Introduction to Vector Spaces

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Space in Mathematics

In Mathematics, **space** is a set (finite or infinite) with some specified structure.

Members of a space have certain properties, and/or have to obey certain rules.

Some examples:
Euclidean Space,
Thermodynamic State-Space,
Vector Space.

Introduction to Vector Spaces

A vector space is a set V of **vectors**[1], with the following axioms.
We assume that $\mathbf{a}, \mathbf{b}, \mathbf{c}$ are members of V, and α, β are real numbers. An addition operator '$+$' is defined, and so is scalar multiplication.[2]

[1] We may keep aside our classical idea of vectors.

[2] We may even keep aside our classical ideas of addition and scalar multiplication.

Vector Space (cont)

The axioms are:

1. V is non-empty.
2. $\alpha \mathbf{a} \in V$. (A member, when scaled by any amount ($+$ve or $-$ve), is a member of V. That is, V is closed under scalar multiplication.)
3. $(\mathbf{a}+\mathbf{b}) \in V$, and $(\mathbf{a}+\mathbf{b})$ is unique. (This means that V is closed under addition.)

   Axioms 2 and 3 together imply that any linear combination of members, *e.g.* $\alpha \mathbf{a} + \beta \mathbf{b} \in V$.

Vector Space (cont)

4. $\mathbf{a} + \mathbf{b} = \mathbf{b} + \mathbf{a}$. (Addition is commutative.)
5. $\mathbf{a} + (\mathbf{b} + \mathbf{c}) = (\mathbf{a} + \mathbf{b}) + \mathbf{c}$. (Addition is associative, so one can write $\mathbf{a} + \mathbf{b} + \mathbf{c}$ without any ambiguity.)
6. $\mathbf{0} \in V$, s.t. $\mathbf{0} + \mathbf{a} = \mathbf{a}$. (Definition of the zero member.)
7. For any $\mathbf{a} \in V$, there exists a $(-\mathbf{a})$, s.t. $\mathbf{a} + (-\mathbf{a}) = \mathbf{0}$. (Definition of the negative of a member.)

Vector Space (cont)

8. $\alpha(\mathbf{a} + \mathbf{b}) = \alpha\mathbf{a} + \alpha\mathbf{b}$.
9. $(\alpha + \beta)\mathbf{a} = \alpha\mathbf{a} + \beta\mathbf{a}$.
10. $\alpha(\beta\mathbf{a}) = (\alpha\beta)\mathbf{a}$. (Thus, one can write $\alpha\beta\mathbf{a}$ without any ambiguity.
11. $1\mathbf{a} = \mathbf{a}$. Multiplication by 1 does not change the member.

We should notice that members of a vector space follow all the rules pertaining to matrices, when we use the standard definition of addition and scalar multiplication.

An Example

Consider the set V_2 of two-component (two-dimensional) vectors $[x_1, x_2]$.
This is a vector set.
$[1, 0], [0, 2], [2, 3]$ are members of this set.

Check that all the axioms hold (are satisfied).

Question: Is the set of real numbers $\mathbb{R}$, a vector set?

Dimension of a Vector Space

The **dimension** of a vector space V, $\dim V$ is the largest number of linearly independent members/vectors in V.

The dimension of a vector set may be finite, or infinite.

The dimension of the vector set V_2 is 2. $\dim(V_2) = 2$.
$[2, 3]$ can be written as a linear combination of $[1, 0]$ and $[0, 2]$.

Basis for a Vector Space

A **basis** for a vector set V is a subset in V that consists of the largest number of linear independent vectors in V.

If we add one or more vectors in V to its basis, then the resulting set of vectors will be a linearly-dependent set.

The number of vectors in a basis for V equals $\dim V$.

Basis : An Example

For the vector set V_2, $[1, 0]$ and $[0, 2]$ form a basis.

The basis is not unique.

$[1, 0]$ and $[2, 3]$ also form a basis.

Span of a Set of Vectors

Let $\mathbf{a}_{(1)}, \mathbf{a}_{(2)}, \cdots, \mathbf{a}_{(p)}$ be any set of p vectors, each with the same number of components.

The set of all possible linear combinations of these vectors is called the **span** of these vectors.

This set is a vector space.

If the set of p vectors is linearly independent, then this will be a basis for that vector space.

Basis – Equivalent Definition

A set of vectors form a basis for V, iff:

1. the vectors form a linearly independent set, and
2. any vector in V can be expressed as a linear combination of the vectors in that set.

When the second condition is satisfied, we say that the set spans the vector space V.

Subspace of a Vector Space

Consider a subset of V that forms a vector space (with respect to the two operations of vector addition and scalar multiplication).
Such a subset is called a **subspace** of V.

Note: V itself is a subspace of V.

Example: Subspace of a Vector Space

For the vector set V_2, consider the set of elements $[x_1, 0]$.

Check whether this is a subspace of V_2.

What about the set of elements $[x_1, 0.5x_1]$?

Theorem

Theorem
A vector space $\mathbb{R}^n$ consists of all vectors with n components.
This vector space has dimension n.

Row Space and Column Space

Given a matrix $\mathbf{A}$,
the span of the row vectors is called the **row space** of $\mathbf{A}$,
and
the span of the column vectors is called the **column space** of $\mathbf{A}$.

Theorem
The row space and column space of a matrix $\mathbf{A}$ have the same dimension, equal to the rank of $\mathbf{A}$.

Example: Row Space, Column Space

Let $\mathbf{A} = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 2 & 3 \end{bmatrix}$.
Its rank is 2.

Its row space is made of all linear combinations of $[1, 0, 2]$ and $[0, 2, 3]$.
This space has dimension 2.

Its column space is made of all linear combinations of $[1, 0]$, $[0, 2]$, and $[2, 3]$.
This space also has dimension 2. **Why?**

Null Space and Nullity

For a matrix $\mathbf{A}$, consider all possible solutions of the homogeneous LAE system: $\mathbf{Ax} = \mathbf{0}$.
The set of solutions is a vector space.
It is called the **null space** of $\mathbf{A}$.
The dimension of the null space is called the **nullity** of $\mathbf{A}$.

The nullity is related to the rank and the number of columns of $\mathbf{A}, (n)$:

$$\text{rank } \mathbf{A} + \text{nullity } \mathbf{A} = n$$

An Example

The homogeneous LAE system $\mathbf{Ax} = \mathbf{0}$, where $\mathbf{A}$ is the 2×3 matrix from the previous example. To determine the solution, we note that the auxiliary matrix $\begin{bmatrix} 1 & 0 & 2 & 0 \\ 0 & 2 & 3 & 0 \end{bmatrix}$ is already in the row echelon form.

Thus the solution is $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -2x_3 \\ -(3/2)x_3 \\ x_3 \end{bmatrix}$.

An Example (cont)

If we use p as a parameter, then the solution can be written down as $\mathbf{x} = \begin{bmatrix} -2p \\ -(3/2)p \\ p \end{bmatrix} = \begin{bmatrix} -2 \\ -(3/2) \\ 1 \end{bmatrix} p$.

Hence, $\mathbf{A}$ has a null space of dimension 1, i.e. $\text{nullity } \mathbf{A} = 1$, and a basis for this null space is $\begin{bmatrix} -2 \\ -(3/2) \\ 1 \end{bmatrix}$.

General Vector Space

Arthur Cayley, James Sylvester, and William Hamilton[3] (19th-century mathematicians) generalised the idea of a vector space.

So as not to confuse with the physical idea of a vector, a vector space is sometimes called **linear space**.

However, this usage is not very common.

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[3] Known for the *Hamiltonian* in Physics, and the *Cayley-Hamilton Theorem* of linear algebra.

Abstractions

- The set of 2-D vectors is a vector set.
- So is the set of 3-D vectors.
- So will be the set of 4-D, or 5-D, or n-D vectors.
- But n-D vectors are represented by an ordered list of n real numbers. Such on ordered list, or sequence, is known as an n-tuple of real numbers.
- Any set of n-tuples that satisfies the axioms will be a vector set.
- The members may not have anything with the idea of vectors at all.

Questions

Which of the following are vector sets?

- the set of complex numbers, $\mathbb{C}$
- the set of first-degree polynomials, $\mathbb{P}_1$
- the set of quadratic polynomials, $\mathbb{P}_2$
- the set of n-th-degree polynomials, $\mathbb{P}_n$
- the set of all polynomials, $\mathbb{P}$
- the set of all $p \times q$ matrices ($p, q :$ given)
- the singleton set $V = \{0\}$
- the singleton set $V = \{1\}$

Abstractions (cont)

Further abstractions are possible by replacing the traditional addition $(+)$ and scalar multiplication $(\cdot$ or no symbol$)$ operations by some other defined operations.

For example, we may define new addition $(\oplus)$ and scalar multiplication $(\odot)$ operations by:

$x \oplus y \equiv xy$,

and

$a \odot x \equiv x^a$,

where a is a real number.

Now, is the set $V = \{x,\ x \in \mathbb{R},\ x > 0\}$, with operations as defined above, a vector set?

Check on Axioms

1. V is non-empty. Axiom 1 is satisfied.
2. Let $a \in \mathbb{R}; x \in V$. Then $a \odot x = x^a \in V$. Axiom 2 is satisfied.
3. Let $x, y \in V$, then $x \oplus y = xy \in V$, and is unique. Axiom 3 is satisfied.
4. Let $x, y \in V$, then $x \oplus y = xy = yx = y \oplus x$. Axiom 4 is satisfied.
5. Let $x, y, z \in V$, then $x \oplus (y \oplus z) = x(yz) = (xy)z = (x \oplus y) \oplus z$. Axiom 5 is satisfied.

Check on Axioms (cont)

6. The zero element. We need a $\mathbf{0} \in V$, s.t. $\mathbf{0} \oplus x = x$, i.e. $\mathbf{0}x = x$. Hence, $\mathbf{0} = 1 \in V$. The zero element is the arithmetic 1. Axiom 6 is satisfied.
7. Existence of a negative of an element. Let $x \in V$, then we need a $\ominus x$ s.t. $x \oplus (\ominus x) = \mathbf{0}$. That is, $x \oplus (\ominus x)\mathbf{0} \implies x(\ominus x) = 1 \implies \ominus x = (1/x)$. Thus the negative of a member is its reciprocal. Axiom 7 is satisfied.

Check on Axioms (cont)

8. Let $a \in \mathbb{R}; x, y \in V$. Then
   $a \odot (x \oplus y) = (xy)^a = (x^a)(y^a) = (a \odot x) \oplus (a \odot y)$.
   Axiom 8 is satisfied.
9. Let $a, b \in \mathbb{R}; x \in V$. Then
   $(a + b) \odot x = x^{a+b} = x^a x^b = a \odot x \oplus a \odot x$. Axiom 9 is satisfied. [Note: the $+$ in $(a + b)$ is the arithmetic $+$, since $a, b \in \mathbb{R}$; $a, b \notin V$.]

Check on Axioms (cont)

10. Let $a, b \in \mathbb{R}; x \in V$. Then $a \odot (b \odot x) = a \odot (x^b) = (x^b)^a = x^{ab} = (ab) \odot x$. Axiom 10 is satisfied.
11. Let $x \in V$, then $1 \odot x = x^1 = x$. Axiom 11 is satisfied.

Since all axioms are satisfied, V, as defined, is a vector set!

Vector Space of Matrices

Matrices with real elements, of size $m \times n$ form an mn-dimensional vector set.
For example, all real 2×2 matrices form a 4-dimenstional vector set.
A basis for this vector set is:

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}.$$

Any 2×2 real matrix is a linear combination of these four matrices.

Vector Space of Polynomials

Consider a set $\mathbb{P}_n$ of all real polynomials of degree less than or equal to n.

This is a vector set of dimension $n+1$. (Why?)

A basis for this vector set is: $1, x, 2x^2, x^3, \cdots, x^n$.

[Why do we say 'a basis', and not 'the basis'?]

Vector Space, Dimension ∞

When a vector set (like $\mathbb{P}_n$) has a basis of n vectors for each n, and this is true for any arbitrarily large n, then we say that the vector set is **infinite-dimensional**.

One example of such a set is the space of all continuous functions $f(x)$ over a range $[a, b]$ on th real x-axis.

Question. Is the set of all real-valued functions, defined, and continuous, on the entire real line, a vector set?

Inner Product Spaces

Some (not all) vector spaces are also **inner product spaces**.

A vector space V will be called an inner product space iff the following axioms hold:
Let $\alpha, \beta \in \mathbb{R}$, and $\mathbf{a}, \mathbf{b}, \mathbf{c} \in V$.

1. For every pair $\mathbf{a}, \mathbf{b}$ there exists a real number denoted by $(\mathbf{a}, \mathbf{b})$, which is called the **inner product** of $\mathbf{a}$ and $\mathbf{b}$.
2. $(\alpha\mathbf{a} + \beta\mathbf{b}, \mathbf{c}) = \alpha(\mathbf{a}, \mathbf{c}) + \beta(\mathbf{b}, \mathbf{c})$ (Linearity).
3. $(\mathbf{a}, \mathbf{b}) = (\mathbf{b}, \mathbf{a})$ (Symmetry).
4. $(\mathbf{a}, \mathbf{a}) \geq 0$; $(\mathbf{a}, \mathbf{a}) = 0$ iff $\mathbf{a} = \mathbf{0}$ (Positive-definiteness).

Inner Product Spaces (cont)

A pair of vectors whose inner product is zero are called **orthogonal** vectors.

The **norm** of a vector $\mathbf{a} \in V$ is defined as

$$\|\mathbf{a}\| = \sqrt{(\mathbf{a}, \mathbf{a})}.$$

The norm is non-negative and is a measure of length.

A **unit vector** is a vector of norm 1.

Some Derivations with Norms

The following relations can be derived:

The Cauchy-Schwarz Inequality:

$$|(\mathbf{a}, \mathbf{b})| \leq \|\mathbf{a}\| \|\mathbf{b}\|.$$

The triangle inequality:

$$\|(\mathbf{a} + \mathbf{b})\| \leq \|\mathbf{a}\| + \|\mathbf{b}\|.$$

The parallelogram equality:

$$\|\mathbf{a} + \mathbf{b}\|^2 + \|\mathbf{a} - \mathbf{b}\|^2 = 2(\|\mathbf{a}\|^2 + \|\mathbf{b}\|^2).$$

Euclidean Space

From our school days we are familiar with Euclidean geometry in two and three dimensions.
We can have $\mathbb{R}^n$, which is the n-dimensional Euclidean space.
Members of this space are vectors (say column vectors) with n components.
If $\mathbf{a}, \mathbf{b} \in \mathbb{R}^n$ $(\mathbf{a}, \mathbf{b}$: column vectors.$)$ then the **Euclidean norm** is:

$$\|\mathbf{a}\| = \sqrt{(\mathbf{a}, \mathbf{a})} = \sqrt{(a_1^2 + a_2^2 + \cdots + a_n^2)}.$$

A Euclidean space is thus an inner product space.
(Check that the axioms are satisfied.)

Function Space

The set of all real-valued functions $f(x), \cdots$ over an interval $\alpha \leq x \leq \beta$ is a vector space, called a **function space**. On this space, an inner products of two members $f(x)$ and $g(x)$ is defined as:

$$(f, g) = \int_{\alpha}^{\beta} f(x) g(x) \, dx,$$

and the norm of $f(x)$ is:

$$\|f(x)\| = \sqrt{(f, f)} = \sqrt{\int_{\alpha}^{\beta} f(x)^2 \, dx}.$$

The function space is also an inner product space.
(Check that the axioms are satisfied.)

Mapping or Transformation

Let X and Y be any two vector spaces. If we assign for any vector $\mathbf{x}$ in X a unique vector $\mathbf{y}$ in Y, then we have defined a **mapping** of X into Y.

A mapping is often represented by a capital letter, *e.g.* F.

F is also known as a **transformation** of $\mathbf{x}$ into $\mathbf{y}$,
or F **operates** on $\mathbf{x}$ to give $\mathbf{y}$.

One may write $\mathbf{y} = F(\mathbf{x})$.

Linear Mapping or Linear Transformation

If a mapping F satisfies the following relations for all $\mathbf{x}, \mathbf{v} \in X$, and $c \in \mathbb{R}$:

$F(\mathbf{x} + \mathbf{v}) = F(\mathbf{x}) + F(\mathbf{v})$, and
$F(c\mathbf{x}) = cF(\mathbf{x})$,

then F is called a **linear mapping** or **linear transformation**.

Lin.Trans. of Space $\mathbb{R}^n$ into Space $\mathbb{R}^m$

Let $X = \mathbb{R}^n$, $Y = \mathbb{R}^m$.

Let A : any real $m \times n$ matrix.

Then $\mathbf{y} = A\mathbf{x}$ will be a linear transformation of X into Y.

It can be shown that:
any linear transformation F of X into Y can be satisfied using a suitable $m \times n$ matrix A.

We then say that A represents F.

Unit Vectors in Euclidean Space

For $\mathbb{R}^n$, we often define a standard basis.

Each member of this basis is a unit vector, such that only one element is 1 and all other elements are 0.

These represent the unit vectors of Euclidean space.

E.g. in 2D Euclidean space $\mathbb{E}^2$, we have the standard basis: $\begin{bmatrix}1\\0\end{bmatrix}$ and $\begin{bmatrix}0\\1\end{bmatrix}$.

In 3D, $\mathbb{E}^3$, we have: $\begin{bmatrix}1\\0\\0\end{bmatrix}$, $\begin{bmatrix}0\\1\\0\end{bmatrix}$, and $\begin{bmatrix}0\\0\\1\end{bmatrix}$.

These are the usual unit vectors $\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$.

Homework

Revisit the transformations we looked at earlier (slides 29–33 in linear.pdf).

Are these linear transformations?

Inverse Transformation

The matrix A in the linear transformation $\mathbf{y} = A\mathbf{x}$ may be a square matrix.

In that case, if A is invertible, then A^{-1} will exist, and so will be the transformation $\mathbf{x} = A^{-1}\mathbf{y}$.

This inverse transformation will also be a linear transformation, since it is represented by a matrix, in this case A^{-1}.

Composition of Linear Transformations

Let X, Y, and Z be three (general) vector spaces.

Let F be a linear transformation from X to Y.
So, if $\mathbf{x}$ is a vector in X, then $\mathbf{y} = F(\mathbf{x})$ is a vector in Y.

Let G be a linear transformation from Y to Z.
So, for any vector $\mathbf{y}$ in Y, $\mathbf{z} = G(\mathbf{y})$ is a vector in Z.

So, we have: $\mathbf{z} = G(\mathbf{y}) = G(F(\mathbf{x}))$.

This transformation of a transformation is known as a composition.

We write $\mathbf{z} = H(\mathbf{x}) = (G \circ F)(\mathbf{x})$.

The transformation $H = (G \circ F)$ is called the composition of G and F.

Exercises

- Show that the composition of two linear transformations is a linear transformation.
- Show that in Euclidean spaces, where a linear transformation is represented by matrix multiplication, the composition of two transformations is equivalent to the multiplication of the associated matrices.