Vector Calculus – 2

Uday N. Gaitonde

IIT Bombay
IIT Dharwad

September 13, 2018

Derivatives is Vector Calculus

In calculus, derivatives map a scalar field onto another scalar field.

In vector calculus, derivatives are use to create the following mappings:

- The gradient maps a scalar field onto a vector field.
- The divergence maps a vector field onto a scalar field.
- The curl maps a vector field onto a vector field.

The Gradient of a Scalar Field

Given: $f(x, y, z)$:a scalar function defined over a domain in 3D space ($\mathbb{R}^3$). f is differentiable.
Then

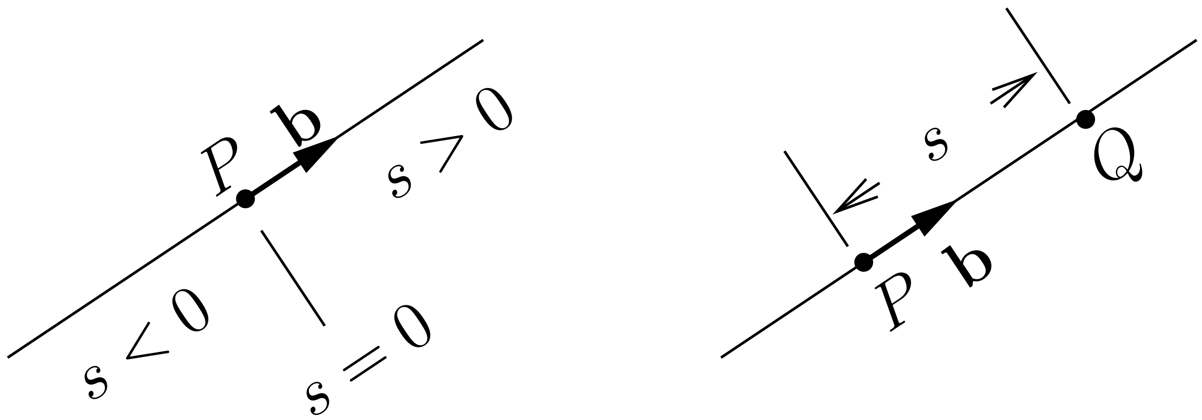
$$\mathrm{grad} f = \nabla f \equiv \underbrace{\frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j} + \frac{\partial f}{\partial z}\mathbf{k}}_{\text{This is a vector.}}$$

$$\underbrace{\nabla}_{\text{"nabla"}} \equiv \underbrace{\frac{\partial}{\partial x}\mathbf{i} + \frac{\partial}{\partial y}\mathbf{j} + \frac{\partial}{\partial z}\mathbf{k}}_{\text{The gradient operator.}}$$

What is the utility of the gradient?

Directional Derivative

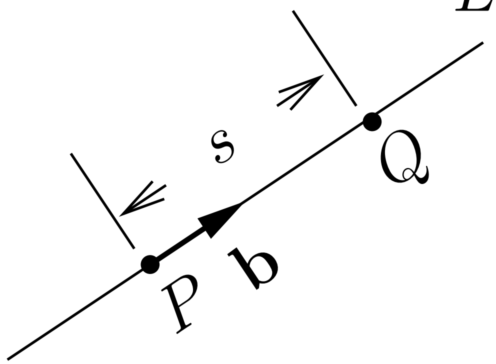
st line in the direction of $\mathbf{b}$



s is measured from P in the direction of $\mathbf{b}$.

$$D_{\mathbf{b}} f = \left.\frac{df}{ds}\right|_{\mathbf{b}} \equiv \lim_{s \to 0} \frac{f(Q) - f(P)}{s}$$

Directional Derivative and Gradient



L: st line through P

Equation of L

$$\begin{aligned}\mathbf{r}(s) &= \mathbf{p}_0 + \mathbf{b}s \\ &= x(s)\mathbf{i} + y(s)\mathbf{i} + z(s)\mathbf{k}\end{aligned}$$

Assume f has partial derivatives which are continuous.

Then $$\frac{d\mathbf{f}}{ds} = \frac{\partial f}{\partial x}\frac{dx}{ds} + \frac{\partial f}{\partial y}\frac{dy}{ds} + \frac{\partial f}{\partial z}\frac{dz}{ds},$$

but $$\frac{d\mathbf{r}}{ds} = \frac{dx}{ds}\mathbf{i} + \frac{dy}{ds}\mathbf{j} + \frac{dz}{ds}\mathbf{k} = \mathbf{b} \quad \text{(why?)}$$

$$\therefore D_{\mathbf{b}} f = \left.\frac{df}{ds}\right|_{\mathbf{b}} = \nabla f \cdot \mathbf{b}$$

Direction of Maximum Increase

Thorem

Let $f(P) = f(x, y, z)$ be a scalar function, with first partial derivatives which are continuous, defined over some domain B. Then ∇f exists, ∇f is a vector, its length independent of the choice of coordinates, and if $\nabla f \neq 0$, then the direction of ∇f is the direction of the maximum rate of increase of f.

Proof

$$D_{\mathbf{b}} f = \nabla f \cdot \mathbf{b} = |\nabla f||\mathbf{b}| \cos \gamma$$

where γ is the included $\angle$ between ∇f and $\mathbf{b}$.

But f and s are scalar functions of the position P.
$\therefore$ their values do not depend on the choice of the coordinate system.

Direction of Maximum Increase (cont)

Proof (cont)

$$\therefore D_{\mathbf{b}} f = \left.\frac{df}{ds}\right|_{\mathbf{b}} : \text{ independent of choice of coordinate system.}$$

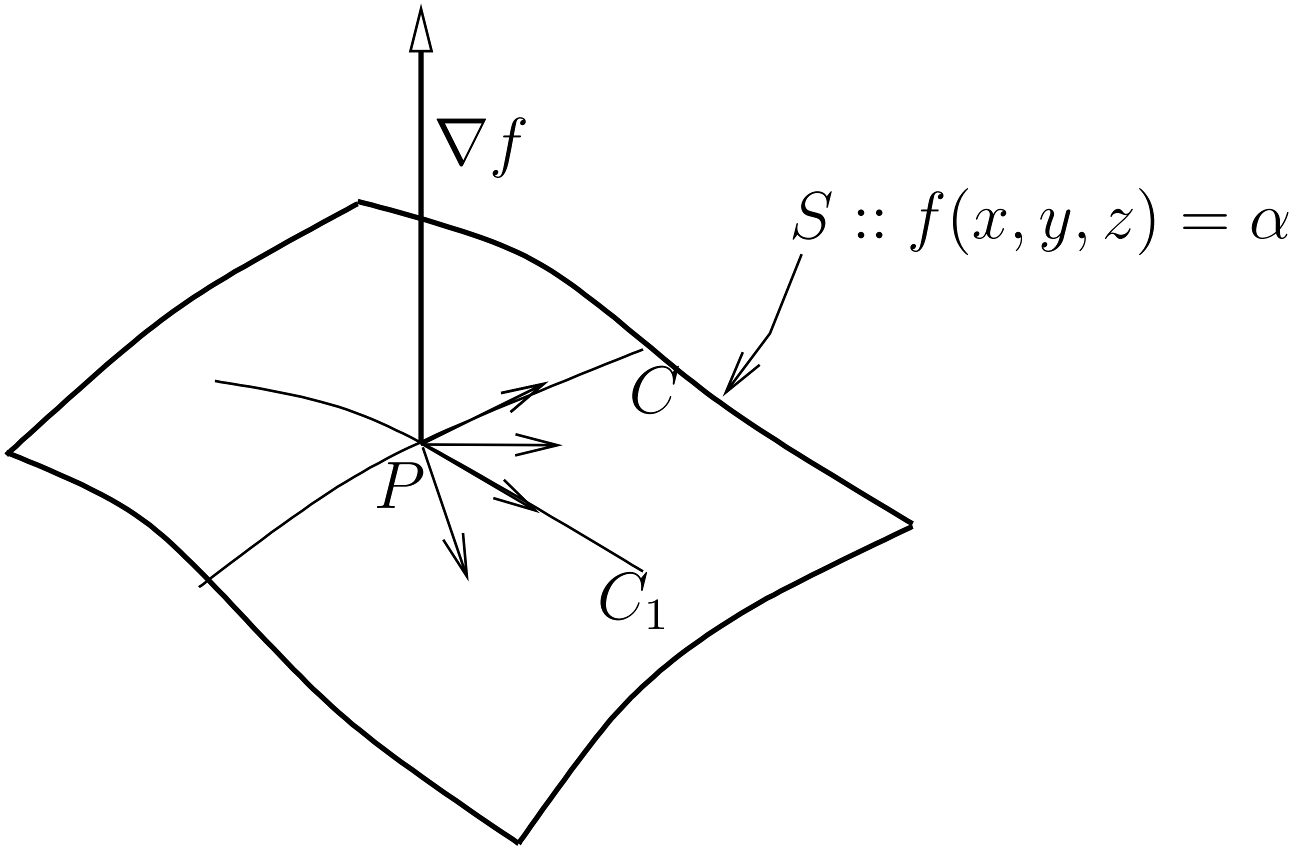
Hence, the length and direction of ∇f does not depend on the choice of the coordinate system; this maked ∇f a proper vector.

$D_{\mathbf{b}} f = \nabla f$ will be max when $\cos\gamma = 1$, *i.e.* when $\mathbf{b}$ is chosen in the direction of ∇f.

$\therefore$, direction of ∇f: direction of maximum rate of increase, and $|\nabla f|$: magnitude of the maximum rate of increase in f.

Gradients and Surfaces

Let $f(x, y, z)$: a scalar function, differentiable.
Then $f(x, y, z) = \alpha$ (a constant) represents a surface, called a "level surface of f".



Gradients and Surfaces (cont)

P: a point on S.
C: a curve on S passing through P, represented by $\mathbf{r}(t) = x(t)(\mathbf{i}) + y(t)(\mathbf{j} + z(t)(\mathbf{k}))$.

$\because C$ lies on S, components of $\mathbf{r}$ satisfy the relation for S:
$[1] \quad f(x(t), y(t), z(T)) = \alpha$.

A tangent vector of C will be $\mathbf{r}'(t) = x'(t)\mathbf{i} + y'(t)\mathbf{j} + z'(t)\mathbf{k}$.

For different curves on S passing through P, the set of tangent vectors at P forms a surface normal of S at P.

Gradients and Surfaces (cont)

Differentiating $[1]$ w.r.t. t:

$$\frac{\partial f}{\partial x}x'(t) + \frac{\partial f}{\partial y}y'(t) + \frac{\partial f}{\partial z}z'(t) = 0$$
$$\therefore \nabla f \cdot \mathbf{r} = 0$$

$\therefore \nabla f$ is normal to all vectors $\mathbf{r}'$ in the tangent plane of S.
$\implies \nabla f$ is the normal vector of S at P.

$\implies$ **Theorem:**
Let $f(x, y, z)$: a differentiable function in space.
Let $f(x, y, z) = \alpha = (const)$: a surface in that space.
Then ∇f at a point P on S is either 0 or a normal vector of S at P.

Potential

Nomenclature in science and engineering:
If $f(P) = f(x, y, z)$ is a scalar field, and
$\mathbf{V}(P) = \mathbf{V}(x, y, z) = \nabla f$ is a vector field, then, often

f is known as the **potential** for the vector field $\mathbf{V}(= \nabla f)$.

Sometimes, a negative sign and a scale factor is included.

The Divergence of a Vector Field

Let $\mathbf{V}(x, y, z) = v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k}$ be a differentiable vector field, with v_1, v_2, v_3 functions of (x, y, z). Then

$$\mathrm{div}\mathbf{V} = \nabla \cdot \mathbf{V} \equiv \frac{\partial v_1}{\partial x} + \frac{\partial v_2}{\partial y} + \frac{\partial v_3}{\partial z},$$

where ∇ is the same scalar-to-vector operator used in the definition of the gradient.

Divergence (cont)

Theorem
Given a vector function $\mathbf{V}(x, y, z)$ on a domain in space, $\mathrm{div}\mathbf{V}(= \nabla \cdot \mathbf{V})$ is a scalar function, whose values depend on $\mathbf{V}$ and the location in space, but <u>not</u> om the choice of the coordinate system.

What is the physical significance of divergence?

Divergence – Significance

In many physical situations, divergence has a meaning:

- In fluid mechanics, $\rho\mathbf{V}$ represents the mass flux.
  $\nabla \cdot (\rho\mathbf{V})$ equals the net outflow of mass per unit volume.
- In heat conduction, $\mathbf{q}$ represents the heat flux.
  $\nabla \cdot (\mathbf{q})$ equals the net heat outflow per unit volume.

The Laplacian

Let $f(x, y, z)$ be a scalar function in space. Its gradient

$$\nabla f = \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j} + \frac{\partial f}{\partial z}\mathbf{k}$$

is a vector function. If we take the divergence of this, we will get another scalar function:

$$\nabla \cdot (\nabla f) = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} = \nabla^2 f.$$

This function is known as the Laplacian of f, and is denoted by $\nabla^2 f$.
∇^2 is known as the Laplacian operator.

Use of the Laplacian

Fourier's Law of heat conduction states that $\vec{q} = -k\nabla T$, where T is the temperature and k the thermal conductivity. The First Law states that "rate of storage of energy equals the net inflow of energy". Hence, we get the heat conduction equation:

$$\rho c_p \frac{\partial T}{\partial t} = -\nabla \cdot \mathbf{q} = -\nabla \cdot (-k\nabla T) = \nabla \cdot (k\nabla T)$$

If k is constant, this reduces to the 'heat conduction equation':

$$\rho c_p \frac{\partial T}{\partial t} = k\nabla^2 T$$