The Banach-Tarski Paradox

Anders O.F. Hendrickson

Department of Mathematics and Computer Science
Concordia College, Moorhead, MN

Math/CS Colloquium Series

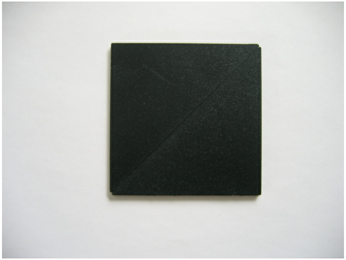
Outline

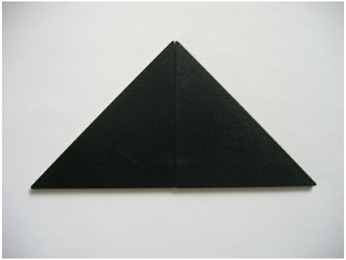
The Banach-Tarski Paradox

Our goal: to dissect a solid ball into finitely many pieces and reassemble the pieces into *two* balls of the original size.

Dissection

We want to know if two shapes are "equidecomposable," meaning that one can be cut up and rearranged to become the other. A common example:





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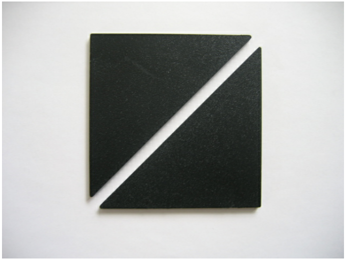

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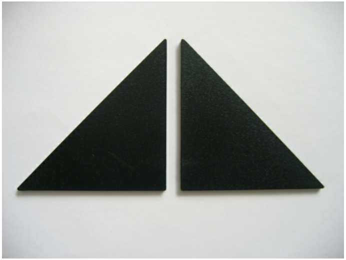

Definitions

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We say that sets X and Y in Euclidean space are ***congruent***, and we write $X \cong Y$, if X can be moved to fill the same space as Y using only rigid motions (sliding and rotating).

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We say that two sets A and B are ***equidecomposable***, and write $A \sim B$, if there are disjoint sets $A_1, \ldots, A_n$ and disjoint sets $B_1, \ldots, B_n$ such that $A = \dot{\bigcup}_{i=1}^{n} A_i$ and $B = \dot{\bigcup}_{i=1}^{n} B_i$ with each $A_i \cong B_i$.

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Transitivity of $\sim$

Theorem

If $A \sim B$ and $B \sim C$, then $A \sim C$.

Proof sketch.

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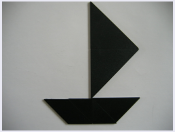

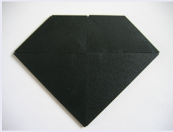

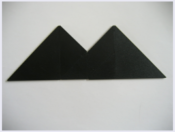

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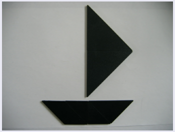

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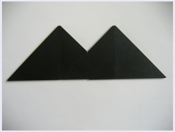

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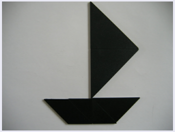

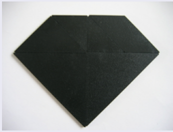

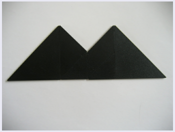

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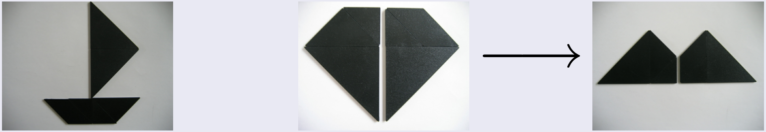

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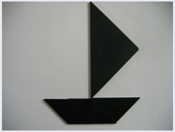

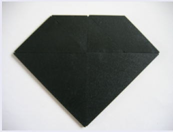

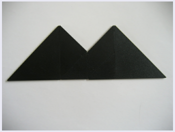

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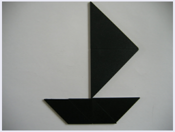

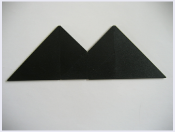

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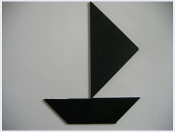

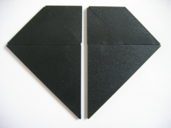

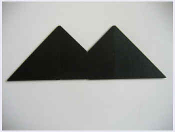

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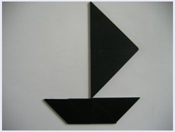

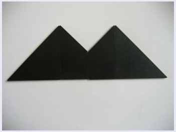

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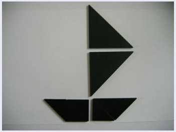

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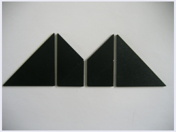

$\square$

Infinity is Weird, Part 1

- Some infinities are more infinite than others.
- If you can list all the elements of an infinite set one by one, it is called ***countably infinite***. Examples:
  - $\mathbb{N} = \{1, 2, 3, 4, 5, 6, \ldots\}$
  - $\mathbb{Z} = \{0, 1, -1, 2, -2, 3, -3, \ldots\}$
- But a *continuum* like $\mathbb{R}$ or $[2, 3]$ or $\mathbb{R}^2$ has more elements than $\mathbb{N}$ does. It is ***uncountably infinite***.
  (Cantor's diagonal argument.)

Infinity is Weird, Part 2

- Suppose you own a (countably) infinite hotel.
- All the rooms are occupied.
- Now Thomas shows up and wants a room.
- Move every existing guest to the right.
- Now there's room for Thomas!

Infinity is Weird, Part 2

Alice	Bob	Carlos	David	Edgar	Fred	$\cdots$

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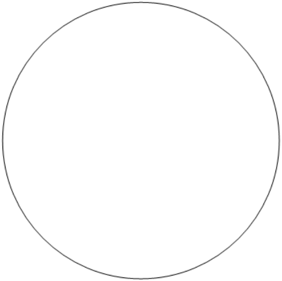

- Pick a starting point and an angle $\theta = \alpha\pi$, where $\alpha \notin \mathbb{Q}$.
- Rotate our starting point by θ repeatedly.
- We never hit the same point twice, since otherwise $m\theta = n\theta + 2k\pi \implies (m-n)\alpha\pi = 2k\pi \implies \alpha = \frac{2k}{m-n} \in \mathbb{Q}$.

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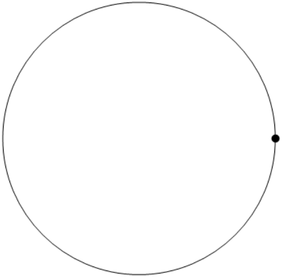

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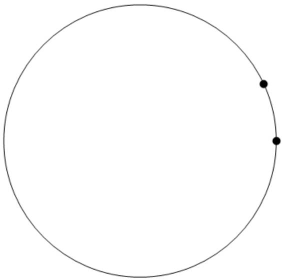

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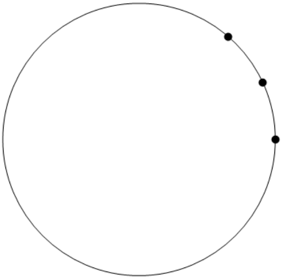

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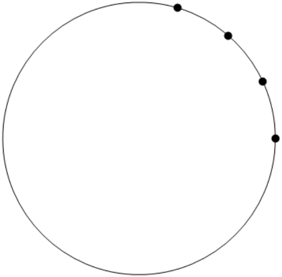

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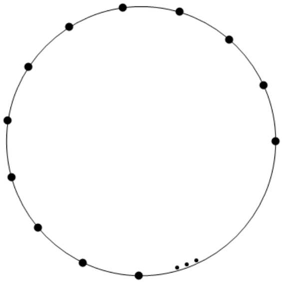

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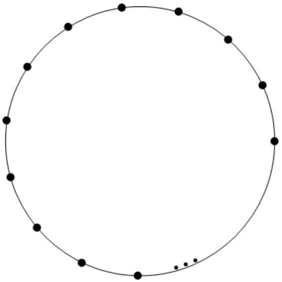

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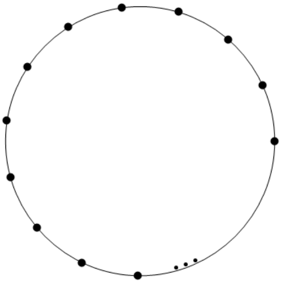
Our first paradox



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- Rotate the points by θ. Each will fit into the next one's slot.

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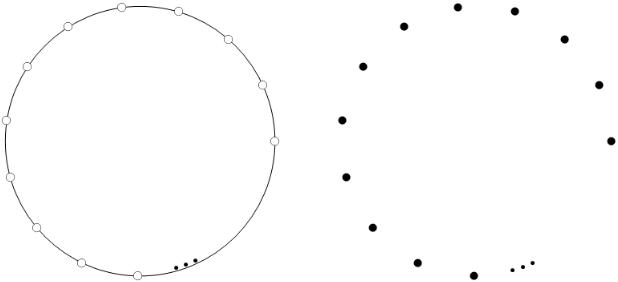

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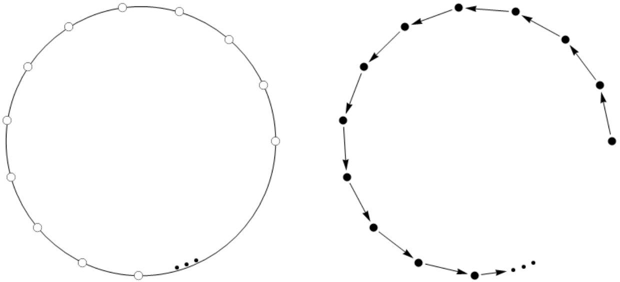

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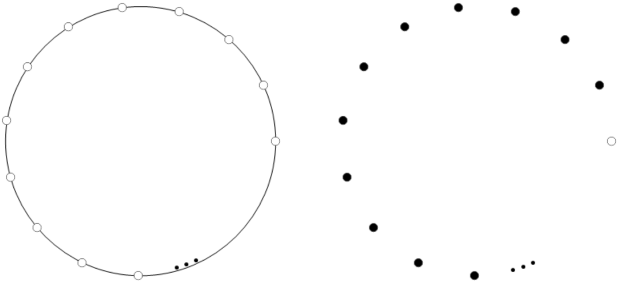

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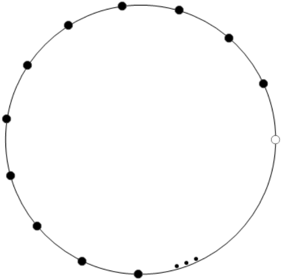

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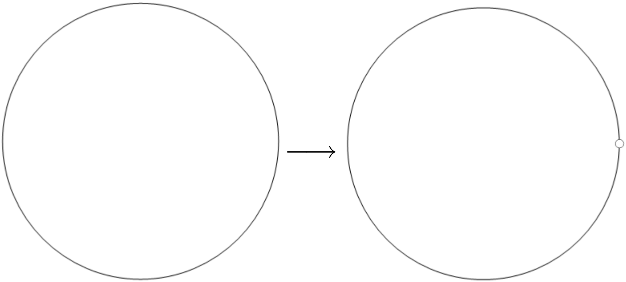

Lemma

$C \sim C - \{point\}$

The free group $\mathbb{F}_2$

$\mathbb{F}_2 =$ the set of all "words" using the "letters" a, b, a^{-1}, and b^{-1}, provided we agree to cancel aa^{-1}, $a^{-1}a$, bb^{-1}, $b^{-1}b$.

- e.g. $abab^{-1}$, a^3b^{-2}, $baa^{-1}b = b^2 \in \mathbb{F}_2$
- Note that $\mathbb{F}_2$ is *countably* infinite

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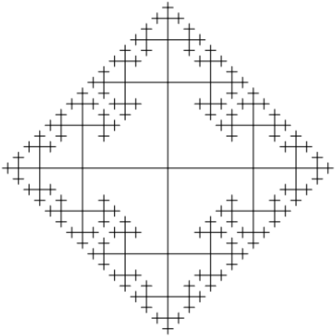

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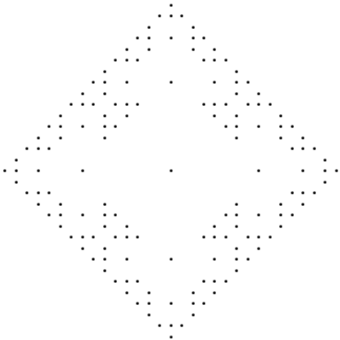

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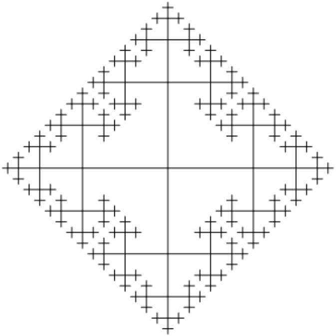

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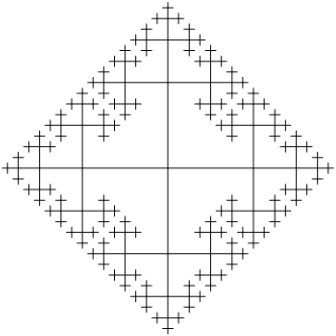

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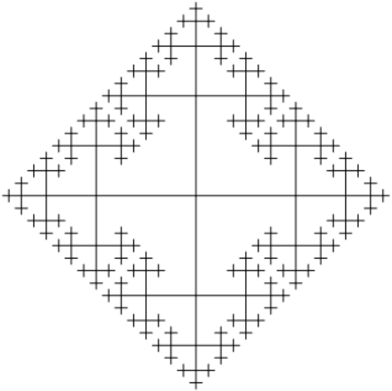

- Dissect $\mathbb{F}_2$ into $R \dot\cup S \dot\cup T \dot\cup U$.

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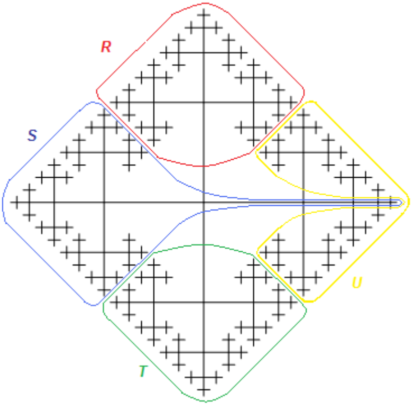

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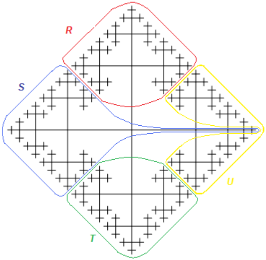
Reassembly of $\mathbb{F}_2$



- Dissect $\mathbb{F}_2$ into $R \dot\cup S \dot\cup T \dot\cup U$.
- Then $b^{-1}R \cup T = \mathbb{F}_2$ and $aS \cup U = \mathbb{F}_2$, so $\mathbb{F}_2 \sim 2\mathbb{F}_2$!
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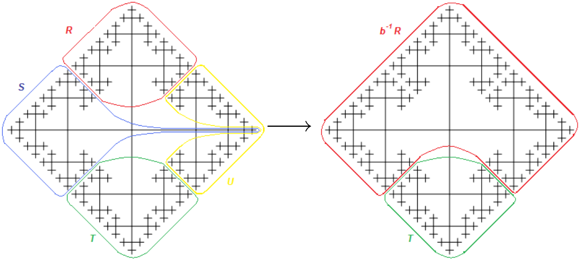

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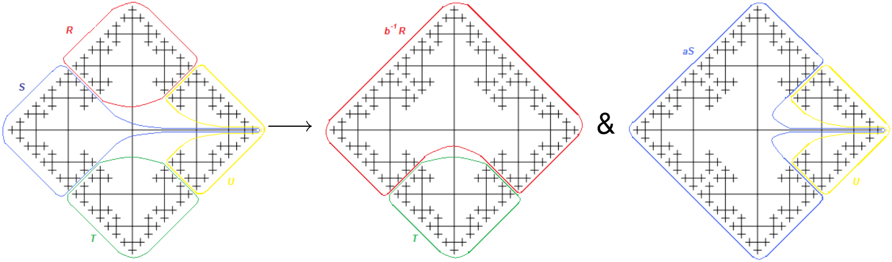

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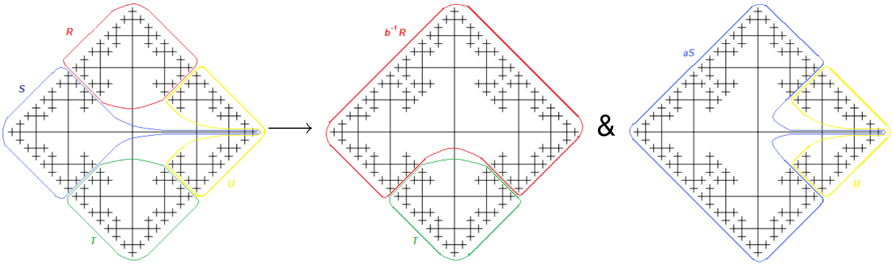

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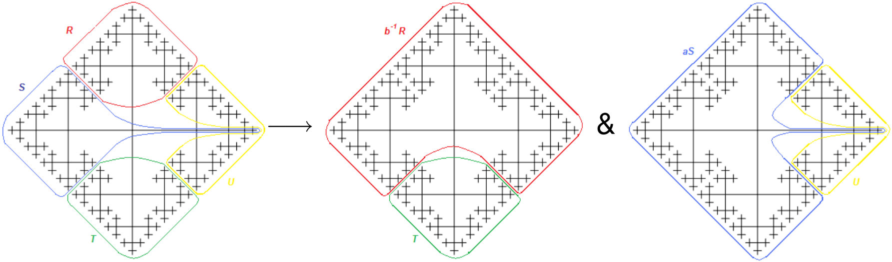

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$\mathbb{F}_2$ on the sphere

- Take $\theta = \cos^{-1} \frac{3}{5}$.
- Consider a sphere, and let $a =$ rotation around x-axis by θ and $b =$rotation around z-axis by θ.
- Like 2 moves of a Rubik's cube,
- Let G be all moves you can get by repeatedly doing any sequence of a, b, a^{-1}, or b^{-1}.

G is isomorphic to $\mathbb{F}_2$

Lemma

No two sequences of rotations in G result in the same rotation of the sphere.

Proof.

A boring computation with matrices. □

Thus G acts just like $\mathbb{F}_2$, and from now on we'll identify the two.

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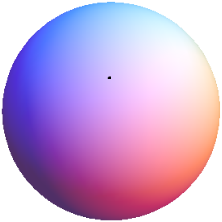

- Pick a point $x \in S$, and subject it to all the rotations.
- *If* we get a different point for each rotation, then we have a "copy" $\mathbb{F}_2 x$ of $\mathbb{F}_2$ on the sphere!
- Then we can divide $\mathbb{F}_2 x$ into four pieces, $\mathbb{F}_2 x = Rx \,\dot\cup\, Sx \,\dot\cup\, Tx \,\dot\cup\, Ux$, move them by rigid motions (rotations), and reassemble them into 2 copies of $\mathbb{F}_2 x$.

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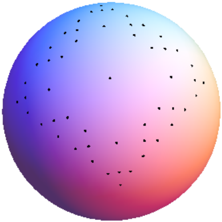

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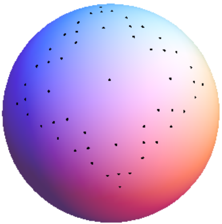

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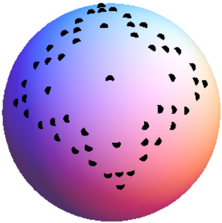

- If we take a set W of such "good" points, then we get lots of copies of $\mathbb{F}_2$.
- So we can divide $\mathbb{F}_2 W$ into 4 pieces, $\mathbb{F}_2 W = RW \dot\cup SW \dot\cup TW \dot\cup UW$, and reassemble them into 2 copies of $\mathbb{F}_2 W$.

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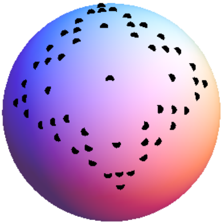

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Bad Points

- What if when we rotate a point x by all the rotations in $\mathbb{F}_2$, we don't get all distinct points?
- If so, then
$$r_1 x = r_2 x \implies (r_2)^{-1} r_1 x = x,$$
so x is unmoved by the rotation $(r_2)^{-1} r_1$—so it's a pole!
- Let $X = \{\text{all bad points}\}$. Every rotation has two poles, so $|X| = 2|\mathbb{F}_2|$ is *countably* infinite.

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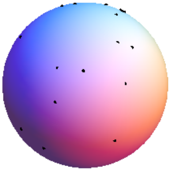

- Since $|X| < |S|$, pick an axis through no point of X.
- Rotate through an angle $\alpha\pi$ so that all points in X vanish.
- If our rotation sends one point of X to another point in X, then that's a bad angle. But there are only countably many bad angles, so we can choose a good angle!

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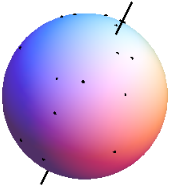

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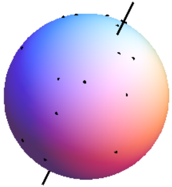

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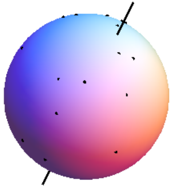

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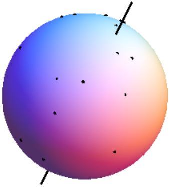
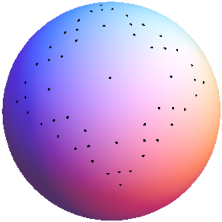


Lemma

$S \sim S - X.$



- Now $S - X$ is the union of infinitely many sets like $\mathbb{F}_2$.
- Pick just one point from each of these sets; call that collection Z. *[Axiom of Choice!]*
- Now $S - X = \mathbb{F}_2 Z = RZ \dot\cup SZ \dot\cup TZ \dot\cup UZ$, and we can reassemble: $S - X = a^{-1}RZ \dot\cup SZ$ and $S - X = bTZ \dot\cup UZ$.

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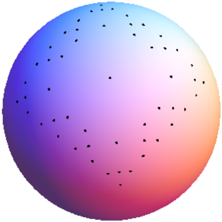

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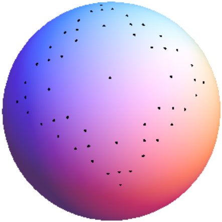

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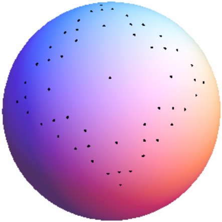

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Doubling the surface

Putting these pieces together, we now can double the surface of a sphere!

Proposition

$S \sim 2S$.

Proof.

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Can we double a solid ball?

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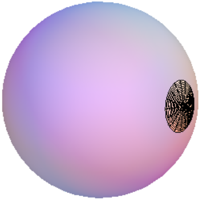

Now, to each piece of the sphere, glue all the points between it and the center!

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$B - \{center\} \sim 2(B - \{center\})$.

The last step: we need to show that $B \sim B - \{\text{center}\}$.

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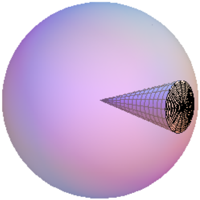

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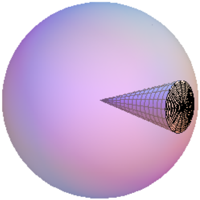

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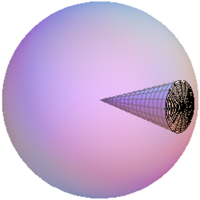

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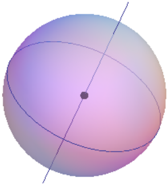

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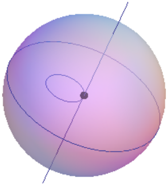

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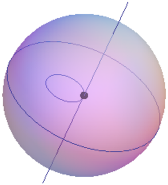

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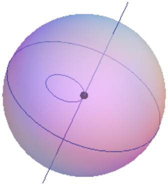

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The Banach-Tarski Theorem, Weaker Form

Theorem (Weak Banach-Tarski)

$B \sim 2B$.

Proof.

$B \sim B - \{\text{center}\} \sim 2(B - \{\text{center}\}) \sim 2B$. □

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The Banach-Tarski Theorem, Full Strength

Theorem (Banach-Tarski)

Let A and B be two bounded subsets of $\mathbb{R}^n$ *with nonempty interiors. Then* $A \sim B$.

Morals of the Story: The Axiom of Choice

- Some people don't like the "fishy" Axiom of Choice, and this is one reason why.
- However,
  - We can still have some paradoxes without it!
  - The Axiom of Choice is needed to prove
    - that every vector space has a basis
    - that every field has an algebraic closure
    - lots of other important results
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Morals of the Story: Measure Theory

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