Integration Methods for Geometry in ACL2(r)

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Theory of differentiability, integration and several algebraic rules of differentiation and integration have been formalized in ACL2(r) by the way of non-standard analysis. As an extension, we present the formalization of Integration by Substitution, and proofs of area of a circle, surface area of a sphere and volume of a sphere using the integration rules formalized in ACL2(r). Integration by substitution, also known as u-substitution, is a method that is often used in calculus to find the integral of a function. We show the proof of u-substitution using the fundamental theorem of calculus and the chain rule which are already proved in ACL2(r). We use the u-substitution proof to verify the area of a circle, we verify the volume of a sphere using the disk method and then we formalize the lateral surface area of a surface of revolution which we use to verify the surface area of a sphere.

1 Introduction

Continuity, differentiablity and proofs of theorems such as the intermediate value and mean value theorems were proved in ACL2 using the non-standard-analysis [6]. The Riemann integral of a continuous function was formalized, and a proof of the First Fundamental Theorem of Calculus (FTC-1) was presented in [12]. The proof of the Second Fundamental Theorem of Calculus (FTC-2) was presented in [8]. The Chain Rule and other algebraic differential rules were presented in [7]. U-substitution method is an application of FTC and chain-rule that is used to change the variable of integration [3]. For example :

$$\int_{x=0}^{x=4} x\sqrt{x^2+9}\,dx = \int_{u=9}^{u=25} \frac{1}{2}\sqrt{u}\,du \ \left[\text{substituting } u \text{ for } x^2+9 \text{ and then, } xdx = \frac{du}{2}\ \right]$$

In section 2 we formalize u-substitution using the FTC and the chain rule which are already proved in ACL2(r). In particular, we show that if f is continuous on I, ψ is defined on an interval, $[a,b]$ whose range is in the interval I, and if ψ has a continuous derivative, then we show that $\int_{\psi(a)}^{\psi(b)} f(x)\,dx = \int_a^b f(\psi(u))\,\psi'(u)\,du$. We use this proof to show the area of a circle with radius r is πr^2.

A solid sphere is a solid of revolution that is obtained by revolving a solid semicircle 360° around the x-axis as shown in figure 1 [16]. In section 3 we use the disk method [13] to prove the volume of a sphere with radius r is $4/3\cdot\pi r^3$. The disk method is used to find the volume of a solid of revolution where we divide the solid into disks of infinitesimal widths and summing up the volumes of all the disks.

In section 4 we prove, the surface area of a sphere with radius r is $4\pi r^2$. A sphere is a surface of revolution [16] obtained when a semicircle is revolved around the x-axis. First, we formalize the lateral surface area of a surface of revolution and then prove the surface area of a sphere by instantiating the formalization. The formalization requires formalizing the length of a rectifiable curve which we have proved and presented in [4]. Summing up, we show the derivation of the lateral surface area of a sphere using the length of a rectifiable of a curve and then, we formalize the lateral surface area of a surface of revolution and prove surface area of a sphere in ACL2(r).

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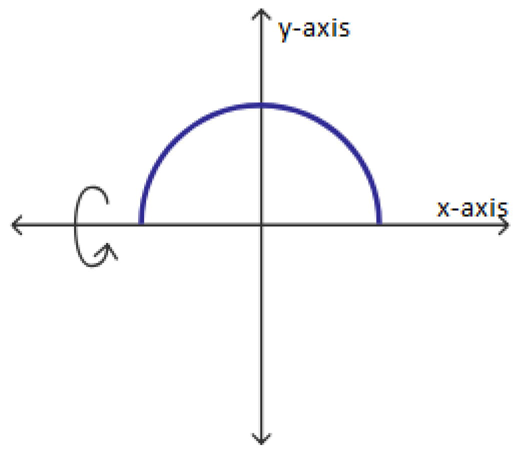

Figure 1

2 Area of a Circle

The area of a sector is directly proportional to the angle made by the sector at the center [10]. So, the area of a circle is 4 times the area of the sector ABC. The equation of a circle with radius r centered at $(0,0)$ is $x^2+y^2=r^2$ and the area of the circle is 4 times the area under the curve $\sqrt{r^2-x^2}$ in the first quadrant which is the green shaded region in the Figure 2 [1]. So, the area of the circle is $4\cdot\int_0^r\sqrt{r^2-x^2}dx$. The integral can be calculated using a method called u-substitution [3] which changes the variable of the integral. Below we show the derivation of the u-substitution rule using the fundamental theorem of calculus and chain rule and we prove the area of a circle using using u-substitution and then we formalize the proofs in ACL2(r).

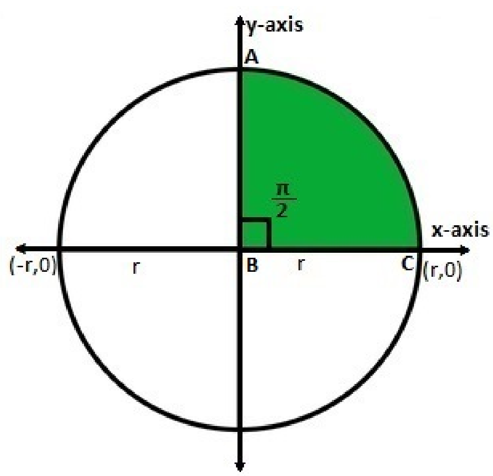

Figure 2

2.1 Derivation of the u-substitution rule

Following [11, 5], we state the principle of u-substitution as follows:

Proposition: Let $I \subseteq R$ be an interval and $\psi : [a,b] \to I$ be a differentiable function with continuous derivative, ψ'. Suppose that $f : I \to R$ is a continuous function . Then,

$$\int_{\psi(a)}^{\psi(b)} f(x)\,dx = \int_a^b f(\psi(u))\psi'(u)\,du$$

Since f and ψ are continuous, the composite function $f(\psi(u))$ is continuous and since the derivative of ψ is continuous, using the product rule of two continuous functions, $f(\psi(u))\psi'(u)$ is continuous. Hence the integral of $f(\psi(u))\,\psi'(u)$ exists. Since f is continuous, there exists an antiderivative of f and let that be F. Using the chain rule:

$$(F \circ \psi)'(u) = F'(\psi(u))\psi'(u) = f(\psi(u))\psi'(u) \tag{1}$$

Applying the chain-rule and the fundamental theorem of calculus to derive the u-substitution rule:

$$\begin{aligned}
\int_a^b f(\psi(u))\psi'(u)\,du &= \int_a^b (F \circ \psi)'(u)\,du && [\ (1)\ ] \\
&= (F \circ \psi)(b) - (F \circ \psi)(a) && [\ \text{FTC-2}\ ] \\
&= F(\psi(b)) - F(\psi(a)) && [\ \text{Definition of } F \circ \psi\ ] \\
&= \int_{\psi(a)}^{\psi(b)} f(x)\,dx && [\ \text{FTC-2}\ ]
\end{aligned} \tag{2}$$

2.2 Deriving the Area of the Circle using U-substitution

Area of the circle is $4 \cdot \int_0^r \sqrt{r^2 - x^2}\,dx$ where $\int_0^r \sqrt{r^2 - x^2}\,dx$ is the area of the circle under the curve in the first quadrant as in Figure 2. If we consider $f(x) = \sqrt{r^2 - x^2}$, $\psi(x) = r\sin x$, $a = 0$, $b = \pi/2$ then, by using the u-substitution rule:

$$\begin{aligned}
4\int_0^r \sqrt{r^2 - x^2}\,dx &= 4\int_{\psi(a)}^{\psi(b)} f(x)\,dx \\
&= 4\int_a^b f(\psi(u))\psi'(u)\,du \qquad [\ (2)\ ] \\
&= 4\int_0^{\pi/2} \sqrt{r^2 - (r\sin u)^2}\, r\cos u\,du \\
&= 4\int_0^{\pi/2} r^2\cos^2 u\,du \\
&= 4r^2\Big(\frac{sin(2u)}{4} + \frac{u}{2}\Big)\Big|_0^{\pi/2} \\
&= \pi r^2
\end{aligned}$$

2.3 U-substitution rule in ACL2(r)

To formalize the proposition in section 2.1, we define two functions f-prime (f in the proposition) and fi (ψ in the proposition). The two domains for the two functions respectively are fi-range and f-o-fi-domain. Derivative of fi defined is fi-prime (ψ' in the proposition).

The constraints required for the proof are shown in the Appendix A. Encapsulated constraints are used to generate the theorems that are used as constraints for the substitution rule. The integral of f-prime, int-f-prime-1, can be defined as shown in [12]. We use FTC-1 to show that the derivative of the integral of f-prime is equal to f-prime:

```
(defthm ftc-1-fprime
  (implies (and (inside-interval-p a (fi-range))
                (inside-interval-p b (fi-range))
                (inside-interval-p c (fi-range))
                (standardp b)
                (i-close b c)
                (not (equal b c)))
           (i-close (/ (- (int-f-prime-1 a b) (int-f-prime-1 a c))
                       (- b c))
                    (f-prime b))))
```

Antiderivative of f-prime, f (F in (1)) is then defined as below and then we show that f-prime is the derivative of f :

```
(defun f(x) (int-f-prime-1 (consta) x))

(defthm f-prime-is-derivative
  (implies (and (standardp x)
                (inside-interval-p x (fi-range))
                (inside-interval-p x1 (fi-range))
                (i-close x x1) (not (= x x1)))
           (i-close (/ (- (f x) (f x1)) (- x x1))
                    (f-prime x)))
  :hints ...)
```

The next lemma we need is the proof of the chain rule (1). We define the composite function $F \circ \psi$ as f-o-fi, derivative-cr-f-o-fi equal to (* (f-prime (fi x)) (fi-prime x)) and differential-cr-f-o-fi is the derivative of f-o-fi. Since f and fi are differentiable, we apply the chain rule proof presented in [7] to prove the lemma.

```
(defthm expand-differential-cr-f-o-fi
  (implies (and (inside-interval-p x (f-o-fi-domain))
                (inside-interval-p (+ x eps) (f-o-fi-domain)))
           (equal (differential-cr-f-o-fi x eps)
              (/ (- (f-o-fi (+ x eps)) (f-o-fi x)) eps)))
  :hints ...)

(defthm differential-cr-f-o-fi-close-1
  (implies (and (standardp x)
                (inside-interval-p x (f-o-fi-domain))
                (inside-interval-p x1 (f-o-fi-domain))
                (i-close x x1) (not (= x x1)))
           (equal (standard-part (differential-cr-f-o-fi x (- x1 x)))
                  (derivative-cr-f-o-fi x)))
  :hints ...)
```

The next step is proving the integral of derivative of f-o-fi between two end points a and b equals to (- (f-o-fi(b)) (f-o-fi(a))). Since f-prime, fi and fi-prime are continuous, f-prime(fi x) is continuous using the composition rule of two continuous functions and ($*$ (f-prime(fi x)) (fi-prime x)) is continuous using the product rule of two continuous functions. f-o-fi-prime is the derivative of f-o-fi which is equal to ($*$ (f-prime(fi x)) (fi-prime x)). Since f-o-fi-prime is continuous and f-o-fi-prime is the derivative of f-o-fi we use FTC-2 to show that the integral of f-o-fi-prime, int-f-o-fi-prime, is equal to (- (f-o-fi(b)) (f-o-fi(a))):

```
(defun derivative-cr-f-o-fi (x)
  (* (f-prime (fi x))
     (fi-prime x)))

(defun f-o-fi-prime (x)
    (if (inside-interval-p x (f-o-fi-domain))
        (derivative-cr-f-o-fi x)
      0))

(defthmd ftc2-usub-equal
  (implies (and (inside-interval-p a (f-o-fi-domain))
                (inside-interval-p b (f-o-fi-domain)))
           (equal (int-f-o-fi-prime a b)
                  (- (f (fi b)) (f (fi a)))))
  :hints ...)
```

The final step is to prove that the integral of f-prime, int-f-prime, with (fi a) and (fi b) being two end points of the integral, is equal to (- (f (fi b)) (f (fi a))). Since f-prime is the derivative of f and is continuous, we apply FTC-2:

```
(defthmd ftc2-usub-1-equal
  (implies (and (inside-interval-p a (f-o-fi-domain))
                (inside-interval-p b (f-o-fi-domain)))
           (equal (int-f-prime (fi a) (fi b))
                  (- (f (fi b)) (f (fi a)))))
  :hints ...)
```

Finally we get the u-substitution rule using the lemmas ftc2-usub-equal and ftc2-usub-1-equal:

```
(defthm usubstitution-f-o-fi
  (implies (and (inside-interval-p a (f-o-fi-domain))
                (inside-interval-p b (f-o-fi-domain)))
           (equal (int-f-prime (fi a) (fi b))
                  (int-f-o-fi-prime a b)))
  :hints (("Goal"
           :use ((:instance ftc2-usub-1-equal)
                 (:instance ftc2-usub-equal))
           :in-theory (disable int-f-o-fi-prime int-f-prime))))
```

2.4 Area of a Circle in ACL2(r)

Quarter area of a circle which is centered at (0,0) with radius r is the area under the curve, f, $\sqrt{r^2-x^2}$ in the first quadrant. The domain of the curve is $[0,r]$. The substitution function, ψ is $r\sin(x)$ and the domain of the function is $[0,\frac{\pi}{2}]$. The derivative of the substitution function, ψ' is $r\cos(x)$.

We first prove the constraints required to use the u-substitution rule for the defined functions. Since sine and cosine are continuous functions as proved in [9], ψ and ψ' are continuous. Then we prove ψ'

is the derivative of ψ [15]. The next lemma required is that f is continuous. Square of a function is continuous and square root of a continuous function is continuous [4]. So we get $\sqrt{r^2-x^2}$ is continuous. We have proved all the constraints required to use the u-substitution rule proved in section 2.1 and we get:

```
(defthm usubstitution-circle
  (implies (and (inside-interval-p a (fi-domain))
                (inside-interval-p b (fi-domain)))
           (equal (int-circle (sub-func a) (sub-func b))
                  (int-circle-sub-prime a b)))
  :hints ...)
```

In the above lemma sub-func is $r \cdot \sin(x)$, int-circle is the integral of $\sqrt{r^2-x^2}$, int-circle-sub-prime is the integral of $f(\psi(x))\psi'(x)$ and fi-domain is the interval $[0, \frac{\pi}{2}]$.

$$\begin{aligned} f(\psi(x))\psi'(x) &= \sqrt{r^2-(r\sin x)^2}\, r\cos x \\ &= r^2\cos^2 x \end{aligned}$$

The area of the circle is equal to the 4 times the integral of $r^2\cos^2(x)$ between 0 to $\pi/2$. First we apply the chain rule to show the derivative of $sin(2x)$ is infinitesimally close to $2cos(2x)$. Then we show the derivative of $r^2 \cdot \left(\frac{sin(2x)}{4}+\frac{x}{2}\right)$ is infinitesimally close to $r^2 cos^2 x$.

$$\begin{aligned} \frac{d}{dx} r^2 \cdot \left(\frac{sin(2x)}{4}+\frac{x}{2}\right) &\approx r^2 \cdot \left(\frac{2cos(2x)}{4}+\frac{1}{2}\right) \\ &\approx r^2 \cdot \left(\frac{2(2cos^2 x-1)}{4}+\frac{1}{2}\right) \\ &\approx r^2 \cdot cos^2 x \end{aligned} \tag{3}$$

Using FTC-2 and (3) :

$$\begin{aligned} 4\int_0^{\pi/2} r^2 \cdot \cos^2 x\, dx &= 4r^2 \cdot \left(\frac{sin(2x)}{4}+\frac{x}{2}\right)\Big|_0^{\pi/2} \\ &= \pi r^2 \end{aligned}$$

3 Volume of a Sphere

A circular disk can be formed by revolving a solid rectangle bar around x-axis 360° as shown in figure 3 from [13]. The disk with width w and radius R has a volume of $\pi R^2 w$. So, a solid of revolution [13] shown in figure 4 which can be divided into circular disks with the infinitesimal width Δx has a volume, $V = \pi\int_a^b [R(x)]^2 dx$ between the interval $[a,b]$ on x-axis where $R(x)$ is function of x equal to the point on the surface of solid of revolution for x on x-axis.

A sphere is a solid of revolution formed by rotating $y = \sqrt{r^2-x^2}$, a solid semicirle, about x-axis. The volume of the each disk in figure 5 [2] is $\Delta V = \pi y^2 \Delta x$ where Δx is the width of the disk and πy^2 is the area of the base of the disk. Total volume of the sphere is equal to two times the volume of the right hemisphere:

$$\begin{aligned} V &= 2\int_0^r \pi y^2\, dx \\ &= 2\pi\int_0^r (r^2-x^2)\, dx \quad [\because \text{y=} \sqrt{r^2-x^2}] \end{aligned}$$

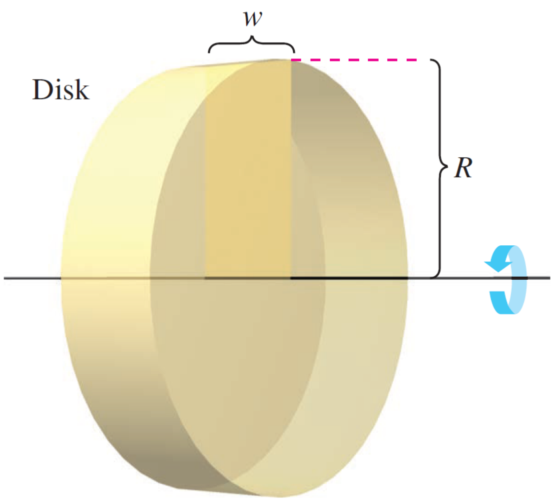

Figure 3

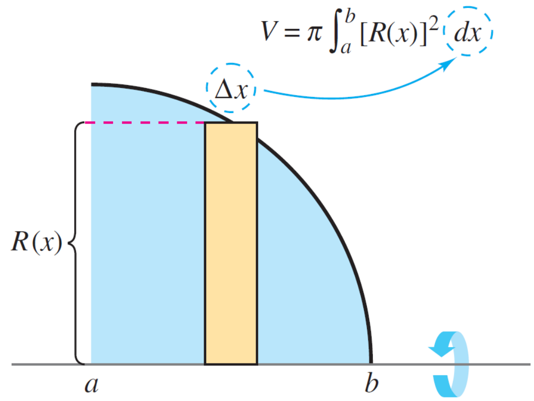

Figure 4

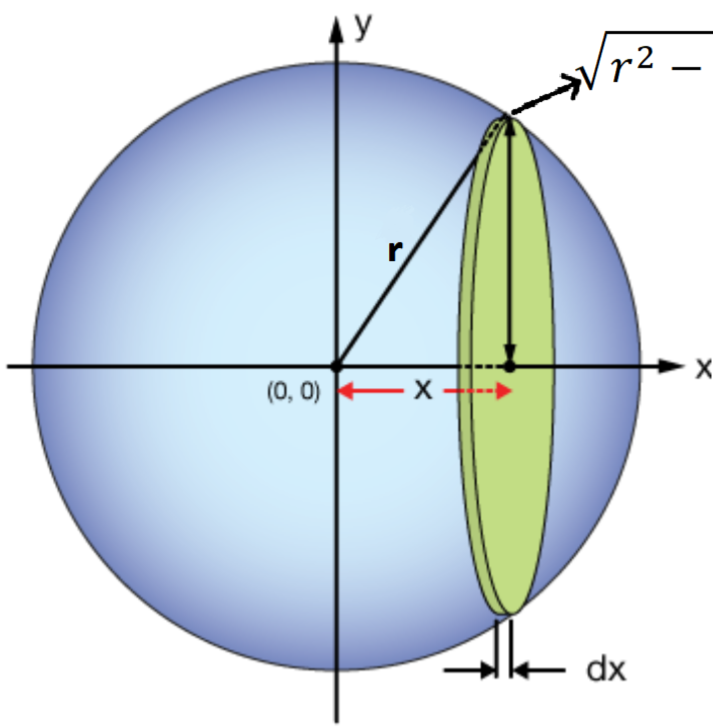

Figure 5

For a standard real number x, $x \approx y$ and $x \neq y$:

$$\begin{aligned} x-y \approx 0 &\implies -\frac{1}{3}x\cdot(x-y) \approx 0 \\ &\implies \frac{1}{3}(y-x)(y+x) - 1/3x(x-y) \approx 0 \;\; [\because \text{y-x} \approx 0] \\ &\implies \frac{1}{3}\cdot(y^2-x^2-x^2+xy) \approx 0 \\ &\implies \frac{1}{3}\cdot(y^2+xy+x^2) \approx x^2 \\ &\implies \frac{x-y}{x-y}\cdot\frac{1}{3}\cdot(y^2+xy+x^2) \approx x^2 \\ &\implies \frac{\frac{x^3}{3}-\frac{y^3}{3}}{x-y} \approx x^2 \end{aligned} \tag{4}$$

For a standard real number $r \geq 0$ which is radius of the sphere, $x \approx y$ and $x \neq y$:

$$r^2 \approx r^2 \implies r^2\cdot\frac{x-y}{x-y} \approx r^2 \implies \frac{r^2x-r^2y}{x-y} \approx r^2 \tag{5}$$

For a standard real number x, $x \approx y$ and $x \neq y$, $(5)-(4)$ gives :

$$\frac{r^2x-\frac{x^3}{3}-(r^2y-\frac{y^3}{3})}{x-y} \approx r^2-x^2 \implies \frac{d}{dx}\left(r^2x-\frac{x^3}{3}\right) \approx r^2-x^2 \tag{6}$$

Using FTC-2 and (6), the volume of the sphere is:

$$\begin{aligned} V = 2\pi\int_0^r (r^2-x^2)\,dx &= 2\pi\cdot\left(r^2x-\frac{x^2}{3}\right)\Big|_0^r \\ &= \frac{4}{3}\pi r^3 \end{aligned}$$

4 Surface Area of a Sphere

A surface of a revolution is a result of revolving a continuous function around an axis of revolution [13]. A sphere is a surface of revolution which is obtained by rotating a semicircle around an axis. The lateral surface area of a surface of a revolution is derived from the surface area of the frustum of a circular cone. The Surface area of the frustum shown in figure 6 which is obtained by revolving the line segment L around the axis of revolution, $S = 2\pi rL$ where $r = \frac{r_1+r_2}{2}$ [14].

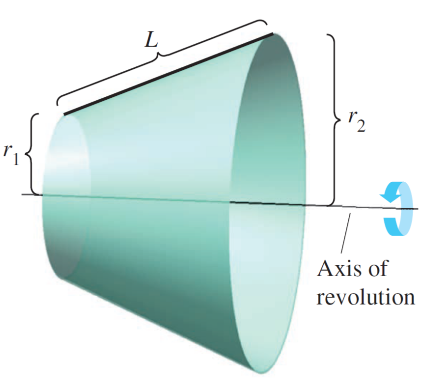

Figure 6

Now, let's consider a continuous function, f which has a continuous derivative in the interval $[a,b]$. The line segment L formed by the function f is a combination of infinitesimal line segments.

The length of the i_{th} infinitesimal line segment in figure 7 is:

$$\Delta L = \sqrt{\Delta x_i^2 + \Delta y_i^2}$$

A surface of a revolution is obtained by revolving f around an axis of revolution as shown in figure 7

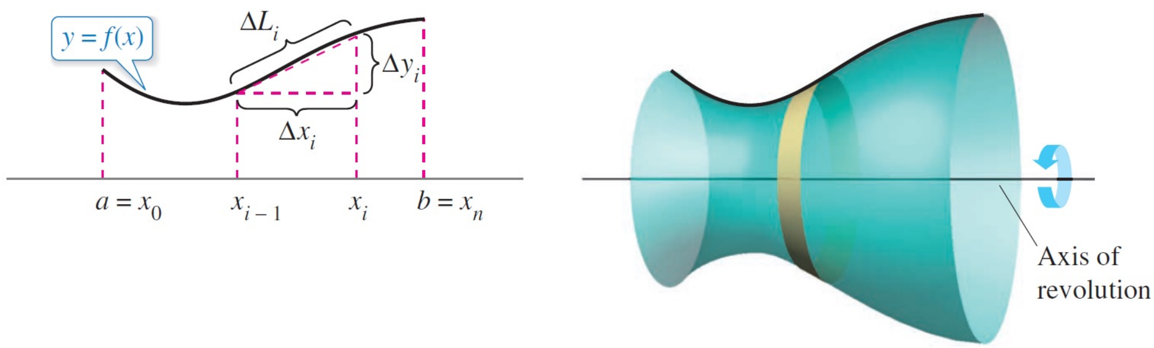

Figure 7

and it is a combination of frustums of circular cones. There exists a point x_i according to the intermediate value theorem on x-axis such that $f(x_i)$ is the average radius of the i_{th} frustum and the lateral surface area of the frustum, ΔS is $2\pi f(x_i)\Delta L$ which is the formula for the lateral surface area of a frustum of a circular cone. The total surface area of the surface of the revolution can be approximated by:

$$\begin{aligned} S &\approx 2\pi \sum_{i=1}^{i=n} f(x_i)\Delta L \\ &\approx 2\pi \sum_{i=1}^{i=n} f(x_i)\sqrt{\Delta x_i^2 + \Delta y_i^2} \\ &\approx 2\pi \sum_{i=1}^{i=n} y_i(t)\sqrt{\left(\frac{\Delta x_i}{\Delta t}\right)^2 + \left(\frac{\Delta y_i}{\Delta t}\right)^2}\Delta t \quad \text{[parameterizing } f \text{ in terms of } t\text{]} \\ &= \int_a^b 2\pi \cdot y(t)\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}dt \qquad [\text{as } n \to \infty] \end{aligned} \tag{7}$$

A continuous function that is differentiable and has a continuous derivative in the interval $[a,b]$ has a total lateral surface area shown in (7). The constraints required to formalize the lateral surface area of a surface of revolution in ACL2(r) are shown in the appendix B. We have shown $\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}$ is continuous in [4]. Using the product rule of continuous functions we prove $y(t) \cdot \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}$ is continuous. And then, we define the integral as the riemann sum of $y(t) \cdot \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}$ between the interval $[a,b]$ which is equal to $\int_a^b 2\pi \cdot y(t)\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}dt$.

Finally, to calculate surface area of a sphere we functionally instantiate (7). A sphere is a surface of revolution obtained by revolving a semicircle with radius r with equation $f(t) = (r\cos t, r\sin t)$, $0 \le t \le \pi$

about x-axis. So, the surface area of a sphere is:

$$\begin{aligned}
S &= \int_0^\pi 2\pi \cdot r\sin(t)\sqrt{\left(\frac{d}{dt}(r\cos(t))\right)^2 + \left(\frac{d}{dt}(r\sin(t))\right)^2}dt \\
&= \int_0^\pi 2\pi \cdot r\sin(t)\sqrt{(-r\sin(t))^2 + (r\cos(t))^2}dt \\
&= \int_0^\pi 2\pi \cdot r\sin(t)\cdot r\cdot dt
\end{aligned}$$

First we show $\frac{d}{dt}(-\cos(t)) \approx \sin(t)$ [15] and then using FTC-2:

$$\begin{aligned}
S &= 2\pi r^2(-cos(t))\Big|_0^\pi \\
&= 4\pi r^2
\end{aligned}$$

5 Conclusion

Integration was formalized in ACL2(r) as riemann sum of the function and then the fundamental theorem of calculus and chain rule were formalized. Integration by substitution is an useful method a lot of times if the integral is of the form $\int_{\psi(a)}^{\psi(b)} f(x)\,dx$. To extend the math library of proofs in ACL2(r) we formalize Integration by Substitution using fundamental theorem of calculus and chain rule. The theory of differentiation and integration were formalized in ACL2(r) by the way of non-standard analysis. We use this theory to prove volume of a sphere and surface area of a sphere. We proved, the volume of a sphere with radius r is $\frac{4}{3}\pi r^3$. We formalized the lateral surface area of a surface of revolution through which we proved the surface of a sphere with radius r is $4\pi r^2$.

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Appendix A Constraints for u-substitution in ACL2(r)

The constraints and the associated naming conventions are as follows:

- **f-prime-real**

```
(realp (f-prime x))
```

- **f-prime-continuous**

```
(implies (and (standardp x)
              (inside-interval-p x (fi-range))
              (i-close x x1)
              (inside-interval-p x1 (fi-range)))
         (i-close (f-prime x) (f-prime x1)))
```

- **fi-real**

```
(realp (fi x))
```

- **fi-prime-real**

```
(realp (fi-prime x))
```

- **fi-prime-is-derivative**

```
(implies (and (standardp x)
              (inside-interval-p x (f-o-fi-domain))
              (inside-interval-p x1 (f-o-fi-domain))
              (i-close x x1) (not (= x x1)))
         (i-close (/ (- (fi x) (fi x1)) (- x x1))
                  (fi-prime x)))
```

- **fi-prime-continuous**

```
(implies (and (standardp x)
              (inside-interval-p x (f-o-fi-domain))
              (i-close x x1)
              (inside-interval-p x1 (f-o-fi-domain)))
         (i-close (fi-prime x) (fi-prime x1)))
```

- **intervalp-fi-range**

```
(interval-p (fi-range))
```

- **fi-range-real**

```
(implies (inside-interval-p x (fi-range))
         (realp x))
```

- **fi-range-non-trivial**

```
(or (null (interval-left-endpoint (fi-range)))
    (null (interval-right-endpoint (fi-range)))
    (< (interval-left-endpoint (fi-range))
    (interval-right-endpoint (fi-range))))
```

- **fi-differentiable**

```
(implies (and (standardp x)
              (inside-interval-p x (f-o-fi-domain))
              (inside-interval-p y1 (f-o-fi-domain))
              (inside-interval-p y2 (f-o-fi-domain))
              (i-close x y1) (not (= x y1))
              (i-close x y2) (not (= x y2)))
         (and (i-limited (/ (- (fi x) (fi y1)) (- x y1)))
         (i-close (/ (- (fi x) (fi y1)) (- x y1))
                  (/ (- (fi x) (fi y2)) (- x y2)))))
```

- **intervalp-f-o-fi-domain**

```
(interval-p (f-o-fi-domain))
```

- **f-o-fi-domain-real**

```
(implies (inside-interval-p x (f-o-fi-domain))
         (realp x))
```

- **f-o-fi-domain-non-trivial**

```
(or (null (interval-left-endpoint (f-o-fi-domain)))
    (null (interval-right-endpoint (f-o-fi-domain)))
    (< (interval-left-endpoint (f-o-fi-domain))
    (interval-right-endpoint (f-o-fi-domain))))
```

- **fi-range-in-domain-of-f-o-fi**

```
(implies (inside-interval-p x (f-o-fi-domain))
         (inside-interval-p (fi x) (fi-range))))
```

Along with the above constraints we defined a constant, *consta* which is a standard number and inside the interval fi-range.

Appendix B Constraints for a continuously differentiable curve

Our witness function for the curve, c is an identity function and so the derivative of the curve, c-drivative is equal to 1 and the domain of the function, der-sum-sqrt-domain, is $[0,1]$. The constraints for the continuously differentiable curve are as follows:

- **c-acl2-numberp**

```
(implies (acl2-numberp x)
         (acl2-numberp (c x)))
```

- **der-sum-sqrt-domain-real**

```
(implies (inside-interval-p x (der-sum-sqrt-domain))
         (realp x))
```

- **der-sum-sqrt-domain-non-trivial**

```
(or (null (interval-left-endpoint (der-sum-sqrt-domain)))
    (null (interval-right-endpoint (der-sum-sqrt-domain)))
    (< (interval-left-endpoint (der-sum-sqrt-domain))
    (interval-right-endpoint (der-sum-sqrt-domain))))
```

- **intervalp-der-sqrt-domain**

```
(interval-p (der-sum-sqrt-domain))
```

- **c-differentiable**

```
(implies (and (standardp x)
              (inside-interval-p x (der-sum-sqrt-domain))
              (inside-interval-p y1 (der-sum-sqrt-domain))
              (inside-interval-p y2 (der-sum-sqrt-domain))
              (i-close x y1) (not (= x y1))
              (i-close x y2) (not (= x y2)))
         (and (i-limited (/ (- (c x) (c y1)) (- x y1)))
              (i-close (/ (- (c x) (c y1)) (- x y1))
                       (/ (- (c x) (c y2)) (- x y2)))))
```

- **c-der-lemma**

```
(implies (and (standardp x)
              (inside-interval-p x (der-sum-sqrt-domain))
              (inside-interval-p y (der-sum-sqrt-domain))
              (i-close x y)
              (not (= x y)))
         (i-close (/ (- (c x) (c y))
                     (- x y))
                  (c-derivative x)))
```

- **c-der-continuous**

```
(implies (and (standardp x)
              (inside-interval-p x (der-sum-sqrt-domain))
              (inside-interval-p y (der-sum-sqrt-domain))
              (i-close x y))
         (i-close (c-derivative x)
                  (c-derivative y)))
```