# Special Square Matrices

A **diagonal matrix** has all non-diagonal elements zero.
$a_{ij} = 0,$ if $i \neq j$.
Example:

$$\mathbf{D} = \begin{bmatrix} a_{11} & 0 & \cdots & 0 \\ 0 & a_{22} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_{nn} \end{bmatrix}$$

# Special Square Matrices (cont)

A **scalar matrix** is a diagonal matrix with all diagonal elements equal to each other.

$a_{ij} = 0,$ if $i \neq j$.
$a_{ii} = \alpha,$ where $\alpha$ is a scalar .

Example:

$$\mathbf{S} = \begin{bmatrix} \alpha & 0 & \cdots & 0 \\ 0 & \alpha & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \alpha \end{bmatrix}$$

# Special Square Matrices (cont)

A **unit matrix** or an **identity matrix** is a scalar matrix with all diagonal elements equal to 1. $a_{ij} = 0,$ if $i \neq j$.
$a_{ii} = 1$.

Or: $a_{ij} = \delta_{ij}$, using the Kronecker[2] Delta.

$$I_{\{nn\}} = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix}$$

[2] Leopold Kronecker (1823–1891), German mathematician. *God made the integers, all else is the work of man.*

# Homework

Given a square matrix $\mathbf{A}$, a scalar matrix $\mathbf{S}$ with each element equal to $\alpha$, and an identity matrix $\mathbf{I}$, all of the same size, show that

- $\mathbf{S} = \alpha\mathbf{I}$
- $\mathbf{SA} = \mathbf{AS} = \alpha\mathbf{A}$
- $\mathbf{AI} = \mathbf{IA} = \mathbf{A}$

# Special Square Matrices (cont)

If all the elements of a square matrix *above* its diagonal are zero, then it is known as an **lower-triangular matrix**.

$a_{ij} = 0$, if $i < j$.

Example:

$$\mathbf{L} = \begin{bmatrix} a_{11} & 0 & \cdots & 0 \\ a_{21} & a_{22} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}$$

# Special Square Matrices (cont)

If all the elements of a square matrix *below* its diagonal are zero, then it is known as an **upper-triangular matrix**.

$a_{ij} = 0,$ if $i > j$.

Example:

$$\mathbf{U} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ 0 & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_{nn} \end{bmatrix}$$

# MATRICES – 2

## Linear Algebraic Equations

# DETERMINANTS

It is expected that students are comfortable with determinants and their properties.
Kreyszig, 10th Ed., Sec. 7.7.

# Watch a Video

Watch the first lecture by Prof. Strang on Linear Algebra.

https://www.youtube.com/watch?v=ZK3O402wf1c

In particular, from 01:00 to 16:00.

# Standard Form of LAEs

$$
\begin{array}{ccccccccc}
a_{11}x_1 & + & a_{12}x_2 & + & \cdots & + & a_{1n}x_n & = & b_1 \\
a_{21}x_1 & + & a_{22}x_2 & + & \cdots & + & a_{2n}x_n & = & b_2 \\
\vdots & & \vdots & & & & \vdots & = & \vdots \\
a_{n1}x_1 & + & a_{n2}x_2 & + & \cdots & + & a_{nn}x_n & = & b_n
\end{array}
$$

We have $n$ equations for $n$ unknowns.

# Matrix Form

$$\underbrace{\mathbf{A}}_{(n\times n)}\underbrace{\mathbf{x}}_{(n\times 1)}=\underbrace{\mathbf{b}}_{(n\times 1)}$$

$$\underbrace{\mathbf{A}=\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}}_{\text{coefficient matrix}}; \quad \underbrace{\mathbf{x}=\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}}_{\text{vector of unknowns}}; \quad \underbrace{\mathbf{b}=\begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}}_{\text{RHS vector}}$$

# The Augmented Matrix

$$\underbrace{\tilde{\mathbf{A}} = \left[\begin{array}{cccc|c} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} & b_m \end{array}\right]}_{\text{augmented matrix}} = [\mathbf{A} \,\vert\, \mathbf{b}]$$

The augmented matrix completely defines the system of equations.

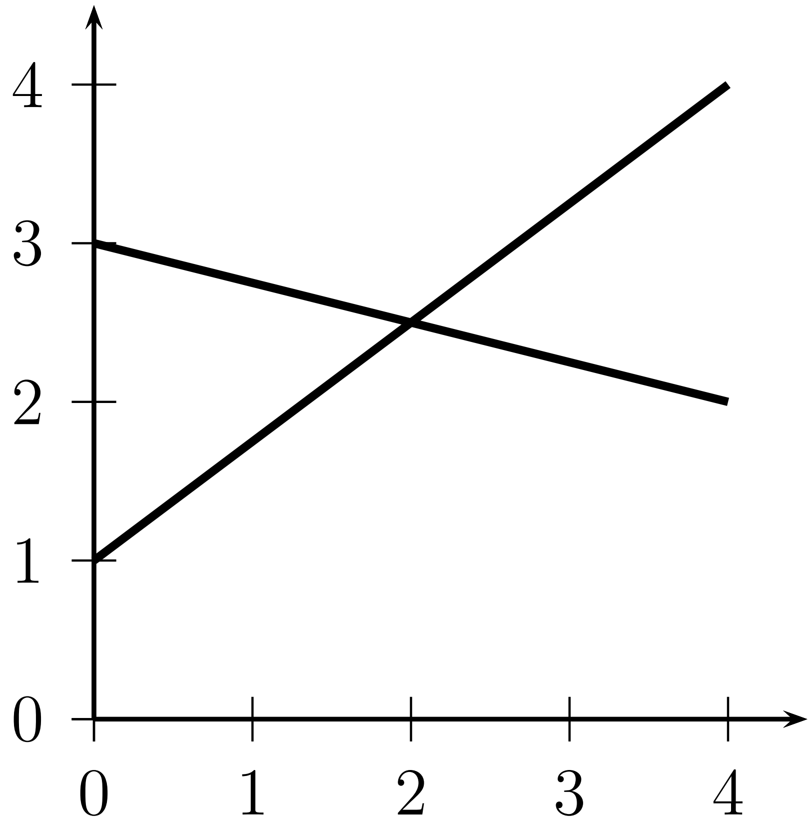
# Classification

- If $\mathbf{b} = \mathbf{0}$, then the system is called a homogeneous system.
- If $\mathbf{b} \neq \mathbf{0}$, then the system is called a non-homogeneous system.

For a homogeneous system of equations:

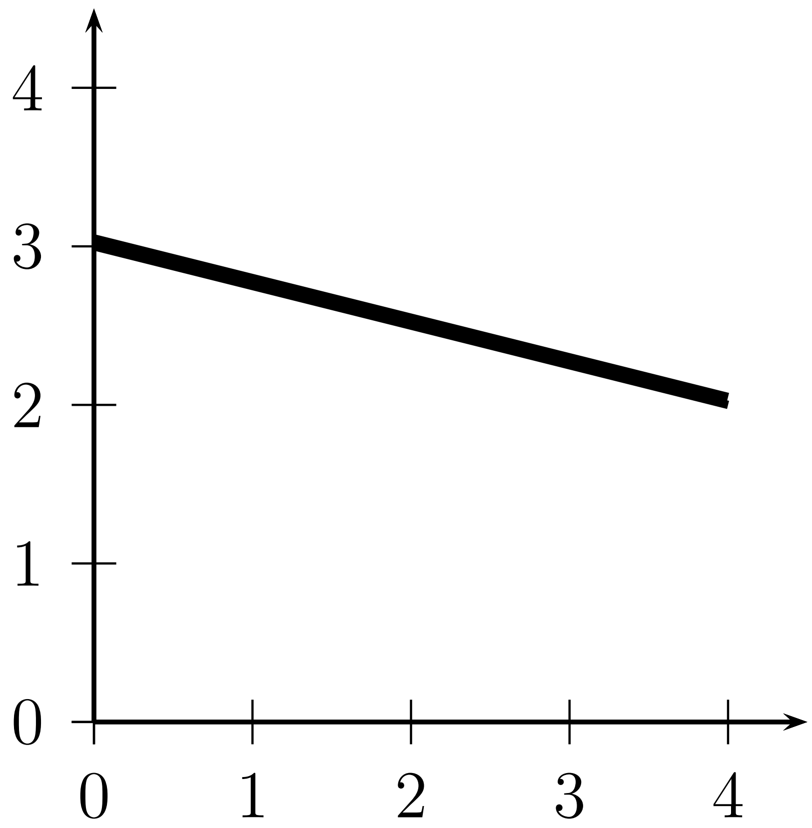
- $b_i = 0$ for each $i$.
- Hence, $\mathbf{x} = \mathbf{0}$ satisfies each equation.
- Hence, $\mathbf{x} = \mathbf{0}$ is a solution of the system.

# A $2 \times 2$ System : Visual Depiction : Case 1



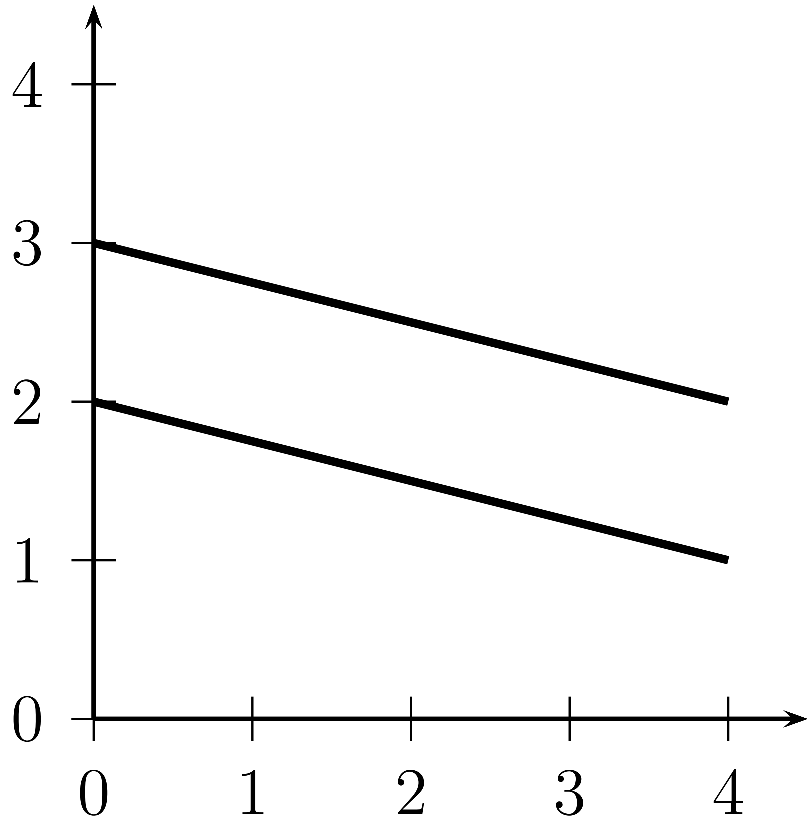

Crossing lines. A unique solution exists.

# A $2 \times 2$ Sytem : Visual Depiction : Case 2



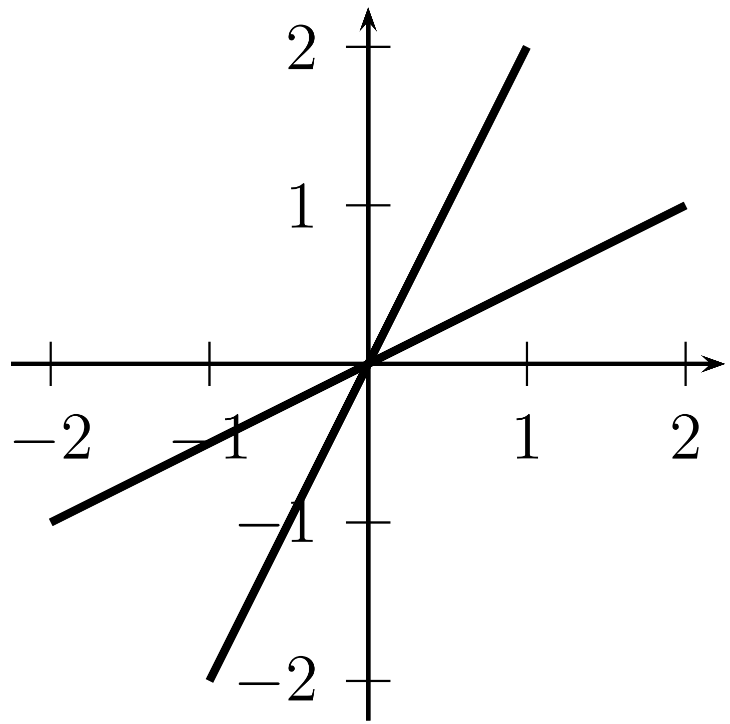

Overlapping lines. Many (infinitely many) solutions exist.

# A $2 \times 2$ Sytem : Visual Depiction : Case 3



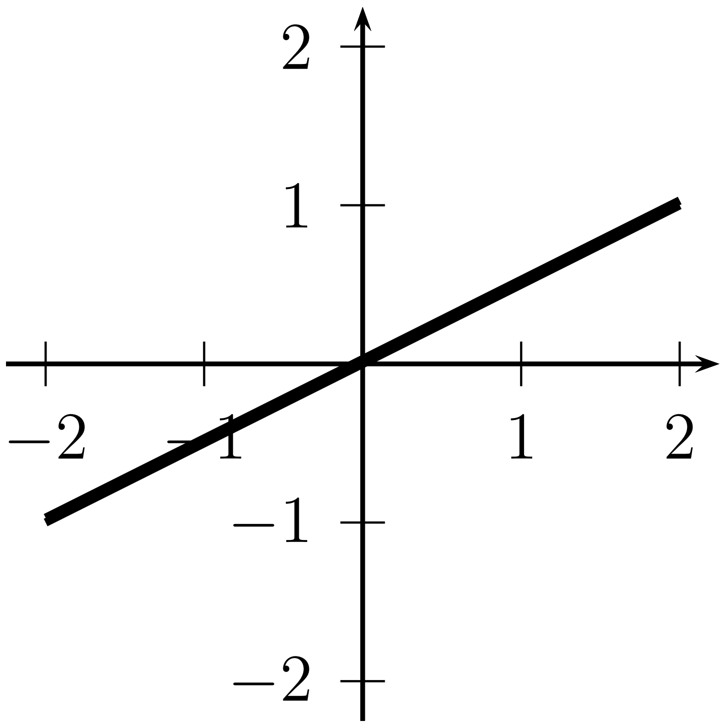

Parallel, non-intersecting lines. No solution exists.

# A $2 \times 2$ Sytem : Visual Depiction : Case 4




Homogeneous system, intersecting lines. The trivial solution is the unique solution.

# A $2 \times 2$ Sytem : Visual Depiction : Case 5




Homogeneous system, overlapping lines. The trivial solution is one of infinitely many solutions.

# Gauss Elimination

**Motivation:**

If the coefficient matrix is upper-triangular, then the solution can be obtained simply by back-substitution.

**Gauss Elimination:**

A process of conversion of the system of LAEs to an equivalent one with an upper-triangular coefficient matrix.

# Gauss Elimination (cont)

This is done using one or more of the following operations, as many times as needed:

- **interchange** of two rows
  (Why do we not interchange columns?)
- **addition** of one row to another
- **multiplication** of a row by a nonzero constant

Such modified LAE systems are called **row-equivalent**.

Row-equivalent systems have the same solution (or set of solutions).

# Some Definitions

A LAE system is **overdetermined** if the number of equations is greater than the number of unknowns: $n_{\mathsf{equ}} > n_{\mathsf{unk}}$,
and **underdetermined** if $n_{\mathsf{equ}} < n_{\mathsf{unk}}$.

A system is **consistent** if it has at least one solution,
**inconsistent** if it has no solution.

# Row-Echelon Form

After Gauss Elimination, the augmented matrix will be in row-echelon form, in which the coefficient matrix part is upper-triangular. So, all elements below the diagonal will be zero.

But the diagonal elements may or may not be zero.

Also, there could be a number of rows at the bottom of the modified coefficient matrix which will all be zero.

# Row-Echelon Form (cont)

The original augmented matrix is
$\tilde{\mathbf{A}} = [\mathbf{A} \,\vdots\, \mathbf{b}]$

The reduced augmented matrix, in row-echelon form is
$\tilde{\mathbf{R}} = [\mathbf{R} \,\vdots\, \mathbf{f}]$

The two matrices are equivalent, and have the same set of solution(s).
They also have the same rank.

# Row-Echelon Form Case 1

If all diagonal elements are non-zero, we have:

$$\tilde{\mathbf{R}} = \left[\begin{array}{ccccc:c} r_{11} & r_{12} & r_{13} & \cdots & r_{1n} & f_1 \\ 0 & r_{22} & r_{23} & \cdots & r_{2n} & f_2 \\ 0 & 0 & r_{33} & \cdots & r_{3n} & f_3 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & r_{nn} & f_n \end{array}\right]$$

where each $r_{ii} \neq 0$.
In this case,
(1) a unique solution exists,
(2) it can be obtained by back-substitution, and
(c) the rank of $\mathbf{A}$ is $n$, or $\mathbf{A}$ is of full rank.

# Row-Echelon Form Case 2

In this case, diagonal elements upto row $r$ are non-zero, and all rows below that have zeros in the modified coefficient matrix:

$$\tilde{\mathbf{R}} = \left[\begin{array}{ccccccc|c} r_{11} & r_{12} & \cdots & r_{1r} & r_{1r+1} & \cdots & r_{1n} & f_1 \\ 0 & r_{22} & \cdots & r_{2r} & r_{2r+1} & \cdots & r_{2n} & f_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & r_{rr} & r_{rr+1} & \cdots & r_{rn} & f_r \\ 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & f_{r+1} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & f_n \end{array}\right]$$

# Row-Echelon Form Case 2 (cont)

Look at the values of $f_{r+1}, \cdots, f_n$.

2A If at least one of these is nonzero, then there is no solution, the system is inconsistent.

2B If all of them are zero, then we have a consistent system with infinitely many solutions. For any assumed set of values for $x_{r+1}, \cdots, x_n$, there is a corresponding solution.

# Row-Echelon Form - Comments

- The leading non-zero coefficient may have any value. However, if we reduce it to 1, it helps (at least with hand calculations).
- Rectangular matrices may also be reduced to row-echelon form.

# Row-Echelon Form – An Example

Consider the LAE system: $\begin{bmatrix} 1 & 2 & 2 \\ 2 & 2 & 1 \\ 3 & 1 & 2 \end{bmatrix} \mathbf{x} = \begin{bmatrix} 11 \\ 9 \\ 11 \end{bmatrix}.$

The augmented matrix is: $\left[\begin{array}{ccc:c} 1 & 2 & 2 & 11 \\ 2 & 2 & 1 & 9 \\ 3 & 1 & 2 & 11 \end{array}\right].$

Elimination of the first column of rows 2 and 3 gives:

$$\left[\begin{array}{ccc:c} 1 & 2 & 2 & 11 \\ 0 & -2 & -3 & -13 \\ 0 & -5 & -4 & -22 \end{array}\right].$$

# Row-Echelon Form – An Example (cont)

We may multiply rows 2 and 3 by $-1$ to get rid of the minus signs: $\left[\begin{array}{ccc|c} 1 & 2 & 2 & 11 \\ 0 & 2 & 3 & 13 \\ 0 & 5 & 4 & 22 \end{array}\right]$.

Elimination of the second column of row 3 leads to:

$$\left[\begin{array}{ccc|c} 1 & 2 & 2 & 11 \\ 0 & 2 & 3 & 13 \\ 0 & 0 & -3.5 & -10.5 \end{array}\right].$$

Or, multiplying the last row by $-1$: $\left[\begin{array}{ccc|c} 1 & 2 & 2 & 11 \\ 0 & 2 & 3 & 13 \\ 0 & 0 & 3.5 & 10.5 \end{array}\right]$.

This is the row-echelon form.

# Row-Echelon Form – An Example (cont)

The solution is obtained by back-substitution:
$x_3 = 3,\ x_2 = 2,\ x_1 = 1.$

Note:

- All diagonal elements in the row-echelon form are non-zero, so we have a unique solution.
- The product of the diagonal elements in the form is $7$, while the determinant of the coefficient matrix is $-7$. Why?

# Row-Echelon Form – An Example (cont)

If the system is: $\begin{bmatrix} 1 & 2 & 2 \\ 2 & 2 & 1 \\ 3 & 4 & 3 \end{bmatrix} \mathbf{x} = \begin{bmatrix} 11 \\ 9 \\ 20 \end{bmatrix}.$

The augmented matrix is: $\left[\begin{array}{ccc:c} 1 & 2 & 2 & 11 \\ 2 & 2 & 1 & 9 \\ 3 & 4 & 3 & 20 \end{array}\right].$

The row-echelon form then turns out to be:

$$\left[\begin{array}{ccc:c} 1 & 2 & 2 & 11 \\ 0 & 2 & 3 & 13 \\ 0 & 0 & 0 & 0 \end{array}\right].$$

# Row-Echelon Form – An Example (cont)

This is a consistent system, with infinitely many solutions.
The determinant of the coefficient matrix is zero.
The solution is:

$x_3 =$ any arbitrary value,
$x_2 = (13 - 3x_3)/2,$
$x_1 = x_3 - 2.$

Since $x_3$ may have any value, there are infinitely many solutions.

# Row-Echelon Form – An Example (cont)

If the system is: $\begin{bmatrix} 1 & 2 & 2 \\ 2 & 2 & 1 \\ 3 & 4 & 3 \end{bmatrix} \mathbf{x} = \begin{bmatrix} 11 \\ 9 \\ \mathbf{21} \end{bmatrix}$.

The augmented matrix is: $\left[\begin{array}{ccc:c} 1 & 2 & 2 & 11 \\ 2 & 2 & 1 & 9 \\ 3 & 4 & 3 & 21 \end{array}\right]$.

The row-echelon form then turns out to be:

$$\left[\begin{array}{ccc:c} 1 & 2 & 2 & 11 \\ 0 & 2 & 3 & 13 \\ 0 & 0 & 0 & \mathbf{1} \end{array}\right].$$

This is an inconsistent system, it has no solution.

# Row-echelon Form: Homework

What are the remaining cases, if any, of the row-echelon form?

What type of solution(s) to they lead to?

# Linear Independence of Vectors

Let $\mathbf{a}_{(1)}, \mathbf{a}_{(2)}, \cdots, \mathbf{a}_{(m)}$ be $m$ vectors of the same length.

Let $c_1, c_2, \cdots, c_m$ be a set of $m$ scalars.

The expression

$$c_1\mathbf{a}_{(1)} + c_2\mathbf{a}_{(2)} + \cdots + c_m\mathbf{a}_{(m)}$$

is a **linear combination of the vectors**.

# Linear Independence of Vectors (cont)

Consider the equation

$$c_1\mathbf{a}_{(1)} + c_2\mathbf{a}_{(2)} + \cdots + c_m\mathbf{a}_{(m)} = \mathbf{0}.$$

One possibility is:
$c_1 = c_2 = \cdots = c_m = 0$,
that is, $\mathbf{c}$ is a zero vector.

If that is the only possibility, then we say that the set of vectors $(\mathbf{a}_{(1)}, \mathbf{a}_{(2)}, \cdots, \mathbf{a}_{(m)})$ is **linearly independent**.

# Linear Independence of Vectors (cont)

If a solution exists other than $\mathbf{c} = \mathbf{0}$, *i.e.* at least one of $(c_1, c_2, \cdots, c_m) \neq 0$, then we say that the set of vectors is **linearly dependent**.

Suppose, if $c_k \neq 0$, then

$$\mathbf{a}_{(k)} = -\frac{1}{c_k} \underbrace{\left(c_1\mathbf{a}_{(1)} + c_2\mathbf{a}_{(2)} + \cdots + c_m\mathbf{a}_{(m)}\right)}_{\text{does not include } c_k\mathbf{a}_{(k)}}$$

that is, $\mathbf{a}_{(k)}$ can be expressed as a linear combination of the other vectors in the set.

# Rank of a Matrix

The **rank** of $r$ of a matrix $\mathbf{A}$ is the maximum number of **linearly independent row vectors** of $\mathbf{a}$.

If $\mathbf{A}$ is of size $m \times n$, then $r = 0$ iff $\mathbf{A} = \mathbf{0}$.

**Definition**

Two matrices $\mathbf{A_1}$ and $\mathbf{A_2}$ are **row-equivalent** if $\mathbf{A_2}$ can be obtained from $\mathbf{A_1}$ by elementary row operations on $\mathbf{A_1}$ (and vice-versa).

# Rank of a Matrix (cont)

**Theorem 1**

Row-equivalent matrices have the same rank.

$\Longrightarrow$ Elementary row operations on a matrix does not change its rank.

# An Example

$$\text{let } \mathbf{A} = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \begin{array}{l} \leftarrow \mathbf{a}_{(1)} \\ \leftarrow \mathbf{a}_{(2)}. \\ \leftarrow \mathbf{a}_{(3)} \end{array}$$

since $(\mathbf{a}_{(3)} - \mathbf{a}_{(2)}) = (\mathbf{a}_{(2)} - \mathbf{a}_{(1)})$,
we have $\mathbf{a}_{(1)} - 2\mathbf{a}_{(2)} + \mathbf{a}_{(3)} = 0$.
$\implies$ $\mathbf{a}_{(1)}, \mathbf{a}_{(2)},$ and $\mathbf{a}_{(3)}$ are **linearly dependent**,
and rank$(\mathbf{A}) < 3$.

but $c_1\mathbf{a}_{(1)} + c_2\mathbf{a}_{(2)} = 0$ has only one solution: $c_1 = c_2 = 0$.
$\implies$ $\mathbf{a}_{(1)}$ and $\mathbf{a}_{(2)}$ are **linearly independent**,
and hence, rank$(\mathbf{A}) = 2$.

# Determination of Rank of a Matrix

Use row operations to reduce it to lower-echelon form. The number of non-zero rows is the rank.

For example, let $\mathbf{A} = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$.

Eliminating the first elements of the second and third rows, we get: $\begin{bmatrix} 1 & 2 & 3 \\ 0 & -3 & -6 \\ 0 & -6 & 12 \end{bmatrix}$, that is, $\begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & 1 & 2 \end{bmatrix}$.

# Determination of Rank of a Matrix (cont)

Now, eliminating element $(3, 2)$, we get: $\begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$.

Since only two rows are non-zero in the reduced (row-echelon) form, rank$(\mathbf{A}) = 2$.

# Rank and Linear Independence

**Theorem 2**[3]

Let there be $m$ vectors with $n$ components each.
Let $\mathbf{A}$ be the matrix (size $m \times n$) formed by these vectors as rows.

The $m$ vectors are linearly independent if $\text{rank}(\mathbf{A}(m \times n)) = m$.

The $m$ vectors are linearly dependent if $\text{rank}(\mathbf{A}(m \times n)) < m$.

[3] Study the proof in Kreyszig.

# Rank and Linear Independence (cont)

**Theorem 3**[4]

The rank of a matrix $\mathbf{A}$ equals the number of linearly independent column vectors of $\mathbf{A}$.

Hence, rank$(\mathbf{A})$=rank$(\mathbf{A}^\mathsf{T})$

[4] Study the proof in Kreyszig.

# An Example

If $\mathbf{A}$ is the matrix in the previous example, then

$$\mathbf{A}^{\mathsf{T}} = \begin{bmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 9 \end{bmatrix} \to \begin{bmatrix} 1 & 4 & 7 \\ 0 & -3 & -6 \\ 0 & -6 & -12 \end{bmatrix} \to \begin{bmatrix} 1 & 4 & 7 \\ 0 & 1 & 2 \\ 0 & 1 & 2 \end{bmatrix}$$

$$\to \begin{bmatrix} 1 & 4 & 7 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}.$$

Hence, $\text{rank}(\mathbf{A}^{\mathsf{T}}) = 2 = \text{rank}\mathbf{A}$.

# Rank and Linear Independence (cont)

**Theorem 4**[5]

Consider $m$ vectors of $n$ components each.
If $m > n$, then these vectors are **linearly dependent**.

$\Longrightarrow$ If $\mathbf{A}$ is of size $m \times n$, then rank $(\mathbf{A}) \leq \mathbf{min}(m, n)$.

[5] Study the proof in Kreyszig.

# Determinants

Students are expected to be knowledgeable about determinants, their properties, and also about Cramer's rule.