \rfloor & \text{if } 3 \le n \le p+3 \\ 0 & \text{if } n > p+3 \end{cases}.$$

2. Let $G = \mathrm{SO}_{2n}(F)$ (relative to the form defined by J_{2n}), $n \ge 3$, with a Type BDI involution θ, given by $\theta(g) = J_{2n}gJ_{2n}$. Then $K \cong \mathrm{S}(\mathrm{O}_n(F) \times \mathrm{O}_n(F))$, which consists of the block diagonal matrices $\mathrm{diag}(g_1, g_2)$, where g_1 and g_2 are in $\mathrm{O}_n(F)$ and $\det(g_1)\det(g_2) = 1$. By Remark 1.2.2, $\Phi_{G,T} = \Phi_G$ (which is of Type D_n) and thus

$$\Phi_{\mathfrak{p}} = \{e_i + e_j, -e_i - e_j \ : \ 1 \le i < j \le n\}$$

with unique maximal weight $\delta = e_1 + e_2$. We also have that $\Phi_{G,T}^+ = \Phi_G^+$, so using the method for obtaining Φ^+ from $\Phi_{G,T}^+$ discussed in Section 1, we have

$$\Phi^+ = \{e_i - e_j \ : \ 1 \le i < j \le n\},$$

and therefore

$$\rho = \left(\frac{n-1}{2}\right)e_1 + \left(\frac{n-3}{2}\right)e_2 + \cdots + \left(\frac{3-n}{2}\right)e_{n-1} + \left(\frac{1-n}{2}\right)e_n.$$

The largest value of $\langle m\delta + \rho, \alpha^\vee\rangle$ for $\alpha \in \Phi^+$ is $m + n - 1$. Thus,

$$N_p = \begin{cases} p - n + 1 & \text{if } 3 \le n \le p+1 \\ 0 & \text{if } \ n > p+1 \end{cases}.$$

4. A multiplicity formula for $k[\mathcal{N}]$

In this section, we derive a formula, first under the assumption that $F = \mathbb{C}$, for the multiplicity $m_n(\lambda)$ of a simple K-module $L(\lambda)$ as a composition factor of $F_n[\mathcal{N}]$. We will then show that this formula also holds when $F = k$ and $n \le N_p$, where N_p is as in Section 3.

4.1. Formal characters

In this subsection, F can be either $\mathbb{C}$ or k. Recalling that T is a fixed maximal torus of K, we can define the formal character of a T-module M as an element in the group ring $\mathbb{Z}[X(T)]$ by

$$\mathrm{ch}(M) = \sum_{\lambda \in X(T)} \dim(M_\lambda)e^\lambda,$$

where the elements e^λ, with $\lambda \in X(T)$, are the basis elements of $\mathbb{Z}[X(T)]$, and $e^\lambda e^\mu = e^{\lambda+\mu}$. If $V = \oplus_{n \ge 0} V_n$ is an $\mathbb{N}$-graded T-module, then we let $\mathrm{ch}_z(V) = \sum_{n\ge 0} \mathrm{ch}(V_n)z^n$.

For T-modules M and M', $\mathrm{ch}(M \otimes M') = \mathrm{ch}(M)\mathrm{ch}(M')$ and $\mathrm{ch}(M \oplus M') = \mathrm{ch}(M) + \mathrm{ch}(M')$. It follows that for graded T-modules V and V', $\mathrm{ch}_z(V \otimes V') = \mathrm{ch}_z(V)\mathrm{ch}_z(V')$.

By Lemma 1.2.1,

$$\mathrm{ch}(\mathfrak{p}) = Re^0 + \sum_{\chi \in \Phi_{\mathfrak{p}} \setminus \{0\}} e^{\chi}, \tag{2}$$

where $R = \mathrm{rank}(G) - \mathrm{rank}(K)$.

Lemma 4.1.1. *For the $\mathbb{N}$-grading of $F[\mathfrak{p}]$ given by homogeneous degree,*

$$\mathrm{ch}_z(F[\mathfrak{p}]) = (1-z)^{-R} \prod_{\chi \in \Phi_{\mathfrak{p}} \setminus \{0\}} (1 - e^{\chi} z)^{-1}.$$

Proof. For a T-module M with symmetric algebra $S(M)$, it is straightforward to show that if $\mathrm{ch}(M) = \sum c_i e^{\lambda_i}$, then $\mathrm{ch}_z(S(M)) = \prod (1 - e^{-\lambda_i} z)^{-c_i}$. Since $\mathfrak{p}$ is self-dual, the formula for $\mathrm{ch}_z(F[\mathfrak{p}])$ now follows immediately from (2). □

By (IT2) in Section 2, $F[\mathfrak{p}]^K$ is generated by r independent homogeneous polynomials, where $r = \dim(\mathfrak{a})$. This, combined with the fact that K acts trivially on $F[\mathfrak{p}]^K$, leads immediately to the following lemma.

Lemma 4.1.2. *Let $d_1, \ldots, d_r$ be the degrees of the* r *polynomials which generate $F[\mathfrak{p}]^K$. Then*

$$\mathrm{ch}_z(F[\mathfrak{p}]^K) = \prod_{i=1}^{r} (1 - z^{d_i})^{-1}.$$

4.2. Multiplicity in characteristic 0

Suppose $F = \mathbb{C}$. For each $n \in \mathbb{N}$,

$$\mathrm{ch}(F_n[\mathcal{N}]) = \sum_{\lambda \in X(T)^+} m_n(\lambda) \mathrm{ch}(L(\lambda)).$$

Thus for a fixed $\lambda \in X(T)^+$, the formula for $m_n(\lambda)$ depends on $\mathrm{ch}_z(F[\mathcal{N}])$.

Proposition 4.2.1. *The formal characters of $F_n[\mathcal{N}]$ for $n \geq 0$ are given by*

$$\mathrm{ch}_z(F[\mathcal{N}]) = \left(\sum_{w \in W_A} z^{l(w)} \right) (1-z)^{r-R} \prod_{\chi \in \Phi_{\mathfrak{p}} \setminus \{0\}} (1 - e^{\chi} z)^{-1}.$$

Proof. Thanks to the K-module isomorphisms $F[\mathfrak{p}]^K \otimes H \cong F[\mathfrak{p}]$ and $H \cong F[\mathcal{N}]$ from Subsection 3.1, $\mathrm{ch}_z(F[\mathcal{N}]) = \mathrm{ch}_z(F[\mathfrak{p}])/\mathrm{ch}_z(F[\mathfrak{p}]^K)$. The proposition thus follows immediately from Lemmas 4.1.1 and 4.1.2 along with (IT2). □

Now define an endomorphism D of $\mathbb{Z}[X(T)]$ by

$$D(e^{\lambda}) = \sum_{w \in W} \det(w) e^{w(\lambda)}.$$

For $\lambda \in X(T)^+$, $\mathrm{ch}(L(\lambda)) = D(e^{\lambda+\rho})/D(e^{\rho})$ by the Weyl Character Formula. We can extend D to an endomorphism of $\mathbb{Z}[X(T)][[z]]$ by defining $w(z) = z$ for all $w \in W$. The following property of D is easily obtained for any $w \in W$ and $\lambda \in X(T)$:

$$w(D(e^{\lambda})) = \det(w) D(e^{\lambda}). \tag{3}$$

Let $W(z) = \prod_{i=1}^{r} \frac{1-z^{d_i}}{1-z}$, where r and d_i are as in (IT2). For $n \geq 0$ and $\lambda \in X(T)$, define the integers $p_n(\lambda)$ by:

$$W(z)(1-z)^{r-R}\prod_{\chi\in\Phi_{\mathfrak{p}}\setminus\{0\}}(1-e^{\chi}z)^{-1}=\sum_{n\geq 0}\sum_{\lambda\in X(T)}p_n(\lambda)e^{\lambda}z^n. \tag{4}$$

The function p_n is analogous to the Kostant partition function and can be described combinatorially given the following setup. For $1\leq i\leq r$, let $X_i=\{0,1,...,d_i-1\}$. For $l\geq 0$, let s_l be the number of maps $\phi:\{1,...,r\}\to\sqcup_{i=1}^{r}X_i$ such that $\phi(i)\in X_i$ for all i and $\sum_{i=1}^{r}\phi(i)=l$.

For $m\geq 0$ and $\lambda\in X(T)$, let $t_m(\lambda)$ be the number of maps $\psi:\Phi_{\mathfrak{p}}\setminus\{0\}\to\mathbb{N}\cup\{0\}$ such that $\lambda=\sum_{\chi\in\Phi_{\mathfrak{p}}}\psi(\chi)\chi$ and $\sum_{\chi\in\Phi_{\mathfrak{p}}}\psi(\chi)=m$. Finally, let $q_n(\lambda)=\sum_{j=0}^{n}s_jt_{n-j}(\lambda)$.

Note that since $\{t\in T_G\ :\ \theta(t)=t^{-1}\}$ is contained in a conjugate of A, and since $\mathfrak{p}_0=\mathrm{Lie}(T_G)\cap\mathfrak{p}$, we have that $r\geq R$.

Proposition 4.2.2. *For $n\geq 0$ and $\lambda\in X(T)$,*

$$p_n(\lambda)=\sum\nolimits_{i=0}^{n}(-1)^i\binom{r-R}{i}q_{n-i}(\lambda).$$

where we follow the convention that $\binom{r-R}{i}=0$ when $i>r-R$.

Proof. Using the fact that $\frac{1-z^{d_i}}{1-z}=1+z+z^2+\cdots+z^{d_i-1}$ for $1\leq i\leq r$, we have that

$$W(z)=\sum_{l=0}^{d_1+\cdots+d_r-r}s_lz^l. \tag{5}$$

By expanding $(1-e^{\chi}z)^{-1}$ as $1+e^{\chi}z+e^{2\chi}z^2+\cdots$ for each $\chi\in\Phi_{\mathfrak{p}}\setminus\{0\}$, we can see that

$$\prod_{\chi\in\Phi_{\mathfrak{p}}\setminus\{0\}}(1-e^{\chi}z)^{-1}=\sum_{m\geq 0}\sum_{\lambda\in X(T)}t_m(\lambda)e^{\lambda}z^m. \tag{6}$$

It follows from (5) and (6) that

$$W(z)\prod_{\chi\in\Phi_{\mathfrak{p}}\setminus\{0\}}(1-e^{\chi}z)^{-1}=\sum_{n\geq 0}\sum_{\lambda\in X(T)}q_n(\lambda)e^{\lambda}z^n. \tag{7}$$

Multiplying both sides of (7) by $(1-z)^{r-R}$ gives the formula for $p_n(\lambda)$. □

Remark 4.2.3. The numbers s_l defined in Proposition 4.2.2 can also be obtained by using the fact that $W(z)=\sum_{w\in W_A}z^{l(w)}$.

Lemma 4.2.4. *For D and p_n as above,*

$$\mathrm{ch}_z(F[\mathcal{N}])D(e^{\rho})=\sum_{n\geq 0}\sum_{\lambda\in X(T)}p_n(\lambda)D(e^{\lambda+\rho})z^n.$$

Proof. Let

$$q=\prod_{\chi\in\Phi_{\mathfrak{p}}\setminus\{0\}}(1-e^{-\chi}z)^{-1}.$$

Since the elements of W permute $\Phi_{\mathfrak{p}}$, $D(q)=q$. We can identify $W(z)$ with the element $W(e^0z)$ in $\mathbb{Z}[X(T)][[z]]$. It is clear then that $D(e^{\rho})W(z)=D(e^{\rho}W(z))$. Thus

$$\begin{aligned}\mathrm{ch}_z(F[\mathcal{N}])D(e^{\rho})&=D(q)D(e^{\rho}W(z))\\&=D(e^{\rho}W(z)q),\end{aligned}$$

and the lemma now follows from the definition of p_n. □

We are now ready for one of our main results.

Theorem 4.2.5. *When $F=\mathbb{C}$, for $n\in\mathbb{N}$ and $\lambda\in X(T)^+$, the multiplicity of the simple K-module $L(\lambda)$ in $F_n[\mathcal{N}]$ is given by*

$$m_n(\lambda)=\sum_{w\in W}\det(w)p_n(w\cdot\lambda),$$

where p_n is defined as in (4).

Proof. Let C_0 be as in Section 1. Lemma 4.2.4 and the property stated in (3) imply that for a fixed n,

$$\begin{aligned}\mathrm{ch}(F_n[\mathcal{N}])&=\sum_{\lambda\in X(T)}p_n(\lambda)D(e^{\lambda+\rho})/D(e^{\rho})\\&=\sum_{\lambda\in C_0}\sum_{w\in W}p_n(w(\lambda+\rho)-\rho)D(e^{w(\lambda+\rho)})/D(e^{\rho})\\&=\sum_{\lambda\in C_0}\sum_{w\in W}\det(w)p_n(w(\lambda+\rho)-\rho)D(e^{\lambda+\rho})/D(e^{\rho}).\end{aligned}$$

Since $\mathrm{ch}(L(\lambda))=D(e^{\lambda+\rho})/D(e^{\rho})=0$ when $\lambda\in C_0\setminus X(T)^+$, we now have

$$\mathrm{ch}(F_n[\mathcal{N}])=\sum_{\lambda\in X(T)^+}\sum_{w\in W}\det(w)p_n(w\cdot\lambda)\mathrm{ch}(L(\lambda)),$$

and the formula for $m_n(\lambda)$ follows. □

4.3. Multiplicity in good characteristic

Now assume $F=k$. The proof of the following theorem very closely follows that of [4, Proposition 4.4(ii)].

Theorem 4.3.1. *Let N_p be as in Section 3, and suppose $n\le N_p$. Then for $\lambda\in X(T)^+$, the formula for the multiplicity of $L(\lambda)$ in $F_n[\mathcal{N}]$ agrees with the formula in the case that $F=\mathbb{C}$. In other words, when $n\le N_p$,*

$$m_n(\lambda)=\sum_{w\in W}\det(w)p_n(w\cdot\lambda).$$

Proof. Let F be any field, and let $\mathscr{K}_F$ be the category of finite-dimensional rational K-modules over F. Let $G_0(\mathscr{K}_F)$ be the corresponding Grothedieck group. The class in $G_0(\mathscr{K}_F)$ containing a K-module V_F is denoted $[V_F]$. The simple K-modules for both $F=k$ and $F=\mathbb{C}$ are precisely the highest weight modules $L(\lambda)_F$ with $\lambda\in X(T)^+$. Thus the free Abelian groups $G_0(\mathscr{K}_k)$ and $G_0(\mathscr{K}_\mathbb{C})$ have bases $\{[L(\lambda)_k]\ :\ \lambda\in X(T)^+\}$ and $\{[L(\lambda)_\mathbb{C}]\ :\ \lambda\in X(T)^+\}$, respectively. The map $[L(\lambda)_k]\mapsto[L(\lambda)_\mathbb{C}]$ for $\lambda\in X(T)^+$ thus defines an isomorphism $G_0(\mathscr{K}_k)\cong G_0(\mathscr{K}_\mathbb{C})$.

Identifying $G_0(\mathscr{K}_k)$ with $G_0(\mathscr{K}_\mathbb{C})$, we can obtain the desired equality of multiplicities for k and $\mathbb{C}$ if we can show that $[k_n[\mathcal{N}]]=[\mathbb{C}_n[\mathcal{N}]]$ for $n\le N_p$. (Recall that in this case, $L(\lambda)_k=H^0(\lambda)$, so that $\mathrm{ch}(L(\lambda)_k)=\mathrm{ch}(L(\lambda)_\mathbb{C})$.) This can be done by induction on n, with the case $n=0$ being trivial since $[k]=[\mathbb{C}]=0$ in $G_0(\mathscr{K}_k)=G_0(\mathscr{K}_\mathbb{C})$. Suppose for some $n\le N_p$ that $[k_j[\mathcal{N}]]=[\mathbb{C}_j[\mathcal{N}]]$ for all $j<n$. As shown in the proof of Proposition 3.2.1, the weights of the composition factors of $k_n[\mathfrak{p}]$ lie in $C_p\cap X(T)^+$, and since Lemma 4.1.1 is true for both $F=k$ and $F=\mathbb{C}$, the formal character of $k_n[\mathfrak{p}]$ coincides with that of $\mathbb{C}_n[\mathfrak{p}]$. Thus $[k_n[\mathfrak{p}]]=[\mathbb{C}_n[\mathfrak{p}]]$. By Proposition 3.2.2, which is stated for $F=k$ but which also holds for $F=\mathbb{C}$, $[F_n[\mathfrak{p}]]=\oplus_{i=0}^n\dim(J_i)_F[F_{n-i}[\mathcal{N}]]$. By (IT3) in Section 2, $\dim(J_i)$ is the same whether $F=k$ or $F=\mathbb{C}$ for all i, and therefore by the induction hypothesis we can conclude that $[k_n[\mathcal{N}]]=[\mathbb{C}_n[\mathcal{N}]]$ for $n\le N_p$. □

Remark 4.3.2. The ideas used in the proof of Theorem 4.3.1 were developed by Friedlander and Parshall in [4] to derive a multiplicity formula for $k_n[\mathcal{N}(\mathfrak{g})]$, where $\mathcal{N}(\mathfrak{g})$ is the nullcone in $\mathfrak{g}$, for all n up to an upper bound when $\mathrm{char}(k)$ is a good prime. In [8, Section 8], Jantzen used the Springer resolution of $\mathcal{N}(\mathfrak{g})$ to extend the multiplicity formula to all n. However, since the proof used there relies on the fact that $\mathcal{N}(\mathfrak{g})$ is a normal variety, it cannot be directly adapted to our current context because it is not necessarily true that $\mathcal{N}$ is normal. For example, when $G = \mathrm{SL}_2(k)$ with $\theta(g) = (g^{-1})^\top$, $\mathcal{N}$ is the variety in k^2 defined by the polynomial $x^2 + y^2$.

Example 4.3.3. Let $F = \mathbb{C}$, and let $G = \mathrm{GL}_4(F)$. Let θ be an involution of Type AII, that is, $\theta(g) = J^{-1}(g^{-1})^\top J$, where

$$J = \begin{bmatrix} 0 & I_2 \\ -I_2 & 0 \end{bmatrix},$$

and I_2 is the 2×2 identity matrix. Then $K = \mathrm{Sp}_4(F)$ and $\mathfrak{p}$ is the space of 4×4 skew-Hamiltonian matrices, which are those matrices of the form

$$\begin{bmatrix} A & B \\ C & A^\top \end{bmatrix},$$

where $B^\top = -B$ and $C^\top = -C$. A maximal θ-split torus in G is $A = \{\mathrm{diag}(a, b, a, b) \ : \ a, b \in F^\times\}$, and the subgroup T_G of invertible diagonal matrices in G is a θ-stable maximal torus which contains A. A maximal torus of K is $T = T_G \cap K = \{\mathrm{diag}(a, b, a^{-1}, b^{-1}) \ : \ a, b \in F^\times\}$.

The root system Φ_G of G relative to T_G is of Type A_3, and the root system Φ of K relative to T is of Type C_2, with Weyl group W isomorphic to the dihedral group D_4 of order 8. Restricting Φ_G to A gives a root system Φ_A of Type A_1 with Weyl group $W_A \cong \mathbb{Z}/2\mathbb{Z}$. We will treat Φ as a subset of $\mathbb{R}^2$ with the standard ordered basis (e_1, e_2), where the simple roots are $e_1 - e_2$ and $2e_2$. The other roots are $\pm 2e_1, \pm(e_1 + e_2), -(e_1 - e_2)$, and $-2e_2$. The half-sum of the positive roots is $\rho = 2e_1 + e_2$, and $X(T)^+ = \{(x, y) \in \mathbb{Z}(e_1 - e_2) \oplus \mathbb{Z}(2e_2) \ : \ 0 \le y \le x\}$.

The set of weights of T on $\mathfrak{g}^{(K)} = \mathfrak{gl}_n(F)$ is $\Phi_{G,T} = \Phi \cup \{0\}$. The elements $\chi \in \Phi \cup \{0\}$ whose weight spaces $\mathfrak{g}_\chi^{(K)}$ intersect $\mathfrak{p}$ nontrivially constitute the set $\Phi_\mathfrak{p}$ of weights of T on $\mathfrak{p}$. Here then, $\Phi_\mathfrak{p} = \{0, \pm(e_1 - e_2), \pm(e_1 + e_2)\}$, and the corresponding weight spaces are

$$\begin{aligned}
\mathfrak{p}_0 &= \mathfrak{a} = \mathrm{Span}(E_{11} + E_{33}, E_{22} + E_{44}), \\
\mathfrak{p}_{e_1 - e_2} &= \mathrm{Span}(E_{14} - E_{23}), \\
\mathfrak{p}_{e_1 + e_2} &= \mathrm{Span}(E_{12} + E_{43}), \\
\mathfrak{p}_{-(e_1 - e_2)} &= \mathrm{Span}(E_{32} - E_{41}), \text{and} \\
\mathfrak{p}_{-(e_1 + e_2)} &= \mathrm{Span}(E_{21} + E_{34}),
\end{aligned}$$

where E_{ij} is a standard basis matrix in $\mathfrak{g}$.

Since $|W_A| = 2$, we have that $W(z) = \sum_{w \in W_A} z^{l(w)} = 1 + z$. Now using Proposition 4.2.2 (with $r = R$ since $\mathfrak{p}_0 = \mathfrak{a}$), it can be shown that if λ is of the form $\lambda = c_1(e_1 - e_2) + c_2(e_1 + e_2) + c_3(-(e_1 - e_2)) + c_4(-(e_1 + e_2))$ for $c_1, c_2, c_3, c_4 \in \mathbb{N} \cup \{0\}$, then

$$p_n(\lambda) = \left\lceil \frac{n+1}{2} \right\rceil - \left\lceil \frac{c_1 + c_2 + c_3 + c_4}{2} \right\rceil. \tag{8}$$

Otherwise, $p_n(\lambda) = 0$ for all n.

Now Theorem 4.2.5 with p_n given by (8) and $W \cong D_4$ yields

$$m_n(\lambda) = \begin{cases} 1 & \text{if } \lambda = n(e_1 + e_2) \\ 0 & \text{otherwise} \end{cases}.$$

Therefore, $F_n[\mathcal{N}] \cong L(n(e_1 + e_2))$ for all $n \geq 0$.

Now suppose $F = k$ with $\mathrm{char}(k) = p$. The unique maximal weight in $\Phi_{\mathfrak{p}}$ is $e_1 + e_2$. For a non-negative integer m, the largest possible value of $\langle m(e_1 + e_2) + \rho, \alpha^\vee \rangle$ for $\alpha \in \Phi^+$ is $2m + 4$. By Theorem 4.3.1, we can thus conclude that $F_n[\mathcal{N}] \cong L(n(e_1 + e_2))$ when $0 \leq n \leq \lfloor \frac{p-4}{2} \rfloor$.

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