MATRICES – 3

Inverse of a Matrix

The Inverse of a Matrix

We consider only square matrices here.

Let $\mathbf{A}$ be a square matrix $(n \times n)$.
If there is another matrix $\mathbf{A}^{-1}$ such that $\mathbf{A}\mathbf{A}^{-1} = \mathbf{I}$,
where $\mathbf{I}$ is a unit (or identity) matrix of the same size as $\mathbf{A}$,
then $\mathbf{A}$ is called **the inverse of $\mathbf{A}$**.[6]

If the inverse exists for a matrix $\mathbf{A}$, then $\mathbf{A}$ is called **invertible** or **nonsingular**.
For such a matrix, $\det \mathbf{A} \neq 0$.

Otherwise, $\mathbf{A}$ is called a **singular** matrix.

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[6] Show that if we have a matrix $\mathbf{B}$ such that $\mathbf{B}\mathbf{A} = \mathbf{I}$, then $\mathbf{B} = \mathbf{A}^{-1}$; and hence, $\mathbf{A}^{-1}\mathbf{A} = \mathbf{I} = \mathbf{A}\mathbf{A}^{-1}$.

The Inverse is Unique

Let $\mathbf{B}$ and $\mathbf{C}$ be inverses of the matrix $\mathbf{A}$. Then

$$\begin{aligned}
\mathbf{B} &= \mathbf{I}\,\mathbf{B} && \text{(multiplication by a unit matrix)} \\
&= (\mathbf{CA})\mathbf{B} && (\mathbf{C} \text{ is an inverse of } \mathbf{A}) \\
&= \mathbf{C}(\mathbf{AB}) && \text{(matrix multiplication is associative)} \\
&= \mathbf{C}\,\mathbf{I} && (\mathbf{B} \text{ is an inverse of } \mathbf{A}) \\
&= \mathbf{C} && \text{(multiplication by a unit matrix)}
\end{aligned}$$

Thus, the inverse of a matrix, if it exists, is unique.

Existence and Rank

Theorem 1[7]

The inverse of a square matrix exists iff it has full rank.

That is:
If $\mathbf{A}$ is of size $n \times n$, $\mathbf{A}^{-1}$ exists iff rank$(\mathbf{A}) = n$.

Thus,
$\mathbf{A}$ is nonsingular if rank$(\mathbf{A}) = n$ and
$\mathbf{A}$ singular if rank$(\mathbf{A}) < n$.

[7] Study the proof of each theorem in Kreyszig.

Gauss-Jordan Elimination

This is a neat and simple method to determine the inverse of a matrix, if it exists.

Let
$\mathbf{A}$: given $(n \times n)$ matrix, hopefully nonsingular,
and $\mathbf{X}$: inverse of $\mathbf{A}$, also $(n \times n)$, to be determined.

We have: $\mathbf{A}\,\mathbf{X} = \mathbf{I}$.

So, we need to solve n sets of $(n \times n)$ LAEs, each with a different RHS vector (representing different column vectors of $\mathbf{I}$), but with the same coefficient matrix $(\mathbf{A})$.

Gauss-Jordan Elimination (cont)

The first of these will be	$\mathbf{A}\,\mathbf{x}_{(1)} = \mathbf{e}_{(1)}$,
the second will be	$\mathbf{A}\,\mathbf{x}_{(2)} = \mathbf{e}_{(2)}$,
the i-th will be	$\mathbf{A}\,\mathbf{x}_{(i)} = \mathbf{e}_{(i)}$,
and the last of these will be	$\mathbf{A}\,\mathbf{x}_{(n)} = \mathbf{e}_{(n)}$,

where $\mathbf{x}_{(i)}$ is the i-th column of $\mathbf{A}^{-1}$, and $\mathbf{e}_i$ is a column vector in which the i-th element is 1, and all other elements are zero.

So, we need to work with n LAE systems with augmented matrices $[\mathbf{A} \mid \mathbf{e}_1], \cdots, [\mathbf{A} \mid \mathbf{e}_i], \cdots, [\mathbf{A} \mid \mathbf{e}_n]$.

Since the coefficient matrix is common, we work with an extended augmented matrix, $[\mathbf{A} \mid \mathbf{I}]$.

Extended Augmented Matrix

$$
[\mathbf{A} \,\vdots\, \mathbf{I}] = \left[\begin{array}{cccc:cccc}
a_{11} & a_{12} & \cdots & a_{1n} & 1 & 0 & \cdots & 0 \\
a_{21} & a_{22} & \cdots & a_{1n} & 0 & 1 & \cdots & 0 \\
\vdots & \vdots & \ddots & a_{1n} & \vdots & \vdots & \ddots & \vdots \\
a_{n1} & a_{n2} & \cdots & a_{nn} & 0 & 0 & \cdots & 1
\end{array}\right]
$$

We use Gauss elimination on this to reduce the left part to an upper-diagonal matrix, to get $[\mathbf{U} \,\vdots\, \mathbf{H}]$, where $\mathbf{U}$ is upper triangular.

If all diagonal terms of $\mathbf{U}$ are non-zero, then $\mathbf{A}$ is nonsingular (full rank)

Otherwise, $\mathbf{A}$ is singular and hence non-invertible.

Computation of Inverse

If $\mathbf{A}$ is nonsingular, then we proceed further with $[\mathbf{U} \mid \mathbf{H}]$, and using row operations reduce it to a matrix where the left side is a unit matrix: $[\mathbf{I} \mid \mathbf{K}]$.

The matrix $\mathbf{K}$ is the inverse of $\mathbf{A}$.[8]

[8]The students should satisfy themselves that this is true.

Illustration of Inversion using Gauss-Jordan

Let us invert the matrix used in the row-echelon form example. The extended auxilimary matrix is:

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 2 & 1 & 0 & 0 \\ 2 & 2 & 1 & 0 & 1 & 0 \\ 3 & 1 & 2 & 0 & 0 & 1 \end{array}\right].$$

Row operations are executed to reduce the left part to row-echelon form with all diagonal elements equal to 1.

R2$/2$, and R3$/3$ leads to:

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 2 & 1 & 0 & 0 \\ 1 & 1 & 1/2 & 0 & 1/2 & 0 \\ 1 & 1/3 & 2/3 & 0 & 0 & 1/3 \end{array}\right].$$

Illustration of Inversion (cont)

The first column can now be brought into the required form by R1 $\leftarrow$ R2$-$R1, and R3 $\leftarrow$ R3$-$R1:

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 2 & 1 & 0 & 0 \\ 0 & -1 & -3/2 & -1 & 1/2 & 0 \\ 0 & -5/3 & -4/3 & -1 & 0 & 1/3 \end{array}\right].$$

R2 and R3 are divided by their leading non-zero elements to get:

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 2 & 1 & 0 & 0 \\ 0 & 1 & 3/2 & 1 & -1/2 & 0 \\ 0 & 1 & 4/5 & 3/5 & 0 & -1/5 \end{array}\right].$$

Illustration of Inversion (cont)

The second column can now be brought into the required form by R3 $\leftarrow$ R3$-$R2:

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 2 & 1 & 0 & 0 \\ 0 & 1 & 3/2 & 1 & -1/2 & 0 \\ 0 & 0 & -7/10 & -2/5 & 1/2 & -1/5 \end{array}\right].$$

Scaling the third row now leads to:

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 2 & 1 & 0 & 0 \\ 0 & 1 & 3/2 & 1 & -1/2 & 0 \\ 0 & 0 & 1 & 4/7 & -5/7 & 2/7 \end{array}\right].$$

Illustration of Inversion (cont)

The Jordan part now comes into operation. R1 $\leftarrow$ R1-2R2 leads to:

$$\left[\begin{array}{ccc:ccc} 1 & 0 & -1 & -1 & 1 & 0 \\ 0 & 1 & 3/2 & 1 & -1/2 & 0 \\ 0 & 0 & 1 & 4/7 & -5/7 & 2/7 \end{array}\right].$$

Finally, R1 $\leftarrow$ R1$+$R3, and R2 $\leftarrow$ R2$-(3/2)$R3 leads to:

$$\left[\begin{array}{ccc:ccc} 1 & 0 & 0 & -3/7 & 2/7 & 2/7 \\ 0 & 1 & 0 & 1/7 & 4/7 & -3/7 \\ 0 & 0 & 1 & 4/7 & -5/7 & 2/7 \end{array}\right].$$

Illustration of Inversion (cont)

Now the left part is a unit matrix, and the right part is the inverse of the original matrix:

$$\begin{bmatrix} -3/7 & 2/7 & 2/7 \\ 1/7 & 4/7 & -3/7 \\ 4/7 & -5/7 & 2/7 \end{bmatrix}.$$

It is left as an exercise to show that:

$$\begin{bmatrix} 1 & 2 & 2 \\ 2 & 2 & 1 \\ 3 & 1 & 2 \end{bmatrix} \begin{bmatrix} -3/7 & 2/7 & 2/7 \\ 1/7 & 4/7 & -3/7 \\ 4/7 & -5/7 & 2/7 \end{bmatrix} = \mathbf{I}.$$

Properties of Inverses

- The inverse of a unit matrix is a unit matrix of the same size.
- The inverse of a diagonal matrix is a diagonal matrix with each diagonal element replaced by the reciprocal of the element of the original matrix.[9]

In case of a 2×2 matrix:

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}; \qquad \mathbf{A}^{-1} = \frac{1}{\det \mathbf{A}} \begin{bmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{bmatrix}$$

[9] Check this out!

Properties of Inverses (cont)

$$
\begin{aligned}
(\mathbf{AB})^{-1} &= \mathbf{B}^{-1}\mathbf{A}^{-1} \\
(\mathbf{ABC})^{-1} &= \mathbf{C}^{-1}\mathbf{B}^{-1}\mathbf{A}^{-1} \\
(\mathbf{A}^{-1})^{-1} &= \mathbf{I} \\
(\mathbf{A}^{\mathsf{T}})^{-1} &= (\mathbf{A}^{-1})^{\mathsf{T}} \\
(k\mathbf{A})^{-1} &= (1/k)\mathbf{A}^{-1}; \quad k \neq 0, \text{ scalar}
\end{aligned}
$$

Some Theorems

Theorem 2

The inverse of a nonsingular $n \times n$ matrix $\mathbf{A} = [a_{ij}]$ is given by

$$\mathbf{A} = \frac{1}{\det \mathbf{A}} [C_{ij}]^{\mathsf{T}},$$

where $[C_{ij}]$ is the cofactor of a_{ij} in $\det \mathbf{A}$.

Some Theorems (cont)

Theorem 3

Let $\mathbf{A}, \mathbf{B}, \mathbf{C}$, be $n \times n$ square matrices. Then

(a) If rank $\mathbf{A} = n$ and $\mathbf{AB} = \mathbf{AC}$, then $\mathbf{B} = \mathbf{C}$.

(b) If rank $\mathbf{A} = n$, then $\mathbf{AB} = \mathbf{0}$ implies $\mathbf{B} = \mathbf{0}$. Hence, if $\mathbf{AB} = \mathbf{0}$, but $\mathbf{A} \neq \mathbf{0}$ as well as $\mathbf{B} \neq \mathbf{0}$, then rank $\mathbf{A} < n$ and rank $\mathbf{B} < n$.

(c) If $\mathbf{A}$ is singular, then $\mathbf{AB}$ and $\mathbf{BA}$ are both singular.

Theorem 4

Let $\mathbf{A}$ and $\mathbf{B}$ be $n \times n$ square matrices. Then,
$\det(\mathbf{AB}) = \det(\mathbf{BA}) = \det \mathbf{A} \det \mathbf{B}$

Very Special Matrices

If $\mathbf{A}^2 = \mathbf{I}$, then $\mathbf{A}$ is called an **involutory** matrix. The inverse of an involutory matrix is the matrix itself. (Prove this.)
Examples of involutory matrices are:

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix},$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}, \text{ and } \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}.$$

All identity matrices are involutory.
The determinant of any involutory matrix is ± 1. (Prove this.)

Very Special Matrices (cont)

If $\mathbf{A}^2 = \mathbf{A}$, then $\mathbf{A}$ is called an **idempotent** matrix.

A square matrix $\mathbf{A}$ is involutory iff $((\mathbf{A} + \mathbf{I})/2)$ is idempotent. (Prove this.)

Create illustrations of idempotent matrices using the involutory matrices in the previous slide. Verify the statement above.

Elementary Matrices

Elementary matrices are obtained from an identity matrix by one elementary row operation.
Pre-multiplying a matrix by an elementary matrix leads to the corresponding row operation on that matrix.
Here is a row-interchange-transformation 5×5 matrix:

$$\mathbf{R}_{ri} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{bmatrix}$$

Given a 5×5 matrix $\mathbf{A}$, $(\mathbf{R}_{ri}\mathbf{A})$ is a matrix which has rows 2 and 5 of $\mathbf{A}$ interchanged.

Elementary Matrices (cont)

Here is a row-multiplication-transformation matrix (also 5×5):

$$\mathbf{R}_m = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & k & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

When premultiplied to $\mathbf{A}$, it transforms it by multiplying its second row by a constant k.

Elementary Matrices (cont)

Here is a row-addition-transformation matrix (also 5×5):

$$\mathbf{R}_a = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & k & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

When premultiplied to $\mathbf{A}$, it transforms it by adding k times its second row to its fourth row.

Elementary Matrices – Questions

- What is the determinant of each of the three elementary-transformation matrices?
- What is the inverse of each of them?
- What is the relation between an elementary-transformation matrix and its inverse?