Vector Calculus

Uday N. Gaitonde

IIT Bombay
IIT Dharwad

September 11, 2018

Vector Algebra

It is assumed that we are comfortable with vector algebra:

- Definition of vectors in 2D and 3D.
- Representation, symbolism, examples.
- Length, magnitude, direction of a vector.
- Unit vectors.
- Equality of two vectors.
- Components of a vector.
- Position vector in physics.
- Mathematical idea of a vector.
- Zero vector.

Vector Algebra (cont)

- Addition of two vectors.
- Properties of vector addition. It is commutative and associative. Property of a zero vector, negative of a vector.
- Multiplication of a scalar and a vector. Properties: distributive, multiplication by 1, multiplication by 0.
- Representation using unit vectors and components.
- All vectors form a real vector space in $\mathbb{R}^3$, with two algebraic operations – addition and scalar multiplication.
- Unit vectors $\mathbf{i}, \mathbf{j}, \mathbf{k}$: standard basis of $\mathbb{R}^3$.

Vector Algebra (cont)

Scalar product or dot product or inner product

- Motivation: work in physics.
- Formal definition, geometric view.
- Expansion in terms of components.
- Range of magnitudes.
- Orthogonal vectors, definition, relation.
- Relation between magnitudes and dot product.
- Properties of dot product.
- Derived properties: triangle inequality, parallelogram inequality.

Vector Algebra (cont)

Vector Product or Cross Product

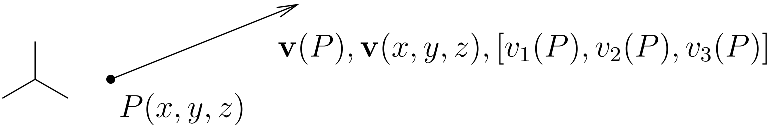
- Motivation: torque or couple.
- Formal definition, geometric view.
- Expansion in terms of components.
- Range of magnitudes.
- Properties: anti-commutative, not associative (in general).

Vector Algebra (cont)

Scalar triple product (or box product)

- Motivation: volume of a parallelepiped.
- Definition.
- Expansion in terms of components.
- Relation to coplanarity and linear dependence.
- Properties.

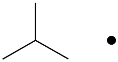
Vector Functions



- A vector function defines a vector field in its domain.
- A domain may be 1D (a curve in space), 2D (a surface in space), or 3D (a volume in space).
- Vector functions may also depend on time.

Examples?

Scalar Functions



$$P(x, y, z); \quad f(P) = f(x, y, z)$$

- A scalar function defines a scalar field in its domain.
- A scalar function may also depend on time.

Examples?

Vector Calculus

Basic concepts are directly linked to those of scalar calculus.

Convergence
Let $\mathbf{a}_{(1)}, \mathbf{a}_{(2)}, \ldots,$ be an infinite sequence of vectors.
If a vector $\mathbf{a}$ exists s.t.

$$\lim_{n\to\infty} \left|\mathbf{a}_{(n)} - \mathbf{a}\right| = 0,$$

then the sequence **converges**, and $\mathbf{a}$ is the limit vector of that sequence. We can then write:

$$\mathbf{a} = \lim_{n\to\infty} \mathbf{a}_{(n)}.$$

Vector Calculus (cont)

A vector function $\mathbf{V}(t)$ of a real scalar variable t may have a limit, $\mathbf{V}_0$:

$$\text{If } \lim_{t\to t_0} |\mathbf{V}(t) - \mathbf{V}_0| = 0, \text{ then we say: } \mathbf{V}_0 = \lim_{t\to t_0} \mathbf{V}(t)$$

For this,
$\mathbf{V}(t)$ needs to be defined over some neighbourhood of t_0.
t_0 may be an interior point or an end point.
$\mathbf{V}(t)$ may or may not be defined at $t = t_0$.

Vector Calculus (cont)

Continuity
If $\mathbf{V}(t)$ is defined in some neighbourhood of t_0, including at t_0 itself, and if

$$\lim_{t \to t_0} \mathbf{V}(t) = \mathbf{V}(t_0)$$

then we say that $\mathbf{V}(t)$ is **continuous** at t_0.

If we represent $V(t)$ in its Cartesian components, then each component needs to be continuous for $\mathbf{V}(t)$ to be continuous.

Derivative of a Vector Function

Motivation: From geometry, the tangent vector of a curve.

If the limit

$$\mathbf{V}'(t) = \lim_{\Delta t \to 0} \frac{\mathbf{V}(t + \Delta t) - \mathbf{V}(t)}{\Delta t} \qquad \text{exists,}$$

then $\mathbf{V}(t)$ is **differentiable** at t
and $\mathbf{V}'(t)$ is the **derivative** of $\mathbf{V}(t)$ with respect to t at t.

In a Cartesian representation, the derivative is obtained by differentiating each component separately.

$$\mathbf{V}'(t) = [v_1'(t), v_2'(t), v_3'(t)]$$

Rules for Vector Derivatives

- $(\alpha \mathbf{V})' = \alpha \mathbf{V}'$ (α: scalar)
- $(\mathbf{U} + \mathbf{V})' = \mathbf{U}' + \mathbf{V}'$
- $(\mathbf{U} \cdot \mathbf{V})' = \mathbf{U}' \cdot \mathbf{V} + \mathbf{U} \cdot \mathbf{V}'$
- $(\mathbf{U} \times \mathbf{V})' = \mathbf{U}' \times \mathbf{V} + \mathbf{U} \times \mathbf{V}'$
- $(\mathbf{U}\ \mathbf{V}\ \mathbf{W})' = (\mathbf{U}'\ \mathbf{V}\ \mathbf{W}) + (\mathbf{U}\ \mathbf{V}'\ \mathbf{W}) + (\mathbf{U}\ \mathbf{V}\ \mathbf{W}')$

Partial Derivatives

Let $\mathbf{V}(t_1, t_2, \ldots, t_n)$ be a vector function of n variables. Then

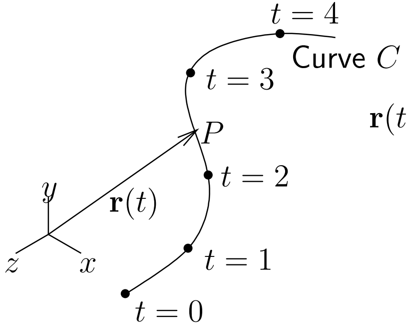
$$\frac{\partial \mathbf{V}}{\partial t_m} = \frac{\partial v_1}{\partial t_m}\mathbf{i} + \frac{\partial v_2}{\partial t_m}\mathbf{j} + \frac{\partial v_3}{\partial t_m}\mathbf{k}$$

We can also define second partial derivatives:

$$\frac{\partial^2 \mathbf{V}}{\partial t_p \partial t_q} = \frac{\partial^2 v_1}{\partial t_p \partial t_q}\mathbf{i} + \frac{\partial^2 v_2}{\partial t_p \partial t_q}\mathbf{j} + \frac{\partial^2 v_3}{\partial t_p \partial t_q}\mathbf{k}$$

Parametric Representation

This is very useful for a curve in space.



$$\begin{aligned} \mathbf{r}(t) &= [x(t), y(t), z(t)] \\ &= x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k} \\ t &: \text{ parameter} \end{aligned}$$

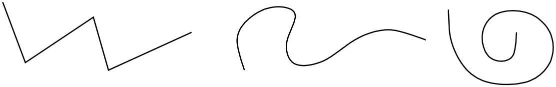
t could be time, arc length, *etc.*.

Increase in $t \implies$ movement in the positive sense on C.

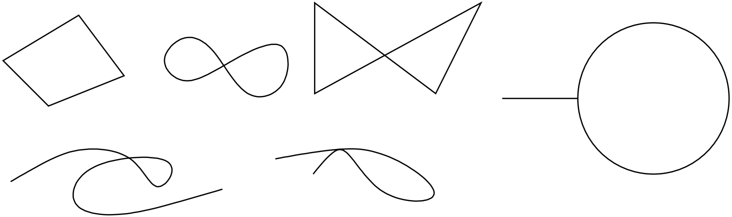
Decrease in $t \implies$ movement in the negative sense on C.

Simple and Non-Simple Curves

Simple Curves

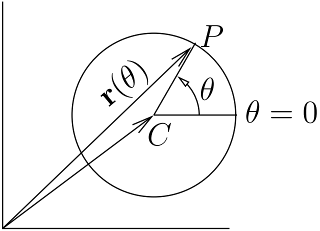


Non-Simple Curves



Parametric Representations

Circle at centre C, radius R

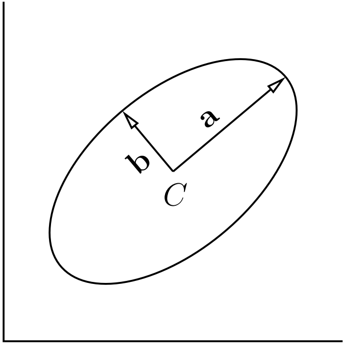


$$\begin{aligned}\mathbf{r} &= \mathbf{r}_C + [R\cos\theta \;\; R\sin\theta] \\ &= \mathbf{r}_C + (R\cos\theta)\mathbf{i} + (R\sin\theta)\mathbf{j}, \quad 0 \le \theta < 2\pi.\end{aligned}$$

As θ goes from $[0, 2\pi)$, the circle is traversed in the anticlockwise direction, starting from the East-most point (3 o'clock). We say that the positive sense is anticlockwise.

Parametric Representations (cont)

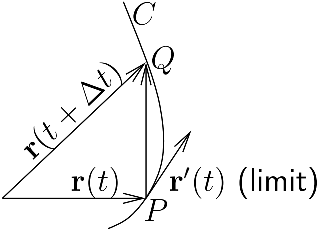
Ellipse with centre C, semi-axes represented by $\mathbf{a}$ and $\mathbf{b}$ (with $\mathbf{a} \cdot \mathbf{b} = 0$)



$$\mathbf{r} = \mathbf{r}_C + \mathbf{a}\cos\theta + \mathbf{b}\sin\theta$$

Homework: What is significance, if any, of θ?

Tangent to a Curve



If $\mathbf{r}(t)$ is differentiable, then

$$\mathbf{r}'(t) = \lim_{\Delta t \to 0} \left[\frac{\mathbf{r}(t+\Delta t) - \mathbf{r}(t)}{\Delta t} \right]$$

is called the tangent vector to the curve C at t.

Tangent to a Curve (cont)

The unit tangent vector at t is

$$\mathbf{u} = \frac{\mathbf{r}'}{|\mathbf{r}'|}.$$

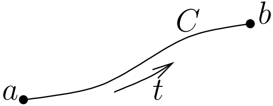
$\mathbf{r}'$ and $\mathbf{u}$ point in the same direction.

The equation of the tangent to curve C at t is given by

$$\mathbf{q}(w) = \mathbf{r}(t) + w\mathbf{r}'(t),$$

where w is a scalar parameter.

Length of a Curve



Take small changes in t as P goes from a to b. Take a limit as the changes become infinitesimal.

$$l = \int_a^b \sqrt{\mathbf{r}' \cdot \mathbf{r}'}\,dt, \text{ where } \mathbf{r}' = \frac{d\mathbf{r}}{dt}$$

If the length can be computed, then the curve C (from a to b) is **rectifiable**.

Arc Length of a Curve (s)

$$s(t) = \int_a^t \sqrt{\mathbf{r}' \cdot \mathbf{r}'}dt, \text{ where } \mathbf{r}' = \frac{d\mathbf{r}}{dt}$$

$$\therefore \frac{ds}{dt} = \sqrt{\frac{d\mathbf{r}}{dt} \cdot \frac{d\mathbf{r}}{dt}}$$

$$\text{or } \left(\frac{ds}{dt}\right)^2 = \frac{d\mathbf{r}}{dt} \cdot \frac{d\mathbf{r}}{dt} = [\mathbf{r}'(t)]^2 = \left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2$$

We often write $d\mathbf{r} = [\ dx\ dy\ dz] = dx\mathbf{i} + dy\mathbf{j} + dz\mathbf{k}$,
then $ds^2 = d\mathbf{r} \cdot d\mathbf{r} = (dx)^2 + (dy)^2 + (dz)^2$.
ds is the elementary arc length, often called the linear element of C.

Length as a parameter

Often, it is useful to use the arc length itself as a parameter.

Then $\mathbf{r}(s)$.

The unit tangent vector then becomes:

$\mathbf{u}(s) = \mathbf{r}'(s)$: differentiation w.r.t. s.

This simplifies some applications significantly.

Movement of a Particle: Position, velocity, acceleration.

Let $\mathbf{r}(t)$: position of a particle at time t.

$$\mathbf{V} \equiv \frac{d\mathbf{r}}{dt} = \mathbf{r}'(t) \text{ :velocity vector.}$$

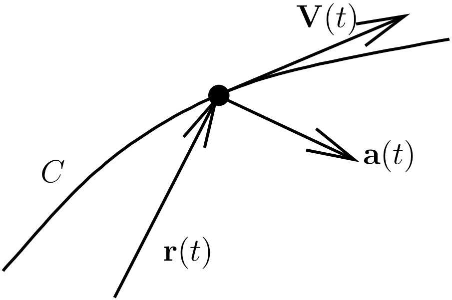
$\mathbf{V}$ is tangent to $\mathbf{r}(t)$.

$$\text{speed } \equiv |\mathbf{V}| = \sqrt{\mathbf{V} \cdot \mathbf{V}} = \sqrt{\mathbf{r}' \cdot \mathbf{r}'} = \frac{ds}{dt}.$$

$$\mathbf{a} \equiv \frac{d\mathbf{V}}{dt} = \frac{d^2\mathbf{r}}{dt^2} \text{ :acceleration vector.}$$

acceleration $\equiv |\mathbf{a}|$.

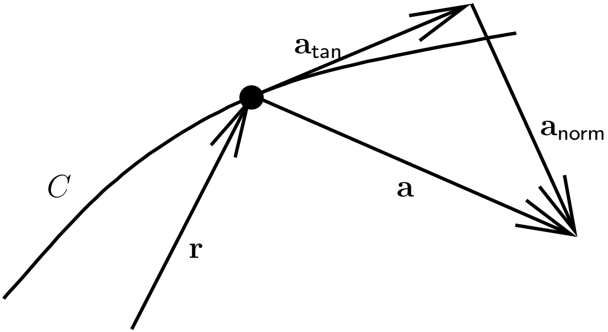
Position, Velocity, Acceleration



$\mathbf{V}(t)$ is always tangent to the curve of motion (locus) C.

$\mathbf{a}(t)$ may not be tangent to C.

Position, Velocity, Acceleration (cont)



$$\mathbf{a} = \mathbf{a}_{\text{tan}} + \mathbf{a}_{\text{norm}}$$

Either component may be zero.

An Extreme Example

Let the movement be such that $\mathbf{V}(t)$ does not change magnitude, changes only direction.

$$\begin{aligned}
\text{Let } |\mathbf{V}(t)| = c = &\text{ a constant} \\
\therefore \mathbf{V} \cdot \mathbf{V} &= c^2 \\
\therefore \frac{d}{dt}(\mathbf{V} \cdot \mathbf{V}) &= 0 \\
\therefore \mathbf{V}' \cdot \mathbf{V} + \mathbf{V} \cdot \mathbf{V}' &= 0 \\
\therefore \mathbf{V} \cdot \mathbf{V}' &= 0
\end{aligned}$$

$\therefore \mathbf{V}$ is either a zero vector or is perpendicular (normal) to $\mathbf{V}$.

A General Case of Acceleration

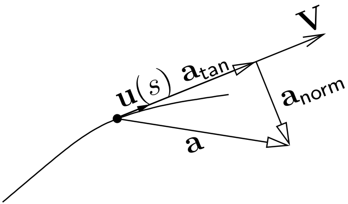
$$\mathbf{V}(t) = \frac{d\mathbf{r}}{dt} = \frac{d\mathbf{r}}{ds}\frac{ds}{dt} = \mathbf{u}(s)\frac{ds}{dt}$$

where $\mathbf{u}(s)$ is the unit tangent vector.

$$\begin{aligned}\mathbf{a}(t) = \frac{d\mathbf{V}(t)}{dt} &= \frac{d}{dt}\left(\mathbf{u}(s)\frac{ds}{dt}\right) \\ &= \underbrace{\frac{d\mathbf{u}(s)}{dt}\frac{ds}{dt}}_{\perp \mathbf{u}(s)} + \underbrace{\mathbf{u}(s)\frac{d^2 s}{dt^2}}_{\parallel \mathbf{u}(s)}\end{aligned}$$

The first term follows from the constant length of $\mathbf{u}(s)$.

A General Case (cont)



$$\begin{aligned}
\text{Thus } \mathbf{a}(t) &= \mathbf{a}_{\mathsf{tan}} + \mathbf{a}_{\mathsf{norm}} \\
&= \mathbf{u}(s)\frac{d^2 s}{dt^2} + \frac{d\mathbf{u}(s)}{dt}\frac{ds}{dt} \\
&= \mathbf{u}(s)\frac{d^2 s}{dt^2} + \frac{d\mathbf{u}(s)}{ds}\left(\frac{ds}{dt}\right)^2
\end{aligned}$$

A General Case (cont)

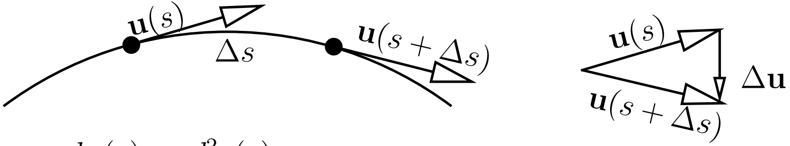
$$
\begin{aligned}
\mathbf{a}_{\text{tan}} &= \text{component (projection) of } \mathbf{a} \text{ in the direction of } \mathbf{V} \\
&= \frac{\mathbf{a} \cdot \mathbf{V}}{|\mathbf{V}|} \\
|\mathbf{a}_{\text{tan}}| &= \text{magnitude of this projection } = \frac{|\mathbf{a} \cdot \mathbf{V}|}{|\mathbf{V}|} \\
\mathbf{a}_{\text{tan}} &= |\mathbf{a}_{\text{tan}}| \times \text{ unit vector in the direction of } \mathbf{V} \\
&= \frac{|\mathbf{a} \cdot \mathbf{V}|}{|\mathbf{V}|} \frac{\mathbf{V}}{|\mathbf{V}|} = \frac{\mathbf{a} \cdot \mathbf{V}}{\mathbf{V} \cdot \mathbf{V}} \mathbf{V}
\end{aligned}
$$

Finally, $\mathbf{a}_{\text{norm}} = \mathbf{a} - \mathbf{a}_{\text{tan}}$

Examples

1. Uniform circular motion of a particle. Centripetal acceleration.
2. Movement on a turntable: constant angular velocity with uniform radial movement. Centripetal acceleration, Coriolis component of acceleration.

Curvature



$$\frac{d\mathbf{u}(s)}{ds} = \frac{d^2\mathbf{r}(s)}{ds^2}$$:measure of curvature

Curvature is defined as: $\kappa(s) \equiv \left|\frac{d\mathbf{u}(s)}{ds}\right|$

κ is always non-negative, has dimension $[L]^{-1}$

$$\rho(s) \equiv \frac{1}{\kappa(s)}$$:radius of curvature

Curvature (cont)

If the locus is a straight line (locally), then
$\kappa(s) = 0, \rho(s) \to \infty$.
$\because \mathbf{u}(s)$ is a unit vector (of constant length),

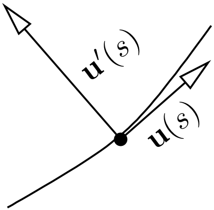
$$\mathbf{u}(s) \perp \left(\frac{d\mathbf{u}(s)}{ds} = \mathbf{u}'(s) \right).$$

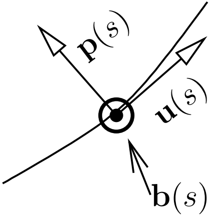
$\therefore \mathbf{u}(s)$ and $\mathbf{u}'(s)$ form a plane, called the osculating plane at that point.

$$\mathbf{p} \equiv \frac{\mathbf{u}'(s)}{|\mathbf{u}'(s)|} = \frac{\mathbf{u}'(s)}{\kappa(s)}$$

is the principal unit normal to the curve at that point.

Curvature (cont)





$\mathbf{b} \equiv \mathbf{u} \times \mathbf{p}$ is called the binormal.
$\mathbf{u}, \mathbf{p}, \mathbf{b}$ form a right-handed triad.

Plane of $\mathbf{u}$ and $\mathbf{p} \equiv$ osculating plane;
plane of $\mathbf{p}$ and $\mathbf{b} \equiv$ normal plane;
plane of $\mathbf{b}$ and $\mathbf{u} \equiv$ rectifying plane.

Torsion

As we traverse along the curve (s increasing),
$\mathbf{u}(s)$ changes direction, and do does $\mathbf{p}(s)$ and $\mathbf{b}(s)$.

Torsion: $\tau(s) \equiv$ variation of $\mathbf{b}(s)$ with $s : |\tau(s)| = |\mathbf{b}'(s)|$.
$\because \mathbf{b}(s) :$ unit vector, $\therefore \mathbf{b}'(s) \perp \mathbf{b}(s)$.

$$\begin{aligned}
\therefore \mathbf{b}(s) \cdot \mathbf{u}(s) &= 0 \\
\therefore (\mathbf{b}(s) \cdot \mathbf{u}(s))' &= 0 \\
\therefore \mathbf{b}'(s) \cdot \mathbf{u}(s) + \underbrace{\mathbf{b}(s) \cdot \mathbf{u}'(s)}_{=0,\ \because \perp} &= 0 \\
\therefore \mathbf{b}'(s) \cdot \mathbf{u}(s) &= 0
\end{aligned}$$

$\therefore \mathbf{b}'(s) \perp \mathbf{u}(s),$ and $\mathbf{b}'(s) \perp \mathbf{b}(s)$.

Torsion (cont)

$\therefore \mathbf{b}'(s)$ must be aligned with $+\mathbf{p}(s)$ or $-\mathbf{p}(s)$.
Let $\mathbf{b}'(s) = -\tau(s)\mathbf{p}(s)$ (note the negative sign!)

$$\begin{aligned} \therefore \mathbf{b}'(s) \cdot \mathbf{p}(s) &= -\tau(s)\mathbf{p}(s) \cdot \mathbf{p}(s) \\ &= -\tau(s) \\ \therefore \tau(s) &= -\mathbf{b}'(s) \cdot \mathbf{p}(s) \end{aligned}$$

The $-$ve sign leads to the tortion (τ) of a right-handed helix positive.
Hence, its use is a matter of convection.