Banach-Tarski Theorem

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Outline

Statement

A solid ball may be separated into a finite number of pieces and reassembled in such a way as to create two solid balls, each identical in shape and volume to the original.

Group Theory

A group is a set G and a binary operator $*$ such that:

1. **Closure**: G is closed under $*$; i.e., if a, b $\in$ G, then a $*$ b $\in$ G
2. **Identity**: There exists an identity element e $\in$ G; i.e., for all a $\in$ G we have a $*$ e = e $*$ a = a
3. **Inverse**: Every element a $\in$ G has an inverse in G; i.e., for all a $\in$ G, there exists an element a0 $\in$ G such that a $*$ a0 = a0 $*$ a = e.
4. **Associativity**: The operator $*$ acts associatively; i.e., for all a, b, c $\in$ G, a $*$ (b $*$ c)=(a $*$ b) $*$ c

Matrices

Let α and β be two matrices. Then,

1. $det(\alpha \cdot \beta) = det(\alpha) \cdot det(\beta)$
2. If α and β are invertible, then $\alpha \cdot \beta$ is invertible and $(\alpha \cdot \beta)^{-1} = \beta^{-1} \cdot \alpha^{-1}$
3. $(\alpha \cdot \beta)^{T} = \beta^{T} \cdot \alpha^{T}$

Rotations

1. **Orthogonal Matrix**: is a square matrix whose transpose equals to its inverse. $A^T = A^{-1}$
2. **Identity Matrix**: $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$
3. **Rotation** : A rotation is an orthogonal matrix whose determinant equals one.

$$p(x) = \begin{bmatrix} p_{11} & p_{12} & p_{13} \\ p_{21} & p_{22} & p_{23} \\ p_{31} & p_{32} & p_{33} \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

Rotations Group

Theorem : Set of rotations form a group.

Proof: If α and β are two rotations:

Closure: Product of any two rotations, $\alpha \cdot \beta$ is still a rotation

$$det(\alpha \cdot \beta) = det(\alpha) \cdot det(\beta) = 1 \cdot 1 = 1$$
$$(\alpha \cdot \beta)^{-1} = \beta^{-1} \cdot \alpha^{-1} = \beta^T \cdot \alpha^T = (\alpha \cdot \beta)^T$$

Identity: Identity is a rotation

$$det(I) = 1$$

$$I^T = I^{-1}$$

Rotations Group (continued)

Inverse: Inverse of a rotation is still a rotation

Let a be a rotation. Then, $a^T = a^{-1}$ and $\det(a) = 1$:

$$\text{orthogonality} : det(a^{-1}) = det(a) = 1$$

$$a = (a^T)^T = (a^{-1})^T = (a^{-1})^{-1}$$

Associativity: Rotations satisfy associativity

Free Group of Rotations

Definition : Free group is a group formed with subset of rotations with infinite order and only having the following relation:

$$\alpha\alpha^{-1} = \alpha^{-1}\alpha = \beta\beta^{-1} = \beta^{-1}\beta = I$$

Free Group Example

Theorem : $\alpha = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{3} & -\frac{2\sqrt{2}}{3} \\ 0 & \frac{2\sqrt{2}}{3} & \frac{1}{3} \end{bmatrix}$ and $\beta = \begin{bmatrix} \frac{1}{3} & -\frac{2\sqrt{2}}{3} & 0 \\ \frac{2\sqrt{2}}{3} & \frac{1}{3} & 0 \\ 0 & 0 & 1 \end{bmatrix}$
generate a free group.

Lemmas required:

1. Group properties
2. If a rotation, w, reduced word of length n formed by α, α^{-1}, β or β^{-1} then, $w(1,0,0)^T$ is of the form $\frac{1}{3^n}(a\sqrt{2}, b, c\sqrt{2})^T$ for integers a, b, c.
3. b is not divisible by 3
$\implies w(1,0,0)^T \neq (1,0,0)^T \implies w \neq I$
4. If $w(\alpha)$ represents a rotation that end with α and if $w \neq I$ then $w(\alpha)$, $w(\alpha^{-1})$, $w(\beta)$ and $w(\beta^{-1})$ represent different rotations.