GROUP ACTIONS – HOMEWORK 1

Exercise 1. Let S^1 be the unit circle in the complex plane, which we can write as $\{e^{2\pi i\theta}: \ \theta \in [0,1]\}$. Consider a map $T : S^1 \to S^1$, which has the following properties:

(a) $T(e^{2\pi i\theta}) = e^{2\pi i f(\theta)}, \ \theta \in [0,1/2]$, where $f : [0,1/2] \to [0,1/2]$ is strictly increasing and continuous, $f(0) = 0$, $f(1/2) = 1/2$, and $f(\theta) > \theta$ for all $\theta \in (0,1/2)$ (draw a picture!)

(b) $T(\overline{z}) = \overline{T(z)}, \ \Im(z) < 0$.

Observe that T is a homeomorphism of S^1, which has two fixed points, 1 and -1.

(i) Prove that the only invariant probability measures for the action of T on S^1 are the convex combinations of Dirac point masses at the fixed points: $c\delta_1 + (1-c)\delta_{-1}$, $c \in [0,1]$.

(ii) Consider a rotation R_α of S^1, where α is irrational modulo π and consider the group $G = \langle T, R_\alpha \rangle \subset \mathrm{Homeo}(S^1)$ (equipped with the discrete metric). Prove that the action of G on X has no invariant measures, that is, $\mathcal{M}^G(S^1) = \emptyset$.

Exercise 2. (Adapted from internet sources.)

Consider two rotation matrices (with their inverses) in $\mathbb{R}^3$: ϕ is a rotation about the z-axis through the angle arccos(3/5), and ρ is rotation about the x-axis through the angle arccos(3/5). (So they are actually elements of the group $SO_3(\mathbb{Q})$.) The matrices are

$$\phi^{\pm 1} = \begin{pmatrix} 3/5 & \mp 4/5 & 0 \\ \pm 4/5 & 3/5 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \rho^{\pm 1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 3/5 & \mp 4/5 \\ 0 & \pm 4/5 & 3/5 \end{pmatrix}.$$

Prove that ϕ and ρ generate a free group.

Suggestion for a proof: we need to show that no reduced word in $\phi^{\pm 1}, \rho^{\pm 1}$ gives an identity. Suppose that such a word w exists. Conjugating, if necessary, we can assume that w ends with ϕ. One can prove by induction on the length of the word w the following

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claim: if $|w| = k$ (this denotes the length of the word), then $w\begin{pmatrix}1\\0\\0\end{pmatrix} = 5^{-k}\begin{pmatrix}a\\b\\c\end{pmatrix}$, where a, b, c are integers and b is not divisible by 5. This is a contradiction, because $w\begin{pmatrix}1\\0\\0\end{pmatrix} \neq \begin{pmatrix}1\\0\\0\end{pmatrix}$. The induction base $|w| = 1$ is clear, because then $w = \phi$. For induction step, consider the cases $w = \phi^{\pm 1}\rho^{\pm 1}v$, $w = \rho^{\pm 1}\phi^{\pm 1}v$, $w = \phi^{\pm 1}\phi^{\pm 1}v$, and $w = \rho^{\pm 1}\rho^{\pm 1}v$.

Exercise 3. Say that two sets A and B in $\mathbb{R}^3$ are *equidecomposable* (notation $A \sim B$) if there exist partitions into disjoint sets $A = \biguplus_{i=1}^n A_i$, $B = \biguplus_{i=1}^n B_i$ and $g_1, \ldots, g_n$, isometries of $\mathbb{R}^3$, such that $B_i = g_i \cdot A_i$. The basic Banach-Tarski Theorem says that a unit ball is equidecomposable with a union of two disjoint unit balls. Write $A \lesssim B$ if $A \sim C$ for some $C \subseteq B$. The Banach-Schröder-Bernstein Theorem says that if $A \lesssim B$ and $B \lesssim A$, then $A \sim B$. Using these results, prove the stronger version of Banach-Tarski, sometimes called the "Pea to Sun paradox": any two sets $A, B \subset \mathbb{R}^3$, bounded and with non-empty interior, are equidecomposable.

Suggestion for a proof: It is enough to show that any set A, as in the problem, is equidecomposable with a subset of a ball B. We can cover the set A by finitely many translates of B (with overlaps). Say we need n copies of B. Using the basic Banach-Tarski Theorem, we deduce that B is equidecomposable with n disjoint copies of B. Translate them into the covering and take the "pieces" of the balls so that they form a partition of the set A. We obtain that A is equidecomposable with a union of the pieces obtained by applying the inverses of the relevant isometries.