Integration by Substitution in ACL2(r)

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Integration by substitution, also known as u-substitution, is a method that is often used in calculus to find the integral of a function. We show a proof of u-substitution in ACL2(r) using the fundamental theorem of calculus and the chain rule, which are already proved in ACL2(r). We use u-substitution to show the area of a circle with radius r is πr^2.

1 Introduction

Continuity, differentiablity and proofs of theorems such as the intermediate value and mean value theorems were presented in [**?**] using non-standard-analysis. The Riemann integral of a continuous function was formalized, and a proof of the First Fundamental Theorem of Calculus (FTC-1) was presented in [**?**]. The proof of the Second Fundamental Theorem of Calculus (FTC-2) was presented in [**?**]. The Chain Rule and other algebraic differential rules were presented in [**?**]. In this paper, we use the FTC and the chain rule to prove u-substitution in ACL2(r). We use this proof to show the area of a circle with radius r is πr^2.

In particular, we show that if f is continuous on I, ψ is defined on an interval, $[a,b]$ whose range is in the interval I, and if ψ has a continuous derivative, then we show that $\int_{\psi(a)}^{\psi(b)} f(x)\,dx = \int_a^b f(\psi(u))\psi'(u)\,du$. In section 2 we show the mathematical derivation of u-substitution rule, and the method is applied to find the area of a circle. In section 3 we formalize in ACL2(r) the derivation shown in section 2 and in section 4 we use the u-substitution rule presented in section 3 to show the area of a circle with radius r is πr^2.

2 Integration by Substitution

Following [**?**], we state the principle of u-substitution as follows:

Proposition: Let $I \subseteq R$ be an interval and $\psi : [a,b] \to I$ be a differentiable function with continuous derivative, ψ'. Suppose that $f : I \to R$ is a continuous function . Then,

$$\int_{\psi(a)}^{\psi(b)} f(x)\,dx = \int_a^b f(\psi(u))\psi'(u)\,du$$

2.1 Derivation of the u-substitution rule

Since f and ψ are continuous, the composite function $f(\psi(u))$ is continuous and since the derivative of ψ is continuous, using the product rule of two continuous functions, $f(\psi(u))\psi'(u)$ is continuous. Hence the integral of $f(\psi(u))\,\psi'(u)$ exists. Since f is continuous, there exists an antiderivative of f and let that be F. Using the chain rule:

$$(F \circ \psi)'(u) = F'(\psi(u))\psi'(u) = f(\psi(u))\psi'(u) \tag{1}$$

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Applying the chain-rule and the fundamental theorem of calculus to derive the u-substitution rule:

$$\begin{aligned}
\int_a^b f(\psi(u))\psi'(u)\,du &= \int_a^b (F\circ\psi)'(u)\,du && [\,(1)\,] \\
&= (F\circ\psi)(b) - (F\circ\psi)(a) && [\text{ FTC-2 }] \\
&= F(\psi(b)) - F(\psi(a)) && [\text{ Definition of } F\circ\psi\ ] \\
&= \int_{\psi(a)}^{\psi(b)} f(x)\,dx && [\text{ FTC-2 }]
\end{aligned} \tag{2}$$

2.2 Deriving the Area of a Circle Using u-substitution

Area of the circle is $4\int_0^r \sqrt{r^2-x^2}\,dx$ where $\int_0^r \sqrt{r^2-x^2}\,dx$ is the area of the circle under the curve in the first quadrant. If we consider $f(x)=\sqrt{r^2-x^2}$, $\psi(x)=r\sin x$, $a=0$, $b=\pi/2$ then, by using the u-substitution rule:

$$\begin{aligned}
4\int_0^r \sqrt{r^2-x^2}\,dx &= 4\int_{\psi(a)}^{\psi(b)} f(x)\,dx \\
&= 4\int_a^b f(\psi(u))\psi'(u)\,du \text{ [u-substitution rule]} \\
&= 4\int_0^{\pi/2} \sqrt{r^2-(r\sin u)^2}\,r\cos u\,du \\
&= 4\int_0^{\pi/2} r^2\cos^2 u\,du \\
&= 4r^2\Big(\frac{sin(2u)}{4}+\frac{u}{2}\Big)\Big|_0^{\pi/2} \\
&= \pi r^2
\end{aligned}$$

3 Proof of the u-substitution rule in ACL2(r)

To formalize the proposition in section 2, we defined two functions f-prime (f in the proposition) and fi (ψ in the proposition). The two domains for the two functions respectively are fi-range and f-o-fi-domain. Derivative of fi defined is fi-prime (ψ' in the proposition).

The constraints required for the proof are shown in the Appendix A. Encapsulated constraints are used to generate the theorems that are used as constraints for the substitution rule. The integral of f-prime, int-f-prime-1, can be defined as shown in [**?**]. We use FTC-1 to show that the derivative of the integral of f-prime is equal to f-prime:

```
(defthm ftc-1-fprime
  (implies (and (inside-interval-p a (fi-range))
                (inside-interval-p b (fi-range))
                (inside-interval-p c (fi-range))
                (standardp b)
                (i-close b c)
                (not (equal b c)))
           (i-close (/ (- (int-f-prime-1 a b) (int-f-prime-1 a c))
                       (- b c))
                    (f-prime b))))
```

Antiderivative of f-prime, f (F in (1)) is then defined as below and then we show that f-prime is the derivative of f :

```
(defun f(x) (int-f-prime-1 (consta) x))

(defthm f-prime-is-derivative
  (implies (and (standardp x)
                (inside-interval-p x (fi-range))
                (inside-interval-p x1 (fi-range))
                (i-close x x1) (not (= x x1)))
           (i-close (/ (- (f x) (f x1)) (- x x1))
                    (f-prime x)))
  :hints ...)
```

The next lemma we need is the proof of the chain rule (1). We define the composite function $F \circ \psi$ as f-o-fi, derivative-cr-f-o-fi equal to (* (f-prime (fi x)) (fi-prime x)) and differential-cr-f-o-fi is the derivative of f-o-fi. Since f and fi are differentiable, we apply the chain rule proof presented in [**?**] to prove the lemma.

```
(defthm expand-differential-cr-f-o-fi
  (implies (and (inside-interval-p x (f-o-fi-domain))
                (inside-interval-p (+ x eps) (f-o-fi-domain)))
           (equal (differential-cr-f-o-fi x eps)
              (/ (- (f-o-fi (+ x eps)) (f-o-fi x)) eps)))
  :hints ...)

(defthm differential-cr-f-o-fi-close-1
  (implies (and (standardp x)
                (inside-interval-p x (f-o-fi-domain))
                (inside-interval-p x1 (f-o-fi-domain))
                (i-close x x1) (not (= x x1)))
           (equal (standard-part (differential-cr-f-o-fi x (- x1 x)))
                  (derivative-cr-f-o-fi x)))
  :hints ...)
```

The next step is proving the integral of derivative of f-o-fi between two end points a and b equals to (- (f-o-fi(b)) (f-o-fi(a))). Since f-prime, fi and fi-prime are continuous, f-prime(fi x) is continuous using the composition rule of two continuous functions and ($*$ f-prime(fi x) (fi-prime x)) is continuous using the product rule of two continuous functions. f-o-fi-prime is the derivative of f-o-fi which is equal to ($*$ f-prime(fi x) (fi-prime x)). Since f-o-fi-prime is continuous and f-o-fi-prime is the derivative of f-o-fi we use FTC-2 to show that the integral of f-o-fi-prime, int-f-o-fi-prime, is equal to (- (f-o-fi(b)) (f-o-fi(a))):

```
(defun derivative-cr-f-o-fi (x)
  (* (f-prime (fi x))
     (fi-prime x)))

(defun f-o-fi-prime (x)
    (if (inside-interval-p x (f-o-fi-domain))
        (derivative-cr-f-o-fi x)
      0))

(defthmd ftc2-usub-equal
  (implies (and (inside-interval-p a (f-o-fi-domain))
```

```
                (inside-interval-p b (f-o-fi-domain)))
           (equal (int-f-o-fi-prime a b)
                  (- (f (fi b)) (f (fi a)))))
  :hints ...)
```

The final step is to prove that the integral of f-prime, int-f-prime, with (fi a) and (fi b) being two end points of the integral, is equal to (- (f (fi b)) (f (fi a))). Since f-prime is the derivative of f and is continuous, we apply FTC-2:

```
(defthmd ftc2-usub-1-equal
  (implies (and (inside-interval-p a (f-o-fi-domain))
                (inside-interval-p b (f-o-fi-domain)))
           (equal (int-f-prime (fi a) (fi b))
                  (- (f (fi b)) (f (fi a)))))
  :hints ...)
```

Finally we get the u-substitution rule using the lemmas ftc2-usub-equal and ftc2-usub-1-equal:

```
(defthm usubstitution-f-o-fi
  (implies (and (inside-interval-p a (f-o-fi-domain))
                (inside-interval-p b (f-o-fi-domain)))
           (equal (int-f-prime (fi a) (fi b))
                  (int-f-o-fi-prime a b)))
  :hints (("Goal"
            :use ((:instance ftc2-usub-1-equal)
                  (:instance ftc2-usub-equal))
            :in-theory (disable int-f-o-fi-prime int-f-prime))))
```

4 Proof of the Area of a Circle in ACL2(r)

Quarter area of a circle which is centered at (0,0) with radius r is the area under the curve, f, $\sqrt{r^2-x^2}$ in the first quadrant. The domain of the curve is $[0,r]$. The substitution function, ψ is $r\sin(x)$ and the domain of the function is $[0,\frac{\pi}{2}]$. The derivative of the substitution function, ψ' is $r\cos(x)$.

We first prove the constraints required to use the u-substitution rule for the defined functions. Since sine and cosine are continuous as proved in [?] ψ and ψ' are continuous. Then we prove ψ' is the derivative of ψ. The next lemma required is that f is continuous. Square of a function is continuous and square root of a continuous function is continuous. So we get $\sqrt{r^2-x^2}$ is continuous. Now we have proved all the constraints required to use the u-substitution rule proved in section 3 and we get:

```
(defthm usubstitution-circle
  (implies (and (inside-interval-p a (fi-domain))
                (inside-interval-p b (fi-domain)))
           (equal (int-circle (sub-func a) (sub-func b))
                  (int-circle-sub-prime a b)))
  :hints ...)
```

In the above proof sub-func is $r\sin(x)$, int-circle is the integral of $\sqrt{r^2-x^2}$, int-circle-sub-prime is the integral of $f(\psi(x))\psi'(x)$ and fi-domain is the interval $[0,\frac{\pi}{2}]$.

$$\begin{aligned} f(\psi(x))\psi'(x) &= \sqrt{r^2-(r\sin x)^2}\, r\cos x \\ &= r^2\cos^2 x \end{aligned}$$

The area of the circle is equal to the 4 times the integral of $r^2\cos^2(x)$ between 0 to $\pi/2$. First we apply the chain rule to show that derivative of $sin(2x)$ equal to $2cos(2x)$. Then we show the derivative of $r^2\left(\frac{sin(2x)}{4}+\frac{x}{2}\right)$ equal to r^2cos^2x.

$$\begin{aligned}\frac{d}{dx}r^2\left(\frac{sin(2x)}{4}+\frac{x}{2}\right) &= r^2\left(\frac{2cos(2x)}{4}+\frac{1}{2}\right)\\ &= r^2\left(\frac{2(2cos^2x-1)}{4}+\frac{1}{2}\right)\\ &= r^2cos^2x\end{aligned}$$

Finally we apply FTC-2:

$$\begin{aligned}4\int_0^{\pi/2} r^2\cos^2 x\,dx &= 4r^2\left(\frac{sin(2x)}{4}+\frac{x}{2}\right)\Big|_0^{\pi/2}\\ &= \pi r^2\end{aligned}$$

5 Conclusions

The paper presents the proof for the integration by substitution rule and we apply the proof to prove the area of a circle with radius r equal to πr^2. The substitution rule presented in this paper can be used to find an integral of any function if the integral of the function can be calculated by applying the u-substitution rule.

Appendix A Constraints for u-substitution in ACL2(r)

The constraints and the associated naming conventions are as follows:

- **f-prime-real**

```
(realp (f-prime x))
```

- **f-prime-continuous**

```
(implies (and (standardp x)
              (inside-interval-p x (fi-range))
              (i-close x x1)
              (inside-interval-p x1 (fi-range)))
         (i-close (f-prime x) (f-prime x1)))
```

- **fi-real**

```
(realp (fi x))
```

- **fi-prime-real**

```
(realp (fi-prime x))
```

- **fi-prime-is-derivative**

```
(implies (and (standardp x)
              (inside-interval-p x (f-o-fi-domain))
              (inside-interval-p x1 (f-o-fi-domain))
              (i-close x x1) (not (= x x1)))
         (i-close (/ (- (fi x) (fi x1)) (- x x1))
                  (fi-prime x)))
```

- **fi-prime-continuous**

```
(implies (and (standardp x)
              (inside-interval-p x (f-o-fi-domain))
              (i-close x x1)
              (inside-interval-p x1 (f-o-fi-domain)))
         (i-close (fi-prime x) (fi-prime x1)))
```

- **intervalp-fi-range**

```
(interval-p (fi-range))
```

- **fi-range-real**

```
(implies (inside-interval-p x (fi-range))
         (realp x))
```

- **fi-range-non-trivial**

```
(or (null (interval-left-endpoint (fi-range)))
    (null (interval-right-endpoint (fi-range)))
    (< (interval-left-endpoint (fi-range))
    (interval-right-endpoint (fi-range))))
```

- **fi-differentiable**

```
(implies (and (standardp x)
              (inside-interval-p x (f-o-fi-domain))
              (inside-interval-p y1 (f-o-fi-domain))
              (inside-interval-p y2 (f-o-fi-domain))
              (i-close x y1) (not (= x y1))
              (i-close x y2) (not (= x y2)))
         (and (i-limited (/ (- (fi x) (fi y1)) (- x y1)))
         (i-close (/ (- (fi x) (fi y1)) (- x y1))
                  (/ (- (fi x) (fi y2)) (- x y2)))))
```

- **intervalp-f-o-fi-domain**

```
(interval-p (f-o-fi-domain))
```

- **f-o-fi-domain-real**

```
(implies (inside-interval-p x (f-o-fi-domain))
         (realp x))
```

- **f-o-fi-domain-non-trivial**

```
(or (null (interval-left-endpoint (f-o-fi-domain)))
    (null (interval-right-endpoint (f-o-fi-domain)))
    (< (interval-left-endpoint (f-o-fi-domain))
    (interval-right-endpoint (f-o-fi-domain))))
```

- **fi-range-in-domain-of-f-o-fi**

```
(implies (inside-interval-p x (f-o-fi-domain))
         (inside-interval-p (fi x) (fi-range))))
```

Along with the above constraints we defined a constant, *consta* which is a standard number and inside the interval fi-range.