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The length of a curve can be defined as the limit of the sum of the chord lengths of infinitesimal curve segments. When this limit exists, the curve is called rectifiable. Using the Theorem Prover ACL2(r), we show that when a complex-valued curve is continuously differentiable, the curve is also rectifiable, and the length of the curve can be computed by taking the integral of the norm of the curve's derivative. We use this result to verify the well-known formula for the circumference of a circle using the second fundamental theorem of calculus. This also serves to show that the constant π in ACL2(r), which is defined algebraically, is actually the same constant π known geometrically to the Greeks.

A MECHANIZED PROOF OF THE CURVE LENGTH OF A RECTIFIABLE CURVE

by

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I dedicate this to all of those who inspired, supported and trusted me.

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Definitions

acl2-numberp :

This is recognizer for numbers. (acl2-numberp x) is true if and only if x is a number, i.e., a rational or complex rational number

defun :

defun is used to define a function symbol

standardp :

ACL2(r) recognizer for standard objects. (Standardp x) is true if and only if x is a "standard" object. This notion of "standard" comes from non-standard analysis and is discussed in Ruben Gamboa's dissertation. In brief, all the familiar objects are standard: e.g., the familiar real numbers are standard, but non-zero infinitesimals are not standard, and the familiar integers are standard, but not those that exceed every integer that you can express in the usual way

defun-std :

The function introduced is accepted only if the body can be shown to produce standard values when all the arguments to the function are standard

defthm :

prove and name a theorem

i-close :

ACL2(r) test for whether two numbers are infinitesimally close. (I-close x y) is true if and only if x-y is an infinitesimal number. This predicate is only defined in ACL2(r)

i-limited :

ACL2(r) recognizer for limited numbers. (I-limited x) is true if and only if x is a number that is not infinitely large.

standard-part :

ACL2(r) function mapping limited numbers to standard numbers. (Standard-part x) is, for a given i-limited number x, the unique real number infinitesimally close (see i-close) to x

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Chapter 1

Introduction

1.1 Motivation

ACL2 is an interactive system in which you can model digital artifacts and guide the system to mathematical proofs about the behavior of those models. It has been used at such places as AMD, Centaur, IBM, and Rockwell Collins to verify interesting properties of commercial designs. It has been used to verify properties of models of microprocessors, microcode, the Sun Java Virtual Machine, operating system kernels, other verifiers, and interesting algorithms [2].

ACL2 supports rational numbers but not real numbers. ACL2(r), a variant of ACL2 supports the real numbers by way of non-standard analysis [3]. Our work shows how ACL2(r) theorem prover can be used prove a continuously differentiable curve is rectifiable. Then we applied the proof to prove that the length of the circle with radius r is equal to $2\pi r$ and thus we want to show that the definition of π that we used in ACL2(r) is equivalent to the π known to the greeks, circumference of the circle over diameter of the circle [4]. We prove circumference of a circle equal to $2\pi r$ and thus π is circumference of a circle divided by diameter of the circle.

1.2 Previous Research

Non-standard analysis in ACL2 was first introduced in [5], and it has been used to verify basic properties of the reals, such as the Fundamental Theorem of Calculus [6]. Non-standard analysis was introduced by Abraham Robinson in about 1960 in order to make rigorous Leibniz's approach to calculus from 17th century [7]. Its main idea is to extend the real number line by adding non-zero infinitesimals, numbers that are less in absolute value than every positive real number.

Continuity and differentiablity including theorems such as intermediate value theorems and mean value theorems were formalized by Ruben Gamboa in [8] using non-standard-analysis. A modular, top-down methodology was presented and Riemann integral of a continuous function was formalized using non-standard-analysis in [6]. The theory of integration in ACL2(r) was first developed in [6], which describes a proof of a version of the Fundamental Theorem of Calculus (FTC). The version of the FTC presented there is sometimes called the First Fundamental Theorem of Calculus. Ruben Gamboa and John Cowles redid this proof, and generalized the result to what is sometimes called the Second Fundamental Theorem of Calculus [9], and we use this to verify circumference of the circle.

1.3 Thesis Overview and Organization

Our work shows a demonstration that a mechanized proof assistant, in particular ACL2(r), can be used to prove that a continuously differentiable curve is rectifiable and an application of the proof to verify the circumference of a circle with radius r is $2\pi r$.

In chapter 2 we show derivation of the formula for length of a rectifiable curve and the formula is applied to find the circumference of a circle with radius r.

In chapter 3 we show a mechanized proof for the formula of length of a complex valued continuously differentiable curve in ACL2(r). First we prove norm of the curve's derivative is continuous if the curve is continuously differentiable. Then we show the length of the curve, which is the sum of all the lengths of the infinitesimal segments of the curve is equal to the integral of norm of the curve's derivative. Thus, if the curve is continuously differentiable, it is rectifiable.

In chapter 4, we use the results from chapter 3 to calculate the circumference of a circle. Thus ACL2-PI defined in ACL2 as first positive root of cosine function is equivalent to the definition of π, circumference of the circle over diameter of the circle.

Chapter 2

Theory

2.1 Continuously Differentiable Function

A continuously differentiable function is a function with continuous first derivative [10]. These are also called smooth curves.

For example let $f(x) = x^2$. Let the derivative of f is $g(x) = 2x$. $g(x)$ is continuous because

$$\lim_{x \to a} g(x) \quad exists \quad for \quad all \quad a$$

$$and$$

$$\lim_{x \to a} g(x) = g(a)$$

Therefore $f(x)$ is continuously differentiable curve. From now on we refer to these curves as smooth curves.

2.2 Length of a Smooth Curve

In this section, we derive a formula for the length of a curve $C(t) = (x(t), y(t))$ on an interval $[a, b]$ where $x(t)$ and $y(t)$ are the real and imaginary parts of the curve $c(t)$ respectively. We will assume that c is continuously differentiable on the interval $[a, b]$.

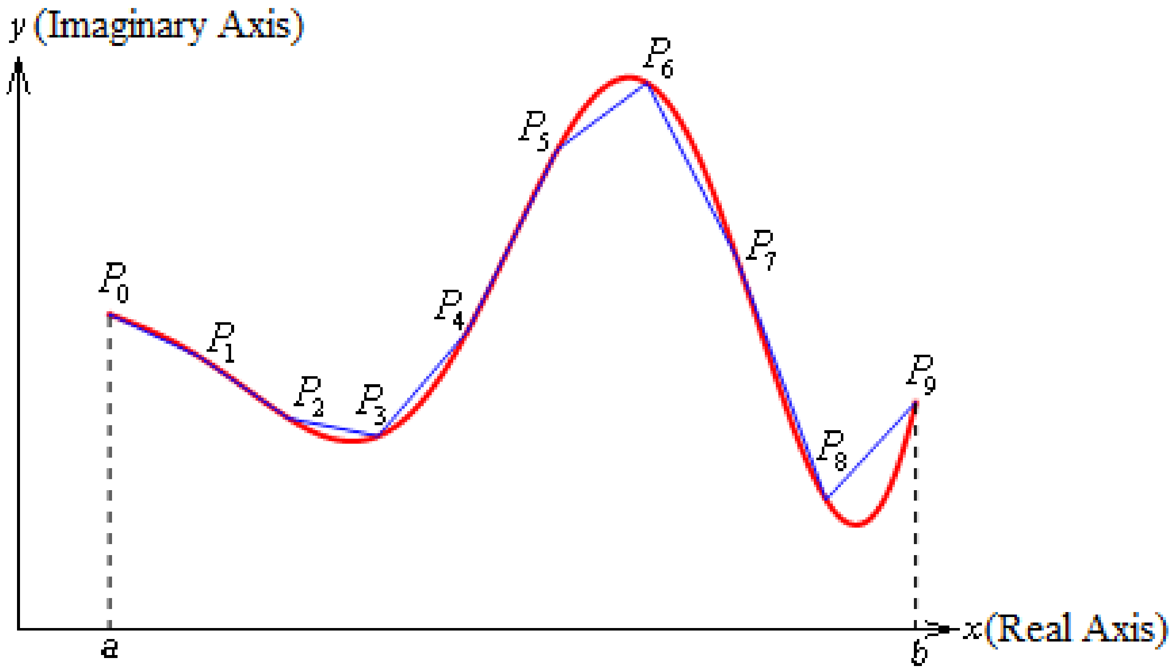

Figure 2.1: Rectifiable Curve [1]

Initially we will need to estimate the length of the curve. We will do this by dividing the interval $[a, b]$ into n equal subintervals each of width Δt and denote the point on the curve at each point by P_i. In the sketch above $a = x(t_0)$ and $b = x(t_n)$. We can then approximate the curve by a series of straight lines connecting the points.

Now, denoting the length of each of these line segments by $\mid P_i \; - \; P_{i-1} \mid$, the length of the curve will approximately be,

$$L \approx \sum_{i=1}^{n} \left| P_i \; - \; P_{i-1} \right|$$

We can get the exact length by making n larger and larger. In other words, the exact length will be,

$$L = \lim_{n \to \infty} \sum_{i=1}^{n} \left| P_i - P_{i-1} \right| = \lim_{n \to \infty} \sum_{i=1}^{n} \left| c(t_i) \; - c(t_{i-1}) \right|$$

$$c(t) = x(t) + i * y(t), t_0 \leq t \leq t_n$$

This implies the following because P_i is $c(t_i)$

$$L = \lim_{n\to\infty} \sum_{i=1}^{n} \Big| c(t_i) \ - c(t_{i-1}) \Big| \ = \lim_{n\to\infty} \sum_{i=1}^{n} \Big| \ \frac{c(t_i) \ - c(t_{i-1})}{\Delta t} \ \Big| \Delta t$$

$\Big| c(t_i) \ - c(t_{i-1}) \Big|$ is the absolute value of the distance between 2 points $c(t_i)$, $(x(t_1), y(t_i))$ and $c(t_{i-1})$, $(x(t_{i-1}), y(t_{i-1}))$ which we can find using pythagorean theorem. As $c'(t)$ is continuous, $\Big| c'(t) \Big|$ is continuous by properties of continuous functions [11] and by definition of integral the above equation is equal to

$$\int_{t_0}^{t_n} \Big| c'(t) \Big| dt = \int_{t_0}^{t_n} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt$$

In the above equation $\frac{dx}{dt}$ is real part of $\frac{dc}{dt}$ and $\frac{dy}{dt}$ is imaginary part of $\frac{dc}{dt}$

2.3 Circumference of a Circle

In this section we show the application of the above proof. A Circle with radius r can be defined as

$$c(t) = r * e^{it} \ \ (0 \leq t \leq 2 * \pi)$$

The derivative of c is equal to $r(-\sin t + i \cos t)$. Since sin and cos are continuous functions, the derivative of c is continuous.

Therefore length of the circle c is equal to integral of the norm of the derivative of the circle over interval 0 to 2π.

circumference of the circle $= \int_0^{2\pi} \sqrt{(r * (-sint))^2 + (r * cost)^2} dt \ = \ \int_0^{2\pi} r dt \ = \ 2\pi r$

Chapter 3

Proof for Length of a Smooth Curve in ACL2(r)

3.1 Proving the Theorem in ACL2(r)

Encapsulate in ACL2 provides a way to execute a sequence of events and then hide some of the resulting effects. We can declare events in encapsulate as local and can be hidden from logical world. We declare an identity function, c(x) in encapsulate as local. We assume c-derivative is difference quotient which is infinitely close to c(x) and c-derivative is continuous. Outside of encapsulate we only know that c-derivative is derivative of c and c-derivative is continuous and definition of c and c-derivative are hidden from logical world. We prove norm of the c-derivative is continuous based on properties of continuous functions.

When proving theorems, we use lemma-instances [12]. Lemma instances are the objects one provides via :use and :by hints to bring to the theorem prover's attention some previously proved or easily provable fact. Once we show derivative is continuous, we can find the length of the curve by taking riemann sum of the lengths infinitely small segments of the curve. If this sum is limited, by definition of the integral length is equal to integral to norm of c-derivative, which is length of the curve c(x).

Then, we take a circle in the complex domain and prove that its derivative is continuous. We use lemma instance of the length of the curve c(x) to get length of the circle.

3.2 Defining Smooth Curve

As we discussed above, a continuously differentiable curve is a curve such that its derivative is continuous. For generalizing, we chose functions *c(x) = x* and *c-derivative(x) = 1*. Constraints we put on c-derivative are c-derivative is infinitely close to difference quotient of *c* and c-derivative is continuous.

```
(encapsulate
  ((c (x) t)
   (c-derivative (x) t))
  ; Our witness continuous function is the identity function.
  (local (defun c(x) x))
  (local (defun c-derivative (x) (declare (ignore x)) 1))
  ;c-derivative is actually derivative of c
   (defthm c-der-lemma
    (implies (and (standardp x)
                  (realp x)
                  (realp y)
                  (i-close x y)
                  (not (= x y)))
          (i-close (/ (- (c x) (c y)) (- x y)) (c-derivative x))))
  ;c-derivative is continuous
  (defthm c-der-continuous
    (implies (and (standardp x)
                  (realp x)
                  (realp y)
                  (i-close x y))
             (i-close (c-derivative x) (c-derivative y))))
   ; The function returns real values or imaginary values.
  (defthm c-acl2num
```

```
    (implies (acl2-numberp x)
             (acl2-numberp (c x))))
  )
```

3.3 Norm of the Derivative of a Smooth Curve is Continuous

To apply integral to norm of the derivative for a function, we have to show that norm of the derivative of the function is continuous. Norm of the derivative of the function is

$$\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt$$

To prove that the above function is continuous we have to prove real part of c-derivative$\left(\frac{dx}{dt}\right)$, imaginary part of c-derivative $\left(\frac{dy}{dt}\right)$, square of continuous functions $\left(\frac{dx}{dt}^2, \frac{dy}{dt}^2\right)$, sum of two continuous functions$\left(\frac{dx}{dt}^2 + \frac{dy}{dt}^2\right)$ and finally the square root of a continuous function ($\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}$) are continuous.

Real part of c-derivative$\left(\frac{dx}{dt}\right)$ and imaginary part of c-derivative $\left(\frac{dy}{dt}\right)$ are continuous because c-derivative $\left(\frac{dc}{dt}\right)$ is continuous. From instance square-is-continuous we can prove $\left(\frac{dx}{dt}^2, \frac{dy}{dt}^2\right)$ are continuous and the proof for sum of 2 continuous functions being continuous is trivial.

Proving square root of a continuous function is continuous is non trivial.

For some real numbers $x1$, $x2$, $x3$ we can prove the following 3 lemmas:

$$if\ x1 \geq 0\ and\ x2 \geq 0\ and\ x1 > x2\ then\ (x1 * x1)\ >\ (x1 * x2) \tag{3.1}$$

$$if\ x1 \geq 0\ and\ x2 \geq 0\ and\ x2 > x1\ then\ (x1 * x2) \geq (x2 * x2) \tag{3.2}$$

$$if\ x1 > x2\ and\ x2 \geq x3\ then\ (x1 > x2) \tag{3.3}$$

Using 3.1, 3.2, 3.3 as hints we can prove

$$if\ x1 \geq 0\ and\ x2 \geq 0\ and\ x1 > x2\ then\ (x1 * x1) > (x2 * x2) \tag{3.4}$$

For real numbers $y1$, $y2$ we can prove the following lemmas

$$if\ (standard-part\ y1) \neq (standard-part\ y2)\ then\ (not\ (i-close\ y1\ y2)) \tag{3.5}$$

$$if\ (y1 \geq 0)\ and\ (y2 \geq 0)\ and\ (not\ (i{-}close\ y1\ y2))\ then\ (standard{-}part\ y1) \neq (standard{-}part\ y2) \tag{3.6}$$

Using 3.6 and 3.4 as hints, we can prove

$$\begin{gathered} if\ (i-limited\ y1)\ and\ (i-limited\ y2)\ and\ (y1>=0)\ and\ (y2>=0)\ and\ (not\ (i-close\ y1\ y2)) \\ then\ (standard-part\ y1)*(standard-part\ y1) \neq (standard-part\ y2)*(standard-part\ y2)) \end{gathered} \tag{3.7}$$

$$(standard-part\ y1)*(standard-part\ y1) = (standard-part\ (y1)^2) \tag{3.8}$$

Using 3.7 and 3.8 as hints, we can prove

$$\begin{aligned} if\ (i-limited\ y1)\ and\ (i-limited\ y2)\ and\ (y1 \geq 0)\ and\ (y2 \geq 0)\ (not\ (i-close\ y1\ y2)) \\ then\ (standard-part\ (y1)^2) \neq (standard-part\ (y2)^2) \end{aligned} \tag{3.9}$$

Using 3.9 and 3.5 as hints,

$$\begin{aligned} if\ (i-limited\ y1)\ and\ (i-limited\ y2)\ (y1 \geq 0)\ and\ (y2 \geq 0)\ and\ (not\ (i-close\ y1\ y2)) \\ then\ (not\ (i-close\ (y1)^2\ (y2)^2)) \end{aligned} \tag{3.10}$$

Finally, we have all the hints required to prove that the square root of a continuous function is continuous using the contrapositive of 3.10

```
(local
 (defthm root-close-f
   (implies (and      (standardp x1)
                      (realp x1)
                      (realp x2)
                      (>= x1 0)
                      (>= x2 0)
                      (i-close x1 x2))
     (i-close (acl2-sqrt x1) (acl2-sqrt x2))
     )
  ;hints omitted
   ))
```

By making use of the above lemma, we can prove that $\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}$ is continuous. The following theorem shows this with domain being all real numbers. In the following theorem der-sum-sqrt is $\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}$

```
(defthm der-sum-sqrt-cont
  (implies (and (standardp x)
                (inside-interval-p x (der-sum-sqrt-domain))
                (inside-interval-p y (der-sum-sqrt-domain))
                (i-close x y)
                )
           (i-close (der-sum-sqrt x)
                    (der-sum-sqrt y)
                    ))
  ; hints omitted
  )
```

3.4 Length of a Smooth Curve

Length of the curve from t_0 to t_n is Riemann sum of these functions $f = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}$ over partition p where t_0 is starting point on the partition and t_n is the last point on the partition is given by *riemann-der-sum-sqrt* function below

```
(defun map-der-sum-sqrt (p)
  (if (consp p)
    (cons (der-sum-sqrt (car p))
          (map-der-sum-sqrt (cdr p)))
    nil))

(defun riemann-der-sum-sqrt (p)
  (dotprod (deltas p)
           (map-der-sum-sqrt (cdr p))))
```

We can prove *riemann-der-sum-sqrt* is limited by functional instantiating *limited-riemann-rcfn-small-partition*.

```
(defthmd limited-riemann-der-sum-sqrt-small-partition
  (implies (and (realp a) (standardp a)
                (realp b) (standardp b)
                (inside-interval-p a (der-sum-sqrt-domain))
                (inside-interval-p b (der-sum-sqrt-domain))
                (< a b))
    (i-limited (riemann-der-sum-sqrt (make-small-partition a b))))
  ;hints omitted
  )
```

From the above theorem we can prove that *riemann-der-sum-sqrt* with standard inputs returns a standard output.

```
  (defun-std strict-int-der-sum-sqrt (a b)
    (if (and (realp a)
             (realp b)
             (inside-interval-p a (der-sum-sqrt-domain))
             (inside-interval-p b (der-sum-sqrt-domain))
             (< a b))
 (standard-part (riemann-der-sum-sqrt (make-small-partition a b)))
      0))
```

The length of the curve is sum these functions for infinitely many small partitions between interval t_0 and t_n on the curve. We can prove this sum is integral of *c-derivative* by using functional instance of *strict-int-rcfn-is-integral-of-rcfn*.

```
(defthm strict-int-der-sum-sqrt-is-integral-of-der-sum-sqrt
  (implies (and (standardp a)
                (standardp b)
                (<= a b)
                (inside-interval-p a (der-sum-sqrt-domain))
                (inside-interval-p b (der-sum-sqrt-domain))
                (partitionp p)
                (equal (car p) a)
                (equal (car (last p)) b)
                (i-small (mesh p)))
           (i-close (riemann-der-sum-sqrt p)
                    (strict-int-der-sum-sqrt a b)))
  ;hints omitted
  )
```

Thus we showed if the curve is continuously differentiable it is rectifiable and the length of the curve in the interval t_0 and t_n given $t_0 \leq t_n$ is equal to $\int_{t_0}^{t_n} |f'(t)|\, dt$.

Chapter 4

Application of the Proof to Verify Circumference of a Circle

In this section, we will apply the proof to show that the circumference of a circle with radius r is equal to $2\pi r$. Radius of a curve which is standard and real can be defined as follows:

```
(encapsulate
  ((rad() t))
  (local (defun rad() 1))
  (defthm rad-det
    (and (realp (rad))
         (standardp (rad))
         (>= (rad) 0)
         (i-limited (rad))
         )
    )
  )
```

Circle in complex domain is represented as $r * e^{ix}$, where r is the radius and $0 \leq x \leq 2\pi$ and it can be defined as follows:

```
(defun circle(x)
  (* (rad) (acl2-exp (* #c(0 1) x)))
  )
```

Let the derivative of the circle be,

```
(defun circle-der(x)
  (* (rad) (complex (- (acl2-sine x)) (acl2-cosine x)))
  )
```

To proceed further we have to prove that the circles derivative is equal to *circle-der(x)* and *circle-der(x)* is continuous.

By using the instances *acl2-sine-derivative*, *acl2-cosine-derivative* [13], *rad-det* and few other lemmas of non-standard analysis as hints, we can prove *circle-der(x)* is infinitely close to difference quotient of *circle(x)*. The second constraint is to prove that *circle-der(x)* is continuous. We can prove this because *acl2-sine* and *acl2-cosine* are continuous functions.

By functional instantiating *der-sum-sqrt-cont* we can prove norm of *circle-der(x)* is continuous. Thus we can define integral for norm of the *circle-der(x)* to get length of the circle c,

```
  (defun-std strict-int-circle-der-sum-sqrt (a b)
    (if (and (realp a)
             (realp b)
             (inside-interval-p a (circle-der-sum-sqrt-domain))
             (inside-interval-p b (circle-der-sum-sqrt-domain))
             (< a b))
      (standard-part
      (riemann-circle-der-sum-sqrt (make-small-partition a b)))
      0))


(defun int-circle-der-sum-sqrt (a b)
  (if (<= a b)
```

```
    (strict-int-circle-der-sum-sqrt a b)
    (- (strict-int-circle-der-sum-sqrt b a))))
```

In the above function, *int-c-der-sum-sqrt* is integral of norm of *circle-der(x)* from a to b and *strict-int-circle-der-sum-sqrt* is strict integral of norm of *circle-der* from a to b. But norm of the *circle-der* is (rad), which is just the radius of the circle. *circle-len* can be defined as below. Derivative of *circle-len* is equal to norm of *circle-der*.

```
(defun circle-len(x)
  (if (realp x)
    (* (rad) x)
    0)
  )
```

By second fundamental theorem of calculus, if $h'(t) = g(t)$ then

$$\int_a^b g(t)dt = h(b) - h(a)$$

So thus we get following theorem by functional instantiating *ftc*-2

```
(defthm apply-ftc-2
  (implies (and (inside-interval-p a (circle-der-sum-sqrt-domain))
                (inside-interval-p b (circle-der-sum-sqrt-domain)))
           (equal (int-circle-der-sum-sqrt a b)
                  (- (circle-len b)
                     (circle-len a))))

  ;hints omitted
  )
```

By letting b as 2π and a as 0 we get the radius of a circle as $2\pi(rad)$.

Thus we proved circumference of a circle from 0 to 2π with radius (rad) is equal to $2\pi(rad)$ where (rad) is radius of the circle which can be any real number and standard.

Chapter 5

Conclusions

We presented in this thesis norm of the derivative of a continuously differentiable complex-valued function is continuous. We proved continuously differentiable curve is rectifiable and length is equal to integral of norm of the curve's derivative. Therefore, the results presented in this paper could also be applied to any other continuously differentiable curve. We used the definition of the Riemann integral to formalize the length of rectifiable curve in ACL2(r).

Using the above result, we verified the formula for the circumference of a circle. Thus, we proved $acl2 - pi$ definition that we use in ACL2(r) is equivalent to circumference of a circle over diameter of the circle.

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