A Digital Signal Processing Approach to Interpolation

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Abstract—In many digital signal processing systems, e.g., vocoders, modulation systems, and digital waveform coding systems, it is necessary to alter the sampling rate of a digital signal. Thus it is of considerable interest to examine the problem of interpolation of bandlimited signals from the viewpoint of digital signal processing. A frequency domain interpretation of the interpolation process, through which it is clear that interpolation is fundamentally a linear filtering process, is presented.

An examination of the relative merits of finite duration impulse response (FIR) and infinite duration impulse response (IIR) digital filters as interpolation filters indicates that FIR filters are generally to be preferred for interpolation. It is shown that linear interpolation and classical polynomial interpolation correspond to the use of the FIR interpolation filter. The use of classical interpolation methods in signal processing applications is illustrated by a discussion of FIR interpolation filters derived from the Lagrange interpolation formula. The limitations of these filters lead us to a consideration of optimum FIR filters for interpolation that can be designed using linear programming techniques. Examples are presented to illustrate the significant improvements that are obtained using the optimum filters

I. Introduction

THE PROCESS of interpolation is familiar to anyone who has had occasion to "read between the lines" in a table of mathematical functions. In digital signal processing, interpolation is required whenever it is necessary to change from one sampling rate to another. For example, in speech processing systems, estimates of speech parameters are often computed at a low sampling rate for low bit-rate storage or transmission; however, for constructing a synthetic speech signal from the low bit-rate representation, the speech parameters are normally required at much higher sampling rates [1], [2]. In such cases, the sampling rate must be increased by a digital interpolation process. As another example, an efficient digital realization of a frequency-multiplexed single-sideband system has been obtained [3] by performing complicated filtering functions at a low sampling rate and simpler functions at the high sampling rate required for grouping several channels into a frequency-multiplexed format. In this process, there is a need for both increasing and decreasing the sampling rate. Another example where sampling rate reduction is required is in converting a delta modulation representation of a waveform to a pulse-code modulation (PCM) representation [4].

In these and other examples, it is important to thoroughly understand the process of interpolation from the point of view of digital signal processing rather than from a numerical analysis viewpoint. For example, tables of mathematical functions are generally constructed so that linear interpolation produces sufficiently accurate results, and for cases where linear interpolation is inadequate, there exists a great variety of higher order polynomial interpolation formulas. In signal processing applications there is a great temptation to try to get along with linear interpolation because it is a simple technique. In this paper we present a frequency-domain interpretation of the interpolation process in which it is clear that interpolation is fundamentally a linear filtering process. This discussion makes it abundantly clear that linear interpolation is generally not appropriate for digital signal processing applications. We discuss the advantages of finite duration impulse response (FIR) over infinite duration impulse response (IIR) digital filters for use as interpolation filters and we discuss the application of recently developed design techniques for FIR filters to the design of optimum interpolation filters. These filters are compared to filters derived from classical polynomial interpolation formulas to illustrate the improvement that can be achieved.

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II. Digital Sampling Rate Alteration

A. Sampling Continuous-Time Signals

Consider a continuous-time signal $\hat{x}(t)$ with Fourier transform

$$\hat{X}(\omega) = \int_{-\infty}^{\infty} \hat{x}(t)e^{-j\omega t}dt.$$

The signal $\hat{x}(t)$ is sampled to produce the sequence

$$x(n) = \hat{x}(nT), \qquad -\infty < n < \infty$$

where T is the sampling period. The z transform of the sequence $x(n)$ is defined as

$$X(z) = \sum_{n=-\infty}^{\infty} x(n)z^{-n}.$$

The z transform evaluated on the unit circle $X(e^{j\omega T})$ will be called the *Fourier transform* of the sequence $x(n)$. It is well known that the Fourier transform of the sequence $x(n)$ is related to the Fourier transform of $\hat{x}(t)$ by [5]

$$X(e^{j\omega T}) = \frac{1}{T}\sum_{k=-\infty}^{\infty} \hat{X}\left(\omega + k\frac{2\pi}{T}\right). \tag{1}$$

If $\hat{x}(t)$ is bandlimited, i.e., $\hat{X}(\omega)=0$ for $|\omega| \geq \Omega$, and if $T \leq \pi/\Omega$, then it can be seen from (1) that

$$X(e^{j\omega T}) = \frac{1}{T}\hat{X}(\omega), \qquad -\frac{\pi}{T} \leq \omega \leq \frac{\pi}{T}$$

as depicted in Fig. 1 where $T=\pi/\Omega$.

Assuming that $\hat{x}(t)$ is bandlimited, the original continuous-time signal can be obtained uniquely from the samples $x(n)$ through the interpolation formula

$$\hat{x}(t) = \sum_{k=-\infty}^{\infty} x(k)\frac{\sin\left[\frac{\pi}{T}(t-kT)\right]}{\frac{\pi}{T}(t-kT)}. \tag{2}$$

In many digital signal processing problems, we are given a sequence $x(n)$, corresponding to sampling period T, and we must obtain from the sequence $x(n)$ a sequence $y(n)=\hat{x}(nT')$; i.e., the sequence $y(n)$ corresponds to sampling $\hat{x}(t)$ at a different sampling rate. If we evaluate (2) for $t=nT'$, we obtain

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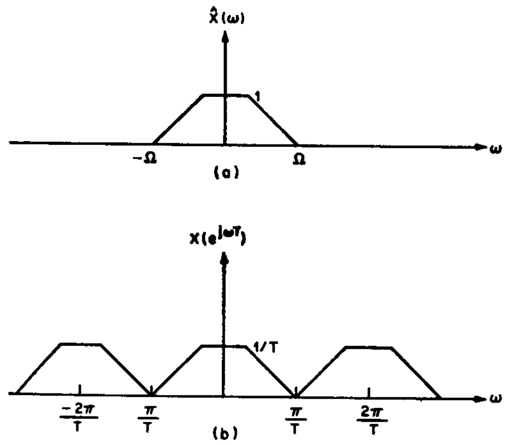

Fig. 1. Illustration of the relationship between the Fourier transform of a continuous-time signal and the Fourier transform of the sequence obtained by sampling with period T.

a direct relationship between $y(n)$ and $x(n)$, but it is clear that such an equation is impossible to evaluate because the functions $\sin [(\pi/T)(t-kT)]/[(\pi/T)(t-kT)]$ are of infinite duration. Rather than simply truncate these functions, it is more reasonable to design finite duration interpolators. To understand how such interpolators can be designed and to understand the limitations of classical interpolators, it is useful to consider the frequency-domain representation of the process of changing the sampling rate.

B. Sampling Rate Reduction—Integer Factors

Suppose that the desired sampling period is $T'=MT$. If M is an integer, this simply implies that the new sequence is

$$\begin{aligned} y(n) &= \hat{x}(nT') = \hat{x}(nMT) \\ &= x(Mn). \end{aligned}$$

That is, the sequence $x(n)$ is "sampled" by retaining only one out of each group of M consecutive samples. The values of the sequence y are samples of the original waveform $\hat{x}(t)$; however, these samples will uniquely determine $\hat{x}(t)$ if and only if $T'\leq\pi/\Omega$. This is clearly just a consequence of the sampling theorem as expressed by (1).

Since we are interested in direct relationships between the sequences $y(n)$ and $x(n)$, it is instructive to derive an equation similar to (1) that relates the Fourier transforms of the two sequences. The derivation of the equation is facilitated by the definition of a new sequence $w(n)$ which is nonzero only at integer multiples of M; that is

$$\begin{aligned} w(n) &= x(n), \qquad n = 0, \quad \pm M, \quad \pm 2M, \cdots \\ &= 0, \qquad \text{elsewhere} \end{aligned}$$

where the sampling period is assumed to be T for both sequences. A convenient representation of $w(n)$ is

$$w(n) = x(n) \left\{ \frac{1}{M} \sum_{l=0}^{M-1} e^{j(2\pi/Mln)} \right\}, \qquad -\infty < n < \infty$$

where the term in brackets may be recognized as a discrete Fourier series representation of a periodic sequence that is one at integer multiples of M and zero otherwise. The sequence $y(n)$, corresponding to sampling period $T'=MT$, is

$$y(n) = w(Mn), \qquad -\infty < n < \infty.$$

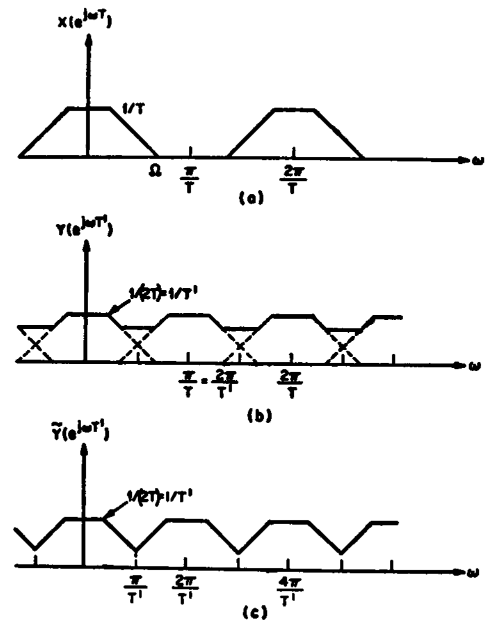

Fig. 2. Sampling rate reduction ($T'=2T$). (a) Fourier transform of original sequence $x(n)$. (b) Fourier transform of sequence $y(n)=x(2n)$ showing aliasing. (c) Fourier transform obtained after sampling rate reduction of a low-pass filtered version of $x(n)$.

Therefore

$$Y(z) = \sum_{n=-\infty}^{\infty} w(Mn)z^{-n}.$$

Since $w(n)$ is zero except at integer multiples of M, we obtain

$$\begin{aligned} Y(z) &= \sum_{n=-\infty}^{\infty} w(n)z^{-n/M} \\ &= \sum_{n=-\infty}^{\infty} x(n) \frac{1}{M} \sum_{l=0}^{M-1} e^{j(2\pi/M)ln}\, z^{-n/M} \\ &= \frac{1}{M} \sum_{l=0}^{M-1} \sum_{n=-\infty}^{\infty} x(n)e^{j(2\pi/M)ln}\, z^{-n/M} \end{aligned}$$

$$Y(z) = \frac{1}{M} \sum_{l=0}^{M-1} X(e^{-j(2\pi/M)l}z^{1/M}). \tag{3}$$

If we evaluate $Y(z)$ on the unit circle, with normalization appropriate for the new sampling rate, we obtain

$$Y(e^{j\omega T'}) = \frac{1}{M} \sum_{l=0}^{M-1} X(e^{j(\omega T'-2\pi l)/M}). \tag{4}$$

There is a clear similarity between (4) and (1).

An example of sampling rate reduction by a factor of 2 is shown in Fig. 2. The Fourier transform of $x(n)$ is depicted in Fig. 2(a) for the case when $\pi/2\Omega < T < \pi/\Omega$ so that

$$X(e^{j\omega T}) = \frac{1}{T} \hat{X}(\omega), \qquad -\frac{\pi}{T} < \omega \leq \frac{\pi}{T}.$$

Fig. 2(b) shows $Y(e^{j\omega T'})$ for $T'=2T$. In this case, aliasing occurs and it is clear that, in general, aliasing will occur in the process of digital sampling rate reduction unless the original sampling period satisfies

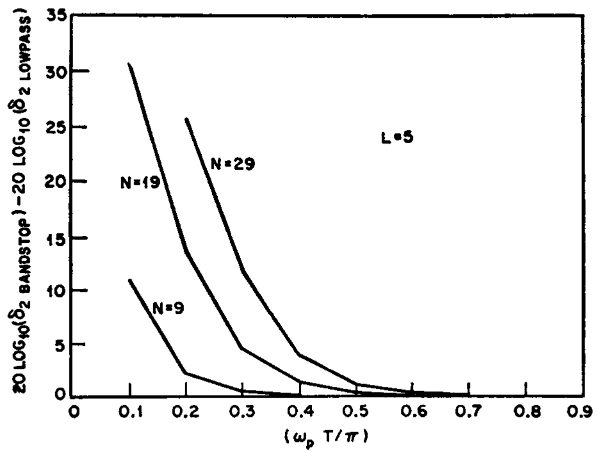

Fig. 15. Comparison of optimum bandstop and optimum low-pass interpolation filters ($L=5$). The curve for $N=29$ is off scale for $\omega_p<0.2\pi/T$.

error. This situation would be a more favorable situation for the classical filters, although it is always possible to design a better filter using the optimum design procedures.

Another interesting comparison is between optimum low-pass filters and optimum bandstop filters. To compare these filters, we set

$$\omega_p = \frac{2\pi}{T} - \omega_{s_1} = \frac{4\pi}{T} - \omega_{s_2}$$

and $\Delta\omega_1 = \Delta\omega_2 = 2\omega_p$ for the bandstop filters and

$$\omega_p = \frac{2\pi}{T} - \omega_{s_1}$$

and $\Delta\omega = \pi/T' - \omega_{s_1}$ for the low-pass filters. That is, both filters were designed to accommodate the same input signal bandwidth. Fig. 15 shows the difference between stopband attenuations for the bandstop filters and the low-pass filters as a function of bandwidth. From these curves we see that for narrow bandwidths the bandstop filters have significantly greater attenuation; however as the bandwidth approaches half the original sampling frequency, there is no difference between the two types of filters.

VI. Conclusions

In this paper we have discussed the process of interpolation as a problem in digital filtering. Most of our discussion has involved frequency-domain representations of the interpolation process and design criteria for digital interpolation filters. We have taken this approach because it is the most reasonable for digital signal processing applications where it is necessary to either raise or lower the sampling rate of a signal. This point of view is in contrast to that of interpolation in tables where one is concerned primarily with minimizing the error in a particular interpolated sample. Because of the variety of factors involved in the design of an interpolation filter, we have not tried to give design formulas and error bounds that would have limited value, but rather we have chosen to attempt to illuminate the important factors involved in the interpolation process and to discuss general design procedures that can be adapted to a variety of situations.

In particular, we have argued that linear-phase FIR filters have many attractive features for discrete-time interpolation and have shown how they may be efficiently utilized. Classical polynomial interpolation has been discussed in the context of digital signal processing. Interpolation filters derived from polynomial interpolation formulas are attractive because the impulse response can be easily computed or looked up in a table. However, we have seen that the frequency response of such systems leaves much to be desired in digital signal processing applications where the original sampling rate may be only slightly above the Nyquist rate.

As an alternative to filters based on classical interpolation formulas, we discussed optimum low-pass and bandstop FIR filters that were designed by linear programming. The bandstop filters have frequency responses that are very similar to the classical interpolators, but are always superior. The bandstop designs appear to be most important for cases when the original sampling rate is several times the Nyquist rate, while the low-pass designs are appropriate when the original sampling rate is close to the Nyquist rate.

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