THE BANACH-TARSKI PARADOX

TOM WESTON

1. INTRODUCTION

The Banach-Tarski paradox is one of the most celebrated paradoxes in mathematics. It states that given any two subsets $\mathcal{A}$ and $\mathcal{B}$ of $\mathbb{R}^3$, which are bounded and have non-empty interior, it is possible to 'cut' $\mathcal{A}$ into a finite number of pieces which can be moved by rigid motions (translations and rotations) to form exactly $\mathcal{B}$. This has many amusing consequences; it is most commonly stated as the assertion that a pea can be cut into a finite number of pieces which could then be reassembled to form the sun.

Intuitively, of course, this is absurd. Rigid motions are supposed to preserve volume. The Banach-Tarski paradox seems to completely contradict this. This is why it is called a paradox and not a theorem. This is also why many people have tried to argue that it must be false. Since the logic of the proof is unquestioned, they argue that one of the underlying assumptions must be flawed.

The assumption that they pick is called the axiom of choice. The axiom of choice is the seemingly innocent statement that given any collection of sets it is possible to pick one element from each of the sets. (Actually, the collection of sets must itself form a set; however, we do not want to delve too deeply into axiomatic set theory here.) Here, however, our intuition seems to cry out that this statement is obviously true. How then are we to reconcile these two conflicting statements?

I have two main goals in writing this paper. The first is to show that the proof of the Banach-Tarski paradox is not difficult. The argument is fairly long and intricate, but never requires anything very deep. In fact, the key step is a simple observation about free groups which is stated and proved in Section 2. The rest of the proof is really just fixing up the little things that do not quite fit. My proof basically follows that given in [3, Chapters 1-3], although I have dispensed with the terminology he introduces. (Until Section 7, when a bit of it is indispensible.) I have also tried to keep things as 'explicit' as possible, to the point where we are eventually writing out decompositions of the unit ball into 52 pieces.

My second goal is to try to show that the Banach-Tarski paradox is not really a paradox. I am aware of several fairly convincing arguments for this. I will save most of them for the final section, but since in Sections 5 and 6 we will need a variant on one, I will present it here. It is an example of a similar 'paradox' in $\mathbb{R}^2$, which exists without any use of the axiom of choice.

So, let $\mathbf{S}^1$ be the unit circle in $\mathbb{R}^2$. Let ℓ be the segment $(0,1)$ along the x-axis; ℓ is simply one of the radii of $\mathbf{S}^1$, without the center. Next, let ρ be a counterclockwise rotation by, say, $\frac{1}{10}$ radians around the origin. So, $\rho(\ell)$ is another radius of $\mathbf{S}^1$, and since 2π is irrational we see that $\rho^n(\ell)$ never comes back to coincide with ℓ. So, define a
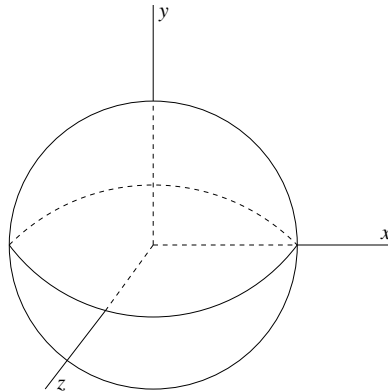
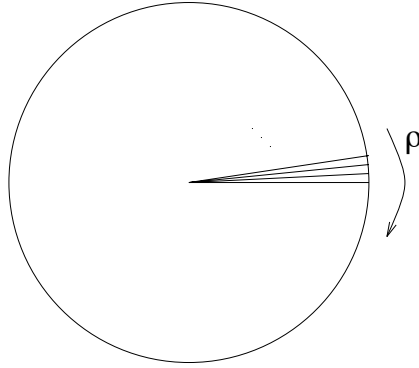

FIGURE 1. The Sphere $\mathbf{S}^2$

FIGURE 2. The Circle Trick

set $\mathcal{C}$ by

$$\mathcal{C} = \biguplus_{n=0}^{\infty} = rho^n(\ell).$$

(Here, and throughout the paper, the symbol $\uplus$ stands for *disjoint union*. We are really using it is as a convenient notation device; $\mathcal{A} \uplus \mathcal{B}$ means that we are considering the set $\mathcal{A} \cup \mathcal{B}$ *and* that $\mathcal{A}$ and $\mathcal{B}$ are disjoint.)

Now, consider the set $\mathbf{S}^1 \uplus \mathcal{C}$; it looks like some sort of wheel with an infinite number of spokes. (See Figure 2.) Now, consider what happens when we rotate $\mathcal{C}$ by ρ^{-1}; that is, clockwise by $\frac{1}{10}$ radians. Since $\rho^n(\ell) \neq \ell$ for any n, we see that

$$\rho^{n-1}(\ell) \neq \rho^{-1}(\ell)$$

for any n. Thus, $\rho^{-1}(\ell)$ does not intersect the set $\mathbf{S}^1 \uplus \mathcal{C}$. That is,

$$\mathbf{S}^1 \uplus \rho^{-1}(\mathcal{C}) = \mathbf{S}^1 \uplus C \uplus \rho^{-1}(\ell).$$

So we have cut $\mathbf{S}^1 \uplus \mathcal{C}$ into two pieces, moved one of them, and obtained the original set *plus* an extra line.

This is certainly not nearly as striking as the Banach-Tarski paradox, but it does illustrate that geometric paradoxes can happen even in 'simple' situations. The Banach-Tarski paradox obtains its additional power from the extra freedom that we get by working in 3 dimensions.

The plan of the paper is as follows: We will begin in Section 2 with a quick review of free groups, and we will then proceed to give the fundamental 'paradoxical' decomposition of the free group of rank 2. In Section 3 we will construct a free group of rank 2 in the group SO_3 of rotations of the sphere $\mathbf{S}^2$. (Note that $\mathbf{S}^2$ is simply the *hollow* sphere, not the solid ball $\mathbf{B}^3$.) In Section 4 we will use the natural group action of SO_3 on $\mathbf{S}^2$ to extend this paradox to most of the sphere; this intermediate result is called the Hausdorff Paradox. In Section 5 we will use a 3-dimensional version of the circle trick above to extend our paradox to the entire sphere $\mathbf{S}^2$. Then in Section 6 we will extend our paradox to the solid ball $\mathbf{B}^3$, which will again require a variant on the circle trick. Finally, in Section 7 we will use a trick due to Banach to extend our paradox to arbitrary bounded subsets of $\mathbb{R}^3$ with interior points. In Section 8 we will return to the underlying 'philosophical' issues behind the Banach-Tarski paradox.

I have tried to keep the prerequisites to a minimum. However, rather than attempt a half-assed introduction to certain subjects myself, I felt that I would refer the reader to Artin's excellent book [1]. In particular, the reader will need to be familiar with group theory and linear transformations at the level of Chapters 1-3 and $4.1, 4.2$. Familiarity with Chapters 5 and 6 is helpful, but not necessary, except for Section 6.7, which is essential. Lastly, some knowledge of the cardinality of sets (countable and uncountable sets) will be needed. The contents of Chapter 2 of [2] are more than sufficient.

2. Free Groups

Let $\mathfrak{F}(x,y)$ be the free group on generators. Recall that $\mathfrak{F}(x,y)$ consists of all possible strings in x, y, x^{-1}, y^{-1} (called *words*) subject only to the relations

(1) $$xx^{-1} = x^{-1}x = yy^{-1} = y^{-1}y = 1,$$

with group operation given simply by concatenation of words. Further recall that every word is equivalent (under the relations (1)) to a unique *reduced word*, in which all possible pairs of inverses have been canceled. We will tend

to identify elements of $\mathfrak{F}(x,y)$ with reduced words, rather than worry about equivalence classes. Despite it's 'simple' definition, the group $\mathfrak{F}(x,y)$ is extremely complicated, and the main step in the proof of the Banach-Tarski paradox will be an application of an easy observation about $\mathfrak{F}(x,y)$, which we will now explain.

Define the set $\mathfrak{W}(x)$ to consist of all of the elements of $\mathfrak{F}(x,y)$ which begin with x. (Remember that we are considering only *reduced* words here.) We can define $\mathfrak{W}(x^{-1})$, $\mathfrak{W}(y)$ and $\mathfrak{W}(y^{-1})$ in the same way. Together with $\{1\}$ this gives a partition of $\mathfrak{F}(x,y)$; that is,

$$(2) \qquad \mathfrak{F}(x,y) = \{1\} \uplus \mathfrak{W}(x) \uplus \mathfrak{W}(x^{-1}) \uplus \mathfrak{W}(y) \uplus \mathfrak{W}(y^{-1}).$$

Now consider what happens when we multiply $\mathfrak{W}(x)$ by x^{-1} on the left; that is, we set

$$x^{-1}\mathfrak{W}(x) = \{x^{-1}\mathfrak{u} \in \mathfrak{F}(x,y) \mid \mathfrak{u} \in \mathfrak{W}(x)\}.$$

Since $\mathfrak{u} \in \mathfrak{W}(x)$, it begins with an x, which we can then cancel with the x^{-1}. We claim that $x^{-1}\mathfrak{W}(x)$ will consist of 1 and all words beginning with x, y or y^{-1}.

Proposition 2.1.

$$x^{-1}\mathfrak{W}(x) = \{1\} \uplus \mathfrak{W}(x) \uplus \mathfrak{W}(y) \uplus \mathfrak{W}(y^{-1}).$$

Proof. Take $\mathfrak{v} \in \mathfrak{W}(x)$. Then $\mathfrak{v}$ is a reduced word beginning with x, so it could not possibly begin xx^{-1}. Therefore $x^{-1}\mathfrak{v}$ can not begin with x^{-1}, so $x^{-1}\mathfrak{W}(x)$ is disjoint from $\mathfrak{W}(x^{-1})$. By (2) this shows that

$$x^{-1}\mathfrak{W}(x) \subseteq \{1\} \uplus \mathfrak{W}(x) \uplus \mathfrak{W}(y) \uplus \mathfrak{W}(y^{-1}).$$

Now take $\mathfrak{v} \in \{1\} \uplus \mathfrak{W}(x) \uplus \mathfrak{W}(y) \uplus \mathfrak{W}(y^{-1})$; for example, suppose $\mathfrak{v} \in \mathfrak{W}(y)$. Then $\mathfrak{v}$ begins with y, so $x\mathfrak{v} \in \mathfrak{W}(x)$, as the x has nothing to cancel with. Therefore $\mathfrak{v} = x^{-1}x\mathfrak{v} \in x^{-1}\mathfrak{W}(x)$. So,

$$x^{-1}\mathfrak{W}(x) \supseteq \mathfrak{W}(y).$$

In exactly the same way we get that $\{1\} \subseteq x^{-1}\mathfrak{W}(x)$, $\mathfrak{W}(x) \subseteq x^{-1}\mathfrak{W}(x)$ and $\mathfrak{W}(y^{-1}) \subseteq x^{-1}\mathfrak{W}(x)$. Thus,

$$x^{-1}\mathfrak{W}(x) \supseteq \{1\} \uplus \mathfrak{W}(x) \uplus \mathfrak{W}(y) \uplus \mathfrak{W}(y^{-1}),$$

which completes the proof. $\qquad\square$

Note that an analogous proof also establishes a similar result with x replaced by x^{-1}, y or y^{-1}. In particular we will need to use

$$y^{-1}\mathfrak{W}(y) = \{1\} \uplus \mathfrak{W}(x) \uplus \mathfrak{W}(x^{-1}) \uplus \mathfrak{W}(y).$$

We can now state the result that will be the main step in the Banach-Tarski paradox.

Corollary 2.2.

$$\mathfrak{F}(x,y) = x^{-1}\mathfrak{W}(x) \uplus \mathfrak{W}(x^{-1})$$

and

$$\mathfrak{F}(x,y) = y^{-1}\mathfrak{W}(y) \uplus \mathfrak{W}(y^{-1}).$$

Proof. This is immediate from (2) and Proposition 2.1. $\qquad\square$

Essentially, we have divided $\mathfrak{F}(x,y)$ into five pieces, and then 'moved' two of these pieces to reassemble two copies of $\mathfrak{F}(x,y)$. The connection with the Banach-Tarski theorem should be clear; we would like to be able to use this construction on the group of rigid motions of a sphere to cut the sphere into five pieces, which we can then shift and reassemble into two copies of the sphere. This is the main step in the Banach-Tarski theorem, although it will not quite work and there will be some cleaning up to do.

We will need one last piece of information about $\mathfrak{F}(x,y)$.

Proposition 2.3. $\mathfrak{F}(x,y)$ *is countable.*

Proof. Let $\mathfrak{F}_n$ be the subset of $\mathfrak{F}(x,y)$ of (reduced) words of length n, where the *length* of a word is the number of symbols in it. Then

$$\mathfrak{F}(x,y) = \biguplus_{n=0}^{\infty} \mathfrak{F}_n.$$

Each $\mathfrak{F}_n$ is certainly finite however; in fact, it can have at most 4^n elements. (It will actually have fewer elements than that for $n \geq 2$, because of cancellation.) Thus $\mathfrak{F}(x,y)$ is a countable union of finite sets, and thus is countable. $\qquad\square$
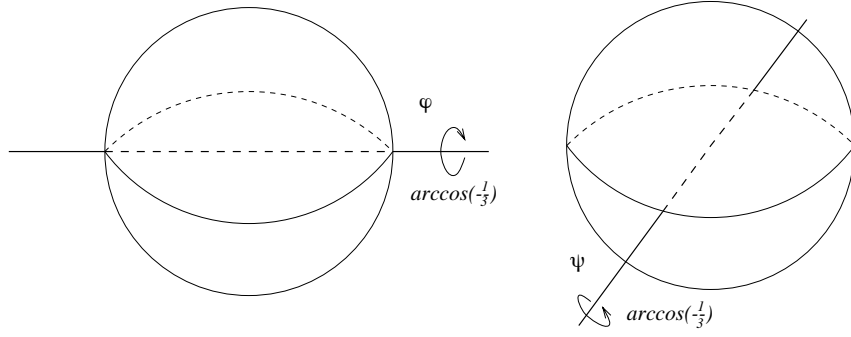
FIGURE 3. The Rotations φ and ψ

3. INDEPENDENT ROTATIONS

In order to apply the preceding construction to the sphere $\mathbf{S}^2$, we first must find a free group of rank 2 in SO_3, the group of rotations of $\mathbf{S}^2$. Recall that SO_3 can be realized as the group of orthogonal 3×3 matrices of determinant 1; that is,

$$SO_3 = \{A \in GL_3(\mathbb{R}) \mid A^t A = I, \det A = 1\}$$

where $GL_3(\mathbb{R})$ is the general linear group of invertible 3×3 matrices, I is the 3×3 identity matrix and A^t denotes the transpose of A.

Also recall that SO_3 has a natural group action on $\mathbf{S}^2$: for any $\rho \in SO_3$ and $p \in \mathbf{S}^2$, $\rho(p)$ is the image of p after rotation by ρ. More concretely, if ρ is represented by a matrix A (with respect to the standard basis of $\mathbb{R}^3$) and $p = (x, y, z)$, then

$$\rho(p) = A \cdot \begin{pmatrix} x \\ y \\ z \end{pmatrix}.$$

We will now give two rotations which satisfy no relations; that is, they generate a free group of rank 2 in SO_3. Let φ be a counterclockwise rotation by $\arccos \frac{1}{3}$ around the x-axis and let ψ be a counterclockwise rotation by $\arccos \frac{1}{3}$ around the z-axis. (See Figure 3.) It is easy to see that $\varphi^{\pm 1}$ and $\psi^{\pm 1}$ are represented by the following matrices (with respect to the usual basis of $\mathbb{R}^3$, of course) :

$$\varphi^{\pm 1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{3} & \mp\frac{2\sqrt{2}}{3} \\ 0 & \pm\frac{2\sqrt{2}}{3} & \frac{1}{3} \end{pmatrix} = \frac{1}{3} \begin{pmatrix} 3 & 0 & 0 \\ 0 & 1 & \mp 2\sqrt{2} \\ 0 & \pm 2\sqrt{2} & 1 \end{pmatrix}$$

$$\psi^{\pm 1} = \begin{pmatrix} \frac{1}{3} & \mp\frac{2\sqrt{2}}{3} & 0 \\ \pm\frac{2\sqrt{2}}{3} & \frac{1}{3} & 0 \\ 0 & 0 & 1 \end{pmatrix} = \frac{1}{3} \begin{pmatrix} 1 & \mp 2\sqrt{2} & 0 \\ \pm 2\sqrt{2} & 1 & 0 \\ 0 & 0 & 3 \end{pmatrix}.$$

Proposition 3.1. φ and ψ are independent as elements of SO_3; that is, they generate a free group of rank 2 inside of SO_3.

Proof. We first must explain what exactly we are proving. When we speak of a word in φ and ψ, we will mean some word in the free group $\mathfrak{F}(x, y)$, with x replaced by φ and y replaced by ψ. We have to show that the only one of these reduced word which evaluates to 1 in the group SO_3 is the empty word. We will do this by showing that $\rho(0, 1, 0) \neq (0, 1, 0)$ for any word ρ in φ and ψ.

First, we observe that if ρ has length n, then $\rho(0, 1, 0)$ has the form

$$\frac{1}{3^n} \left(a\sqrt{2}, b, c\sqrt{2} \right)$$

for integers a, b, c. We prove this by induction on n. If $n = 0$, then $\rho = 1$ and $\rho(0, 1, 0) = (0, 1, 0)$, which has the correct form. So suppose that $n > 1$. Let ρ be a reduced word of length n. Then we can write ρ in one of the forms

$$\varphi\rho', \varphi^{-1}\rho', \psi\rho', \psi^{-1}\rho'$$

for some reduced word ρ' of length $n - 1$. Then by the induction hypothesis we can write

$$\rho'(0, 1, 0) = \frac{1}{3^{n-1}} \left(a\sqrt{2}, b, c\sqrt{2} \right)$$

for some integers a, b, c. Therefore $\rho(0, 1, 0)$ is one of

$$\varphi\rho'(0, 1, 0) = \varphi \left(\frac{1}{3^{n-1}} \left(a\sqrt{2}, b, c\sqrt{2} \right) \right)$$

$$= \frac{1}{3^n} \begin{pmatrix} 3 & 0 & 0 \\ 0 & 1 & -2\sqrt{2} \\ 0 & 2\sqrt{2} & 1 \end{pmatrix} \begin{pmatrix} a\sqrt{2} \\ b \\ c\sqrt{2} \end{pmatrix}$$

$$= \frac{1}{3^n} \left(3a\sqrt{2}, b - 4c, (2b + c)\sqrt{2} \right)$$

$$\varphi^{-1}\rho'(0, 1, 0) = \varphi^{-1} \left(\frac{1}{3^{n-1}} \left(a\sqrt{2}, b, c\sqrt{2} \right) \right)$$

$$= \frac{1}{3^n} \begin{pmatrix} 30 & 0 \\ 0 & 1 & 2\sqrt{2} \\ 0 & -2\sqrt{2} & 1 \end{pmatrix} \begin{pmatrix} a\sqrt{2} \\ b \\ c\sqrt{2} \end{pmatrix}$$

$$= \frac{1}{3^n} \left(3a\sqrt{2}, b + 4c, (-2b + c)\sqrt{2} \right)$$

$$\psi\rho'(0, 1, 0) = \psi \left(\frac{1}{3^{n-1}} \left(a\sqrt{2}, b, c\sqrt{2} \right) \right)$$

$$= \frac{1}{3^n} \begin{pmatrix} 1 & -2\sqrt{2} & 0 \\ 2\sqrt{2} & 1 & 0 \\ 0 & 0 & 3 \end{pmatrix} \begin{pmatrix} a\sqrt{2} \\ b \\ c\sqrt{2} \end{pmatrix}$$

$$= \frac{1}{3^n} \left((a - 2b)\sqrt{2}, 4a + b, 3c\sqrt{2} \right)$$

$$\psi^{-1}\rho'(0, 1, 0) = \psi^{-1} \left(\frac{1}{3^{n-1}} \left(a\sqrt{2}, b, c\sqrt{2} \right) \right)$$

$$= \frac{1}{3^n} \begin{pmatrix} 1 & 2\sqrt{2} & 0 \\ -2\sqrt{2} & 1 & 0 \\ 0 & 0 & 3 \end{pmatrix} \begin{pmatrix} a\sqrt{2} \\ b \\ c\sqrt{2} \end{pmatrix}$$

$$= \frac{1}{3^n} \left((a + 2b)\sqrt{2}, -4a + b, 3c\sqrt{2} \right)$$

which all have the required form. This completes the induction.

Now, suppose that we have a non-identity reduced word ρ in φ and ψ of length n with $\rho(0, 1, 0) = (0, 1, 0)$. Writing $\rho(0, 1, 0)$ in the form above, we see that we must have $a = c = 0$ and $b = 3^n$. In particular, since $\rho \neq 1$, we must have $n \geq 1$, so that

$$a \equiv b \equiv c \equiv 0 \pmod{3}.$$

We will show that this is not possible.

To do this, let us define, for any $\rho \in \mathfrak{F}(\varphi, \psi)$, $N(\rho)$ to be the triple (a, b, c) above, considered modulo 3. This makes sense since a, b, c are all integers. We must show that for any non-identity reduced word ρ in φ and ψ we always have $N(\rho) \neq (0, 0, 0)$.

Now, if $N(\rho) = (a, b, c)$, then our above computations show that

$$\begin{array}{llll}
N(\varphi\rho) & = & (3a, b - 4c, 2b + c) & = & (0, b - c, c - b) \\
N(\varphi^{-1}\rho) & = & (3a, b + 4c, -2b + c) & = & (0, b + c, b + c) \\
N(\psi\rho) & = & (a - 2b, 4a + b, 3c) & = & (a + b, a + b, 0) \\
N(\psi^{-1}\rho) & = & (a + 2b, b - 4a, 3c) & = & (a - b, b - a, 0).
\end{array}$$

Iterating these, it is easy to see that for $n > 0$,

$$N(\varphi^n \rho) = \begin{cases} (0, b - c, c - b), & n \text{ odd}; \\ (0, c - b, b - c), & n \text{ even}; \end{cases}$$

$$N(\varphi^{-n} \rho) = \begin{cases} (0, b + c, b + c) & n \text{ odd}; \\ (0, -b - c, -b - c), & n \text{ even}; \end{cases}$$

$$N(\psi^n \rho) = \begin{cases} (a + b, a + b, 0) & n \text{ odd}; \\ (-a - b, -a - b, 0) & n \text{ even}; \end{cases}$$

$$N(\psi^{-n} \rho) = \begin{cases} (a - b, b - a, 0) & n \text{ odd}; \\ (b - a, a - b, 0) & n \text{ even}. \end{cases}$$

(For example, $N(\varphi^2 \rho) = N(\varphi(\varphi\rho))$, where $N(\varphi\rho) = (0, b - c, c - b)$. Thus

$$N(\varphi^2 \rho) = (0, (b - c) - (c - b), (c - b) - (b - c)) = (0, 2b - 2c, 2c - 2b) = (0, c - b, b - c).$$

Similarly, $N(\varphi^3 \rho) = (0, b - c, c - b)$, and it alternates from there. The other checks are done in the same way.)

So, let ρ be a non-identity reduced word in φ and ψ. Then ρ alternates non-zero powers of φ and ψ. Suppose first that ρ ends (on the right) with a power of φ, so that it has the form

$$\cdots \varphi^{n_3} \psi^{n_2} \varphi^{n_1}$$

where the n_i are non-zero. Then $N(\varphi^{n_1}) = N(\varphi^{n_1} \cdot 1)$, where $N(1) = (0, 1, 0)$. Thus, plugging into our above formulas,

(3) $$N(\varphi^{n_1}) \in \{(0, 1, 2), (0, 2, 1), (0, 1, 1), (0, 2, 2)\}.$$

Again using our above formulas,

$$N(\psi^{n_2} \varphi^{n_1}) \in \{(1, 1, 0), (2, 2, 0), (2, 1, 0), (1, 2, 0)\}.$$

(We just have to plug the 4 values in (3) into the equations for $N(\psi^{\pm n} \rho)$. Although there are 16 different possibilities to check, we only get these 4 triples.) Continuing, we find that

$$N(\varphi^{n_3} \psi^{n_2} \varphi^{n_1}) \in \{(0, 1, 2), (0, 2, 1), (0, 1, 1), (0, 2, 2)\}$$

again. Pictorially,

$$\{(0, 1, 0)\} \xrightarrow{\varphi^{n_1}} \{(0, 1, 2), (0, 2, 1), (0, 1, 1), (0, 2, 2)\} \xrightarrow{\psi^{n_2}} \{(1, 1, 0), (2, 2, 0), (2, 1, 0), (1, 2, 0)\}$$

$$\xrightarrow{\varphi^{n_3}} \{(0, 1, 2), (0, 2, 1), (0, 1, 1), (0, 2, 2)\} \longrightarrow \qquad \cdots$$

We thus see that we could not have $N(\rho) = (0, 0, 0)$. Thus $\rho(0, 1, 0) \neq (0, 1, 0)$, so ρ is not the identity rotation.

Similarly, if ρ starts with a power of ψ, we get

$$\{(0, 1, 0)\} \xrightarrow{\psi^{n_1}} \{(1, 1, 0), (2, 2, 0), (2, 1, 0), (1, 2, 0)\} \xrightarrow{\varphi^{n_2}} \{(0, 1, 2), (0, 2, 1), (0, 1, 1), (0, 2, 2)\}$$

$$\xrightarrow{\psi^{n_3}} \{(1, 1, 0), (2, 2, 0), (2, 1, 0), (1, 2, 0)\} \longrightarrow \qquad \cdots$$

So again $N(\rho) \neq (0, 0, 0)$, so $\rho(0, 1, 0) \neq (0, 1, 0)$. This completes the proof. $\square$

4. The Hausdorff Paradox

Now that we have realized the free group on 2 generators as a subgroup of SO_3, namely $\langle \varphi, \psi \rangle = \mathfrak{F}(\varphi, \psi)$, we can attempt to pass our 'paradoxical' construction of Section 2 to $\mathbf{S}^2$. This is accomplished through the natural action of SO_3 on $\mathbf{S}^2$ discussed in Section 3. This SO_3-action induces an equivalence relation on $\mathbf{S}^2$, where $p, q \in \mathbf{S}^2$ are equivalent if there is a $\rho \in SO_3$ with $\rho(p) = q$. The equivalence classes are called the SO_3-orbits; if $p \in \mathbf{S}^2$, its orbit is simply the set

$$\{\rho(p) \mid \rho \in SO_3\}.$$

Of course, the orbits of the SO_3-action are not very interesting; for any points $p, q \in \mathbf{S}^2$ there are lots of rotations in SO_3 sending p to q. Therefore all points in SO_3 are equivalent, and $\mathbf{S}^2$ itself is the only orbit.

The orbits get considerably more interesting if we look only at the $\mathfrak{F}(\varphi, \psi)$-action. In fact, since $\mathfrak{F}(\varphi, \psi)$ is countable and the orbit of a point $p \in \mathbf{S}^2$ is just

$$\{\rho(p) \mid \rho \in \mathfrak{F}(\varphi, \psi)\},$$

the orbits must also be countable. Since $\mathbf{S}^2$ is uncountable, there must be uncountably many of these $\mathfrak{F}(\varphi, \psi)$-orbits. (This is because a countable union of countable sets is still countable.)

We now need to invoke the axiom of choice. We want to pick exactly one representative of each $\mathfrak{F}(\varphi, \psi)$-orbit. This is precisely what the axiom of choice says that we can do. So let $\mathcal{M}_0$ be the set of all of these. We will now try to use this set $\mathcal{M}_0$ to pass our paradox from $\mathfrak{F}(\varphi, \psi)$ to $\mathbf{S}^2$. This is not quite going to work, for reasons that we will see in a moment, but with a slight modification we will obtain the key step in the Hausdorff paradox.

Our plan is as follows: We have our partition

$$(4) \qquad \mathfrak{F}(\varphi, \psi) = 1 \uplus \mathfrak{W}(\varphi) \uplus \mathfrak{W}(\varphi^{-1}) \uplus \mathfrak{W}(\psi) \uplus \mathfrak{W}(\psi^{-1})$$

of $\mathfrak{F}(\varphi, \psi)$. Now, for any $\rho \in \mathfrak{F}(\varphi, \psi)$, we can form the set

$$\rho(\mathcal{M}_0) = \{\rho(p) \mid p \in \mathcal{M}_0\}.$$

Then,

$$\mathfrak{F}(\varphi, \psi)\mathcal{M}_0 = \{\rho(p) \mid \rho \in \mathfrak{F}(\varphi, \psi), p \in \mathcal{M}_0\}$$

will be all of $\mathbf{S}^2$, since $\mathcal{M}_0$ contains an element of each $\mathfrak{F}(\varphi, \psi)$-orbit. Therefore, applying (4) to $\mathcal{M}_0$, we obtain

$$\mathbf{S}^2 = \mathfrak{F}(\varphi, \psi)\mathcal{M}_0 = \mathcal{M}_0 \cup \mathfrak{W}(\varphi)\mathcal{M}_0 \cup \mathfrak{W}(\varphi^{-1})\mathcal{M}_0 \cup \mathfrak{W}(\psi)\mathcal{M}_0 \cup \mathfrak{W}(\psi^{-1})\mathcal{M}_0.$$

Unfortunately, these unions need no longer be disjoint. For example, if by some chance $(1, 0, 0) \in \mathcal{M}_0$ then, since $\varphi(1, 0, 0) = (1, 0, 0)$, $(1, 0, 0)$ will be both in $\mathcal{M}_0$ and $\mathfrak{W}(\varphi)\mathcal{M}_0$. More generally, if p is a fixed point of any $\rho \in \mathfrak{F}(\varphi, \psi)$, then p could lie in two of the sets of the decomposition.

We will remedy this in the most naive possible way: if fixed points of rotations in $\mathfrak{F}(\varphi, \psi)$ are causing problems, we will simply get rid of them. So let $\mathcal{D}$ be the set of all points of $\mathbf{S}^2$ fixed by any non-identity $\rho \in \mathfrak{F}(\varphi, \psi)$. That is,

$$\mathcal{D} = \{p \in \mathbf{S}^2 \mid \text{There is a } \rho \in \mathfrak{F}(\varphi, \psi), \rho \neq 1, \text{ with } \rho(p) = p.\}.$$

Since a non-identity rotation fixes exactly its axis of rotation, $\mathcal{D}$ will consist exactly of the points of $\mathbf{S}^2$ which lie on axes of rotations in $\mathfrak{F}(\varphi, \psi)$. In particular, since each non-identity element of $\mathfrak{F}(\varphi, \psi)$ will contribute only 2 such points, $\mathcal{D}$ is countable.

Now, we would like to consider our $\mathfrak{F}(\varphi, \psi)$-action on $\mathbf{S}^2 - \mathcal{D}$. We first must check that we even have a well-defined action here. That is, we must check that if $p \in \mathbf{S}^2 - \mathcal{D}$, then $\rho(p) \in \mathbf{S}^2 - \mathcal{D}$ for all $\rho \in \mathfrak{F}(\varphi, \psi)$. We will prove the contrapositive. So suppose we have a point $p \in \mathbf{S}^2$ and a rotation $\rho \in \mathfrak{F}(\varphi, \psi)$ with $\rho(p) \in \mathcal{D}$. Then by the definition of $\mathcal{D}$ there is a non-identity $\sigma \in \mathfrak{F}(\varphi, \psi)$ with $\sigma(\rho(p)) = \rho(p)$. But then $\rho^{-1}\sigma\rho$ is also a non-identity rotation, and $\rho^{-1}\sigma\rho(p) = p$. So $p \in \mathcal{D}$.

Now, as before, by the axiom of choice we can choose a representative of each $\mathfrak{F}(\varphi, \psi)$-orbit on $\mathbf{S}^2 - \mathcal{D}$. Let $\mathcal{M}$ be the set of these representatives. This time our partition of $\mathfrak{F}(\varphi, \psi)$ will give us a partition of $\mathbf{S}^2 - \mathcal{D}$. In fact, suppose $\rho_1, \rho_2 \in \mathfrak{F}(\varphi, \psi)$ and $\rho_1(\mathcal{M}) \cap \rho_2(\mathcal{M}) \neq \emptyset$. Thus there is a point p with $p \in \rho_1(\mathcal{M})$ and $p \in \rho_2(\mathcal{M})$. Then there are points $p_1, p_2 \in \mathcal{M}$ with $p = \rho_1(p_1)$ and $p = \rho_2(p_2)$. But then

$$\rho_1^{-1}\rho_2(p_2) = \rho_1^{-1}(p)$$
$$= p_1$$

so p_1 and p_2 lie in the same orbit. Since $\mathcal{M}$ contains exactly one representative of this equivalence class, we must have $p_1 = p_2$. But then p_1 is a fixed point of $\rho_1^{-1}\rho_2$. Since $p_1 \notin \mathcal{D}$, we must have $\rho_1^{-1}\rho_2 = 1$, so $\rho_1 = \rho_2$. So we have shown that if $\rho_1 \neq \rho_2$, then $\rho_1(\mathcal{M})$ and $\rho_2(\mathcal{M})$ are disjoint.

From this it follows that our partition (4) gives rise to a *partition* of $\mathbf{S}^2 - \mathcal{D}$:

$$(5) \qquad \mathbf{S}^2 - \mathcal{D} = \mathfrak{F}(\varphi, \psi)\mathcal{M} = \mathcal{M} \uplus \mathfrak{W}(\varphi)\mathcal{M} \uplus \mathfrak{W}(\varphi^{-1})\mathcal{M} \uplus \mathfrak{W}(\psi)\mathcal{M} \uplus \mathfrak{W}(\psi^{-1})\mathcal{M}.$$

We are now in a position to apply Corollary 2.2. By our disjointness above, we also get disjoint unions

$$(6) \qquad \mathbf{S}^2 - \mathcal{D} = \varphi^{-1}(\mathfrak{W}(\varphi)\mathcal{M}) \uplus \mathfrak{W}(\varphi^{-1})\mathcal{M}$$

and

$$(7) \qquad \mathbf{S}^2 - \mathcal{D} = \psi^{-1}(\mathfrak{W}(\psi)\mathcal{M}) \uplus \mathfrak{W}(\psi^{-1})\mathcal{M}.$$

Let us write this more suggestively. Define four subsets of $\mathbf{S}^2$ as follows :

$$
\begin{aligned}
\mathcal{W}_\varphi &= \mathfrak{M}(\varphi)\mathcal{M} \\
\mathcal{W}_{\varphi^{-1}} &= \mathfrak{M}(\varphi^{-1})\mathcal{M} \\
\mathcal{W}_\psi &= \mathfrak{M}(\psi)\mathcal{M} \\
\mathcal{W}_{\psi^{-1}} &= \mathfrak{M}(\psi^{-1})\mathcal{M}.
\end{aligned}
$$

Then (5) tells us that

$$\mathbf{S}^2 - \mathcal{D} = \mathcal{M} \uplus \mathcal{W}_\varphi \uplus \mathcal{W}_{\varphi^{-1}} \uplus \mathcal{W}_\psi \uplus \mathcal{W}_{\psi^{-1}}$$

and (6) and (7) tell us that we can rotate $\mathcal{W}_\varphi$ and $\mathcal{W}_\psi$ to obtain

$$\mathbf{S}^2 - \mathcal{D} = \varphi^{-1}\mathcal{W}_\varphi \uplus \mathcal{W}_{\varphi^{-1}}$$

and

$$\mathbf{S}^2 - \mathcal{D} = \psi^{-1}\mathcal{W}_\psi \uplus \mathcal{W}_{\psi^{-1}}.$$

This is known as the Hausdorff paradox.

Theorem 4.1 (Hausdorff Paradox). *There is a countable set $\mathcal{D} \subseteq \mathbf{S}^2$ such that $\mathbf{S}^2 - \mathcal{D}$ can be divided into 5 pieces which can be rotated to form 2 copies of $\mathbf{S}^2 - \mathcal{D}$.*

Let us pause to consider the sets we have constructed. $\mathcal{D}$ is an explicitly given countable set: it is simply the points on the axes of a known set of rotations $\mathfrak{F}(\varphi, \psi)$. The other 5 sets, $\mathcal{M}, \mathcal{W}_\varphi, \mathcal{W}_{\varphi^{-1}}, \mathcal{W}_\psi, \mathcal{W}_{\psi^{-1}}$, are rather mysterious. They are certain uncountable sets which only exist due to the axiom of choice, and there is very little more that we can say about them.

5. The Banach-Tarski Paradox on $\mathbf{S}^2$

We must now use our 'paradoxical' decomposition of $\mathbf{S}^2 - \mathcal{D}$ to construct one on all of $\mathbf{S}^2$. First we make three simple observations about disjoint unions.

(1) If a set $\mathcal{A}$ is a disjoint union of subsets $\mathcal{A}_i$,

(8)
$$\mathcal{A} = \biguplus_i \mathcal{A}_i,$$

and if each A_i in turn is a disjoint union of subsets $\mathcal{A}_{ij}$,

$$\mathcal{A}_i = \biguplus_j \mathcal{A}_{ij},$$

then $\mathcal{A}$ is the disjoint union of the sets $\mathcal{A}_{ij}$:

$$\mathcal{A} = \biguplus_{i,j} \mathcal{A}_{ij}.$$

(2) If a set $\mathcal{A} \in \mathbf{S}^2$ is a disjoint union of subsets $\mathcal{A}_i$, as in (8), and if $\rho \in SO_3$ is any rotation, then $\rho(\mathcal{A})$ is the disjoint union of the sets $\rho(\mathcal{A}_i)$:

$$\rho(\mathcal{A}) = \biguplus_i \rho(\mathcal{A}_i).$$

(3) If a set $\mathcal{A} \in \mathbf{S}^2$ is a disjoint union of subsets $\mathcal{A}_i$, as in (8), and if $\mathcal{B} \in \mathbf{S}^2$ is some other set, then $\mathcal{A} \cap \mathcal{B}$ is the disjoint union of the (possibly empty) sets $\mathcal{A}_i \cap \mathcal{B}$:

$$\mathcal{A} \cap \mathcal{B} = \biguplus_i (\mathcal{A}_i \cap \mathcal{B}).$$

These three facts will be enough to insure that all of the operations we will perform below preserve disjoint unions. Since there will be enough else going on to keep us busy, we will not comment on these issues again.

We would now like to find a three-dimensional analogue of the circle construction in the introduction. We must find a rotation $\sigma \in SO_3$ which we can somehow use to 'erase' the set $\mathcal{D}$. To begin, select a line ℓ through the origin which does not intersect $\mathcal{D}$; this is certainly possible since $\mathcal{D}$ is countable and there are uncountably many lines through the origin in $\mathbb{R}^3$. Define $\ell_\theta \in SO_3$ to be the counterclockwise rotation by θ radians around the line ℓ. (See Figure 4.) We would like σ to be one of these rotations, with the additional property that no matter how many times we apply it to a point in $\mathcal{D}$ we never again land in $\mathcal{D}$. So, define a set T by

$$T = \{\theta \in [0, 2\pi) \mid \text{There is a } p \in \mathcal{D} \text{ and a positive integer } n \text{ with } \ell_{n\theta}(p) \in \mathcal{D}\}.$$
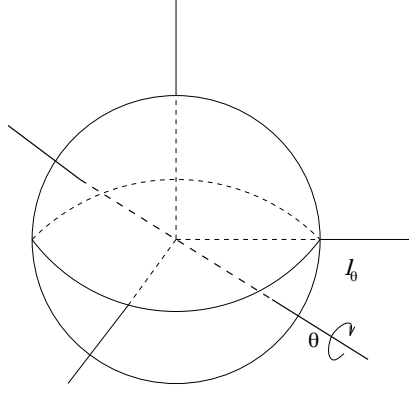
FIGURE 4. The rotations ℓ_θ

Now, for each of the countably many pairs of points $(p, q) \in \mathcal{D} \times \mathcal{D}$ we get at most countably many angles to put in T, so T will be countable. So, since $[0, 2\pi)$ is uncountable we may choose some $\theta_0 \notin T$. Then set

$$\sigma = \ell_{\theta_0}.$$

By its construction we have that

$$\sigma^n(\mathcal{D}) \cap \mathcal{D} = \emptyset$$

for all positive integers n. In fact, applying σ^m to this, we find that

$$\sigma^{m+n}(\mathcal{D}) \cap \sigma^m(\mathcal{D}) = \emptyset,$$

and then replacing n by $n - m$ we have

$$\sigma^n(\mathcal{D}) \cap \sigma^m(\mathcal{D}) = \emptyset$$

for all integers m, n with $m \neq n$. Now, define

$$\mathcal{E} = \biguplus_{n=0}^{\infty} \sigma^n(\mathcal{D}) \subseteq \mathbf{S}^2.$$

Then $\mathcal{D} \subsetneq \mathcal{E}$.

Next, consider the disjoint union

$$\mathbf{S}^2 = (\mathbf{S}^2 - \mathcal{E}) \uplus \mathcal{E}.$$

Since

$$\begin{aligned}
\sigma \mathcal{E} &= \sigma \biguplus_{n=0}^{\infty} \sigma^n(\mathcal{D}) \\
&= \biguplus_{n=0}^{\infty} \sigma \sigma^n(\mathcal{D}) \\
&= \biguplus_{n=0}^{\infty} \sigma^{n+1}(\mathcal{D}) \\
&= \biguplus_{n=1}^{\infty} \sigma^n(\mathcal{D}) \\
&= \mathcal{E} - \mathcal{D},
\end{aligned}$$

we have

$$\mathbf{S}^2 - \mathcal{D} = (\mathbf{S}^2 - \mathcal{E}) \uplus (\mathcal{E} - \mathcal{D}) = (\mathbf{S}^2 - \mathcal{E}) \uplus \sigma(\mathcal{E}).$$

That is, rotating $\mathcal{E}$ by σ and leaving the rest of $\mathbf{S}^2$ fixed we have 'erased' $\mathcal{D}$ and obtained $\mathbf{S}^2 - \mathcal{D}$. Looking at this the other way, starting with a decomposition of $\mathbf{S}^2 - \mathcal{D}$ we will now be able to obtain one of $\mathbf{S}^2$. We will now carry this out with our decompositions (5), (6) and (7) of $\mathbf{S}^2 - \mathcal{D}$.

First we make two simple observations that will make our manipulations a lot easier to follow. Since $\mathcal{D} \subsetneq \mathcal{E}$, we have $\mathbf{S}^2 - \mathcal{E} \subsetneq \mathbf{S}^2 - \mathcal{D}$, so

$$\mathbf{S}^2 - \mathcal{E} = (\mathbf{S}^2 - \mathcal{D}) \cap (\mathbf{S}^2 - \mathcal{E}).$$

Also,

$$\begin{aligned}
\mathcal{E} &= \sigma^{-1}\sigma(\mathcal{E}) \\
&= \sigma^{-1}(\mathcal{E} - \mathcal{D}) \\
&= \sigma^{-1}\left[(\mathbf{S}^2 - \mathcal{D}) \cap \mathcal{E}\right].
\end{aligned}$$

Thus, we have

$$\begin{aligned}
\mathbf{S}^2 &= (\mathbf{S}^2 - \mathcal{E}) \uplus \mathcal{E} \\
&= \left[(\mathbf{S}^2 - \mathcal{D}) \cap (\mathbf{S}^2 - \mathcal{E})\right] \uplus \sigma^{-1}\left[(\mathbf{S}^2 - \mathcal{D}) \cap \mathcal{E}\right].
\end{aligned}$$

We now want to plug in our three decompositions of $\mathbf{S}^2 - \mathcal{D}$. So,

$$\begin{aligned}
\mathbf{S}^2 &= \left[(\mathcal{M} \uplus \mathcal{W}_\varphi \uplus \mathcal{W}_{\varphi^{-1}} \uplus \mathcal{W}_\psi \uplus \mathcal{W}_{\psi^{-1}}) \cap (\mathbf{S}^2 - \mathcal{E})\right] \uplus \sigma^{-1}\left[(\mathcal{M} \uplus \mathcal{W}_\varphi \uplus \mathcal{W}_{\varphi^{-1}} \uplus \mathcal{W}_\psi \uplus \mathcal{W}_{\psi^{-1}}) \cap \mathcal{E}\right] \\
&= \left[(\varphi^{-1}\mathcal{W}_\varphi \uplus \mathcal{W}_{\varphi^{-1}}) \cap (\mathbf{S}^2 - \mathcal{E})\right] \uplus \left[(\varphi^{-1}\mathcal{W}_\varphi \uplus \mathcal{W}_{\varphi^{-1}}) \cap \mathcal{E}\right] \\
&= \left[(\psi^{-1}\mathcal{W}_\psi \uplus \mathcal{W}_{\psi^{-1}}) \cap (\mathbf{S}^2 - \mathcal{E})\right] \uplus \left[(\psi^{-1}\mathcal{W}_\psi \uplus \mathcal{W}_{\psi^{-1}}) \cap \mathcal{E}\right].
\end{aligned}$$

This suggests that we should define

$$\begin{aligned}
\mathcal{M}^0 &= \mathcal{M} \cap (\mathbf{S}^2 - \mathcal{E}) & \mathcal{M}^1 &= \mathcal{M} \cap \mathcal{E} \\
\mathcal{W}_\varphi^0 &= \mathcal{W}_\varphi \cap (\mathbf{S}^2 - \mathcal{E}) & \mathcal{W}_\varphi^1 &= \mathcal{W}_\varphi \cap \mathcal{E} \\
\mathcal{W}_{\varphi^{-1}}^0 &= \mathcal{W}_{\varphi^{-1}} \cap (\mathbf{S}^2 - \mathcal{E}) & \mathcal{W}_{\varphi^{-1}}^1 &= \mathcal{W}_{\varphi^{-1}} \cap \mathcal{E} \\
\mathcal{W}_\psi^0 &= \mathcal{W}_\psi \cap (\mathbf{S}^2 - \mathcal{E}) & \mathcal{W}_\psi^1 &= \mathcal{W}_\psi \cap \mathcal{E} \\
\mathcal{W}_{\psi^{-1}}^0 &= \mathcal{W}_{\psi^{-1}} \cap (\mathbf{S}^2 - \mathcal{E}) & \mathcal{W}_{\psi^{-1}}^1 &= \mathcal{W}_{\psi^{-1}} \cap \mathcal{E}.
\end{aligned}$$

Then,

$$\begin{aligned}
\mathbf{S}^2 &= (\mathcal{M}^0 \uplus \mathcal{W}_\varphi^0 \uplus \mathcal{W}_{\varphi^{-1}}^0 \uplus \mathcal{W}_\psi^0 \uplus \mathcal{W}_{\psi^{-1}}^0) \uplus \sigma^{-1}(\mathcal{M}^1 \uplus \mathcal{W}_\varphi^1 \uplus \mathcal{W}_{\varphi^{-1}}^1 \uplus \mathcal{W}_\psi^1 \uplus \mathcal{W}_{\psi^{-1}}^1) \\
&= \mathcal{M}^0 \uplus \sigma^{-1}\mathcal{M}^1 \uplus \mathcal{W}_\varphi^0 \uplus \sigma^{-1}\mathcal{W}_\varphi^1 \uplus \mathcal{W}_{\varphi^{-1}}^0 \uplus \sigma^{-1}\mathcal{W}_{\varphi^{-1}}^1 \uplus \mathcal{W}_\psi^0 \uplus \sigma^{-1}\mathcal{W}_\psi^1 \uplus \mathcal{W}_{\psi^{-1}}^0 \uplus \sigma^{-1}\mathcal{W}_{\psi^{-1}}^1.
\end{aligned}$$

In fact, in order to also use the decompositions (6) and (7), we further break up $\mathcal{W}_\varphi$ and $\mathcal{W}_\psi$ as follows:

$$\begin{aligned}
\mathcal{W}_\varphi^{00} &= \mathcal{W}_\varphi^0 \cap \varphi(\mathbf{S}^2 - \mathcal{E}) &= \mathcal{W}_\varphi \cap (\mathbf{S}^2 - \mathcal{E}) \cap \varphi(\mathbf{S}^2 - \mathcal{E}) \\
\mathcal{W}_\varphi^{01} &= \mathcal{W}_\varphi^0 \cap \varphi(\mathcal{E}) &= \mathcal{W}_\varphi \cap (\mathbf{S}^2 - \mathcal{E}) \cap \varphi(\mathcal{E}) \\
\mathcal{W}_\varphi^{10} &= \mathcal{W}_\varphi^1 \cap \varphi(\mathbf{S}^2 - \mathcal{E}) &= \mathcal{W}_\varphi \cap \mathcal{E} \cap \varphi(\mathbf{S}^2 - \mathcal{E}) \\
\mathcal{W}_\varphi^{11} &= \mathcal{W}_\varphi^1 \cap \varphi(\mathcal{E}) &= \mathcal{W}_\varphi \cap \mathcal{E} \cap \varphi(\mathcal{E}) \\
\mathcal{W}_\psi^{00} &= \mathcal{W}_\psi^0 \cap \psi(\mathbf{S}^2 - \mathcal{E}) &= \mathcal{W}_\psi \cap (\mathbf{S}^2 - \mathcal{E}) \cap \psi(\mathbf{S}^2 - \mathcal{E}) \\
\mathcal{W}_\psi^{01} &= \mathcal{W}_\psi^0 \cap \psi(\mathcal{E}) &= \mathcal{W}_\psi \cap (\mathbf{S}^2 - \mathcal{E}) \cap \psi(\mathcal{E}) \\
\mathcal{W}_\psi^{10} &= \mathcal{W}_\psi^1 \cap \psi(\mathbf{S}^2 - \mathcal{E}) &= \mathcal{W}_\psi \cap \mathcal{E} \cap \psi(\mathbf{S}^2 - \mathcal{E}) \\
\mathcal{W}_\psi^{11} &= \mathcal{W}_\psi^1 \cap \psi(\mathcal{E}) &= \mathcal{W}_\psi \cap \mathcal{E} \cap \psi(\mathcal{E})
\end{aligned}$$

Now,

$$\begin{aligned}
\mathbf{S}^2 = {} &\mathcal{M}^0 \uplus \sigma^{-1}\mathcal{M}^1 \uplus \mathcal{W}_\varphi^{00} \uplus \mathcal{W}_\varphi^{01} \uplus \sigma^{-1}\mathcal{W}_\varphi^{10} \uplus \sigma^{-1}\mathcal{W}_\varphi^{11} \uplus \mathcal{W}_{\varphi^{-1}}^0 \uplus \sigma^{-1}\mathcal{W}_{\varphi^{-1}}^1 \uplus \\
&\mathcal{W}_\psi^{00} \uplus \mathcal{W}_\psi^{01} \uplus \sigma^{-1}\mathcal{W}_\psi^{10} \uplus \sigma^{-1}\mathcal{W}_\psi^{11} \uplus \mathcal{W}_{\psi^{-1}}^0 \uplus \sigma^{-1}\mathcal{W}_{\psi^{-1}}^1.
\end{aligned}$$

Also,

$$\begin{aligned}
\mathbf{S}^2 &= \left[(\varphi^{-1}\mathcal{W}_\varphi \uplus \mathcal{W}_{\varphi^{-1}}) \cap (\mathbf{S}^2 - \mathcal{E})\right] \uplus \sigma^{-1}\left[(\varphi^{-1}\mathcal{W}_\varphi \uplus \mathcal{W}_{\varphi^{-1}}) \cap \mathcal{E}\right]\\
&= \left[(\varphi^{-1}\mathcal{W}_\varphi \cap (\mathbf{S}^2 - \mathcal{E})) \uplus (\mathcal{W}_{\varphi^{-1}} \cap (\mathbf{S}^2 - \mathcal{E}))\right] \uplus \sigma^{-1}\left[(\varphi^{-1}\mathcal{W}_\varphi \cap \mathcal{E}) \uplus (\mathcal{W}_{\varphi^{-1}} \cap \mathcal{E})\right]\\
&= \left[\varphi^{-1}\left(\mathcal{W}_\varphi \cap \varphi(\mathbf{S}^2 - \mathcal{E})\right) \uplus (\mathcal{W}_{\varphi^{-1}} \cap (\mathbf{S}^2 - \mathcal{E}))\right] \uplus \sigma^{-1}\left[\varphi^{-1}\left(\mathcal{W}_\varphi \cap \varphi(\mathcal{E})\right) \uplus (\mathcal{W}_{\varphi^{-1}} \cap \mathcal{E})\right]\\
&= \left[\varphi^{-1}(\mathcal{W}_\varphi^{00} \uplus \mathcal{W}_\varphi^{10}) \uplus \mathcal{W}_{\varphi^{-1}}^0\right] \uplus \sigma^{-1}\left[\varphi^{-1}(\mathcal{W}_\varphi^{01} \uplus \mathcal{W}_\varphi^{11}) \uplus \mathcal{W}_{\varphi^{-1}}^1\right]\\
&= \varphi^{-1}\mathcal{W}_\varphi^{00} \uplus \varphi^{-1}\mathcal{W}_\varphi^{10} \uplus \mathcal{W}_{\varphi^{-1}}^0 \uplus \sigma^{-1}\varphi^{-1}\mathcal{W}_\varphi^{01} \uplus \sigma^{-1}\varphi^{-1}\mathcal{W}_\varphi^{11} \uplus \sigma^{-1}\mathcal{W}_{\varphi^{-1}}^1.
\end{aligned}$$

Similarly,

$$\mathbf{S}^2 = \psi^{-1}\mathcal{W}_\psi^{00} \uplus \psi^{-1}\mathcal{W}_\psi^{10} \uplus \mathcal{W}_{\psi^{-1}}^0 \uplus \sigma^{-1}\psi^{-1}\mathcal{W}_\psi^{01} \uplus \sigma^{-1}\psi^{-1}\mathcal{W}_\psi^{11} \uplus \sigma^{-1}\mathcal{W}_{\psi^{-1}}^1.$$

We will rewrite this one last time. Define

$$\begin{aligned}
\mathcal{A}_1 &= \mathcal{M}^0 & \mathcal{A}_2 &= \sigma^{-1}\mathcal{M}^1\\
\mathcal{A}_3 &= \mathcal{W}_\varphi^{00} & \mathcal{A}_4 &= \sigma^{-1}\mathcal{W}_\varphi^{10}\\
\mathcal{A}_5 &= \mathcal{W}_\varphi^{01} & \mathcal{A}_6 &= \sigma^{-1}\mathcal{W}_\varphi^{11}\\
\mathcal{A}_7 &= \mathcal{W}_{\varphi^{-1}}^0 & \mathcal{A}_8 &= \sigma^{-1}\mathcal{W}_{\varphi^{-1}}^1\\
\mathcal{A}_9 &= \mathcal{W}_\psi^{00} & \mathcal{A}_{10} &= \sigma^{-1}\mathcal{W}_\psi^{10}\\
\mathcal{A}_{11} &= \mathcal{W}_\psi^{01} & \mathcal{A}_{12} &= \sigma^{-1}\mathcal{W}_\psi^{11}\\
\mathcal{A}_{13} &= \mathcal{W}_{\psi^{-1}}^0 & \mathcal{A}_{14} &= \sigma^{-1}\mathcal{W}_{\psi^{-1}}^1.
\end{aligned}$$

Then,

$$\begin{aligned}
\mathbf{S}^2 &= \mathcal{A}_1 \uplus \mathcal{A}_2 \uplus \cdots \uplus \mathcal{A}_{14}\\
\mathbf{S}^2 &= \varphi^{-1}\mathcal{A}_3 \uplus \varphi^{-1}\sigma\mathcal{A}_4 \uplus \sigma^{-1}\varphi^{-1}\mathcal{A}_5 \uplus \sigma^{-1}\varphi^{-1}\sigma\mathcal{A}_6 \uplus \mathcal{A}_7 \uplus \mathcal{A}_8\\
\mathbf{S}^2 &= \psi^{-1}\mathcal{A}_9 \uplus \psi^{-1}\sigma\mathcal{A}_{10} \uplus \sigma^{-1}\psi^{-1}\mathcal{A}_{11} \uplus \sigma^{-1}\psi^{-1}\sigma\mathcal{A}_{12} \uplus \mathcal{A}_{13} \uplus \mathcal{A}_{14}.
\end{aligned}$$

This, finally, proves the Banach-Tarski paradox for $\mathbf{S}^2$.

Theorem 5.1 (The Banach-Tarski Paradox, Version 1). *The sphere $\mathbf{S}^2$ can be partitioned into a finite number of pieces which can be rotated to form two copies of $\mathbf{S}^2$.*

6. THE BANACH TARSKI PARADOX ON $\mathbf{B}^3$

The next step, then, is to extend this construction to the solid ball $\mathbf{B}^3$. Note that this means that we no longer must work exclusively with rotations in SO_3; we can also work with some rotations around axes not containing the origin. To avoid getting completely lost in notation, we define

$$\begin{aligned}
\rho_3 &= \varphi^{-1} & \rho_4 &= \varphi^{-1}\sigma\\
\rho_5 &= \sigma^{-1}\varphi^{-1} & \rho_6 &= \sigma^{-1}\varphi^{-1}\sigma\\
\rho_7 &= 1 & \rho_8 &= 1\\
\rho_9 &= \psi^{-1} & \rho_{10} &= \psi^{-1}\sigma\\
\rho_{11} &= \sigma^{-1}\psi^{-1} & \rho_{12} &= \sigma^{-1}\psi^{-1}\sigma\\
\rho_{13} &= 1 & \rho_{14} &= 1.
\end{aligned}$$

Thus, we can now write

$$\mathbf{S}^2 = \biguplus_{i=1}^{14} \mathcal{A}_i = \biguplus_{i=3}^{8} \rho_i\mathcal{A}_i = \biguplus_{i=9}^{14} \rho_i\mathcal{A}_i.$$

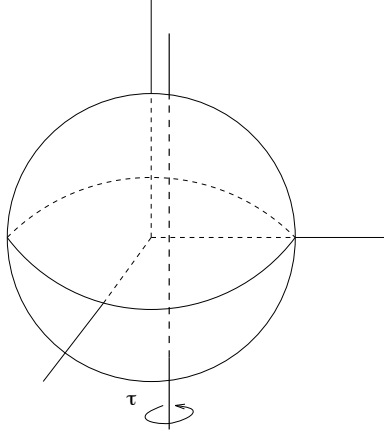We begin our extension in the obvious way: Define the sets

$$\mathcal{B}_i \subseteq \mathbf{B}^3, 1 \leq i \leq 14,$$

by extending each $\mathcal{A}_i$ radially inward. That is, using spherical coordinates,

$$\mathcal{B}_i = \{(r, \theta, \phi) \mid (1, \theta, \phi) \in \mathcal{A}_i, 0 < r \leq 1\}.$$

This gives us partitions of all of $\mathbf{B}^3$ except for the origin 0:

$$\mathbf{B}^3 - 0 = \biguplus_{i=1}^{14} \mathcal{B}_i = \biguplus_{i=3}^{8} \rho_i\mathcal{B}_i = \biguplus_{i=9}^{14} \rho_i\mathcal{B}_i.$$

FIGURE 5. The Rotation τ

We must once again find a way to deal with this little complication. Luckily, essentially the same trick that we used to deal with $\mathcal{D}$ will work.

There are many ways to do this, and all of our choices here are pretty much arbitrary. First, choose a point near 0, say $(\frac{1}{10}, 0, 0)$. Next take the line parallel to the y-axis through this point. Let τ be a rotation around this line such that $\tau^n(0) \neq 0$ for all $n > 0$. (If we are willing to use the fact that π is irrational, then an angle of 1 radian will do. Otherwise, we can still use some angle like $\sqrt{2}^{\circ}$, since no multiple of $\sqrt{2}$ can give 360.) Note that $\tau^n(0) \in \mathbf{B}^3$ for all n.

Now, since $\tau^n(0) \neq 0$ for all n, as before we find that

$$\tau^n(0) \neq \tau^m(0)$$

for all $m \neq n$. So define

$$\mathfrak{F} = \biguplus_{n=0}^{\infty} \tau^n(0).$$

Then $\tau(\mathfrak{F}) = \mathfrak{F} - 0$. So, since

$$\mathbf{B}^3 = (\mathbf{B}^3 - \mathfrak{F}) \uplus \mathfrak{F}$$

we have

$$\mathbf{B}^3 - 0 = (\mathbf{B}^3 - \mathfrak{F}) \uplus \tau(\mathfrak{F}).$$

Again, as before we find that

$$\mathbf{B}^3 = \left[(\mathbf{B}^3 - 0) \cap (\mathbf{B}^3 - \mathfrak{F}) \right] \uplus \tau^{-1} \left[(\mathbf{B}^3 - 0) \cap \mathfrak{F} \right].$$

Thus,

$$\begin{aligned}
\mathbf{B}^3 &= \left[\biguplus_{i=1}^{14} \mathcal{B}_i \cap (\mathbf{B}^3 - \mathfrak{F}) \right] \uplus \tau^{-1} \left[\biguplus_{i=1}^{14} \mathcal{B}_i \cap \mathfrak{F} \right] \\
&= \left[\biguplus_{i=3}^{8} \rho_i \mathcal{B}_i \cap (\mathbf{B}^3 - \mathfrak{F}) \right] \uplus \tau^{-1} \left[\biguplus_{i=3}^{8} \rho_i \mathcal{B}_i \cap \mathfrak{F} \right] \\
&= \left[\biguplus_{i=9}^{14} \rho_i \mathcal{B}_i \cap (\mathbf{B}^3 - \mathfrak{F}) \right] \uplus \tau^{-1} \left[\biguplus_{i=9}^{14} \rho_i \mathcal{B}_i \cap \mathfrak{F} \right].
\end{aligned}$$

So, finally, as before we define

$$\begin{aligned}
\mathcal{B}_i^0 &= \mathcal{A}_i \cap (\mathbf{B}^3 - \mathfrak{F}) \\
\mathcal{B}_i^1 &= \mathcal{A}_i \cap \mathfrak{F}
\end{aligned}$$

for $i = 1, 2$, and

$$\mathcal{B}_i^{00} = \mathcal{B}_i \cap (\mathbf{B}^3 - \mathfrak{F}) \cap \rho_i^{-1}(\mathbf{B}^3 - \mathfrak{F})$$
$$\mathcal{B}_i^{01} = \mathcal{B}_i \cap (\mathbf{B}^3 - \mathfrak{F}) \cap \rho_i^{-1}(\mathfrak{F})$$
$$\mathcal{B}_i^{10} = \mathcal{B}_i \cap \mathfrak{F} \cap \rho_i^{-1}(\mathbf{B}^3 - \mathfrak{F})$$
$$\mathcal{B}_i^{11} = \mathcal{B}_i \cap \mathfrak{F} \cap \rho_i^{-1}(\mathfrak{F})$$

for $i = 3, \ldots, 14$. (Many of these may be empty, but it does not matter.)

Thus,

$$
\begin{aligned}
\mathbf{S}^2 &= \mathcal{B}_1^0 \uplus \tau^{-1}\mathcal{B}_1^1 \uplus \mathcal{B}_2^0 \uplus \tau^{-1}\mathcal{B}_2^1 \uplus \biguplus_{i=3}^{14} \mathcal{B}_i^{00} \uplus \biguplus_{i=3}^{14} \mathcal{B}_i^{01} \uplus \biguplus_{i=3}^{14} \tau^{-1}\mathcal{B}_i^{10} \uplus \biguplus_{i=3}^{14} \tau^{-1}\mathcal{B}_i^{11} \\
&= \biguplus_{i=3}^{8} \rho_i\mathcal{B}_i^{00} \uplus \biguplus_{i=3}^{8} \rho_i\mathcal{B}_i^{10} \uplus \biguplus_{i=3}^{8} \tau^{-1}\rho_i\mathcal{B}_i^{01} \uplus \biguplus_{i=3}^{8} \tau^{-1}\rho_i\mathcal{B}_i^{11} \\
&= \biguplus_{i=9}^{14} \rho_i\mathcal{B}_i^{00} \uplus \biguplus_{i=9}^{14} \rho_i\mathcal{B}_i^{10} \uplus \biguplus_{i=9}^{14} \tau^{-1}\rho_i\mathcal{B}_i^{01} \uplus \biguplus_{i=9}^{14} \tau^{-1}\rho_i\mathcal{B}_i^{11} .
\end{aligned}
$$

This, then, is our final paradoxical decomposition of $\mathbf{B}^3$. Of course, there is nothing special about the ball of radius 1 centered at the origin; using rotations around axes not containing the origin we can obtain a similar decomposition of any solid ball in $\mathbb{R}^3$.

Theorem 6.1 (The Banach-Tarski Paradox, Version 2). *Any solid ball* $\mathbf{B}$ *in* $\mathbb{R}^3$ *can be cut into finitely many pieces which can be rotated to form 2 copies of* $\mathbf{B}$.

7. The Generalized Banach-Tarski Paradox

To get the final version of the Banach-Tarski paradox we will have to consider subsets of $\mathbb{R}^3$ much more general than $\mathbf{B}^3$. It will thus no longer be at all practical to carry around the sort of 'explicit' decompositions that we have been. We will therefore have to formalize our notions somewhat. First, since we will need translations as well as rotations to do our decompositions, let $\mathbf{M}_3$ be the group of translations and rotations in $\mathbb{R}^3$. (For an analysis of this group, see [1, Chapter 5]. We will not need any deep facts about it.)

Definition 1. Let $\mathcal{A}$ and $\mathcal{B}$ be two subsets of $\mathbb{R}^3$. We will say that $\mathcal{A}$ and $\mathcal{B}$ are *equidecomposable*, written $\mathcal{A} \sim \mathcal{B}$, if there exist partitions of $\mathcal{A}$ and $\mathcal{B}$ into the same number of pieces,

(9)
$$\mathcal{A} = \biguplus_{i=1}^{n} \mathcal{A}_i, \qquad \mathcal{B} = \biguplus_{i=1}^{n} \mathcal{B}_i$$

and $g_i \in \mathbf{M}_3$ such that

$$g_i\mathcal{A}_i = \mathcal{B}_i$$

for all i.

Roughly, $\mathcal{A}$ and $\mathcal{B}$ are equidecomposable if we can cut $\mathcal{A}$ into finitely many pieces which we can then rearrange by rigid motions to form exactly $\mathcal{B}$. Let us restate the Banach-Tarski paradox in this language.

Corollary 7.1. *Let* $\mathbf{B}$ *be a ball in* $\mathbb{R}^3$ *of radius* r. *Then there is a subset* $\mathbf{B}_0$ *of* $\mathbf{B}$ *such that* $\mathbf{B} - \mathbf{B}_0$ *is equidecomposable with a disjoint union of two balls of radius* r.

Proof. Simply take $\mathbf{B}_0 = \mathcal{B}_1^0 \uplus \tau^{-1}\mathcal{B}_1^1 \uplus \mathcal{B}_2^0 \uplus \tau^{-1}\mathcal{B}_2^1$. This is then precisely what we proved in Section 6. $\square$

We now check that the relation of equidecomposability is an equivalence relation.

(1) Reflexivity: Take $\mathcal{A}_1 = \mathcal{A}$, $g_1 = 1$. Then $g_1\mathcal{A}_1 = \mathcal{A}_1$, so $\mathcal{A} \sim \mathcal{A}$.

(2) Symmetry: Suppose $\mathcal{A} \sim \mathcal{B}$. Then we can write $\mathcal{A}$ and $\mathcal{B}$ in the form (9) with $g_i \in \mathbf{M}_3$ such that $g_i\mathcal{A}_i = \mathcal{B}_i$ for all i. Then $\mathcal{A}_i = g_i^{-1}\mathcal{B}_i$, so $\mathcal{B} \sim \mathcal{A}$.

(3) Transitivity: Suppose $\mathcal{A} \sim \mathcal{B}$ and $\mathcal{B} \sim \mathcal{C}$. Then we can write

$$\mathcal{A} = \biguplus_{i=1}^{n} \mathcal{A}_i, \qquad \mathcal{B} = \biguplus_{i=1}^{n} \mathcal{B}_i = \biguplus_{j=1}^{n'} \mathcal{B}_j', \qquad \mathcal{C} = \biguplus_{j=1}^{n'} \mathcal{C}_j$$

with $g_i, g_j' \in \mathbf{M}_3$ satisfying $g_i \mathcal{A}_i = \mathcal{B}_i$, $g_j' \mathcal{B}_j' = \mathcal{C}_j$ for all i and j.

Set $\mathcal{A}_{ij} = g_i^{-1}(\mathcal{B}_i \cap \mathcal{B}_j')$. Then

$$
\biguplus_{i,j} \mathcal{A}_{ij} = \biguplus_{i=1}^{n} \biguplus_{j=1}^{n'} \mathcal{A}_{ij}
$$

$$
= \biguplus_{i=1}^{n} \biguplus_{j=1}^{n'} g_i^{-1}(\mathcal{B}_i \cap \mathcal{B}_j')
$$

$$
= \biguplus_{i=1}^{n} g_i^{-1} \left(\mathcal{B}_i \cap \biguplus_{j=1}^{n'} \mathcal{B}_j \right)
$$

$$
= \biguplus_{i=1}^{n} g_i^{-1}(\mathcal{B}_i \cap \mathcal{B})
$$

$$
= \biguplus_{i=1}^{n} g_i^{-1}(\mathcal{B}_i)
$$

$$
= \biguplus_{i=1}^{n} \mathcal{A}_i
$$

$$
= \mathcal{A}
$$

so the $\mathcal{A}_{ij}$ partition $\mathcal{A}$. Further,

$$
\biguplus_{i,j} g_j' g_i \mathcal{A}_{ij} = \biguplus_{i,j} g_j' g_i g_i^{-1}(\mathcal{B}_i \cap \mathcal{B}_j')
$$

$$
= \biguplus_{j=1}^{n'} \biguplus_{i=1}^{n} g_j'(\mathcal{B}_i \cap \mathcal{B}_j')
$$

$$
= \biguplus_{j=1}^{n'} g_j' \left(\mathcal{B}_j' \cap \biguplus_{i=1}^{n} \mathcal{B}_i \right)
$$

$$
= \biguplus_{j=1}^{n'} g_j'(\mathcal{B}_j' \cap \mathcal{B})
$$

$$
= \biguplus_{j=1}^{n'} g_j'(\mathcal{B}_j)
$$

$$
= \biguplus_{j=1}^{n'} \mathcal{C}_j
$$

$$
= \mathcal{C}.
$$

So $\mathcal{A} \sim \mathcal{C}$.

In fact, the relation $\sim$ has two other important properties.

(4) If $\mathcal{A} \sim \mathcal{B}$, then there is a bijection $\alpha : \mathcal{A} \to \mathcal{B}$ such that $\mathcal{C} \sim \alpha(\mathcal{C})$ for any $\mathcal{C} \subseteq \mathcal{A}$.

Proof. Since $\mathcal{A} \sim \mathcal{B}$ we have decompositions as in (9) together with $g_i \in \mathbf{M}_3$ satisfying $g_i \mathcal{A}_i = \mathcal{B}_i$. Now, for any $p \in \mathcal{A}$, let $\mathcal{A}_{i_p}$ be the subset in which p lies. Then define $\alpha(p) = g_{i_p}(p)$. α will be a bijection since the g_i are all bijections. Further, if $\mathcal{C} \subseteq \mathcal{A}$, write

$$
\mathcal{C} = \biguplus_{i=1}^{n} (\mathcal{A}_i \cap \mathcal{C}).
$$

Then by the definition of α we have

$$\alpha(\mathcal{C}) = \biguplus_{i=1}^{n} g_i(\mathcal{A}_i \cap \mathcal{C}),$$

so $\mathcal{C} \sim \alpha(\mathcal{C})$. $\qquad\square$

(5) If $\mathcal{A} \cap \mathcal{B} = \mathcal{A}' \cap \mathcal{B}' = \emptyset$, and if $\mathcal{A} \sim \mathcal{A}'$ and $\mathcal{B} \sim \mathcal{B}'$, then $\mathcal{A} \uplus \mathcal{B} \sim \mathcal{A}' \uplus \mathcal{B}'$.

Proof. Write

$$\mathcal{A} = \biguplus_{i=1}^{n} \mathcal{A}_i, \qquad \mathcal{A}' = \biguplus_{i=1}^{n} \mathcal{A}_i', \qquad \mathcal{B} = \biguplus_{j=1}^{n'} \mathcal{B}_j, \qquad \mathcal{B}' = \biguplus_{j=1}^{n'} \mathcal{B}_j',$$

with $g_i, g_i' \in \mathbf{M}_3$ satisfying $g_i \mathcal{A}_i = \mathcal{A}_i'$, $g_j' \mathcal{B}_j = \mathcal{B}_j'$. Then,

$$\mathcal{A} \uplus \mathcal{B} = \biguplus_{i=1}^{n} \mathcal{A}_i \uplus \biguplus_{j=1}^{n'} \mathcal{B}_j$$

and

$$\mathcal{A}' \uplus \mathcal{B}' = \biguplus_{i=1}^{n} g_i \mathcal{A}_i \uplus \biguplus_{j=1}^{n'} g_j' \mathcal{B}_j$$

so $\mathcal{A} \uplus \mathcal{B} \sim \mathcal{A}' \uplus \mathcal{B}'$. $\qquad\square$

We can also use $\sim$ to define another relation on subsets of $\mathbb{R}^3$.

Definition 2. Let $\mathcal{A}$ and $\mathcal{B}$ be subsets of $\mathbb{R}^3$. We write $\mathcal{A} \preccurlyeq \mathcal{B}$ if there is a $\mathcal{B}' \subseteq \mathcal{B}$ with $\mathcal{A} \sim \mathcal{B}'$.

That is, $\mathcal{A} \preccurlyeq \mathcal{B}$ if $\mathcal{A}$ is equidecomposable with a subset of $\mathcal{B}$. It is easy to see that $\preccurlyeq$ is reflexive and transitive. Also, if $\mathcal{A} \subseteq \mathcal{B}$ and $\mathcal{B} \preccurlyeq \mathcal{C}$, then $\mathcal{A} \preccurlyeq \mathcal{C}$. Similarly, if $\mathcal{A} \preccurlyeq \mathcal{B}$ and $\mathcal{B} \subseteq \mathcal{C}$, then $\mathcal{A} \preccurlyeq \mathcal{C}$.

We are now ready to begin the proof of the final version of the Banach-Tarski paradox. First let us consider what properties of subsets of $\mathbb{R}^3$ are preserved by equidecomposability. The Banach-Tarski paradox shows that 'volume', whatever precisely that means, is not so preserved. However, there are at least two properties which are. First, if a set $\mathcal{A}$ has non-empty interior (that is, if the $\mathcal{A}$ contains any solid ball of any non-zero radius), then any set equidecomposable with $\mathcal{A}$ must also have non-empty interior. (This is because we are only cutting $\mathcal{A}$ into finitely many pieces; one of these will still have to contain some, possibly smaller, ball.) Second, if $\mathcal{A}$ is bounded (that is, if $\mathcal{A}$ is contained in some ball of finite radius), then any set equidecomposable with $\mathcal{A}$ will also be bounded. (This is because the finitely many $g_i \in \mathbf{M}_3$ can all only move $\mathcal{A}$ by a finite distance.) The generalized Banach-Tarski paradox states that any two sets having these properties are equidecomposable. We will now prove this using properties of $\preccurlyeq$.

Proposition 7.2. *Let $\mathcal{A}$ and $\mathcal{B}$ be bounded subsets of $\mathbb{R}^3$ with non-empty interior. Then $\mathcal{A} \preccurlyeq \mathcal{B}$.*

Proof. By our hypotheses, we can choose solid balls $\mathbf{B}_\mathcal{A}$ and $\mathbf{B}_\mathcal{B}$ with $\mathcal{A} \subseteq \mathbf{B}_\mathcal{A}$ and $\mathbf{B}_\mathcal{B} \subseteq \mathcal{B}$. Now, choose n large enough so that $\mathbf{B}_\mathcal{A}$ can be covered by n (overlapping) copies of $\mathbf{B}_\mathcal{B}$. Let $\mathcal{S} \in \mathbb{R}^3$ be a disjoint union of n copies of $\mathbf{B}_\mathcal{B}$. Then, by the definition of n we can move pieces of $\mathcal{S}$ to cover $\mathbf{B}_\mathcal{A}$. Taking appropriate subsets of these pieces, we can then exactly cover $\mathbf{B}_\mathcal{A}$. Thus, $\mathbf{B}_\mathcal{A} \preccurlyeq \mathcal{S}$.

Next, by the Banach-Tarski paradox we can repeatedly duplicate a subset of $\mathbf{B}_\mathcal{B}$ to form as many copies of $\mathbf{B}_\mathcal{B}$ as we want. In particular, we can form n copies of $\mathbf{B}_\mathcal{B}$, so $\mathcal{S} \preccurlyeq \mathbf{B}_\mathcal{B}$. Thus,

$$\mathcal{A} \subseteq \mathbf{B}_\mathcal{A} \preccurlyeq \mathcal{S} \preccurlyeq \mathbf{B}_\mathcal{B} \subseteq \mathcal{B},$$

so $\mathcal{A} \preccurlyeq \mathcal{B}$. $\qquad\square$

Of course, we can swap $\mathcal{A}$ and $\mathcal{B}$ above, so $\mathcal{B} \preccurlyeq \mathcal{A}$. Thus, the last step is to show that $\mathcal{A} \preccurlyeq \mathcal{B}$ and $\mathcal{B} \preccurlyeq \mathcal{A}$ implies that $\mathcal{A} \sim \mathcal{B}$.

Proposition 7.3 (The Banach-Schröder-Bernstein Theorem). *Suppose $\mathcal{A} \preccurlyeq \mathcal{B}$ and $\mathcal{B} \preccurlyeq \mathcal{A}$. Then $\mathcal{A} \sim \mathcal{B}$.*

Proof. By our hypotheses, we can choose $\mathcal{A}' \subseteq \mathcal{A}$ and $\mathcal{B}' \subseteq \mathcal{B}$ such that $\mathcal{A} \sim \mathcal{B}'$ and $\mathcal{A}' \sim \mathcal{B}$. Let $\alpha : \mathcal{A} \to \mathcal{B}'$ and $\beta : \mathcal{A}' \to \mathcal{B}$ be the bijections of Property 4 of $\sim$ above. Now, set $\mathcal{C}_0 = \mathcal{A} - \mathcal{A}'$ and for $n \geq 1$ set

$$\mathcal{C}_n = \beta^{-1}\alpha(\mathcal{C}_{n-1}) \subseteq \mathcal{A}.$$

Let $\mathcal{C}$ be the (not necessarily disjoint) union of the $\mathcal{C}_n$:

$$\mathcal{C} = \bigcup_{n=0}^{\infty} \mathcal{C}_n \subseteq \mathcal{A}.$$

Now, since $\mathcal{C}_0 \subseteq \mathcal{C}$ and $\mathcal{C}_0 \cap \mathcal{A}' = \emptyset$, we have

$$\mathcal{A} - \mathcal{C} = \mathcal{A}' - (\mathcal{C} \cap \mathcal{A}') = \mathcal{A}' - \bigcup_{n=1}^{\infty} \mathcal{C}_n.$$

Thus, since α and β are bijections,

$$\begin{aligned}
\beta(\mathcal{A} - \mathcal{C}) &= \beta \left(\mathcal{A}' - \bigcup_{n=1}^{\infty} \mathcal{C}_n \right) \\
&= \beta(\mathcal{A}') - \beta \left(\bigcup_{n=1}^{\infty} \beta^{-1}\alpha(\mathcal{C}_{n-1}) \right) \\
&= \mathcal{B} - \alpha \left(\bigcup_{n=1}^{\infty} \mathcal{C}_{n-1} \right) \\
&= \mathcal{B} - \alpha \left(\bigcup_{n=0}^{\infty} \mathcal{C}_n \right) \\
&= \mathcal{B} - \alpha(\mathcal{C}).
\end{aligned}$$

So by Property 4,

$$\mathcal{A} - \mathcal{C} \sim \mathcal{B} - \alpha(\mathcal{C}).$$

Also by Property 4,

$$\mathcal{C} \sim \alpha(\mathcal{C}).$$

Thus, by Property 5,

$$(\mathcal{A} - \mathcal{C}) \uplus \mathcal{C} \sim (\mathcal{B} - \alpha(\mathcal{C})) \uplus \alpha(\mathcal{C})$$
$$\mathcal{A} \sim \mathcal{B}.$$

$\square$

Corollary 7.4 (The Banach-Tarski Paradox, Version 3). *Any two bounded subsets of $\mathbb{R}^3$ with interior points are equidecomposable.*

8. Final Remarks

I will conclude with a few remarks on the alleged paradoxical nature of the Banach-Tarski paradox. I would argue that this result is in no way a paradox. It does go against our intuition, but very often our intuition is flawed; that is why mathematics requires rigor.

The reason that the Banach-Tarski paradox is so disturbing to people, of course, is that it assaults our intuition about 3-dimensional geometry, a subject in which we have a great deal of first-hand experience. The corresponding paradox for free groups, while somewhat surprising, is certainly not something that anyone would argue with. However, as I hope the reader was able to determine from the above proof before it completely degenerated into a mess of strange sets, the result on free groups is the *key step* in the proof of the Banach-Tarski paradox. From this point of view, the Banach-Tarski paradox is not a statement about $\mathbb{R}^3$ so much as it is a statement about the complexity of the group $\mathbf{M}_3$.

What, then, of our intuition about the preservation of volume by rigid motions? Well, as any student of measure theory is aware, when one attempts to rigorously define a notion of volume, complications immediately arise. Any hope of being able to give a consistent definition of volume to *every* subset of $\mathbb{R}^3$ must quickly be abandoned; indeed, when one looks at the definitions and theorems of measure theory it is immediately apparent that there is absolutely

no reason why we should be able to assign a volume to every set. This is precisely what happens in the Banach-Tarski paradox; most of the sets we have constructed have no volume to preserve, so when put together in strange ways they are able to change their collective volume.

I would like to make one final argument about the Banach-Tarski paradox. Since subsets of $\mathbb{R}^3$ with interior points all have the same cardinality, the following statement is an immediate consequence of the definition of bijection:

> If $\mathcal{A}$ and $\mathcal{B}$ are bounded subsets of $\mathbb{R}^3$ with interior points, then $\mathcal{A}$ can be cut into uncountably many pieces which can be moved by rigid motions to form exactly $\mathcal{B}$.

The Banach-Tarski paradox, then, is simply this statement with *uncountably* replaced by *finitely*. That this replacement can be made, then, is really a statement about the size and complexity of the group $\mathbf{M}_3$; it is large enough that we can group the uncountably many points together in such a way that only finitely many distinct rigid motions are necessary. For all of these reasons, it is my opinion that the Banach-Tarski paradox is in no way a convincing argument against the axiom of choice; the axiom of choice may or may not be a reasonable assumption in mathematics, but the Banach-Tarski paradox is no reason to reject it.

References

[1] Michael Artin, *Algebra*. Prentice Hall, Englewood Cliffs, New Jersey, 1991.
[2] Walter Rudin, *Principles of Mathematical Analysis, 3rd edition*. McGraw Hill Book Company, New York, 1976.
[3] Stanley Wagon, *The Banach-Tarski Paradox*. Cambridge University Press, Cambridge, 1985.