How to Find an Example

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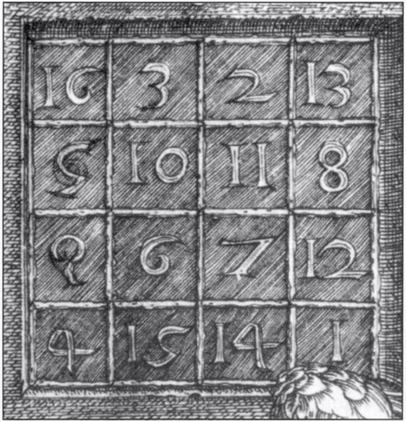
Outline

Be creative!

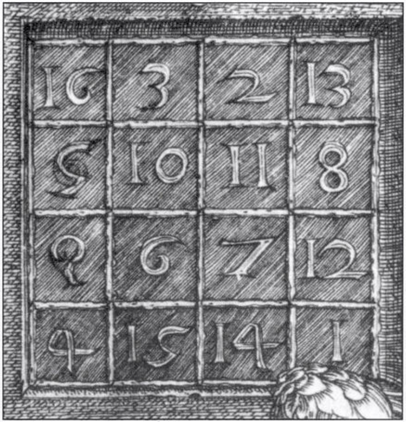
Albrecht Duerer,
Melancholia, 1514

Magic Square: Definition



Magic Square: Definition

- $1, 2, 3, \ldots, 15, 16$



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- $1, 2, 3, \ldots, 15, 16$
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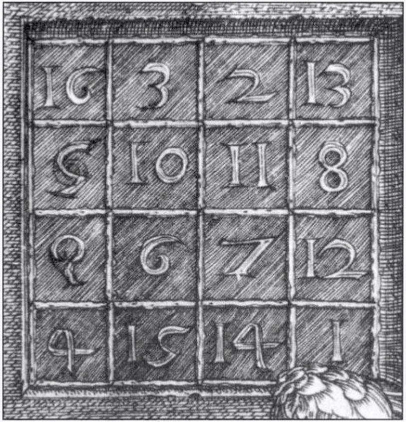
  for $n \times n$



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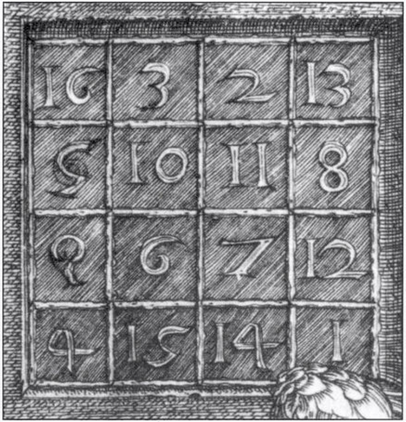
- $1, 2, 3, \ldots, n^2$ for $n \times n$
- the same sum in columns, rows, diagonals



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- what about size 3 – made of $1, 2, 3, \ldots, 9$?

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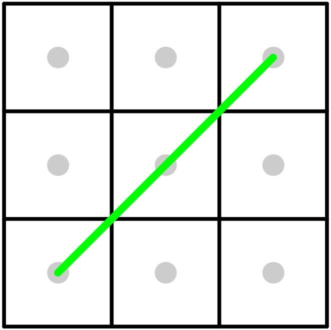
- $45/3 = 15$

Hint: the Center



summing up four lines…

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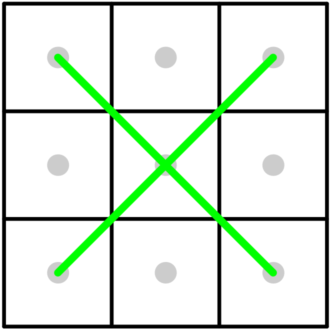

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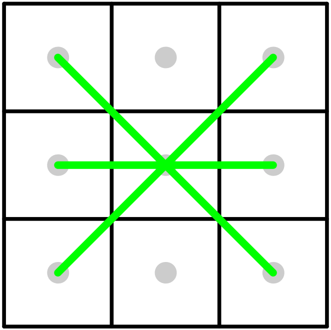

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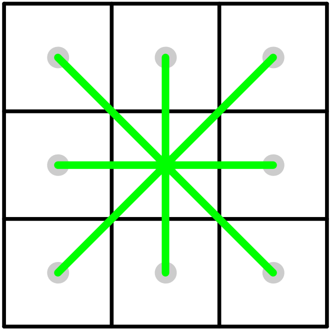

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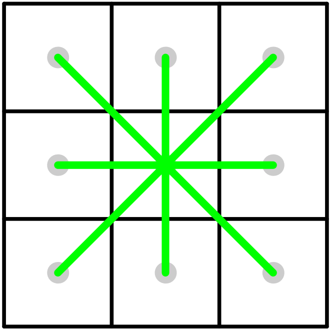

summing up four lines…

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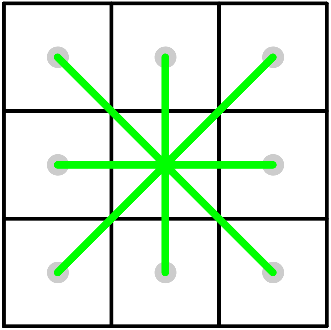

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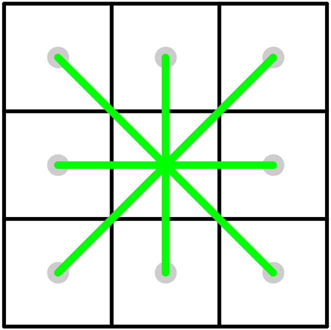

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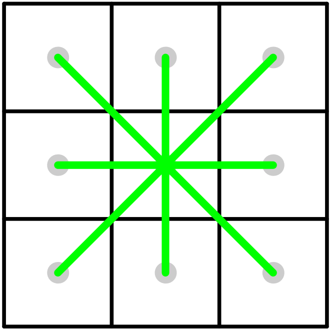

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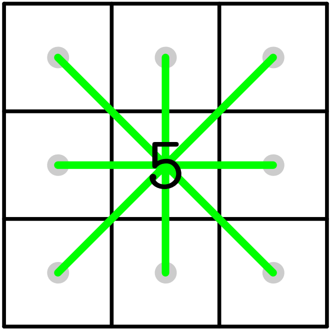

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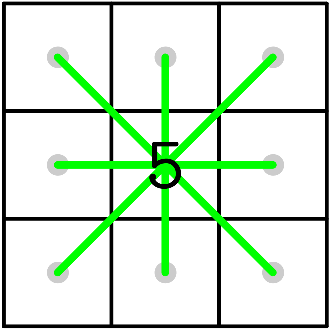

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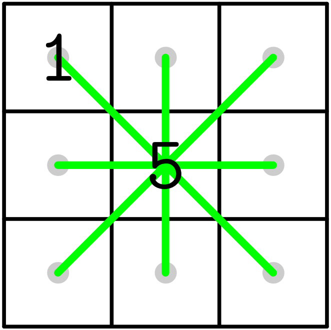

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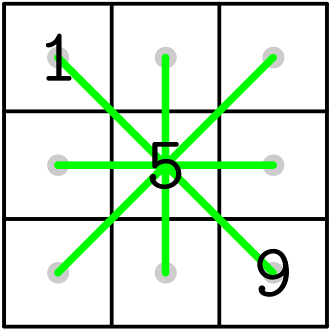

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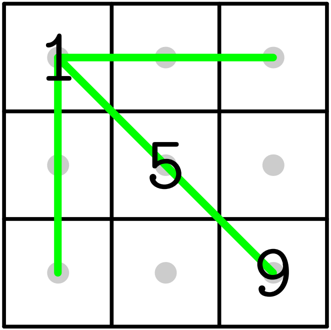

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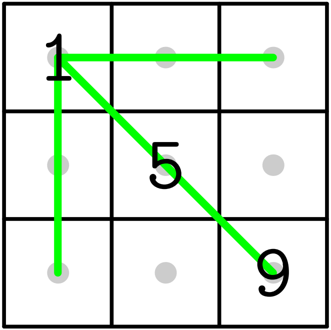

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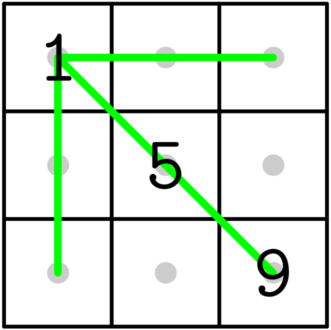

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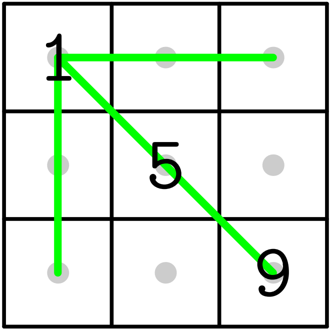

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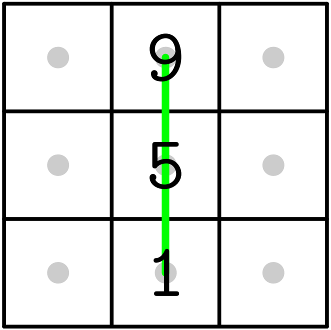

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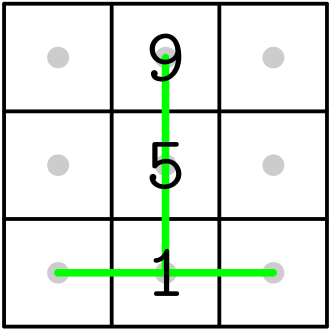

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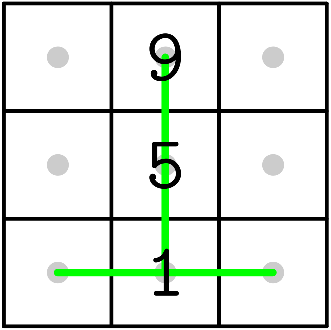

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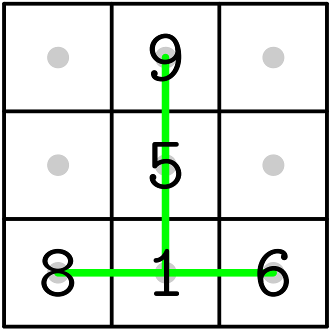

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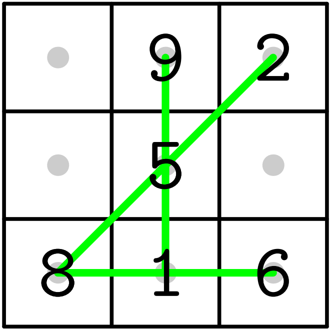

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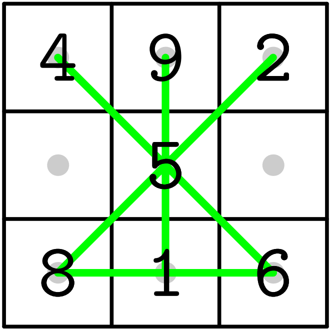

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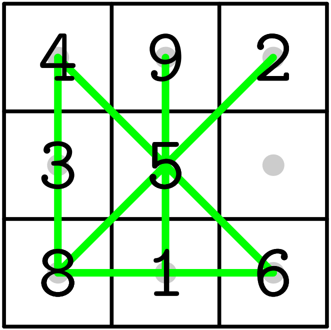

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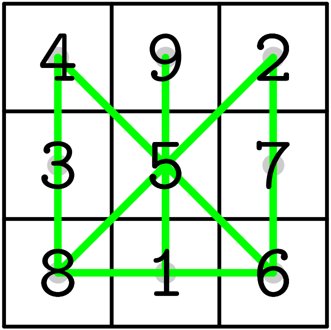

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- is it possible?

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Spoiler–2

1	2	0
0	1	2
2	0	1

$\rightarrow$

2	4	1
1	2	4
4	1	2

0	2	1
2	1	0
1	0	2

$\rightarrow$

1	9	3
9	3	1
3	1	9

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$$\times$$

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$$\begin{array}{|c|c|c|}\hline 2 & 36 & 3 \\ \hline 9 & 6 & 4 \\ \hline 12 & 1 & 18 \\ \hline \end{array}$$

Outline

Magic Squares

Narrowing the Search

Multiplicative Magic Squares

More Puzzles

Integer linear combinations

Paths in a Graph

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- $11 \times 9127 = 100\,397$
- next one is too big

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- $2 \times 2 \times 3 \times 5 \times 7 = 420$

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- 420×3 is too big

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- $\sqrt{31415} = 177.242771\ldots$
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- answer: 177 243 and 560 491.

Outline

- Magic Squares
- Narrowing the Search
- Multiplicative Magic Squares
- More Puzzles
- Integer linear combinations
- Paths in a Graph

7 **and** 13

Imagine a country with 7/13 florins coins.
Two people, each has many coins of each type.
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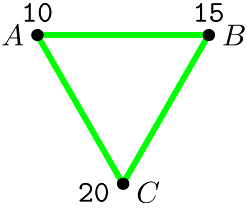
- the equation $15x + 21y = c$ has integer solutions $\Leftrightarrow$ c is a multiple of 3

Outline

Hotels and Paths

- Hotels $A(10)$, $B(15)$, $C(20)$; change every night for $10 + 15 + 20 = 45$ nights



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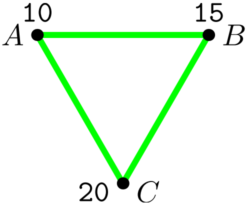

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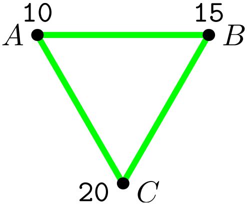

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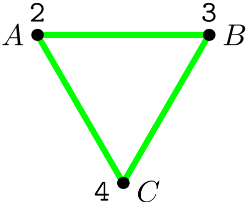

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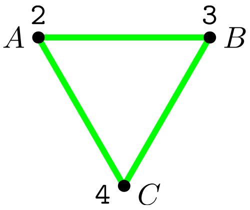

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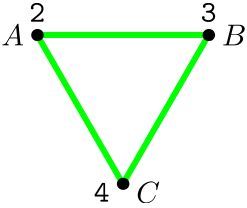

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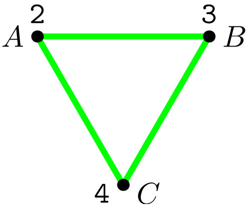

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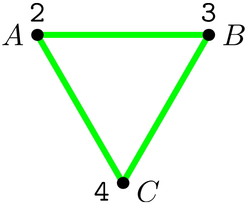

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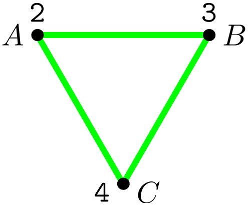

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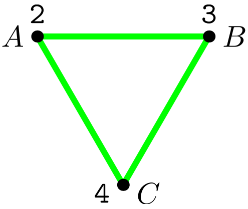

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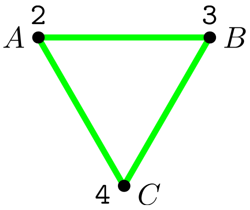

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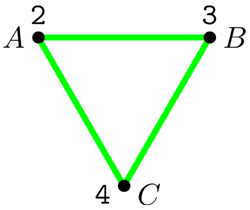

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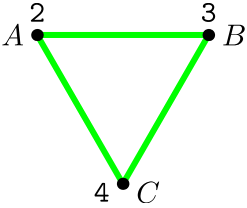

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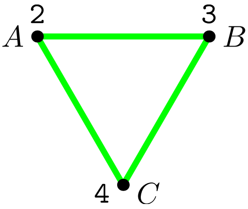

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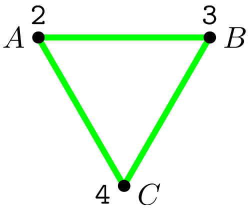

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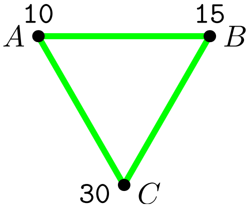
- $ACACBCBCB$ (5 times)

When a Path Does Not Exist

- Hotels $A(10)$, $B(15)$, $C(30)$; change every night for $10 + 15 + 30 = 55$ nights



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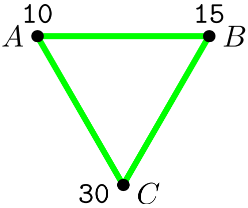

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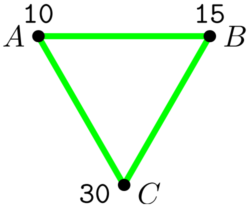

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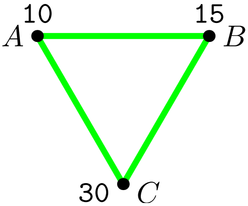

- possible or not?
- obstacle: too many vouchers for C
- to use them all we need at least 29 others