# Introduction to Linear Algebra

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# Numbers and More Numbers

- Simple numbers: scalars
  0, 2.4, $\pi$, $4/3$, ...
- Ordered pairs of numbers: *e.g.* complex numbers
  (1.2, 0.8)
- Ordered triplets of numbers: *e.g.* vectors
  (1.2, 3.4, -1)

# Other Ordered Sets of Numbers

- Lists
- Arrays
- Images
- Matrices
- ...

# Visual Confusion

- Ordered pair of numbers: $A = (a_1, a_2), B = (b_1, b_2)$.
  Complex numbers? 2-D vectors?
- Visually similar on a 2-D plane $(xy)$.
- In either case, $A + B = (a_1 + b_1, a_2 + b_2)$.
- If they are complex numbers, then
  $AB = (a_1 b_1 - a_2 b_2, a_1 b_2 + a_2 b_1)$.
- If they are vectors, then
  $A \cdot B = a_1 b_1 + a_2 b_2$ (scalar), and
  $A \times B = (0, 0, a_1 b_2 - a_2 b_1)$ (vector).

# MATRICES – 1

## Elementary Topics

# Matrices

- Rectangular arrays of numbers
- Obey certain rules of operations on them
- Come in various sizes
- Come in various types (diagonal, banded, sparse, triangular, ... )

# Matrix $\mathbf{A}$ of size $m \times n$

$$\mathbf{A} = \underbrace{\begin{bmatrix} a_{1,1} & a_{1,2} & \cdots & a_{1,n} \\ a_{2,1} & a_{2,2} & \cdots & a_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m,1} & a_{m,2} & \cdots & a_{m,n} \end{bmatrix}}_{n \text{ columns}} \left.\vphantom{\begin{matrix} a \\ a \\ \vdots \end{matrix}}\right\} m \text{ rows}$$

- If $m = n$, then $\mathbf{A}$ is a square matrix, and it has a (main/primary) diagonal, and, a cross or secondary diagonal.
- Square matrices are special.

# Some Types of Matrices

- Row matrix: just one row: $1 \times n$ : aka 'row vector'
- Column matrix: just one column: $n \times 1$ : aka 'column vector'
- Note: 'Vector' here has a different meaning! These are linear-algebra-vectors!
- Triangular matrices:
  - upper triangular
  - lower triangular

# Equality of Matrices

Two matrices, $\mathbf{A}$ and $\mathbf{B}$ are **equal** iff:

- size of $\mathbf{A}$ = size of $\mathbf{B}$.
- $a_{ij} = b_{ij}$, for each **appropriate** $i$ and $j$.

We then write: $\mathbf{A} = \mathbf{B}$.

Note:

- Equality is a symmetric property: $\mathbf{A} = \mathbf{B} \implies \mathbf{B} = \mathbf{A}$.
- It is also transitive: $\mathbf{A} = \mathbf{B}$ and $\mathbf{B} = \mathbf{C} \implies \mathbf{A} = \mathbf{C}$.

(Remember the Zeroth Law of Thermodynamics?)

# Addition of Matrices

Two matrices $\mathbf{A}$ and $\mathbf{B}$ can be added iff they are of the same size. In that case:

- $\mathbf{C} = \mathbf{A} + \mathbf{B}$
- $c_{ij} = a_{ij} + b_{ij}$ for each $i, j$
- The operation is commutative: $\mathbf{A} + \mathbf{B} = \mathbf{B} + \mathbf{A}$.

# Multiplication by a Scalar

A matrix $\mathbf{A}$ can be multiplied by a scalar, $\alpha$, and we obtain a matrix $\mathbf{D}$ of the same size.

- $\mathbf{D} = \alpha \mathbf{A}$
- $d_{ij} = \alpha\, a_{ij}$ for each $i, j$
- If $\alpha = -1$, then $\mathbf{D} = (-1)\mathbf{A} \equiv -\mathbf{A}$ is the negative of matrix $\mathbf{A}$.
  We can now define $\mathbf{A} - \mathbf{B} \equiv \mathbf{A} + (-1)\mathbf{B}$
- If $\alpha = 0$ then $0\mathbf{A} = \mathbf{0}$, where $\mathbf{0}$ is the zero matrix of the size of $\mathbf{A}$, each element of which is $0$.

# Some Rules

$$
\begin{array}{rcll}
\mathbf{A}+\mathbf{B} & = & \mathbf{B}+\mathbf{A} & \text{(commutative)} \\
(\mathbf{A}+\mathbf{B})+\mathbf{C} & = & \mathbf{A}+(\mathbf{B}+\mathbf{C}) & \text{(associative)} \\
 & = & \mathbf{A}+\mathbf{B}+\mathbf{C} & \text{(no confusion!)} \\
\mathbf{A}+\mathbf{0} & = & \mathbf{A}=\mathbf{0}+\mathbf{A} & \\
\mathbf{A}+(-\mathbf{A}) & = & \mathbf{A}-\mathbf{A}=\mathbf{0} & \\
\alpha(\mathbf{A}+\mathbf{B}) & = & \alpha\mathbf{A}+\alpha\mathbf{B} & \text{(distributive)} \\
(\alpha+\beta)\mathbf{A} & = & \alpha\mathbf{A}+\beta\mathbf{A} & \text{(distributive)} \\
\alpha(\beta\mathbf{A}) & = & (\alpha\beta)\mathbf{A} & \\
 & = & \alpha\beta\mathbf{A} & \text{(no confusion!)} \\
0\mathbf{A} & = & \mathbf{0} & \\
1\mathbf{A} & = & \mathbf{A} &
\end{array}
$$

# Column Matrix, Row Matrix

Column Matrix: Size: $m \times 1$.

$$\mathbf{a} = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_m \end{bmatrix}$$

Also known as a column **vector** of length $m$.

Row Matrix: Size: $1 \times n$.

$$\mathbf{b} = \begin{bmatrix} b_1 & b_2 & \cdots & b_n \end{bmatrix}$$

Also known as a row **vector** of length $n$.

# Inner Product

The inner product of two 'vectors' $\mathbf{a}$ and $\mathbf{b}$ of the same length $n$ is a scalar:

$$
\begin{aligned}
c &= a_1 b_1 + a_2 b_2 + \cdots + a_n b_n \\
&= \sum_{i=1}^{n} a_i b_i
\end{aligned}
$$

The inner product is sometimes represented as $\mathbf{a} \cdot \mathbf{b}$[1].
It is a generalisation of the dot product of (real-world) vectors.

[1] In tensor notation, this is simply written as $a_i b_i$.

# Multiplication of Matrices

- This is restrictive, compared to our old ideas of multiplication.
- $\mathbf{AB}$ is defined iff the sizes of $\mathbf{A}$ and $\mathrm{B}$ are **compatible**.
- The number of columns $\mathbf{A}$ must equal the number of rows of $\mathbf{B}$.
- That is, if the size of $\mathbf{A}$ is $m \times n$, then that of $\mathbf{B}$ should be $n \times p$.
- The product matrix, $\mathbf{C} = \mathbf{AB}$ will be of size $m \times p$.

# Multiplication – Definition

- Let $\mathbf{A}$ be of size $m \times n$, and $\mathbf{B}$ of size $n \times p$.
- Then $\mathbf{C} = \mathbf{AB}$ has elements

$$c_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj}$$

for $i = 1, \cdots, m$ and $j = 1, \cdots, p$.

# Visual Depiction

$$
\begin{array}{r|cccc|ccc|c}
\multicolumn{1}{r}{} & & & & \uparrow & \multicolumn{3}{c}{\text{column } j \text{ of } \mathbf{B}} & \\
\multicolumn{1}{r}{} & & & & \uparrow & \cdots & b_{1j} & \cdots & \uparrow \\
\multicolumn{1}{r}{} & & & & n & \cdots & b_{2j} & \cdots & n \\
\multicolumn{1}{r}{} & & & & \downarrow & \mathbf{B} & \vdots & \ddots & \downarrow \\
\multicolumn{1}{r}{} & \multicolumn{4}{c}{\leftarrow n \rightarrow} & \cdots & b_{nj} & \cdots & \\
\hline
 & \vdots & \vdots & \mathbf{A} & \vdots & \mathbf{C} & \vdots & \ddots & \uparrow \\
\text{row } i \text{ of } \mathbf{A} & a_{i1} & a_{i2} & \cdots & a_{in} & \cdots & \boxed{c_{ij}} & \cdots & m \\
 & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \downarrow \\
\hline
\multicolumn{1}{r}{} & \multicolumn{4}{c}{\leftarrow n \rightarrow} & \multicolumn{3}{c}{\leftarrow p \rightarrow} & \\
\end{array}
$$

$c_{ij}$ is the inner product of (row $i$ of $\mathbf{A}$) and (column $j$ of $\mathbf{B}$).
Note: their sizes are equal.

# Matrix Product

- Generalisation of the inner product to matrices.
- The first matrix $(\mathbf{A})$ is considered a vertical stack of row vectors.
- The second matrix $(\mathbf{B})$ is considered a horizontal stack of column vectors.
- The product matrix $(\mathbf{C})$ is an array of the inner products of corresponding vectors.

# Matrix Product: Commutation?

If $\underbrace{\mathbf{A}}_{m\times n} \times \underbrace{\mathbf{B}}_{n\times p} = \underbrace{\mathbf{C}}_{m\times p}$ is possible

then $\underbrace{\mathbf{B}}_{n\times p} \times \underbrace{\mathbf{A}}_{m\times n}$ is not possible, unless $p = m$.

# Matrix Product - Properties 1

$$\begin{array}{cccc}
\mathbf{A} & \mathbf{B} & \rightarrow & \mathbf{C} \\
(3 \times \mathbf{4}) & (\mathbf{4} \times 5) & \rightarrow & (3 \times 5) \\
 & & & \\
\mathbf{B} & \mathbf{A} & : & \text{Not possible!} \\
(4 \times \mathbf{5}) & (\mathbf{3} \times 4) & : & (\mathbf{5} \neq \mathbf{3})
\end{array}$$

# Matrix Product - Properties 2

- For both $\mathbf{AB}$ and $\mathbf{BA}$ to exist, the shapes of $\mathbf{A}$ and $\mathbf{B}$ must be 'flipped'. If $\mathbf{A}$ is $m \times n$, then $\mathbf{B}$ must be $n \times m$.
- $\mathbf{AB}$ will be $m \times m$, $\mathbf{BA}$ will be $n \times n$. Unless $m = n$, their sizes will be different.

# Matrix Product is Not Commutative

If $m = n$, i.e. if $\mathbf{A}$ and $\mathbf{B}$ are square, even then $\mathbf{AB} \neq \mathbf{BA}$, in general.

$$\mathbf{A} = \begin{bmatrix} 2 & 1 \\ 0 & 0 \end{bmatrix}, \mathbf{B} = \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}.$$

$$\mathbf{AB} = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, \mathbf{BA} = \begin{bmatrix} 0 & 0 \\ 2 & 1 \end{bmatrix}.$$

# Matrix Product - Illustration

If $\mathbf{A}$ and $\mathbf{B}$ are square and $\mathbf{AB} = \mathbf{0}$, then this does not imply that $\mathbf{A} = \mathbf{0}$ or $\mathbf{B} = \mathbf{0}$ or $\mathbf{BA} = \mathbf{0}$ in general.

$$\mathbf{A} = \begin{bmatrix} 4 & 2 \\ 2 & 1 \end{bmatrix}, \mathbf{B} = \begin{bmatrix} 2 & -1 \\ -4 & 2 \end{bmatrix}.$$

$$\mathbf{AB} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \mathbf{BA} = \begin{bmatrix} 6 & 3 \\ -12 & -6 \end{bmatrix}.$$

# Some More Rules

$$
\begin{aligned}
(\alpha\mathbf{A})\mathbf{B} &= \alpha(\mathbf{AB}) = \mathbf{A}(\alpha\mathbf{B}) = \alpha\mathbf{AB} \\
&\qquad\qquad\text{(maintain order of matrices)} \\
\mathbf{A}(\mathbf{BC}) &= (\mathbf{AB})\mathbf{C} = \mathbf{ABC} \qquad \text{(associative)} \\
(\mathbf{A}+\mathbf{B})\mathbf{C} &= \mathbf{AC} + \mathbf{BC} \qquad \text{(distributive)} \\
\mathbf{C}(\mathbf{A}+\mathbf{B}) &= \mathbf{CA} + \mathbf{CB} \qquad \text{(distributive)}
\end{aligned}
$$

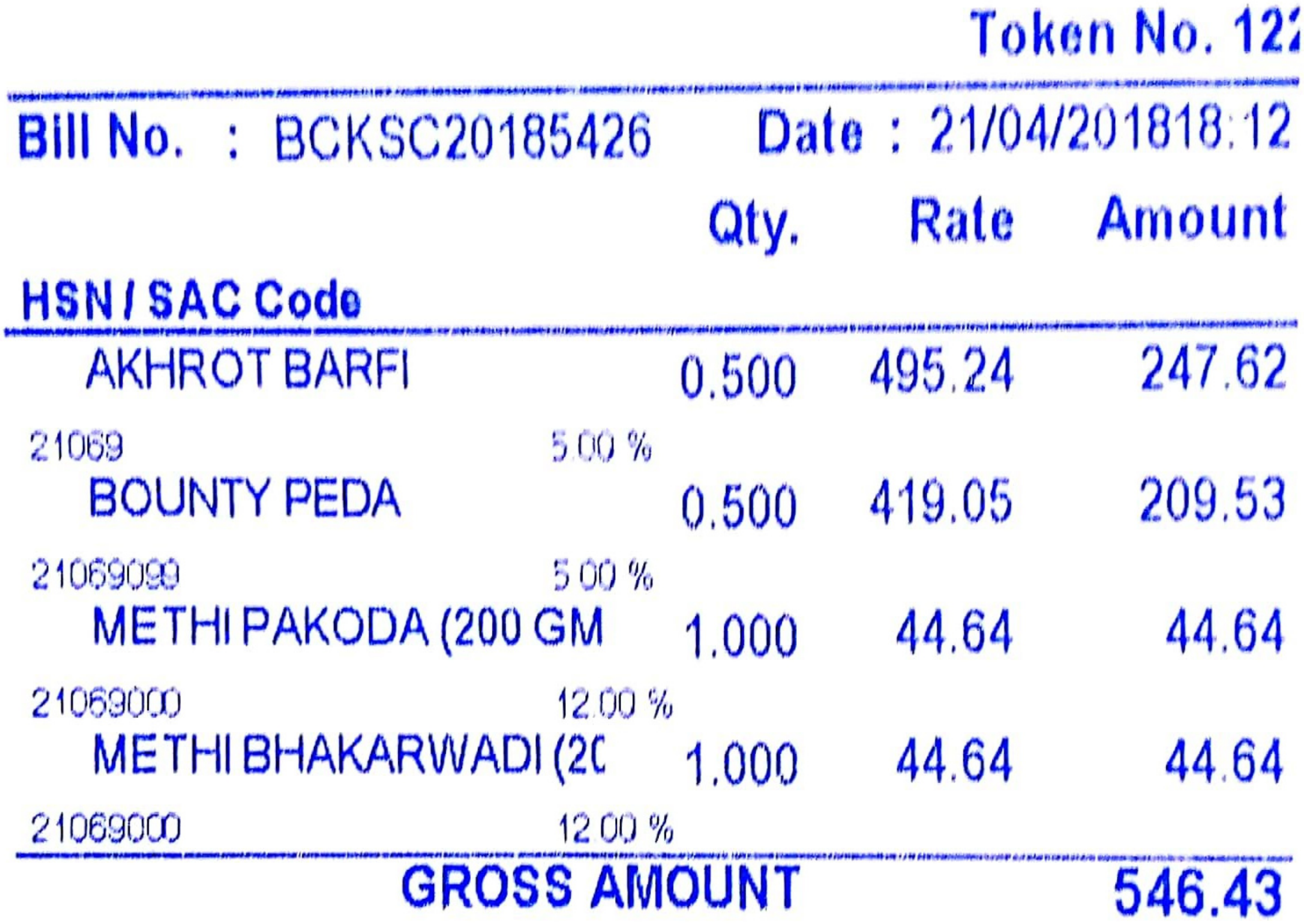
# Matrix and Vector

A matrix and vector (of appropriate sizes) can be multiplied.
For example:

$$\begin{array}{cccc} \mathbf{A} & \mathbf{x} & \rightarrow & \mathbf{b} \\ (m \times n) & (n \times 1) & & (m \times 1) \end{array}$$

$$\begin{array}{cccc} \mathbf{y} & \mathbf{B} & \rightarrow & \mathbf{c} \\ (1 \times m) & (m \times n) & & (1 \times n) \end{array}$$

# A Shopping List



Token No. 12?

Bill No. : BCKSC20185426 Date : 21/04/201818:12

| HSN / SAC Code | | Qty. | Rate | Amount |
|---|---|---|---|---|
| AKHROT BARFI | | 0.500 | 495.24 | 247.62 |
| 21069 | 5.00 % | | | |
| BOUNTY PEDA | | 0.500 | 419.05 | 209.53 |
| 21069099 | 5.00 % | | | |
| METHI PAKODA (200 GM | | 1.000 | 44.64 | 44.64 |
| 21069000 | 12.00 % | | | |
| METHI BHAKARWADI (2( | | 1.000 | 44.64 | 44.64 |
| 21069000 | 12.00 % | | | |
| GROSS AMOUNT | | | | 546.43 |

# A Shopping List - Inner Product

| | Price |
|---|---|
| AB | 495.24 |
| BP | 419.05 |
| MP | 44.64 |
| MB | 44.64 |

| | AB | BP | MP | MB | |
|---|---|---|---|---|---|
| Quantity | 0.5 | 0.5 | 1 | 1 | 546.43 |

# Matrix Multiplication Example

Price Matrix

| | GWS | GES |
|---|---|---|
| Bread | 5 | 3 |
| Meat | 5 | 4 |
| Veg | 3 | 4 |
| Fruit | 2 | 4 |

| Req Matrix | Bread | Meat | Veg | Fruit |
|---|---|---|---|---|
| A | 4 | 3 | 2 | 1 |
| B | 2 | 1 | 4 | 3 |
| C | 4 | 1 | 2 | 3 |

Cost Matrix

| | |
|---|---|
| 43 | 36 |
| 33 | 38 |
| 37 | 36 |

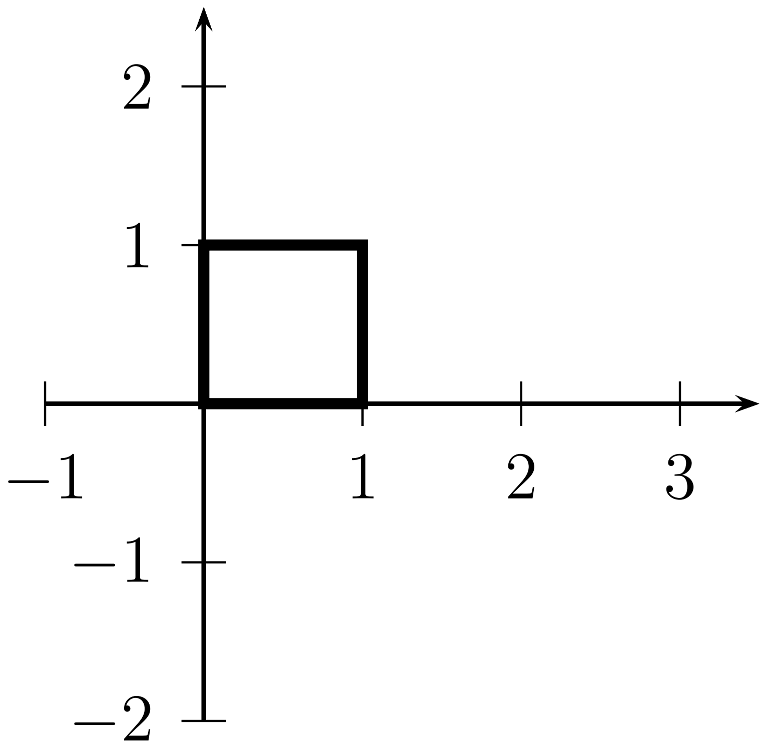
# Coordinate Transformations

- $\mathbf{x_2} = \mathbf{A}\mathbf{x_1}$ represents a linear transformation.
- It can scale, rotate, shear a set of points (or a figure).
- $\mathbf{x_1}$ is the original point in a column-vector format.
- $\mathbf{A}$ is the transformation matrix.
- $\mathbf{x_2}$ is the transformed location of the point (in a column-vector format).

Some examples follow.

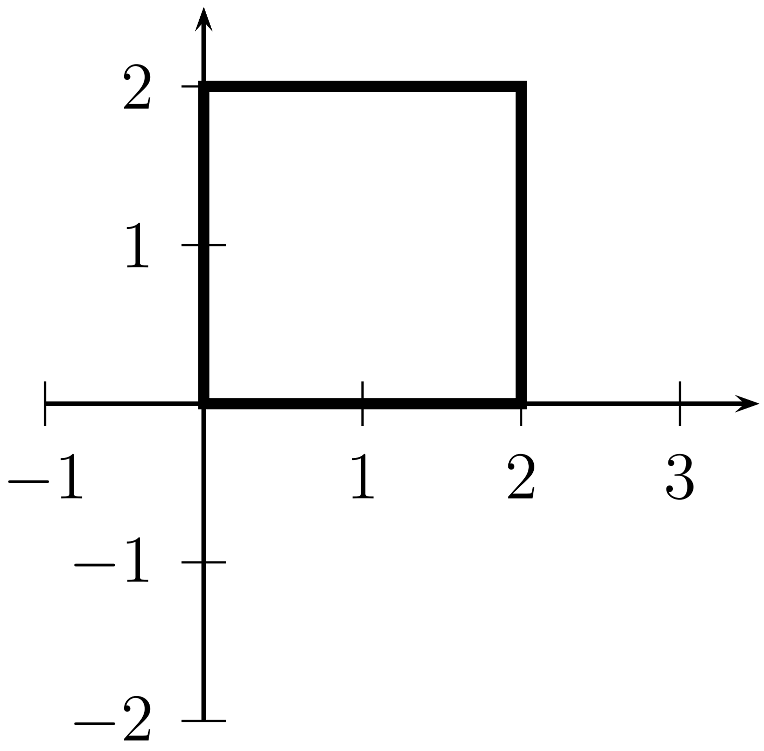
# The original figure - a unit square

$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ does not change anything!



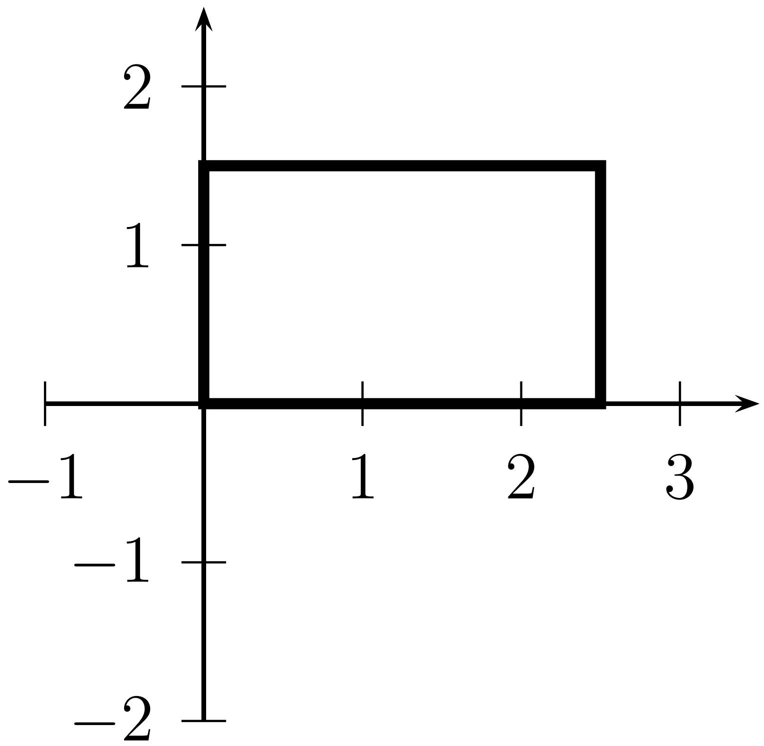
# Scaling - Uniform

$\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$ scales by 2 in each direction.



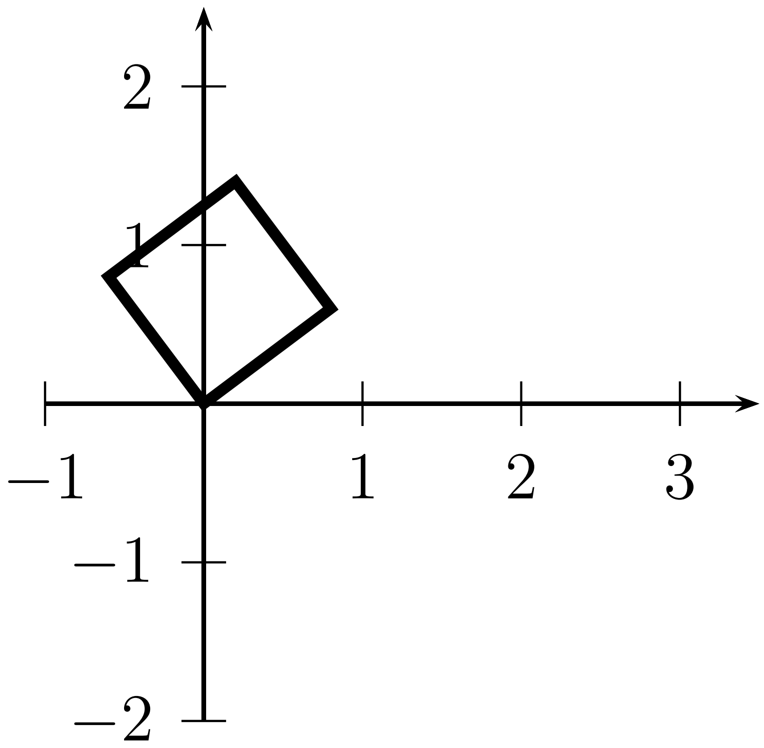
# Scaling - Nonuniform

$\begin{bmatrix} 2.5 & 0 \\ 0 & 1.5 \end{bmatrix}$ does unequal scaling.



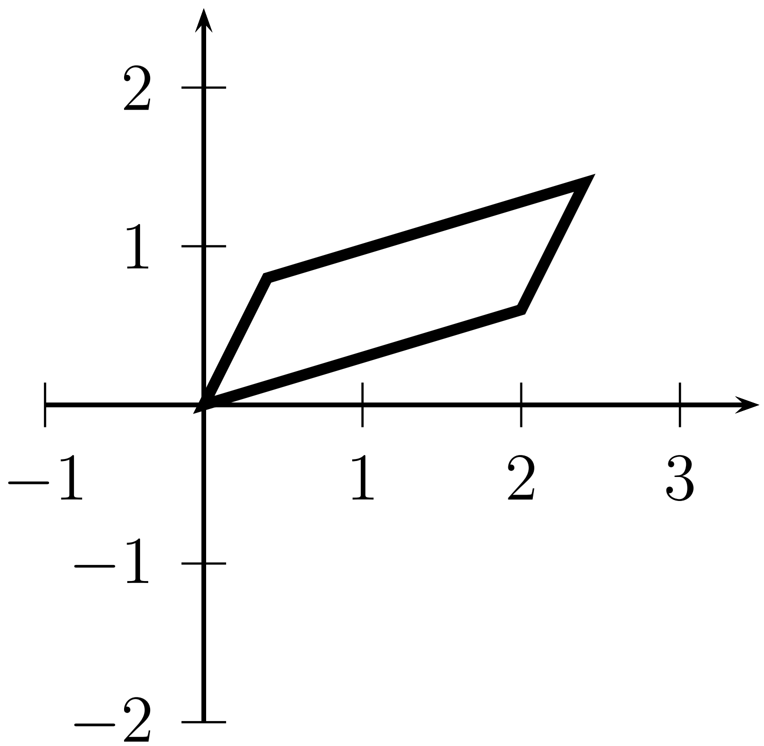
# Rotation

$$\begin{bmatrix} 0.8 & -0.6 \\ 0.6 & 0.8 \end{bmatrix}$$ rotates by $\arctan(\mathbf{0.6}/\mathbf{0.8})$.



# Arbitrary Transformation

$\begin{bmatrix} 2.0 & 0.4 \\ 0.6 & 0.8 \end{bmatrix}$ does rotation, shear, stretch!




Can you set up *reflections* of various kinds?

# Homework

Go through the examples in Kreyszig, 10th Ed, pp 269–270.

Create your own examples.

# Transpose of a Matrix

- Transposition is a **unary** operation.
- If $\mathbf{A}$ is of size $(m \times n)$, then $\mathbf{B} = \mathbf{A}^\mathsf{T}$ is of size $(n \times m)$, and
- $b_{ij} = a_{ji}$ for all appropriate $(i, j)$.
- The transpose operation is similar to a reflection across the diagonal.
- Transpose of a row vector is a column vector of the same length.
- Transpose of a column vector is a row vector of the same length.

# Illustration

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix}; \qquad \mathbf{B} = \mathbf{A}^{\mathsf{T}} = \begin{bmatrix} a_{11} & a_{21} \\ a_{12} & a_{22} \\ a_{13} & a_{23} \end{bmatrix}$$

# Rules for Transposition

$$\begin{aligned}
(\mathbf{A}^{\mathsf{T}})^{\mathsf{T}} &= \mathbf{A} \\
(\mathbf{A}+\mathbf{B})^{\mathsf{T}} &= \mathbf{A}^{\mathsf{T}}+\mathbf{B}^{\mathsf{T}} \\
(\alpha\mathbf{A})^{\mathsf{T}} &= \alpha\mathbf{A}^{\mathsf{T}} \\
(\mathbf{A}\mathbf{B})^{\mathsf{T}} &= \mathbf{B}^{\mathsf{T}}\mathbf{A}^{\mathsf{T}} \quad \text{(note order-reversal)}
\end{aligned}$$

# Special Type of Square Matrices

- If a matrix $\mathbf{A}$ is square, then $\mathbf{A}$ and $\mathbf{A}^\mathsf{T}$ have the same size.
- $\mathbf{A}$ is called symmetric iff $\mathbf{A} = \mathbf{A}^\mathsf{T}$.
- $\mathbf{A}$ is called skew-symmetric iff $\mathbf{A} = -\mathbf{A}^\mathsf{T}$.

# Some Properties

Show that:

- The diagonal elements of any skew-symmetric matrix are zero.
- Any square matrix can be shown to be the sum of two matrices, a symmetric one, and a skew-symmetric one. Derive expressions for these matrices.