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A CONTINUOUS VERSION OF THE HAUSDORFF–BANACH–TARSKI PARADOX

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We come up with a simple proof for a continuous version of the Hausdorff–Banach–Tarski paradox, which does not make use of Robinson's method of compatible congruences and fits in the case of finite and countable paradoxical decompositions. It is proved that there exists a free subgroup whose rank is of the power of the continuum in a rotation group of a three-dimensional Euclidean space. We also argue that unbounded subsets of Euclidean space containing inner points are denumerably equipollent.

Hausdorff [1, pp. 5-10] proved that there is no finitely additive measure that is defined on all subsets of a Euclidean space $\mathbb{R}^3$ and is invariant under space motions. The main reason is the existence of nontrivial free products and free non-Abelian subgroups in a space rotation group. Banach and Tarski [2], developing Hausdorff's construction, demonstrated that every two bounded sets containing inner points are equipollent; that is, such sets can be decomposed into a finite and equal number of parts so that the parts of the two sets are mutually isometric under some mapping. In other words, a first set may be divided into a finite number of parts in a way that after the parts are moved in space and joined together we obtain a second set. In particular, any sphere and polyhedron are equipollent. If their measures are different, then all parts involved in partitioning are not measurable. This is properly the well-known Hausdorff–Banach–Tarski paradox.

Robinson [3] minimized the number of parts in a paradoxical decomposition of a sphere using the method of compatible congruencies. That method was widely used in subsequent research—in particular, in cases where continuous decompositions were allowed, but along with finite assemblies only. Mycielski [4], using Robinson's method and Sierpinski's construction [5] of a free subgroup of power-of-the-continuum rank in a rotation group of a three-dimensional Euclidean space, derived

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a continuous paradoxical decomposition of a sphere. A similar result was independently obtained by Dekker and de Groot [6]. A case study and further development of the theory of paradoxical decompositions are reproduced in [7].

In this paper, continuous paradoxical decompositions are derived without using Robinson's method. To be more precise, we prove that any ball in a three-dimensional Euclidean space may be reconstructed into a union of continuum many pairwise disjoint balls, each of which is isometric to the initial, by using a finite sequence of finite decomposition-based assemblies and a single continuous decomposition (see Thm. 2). The main difficulty—removing uncountable nonfree orbits of a free rotation group under the action on a sphere—is coped with purely algebraically. The proof is extended without changes to the case of countable or finite decompositions. Also we prove briefly that a rotation group of a three-dimensional Euclidean space contains a free subgroup whose rank has the power of the continuum, which cannot be included as a free factor into a larger free subgroup (Thm. 1). For completeness, we present corollaries on a paradoxical continuous decomposition of a space, a countable paradoxical decomposition of a sphere, and on decompositions in dimension n not less than 3. Lastly, it is shown that every two unbounded subsets of Euclidean space containing inner points are denumerably equipollent (Thm. 3). Our reasoning is closed modulo elementary facts in algebraic geometry and group theory.

THEOREM 1. A rotation group SO_3 of a three-dimensional Euclidean space contains a free subgroup of the power of the continuum, whose basis is maximal with respect to inclusion in the class of bases for free subgroups.

Proof. It is known that SO_3 contains a subgroup isomorphic to the free product $Z_3 * Z_2$. In fact, let

$$g = \frac{1}{2}\begin{pmatrix} 2 & 0 & 0 \\ 0 & -1 & -\sqrt{3} \\ 0 & \sqrt{3} & -1 \end{pmatrix}, \quad h = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

Then $g, h \in SO_3$ and $g^3 = h^2 = 1$. By induction on k, we show that $v_k = g^{\varepsilon_1} h g^{\varepsilon_2} h \dots g^{\varepsilon_k} h \neq 1$ for $k \geqslant 1$, with $\varepsilon_i = \pm 1$. It is easy to verify that

$$g^{\pm 1} h = \frac{1}{2}\begin{pmatrix} 0 & 2 & 0 \\ -1 & 0 & \pm\sqrt{3} \\ \pm\sqrt{3} & 0 & 1 \end{pmatrix} \equiv \frac{1}{2}\begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & \sqrt{3} \\ \sqrt{3} & 0 & 1 \end{pmatrix} \bmod 2.$$

Suppose

$$v_k \equiv \frac{1}{2^k}\begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & \sqrt{3} \\ \sqrt{3} & 0 & 1 \end{pmatrix} \bmod 2.$$

Then

$$v_{k+1} = v_k g^{\pm 1} h \equiv \frac{1}{2^k} \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & \sqrt{3} \\ \sqrt{3} & 0 & 1 \end{pmatrix} \cdot \frac{1}{2} \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & \sqrt{3} \\ \sqrt{3} & 0 & 1 \end{pmatrix}$$
$$\equiv \frac{1}{2^{k+1}} \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & \sqrt{3} \\ \sqrt{3} & 0 & 1 \end{pmatrix} \bmod 2,$$

and the last matrix is incomparable with a unit matrix modulo 2. Hence g and h generate the free product $Z_3 * Z_2$. It is not hard to verify that the commutator subgroup of $Z_3 * Z_2$ is the free group F_2 of rank 2 generated by commutators $a = [g, h] = ghg^{-1}h^{-1} = ghg^2h$ and $b = [g^2, h] = g^2hgh$. In turn, F_2 contains free subgroups F_ω of countable rank—for instance, a subgroup freely generated by elements a^kba^{-k}, $k \in \mathbb{Z}$. Since $SO_3 \simeq SU_2/Z_2$, SU_2 also contains a free subgroup of countable rank generated, for example, by preimages of a^kba^{-k}, $k \in \mathbb{Z}$.

We fix a denumerable set B of a free subgroup F_ω in SU_2, letting $\mathcal{X}$ be the set of bases for free subgroups of SU_2 containing B. A partial order on $\mathcal{X}$ is defined by an inclusion relation. Every ascending chain in $\mathcal{X}$ has a maximal element, which is the union of all bases in the chain, since the condition of being free for a basis is formulated for finite subsets. By Zorn's lemma, $\mathcal{X}$ contains a maximal element A. We claim that A is a basis having the power of the continuum.

Assume to the contrary that A is a countable basis containing the basis B. By maximality, the set $A \cup \{g\}$ does not form a basis of a free subgroup, for every element $g \neq 1$ of SU_2. Consequently, there exist elements $a_1, \ldots, a_n$ in A and a nontrivial noncancellable word w in variables $x_0, x_1, \ldots, x_n$ such that $w(g, a_1, \ldots, a_n) = 1$. Obviously, $w(a_0, a_1, \ldots, a_n) \neq 1$ for any $a_0 \in A$, with $a_0 \neq a_1$, …, $a_0 \neq a_n$. Such a_0 exist since the basis A is countable. Hence the equation $w(X, a_1, \ldots, a_n) = 1$ defines a proper algebraic submanifold $S(w)$ on the real sphere $SU_2 \simeq S^3$ containing an element g. Manifolds of type $S(w)$ cover the sphere S^3, and there are countably many such manifolds, since a set of words of type w is denumerable. We are led to a contradiction, for a sphere of dimension at least 1 cannot be covered by a denumerable set of proper algebraic manifolds.

Indeed, every algebraic manifold is a union of finitely many irreducible algebraic manifolds. Therefore, manifolds in a family $S(w, i)$ covering a sphere may be conceived of as proper irreducible, provided that countability of the family is preserved. Let $H(c)$ be a cross-section of a sphere S^3 by a plane $x_4 = c$, where $|c| < 1$. Clearly, $H(c)$ is a sphere of lesser dimension. Obviously, cross-sections $H(c)$ are pairwise disjoint and are irreducible (cf. remark below) algebraic submanifolds of codimension 1 in S^3. If $S(w, i) \supseteq H(c)$, then $S(w, i) = H(c)$, since our manifolds are irreducible and their dimensions are equal. Consequently, $S(w, i)$ cannot cover other cross-sections $H(c')$, $c' \neq c$. There are a continuum of such cross-sections $H(c)$, and the family $S(w, i)$ is countable. Therefore, there exists a cross-section $H(c)$ such that $S(w, i) \cap H(c) \subsetneq H(c)$ for all $S(w, i)$. Thus $H(c)$ is covered by a countable family of proper algebraic submanifolds $S(w, i) \cap H(c)$. Thereby the covered sphere diminishes in dimension. Proceeding further with this argument, we will arrive at a circumference covered by a countable family of proper irreducible algebraic manifolds, i.e., by a countable family of points, which leads us to a contradiction.

Consequently, A is a maximal basis of the power of the continuum in a free subgroup of SU_2. Obviously, its image in the factor group $SO_3 \simeq SU_2/Z_2$ possesses the same property. The theorem is proved.

Remark 1. A cross-section $H(c)$ is reduced since I is a prime ideal generated by polynomials $x_4 - c$ and $x_1^2 + \ldots + x_4^2 - 1$ in the algebra $\mathbb{R}[x_1, \ldots, x_4]$. In fact, suppose

$$f \cdot g \in I,\ f(X)g(X) = (x_4 - c)u(X) + (x_1^2 + \ldots + x_4^2 - 1)v(X).$$

Applying a specialization homomorphism $h(X) \mapsto h(X)|_{x_4=c}$, we arrive at the equality

$$\overline{f(X)} \cdot \overline{g(X)} = (x_1^2 + \ldots + x_3^2 + c^2 - 1)\overline{v(X)}.$$

A polynomial $p(X) = x_1^2 + \ldots + x_3^2 + c^2 - 1$ is indecomposable in $\mathbb{R}[x_1, \ldots, x_3]$; otherwise it is a product of two polynomials of degree 1 and the sphere contains a plane. Now $p \mid \overline{f}$ or $p \mid \overline{g}$ since factorization in polynomial algebra is unique. Suppose $\overline{f} = p\overline{w}$. Then $f(X) = (x_4 - c)h(X) + (x_1^2 + \ldots + x_4^2 - 1)w(X)$ for suitable $h, w \in \mathbb{R}[x_1, \ldots, x_4]$, i.e., $f \in I$.

Remark 2. Theorem 1 holds true for algebraic groups over a field of power continuum if those groups contain a free non-Abelian subgroup.

Recall that two subsets X and Y in a Euclidean space are said to be *equipollent* if they admit finite decompositions $X = \bigsqcup X_i$ and $Y = \bigsqcup Y_i$ for which there are motions (isometries) f_i on the Euclidean space such that $f_i(X_i) = Y_i$ for all i. A corresponding map $f : X \to Y$, $f|_{X_i} = f_i$, is called a *piecewise motion*. We also say that Y is obtained from X via a finite decomposition-based assembly, and write $X \sim Y$. Equipollence is an equivalence relation. Reflexivity and symmetry are obvious. Transitivity is obtained as follows. Let, in addition, finite decompositions $Y = \bigsqcup\limits_{j\in J} Y_j'$ and $Z = \bigsqcup\limits_{j\in J} Z_j$ and motions $g_j : Y_j' \to Z_j$ be given. Put $Y_{ij} = Y_i \cap Y_j'$, $X_{ij} = f_i^{-1}(Y_{ij})$, and $Z_{ij} = g_j(Y_{ij})$. Then $X = \bigsqcup\limits_{i\in I} \bigsqcup\limits_{j\in J} X_{ij}$ and $Z = \bigsqcup\limits_{i\in I} \bigsqcup\limits_{j\in J} Z_{ij}$ are finite decompositions, and $Z_{ij} = g_j(f_i(X_{ij}))$, where $g_j f_i$ is a motion. Here, of course, we can remove empty intersections $Y_{ij} = Y_i \cap Y_j'$.

If we admit finite decompositions we also obtain an equivalence relation (denumerable equipollence $X \sim_\omega Y$). If, however, continuous decompositions are allowed then sets of power continuum may be scattered into points, and so each such set can be obtained from another via a continuous decomposition and a continuous assembly. What will be produced if any decompositions but only finite assemblies are allowed?

THEOREM 2. Every ball in a three-dimensional Euclidean space may be reconstructed into a union of continuum many pairwise disjoint balls, each of which is isometric to the initial, by using a finite sequence of finite decomposition-based assemblies and a single continuous decomposition.

Proof of Theorem 2. We rely on Theorem 1 and the following well-known lemma.

LEMMA 1. Let X be a set in Euclidean space, h a motion, M a subset of X, $h^k M \subseteq X$, and $h^k M \cap M = \varnothing$ for all natural k. Then X and $X - M$ are equipollent.

Proof of Lemma 1. Obviously, $h^k M \cap h^l M = \varnothing$ for $k > l$. Let $Z = \bigcup_{k \geqslant 0} h^k M$. Then $X = Z \bigsqcup (X - Z) \sim hZ \bigsqcup (X - Z) = X - M$.

We divide the proof of Theorem 2 into several stages.

Stage 1. Removal of center.

It is stated that a ball with removed center is equipollent to an initial ball. In fact, let a line L be half a ball radius apart from the center o of the ball X and h be a space rotation around the line L by an angle incommensurable with π, i.e., distinct from $2k\pi/n$, where $k \in \mathbb{Z}$ and $n \in \mathbb{N}$. Then $h^k o \in X$, $h^k o \neq o$, for $k > 0$. By Lemma 1, the ball X and the punctured ball $X - \{o\}$ are equipollent.

Stage 2. We need only prove Theorem 2 for a sphere and rotation motions around axes passing through the center of the sphere.

Indeed, decompositions of a ball's surface are radially projected into the interior of a center-free ball. Rotations of a ball's surface induce consistent rotations of a center-free ball.

By Theorem 1, the rotation group SO_3 of the sphere S contains a free subgroup G with basis g_i, $i \in I$, having the power of the continuum. If we separate one free generator in advance we may fix a rotation h of infinite order such that a group generated by G and h is the free product $G * \langle h \rangle$; in particular, h does not commute with nontrivial elements of the group G.

Stage 3. Decomposition into free and nonfree orbits.

A sphere S is decomposed into orbits under the action of a group G. The orbit

$$X = \{gx \mid g \in G\}$$

of an element $x \in S$ is said to be *free* if it does not contain fixed points with respect to nonidentity elements of the group G. Other orbits are said to be *nonfree.*

Note that every point of a nonfree orbit is fixed with respect to a suitable nonidentity element of the group: if $f \in G$, $f \neq 1$, and $fx = x$ then $gfg^{-1} \in G$, $gfg^{-1} \neq 1$, and $(gfg^{-1})(gx) = gx$. Of course, nonfree orbits exist. It suffices to take orbits of intersection points between the sphere and axes of nontrivial rotations in G.

On the other hand, free orbits exist as well. Otherwise, for any $x \in S$ there is $g \in G$ such that $g \neq 1$ and $gx = x$. If x is a fixed point in the initially selected rotation h, then g and h are rotations around a common fixed axis, and $gh = hg$. We have arrived at a contradiction with the fact that a group generated by G and h is a free product; i.e., $\langle G, h \rangle = G * \langle h \rangle$.

Stage 4. Removal of nonfree orbits.

Let N be a union of all nonfree orbits. It is stated that $h^k N \cap N = \varnothing$ for $k > 0$. Assume the contrary. Then there exists a point $x \in N$ such that $h^k x = y \in N$. The orbits in N are not free, and so there are $f, g \in G$ such that $f \neq 1$, $g \neq 1$, $fx = x$, and $gy = y$. Hence

$$gh^k x = h^k x,\ h^{-k} g h^k x = x,\ f \cdot h^{-k} g h^k = h^{-k} g h^k \cdot f,$$

since the rotations f and $h^{-k} g h^k$ have a common fixed point x. The last equality clashes with the uniqueness of a reduced normal form for the free product $G * \langle h \rangle$.

Since $h^k N \subseteq S$ for $k > 0$, both of the conditions of Lemma 1 are satisfied. Hence S and $S - N$ are equipollent.

Stage 5. Simultaneous removal of representatives of free orbits.

Let M be a union of representatives of all free orbits. Fix a nonidentity element g of a group G. It is stated that the conditions of Lemma 1 are met: i.e.,

$$g^k M \subset S - N,\ g^k M \cap M = \varnothing \text{ for all } k > 0.$$

The first claim is obvious. The second claim follows from the fact that the action on orbits in $S - N$ is free. This, combined with Lemma 1, implies that $S - N$ and $S - (N \cup M)$ are equipollent.

Stage 6. Paradoxical decomposition of a punctured free orbit.

Let $X = \{gx \mid g \in G\}$ be a free orbit. A group G is free with a basis g_i, $i \in I$, of the power of the continuum, every element of G is representable as a unique finite noncancellable word written in the generators g_i, $i \in I$. Let

$$X_i^+ = \{g_i w x \mid \text{the word } g_i w \text{ is noncancellable}\},$$
$$X_i^- = \{g_i^{-1} w x \mid \text{the word } g_i^{-1} w \text{ is noncancellable}\}.$$

Then

$$X = \bigsqcup_{i \in I} (X_i^+ \bigsqcup X_i^-) \bigsqcup \{x\}.$$

Clearly,

$$g_i X_i^- \bigsqcup X_i^+ = X \text{ for all } i \in I.$$

Hence the punctured orbit $X - \{x\}$ admits a continuous decomposition, from which a continuum of orbits X may be obtained by applying pair assemblies. Having realized such a decomposition in each punctured free orbit with respect to elements of the basis g_i, $i \in I$, for G, we may assert that the union $S - (N \cup M)$ of all punctured free orbits admits a continuous decomposition to which pair assemblies can be applied to produce a continuum of sets $S - N$, which are equipollent to the entire sphere S in view of Stage 4. The theorem is proved.

COROLLARY 1. A three-dimensional Euclidean space can be reconstructed into a union of continuum many pairwise disjoint three-dimensional Euclidean spaces by applying a finite sequence of finite decomposition-based assemblies and a single continuous decomposition.

Proof. The required statement follows from the argument used in proving Theorem 2 for a ball once we have noted that a punctured space is a union of concentric spheres.

COROLLARY 2. A ball in Euclidean space is denumerably equipollent to a disjoint union of a denumerable set of balls isometric to the initial.

The proof of Theorem 2 presented above is also true for countable or only finite decompositions, if as the group G we choose a free subgroup of countable or finite rank.

COROLLARY 3. An analog of Theorem 2, and also of Corollaries 1 and 2, is valid in Euclidean spaces of dimension at least 3.

Proof. It suffices to prove the statement for spheres S^n using induction on $n \geqslant 2$. In view of Lemma 1, the sphere S^n is equipollent to a twice-punctured sphere $S^n - \{x, -x\}$, if we exercise rotations around an axis orthogonal to the vector x through an angle incommensurable with π. The twice-punctured sphere is partitioned into a union of flat cross-sections orthogonal to the vector x, which are isometric to spheres of dimension $n-1$. The paradoxical decomposition of an equatorial cross-section S^{n-1} can be extended "along meridians" to other cross-sections $S^n - \{x, -x\}$, and hence also to S^n. The motions of S^{n-1} are extended to the sphere S^n with preservation of the specified flat cross-sections.

THEOREM 3. Every two unbounded subsets of Euclidean space possessing inner points are denumerably equipollent.

The **proof** is similar to a known proof of the version of the theorem for the case of bounded subsets of Euclidean space possessing inner points. Furthermore, we need the following:

LEMMA 2 (Banach–Schroeder–Bernstein theorem). If a set X is equipollent to a subset Y' of Y, and inversely, if a set Y is equipollent to a subset X' of X, then X and Y are equipollent.

Proof. Let $f : X \to Y'$ and $g : Y \to X'$ be piecewise motions realizing an equivalence. We may assume that X' and Y' are proper subsets of X and Y, respectively. Let $Z_0 = X - X'$ and $Z_{k+1} = gf(Z_k)$ for $k \geqslant 0$. Then $g^{-1}(Z_{k+1}) = f(Z_k)$. Let $Z = \bigcup\limits_{k\geqslant 0} Z_k$. Obviously, $Z \sim f(Z)$. On the other hand, $X - Z \subset X'$ and

$$g^{-1}(X-Z) = g^{-1}\left(X' - \bigcup_{k\geqslant 0} Z_{k+1}\right) = g^{-1}X' - \bigcup_{k\geqslant 0} g^{-1}(Z_{k+1})$$
$$= Y - \bigcup_{k\geqslant 0} f(Z_k) = Y - f(Z).$$

Consequently,

$$X = Z \bigsqcup (X-Z) \sim f(Z) \bigsqcup g^{-1}(X-Z) = f(Z) \bigsqcup (Y - f(Z)) = Y.$$

The lemma is proved.

Now let X and Y be two unbounded subsets of Euclidean space having inner points. Choosing an appropriate radius, we may assume that X and Y contain isometric balls B and B', respectively. The set X can be covered by a union of countably many balls B_i isometric to the ball B. Hence

$$B \subseteq X \subseteq \bigcup B_i \preccurlyeq_\omega \bigsqcup B_i \sim_\omega B.$$

Here the last denumerable equipollence follows from Corollary 2 given after the proof of Theorem 2, and the last but one embedding is a consequence of partitioning $\bigcup B_i$ into subsets contained in the balls B_i. Applying Lemma 2, we see that B and X are denumerably equipollent. Similarly, B' and Y have the same property. Hence X and Y, too, are denumerably equipollent. Theorem 3 is proved.

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