$$F(s) = \frac{1}{s}\left[\frac{-477s}{(s+3.8)^2+(380)^2} + \frac{170(s+1)}{(s+0.5)^2+(7.27)^2} + \frac{308s}{(s+8.2)^2+(820)^2}\right]$$

and assume the sampling interval is $T=0.02$. Each of the three terms in the above expression is a body-bending mode of a missile and the $1/s$ term comes from a zero-order-hold circuit. Using (7), $F(z)$ can be written as

$$F(z) = \alpha_0 z^6 + \alpha_5 z^5 + \cdots + \alpha_1 z / [(z-1) \cdot (z^2 - 2ze^{-3.8T}\cos 380T + e^{-7.6T}) \cdot (z^2 - 2ze^{-0.5T}\cos 7.27T + e^{-T}) \cdot (z^2 - 2ze^{-8.2T}\cos 820T + e^{-16.4T})]. \quad (11)$$

For this example it is not difficult to find $f(t)$ in closed form because $F(s)$ is given as three separate terms.

A computer program was written to do the following: 1) compute $f(T)$, $f(2T)$, $\cdots$, $f(6T)$; 2) multiply out the denominator of (11); and 3) use (10) to compute α_i. The results are

$$\begin{aligned}
\alpha_1 &= -1.0885516 & \beta_0 &= -0.60653066\\
\alpha_2 &= -2.0658531 & \beta_1 &= 1.0472703\\
\alpha_3 &= -0.3831613 & \beta_2 &= -0.46870904\\
\alpha_4 &= -0.3240850 & \beta_3 &= 0.31790131\\
\alpha_5 &= 2.0865444 & \beta_4 &= -0.59262253\\
\alpha_6 &= 2.0585400 & \beta_5 &= 1.4206653\\
& & \beta_6 &= -2.1179747\\
& & \beta_7 &= 1.0000000.
\end{aligned}$$

Conclusions

A computer method has been given for finding the z transform from a given Laplace transform. The major difficulty with this method is that it is necessary to invert the Laplace transform but for some problems this is not difficult as shown by the example considered in the last section. However, in general, it will be necessary to invert the Laplace transform numerically. One advantage of this method is that there is no added computation for multiple poles.

Prentiss N. Robinson
Dep. Elec. Eng.
Univ. Maryland
College Park, Md. 20742

Guner S. Robinson
Communications Satellite Corp.
COMSAT Labs.
Clarksburg, Md. 20734

References

[1] E. I. Jury, *Theory and Application of the Z-Transform Method.* New York: Wiley, 1964.
[2] D. P. Lindorff, *Theory of Sampled-Data Control Systems.* New York: Wiley, 1965.
[3] M. Cuenod and A. Durling, *A Discrete-Time Approach for System Analysis.* New York: Academic Press, 1969.
[4] A. Papoulis, "A new method of inversion of the Laplace transform," *Quart. Appl. Math.*, vol. 14, pp. 405–414, 1956.
[5] C. Lancsoz, *Applied Analysis.* Englewood Cliffs, N. J.: Prentice-Hall, 1957.
[6] W. T. Weeks, "Numerical inversion of Laplace transforms using Laguerre functions," *J. Ass. Comput. Mach.*, vol. 13, pp. 419–426, July 1966.
[7] L. A. Schmittroth, "Numerical inversion of Laplace transforms," *Commun. Ass. Comput. Mach.*, pp. 171–173, 1960.
[8] R. Piessens, "New quadrature formulas for the numerical inversion of the Laplace transform," *BIT*, vol. 9, pp. 1–11, 1969.
[9] R. Bellman, R. Kalaba, and J. Lockett, *Numerical Inversion of the Laplace Transform.* New York: American Elsevier, 1966.
[10] R. Piessens, "Numerical inversion of the Laplace transform," *IEEE Trans. Automat. Contr.* (Corresp.), vol. AC-14, pp. 299–301, June 1969.
[11] H. Dubner and J. Abate, "Numerical inversion of Laplace transforms by relating them to finite Fourier cosine transform," *J. Ass. Comput. Mach.*, vol. 15, pp. 115–123, Jan. 1969.
[12] J. Stehfest, "Numerical inversion of Laplace transforms," *Commun. Ass. Comput. Mach.*, vol. 13, pp. 47–49, Jan. 1970.
[13] C. D. Han, "A note on the numerical inversion of the Laplace transform by Legendre polynomials," *IEEE Trans. Automat. Contr.* (Corresp.), vol. AC-12, pp. 230–231, Apr. 1967.
[14] G. E. Cook, E. D. Denman, and H. M. Carr, "Numerical inversion of Laplace transforms by the Laguerre–Gauss quadrature approximation," *IEEE Trans. Automat. Contr.* (Corresp.), vol. AC-12, pp. 623–624, Oct. 1967.
[15] W. J. Felts and G. E. Cook, "On the numerical approximation of the Laplace transform," *IEEE Trans. Automat. Contr.* (Corresp.), vol. AC-14, pp. 297–299, June 1969.
[16] H. M. Carr and G. E. Cook, "Explicit inversion of the Laplace transform by a Jacobi–Gauss quadrature approximation," *IEEE Trans. Automat. Contr.* (Corresp.), vol. AC-14, pp. 777–779, Dec. 1969.
[17] M. T. Stroot and G. H. Weiss, "Some test problems with a Laplace transform inversion technique," Div. Comput. Res. Technol., National Institutes of Health, Dep. Health, Education and Welfare, Bethesda, Md., unpublished rep.
[18] M. L. Liou, "A novel method of evaluating transient response," *Proc. IEEE*, vol. 54, pp. 20–23, Jan. 1966.
[19] "Laplace transform inversion," *Program Library Users Guide, Mark I*, Time-Sharing Service, General Electric Co., 1969.
[20] C. F. Chen and R. E. Yates, "A new matrix formula for the inverse Laplace transformation," *Trans. ASME, J. Bas. Eng.*, pp. 269–272, June 1967.
[21] C. F. Chen and L. S. Shieh, "Generalization and computerization of Heaviside's expansion for performing the inverse Laplace transform of high order systems," in *Proc. IEEE, Region III Conv. Rec.*, Jackson, Fla., 1967, pp. 285–294.
[22] R. Piessens, "Partial fraction expansion and inversion of rational Laplace transforms," IEEE Program Library No. (CP5).

A Note on "Discrete Hilbert Transform"

Abstract

An expression for the discrete Hilbert transform (DHT) is given for the case of an odd number of points.

In the above paper,[1] Čížek gave the following formulas for the DHT:

$$g_i = \frac{2}{N} \sum_{v=0,2,4,\cdots} f_v \cot (v-i)\frac{\pi}{N}, \qquad \text{for } i \text{ odd} \quad (19)^2$$

$$g_i = \frac{2}{N} \sum_{v=1,3,5,\cdots} f_v \cot (v-i)\frac{\pi}{N}, \qquad \text{for } i \text{ even} \quad (20)$$

where f_v is the original sequence, g_i is the transformed sequence, and N is the number of points in each sequence.

In order to correspond with the usual definition of the Hilbert transform,[3,4] the g_i above should be taken with the opposite sign.

Since

$$\mathcal{H}\{f(t)\} = \frac{1}{\pi}\int_{-\infty}^{+\infty} f(\tau)\cdot\frac{1}{t-\tau}\,d\tau = \frac{1}{\pi t} * f(t)$$

and therefore

$$g_i = \frac{2}{N} \sum_{v=0,2,4,\cdots} f_v \cdot \cot (i-v)\frac{\pi}{N}, \qquad \text{for } i \text{ odd}$$

$$g_i = \frac{2}{N} \sum_{v=1,3,5,\cdots} f_v \cdot \cot (i-v)\frac{\pi}{N}, \qquad \text{for } i \text{ even.}$$

These expressions, however, are valid only for N even, although this fact is not explicitly stated in the paper.[1]

Manuscript received July 7, 1971.
[1] V. Čížek, *IEEE Trans. Audio Electroacoust.*, vol. AU-18, pp. 340–343, Dec. 1970.
[2] The unprimed numbers refer to the ones in the original paper.
[3] H. E. Rowe, *Signals and Noise in Communication Systems.* Princeton, N. J.: Van Nostrand, 1965, p. 17.
[4] A. Erdélyi, *Tables of Integral Transforms.* New York: McGraw-Hill, 1954.

For N odd the sequence H_k (the discrete representation of $-j$ sign ω) becomes

$$H_k = \begin{cases} -j, & k = 1, 2, \cdots, N/2 - 1/2 \\ 0, & k = 0 \\ +j, & k = N/2 + 1/2, \cdots, N-1. \end{cases} \qquad (14')$$

By some mathematical manipulations similar to the ones used by Čížek,[1] one obtains the following expression for DHT for an odd number of points:

$$g_i = \frac{1}{N} \sum_{v=0}^{N-1} f_v \cdot c(i-v), \qquad i = 0, 1, \cdots, N-1 \qquad (18')$$

where

$$c(i-v) = \begin{cases} 0, & \text{for } i - v = 0 \\ \cot (i-v) \dfrac{\pi}{N} - \dfrac{(-1)^{i-v}}{\sin (i-v) \dfrac{\pi}{N}}, & \text{for } i - v \neq 0. \end{cases}$$

Federico Bonzanigo
Inst. Appl. Phys.
Federal Inst. Tech. (ETH)
Zurich, Switzerland

A Note on the Spectral Analysis of Periodic Functions

Abstract

An analysis of the Fourier transform for a truncated cosine waveform is given that illustrates the dependence of the transform on the phase as the period of the waveform becomes large with respect to the record length. The effects of sampling on the transform are then discussed and illustrated.

The use of the discrete Fourier transform for the detection of a periodic signal is based on the fact that the Fourier transform of a sinusoidal waveform consists of two impulse functions, each having a weight of one-half the waveform magnitude. The discrete Fourier transform of a sampled finite length record provides an approximation to the impulses that is dependent on both the record length and the sampling rate. This correspondence describes the effects of these dependencies.

Given a finite length signal record

$$x(t) = [A \cos (\omega_o t + \theta)][u(t) - u(t - 2R)] \qquad (1)$$

as shown in Fig. 1, the Fourier transform of the record is

$$X(\omega) = e^{-j\omega R} \left[A e^{j(\theta + \omega_o R)} \frac{\sin [(\omega - \omega_o) R]}{\omega - \omega_o} + A e^{-j(\theta + \omega_o R)} \frac{\sin [(\omega + \omega_o) R]}{\omega + \omega_o} \right]. \qquad (2)$$

Manuscript received August 9, 1971.

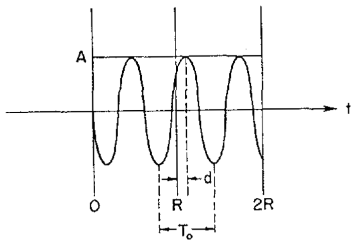

Fig. 1. Finite length sinusoidal record. Total record length $= 2R$; peak amplitude $= A$; period of sinusoid $= T_o$; angular frequency of sinusoid $= \omega_o = 2\pi/T_o$; phase angle (in degrees) $= \theta = d/T_o \times 360$.

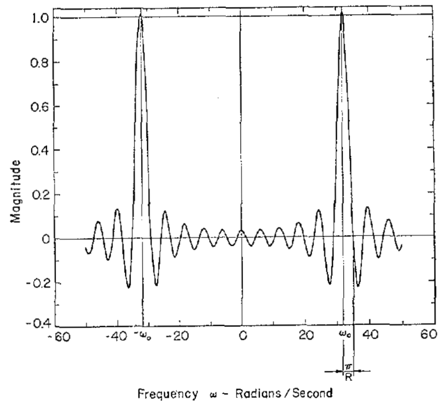

Fig. 2. Magnitude plot of the Fourier transform of the function $X(t) = [\cos 32t]\,[u(t) - u(t-2)]$.

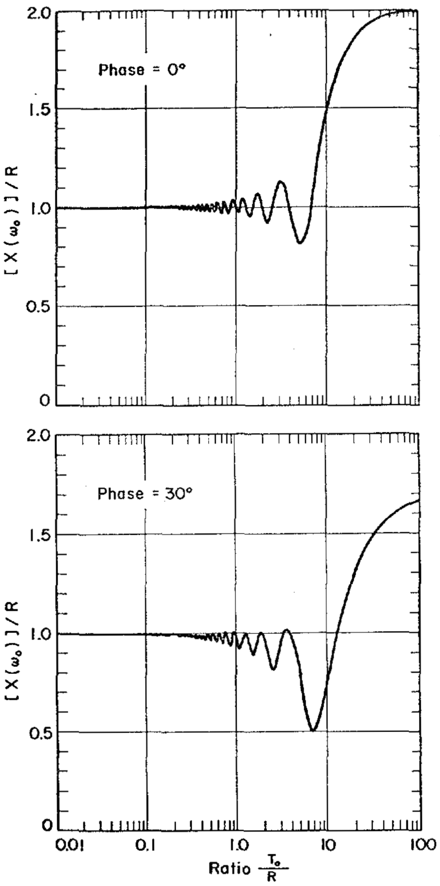

Fig. 3. Plots of the normalized magnitudes of the transform of $X(t)$ evaluated at $\omega = \omega_o$ and $\theta = 0°$ and $30°$. The magnitudes are plotted as a function of the ratio of the sinusoid period to the record semilength.

To illustrate the approximation to the impulses, a plot of $|X(\omega)|$ for $\omega_o = 32$, $A = R = 1$, and $\theta = 0$ is shown in Fig. 2. The dependence on the amplitude of $|X(\omega)|$ at the frequencies

$$\omega = \pm \omega_o \qquad (3)$$

on the period T_o and the phase angle θ for a fixed record length is shown by evaluating the absolute value of (2) at $\omega = \omega_o$. This results in

$$|X(\omega_o)| = A \left[1 + \frac{\sin^2 \left(4\pi \dfrac{R}{T_o}\right)}{16\pi^2 \left(\dfrac{R}{T_o}\right)^2} + \frac{\sin \left(4\pi \dfrac{R}{T_o}\right)}{2\pi \dfrac{R}{T_o}} \cos \left(2\theta + 4\pi \frac{R}{T_o}\right) \right]^{\frac{1}{2}}. \qquad (4)$$

Plots of (4), normalized by R, as a function of T_o/R for $\theta = 0$, 30°, 60°, and 90° are given in Figs. 3 and 4 for $A = 1$. These curves show that the amplitude of $|X(\omega_o)|$ is highly phase dependent for $0.1R < T_o < \infty$. For T_o less than $0.1R$, the magnitude of $|X(\omega_o)|/R$ is approximately one with θ having less effect as T_o becomes small. For T_o greater than $0.1R$ the value of $|X(\omega_o)|/R$ oscillates as T_o increases and as T_o becomes large:

$$|X(\omega_o)|/R \rightarrow 2 \cos \theta. \qquad (5)$$

Thus for T_o large with respect to R, the magnitude becomes directly proportional to the cosine of the phase.

The Fourier transform of the sampled function

$$x^*(t) = [A \cos (\omega_o t + \theta)][u(t) - u(t - 2R)] \cdot \left[\sum_{n=-\infty}^{\infty} \delta(t - nT) \right] \qquad (6)$$

is equal to the discrete Fourier transform at a discrete set of points and is given by