Vector Calculus – 2

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Derivatives in Vector Calculus

In calculus, derivatives map a scalar field onto another scalar field.

In vector calculus, derivatives are use to create the following mappings:

- The gradient maps a scalar field onto a vector field.
- The divergence maps a vector field onto a scalar field.
- The curl maps a vector field onto a vector field.

The Gradient of a Scalar Field

Given: $f(x,y,z)$:a scalar function defined over a domain in 3D space $(\mathbb{R}^3)$. f is differentiable.
Then

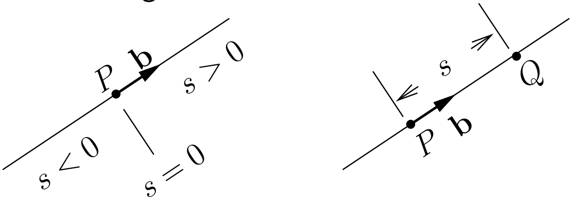
$$\text{grad} f = \nabla f \equiv \underbrace{\frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j} + \frac{\partial f}{\partial z}\mathbf{k}}_{\text{This is a vector.}}$$

$$\underbrace{\nabla}_{\text{"nabla"}} \equiv \underbrace{\frac{\partial}{\partial x}\mathbf{i} + \frac{\partial}{\partial y}\mathbf{j} + \frac{\partial}{\partial z}\mathbf{k}}_{\text{The gradient operator.}}$$

What is the utility of the gradient?

Directional Derivative

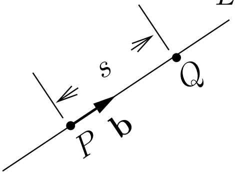
straight line in the direction of $\mathbf{b}$



s is measured from P in the direction of $\mathbf{b}$.

$$D_{\mathbf{b}} f = \left.\frac{df}{ds}\right|_{\mathbf{b}} \equiv \lim_{s \to 0} \frac{f(Q) - f(P)}{s}$$

Directional Derivative and Gradient



L: st line through P

Equation of L

$$\begin{aligned} \mathbf{r}(s) &= \mathbf{p}_0 + \mathbf{b}s \\ &= x(s)\mathbf{i} + y(s)\mathbf{i} + z(s)\mathbf{k} \end{aligned}$$

Assume f has partial derivatives which are continuous.

Then $\quad \dfrac{df}{ds} = \dfrac{\partial f}{\partial x}\dfrac{dx}{ds} + \dfrac{\partial f}{\partial y}\dfrac{dy}{ds} + \dfrac{\partial f}{\partial z}\dfrac{dz}{ds},$

but $\quad \dfrac{d\mathbf{r}}{ds} = \dfrac{dx}{ds}\mathbf{i} + \dfrac{dy}{ds}\mathbf{j} + \dfrac{dz}{ds}\mathbf{k} = \mathbf{b} \quad$ (why?)

$$\therefore D_{\mathbf{b}} f = \left.\frac{df}{ds}\right|_{\mathbf{b}} = \nabla f \cdot \mathbf{b}$$

Direction of Maximum Increase

Thorem

Let $f(P) = f(x, y, z)$ be a scalar function, with first partial derivatives which are continuous, defined over some domain B. Then ∇f exists, ∇f is a vector, its length independent of the choice of coordinates, and if $\nabla f \neq 0$, then the direction of ∇f is the direction of the maximum rate of increase of f.

Proof

$$D_{\mathbf{b}} f = \nabla f \cdot \mathbf{b} = |\nabla f||\mathbf{b}| \cos \gamma$$

where γ is the included $\angle$ between ∇f and $\mathbf{b}$.

But f and s are scalar functions of the position P.
$\therefore$ their values do not depend on the choice of the coordinate system.

Direction of Maximum Increase (cont)

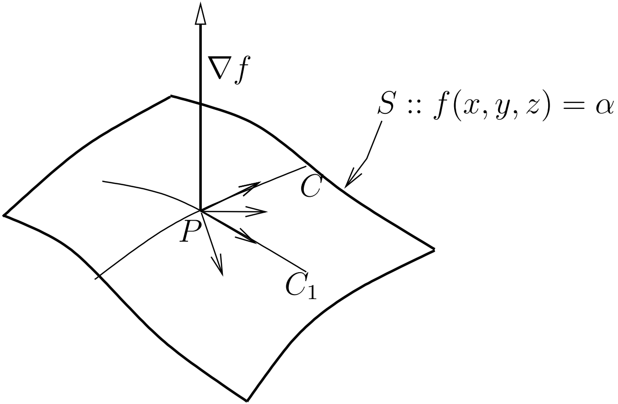
Proof (cont)

$$\therefore D_{\mathbf{b}} f = \left.\frac{df}{ds}\right|_{\mathbf{b}}$$: independent of choice of coordinate system.

Hence, the length and direction of ∇f does not depend on the choice of the coordinate system; this makes ∇f a proper vector.
$D_{\mathbf{b}} f = \nabla f$ will be maximum when $\cos\gamma = 1$, *i.e.* when $\mathbf{b}$ is chosen in the direction of ∇f.
$\therefore$, direction of ∇f: direction of maximum rate of increase, and $|\nabla f|$: magnitude of the maximum rate of increase in f.

Gradients and Surfaces

Let $f(x, y, z)$: a scalar function, differentiable.
Then $f(x, y, z) = \alpha$ (a constant) represents a surface, called a "level surface of f".



Gradients and Surfaces (cont)

P: a point on S.
C: a curve on S passing through P, represented by
$\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}$.

$\because C$ lies on S, components of $\mathbf{r}$ satisfy the relation for S:
$[1] \quad f(x(t), y(t), z(T)) = \alpha$.

A tangent vector of C will be $\mathbf{r}'(t) = x'(t)\mathbf{i} + y'(t)\mathbf{j} + z'(t)\mathbf{k}$.

For different curves on S passing through P, the set of tangent vectors at P forms a surface tangent to S at P.
A normal to this plane at P will then be the surface-normal of S at P.

Gradients and Surfaces (cont)

Differentiating $[1]$ w.r.t. t:

$$\frac{\partial f}{\partial x}x'(t) + \frac{\partial f}{\partial y}y'(t) + \frac{\partial f}{\partial z}z'(t) = 0$$
$$\therefore \nabla f \cdot \mathbf{r}' = 0$$

$\therefore \nabla f$ is normal to all vectors $\mathbf{r}'$ in the tangent plane of S.
$\implies \nabla f$ is the normal vector of S at P.

$\implies$ **Theorem:**
Let $f(x, y, z)$: a differentiable function in space.
Let $f(x, y, z) = \alpha = (const)$: a surface in that space.
Then ∇f at a point P on S is either 0 or a normal vector of S at P.

Potential

Nomenclature in science and engineering:

If $f(P) = f(x, y, z)$ is a scalar field, and
$\mathbf{V}(P) = \mathbf{V}(x, y, z) = \nabla f$ is a vector field, then, often

f is known as the **potential** for the vector field $\mathbf{V}(= \nabla f)$.

Sometimes, a negative sign and a scale factor is included.

The Divergence of a Vector Field

Let $\mathbf{V}(x, y, z) = v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k}$ be a differentiable vector field, with v_1, v_2, v_3 functions of (x, y, z). Then

$$\mathrm{div}\mathbf{V} = \nabla \cdot \mathbf{V} \equiv \frac{\partial v_1}{\partial x} + \frac{\partial v_2}{\partial y} + \frac{\partial v_3}{\partial z},$$

where ∇ is the same scalar-to-vector operator used in the definition of the gradient.

Divergence (cont)

Theorem
Given a vector function $\mathbf{V}(x, y, z)$ on a domain in space, $\mathrm{div}\mathbf{V}(= \nabla \cdot \mathbf{V})$ is a scalar function, whose values depend on $\mathbf{V}$ and the location in space, but <u>not</u> on the choice of the coordinate system.

What is the physical significance of divergence?

Divergence – Significance

In many physical situations, divergence has a meaning:

- In fluid mechanics, $\rho\mathbf{V}$ represents the mass flux.
  $\nabla \cdot (\rho\mathbf{V})$ equals the net outflow of mass per unit volume.
- In heat conduction, $\mathbf{q}$ represents the heat flux.
  $\nabla \cdot (\mathbf{q})$ equals the net heat outflow per unit volume.

The Laplacian

Let $f(x, y, z)$ be a scalar function in space. Its gradient

$$\nabla f = \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j} + \frac{\partial f}{\partial z}\mathbf{k}$$

is a vector function. If we take the divergence of this, we will get another scalar function:

$$\nabla \cdot (\nabla f) = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} = \nabla^2 f.$$

This function is known as the Laplacian of f, and is denoted by $\nabla^2 f$.
∇^2 is known as the Laplacian operator.

Use of the Laplacian

Fourier's Law of heat conduction states that $\mathbf{q} = -k\nabla T$, where T is the temperature and k the thermal conductivity. The First Law states that "rate of storage of energy equals the net inflow of energy". Hence, we get the heat conduction equation:

$$\rho c_p \frac{\partial T}{\partial t} = -\nabla \cdot \mathbf{q} = -\nabla \cdot (-k\nabla T) = \nabla \cdot (k\nabla T)$$

If k is constant, this reduces to the 'heat conduction equation':

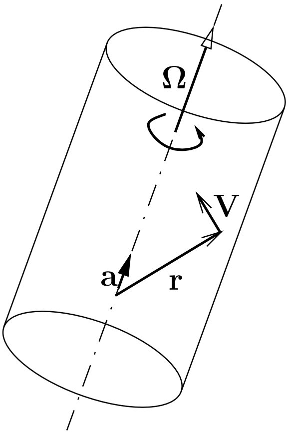
$$\rho c_p \frac{\partial T}{\partial t} = k\nabla^2 T$$

Curl of a Vector Field

Let $\mathbf{V}(x, y, z) = v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k}$ be a differentiable vector field, with v_1, v_2, v_3 functions of (x, y, z). Then

$$
\begin{aligned}
\mathrm{curl}\mathbf{V} &\equiv \nabla \times \mathbf{V} \\
&= \left(\frac{\partial v_3}{\partial y} - \frac{\partial v_2}{\partial z}\right)\mathbf{i} + \left(\frac{\partial v_1}{\partial z} - \frac{\partial v_3}{\partial x}\right)\mathbf{j} + \left(\frac{\partial v_2}{\partial x} - \frac{\partial v_1}{\partial y}\right)\mathbf{k} \\
&= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ v_1 & v_2 & v_3 \end{vmatrix}
\end{aligned}
$$

Rotation of a Solid Body



$$
\begin{aligned}
\boldsymbol{\Omega} &= \omega\mathbf{a} \\
\mathbf{V} &= \boldsymbol{\Omega}\times\mathbf{r}
\end{aligned}
$$

If $\mathbf{a}=\mathbf{k}$ (choice of direction),

then $\boldsymbol{\Omega} = [0,\ 0,\ \omega]$.

If $\mathbf{r} = [x,\ y,\ z]$,

$$
\text{then } \mathbf{V} = \boldsymbol{\Omega}\times\mathbf{r} = [-\omega y,\ \omega x,\ 0] = -\omega y\mathbf{i} + \omega x\mathbf{j}.
$$

Rotation of a Solid Body (cont)

$$\text{Thus, } \operatorname{curl}\mathbf{V} \equiv \nabla \times \mathbf{V} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ -\omega y & \omega x & 0 \end{vmatrix}$$

$$= 2\omega\mathbf{k} = 2\mathbf{\Omega}$$

Thus, solid body rotation leads to a $\mathbf{V}$-field such that
$\nabla \times \mathbf{V}$ is in the direction of $\mathbf{\Omega}$
and its magnitude $= |2\mathbf{\Omega}| = 2\omega$.

Properties of Curl

1. Curl of a gradient

Let f be a scalar field, differentiable.
Let ∇f, a vector field, be also differentiable.
Then $\nabla \times (\nabla f) = \mathbf{0}$ (vector).
That is, the gradient field of a scalar field has zero curl.
We say that that the gradient field is irrotational.

Note: $\nabla \cdot (\nabla f) = \nabla^2 f$, is a scalar field, and need not be zero.

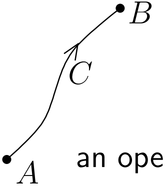
Properties of Curl (cont)

2. Divergence of a curl

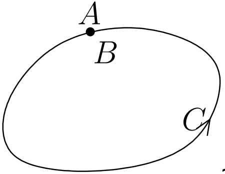
Let $\mathbf{V}$ be a vector field, twice differentiable.
Then $\nabla \times \mathbf{V}$ is another vector field.
Then $\nabla \cdot (\nabla \times \mathbf{V}) = 0$.
We say that the divergence of the curl of a vector field is zero.

Integral Calculus of Vectors

Integration of vectors is a generalisation of ordinary integrals like $\int_a^b f(x)dx$. Line integrals are defined over a curve in space:



an open curve



a closed curve

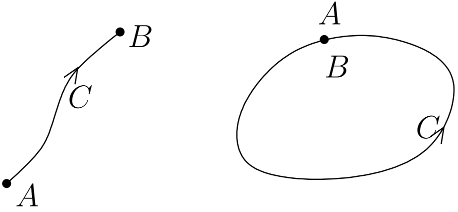
The curve C can be parametrically represented:
$\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}; \quad a \leq t \leq b$,
where A at $\mathbf{r}(a)$ is the initial point,
and B at $\mathbf{r}(b)$ is the terminal point.
t increases from a to b: the positive direction of C.
If points A and B coincide, then we have a closed curve.

Smooth Curves

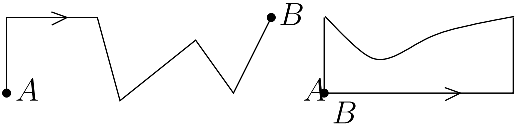
C is called a smooth curve if:

1: at each point, a unique tangent exists, and

2: the direction of this tangent varies continuously.



smooth curves



non-smooth,
but piecewise-smooth,
curves

Henceforth, all curves are assumed piecewise smooth.

Line Integral

Line integral of a vector function:

$$\int_C \mathbf{F}(\mathbf{r}) \cdot d\mathbf{r} = \int_a^b \mathbf{F}(\mathbf{r}(t)) \cdot \underbrace{\frac{d\mathbf{r}}{dt}}_{\mathbf{r}'(t)} \, dt$$

$$= \int_C (F_1 dx + F_2 dy + F_3 dz)$$

$$= \int_C (F_1 x' + F_2 y' + F_3 z') dt$$

If C is closed, $\displaystyle\int_C$ may be written as $\displaystyle\oint_C$.

Line Integral (cont)

The integral is scalar ($\because$ dot product).

$$\int_C \mathbf{F}(\mathbf{r}) \cdot d\mathbf{r} = \int_a^b \underbrace{\mathbf{F}(\mathbf{r}(t)) \cdot \frac{\mathbf{r}'}{|\mathbf{r}'|}}_{F_{\mathsf{tan}}} \underbrace{|\mathbf{r}'|dt}_{ds}$$

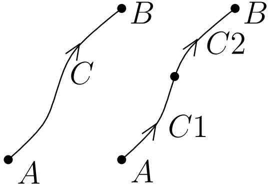
Rules for line integrals:

$$\int_C \alpha\mathbf{F}(\mathbf{r}) \cdot d\mathbf{r} = \alpha \int_C \mathbf{F}(\mathbf{r}) \cdot d\mathbf{r}, \quad (\alpha : \mathsf{const})$$

$$\int_C (\mathbf{F}(\mathbf{r}) + \mathbf{G}(\mathbf{r})) \cdot d\mathbf{r} = \int_C \mathbf{F}(\mathbf{r}) \cdot d\mathbf{r} + \int_C \mathbf{G}(\mathbf{r}) \cdot d\mathbf{r}$$

Line Integral (cont)

If C is partitioned into $C1$ and $C2$, and all three have the same direction, then:



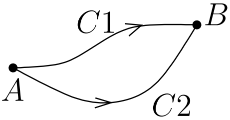
$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_{C1} \mathbf{F} \cdot d\mathbf{r} + \int_{C2} \mathbf{F} \cdot d\mathbf{r}$$

If the direction of traverse (orientation) is reversed, then the value of the integral changes sign.

Vector Line Integral

$$\int_C \mathbf{F}(\mathbf{r})dt = \left[\int_C F_1(\mathbf{r})dt\right]\mathbf{i} + \left[\int_C F_2(\mathbf{r})dt\right]\mathbf{j} + \left[\int_C F_3(\mathbf{r})dt\right]\mathbf{k}$$

Path Dependence

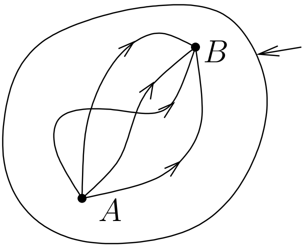


Generally, $\int_A^B \mathbf{F} \cdot d\mathbf{r}$ depends on (1) the end points A and B, and (2) the path from A to B.
Hence, different paths (curves) from A to B may lead to different values of the integral.

Some integrals may be path-independent.

These are important items in physics.

Path-Independence of Line Integrals



domain D over which $\mathbf{F}(\mathbf{r})$ is defined ($A, B,$ any curve, must be inside D)

Theorem

The following statements are equivalent.

1. $\int_A^B \mathbf{F}(\mathbf{r}) \cdot d\mathbf{r}$ is path-independent.
2. $\mathbf{F} = \nabla f$; *i.e.* a potential (f) exists.
3. $\oint \mathbf{F}(\mathbf{r}) \cdot d\mathbf{r} = 0$ for any closed curve in D.
4. If D is simply-connected, $\nabla \times \mathbf{F} = 0$.

Path-Independence of Line Integrals (cont)

From [2.], we can deduce:

$$\int_B^A \mathbf{F} \cdot d\mathbf{r} = f(B) - f(A), \text{ where } \mathbf{F} = \nabla f.$$

Note the similarity to definite integrals.

[3.] is used interchangeably with [1.], as in Thermodynamics (where it is used twice — once for the First Law, and once for the Second Law).

Path-Independence of Line Integrals (cont)

[4.] is based on the Pfaffian[1] form:
$\mathbf{F} \cdot d\mathbf{r} = F_1 dx + F_2 dy + F_3 dz.$
This form is called **exact**; the RHS is called an **exact differential** (or Pfaffian differential), if we do have a differentiable function $f(x, y, z)$ s.t.

$$\begin{aligned} \mathbf{F} \cdot d\mathbf{r} &= F_1 dx + F_2 dy + F_3 dz \\ &= df = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy + \frac{\partial f}{\partial z} dz \\ &= \nabla f \cdot d\mathbf{r} \\ \text{That means, } \mathbf{F} &= \nabla f, \\ \therefore \; F_1 &= \frac{\partial f}{\partial x}, F_2 = \frac{\partial f}{\partial y}, F_3 = \frac{\partial f}{\partial z}. \end{aligned}$$

[1] Johann Friedrich Pfaff (1765–1825), German mathematician.

Path-Independence of Line Integrals (cont)

For simply connected domains[2]

$$\text{if } \mathbf{F} = \nabla f$$
$$\text{then } \nabla \times \mathbf{F} = \nabla \times (\nabla f) = 0$$

By this, or by cross-differentiation,

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x} \implies \frac{\partial F_2}{\partial x} = \frac{\partial F_1}{\partial x} \implies \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial x} = 0,$$

etc.. $\therefore$ each component of $\nabla \times F = 0$.
$\therefore \; \nabla \times F = 0$.

[2]A SCD is one in which any closed curve can be continuously shruck to a point without leaving that domain.

Path-Independence of Line Integrals (cont)

A spacial case: *Flatland*, which is a domain in (x, y)-plane.

$$\int_C \mathbf{F}(\mathbf{r}) \cdot d\mathbf{r} = \int_C (F_1 dx + F_2 dy)$$

will be path independent

iff $F_1 dx + F_2 dy$ is an exact differential,

i.e. iff $\dfrac{\partial F_1}{\partial y} = \dfrac{\partial F_2}{\partial x}$,

i.e. iff $\dfrac{\partial F_1}{\partial y} - \dfrac{\partial F_2}{\partial x} = 0$,

i.e. iff $\nabla \times \mathbf{F} = \left(\dfrac{\partial F_1}{\partial y} - \dfrac{\partial F_2}{\partial x} \right) \mathbf{k} = 0.$

Vector Integral Theorems

- Help us convert integrals from one form to another.
- volume integral $\longleftrightarrow$ surface integral.
- surface integral $\longleftrightarrow$ volume integral.

These are useful

- for their convenience — often lead to ease of evaluation, and
- by making all integrals in an equation of the same type, often help convert integral equations into differential equations.

Green's Theorem in the Plane (x, y)

R: closed, bounded region in (x, y) plane (need not be simply-connected).

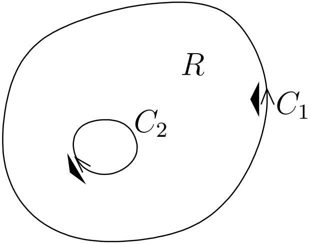
C: boundary of R, consisting of finitely many smooth curves.

$F_1(x, y)$ and $F_2(x, y)$: continuous functions, s.t. the partial derivatives $\partial F_1/\partial y$ and $\partial F_2/\partial x$ exist and are continuous everywhere in R.

Then

$$\iint_R (\partial F_2/\partial x - \partial F_1/\partial y)\, dx\, dy = \oint_C (F_1 dx + F_2 dy).$$

Green's Theorem (cont)

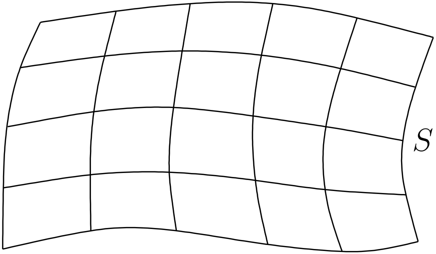


The line-integral on the RHS is on the entire C, traversed in such a way that R is on the left as C is traversed.

If $\mathbf{F} \equiv F_1\mathbf{i} + F_2\mathbf{j}$, then we get the vector form of Green's Theorem:

$$\iint_R (\nabla \times \mathbf{F}) \cdot \mathbf{k}\, dx\, dy = \oint_C \mathbf{F} \cdot d\mathbf{r}.$$

Surface Integrals in 3-D



Surface in 3-D
(assumed piecewise smooth)
$\mathbf{r}(u, v)$:definition of the surface

$\mathbf{r}(u, v) = x(u, v)\mathbf{i} + y(u, v)\mathbf{j} + z(u, v)\mathbf{k}$,
where (u, v) : defined over R in the u, v plane.

Surface Integrals in 3-D (cont)

At any point on S,

$$\mathbf{r}_u = \left(\frac{\partial \mathbf{r}}{\partial u}\right)_v \quad \text{:tangent vector along a const-}u\text{ line,}$$

$$\mathbf{r}_v = \left(\frac{\partial \mathbf{r}}{\partial v}\right)_y \quad \text{:tangent vector along a const-}v\text{ line}$$

$$\therefore\ \mathbf{N} = \mathbf{r}_u \times \mathbf{r}_v \text{ is a normal vector, and}$$

$$\text{and } \mathbf{n} = \frac{\mathbf{N}}{|\mathbf{N}|} \text{ is a unit normal vector.}$$

All of these will exist, except perhaps at edges and corners.

Surface Integrals in 3-D (cont)

For $\mathbf{F}$: a vector function,

$$\iint\limits_S \mathbf{F} \cdot \mathbf{n}\, dA \equiv \iint\limits_R \mathbf{F}(\mathbf{r}(u,v)) \cdot \mathbf{N}(u,v)\, du\, dv$$

is the definitition of the surface integral of $\mathbf{F}$ over S.

$|\mathbf{r}_u \times \mathbf{r}_v| = |\mathbf{N}| =$ area of parallelogram of sides $\mathbf{r}_u$ and $\mathbf{r}_v$.

$\therefore \mathbf{n}\, dA = \mathbf{n}|\mathbf{N}|\, du\, dv = \mathbf{N}\, du\, dv$.

$\mathbf{F} \cdot \mathbf{n} =$ normal component of $\mathbf{F}$ (normal to surface S).

Surface Integrals in 3-D (cont)

$$\begin{aligned}
\mathbf{F} &= F_1\mathbf{i} + F_2\mathbf{j} + F_3\mathbf{k}, \\
\mathbf{N} &= N_1\mathbf{i} + N_2\mathbf{j} + N_3\mathbf{k}, \\
\mathbf{n} &= \cos\alpha\mathbf{i} + \cos\beta\mathbf{j} + \cos\gamma\mathbf{k}.
\end{aligned}$$

$\cos\alpha = \mathbf{n}\cdot\mathbf{i}, \cos\beta = \mathbf{n}\cdot\mathbf{j}, \cos\gamma = \mathbf{n}\cdot\mathbf{k}$, are the direction cosines for $\mathbf{n}$.

Surface Integrals in 3-D (cont)

Thus,

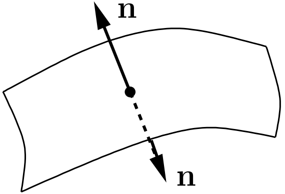
$$\iint_S \mathbf{F}\cdot\mathbf{n}\,dA = \iint_S (F_1N_1 + F_2N_2 + F_3N_3)\,du\,dv$$

$$= \iint_S (F_1\cos\alpha + F_2\cos\beta + F_3\cos\gamma)\,dA$$

$$\text{But}\quad \begin{aligned} \cos\alpha\,dA &= dy\,dz \\ \cos\beta\,dA &= dz\,dx \\ \cos\gamma\,dA &= dx\,dy \end{aligned}$$

$$\iint_S \mathbf{F}\cdot\mathbf{n}\,dA = \iint_S (F_1\,dydz + F_2\,dzdx + F_3\,dxdy)$$

Effect of Orientation



We have a choice of normals
$\mathbf{n}$ or $-\mathbf{n}$ ($\mathbf{N}$ or $-\mathbf{N}$).
This changes the value of the integral by a factor of (-1)

A smooth surface S is **orientable** if the positive normal direction at some point P on S can be moved in a continuous and unique way over the entire surface.

We assume our surfaces to be smooth and orientable.

Triple Integrals

- These are integrals over a volume $(dV = dx\,dy\,dz)$
- We consider a 3-D region T

Divergence Theorem of Gauss

Perhaps the most important and celebrated theorem in Vector Calculus.

Transforms a **triple integral over a volume** into a **surface integral over the surface bounding the volume**, and *vice-versa*.

Divergence Theorem of Gauss (cont)

T : a closed, bounded region in space.

S : the bounding surface of T, piecewise smooth and orientable.

$\mathbf{F}(x, y, z)$: a vector function, continuous, with continuous first partial derivatives all over T (and S, and maybe beyond).

Then

$$\iiint\limits_T \nabla \cdot \mathbf{F}\, dV = \iint\limits_S \mathbf{F} \cdot \underbrace{\mathbf{n}\, dA}_{d\mathbf{S}} = \iint\limits_S \mathbf{F} \cdot d\mathbf{S}$$

Divergence Theorem of Gauss (cont)

If $\mathbf{F} = [F_1, F_2, F_3]$ and $\mathbf{n} = [\cos\alpha, \cos\beta, \cos\gamma]$, then the theorem becomes:

$$\iiint\limits_T \left(\frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z} \right) dV$$

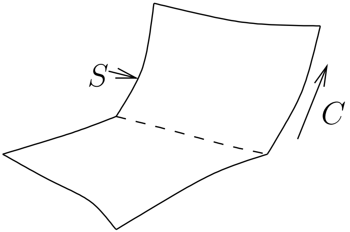
$$= \iint\limits_S (F_1 \cos\alpha + F_2 \cos\beta + F_3 \cos\gamma) dA$$

$$= \iint\limits_S (F_1\, dydz + F_2\, dzdx + F_3\, dxdy)$$

Stokes' Theorem

Stokes' Theorem is the 3-dimensional version of Green's Theorem.

It relates a surface-integral over a 3-D surface to a line-integral over the bounding line in 3-D.



S :surface in 3-D, piecewise smooth, and oriented

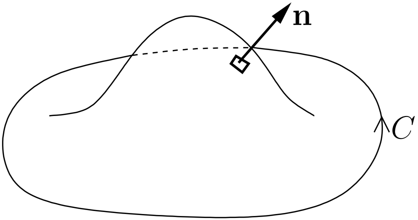
C :boundary of S, piecewise smooth, simple, closed

$\mathbf{F}(x, y, z)$:vector function, defined over $S{+}{+}$, continuous, with continuous (first) partial derivatives.

Stokes' Theorem (cont)

Then

$$\iint_S (\nabla \times \mathbf{F}) \cdot \underbrace{\mathbf{n}\, dA}_{d\mathbf{S}} = \oint_C \mathbf{F} \cdot \mathbf{r}'(s) ds$$



$\mathbf{n}$:unit normal vector to S
$\mathbf{r}'(s)$:unit tangential vector to C

Direction of $\oint$: s.t. topologically, a vertically upwards $\mathbf{n}$ is to the left of the traverse.

Stokes' Theorem (cont)

Stokes' Theorem relates directly to path independence:

If the RHS equals zero everywhere ($\because$ path independent),

then the LHS equals zero everywhere.

$\implies \nabla \times F = 0$ everywhere.