Banach-Tarski Paradox and Amenability

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Banach-Tarski

What is Banach Tarski

- Start with a sphere
- Partition it into finitely many sets
- Rotate and translate these sets
- Get two spheres of the same volume as the first

Some Quick Observations/History

Let S^2 be the sphere. Let $(A_i)_{i=1}^n$ be a partition of S that allows the Banach-Tarski Paradox to occur.

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- At least one of these subsets A_i has to be nonmeasurable.
- The standard tool we use for constructing nonmeasurable sets is the Axiom of Choice(although we don't require the full power of AC).
- This was one of the reasons Axiom of Choice was controversial in the early 20th century.

Construction

Let G be a group and A an arbitrary set.
Fix a group action $G \curvearrowright A$.

Paradoxicality

The set A is G-**paradoxical** if there exist disjoint subsets $A_1, \ldots, A_n, B_1, \ldots, B_m$ of G and elements $a_1, \ldots, a_n, b_1, \ldots, b_m$ of G such that $A = \bigcup_i a_i \cdot A_i = \bigcup_j b_j \cdot B_j$.

This should look similar to the statement of Banach-Tarski, since Banach-Tarski can be restated as the sphere being $SO(3)$-paradoxical.

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Paradoxicality of $F_2 = \langle a, b \rangle$
For a generator $g \in \{a, a^{-1}, b, b^{-1}\}$, set $S(g)$ to be the set of all words that begin with the letter g.

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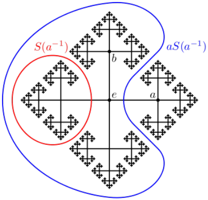
Notice that $a \cdot S(a^{-1}) = \{e\} \sqcup S(a^{-1}) \sqcup S(b) \sqcup S(b^{-1})$



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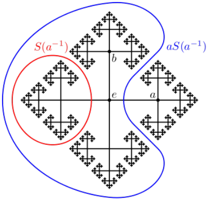

So F_2 is F_2-paradoxical.

Construction

An action of F_2 on the sphere S^2

Consider the matrices

$$A = \begin{pmatrix} \frac{1}{3} & \frac{2\sqrt{2}}{3} & 0 \\ -\frac{2\sqrt{2}}{3} & \frac{1}{3} & 0 \\ 0 & 0 & 1 \end{pmatrix}, B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{3} & \frac{2\sqrt{2}}{3} \\ 0 & -\frac{\sqrt{2}}{3} & \frac{1}{3} \end{pmatrix}.$$

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So, in fact, we have a free action $F^2 \curvearrowright S^2 \setminus C$ where C is a countable set.

Construction

Pushing forward the paradoxicality of F^2

Recall $F_2 = S(a) \sqcup a \cdot S(a^{-1}) = S(b) \sqcup b \cdot S(b^{-1})$.

- Use Axiom of Choice to choose a single point in every orbit. Think of this set of points as the identity of F_2.
- Now simultaneously find sets $T(a)$, $T(a^{-1})$, $T(b)$, $T(b^{-1})$ analogously to $S(a), \ldots$
- $S^2 \setminus C = T(a) \sqcup a \cdot T(a^{-1}) = T(b) \sqcup b \cdot T(b^{-1})$.

So, $S^2 \setminus C$ is, in fact, F^2-paradoxical.

Definitions

Amenability

A countable discrete group G is **amenable** if there exists a sequence of nonempty finite sets $(F_n)_{n\in\mathbb{N}}$ such that for every $g \in G$

$$\frac{|F_n \Delta (g \cdot F_n)|}{|F_n|} \to 0 \text{ as } n \to \infty$$

Such a sequence is called a Følner sequence.

Examples

- finite groups
- $\mathbb{Z}^n$
- abelian groups
- solvable groups

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Proof. Take a left-invariant finitely additive probability measure m on G. If G were paradoxical, then
$G = \bigsqcup A_i \sqcup \bigsqcup B_j = \bigsqcup a_i \cdot A_i = \bigsqcup b_j \cdot B_j$. But we now get that
$1 = m(G) = m(G) + m(G) = 2$ due to the properties of m.

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Other directions are also true but not as easy.

Ergodicity

Probability Measure Preserving

A group action on a standard measure space $G \curvearrowright ([0,1], \lambda)$ is **probability measure preserving(p.m.p.)** if for every $g \in G$ and measurable $A \subseteq X$, the measure $\lambda(A) = \lambda(g \cdot A)$.

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Ergodic

A p.m.p. group action $G \curvearrowright ([0,1],\lambda)$ is **ergodic** if for every G-invariant set $A \subseteq X$, $\lambda(A) = 10$ or $\lambda(A) = 1$.

Example Irrational translation(fix $q \in \mathbb{R} \setminus \mathbb{Q}$) by $\mathbb{Z}$,

$$n \cdot x \mapsto x + nq \mod 1$$

Orbit Equivalence

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Two group actions on a standard measure space $G, H \curvearrowright ([0,1], \lambda)$ are **Orbit Equivalent** if there is a measure space isomorphism $f : ([0,1], \lambda) \to ([0,1], \lambda)$ such that

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Dye's Theorem

If G, H are amenable, then any 2 free p.m.p. ergodic actions on a standard measure space are orbit equivalent.

So, up to orbit equivalence, there is one action of amenable groups.

What about nonamenable groups?

Hjorth 2005

Every nonamenable group has at least 2 free p.m.p ergodic non-orbit equivalent actions.

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Epstein 2007

Every nonamenable group has at least $\mathfrak{c} \geq 2^{\aleph_0}$ free p.m.p ergodic non-orbit equivalent actions.

So, nonamenable groups have a lot of actions.