Integration by substitution in ACL2(r)

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November 6, 2019

Abstract

Integration by substitution also known as u-substitution is a method that is often used in calculus to find the integral of a function. We show a proof of u-substitution in ACL2(r) using fundamental theorem of calculus and chain rule which are already proved in ACL2(r). We use the proof to show the area of a circle with radius r is πr^2.

1 Introduction

Continuity, differentiablity and proofs of theorems such as intermediate value and mean value were presented in [1] using non-standard-analysis. Riemann integral of a continuous function was formalized and a proof of a version of the Fundamental Theorem of Calculus (FTC) presented in [2]. The version of the FTC presented is sometimes called the First Fundamental Theorem of Calculus (FTC-1). Ruben Gamboa and John Cowles redid this proof and generalized the result to what is sometimes called the Second Fundamental Theorem of Calculus (FTC-2) [3]. The Chain rule and other algebraic differential rules were presented in [4]. We use FTC and chain rule to prove the u-substitution in ACL2(r). We use this proof to show the area of a circle with radius r is πr^2.

If f is continuous on I and ψ is defined on an interval $[a, b]$ whose range is in the interval I and if ψ has a continuous derivative then we show that $\int_{\psi(a)}^{\psi(b)} f(x)\,dx = \int_a^b f(\psi(u))\psi'(u)\,du$ which is called as u-substitution rule in ACL2(r). In section 2 we show the mathematical derivation of u-substitution rule and the method is applied to find the area of a circle. There are several ways to prove the area of a circle but we show a proof that uses the u-substitution rule. In section 3 we show a mechanized proof of the u-substitution using ACL2(r) theorem prover and in section 4 we use the u-substitution rule presented in section 3 to show the area of a circle with radius r is πr^2.

2 Integration by substitution

Proposition: Let $I \subseteq R$ be an interval and $\psi : [a, b] \to I$ be a differentiable function with continuous derivative, ψ'. Suppose that $f : I \to R$ is a continuous function [5]. Then,

$$\int_{\psi(a)}^{\psi(b)} f(x)\,dx = \int_a^b f(\psi(u))\psi'(u)\,du$$

2.1 Derivation of the u-substitution rule

Since f and ψ are continuous the composite function $f(\psi(u))$ is continuous and since derivative of ψ is continuous, by product rule of two continuous functions, $f(\psi(u))\psi'(u)$ is continuous. Hence the integral of $f(\psi(u))\,\psi'(u)$ exists. Since f is continuous, there exists an antiderivative of f and let that be F. Using the chain rule:

$$(F \circ \psi)'(u) \;=F'(\psi(u))\psi'(u) \;= f(\psi(u))\psi'(u) \tag{2.1.1}$$

Applying chain-rule and fundamental theorem of calculus to derive the u-substitution rule:

$$\begin{aligned}
\int_a^b f(\psi(u))\psi'(u)\,du &= \int_a^b (F\circ\psi)'(u)\,du \;\;[\;(2.1.1)\;] \\
&= (F\circ\psi)(b) \;-\; (F\circ\psi)(a) \;\;[\text{ FTC-2 }] \\
&= F(\psi(b)) \;-\; F(\psi(a)) \;\;[\text{ Definition of } F\circ\psi \;] \\
&= \int_{\psi(a)}^{\psi(b)} f(x)\,dx \;\;[\text{ FTC-2 }]
\end{aligned} \tag{2.1.2}$$

2.2 Deriving the Area of a Circle using u-substitution

The equation of the circle with radius r centered at $(0,0)$ is $y = \sqrt{r^2 - x^2}$. Area of the circle is $4 * \int_0^r \sqrt{r^2-x^2}\,dx$ where $\int_0^r \sqrt{r^2-x^2}\,dx$ is the quarter circle area. If we consider $f(x) = \sqrt{r^2-x^2}$, $\psi(x) = r\sin x$, $a = 0$, $b = \pi/2$ then, by using the u-substitution rule:

$$\begin{aligned}
4 * \int_0^r \sqrt{r^2-x^2}\,dx &= 4 * \int_{\psi(a)}^{\psi(b)} f(x)\,dx \\
&= 4 * \int_a^b f(\psi(u))\psi'(u)\,du \;\;[\text{ u-substitution rule}] \\
&= 4 * \int_0^{\pi/2} \sqrt{r^2-(r\sin u)^2} * r\cos u\,du \\
&= 4 * \int_0^{\pi/2} r^2\cos^2 u\,du \\
&= 4 * (r^2) * \Big(\frac{sin(2x)}{4} + \frac{x}{2}\Big)\Big|_0^{\pi/2} \\
&= \pi r^2
\end{aligned}$$

3 Proof of the u-substitution rule in ACL2(r)

Going by the proposition in section 2 we defined two functions f-prime (f in the proposition) and fi (ψ in the proposition). The two domains for the two functions respectively are fi-range and f-o-fi-domain. Derivative of the fi defined is fi-prime (ψ' in the proposition).

Defined Macros are used to generate the theorems that are used as constraints for the substitution rule. The Macros and the associated naming conventions are as follows:

- **f-prime-real**

```
(realp (f-prime x))
```

- **f-prime-continuous**

```
(implies (and (standardp x)
              (inside-interval-p x (fi-range))
              (i-close x x1)
              (inside-interval-p x1 (fi-range)))
    (i-close (f-prime x)
             (f-prime x1)))
```

- **fi-real**

```
(realp (fi x))
```

- **fi-prime-real**

```
(realp (fi-prime x))
```

- **fi-prime-is-derivative**

```
(implies (and (standardp x)
              (inside-interval-p x (f-o-fi-domain))
              (inside-interval-p x1 (f-o-fi-domain))
              (i-close x x1) (not (= x x1)))
    (i-close (/ (- (fi x) (fi x1)) (- x x1))
             (fi-prime x)))
```

- **fi-prime-continuous**

```
(implies (and (standardp x)
              (inside-interval-p x (f-o-fi-domain))
              (i-close x x1)
              (inside-interval-p x1 (f-o-fi-domain)))
    (i-close (fi-prime x)
             (fi-prime x1))))
```

- **intervalp-fi-range**

```
(interval-p (fi-range))
```

- **fi-range-real**

```
(implies (inside-interval-p x (fi-range))
         (realp x))
```

- **fi-range-non-trivial**

```
(or (null (interval-left-endpoint (fi-range)))
    (null (interval-right-endpoint (fi-range)))
    (< (interval-left-endpoint (fi-range))
    (interval-right-endpoint (fi-range))))
```

- **fi-differentiable**

```
(implies (and (standardp x)
              (inside-interval-p x (f-o-fi-domain))
              (inside-interval-p y1 (f-o-fi-domain))
              (inside-interval-p y2 (f-o-fi-domain))
              (i-close x y1) (not (= x y1))
              (i-close x y2) (not (= x y2)))
     (and (i-limited (/ (- (fi x) (fi y1)) (- x y1)))
              (i-close (/ (- (fi x) (fi y1)) (- x y1))
                       (/ (- (fi x) (fi y2)) (- x y2)))))
```

- **intervalp-f-o-fi-domain**

```
(interval-p (f-o-fi-domain))
```

- **f-o-fi-domain-real**

```
(implies (inside-interval-p x (f-o-fi-domain))
              (realp x))
```

- **f-o-fi-domain-non-trivial**

```
(or (null (interval-left-endpoint (f-o-fi-domain)))
    (null (interval-right-endpoint (f-o-fi-domain)))
    (< (interval-left-endpoint (f-o-fi-domain))
    (interval-right-endpoint (f-o-fi-domain))))
```

- **fi-range-in-domain-of-f-o-fi**

```
(implies (inside-interval-p x (f-o-fi-domain))
              (inside-interval-p (fi x) (fi-range))))
```

Along with the above theorems we defined a constant, *consta* which is a standard number and inside the interval fi-range. The integral of f-prime, int-f-prime-1, can be defined as shown in [2]. We use the proof of the first fundamental theorem of calculus to show that the derivative of the integral of f-prime is equal to f-prime:

- **ftc-1-fprime**

```
(implies (and (inside-interval-p a (fi-range))
                   (inside-interval-p b (fi-range))
                   (inside-interval-p c (fi-range))
```

```
            (standardp b)
            (i-close b c)
            (not (equal b c)))
       (i-close (/ (- (int-f-prime-1 a b) (int-f-prime-1 a c
          ))
                   (- b c))
                (f-prime b)))
```

Antiderivative of f-prime, f (F in (2.1.1)) is then defined as below and then we show that f-prime is the derivative of f :

```
(defun f(x) (int-f-prime-1 (consta) x))
```

- **f-prime-is-derivative**

```
(implies    (and   (standardp x)
                   (inside-interval-p x (fi-range))
                   (inside-interval-p x1 (fi-range))
                   (i-close x x1) (not (= x x1)))
            (i-close (/ (- (f x) (f x1)) (- x x1))
                            (f-prime x)))
```

The next lemma we need is the proof of the chain rule (2.1.1). We define the composite function $F \circ \psi$ as f-o-fi, derivative-cr-f-o-fi equal to ((f-prime (fi x)) $*$ (fi-prime x)) and differential-cr-f-o-fi is the derivative of f-o-fi. Since f and fi are differentiable, we apply the the proof of the chain rule proof presented in [4] to prove the lemma.

- **expand-differential-cr-f-o-fi-1**

```
(implies    (and    (inside-interval-p x (f-o-fi-domain))
                    (inside-interval-p x1 (f-o-fi-domain)))
            (equal  (differential-cr-f-o-fi x1 (- x x1))
                    (/ (- (f-o-fi x) (f-o-fi x1)) (- x x1)))))
```

- **differential-cr-f-o-fi-close-1**

```
(implies (and   (standardp x)
                (inside-interval-p x (F-o-fi-domain))
                (inside-interval-p x1 (F-o-fi-domain))
                (i-close x x1) (not (= x x1)))
     (equal (standard-part (differential-cr-f-o-fi x (- x1 x)))
                (derivative-cr-f-o-fi x)))
```

The next step is proving the integral of derivative of f-o-fi between two end points a and b is equal to f-o-fi(b) - f-o-fi(a). f-o-fi-prime is the derivative of f-o-fi. Since f-prime, fi and fi-prime are continuous, f-prime(fi x) is continuous by the composition rule of two continuous functions and f-prime(fi x)$*$(fi x) is continuous by the product rule of two continuous functions. Since f-o-fi-prime is continuous and f-o-fi-prime is the derivative of f-o-fi we use the proof of FTC-2 to show that the integral of f-o-fi-prime is equal to f-o-fi(b) - f-o-fi(a):

```
(defun F-o-fi-prime (x)
  (if (inside-interval-p x (f-o-fi-domain))
      (derivative-cr-f-o-fi x)
    0))
```

- **ftc2-usub-equal**

```
(implies (and (inside-interval-p a (f-o-fi-domain))
              (inside-interval-p b (f-o-fi-domain)))
         (equal (int-f-o-fi-prime a b)
                (- (f (fi b)) (f (fi a)))))
```

The final step is to prove that the integral of f-prime with (fi a) and (fi b) being two end points of the integral equal to f-o-fi((fi b)) - f-o-fi((fi a)). Since f-prime is the derivative of f and is continuous, we apply FTC-2:

- **ftc2-usub-1-equal**

```
(implies (and (inside-interval-p a (f-o-fi-domain))
              (inside-interval-p b (f-o-fi-domain)))
         (equal (int-f-prime (fi a) (fi b))
                (- (f (fi b)) (f (fi a)))))
```

Finally we get the u-substitution rule using the lemmas ftc2-usub-equal and ftc2-usub-1-equal:

```
(defthm usubstitution-f-o-fi
  (implies (and (inside-interval-p a (f-o-fi-domain))
                (inside-interval-p b (f-o-fi-domain)))
           (equal (int-f-prime (fi a) (fi b))
                  (int-f-o-fi-prime a b)))
  :hints (("Goal"
           :use ((:instance ftc2-usub-1-equal)
                 (:instance ftc2-usub-equal))
           :in-theory (disable int-f-o-fi-prime int-f-prime))))
```

4 Proof of the Area of a Circle in ACL2(r)

If the circle, f is centered at $(0,0)$ and if the radius of the circle is r, then the circle can be defined as $\sqrt{(r)^2 - x^2}$. The domain of the circle is $[0, r]$. The substitution function, ψ is $r\sin(x)$ and the domain of the function is $[0, \frac{\pi}{2}]$. The derivative of the substitution function, ψ' is $r\cos(x)$.

We first prove the constraints required to use the u-substitution rule for the defined functions. Since sine and cosine are continuous as proved in [6] ψ and ψ' are continuous. Then we prove ψ' is the derivative of ψ. The next lemma required is f is continuous. Square of a function is continuous and square root of a continuous function is continuous. So we get $\sqrt{(r)^2 - x^2}$ is continuous. Now we have proved all the constraints required to use the u-substition rule proved in section 3 and we get:

- **usubstitution-circle**

```
(implies (and (inside-interval-p a (fi-domain))
              (inside-interval-p b (fi-domain)))
         (equal (int-circle (sub-func a) (sub-func b))
                (int-circle-sub-prime a b)))
```

In the above proof sub-func is $r\sin(x)$, int-circle is the integral of $\sqrt{(r)^2 - x^2}$, int-circle-sub-prime is the integral of $f(\psi(x))\psi'(x)$ and fi-domain is the interval $[0, \frac{\pi}{2}]$.

$$\begin{aligned} f(\psi(x))\psi'(x) &= \sqrt{r^2 - (r\sin x)^2} * r\cos x \\ &= r^2\cos^2 x \end{aligned}$$

The area of the circle is equal to the 4 times the integral of $r^2\cos^2(x)$ between 0 to $\pi/2$. First we apply the chain rule to show that derivative of $sin(2x)$ equal to $2cos(2x)$. Then we show the derivative of $(r^2)\Big(\frac{sin(2x)}{4} + \frac{x}{2}\Big)$ equal to r^2cos^2x.

$$\begin{aligned} \frac{d}{dx}(r^2)\Big(\frac{sin(2x)}{4} + \frac{x}{2}\Big) &= r^2\Big(\frac{2cos(2x)}{4} + \frac{1}{2}\Big) \\ &= r^2\Big(\frac{2(2cos^2x - 1)}{4} + \frac{1}{2}\Big) \\ &= r^2cos^2x \end{aligned}$$

Finally we apply FTC-2:

$$\begin{aligned} 4 * \int_0^{\pi/2} r^2\cos^2 x\, du &= 4 * (r^2) * \Big(\frac{sin(2x)}{4} + \frac{x}{2}\Big)\Big|_0^{\pi/2} \\ &= \pi r^2 \end{aligned}$$

5 Conclusions

The paper presents the proof for the integration by substitution rule and we apply the proof to prove the area of a circle with radius r is equal to πr^2. The substitution rule presented in this paper can be used to find an integral of any function if the integral of the function can be calculated by applying the u-substitution rule.

References

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[5] Integration by substitution. Integration by substitution — Wikipedia, the free encyclopedia, 2019. [Online; accessed 06-November-2019].

[6] Ruben Antonio Gamboa. *Mechanically Verifying Real-valued Algorithms in Acl2*. PhD thesis, 1999. AAI9948349.