Mechanized Proof of the Curve Length of a Rectifiable Curve

Jagadish Bapanapally

May 23, 2017

Abstract

Infinitesimal calculus has led to prove length of a rectifiable curve. This work presents a formalization of a rectifiable curve in the logic of ACL2(r), which is a variant of ACL2 that supports reasoning about the real and complex numbers by way of non-standard analysis. The length of a curve can be defined as the limit of the sum of the chord lengths in a partition. When this limit exists, the curve is called rectifiable. We show that when a complex-valued curve is continuously differentiable, the curve is also rectifiable, and the length of the curve can be computed by taking the integral of the norm of the curve's derivative. We use this result to verify the well-known formula for the circumference of a circle using the second fundamental theorem of calculus.

The approach is to prove that function, norm of the curve's derivative, is continuous if the curve is continuously differentiable and show that the length of the curve, which is the sum of all these functions over infinitely small partitions of the curve is equal to the integral of the function. Thus, if the curve is continuously differentiable, it is rectifiable.

1 Introduction

Logic can be defined as the formal study of reasoning. Automated deduction is concerned with the mechanization of formal reasoning, following the laws of logic. Therefore, if we replace "formal" by "mechanical" we can place almost the entire set of methodologies used in the field of automated theorem proving (ATP) within the scope of logic.

A Rectifiable curve is a curve which has finite length. Let l be a simple curve , and let (P_n) be a sequence of polygonal approximations of l such that the maximum length of their line segments converges to 0. The length of l is is by definition the limit of the sequence of the lengths of (P_n):

$L(P_n) = \limsup_{n \to \infty} L(P_n)$.

We present our efforts in formalizing curve length of a rectifiable curve in the logic of ACL2(r), which is a variant of ACL2 that supports reasoning about the real and complex numbers via non-standard analysis.

We start with the theory behind rectifiable curve followed by formalizing the length of a rectifiable curve in ACL2(r) and application of this proof.

2 Theory

In this section, we derive a formula for the length of a curve $(x(t), y(t)) = f(t)$ on an interval $[a, b]$ where $x(t)$ and $y(t)$ are the real and imaginary parts of the curve $f(t)$ respectively. We will assume that f is continuously differentiable on the interval $[a, b]$. We use Riemann sums to approximate the length of the curve over the interval and then take the limit to get an integral.

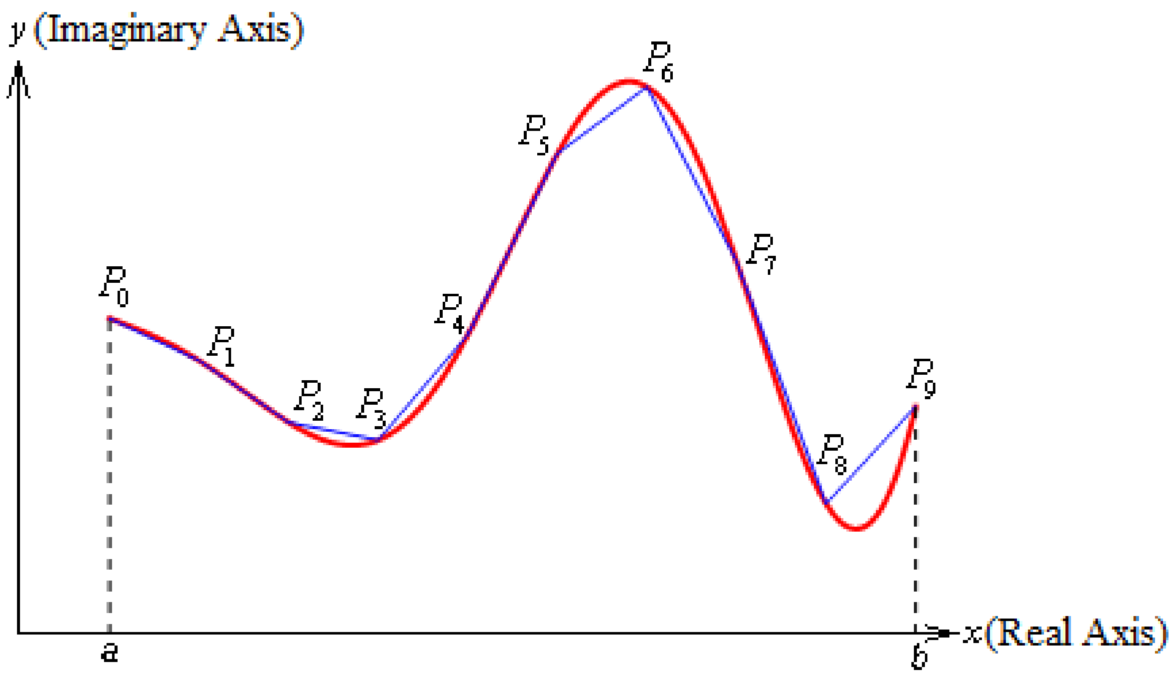

Figure 1: Rectifiable Curve

Initially we will need to estimate the length of the curve. We will do this by dividing the interval into n equal subintervals each of width Δx and denote the point on the curve at each point by P_i. In the sketch above $a = x(t_0)$ and $b = y(t_n)$. We can then approximate the curve by a series of straight lines connecting the points.

Now, denoting the length of each of these line segments by $\mid P_i \quad P_{i-1} \mid$, the length of the curve will approximately be,

$$L \approx \sum_{i=1}^{n} \mid P_i - P_{i-1} \mid$$

and we can get the exact length by making n larger and larger. In other words, the exact length will be,

$$L = \lim_{n \to \infty} \sum_{i=1}^{n} \mid P_i - P_{i-1} \mid = \lim_{n \to \infty} \sum_{i=1}^{n} \mid f(t_i) \ - f(t_{i-1}) \mid ; f(t) = x(t) + i * y(t), t_0 \leq t \leq t_n$$

This implies the following because P_i is $f(t_i)$

$$L = \lim_{n \to \infty} \sum_{i=1}^{n} |\ f(t_i)\ - f(t_{i-1})\ | = \lim_{n \to \infty} \sum_{i=1}^{n} |\ \frac{f(t_i)\ - f(t_{i-1})}{\Delta t}\ | \ \Delta t$$

By definition of integral the above equation is equal to

$$\int_{t_0}^{t_n} |f'(t)|\, dt = \int_{t_0}^{t_n} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt$$

In the above equation $\frac{dx}{dt}$ is real part of $\frac{df}{dt}$ and $\frac{dy}{dt}$ is imaginary part of $\frac{df}{dt}$

3 Formalizing Length Of A Rectifiable Curve

In this section we show that continuously differentiable curve is rectifiable and its length is equal to norm of the derivative of the curve.

3.1 Defining Continuously Differentiable curve

If a curve is continuously differentiable, it implies that its derivative is continuous.

```
(encapsulate
  ((f (x) t)
   (f-derivative (x) t))
  ;; Our witness continuous function is the identity function.
  (local (defun f(x) x))
  (local
   (defun f-derivative (x) (declare (ignore x)) 1
     ))
     ;lemma to show f-derivative is actually derivative of f
  (defthmd f-der-lemma
    (implies (and (standardp x)
                  (realp x)
                  (realp y)
                  (i-close x y)
                  (not (= x y))
                  )
             (i-close (/ (- (f x) (f y)) (- x y)) (f-derivative x))
             )
    )
    ;f is continuously differentiable which implies f-derivative is continuous
  (defthmd f-der-continuous
    (implies
     (and
      (standardp x)
      (realp x)
```

```
    (realp y)
    (i-close x y)
    )
   (i-close (f-derivative x) (f-derivative y))
   )
  )
 )
```

3.2 Proving norm of the derivative of the function is Continuous

Norm of the derivative of the function is

$$\int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

To prove that the above function is continuous we have to prove that the real part of f-derivative($\frac{dx}{dt}$), the imaginary part of f-derivative ($\frac{dy}{dt}$), the square of continuous functions ($\frac{dx}{dt}^2, \frac{dy}{dt}^2$), the sum of two continuous functions($\frac{dx}{dt}^2 + \frac{dy}{dt}^2$) and finally the square root of a continuous function ($\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}$) are continuous.

$\frac{dx}{dt}$ and $\frac{dy}{dt}$ are continuous because $\frac{df}{dt}$ is continuous. From instance square-is-continuous we can prove ($\frac{dx}{dt}^2, \frac{dy}{dt}^2$) are continuous and the proof for sum of 2 continuous functions being continuous is trivial.

Proving square root of a continuous function is continuous is non trivial.

For real numbers $x1, x2, x3$ we can prove following lemmas:

$$if\ x1 >= 0\ and\ x2 >= 0\ and\ x1 > x2\ then\ (x1 * x1)\ >\ (x1 * x2) \tag{3.2.1}$$

$$if\ x1 >= 0\ and\ x2 >= 0\ and\ x2 > x1\ then\ (x1 * x2) >= (x2 * x2) \tag{3.2.2}$$

$$if\ x1 > x2\ and\ x2 >= x3 then\ (x1 > x2) \tag{3.2.3}$$

Using 3.2.1, 3.2.2, 3.2.3 we can prove

$$if\ x1 >= 0\ and\ x2 >= 0\ and\ x1 > x2\ then\ (x1 * x1) > (x2 * x2) \tag{3.2.4}$$

For real numbers $y1, y2$ we can prove the following lemmas

$$if\ (standard - party1) \neq (standard - party2)\ then\ (not\ (i - close\ y1\ y2)) \tag{3.2.5}$$

$$if\ (y1 >= 0)\ and\ (y2 >= 0)\ and\ (not\ (i{-}close\ y1\ y2))\ then\ (standard{-}party1) \neq (standard{-}party2) \tag{3.2.6}$$

Using 3.2.6 and 3.2.4, we can prove

$$\begin{gathered} if\ (i - limited\ y1)\ and\ (i - limited\ y2)\ and\ (y1 >= 0)\ and\ (y2 >= 0)\ and\ (not\ (i - close\ y1\ y2)) \\ then\ (standard - part\ y1) * (standard - part\ y1) \neq (standard - part\ y2) * (standard - part\ y2)) \end{gathered} \tag{3.2.7}$$

$$(standard - part\ y1) * (standard - part\ y1) = (standard - part\ (y1^2)) \tag{3.2.8}$$

Using 3.2.7 and 3.2.8, we can prove

$$\begin{gathered} if\ (i - limited\ y1)\ and\ (i - limited\ y2)\ and\ (y1 >= 0)\ and\ (y2 >= 0)\ (not\ (i - close\ y1\ y2)) \\ then\ (standard - part\ y1^2) \neq (standard - part\ y2^2) \end{gathered} \tag{3.2.9}$$

Using 3.2.9 and 3.2.5,

$$\begin{gathered} if\ (i - limited\ y1)\ and\ (i - limited\ y2)\ (y1 >= 0)\ and\ (y2 >= 0)\ and\ (not\ (i - close\ y1\ y2)) \\ then\ (not\ (i - close\ (y1^2)\ (y2^2))) \end{gathered} \tag{3.2.10}$$

Finally, we have all the lemmas to prove that the square root of a continuous function is continuous using 3.2.10 by the logic,

$$\neg p \to \neg q \Leftrightarrow q \to p$$

```
(defthmd root-close-lemma
  (implies
   (and (standardp x1)
        (realp x1)
        (realp x2)
        (>= x1 0)
        (>= x2 0)
        (i-close x1 x2)

        )
   (i-close (acl2-sqrt x1) (acl2-sqrt x2))
   )
  ;:hints omitted
  )
```

By making use of the above lemma, we can prove that $\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}$ is continuous. The following theorem shows this. In the following theorem f-der-sum-sqrt is $\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}$

```
(defthmd f-der-sum-sqrt-cont
  (implies
    (and (standardp x)
         (inside-interval-p x (interval 0 nil))
         (inside-interval-p y (interval 0 nil))
         (i-close x y)
      )
     (i-close
      (f-der-sum-sqrt x)
      (f-der-sum-sqrt y)
      )
    )
    ;hints omitted
  )
```

3.3 Length of the Curve

Riemann sum of these functions $f = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}$ over partition p is given by riemann-f function below

```
    (defun map-f (p)
      (if (consp p)
        (cons (f (car p))
              (map-f (cdr p)))
       nil))

    (defun riemann-f (p)
      (dotprod (deltas p)
               (map-f (cdr p))))
```

We can prove riemann-f is limited using functional-instance of limited-riemann-rcfn-small-partition on the interval 0 to infinity.

```
    (defthmd limited-riemann-f-small-partition
      (implies (and (realp a) (standardp a)
                    (realp b) (standardp b)
                    (inside-interval-p a (interval 0 nil))
                    (inside-interval-p b (interval 0 nil))
                    (< a b))
               (i-limited (riemann-f (make-small-partition a b))))
               ;:hints omitted
      )
```

From the above theorem we can prove that riemann-f with standard inputs returns a standard output

```
    (defun-std strict-int-f (a b)
    (if (and (realp a)
             (realp b)
             (inside-interval-p a (interval 0 nil))
             (inside-interval-p b (interval 0 nil))
             (< a b))
      (standard-part (riemann-f (make-small-partition a b)))
      0)
        ;:hints omitted
      )
```

The length of the curve is sum these functions for infinitely many small partitions between interval t_0 and t_n on the curve. We can prove this sum is integral of f by using functional instance of $strict-int-rcfn-is-integral-of-rcfn$. By using the above theorem, we showed if the curve is continuously differentiable it is rectifiable and the length of the curve in the interval t_0 and t_n given $t_0 <= t_n$ is equal to $\int_{t_0}^{t_n} |f'(t)|\, dt$.

4 Application of the Proof

In this section, we will apply the proof to show that the circumference of a circle with radius r is equal to $2 * \pi * r$. Radius of a curve which is standard and real can be defined as follows:

```
(encapsulate ((rad() t))
  (local (defun rad() 1))
  (defthmd rad-det
    (and (realp (rad))
         (standardp (rad))
         (>= (rad) 0)
         (i-limited (rad))
         )
    )
  )
```

Circle in complex domain is represented as $r * e^{ix}$, where r is the radius and $0 \leq x \leq 2 * \pi$ and it can be defined as follows:

```
(defun c(x)
    (* (rad) (acl2-exp (* #c(0 1) x)))
  )
```

Let the derivative of the curve be,

```
(defun c-der(x)
  (* (rad) (complex (- (acl2-sine x)) (acl2-cosine x)))
  )
```

To proceed further we have to prove that the curves derivative is close to $c-der(x)$ and $c-der(x)$ is continuous.

By using the instances $acl2-sine-derivative$, $acl2-cosine-derivative$, $rad-det$ and few other lemmas of non-standard analysis we can prove $c-der(x)$ is infinitely close to derivative of $c(x)$. The second constraint is to prove that $c-der(x)$ is continuous. We can prove this because $acl2-sine$ and $acl2-cosine$ are continuous functions and using non-standard analysis lemmas.

By using functional instantiating $f-der-sum-sqrt-cont$ we can prove norm of $c-der(x)$ is continuous. Thus we can define Riemann integral for norm of the $c-der(x)$ to get length of the circle c,

```
(defun int-c-der-sum-sqrt (a b)
(if (<= a b)
  (strict-int-c-der-sum-sqrt a b)
  (- (strict-int-c-der-sum-sqrt b a))))
```

In the above function, $int-c-der-sum-sqrt$ is integral of norm of $c-der(x)$ from a to b and $strict-int-c-der-sum-sqrt$ is strict integral of norm of $c-der$ from a to b. But norm of the $c-der(x)$ is (rad), which is just the radius of the circle. $f-len(x)$ can be defined as below.Derivative of $f-len$ is equal to norm of $c-der(x)$.

```
(defun f-len(x)
  (if (realp x)
    (* (rad) x)
    0)
  )
```

By second fundamental theorem of calculus, if $h'(t) = g(t)$ then

$$\int_a^b g(t)dt = h(b) - h(a)$$

So thus we get following theorem by functional instantiating $ftc-2$

```
(defthmd apply-ftc-2
 (implies (and (inside-interval-p a (f-der-sum-sqrt-domain))
               (inside-interval-p b (f-der-sum-sqrt-domain)))
          (equal (int-f-der-sum-sqrt a b)
                 (- (f-len b)
                    (f-len a))))
     ;hints omitted
    )
```

By letting b as 2π and a as 0 we get the radius of a circle as $2\pi(rad)$.

Thus we proved circumference of a circle from 0 to 2π with radius (rad) is equal to $2*\pi*(rad)$ where (rad) is radius of the circle which can be any real number and standard.

5 Conclusion

In this paper, we proved that the norm of the curve's derivative is continuous if the curve is continuously differentiable. We used the definition of the integral to formalize the length of rectifiable curve in ACL2(r).

Using the above result, we verified the formula for the circumference of a circle. Therefore, the results presented in this paper could also be applied to any other continuously differentiable curves.