Discrete Hilbert Transform

VÁCLAV ČÍŽEK
Institute of Radioengineering and Electronics
Czechoslovak Academy of Sciences
Prague, Czechoslovakia

Abstract

The Hilbert transform $\mathcal{H}\{f(t)\}$ of a given waveform $f(t)$ is defined with the convolution $\mathcal{H}\{f(t)\}=f(t)*(1/\pi t)$. It is well known that the second type of Hilbert transform $\mathcal{H}_0\{f(x)\}=\phi(x)*(1/2\pi)\cot\frac{1}{2}x$ exists for the transformed function $f(tg\frac{1}{2}x)=\phi(x)$. If the function $f(t)$ is periodic, it can be proved that one period of the $\mathcal{H}$ transform of $f(t)$ is given by the $\mathcal{H}_1$ transform of one period of $f(t)$ without regard to the scale of the variable.

On the base of the discrete Fourier transform (DFT), the discrete Hilbert transform (DHT) is introduced and the defining expression for it is given. It is proved that this expression of DHT is identical to the relation obtained by the use of the trapezoidal rule to the cotangent form of the Hilbert transform.

Manuscript received March 13, 1970.

This paper was presented at the International Seminar on Digital Processing of Analog Signals, Zürich, Switzerland, March 11–13, 1970.

Many papers concerning FFT, e.g., [2], [3], explain it by the use of discrete Fourier transform (DFT), some properties of which are presented in [1]. The need for an efficient Hilbert transform and its simulation generated interest in applying the FFT for this purpose. The simplest method of such an application of FFT consists of one direct FFT, one multiplication, and one inverse FFT. However, this direct procedure is considerably complicated. Fortunately, the DFT allows the direct expression of the Hilbert transform in a simple way, which will be described further.

This procedure is closely related to the theory of conjugate trigonometric series [5] and to the application of Runge's method of harmonic analysis for approximative Hilbert transform [4].

In this paper the DFT will be use to explain the concept of the discrete Hilbert transform (DHT), and some of its interesting properties will be presented. Further, the relation of DHT to the cotangent form of Hilbert transform and to the Hilbert transform of periodic function will be given.

I. Hilbert Transform

The Hilbert transform ($\mathcal{H}$ transform) of the given function is defined with (1).

$$\mathcal{H}\{f(t)\} = -\frac{1}{\pi}\int_{-\infty}^{+\infty} f(\tau)\,\frac{d\tau}{\tau - t} = -\frac{1}{\pi t} * f(t). \qquad (1)$$

The Hilbert transform can be interpreted from this relation as a convolution between $f(t)$ and $-(1/\pi t)$. If we denote the Fourier transform of $f(t)$ by $F(j\omega)$,

$$\mathcal{F}\{f(t)\} = \int_{-\infty}^{+\infty} f(t)e^{-j\omega t}dt = F(j\omega) \qquad (2)$$

and if we use (3)

$$\mathcal{F}\left\{-\frac{1}{\pi t}\right\} = j \operatorname{sgn} \omega, \qquad (3)$$

then we obtain for the Fourier transform of the Hilbert transform the following relation:

$$\mathcal{F}\{\mathcal{H}\{f(t)\}\} = \operatorname{sgn} \omega F(j\omega). \qquad (4)$$

The function $j \operatorname{sgn} \omega$ is defined as

$$\operatorname{sgn} \omega = \begin{cases} -1 & \omega < 0 \\ 0 & \omega = 0 \\ +1 & \omega > 0. \end{cases}$$

If we substitute in the fundamental relation (1)

$$t = tg\tfrac{1}{2}\phi; \qquad \tau = tg\tfrac{1}{2}\psi, \qquad (5)$$

and denote

$$f(t) = f(tg\tfrac{1}{2}\phi) = f_0(\phi), \qquad (6)$$

then from (1), (7) follows:

$$\mathcal{H}\{f(t)\} = \frac{1}{2\pi}\int_{-\pi}^{+\pi} f_0(\psi)\ tg\,\frac{\psi}{2}\,d\psi + \frac{1}{2\pi}\int_{-\pi}^{+\pi} f_0(\psi)\ \cot\ \frac{\psi-\phi}{2}\,d\psi. \tag{7}$$

In this paper we shall note the Hilbert transform in cotangent form (or $\mathcal{H}_0$ transform) by $\mathcal{H}_0\{f_0(\phi)\}$ which is given by definition (8).

$$\mathcal{H}_0\{f_0(\phi)\} = \frac{1}{2\pi}\int_{-\pi}^{+\pi} f_0(\psi)\ \cot\ \frac{\psi-\phi}{2}\,d\psi. \tag{8}$$

Then the relation between $\mathcal{H}$ transform and $\mathcal{H}_0$ transform can be expressed in the following manner:

$$\mathcal{H}\{f(t)\} = \mathcal{H}_0\{f_0(\phi)\} + \frac{1}{2\pi}\int_{-\pi}^{+\pi} f_0(\psi)\ tg\,\frac{\psi}{2}\,d\psi. \tag{9}$$

From (9) it follows that the resulting functions defined with $\mathcal{H}$ and $\mathcal{H}_0$ transforms differ in a constant value only. This constant can be expressed in various forms.

The $\mathcal{H}_0$ transform can be modified for functions defined on the finite interval $[-(T/2), +(T/2)]$. If we substitute in $f_0(\varphi)$. $\varphi=(2\pi/T)x$ we obtain the function $f_1(x)=f_0(2\pi x/T)=f_0(\varphi)$. On this function the Hilbert transform in cotangent form may be applied and this modified Hilbert transform ($\mathcal{H}_1$ transform) is expressed by (10).

$$\mathcal{H}_0\{f_0(\phi)\} = \frac{1}{T}\int_{-T/2}^{+T/2} f_1(\xi)\ \cot\ \frac{\pi}{T}\,(\xi - x)d\xi = \mathcal{H}_1\{f_1(x)\} = \mathcal{H}_1\left\{f_0\left(\frac{2\pi}{T}x\right)\right\}. \tag{10}$$

II. Discrete Hilbert Transform

To introduce the DFT we use the relations presented in [1]. Let x_i, $i=0, 1, \cdots, N-1$ denote the sequence of N complex finite numbers. Then the DFT of this sequence is a sequence X_k, $k=0, 1, \cdots, N-1$ defined with (11).

$$X_k = \frac{1}{N}\sum_{i=0}^{N-1} x_i e^{-jik(2\pi/N)}. \tag{11}$$

For the inverse DFT the sequence X_i is defined by (12).

$$x_i = \sum_{k=0}^{N-1} X_k e^{jik(2\pi/N)}. \tag{12}$$

We shall denote the DFT in a symbolic way by $x_i \leftrightarrow X_k$. From this defining relation it follows that x_i and X_k are periodic sequences.

The well known relation (4) holds for the Fourier transform of the Hilbert transform. Similarly, we introduce the discrete Hilbert transfrom (DHT) by defining the DHT of the sequence f_i as the inverse DFT of the sequence G_k

$$G_k = F_k \cdot H_k \tag{13}$$

where H_k is the sequence described by (14).

$$H_k = \begin{cases} -j, & k = 1, 2, \cdots, \dfrac{N}{2} - 1 \\ 0, & k = 0, N/2 \\ +j, & k = \dfrac{N}{2}+1, \dfrac{N}{2}+2, \cdots, N-1. \end{cases} \tag{14}$$

The sequence H_k can be interpreted as a discrete representation of the function j sgn ω. Its properties for $k=0$ and $k=N/2$ follow from the simultaneous validity of (15) which describes the general properties of the DFT of real functions.

$$X_N = X_0$$
$$X_k = \overline{X}_{-k} = \overline{X}_{N-k}. \tag{15}$$

Following the definition of DHT, we obtain (16) from (11) and (13) for the DHT g_i of the sequence f_i.

$$g_i \leftrightarrow G_k$$
$$g_i = \frac{1}{N}\sum_{k=0}^{N-1} H_k \sum_{v=0}^{N-1} f_v e^{jkv(2\pi/N)} \cdot e^{-jik(2\pi/N)} = \frac{1}{N}\sum_{v=0}^{N-1} fv \sum_{k=0}^{N-1} H_k e^{jk(v-i)(2\pi/N)}. \tag{16}$$

If we substitute for H_k from (14) we obtain (17).

$$g_i = \frac{j}{N}\sum_{v=0}^{N-1} f_v \left[-\sum_{k=1}^{(N/2)-1} e^{jk(v-i)(2\pi/N)} + \sum_{k=(N/2)+1}^{N-1} e^{jk(v-i)(2\pi/N)}\right] = -\frac{j}{N}\sum_{v=0}^{N-1} f_v[1-(-1)^{v-i}] \sum_{k=1}^{(N/2)-1} e^{jk(v-i)(2\pi/N)} \tag{17}$$

from which the resulting expression for DHT is

$$g_i = \frac{1}{N}\sum_{v=0}^{N-1} f_v[1-(-1)^{v-i}]\ \cot\ (v-i)\,\frac{\pi}{N}. \tag{18}$$

This relation can be modified for i even or odd in the following way:

$$g_i = \frac{2}{N}\sum_{v=0,2,4\cdots} f_v\ \cot\ (v-i)\,\frac{\pi}{N}, \quad \text{for } i \text{ odd} \tag{19}$$

$$g_i = \frac{2}{N}\sum_{v=1,3,5\cdots} f_v\ \cot\ (v-i)\,\frac{\pi}{N}, \quad \text{for } i \text{ even.} \tag{20}$$

From the foregoing facts it follows that the procedure composed of DFT, multiplication, and inverse DFT corresponds to the simple expression for discrete convolution. It is evident that this expression can be interpreted in various ways (e.g., as matrix multiplication).

Next we shall demonstrate an important relationship between the DHT and the $\mathcal{H}_1$ transform.

III. Relation Between DHT and $\mathcal{H}_1$ Transform

The DFT has one important property, consisting of periodic continuation of the given and transformed sequences. Let us therefore discuss the Hilbert transform of a periodic function.

The given periodic function $f(t)$ with the periodic T can be expressed by the use of the function $f_T(t)$ which represents one period of $f(t)$:

$$f(t) = \sum_{-\infty}^{+\infty} f_T(t - kT) = f(t) * \sum_{-\infty}^{+\infty} \delta(t - kT)$$

$$f_T(t) = \begin{cases} f(t) & |t| < T/2 \\ 0. \end{cases} \tag{21}$$

For the Fourier transform of $f(t)$ we obtain (22).

$$\begin{aligned} F_T(j\omega) &= \mathfrak{F}\{f_T(t)\} \\ F(j\omega) &= F_T(j\omega) \cdot \sum_{-\infty}^{+\infty} e^{-j\omega i T} \\ &= \frac{2\pi}{T} F_T(j\omega) \sum_{-\infty}^{+\infty} \delta\left(\omega - k\frac{2\pi}{T}\right). \end{aligned} \tag{22}$$

It is known that the following relation holds.

$$\sum_{i=-\infty}^{+\infty} e^{-ji\omega T} = \frac{2\pi}{T} \sum_{-\infty}^{+\infty} \delta\left(\omega - k\frac{2\pi}{T}\right).$$

For the Fourier transform of the Hilbert transform one obtains the relation (23).

$$\begin{aligned} \mathfrak{F}\{\mathcal{H}\{f(t)\}\} &= j \operatorname{sgn} F(j\omega) \\ &= \frac{2\pi}{T} F_T(j\omega) \cdot j \operatorname{sgn} \omega \sum_{-\infty}^{+\infty} \delta\left(\omega - k\frac{2\pi}{T}\right) \end{aligned} \tag{23}$$

from which

$$\begin{aligned} \mathcal{H}\{f(t)\} &= \frac{1}{T} f_T(t) * j\left[-\sum_{-\infty}^{-1} e^{jk(2\pi/T)t} + \sum_{-1}^{\infty} e^{jk(2\pi/T)t}\right] \\ &= -\frac{1}{T} f_T(t) * 2\sum_{k=1}^{\infty} \sin k\frac{2\pi}{T} t \end{aligned}$$

for the Hilbert transform follows.

While in the distribution sense, the following holds

$$\sum_{k=1}^{\infty} \sin kx = \frac{1}{2} \cot \frac{1}{2} x \tag{24}$$

and the Hilbert transform can be expressed by the integral (25).

$$\begin{aligned} \mathcal{H}\{f(t)\} &= -\frac{1}{T} f_T(t) * \cot \frac{\pi}{T} t \\ &= \frac{1}{T} \int_{-t/2}^{+t/2} f_T(t) \cot \frac{\pi}{T} (\tau - t) d\tau \\ &= \mathcal{H}_1\{f_T(t)\}. \end{aligned} \tag{25}$$

From this an important conclusion may be stated: The $\mathcal{H}$ transform of periodic function $f(t)$ is given by the $\mathcal{H}_1$ transform of the function $f_T(t)$ which expresses one period of $f(t)$.

We obtain another important result if we express the integral for the $\mathcal{H}_1$ transform using the trapezoidal rule in the tangent form. If we divide the period T into N elements of the width I/N then the elementary area is f_v cot $(\pi/N)(v(T/N)-t)$, $v = 1,3,5, \cdots$. For $t = i(T/N)$, $i = 0, 2, \cdots$ the following holds:

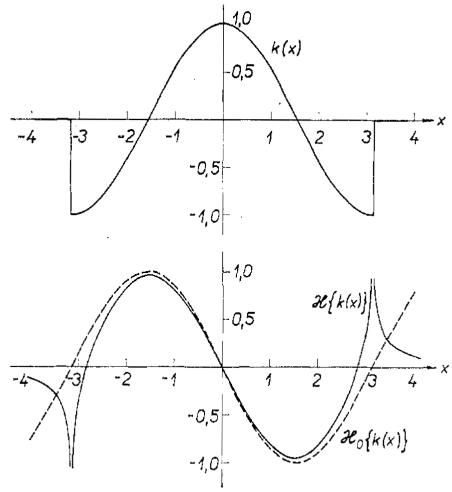

Fig. 1. $\mathcal{H}$ transform and $\mathcal{H}_0$ transform of $k(x)$.

$$\mathcal{H}_1\{f(t)\}_{\substack{t=i(T/N) \\ i \text{ even}}} \approx \frac{2}{N} \sum_{v=1,3,5,\ldots} f_v \cot (v - i) \frac{\pi}{N} \cdot$$

Similarly, if we choose the elementary areas with the center in points $v(T/N)$, $v = 0,2, \cdots$, it holds for $t = i(T/N)$, $i = 1,3,5$:

$$\mathcal{H}_1\{f(t)\}_{\substack{t=i(T/N) \\ i \text{ odd}}} \approx \frac{2}{N} \sum_{v=0,2,4\cdots} f_v \cot (v - i) \frac{\pi}{N} \cdot$$

The discrete Hilbert transform of $f_T(t)$ is given by the integral for the $\mathcal{H}_1$ transform of $f_T(t)$ expressed numerically and is an approximative value of $\mathcal{H}\{f(t)\}$.

IV. Examples

In Fig. 1 the function

$$k(x) = \begin{cases} \cos x & (-\pi + \pi) \\ 0 \end{cases}$$

and its $\mathcal{H}$ and $\mathcal{H}_0$ transforms are shown. For these equations

$$\begin{aligned} \mathcal{H}\{k(x)\} &= \frac{1}{\pi}\{[Ci(\pi - x) - Ci(\pi + x)] \cos x \\ &\quad - [Si(\pi + x) + Si(\pi - x)] \sin x\} \\ \mathcal{H}_0\{k(x)\} &= -\sin x. \end{aligned}$$

The application of DHT for $N > 2$ gives values identical to those of $\mathcal{H}_0\{k(x)\}$.

Fig. 2 presents the right-hand half of the even function

$$f_T(t) = \begin{cases} 1 & |t| < \frac{1}{6}T \\ 0 \end{cases}$$

and its $\mathcal{H}$ and $\mathcal{H}_1$ transforms, which are given by the following relations:

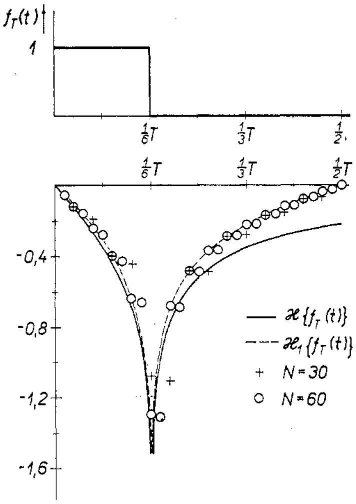

Fig. 2. Calculation of DHT of even $f_T(t)$ with $N=30$ and $N=60$ and comparison with odd functions $\mathcal{H}\{f_T(t)\}$ and $\mathcal{H}_1\{f_T(t)\}$.

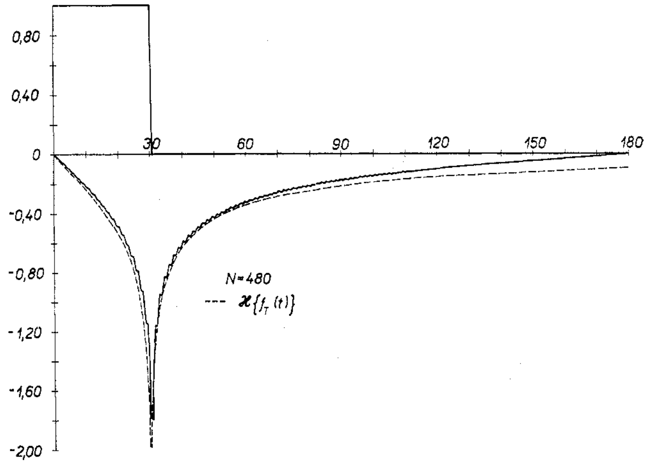

Fig. 3. Calculation of DHT of $f_T(t)$ with $N=480$. Function $f_T(t)$ is even and the Hilbert transform is odd.

$$\mathcal{H}\{f_T(t)\} = \frac{1}{\pi}\ln\left|\frac{t-\frac{1}{6}T}{t+\frac{1}{6}T}\right|$$

$$\mathcal{H}_1\{f_T(t)\} = \frac{1}{\pi}\ln\left|\frac{\sin\frac{\pi}{T}\left(t-\frac{1}{6}T\right)}{\sin\frac{\pi}{T}\left(t+\frac{1}{6}T\right)}\right|.$$

The dots and crosses point out the values of DHT for $N=30$ and $N=60$, respectively.

In Fig. 3 we see the function

$$f_T(t) = \begin{cases} 1 & |t| < 30 \quad T = 360 \\ 0 \end{cases}$$

and its DHT for $N=480$. The values of DHT are not given by points, but by a curve that consists of lines connecting pairs of adjacent values.

V. Additional Comments

The DHT can also be obtained by another procedure, e.g., by direct sampling of the function $f(t)$. Analogously, to the inverse $\mathcal{H}$ transform we may construct the inverse DHT which is given by an expression indentical in form to (18).

From (26) and (27) it follows that the DHT can also be indirectly used for functions given in an infinite interval [4]. In this case it is only necessary to transform the finite interval by the substitution of (5) into a finite interval.

References

[1] J. W. Cooley, P. A. W. Lewis, and P. D. Welch, "The finite Fourier transform," *IEEE Trans. Audio Electroacoust.*, vol. AU-17, pp. 77–85, June 1969.

[2] W. T. Cochran, J. W. Cooley, D. L. Favin, H. D. Helms, R. A. Kaenel, W. W. Lang, G. C. Maling, Jr., D. E. Nelson, C. M. Rader, and P. D. Welch, "What is the fast Fourier transform?" *IEEE Trans. Audio Electroacoust.* vol. AU-15, pp. 45–55, June 1967.

[3] J. W. Cooley, P. A. Lewis, and P. D. Welch, "Application of the fast Fourier transform to computation of Fourier integrals, Fourier series, and convolution integrals," *IEEE Trans. Audio Electroacoust.* vol. AU-15, pp. 79–84, June 1967.

[4] V. Čížek, "Numerische Hilbert-Transformation," *Proc. Inst. Radio Eng. Electron., Czechoslovak Academy of Sciences*, no. 11, Praha. 1961.

[5] I. K. Bari, *Trigonometric Series* (in Russian). Moscow: Gos. Izdat. Fiz-mat. Lit., 1961.

Václav Čížek was born in Dobruška, Czechoslovakia, on April 21, 1932. He received the Ing. Degree in 1954 from the Technical University, Prague, Czechoslovakia. He did graduate work from 1955 to 1968 at the Institute of Radio Engineering and Electronics, Czechoslovak Academy of Sciences, Prague. In 1961 he received the C.Sc. degree of technical sciences from the Technical University, Prague.

From 1954 to 1965 he worked in the field of impulse techniques. Since 1959 he has been engaged in circuit theory and frequency domain analysis. He is with the Institute of Radio Engineering and Electronics, Czechoslovak Academy of Sciences, Prague, where his research interests are in signal theory with regard to nonconventional filtering methods. He is the author of *Theory of Frequency Characteristics of Linear Systems*.

Dr. Čížek is a member of the Czechoslovak National Committee URSI.