showed that for $N = 256$ a factored autocorrelation was computed 26 percent faster than the unfactored, over the range of $14 < M < 50$.

Conclusion

Factoring provides a means of significantly reducing the number of multiplications required to compute a limited number of short-term autocorrelation coefficients. The most obvious benefit is reduction in computation time. Ongoing research [2] indicates that the factored form may have a secondary advantage of reduced truncation or roundoff errors in finite word length computations.

References

[1] J. D. Markel, "Formant trajectory estimation from a linear least-squares inverse filter formulation," Speech Commun. Res. Lab., Santa Barbara, Calif., Mono. 7, Aug. 1971.
[2] ——, private communication.

A Note on Exact Discrete Fourier Transforms

C. M. RADER

Abstract—**It is proved that the z transform of a sequence of data values cannot be exactly computed using a binary number representation for values of z on the unit circle, except $z = \pm 1$, $z = \pm j$. It is also proved that the discrete Fourier transform (DFT) of a sequence of data values cannot be evaluated with rational numbers.**

When we compute the value of a z transform, using a numerical evaluation rather than an algebraic expression, we are generally satisfied with an approximate result. However, it may be of interest to inquire whether an exact result is possible. This note contains proofs of two theorems. They are as follows.

Theorem 1: It is impossible to evaluate the z transform of a sequence, using binary arithmetic (although it is sometimes possible with decimal arithmetic), for values of z on the unit circle.

Theorem 2: It is impossible to evaluate the discrete Fourier transform (DFT) of a sequence using rational numbers.

Both theorems depend on the simplest ideas of number theory. Both theorems exclude the trivial case of $z = \pm 1$ and $z = \pm j$.

Proof of Theorem 1: For a general sequence, we obviously require that z itself be expressable as a binary number. Let the binary number be $z = 2^{-K}(a + jb)$ where K is the word length and a and b are integers. If z is on the unit circle we have

$$a^2 + b^2 = 2^{2K} = 4^K. \tag{1}$$

Clearly if a is odd and b is even, or if a is even and b is odd, then $a^2 + b^2$ would be odd whereas (1) asserts that it is even. If a and b are odd, it can be seen that $a^2 + b^2$ is even but is not divisible by 4 since $(2a_1 + 1)^2 + (2b_1 + 1)^2 = 4(a_1^2 + b_1^2 + a_1 + b_1) + 2$. Therefore, the only possibility consistent with (1) is a even and b even. Let $K_1 = K - 1$, $a = 2a_1$, $b = 2b_1$. Then (1) becomes

$$a_1^2 + b_1^2 = 4^{K_1}. \tag{2}$$

The same reasoning will show that a_1, b_1 are both even, thus, letting $K_2 = K_1 - 1$, $a_1 = 2a_2$, $b_1 = 2b_2$, we get

$$a_2^2 + b_2^2 = 4^{K_2}. \tag{3}$$

and we can clearly repeat the process until we get

$$a_K^2 + b_K^2 = 4^0 = 1. \tag{4}$$

This implies that either a_K or b_K is zero, which would get us back to the four originally excluded cases, $z = \pm 1$, $z = \pm j$. Therefore, all other cases are impossible.

Note, with decimal representations we can have $z_0 = 0.6 + 0.8j$, for example. Other possible decimal exact representations can be found by computing powers of z_0.

Proof of Theorem 2: We shall prove that any value of $W = j(2\pi/N)$ cannot have both $\cos(2\pi/N)$ and $\sin(2\pi/N)$ expressed as rational numbers. It is sufficient to show that this is impossible for $N = p$, a prime.[1]

Without loss of generality, let $W = (a/c) + j(b/c)$, where a, b, and c are not all divisible by the same integer. This is like the restriction that they be in lowest terms. We express W^p by the binomial theorem:

$$W^p = \sum_{k=0}^{p} \left(\frac{a}{c}\right)^k j^{p-k} \left(\frac{b}{c}\right)^{p-k} \binom{p}{k}. \tag{5}$$

Multiplying both sides by c^p we get

$$\sum_{k=0}^{p} a^k j^{p-k} \binom{p}{k} = c^p. \tag{6}$$

It is important to recognize that all the binomial coefficients, $\binom{p}{k}$, in this expression are divisible by p except $\binom{p}{0}$ and $\binom{p}{p}$. This may be seen from the definition

$$\binom{p}{k} = \frac{p!}{(p-k)!\,k!}. \tag{7}$$

From the definition of a prime, no term in the denominator divides p in the numerator. Since $\binom{p}{k}$ is an integer, it is divisible by p. It is also useful to realize that the even numbered terms of the sum in (6) give the imaginary part, which is zero, while the odd numbered terms give the real part which is c^p. We first take the imaginary part modulo p. There is only one term not zero; therefore,

$$b^p \equiv 0 \bmod p. \tag{8}$$

Thus,

$$b \equiv 0 \bmod p.$$

We now examine the real part of (6) modulo p, this time obtaining

$$a^p \equiv c^p \bmod p. \tag{9}$$

If $a \equiv 0$ then (9) would imply that $c \equiv 0$ and then all three of a, b, c would be divisible by p, contrary to our assumption of lowest terms. Therefore,

$$a \not\equiv 0 \bmod p. \tag{10}$$

At this point we use (8) to justify the substitution

$$b = b_1 p \tag{11}$$

into (6), giving

$$\sum_{k=0}^{p} a^k j^{p-k} b_1^{p-k} p^{p-k} \binom{p}{k} = c^p. \tag{12}$$

We take the imaginary part of (12) modulo p^3. Only one term is nonzero, for $k = p - 1$, which gives

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[1] This is sufficient because if $W^N = 1$ for $N = pQ$, p a prime, then (W^Q) could not be rational. But if W is rational, so are all of its integer powers.

$$a^{p-1}\, b_1 p^2 \equiv 0 \bmod p^3,$$

$$\text{thus,} \quad b_1 \equiv 0 \bmod p, \tag{13}$$

$$b_1 = b_2 p.$$

We make a substitution of (13) into (12) to get

$$\sum_{k=0}^{p} a^k j^{p-k} b_2^{p-k} (p^2)^{p-k} \binom{p}{k} = c^p \tag{14}$$

and take the imaginary part modulo p^4. This gives

$$a^{p-1}\, b_2 p^3 \equiv 0 \bmod p^4,$$

$$\text{thus,} \quad b_2 \equiv 0 \bmod p, \tag{15}$$

$$b_2 = b_3 p.$$

It should be clear that we can iterate this procedure as many times as we like, without limit, to show that any power of p must divide b. Of course there is no finite number b satisfying such a constraint, and therefore our original hypothesis was in error. This proves that $\cos(2\pi/N)$ and $\sin(2\pi/N)$ cannot both be rational numbers, except for $N = 2$, $N = 4$, which were excluded initially. The proof holds for odd primes only, but the case of N a power of 2 is easily excluded separately.

This note has demonstrated mathematically what most people accepted intuitively, namely that the numerical evaluation of a z transform or a DFT by exact techniques is an impossibility.

Loudspeaker System Figures of Merit

RICHARD H. SMALL

Abstract—**The loudspeaker system small-signal figure of merit is a useful performance criterion that must be evaluated and used with some care. In practice, this figure of merit is significantly reduced by enclosure dissipation, including that contributed by filling materials.**

The expression for small-signal figure of merit presented by Lampton [1] features an improvement on a similar expression developed by Small [2], [3], and some of the data presented lead to an important cautionary conclusion only lightly touched on by Lampton.

By separating the physical constant and the system-response factor K, Lampton provides a noteworthy improvement in comprehensibility and utility of the figure-of-merit expression compared to the form used by Small in which these are left combined as $k_{\eta(G)}$ [3]–[5]. The improvement is twofold: first because K is a dimensionless number of comfortable magnitude that depends only on the system alignment, and second because K is independent of the system of units used (e.g., SI or English) and of the radiation angle of the system acoustic load; these conditions affect only the physical constant, which is easily adjusted for each situation.

There is a very interesting way of rearranging the basic efficiency expression that was pointed out to the author by Franklin Montgomery of the National Bureau of Standards following a lecture given at Washington, D.C., in June 1972. If the system half-power or −3-dB cutoff frequency f_3 (Lampton's f_c) is expressed in terms of its wavelength λ_3, the efficiency expression becomes

$$\eta_0 = 4\pi^2\, K_\eta\, [V_B/\lambda_3{}^3] \tag{1}$$

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where

- η_0 asymptotic efficiency for radiation into half-space (Lampton's E),
- K_η complete efficiency constant or figure of merit (Small's k_η divided by $4\pi^2/c^3$, or Lampton's product of K, $(1 - Q/Q_0)$, and $1.4/\gamma$), and
- V_B net internal volume of the enclosure (Lampton's V).

Every factor on the right-hand side of (1) is now dimensionless, including the final factor in brackets, provided that the enclosure dimensions and the wavelength are expressed in the same units.

In common with many other transducers and radiators, it is the small dimensions of the system in relation to a wavelength that restrict the radiation efficiency to a very low value; the magnitude of the bracketed factor for typical domestic high-fidelity sound reproducers is restricted to the order of 10^{-4}.

The expressions for and values of the small-signal figure-of-merit response factors given by Lampton [1] and by Small [3]–[5] for both sealed enclosure and maximally-flat vented enclosure systems are in agreement when differences in normalization and notation are allowed for. It is particularly interesting, however, that the values of K for Lampton's bounded-ripple alignments exceed those reported by Small [3], [5] for the comparable Chebyshev alignments of the type used by Thiele [6]. The maximum value of K attainable with the Chebyshev alignments is about 4.1, occurring for a ripple of approximately 0.2 dB. (The Thiele alignment 9 used as an example by Lampton has about 0.5-dB ripple, not 1.8 dB as originally given by Thiele; errors in the ripple calculations were not discovered until after the publication of Thiele's paper.)

The difference in maximum K values arises from the fact that the Chebyshev alignments have positive ripple only, while the bounded-ripple alignments permit negative ripple as well. By extension, the "degenerated Chebyshev type I" alignments used by Nomura [7], which have negative ripple only, should exhibit even higher values of K. The "high-K" alignment given by Lampton in [1] has this property, exhibiting 3 dB of negative ripple that depresses the response over a broad frequency range (Fig. 1). This alignment might well be subjectively disappointing, regardless of its high value of K. Lampton's goal of seeking alignments "for which the response curve is acceptable" thus deserves greater cautionary emphasis in relation to the additional goal of seeking a high figure of merit.

As pointed out by Lampton, alignments with ripple generally appear to offer a modest increase in figure of merit over flatter alignments. However, analysis of vented-box systems containing normal amounts of enclosure dissipation [3], [5] shows that this dissipation significantly reduces the figure of merit for all alignments but especially for alignments with ripple. The result is effectively a loss of the figure-of-merit advantage of these alignments.

It should be noted that the statements in [1] concerning the effects of enclosure filling material are somewhat oversimplified and could lead to difficulty or disappointment in practical applications. The factor γ in the denominator of [1, eq. (4)] is seldom reduced by more than 20 percent when a practical filling material is added to a loudspeaker enclosure. In addition, γ is not the only factor that may be affected. Depending on alterations made to the system at the time of filling (such as reducing V to keep S constant as γ falls), the material and its losses also generally affect S, G, the effective value of Q_0 and thus Q.

The overall effects of filling material on the small-signal figure of merit for sealed systems are exhibited more clearly when the efficiency expression is renormalized in terms of system parameters. This procedure is followed in [4], where the effects are analyzed and discussed in some detail; a very useful large-signal figure of merit is also developed in this paper.

Filling materials are seldom employed in vented enclosures. The effects are analyzed in [3, ch. 7], where it is shown that