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On a simple algorithm to calculate the 'energy' of a signal

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ABSTRACT

A simple algorithm is derived that enables on-the-fly calculation of the 'energy' required to generate, in a certain sense, a signal. The results of applying this algorithm to a number of well-known signals are shown. Some of the invariance and noise properties of the algorithm are derived and then verified by simulation. The implementation of the algorithm and its application to speech processing are briefly discussed.

I. INTRODUCTION

In the traditional signal processing literature when one speaks about the 'energy' in a signal the usual tendency is to talk about the average of the sum of the squares of the magnitude of that signal as representing the energy. This is very much the case in most of the speech literature where the energy in a certain segment of a speech signal is calculated in this way. An alternate representation also commonly used is to take the discrete Fourier transform (DFT) of that same signal segment where now the squares of the magnitudes of the frequency samples of the computed transform are assumed to represent the energy in the respective frequency components. Notice for example that a unit 10 Hz signal is said to have the same energy as a unit 1000 Hz signal. However the energy required to *generate* the acoustic signal of 1000 Hz is much greater than that for the 10 Hz signal. To begin to understand the difference we first focus attention on the generation process. In the case of simple harmonic motion, i.e. the fundamental sinusoidal oscillation, energy considerations show that the energy required to generate the oscillating signal is given by the square of the product of the signal's amplitude *and* the signal's frequency.

In this paper we discuss the energy in signals in light of the above observations. In the first section we derive the equation for the energy of a simple oscillator from the basic physics of the motion. This energy is shown to be directly proportional to the square of the product of amplitude and frequency. In the next section a simple algorithm to obtain a running estimate of the energy of the signal is derived. This algorithm operates on only three sequential samples of the signal at a time and hence can be applied on-the-fly. The properties of the algorithm are considered next by analyzing the output of the algorithm when it is applied to different types of signals. The correctness of the analysis is confirmed by computer simulations.

II. SIMPLE HARMONIC MOTION

Applying Newton's law of motion to the motion of a mass m suspended by a spring of force constant k, yields the well-known second-order differential equation

$$\frac{d^2x}{dt^2} + \frac{k}{m}x = 0 \tag{1}$$

whose solution is the simple harmonic motion given by $x(t) = A\cos(\omega t + \phi)$ where A is the amplitude of the oscillation, ω is the frequency of the oscillation and is equal to $(k/m)^{1/2}$, and ϕ is the arbitrary initial phase.

The total energy E of the system is the sum of the potential energy in the spring and the kinetic energy of the mass and is

$$E = \frac{1}{2}k x^2 + \frac{1}{2}m \dot{x}^2. \tag{2}$$

Substituting for x and solving, we obtain

$$E = \frac{1}{2}m \omega^2 A^2 \tag{3}$$

or

$$E \propto A^2\omega^2 . \tag{4}$$

Thus the energy of a simple oscillation is proportional not only to the square of the amplitude but also to the square of the frequency of the oscillation. This fact has been known for some time [1], but its significance seems to have been consistently overlooked. We now look for simple means for calculating a running estimate of this quantity, the energy E.

III. THE ALGORITHM

Let x_n be the samples of the signal representing the motion of the oscillatory body. Therefore

$$x_n = A\cos(\Omega n + \phi) \tag{5}$$

where Ω is the digital frequency in radians/sample and is given by $\Omega = 2\pi f / f_s$ where f is the analog frequency and f_s is the sampling frequency. ϕ is the arbitrary initial phase in radians.

Noting that there are three parameters in (5), A, Ω, and ϕ, then it follows that we should be able, in principle and subject to certain restrictions, to obtain the values of these parameters from three samples of the signal x_n. For convenience we choose three adjacent equally-spaced samples of x_n from which to make the determination. The signal values at these points are then given by

$$\begin{aligned} x_n &= A\cos(\Omega n + \phi) \\ x_{n+1} &= A\cos[(n+1)\Omega + \phi] \\ x_{n-1} &= A\cos[(n-1)\Omega + \phi] . \end{aligned} \tag{6}$$

Using the trigonometric identity

$$\cos(\alpha + \beta)\cos(\alpha - \beta) = \frac{1}{2}[\cos(2\alpha) + \cos(2\beta)] \tag{7}$$

we obtain

$$x_{n+1}x_{n-1} = \frac{A^2}{2}[\cos(2\Omega n + 2\phi) + \cos(2\Omega)] \tag{8}$$

and upon using the identity

$$\cos 2\alpha = 2\cos^2\alpha - 1 = 1 - 2\sin^2\alpha \tag{9}$$

there results

$$x_{n+1}x_{n-1} = A^2\cos^2(\Omega n + \phi) - A^2\sin^2(\Omega) . \tag{10}$$

But we notice that the first term on the right-hand side in (10) is

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simply the square of x_n. Thus with the substitution of the definition of x_n we obtain

$$x_{n+1}x_{n-1} = x_n^2 - A^2\sin^2(\Omega) \quad (11)$$

or

$$A^2\sin^2(\Omega) = x_n^2 - x_{n+1}x_{n-1} \, . \quad (12)$$

It is important to note that this expression, (12), is exact and is unique provided that the value of Ω is restricted to values less than $\pi/2$, the equivalent of one-fourth of the sampling frequency. It is almost in the desired form of (4).

We also know that for small values of Ω, $\sin(\Omega) \approx \Omega$. Now if we limit the value of Ω to $\Omega < \pi/4 = 0.7854$, i.e. $f/f_s < 1/8$ then the relative error is always below 11%. There results

$$A^2\Omega^2 \approx x_n^2 - x_{n+1}x_{n-1} \, . \quad (13)$$

The above expression gives us a good measure of the energy of the oscillating signal when the sampling rate of the signal is greater than eight times the frequency of oscillation of the signal, i.e. at least two sample points in each quarter cycle of the sinusoidal oscillation. Thus the above expression forms a simple algorithm to obtain a measure of the energy in any single-component signal:

$$E_n = x_n^2 - x_{n+1}x_{n-1} = A^2\sin^2(\Omega) \approx A^2\Omega^2 \quad (14)$$

where E_n is the output of the algorithm and x_n is the signal being analyzed. Teager [2] had related this algorithm cryptically to the author earlier but had provided few details regarding its derivation or properties. Hence we refer to it as **Teager's Algorithm** for estimating a measure of the energy in a single-component signal.

It is interesting to note that this expression, (14), a) is independent of the initial phase, ϕ, of the oscillation, b) is symmetric in the sense that changing $x_n \to -x_n$ or $n \to -n$, i.e. reversing the signal in time, does not change the resulting value, c) is robust even if the signal passes through zero, as no division operation is required, and d) is capable of responding very rapidly (in two sampling instants) to changes in both A and Ω.

A close look at the algorithm shows that it involves nonlinear operations on the signal, i.e. multiplication of the signal by itself and by itself shifted in time. It is a simple algorithm involving only *two* multiplications and *one* subtraction per point. This algorithm can be easily implemented in parallel. We next look at the application of Teager's algorithm to common signals.

IV. SOME PROPERTIES OF THE ALGORITHM

The derivation of the algorithm assumed that the oscillation was sinusoidal and that the parameters, A, Ω, and ϕ were essentially constant. In this section we observe the nature of the output of this algorithm when it is applied to signals of this type when these parameters are now functions of time. These signal types include exponentially-damped sinusoids, 'chirp' signals, and composite of sinusoids.

a. An exponentially-damped sinusoidal signal

Let

$$x_n = Ae^{-an}\sin(\Omega n + \phi) \, . \quad (15)$$

This represents an exponentially-damped signal with the time constant of the exponential term being proportional to $1/a$. The application of the algorithm on this signal yields the result

$$E_n = (Ae^{-an})^2\sin^2(\Omega) \, . \quad (16)$$

Again making the assumption that the sampling frequency is greater than eight times the frequency of the analog signal, i.e. $\Omega < \pi/4$,

$$E_n \approx (Ae^{-an})^2\Omega^2 = A^2e^{-2an}\Omega^2 \, . \quad (17)$$

Thus for an exponentially-damped signal the output of the algorithm decreases at an exponential rate which is *twice* that of the input signal. This has the physical interpretation that it represents the decaying energy of the process generating this signal. This result is shown in Figure 1.

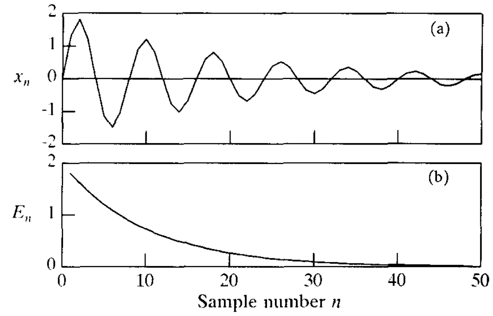

Fig. 1 The algorithm applied to a damped sinusoidal signal. (a) The damped sinusoidal signal $2e^{-0.05n}\sin(n\pi/4)$. (b) The output of the algorithm which varies approximately as $4e^{-0.1n}(\pi/4)^2$.

If we set $\Omega = 0$ in (15), the signal becomes the simple damped exponential and the output of Teager's algorithm gives $E_n = 0$. Likewise if the signal is simply a constant, the output of the algorithm is again 0. That this is so can also be obtained using the fact that the samples of these signals, the simple exponential and the constant, form a geometric progression, i.e. the ratio of adjacent terms is a constant, and thus any three consecutive samples of these signals satisfy the equation

$$\frac{x_n}{x_{n-1}} = \frac{x_{n+1}}{x_n} \quad (18)$$

and thus

$$x_n^2 - x_{n+1}x_{n-1} = 0 \, . \quad (19)$$

For a simple ramp signal defined by $x_n = Cn + D$, the application of the algorithm on the signal yields $E_n = C^2$ and for a quadratic signal $x_n = Bn^2 + Cn + D$ the algorithm gives the more general result

$$E_n = (Bn + C)^2 + B^2(n^2 - 1) - 2BD \, . \quad (20)$$

b. A chirp signal

Let the frequency of the signal at the beginning of the time window be Ω_1 and let its frequency decrease by $\Delta\Omega$ in N samples of the signal. Therefore, the instantaneous frequency, $\Omega(n)$, of the signal is given by

$$\Omega(n) = \Omega_1 - \frac{\Delta\Omega}{N}n \, ; \quad n = 0, 1, 2, \cdots, N-1 \, . \quad (21)$$

Integrating $\Omega(n)$ gives

$$\theta(n) = \int_0^n \Omega(n)dn \quad (22)$$

$$\theta(n) = \Omega_1 n - \frac{\Delta\Omega}{2N}n^2 \quad (23)$$

where $\theta(n)$ is the instantaneous phase of the signal and is quadratic in n as expected.

Therefore, the equation of the chirp signal is

$$x_n = A\cos[\Omega_1 n - \frac{\Delta\Omega}{2N}n^2] \, . \quad (24)$$

If the assumption is made that the rate of change in frequency is small enough so that the following substitutions can be made

$$\cos(\frac{\Delta\Omega}{N}) \approx 1 \quad (25)$$

and

$$\sin(\frac{\Delta\Omega}{N}) \approx \frac{\Delta\Omega}{N} \tag{26}$$

then, after simplification, the algorithm gives the following result

$$E_n = A^2\sin^2[\Omega(n)] + \frac{A^2}{2}(\frac{\Delta\Omega}{N})\sin[2\theta(n)] \,. \tag{27}$$

If the frequency change is sufficiently small then the second term in the above expression can be neglected yielding the following result

$$E_n \approx A^2\sin^2[\Omega(n)] \tag{28}$$

and again if we assume that the sampling frequency is greater than eight times the highest instantaneous frequency of the signal then the above equation becomes

$$E_n \approx A^2\Omega^2(n) \,. \tag{29}$$

Thus this algorithm is able to track the change in frequency of the chirp signal very effectively. Figure 2 dramatically shows this effect for a chirp signal. The small ripple that appears in the output is accurately explained by the second term in the E_n expression, (27), above. This ripple component can be easily filtered out if so desired because its frequency is large compared to that of the energy signal.

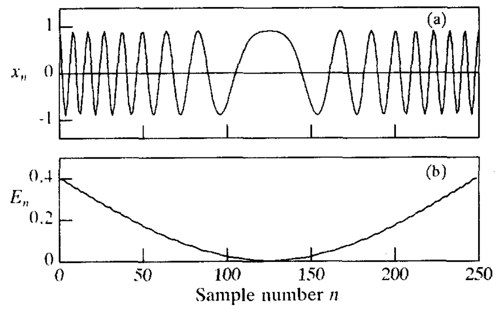

Fig. 2 The application of the algorithm to a chirp signal. (a) The chirp signal of amplitude 0.9 with frequency decreasing linearly from $\pi/4$ to zero and then increasing linearly back to $\pi/4$ in 251 samples. (b) The output of the algorithm which decreases as $\sin^2[\Omega(n)]$ from 0.398 to 0 and then increases similarly back to 0.398.

c. A signal consisting of two frequency components

For a single sinusoidal signal, $x_n = A\cos(\Omega n + \phi)$, the Teager algorithm obviously gives us $E_n \approx A^2\Omega^2$ which is a constant if A and Ω are constant. Figure 3 shows the results of applying the algorithm to this signal.

For signals consisting of two different frequency components we observe a different result. Let

$$x_n = A_1\sin(\Omega_1 n + \phi_1) + A_2\sin(\Omega_2 n + \phi_2) \,. \tag{30}$$

The amplitudes and frequencies of the two components are A_1 and A_2 and Ω_1 and Ω_2, respectively. When this signal is the input to the algorithm, the output on simplification of the mathematics yields

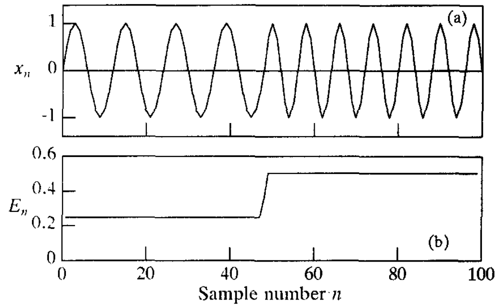

Fig. 3 The algorithm applied to sinusoidal signals. (a) A unit amplitude sinusoidal signal first of frequency $\pi/6$ then of frequency $\pi/4$. (b) The output of the algorithm applied to (a) has a constant magnitude when the signal amplitude and frequency are constant.

$$\begin{aligned} E_n &= A_1^2\sin^2(\Omega_1) + A_2^2\sin^2(\Omega_2) \\ &+ 2A_1A_2\sin^2[(\Omega_1+\Omega_2)/2]\cos[(\Omega_1-\Omega_2)n + (\phi_1-\phi_2)] \\ &+ 2A_1A_2\sin^2[(\Omega_1-\Omega_2)/2]\cos[(\Omega_1+\Omega_2)n + (\phi_1+\phi_2)] \,. \end{aligned} \tag{31}$$

Again if Ω_1 and $\Omega_2 < \pi/4$ and $|\Omega_1-\Omega_2| \approx 0$, then

$$\begin{aligned} E_n &\approx A_1^2\Omega_1^2 + A_2^2\Omega_2^2 \\ &+ A_1A_2[1 - \cos(\Omega_1+\Omega_2)]\cos[(\Omega_1-\Omega_2)n + (\phi_1-\phi_2)] \,. \end{aligned} \tag{32}$$

Thus the output is not only the sum of the energies of the two components but has also an additional cross term having a cosinusoidal variation at the difference frequency of the two frequency components. The effect of the presence of this cross term is clearly shown in Figure 4 for where it is seen that the effect is sizable and reminiscent of a beating phenomenon. More will be said of this effect later.

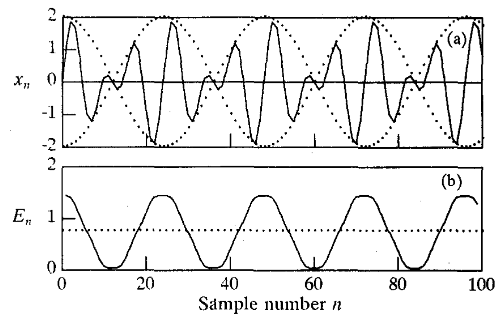

Fig. 4 The algorithm applied to a composite of sinusoids. (a) Sum of two sinusoids of frequency $\pi/4$ and $\pi/6$, each of unit amplitude. (b) The output of the algorithm when applied to (a) oscillates at the difference frequency, $\pi/12$, about the sum of the energies in the two sinusoids as per (31).

V. INVARIANCE PROPERTIES OF THE ALGORITHM

The invariance of an algorithm under transformation is a property that indicates the robustness of the algorithm. Hamming, (see [3] p. 73), notes "· · · an algorithm which transforms properly with respect to a class of transformations is more basic than one that does not". A number of different transformations such as scaling, addition of a bias, and superposition of several signals are considered.

If the input signal is uniformly scaled by a factor C, it can be easily seen that the output of the algorithm becomes scaled by C^2.

If an arbitrary constant is added to the input signal i.e. $x'_n = x_n + C$, then the new output is given by

$$E'_n = E_n - C(x_{n+1} - 2x_n + x_{n-1}) . \tag{33}$$

Thus the algorithm is not invariant under this type of transformation. The additional term in (33) is recognized as $-C$ times the second central difference of the signal which, if it is a sinusoid for example, is also a component that varies with time. Thus if $x'_n = A \sin(\Omega n + \phi) + C$ there results

$$E'_n = A^2\sin^2(\Omega) + 4AC\sin^2(\Omega/2)^2\sin(\Omega n + \phi) . \tag{34}$$

a. The effect of superposition of two signals

Let the input signal now consist of two different components x_1 and x_2. Also let E_1 and E_2 be the output from the algorithm when these signals are input independently to the algorithm. Hence the superposition of these two signals gives

$$x_n = x_{1_n} + x_{2_n} . \tag{35}$$

The output of the algorithm in this case is given by

$$E_n = E_{1_n} + E_{2_n} - [x_{1_{n-1}}x_{2_{n-1}} - 2x_{1_n}x_{2_n} + x_{1_{n-1}}x_{2_{n+1}}] . \tag{36}$$

In addition to the sum of the energies of the two components the result contains also cross-correlation-type terms between the two signals. Thus in order to use this algorithm to analyze signals which consist of more than one component, it is essential to separate the two components of the signal by some form of filtering before applying the algorithm. For example for speech signals the different components are the different formants and these can be separated routinely by the use of a bank of bandpass filters with appropriate characteristics.

VI. THE EFFECT OF NOISE ON THE ALGORITHM

Since the algorithm involves the multiplication of the samples of the input signal both with itself and with displaced samples of itself, the algorithm is expected to be sensitive to noise in the signal. Let the clean input signal, s_n, be corrupted by an additive zero-mean uncorrelated Gaussian noise, ν_n, of variance σ^2. The noisy signal is then given by $x_n = s_n + \nu_n$. Now the output of the algorithm is given by

$$E_n = x_n^2 - x_{n+1}x_{n-1} . \tag{37}$$

Taking the expectation of this expression (where $\mathbf{E}[.]$ is the usual expectation operator) gives after simplification

$$\mathbf{E}[E_n] = s_n^2 - s_{n+1}s_{n-1} + \sigma^2 . \tag{38}$$

Thus the expected value of the output of the algorithm is offset from its value for the clean signal by the variance of the noise. This shows that the algorithm is sensitive to the noise, especially wideband noise or noise with energy as large or larger than the signal being followed.

However it is very important to note that the bandwidth of the energy function, E_n, for many physical signals, including speech, is much lower than that of the components of the signal itself and hence the energy signal may be lowpass filtered to greatly reduce the effects of the noise. Simulation results clearly illustrate this.

VII. CONCLUSIONS

In this paper we have derived and discussed the properties of an algorithm known as Teager's Algorithm which, by operating on-the-fly on signals composed of a single time-varying frequency, is able to extract a measure of the energy of the mechanical process that generated this signal; this energy measure is equal to the product of the square of the amplitude and the square of the frequency of the signal. The algorithm uses only three adjacent samples of the signal and requires only three arithmetic operations per each time shift. In order to use the algorithm effectively when the signal consists of several different frequency components it is important to pass the signal through a bank of bandpass filters first; the algorithm is then applied to the outputs from each of these bandpass filters.

The noise characteristics of the algorithm also show that it is sensitive to noise; hence care must be taken when this algorithm is applied to signals which have noise added to them. The effects of noise can be significantly reduced by proper linear filtering and by taking advantage of the fact that the bandwidths of the modulations of the signals are usually much smaller than that of the noise.

The application of the algorithm to signals composed of two or more frequency components does not give the energy of the system generating this composite signal. Nonetheless, it is interesting to observe that in this case the output of the algorithm is maximum when the composite signal is at its peak and a minimum when at zero; it is as if the algorithm is able to extract the envelope function of the signal.

The most important characteristic of this algorithm is that in those cases where there are single sinusoidal components that are either constant or rapidly changing in frequency, for example in the analysis of chirp signals, the algorithm is able to track very effectively and efficiently the instantaneous frequency of the signal. This property makes the algorithm potentially useful for rapid formant tracking in speech analysis.

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