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BANACH-TARSKI DECOMPOSITIONS USING SETS WITH THE PROPERTY OF BAIRE

RANDALL DOUGHERTY AND MATTHEW FOREMAN

1. INTRODUCTION

Perhaps the most strikingly counterintuitive theorem in mathematics is the "Banach-Tarski paradox": A ball in $\mathbf{R}^3$ can be decomposed into finitely many pieces which can be rearranged by rigid motions and reassembled to form two balls of the same size as the original. The Axiom of Choice is used to construct the decomposition; the "paradox" is resolved by noting that the pieces cannot be Lebesgue measurable.

In this paper, we solve a problem posed by Marczewski in 1930 by showing that there are paradoxical decompositions of the unit ball using pieces which have the property of Baire.

We also prove related "paradoxes" which involve only open sets; these proofs are entirely constructive and make no use of the Axiom of Choice. One such result is the following: one can find a finite collection of disjoint open subsets of the unit ball which can be rearranged by suitable isometries to form a set whose closure is a solid ball of radius 10^{10}.

For both kinds of decomposition, the results are not limited to $\mathbf{R}^3$. In fact, we produce such decompositions for the sphere S^n ($n \geq 2$) and for the unit ball in $\mathbf{R}^n$ ($n \geq 3$), as well as related spaces. (See Theorem 2.6, among others.) It follows (Proposition 2.11) that, for $n \geq 2$, there is no isometry-invariant, finitely additive measure on the subsets of S^n ($\mathbf{R}^{n+1}$) with the property of Baire which gives measure 1 to S^n (the unit ball in $\mathbf{R}^{n+1}$). For other equivalent forms of Marczewski's problem and for additional references, see Mycielski [6] or chapter 9 of Wagon [9].

From the above, standard arguments show that, if A and B are bounded subsets of $\mathbf{R}^n$ ($n \geq 3$) with the property of Baire and with nonempty interior, then A can be decomposed into finitely many sets with the property of Baire which can be rearranged to form B (Corollary 2.7).

All of these results are consequences of a theorem about decompositions of open sets which is quite general: it holds in any Polish space (complete separable

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metric space) with a sufficiently free group of homeomorphisms. (A review of terminology is given at the end of this section.)

Theorem 3.1. *Suppose $\mathscr{X}$ is a Polish space, and G is a countable group of homeomorphisms of $\mathscr{X}$ which acts freely on a comeager subset of $\mathscr{X}$. Let $N \geq 1$ be given, and suppose that G has a free subgroup of rank $3N$ with free generators f_{ij} $(i = 1, 2, 3;\ 1 \leq j \leq N)$. Then there are disjoint open subsets A_{ij} of $\mathscr{X}$ such that, for each $j \leq N$, $f_{1j}(A_{1j}) \cup f_{2j}(A_{2j}) \cup f_{3j}(A_{3j})$ is dense in $\mathscr{X}$.*

In fact, all of our results about sets with the property of Baire can be obtained by combining a known fact about arbitrary sets (the Banach-Tarski paradox or a related result) with an analogous fact about open sets (a consequence or variant of Theorem 3.1). For example, one of the preceding results has the following analogue for open sets (Corollary 2.8): if A and B are nonempty bounded open subsets of $\mathbf{R}^n$ $(n \geq 3)$, then there are finitely many disjoint open subsets of A with union dense in A which can be rearranged by rigid motions to form a dense open subset of B. In particular, there are finitely many disjoint open subsets of the unit ball that can be rearranged to form a set whose closure is a very large cube.

Variants of Theorem 3.1 are used to prove the following three results, among others:

If A and B are subsets of S^2 with the property of Baire having nonempty interior, then A can be decomposed into finitely many pieces which can be rearranged to form B (Corollary 5.2). (This is not as easy as the corresponding result for $\mathbf{R}^3$, because symmetrical neighborhoods in S^2 do not have free groups of isometries on more than one generator.)

For any finite $N \geq 3$, the sphere S^2 can be decomposed into N congruent pieces having the property of Baire (Theorem 5.4). (One cannot just divide the sphere into N identical sectors, because then the two poles would be left over. It is still open whether a sphere can be divided into three congruent Borel pieces, or even into three congruent Lebesgue-measurable pieces [9, p. 47].)

The sphere S^2 can be decomposed into six pieces with the property of Baire which can be rearranged to form two copies of S^2, and the number six is optimal; the same holds for the unit ball in $\mathbf{R}^3$ (Theorems 5.7, 5.9, and 5.11). (This contrasts with the situation for unrestricted pieces, where four suffice for S^2 but five are needed for a ball in $\mathbf{R}^3$ [9, p. 40].)

As noted earlier, the proofs of our results about open sets are completely constructive (if performed carefully; see the end of §3) and make no use of the Axiom of Choice; hence, one cannot explain away their paradoxical nature by blaming it on that axiom. Instead, one can note that the open sets resulting from our construction have boundaries of positive measure. When the open sets are packed into a small space, these boundaries overlap greatly; when the open sets are rearranged within a larger set, the boundary overlap is less extreme, so the total measure of the closures is greater.

This paper is organized as follows: Section 1 includes an overview of the paper and some background material. Section 2 gives a review of the Banach-

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Tarski theorem and shows how to combine it with our main theorem on open sets (Theorem 3.1) to get decompositions into sets with the property of Baire; §3 comprises the proof of the main theorem. (These sections are self-contained, except that we omit the proof that free groups of rotations of S^2 exist.) Section 4 shows how to modify the proof in §3 to get related results about open sets, and §5 gives applications of these results. The paper concludes with a brief list of open problems.

We now give a very brief summary of the method used to prove the main theorem (3.1). The required open sets are built up in stages, as increasing unions of smaller sets. The density of the images of the open sets under the relevant homeomorphisms can be viewed as a countable list of requirements: each basic open set must be met by one of the images. Each stage of the construction increases the current open sets to meet one of these requirements. Each stage has two parts: first, we add finitely many new generic points (i.e., points that lie in all relevant comeager sets) to the open sets, one of which is in the proper location to meet the current requirement; then we "inflate" these new points to open neighborhoods so small that anything which happens within the neighborhood must happen at the point itself. The most important part of the proof is a list of inductive hypotheses on the current open sets which allows the above two steps to proceed at each stage.

To combine the open set results with the arbitrary set results from the Banach-Tarski theorem, first note that the open sets handle all points in the space except for a nowhere dense set. At the cost of increasing the nowhere dense set of exceptions to a meager set, we can replace the open sets with Borel sets which are invariant under all of the relevant functions (both from the open set decomposition and from the Banach-Tarski decomposition). This Borel decomposition of an invariant set is then combined with the Banach-Tarski decomposition restricted to the meager exceptional set; the pieces used in the latter are meager and hence have the property of Baire.

We conclude this section with a summary of results from topology and group theory which will be needed later.

Suppose $\mathscr{X}$ is a Polish space (a separable complete metric space, or a topological space which admits a complete separable metric). The closure of a set $A \subseteq \mathscr{X}$ is denoted $\overline{A}$. A set $A \subseteq \mathscr{X}$ has the *property of Baire* if there is an open set $O \subseteq \mathscr{X}$ such that the symmetric difference $A \triangle O$ is meager (a countable union of nowhere dense sets). A comeager set is a set whose complement is meager; in particular, dense open sets are comeager. (The Baire Category Theorem states that any comeager set is dense.) Clearly the meager sets form a σ-ideal; the collection $\mathscr{B}$ of sets with the property of Baire contains all open sets and is easily seen to be closed under countable unions and complements [4, 2H.3], so all Borel sets have the property of Baire.

Let G be a group acting on a set X. We say G acts *freely* on X if no nonidentity element of G has a fixed point in X. (This has no relation to whether G is itself a free group.) We similarly define "G acts freely on S" for a subset S of X, but S is required to be invariant under the action of G. If G has elements $g_1, \ldots, g_n$ such that the unique homomorphism from the free group on generators $a_1, \ldots, a_n$ to G which sends a_i to g_i for all i is in fact an

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isomorphism, then we say that the elements $g_1, \ldots, g_n$ are *free generators* for G. We will use the well-known fact that a free group on two generators σ, τ has, for any finite n, a subgroup which is free on n generators; for example, the elements $\sigma^k \tau^k$ for $1 \leq k \leq n$ are free generators for such a subgroup [3, p. 43]. The subgroup generated by a given subset S of a group will be denoted by $\langle S \rangle$.

An excellent source for background material on paradoxical decompositions is Wagon's book, *The Banach-Tarski Paradox* [9].

2. Old and new decompositions

In this section, we will review the basic definitions and results about Banach-Tarski decompositions (from Wagon [9]) and show how to combine these results with Theorem 3.1 to get decompositions using sets with the property of Baire.

Definition 2.1 [9, pp. 23–24]. Suppose G is a group acting on a set X. Subsets A and B of X are called *G-congruent* if there is an element g of G such that $g(A) = B$. Sets $A, B \subseteq X$ are called *G-equidecomposable* (in symbols, $A \sim_G B$) if A and B can be partitioned into sets $A_1, \ldots, A_n$ and $B_1, \ldots, B_n$, respectively, so that, for each $i \leq n$, A_i is G-congruent to B_i. The set A is called *G-paradoxical* if there are disjoint sets $B, B' \subseteq A$ such that $A \sim_G B$ and $A \sim_G B'$.

The G will be suppressed if the group is clear from context. If G is a group of homeomorphisms of a Polish space $\mathscr{X}$, and A and B are subsets of $\mathscr{X}$ with the property of Baire, then A and B are *Baire G-equidecomposable* ($A \sim_G B$ in $\mathscr{B}$) if the pieces $A_1, \ldots, A_n$ can be chosen to have the property of Baire.

The statement "A and B are G-equidecomposable" is equivalent to: there is a bijection $f: A \to B$ which is piecewise in G (i.e., there are sets $A_1, \ldots, A_n$ with union A such that, for each i, f agrees with some element of G on A_i). For Baire equidecomposability, the sets A_i are required to have the property of Baire; note that, in this case, for any $C \subseteq A$, C has the property of Baire if and only if $f(C)$ has the property of Baire.

It is not hard to see that the composition of two functions which are piecewise in G is also piecewise in G. (If $f: A \to B$ and $g: B \to C$ are piecewise in G, with pieces $A_1, \ldots, A_n$ and $B_1, \ldots, B_m$, respectively, then the mn pieces $A_i \cap f^{-1}(B_j)$ witness that $g \circ f$ is piecewise in G.) Hence, $\sim_G$ is an equivalence relation.

Intuitively, "X is G-paradoxical" means that one can find finitely many disjoint subsets of X which can be rearranged by elements of G to form two copies of X. (This can be formalized by moving to a space consisting of infinitely many copies of X, where one can form an equidecomposability type semigroup [9, Chapter 8].) Once one gets two copies, it is easy to get any finite number of copies: if $f: X \to A$ and $g: X \to B$ are bijections and piecewise in G, where A and B are disjoint, then the four maps $f \circ f$, $f \circ g$, $g \circ f$, and $g \circ g$ are piecewise in G and have disjoint ranges, and the same method gives 2^N such maps for any given N.

We define $A \preceq_G B$ to mean that A is G-equidecomposable with some subset

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of B (equivalently, there is a one-to-one function $f: A \to B$ which is piecewise in G); "$A \preceq_G B$ in $\mathscr{B}$" is defined similarly. Clearly $\preceq_G$ is reflexive and transitive; the following result shows that $\preceq_G$ is in fact a partial ordering on $\sim_G$-equivalence classes.

Proposition 2.2 (Banach-Schröder-Bernstein [9, p. 25]). *If $A \preceq_G B$ and $B \preceq_G A$, then $A \sim_G B$. If G is a group of homeomorphisms, $A \preceq_G B$ in $\mathscr{B}$, and $B \preceq_G A$ in $\mathscr{B}$, then $A \sim_G B$ in $\mathscr{B}$.*

Proof. Suppose $f: A \to B$ and $g: B \to A$ are one-to-one functions which are piecewise in G; for the second part, assume that the pieces involved have the property of Baire. Let $C_0 = A \setminus g(B)$ and, for $n > 0$, $C_n = g(f(C_{n-1}))$; let $C = \bigcup_{n=0}^{\infty} C_n$. Then it is easy to check that $A \setminus C = g(B \setminus f(C))$, so $A \setminus C \sim_G B \setminus f(C)$. But f witnesses $C \sim_G f(C)$, so we can put these together to get $A \sim_G B$. For the second part, since the pieces have the property of Baire, f and g preserve the property of Baire, so the sets C_n all have this property. Hence, C has the property of Baire, so we have $A \setminus C \sim_G B \setminus f(C)$ in $\mathscr{B}$ and $C \sim_G f(C)$ in $\mathscr{B}$; therefore, $A \sim_G B$ in $\mathscr{B}$. □

Corollary 2.3 [9, p. 25]. *Suppose $A \subseteq X$ and G acts on X. If there are disjoint sets $A_1, \dots, A_n \subseteq A$ such that, for all i, $A_i \sim A$, then there are such sets A_i whose union is all of A. The same holds for $\sim_G$ in $\mathscr{B}$ if G is a group of homeomorphisms.*

Proof. Suppose $A_i \sim_G A$ where the sets A_i are disjoint subsets of A. Let $A_1' = A \setminus \bigcup_{i=2}^{n} A_i$ and $A_i' = A_i$ for $2 \le i \le n$. Then $A \sim_G A_1 \preceq_G A_1' \preceq_G A$, so $A \sim_G A_1'$ by Proposition 2.2; hence, the sets A_i' have the desired properties. The same argument works for the Baire version. □

If G is the free group on two generators σ and τ, then G acts freely on itself by left translation. Let $W(a)$ be the set of reduced words in G beginning with a, where a is σ, σ^{-1}, τ, or τ^{-1}. Then $W(\sigma) \cup \sigma W(\sigma^{-1}) = G$ and $W(\tau) \cup \tau W(\tau^{-1}) = G$. Thus, both $W(\sigma) \cup W(\sigma^{-1})$ and $W(\tau) \cup W(\tau^{-1})$ are equicomposable with G, so G is G-paradoxical.

If this free group G acts freely on a set X, we can choose one element x_O from each orbit O; let V be the set of these chosen points. Let $A = (W(\sigma) \cup W(\sigma^{-1}))V$ and $B = (W(\tau) \cup W(\tau^{-1}))V$; then A and B are disjoint subsets of X such that $A \sim X$ and $B \sim X$, so X is G-paradoxical. (See Wagon [9, p. 11].) It now follows that, for any $N \ge 1$, we can partition X into sets $A_1, \dots, A_N$ such that each A_i is equidecomposable with X. (Of course, we are using the full Axiom of Choice here; the standard Vitali arguments show that, in cases such as a countably infinite group of isometries of S^n, the set V cannot be measurable or have the property of Baire.)

It is a well-known result of Hausdorff that the group G of rotations of S^2 includes a subgroup G' which is free on two generators [9, p. 15]. This subgroup does not act freely on S^2; any nontrivial rotation has exactly two fixed points. However, if we define D to be the set of all points which are images under elements of G' of fixed points of nonidentity elements of G', then D is a countable G'-invariant set and G' acts freely on $S^2 \setminus D$, so we get the result

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of Hausdorff that $S^2 \setminus D$ is G-paradoxical for some countable D [9, p. 18].

It is not hard to show that $S^2 \sim_G S^2 \setminus D$ for any countable set $D \subseteq S^2$ [9, p. 27]. To see this, find an axis of rotation (i.e., a line through the center of the sphere) which does not meet D, and find a rotation ρ around this axis such that no iterate ρ^n ($n > 0$) sends a point of D to another point of D (this excludes only countably many of the uncountably many rotations around this axis, so such a ρ exists). Let $E = \bigcup_{n=0}^{\infty} \rho^n(D)$; then $S^2 = (S^2 \setminus E) \cup E \sim_G (S^2 \setminus E) \cup \rho(E) = S^2 \setminus D$. The same proof works for any sphere S^m ($m \geq 1$), if the phrase "axis of rotation" is suitably interpreted (start by finding an $(m-1)$-dimensional flat through the center of the sphere which does not meet D).

We can now prove the Banach-Tarski paradox: for any $n \geq 2$, the sphere S^n is paradoxical using rotations, and any ball in $\mathbf{R}^{n+1}$ is paradoxical using isometries of $\mathbf{R}^{n+1}$ [9, p. 53]. For S^2 this follows immediately from the preceding two paragraphs. Given the result for S^n, we get the same result for $S^{n+1} \setminus \{p, p'\}$ where p and p' are antipodal points of S^{n+1}, since $S^{n+1} \setminus \{p, p'\}$ is a union of copies of S^n and we can use the same rotations and decompositions on each copy. By the preceding paragraph, S^{n+1} and $S^{n+1} \setminus \{p, p'\}$ are equidecomposable, so S^{n+1} is paradoxical. Similarly, if S^n is paradoxical, then so is $B \setminus \{o\}$ where B is a ball in $\mathbf{R}^{n+1}$ with center o, since $B \setminus \{o\}$ is a union of copies of S^n. If S is one of these copies, and $p \in S$, then $S \setminus \{p\} \sim S$; but clearly $\{p\} \sim \{o\}$, so $S \sim S \cup \{o\}$, so $B \setminus \{o\} \sim B$, so B is paradoxical. The same argument applies to a "ball" (the set of points within a certain distance of a given center point) in S^{n+1} for $n \geq 2$, since the group of isometries of this ball is the same as the group of isometries of a ball in $\mathbf{R}^{n+1}$.

We now show that any two bounded subsets of $\mathbf{R}^n$ ($n \geq 3$) with nonempty interior, or any two subsets of S^n ($n \geq 3$) with nonempty interior, are equidecomposable using isometries [9, p. 53]. By Proposition 2.2, it suffices to show that, if A and B are two such sets, then $A \preceq B$. Let C be a ball included in B. Since A is bounded, it can be covered by finitely many translates of C. Since C is paradoxical, there are finitely many disjoint subsets of C which can be rearranged by isometries to cover all of these translates; these subsets can be cut down so that their images under the relevant isometries cover A without overlap. This shows that $A \preceq C$, so $A \preceq B$, as desired. (The same fact holds for subsets of S^2, but the proof is more difficult; see §5.)

This completes the review of the Banach-Tarski paradox and related results. We now have to see how to combine these results using arbitrary pieces with Theorem 3.1, which uses open pieces but does not produce a decomposition of the entire space. To do this, we use the following lemma.

Lemma 2.4. *Suppose G is a group of homeomorphisms on a topological space $\mathscr{X}$.*

(a) *If A and B are subsets of $\mathscr{X}$ with the property of Baire, $A \preceq_G B$, and there is a meager set D such that $A \setminus D \preceq_G B$ in $\mathscr{B}$, then $A \preceq_G B$ in $\mathscr{B}$.*

(b) *If $B \subseteq \mathscr{X}$ has the property of Baire, there are N disjoint subsets of B each of which is G-equidecomposable with B, and there are N disjoint subsets*

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of B with the property of Baire each of which is Baire G-equidecomposable with some set $B \setminus D$ where D is meager, then there are N disjoint subsets of B with the property of Baire each of which is Baire G-equidecomposable with B.

Proof. For both parts, the hypotheses only involve finitely many elements of G, and these elements generate a countable subgroup of G, so we may assume that G is countable.

For (a), let E be the union of the sets $g(D)$ for $g \in G$; then E is a G-invariant meager set. If $f: A \to B$ and $f': A \setminus D \to B$ are the piecewise-in-G functions witnessing the hypotheses of (a), then define $f'': A \to B$ by letting $f''(x)$ be $f(x)$ if $x \in A \cap E$, $f'(x)$ if $x \in A \setminus E$. Then f'' is piecewise in G. Furthermore, the pieces are of two sorts: the pieces from f restricted to $A \cap E$, which have the property of Baire because they are subsets of the meager set E, and the pieces from f' restricted to $A \setminus E$, which have the property of Baire because the pieces from f' do. It only remains to show that f'' is one-to-one. The function f'' is composed of the two parts $f \restriction (A \cap E)$ and $f' \restriction (A \setminus E)$, each of which is one-to-one, so we must see that these parts have disjoint ranges. But f and f' are piecewise in G, and E is G-invariant; hence, $f(A \cap E) \subseteq E$ and $f'(A \setminus E) \subseteq \mathscr{X} \setminus E$, so these two sets are disjoint, as desired.

The proof of (b) is similar. Suppose the functions $f_i: B \to A_i$ and $f_i': B \setminus D_i \to A_i'$ for $1 \leq i \leq N$ witness the hypotheses of (b); in particular, the functions f_i and f_i' are bijections, the sets $A_i \subseteq B$ are disjoint, the sets $A_i' \subseteq B$ are also disjoint, and D_i is meager. Let E be the union of $g(D_i)$ for $g \in G$ and $i \leq N$; then E is G-invariant and meager. Define $f_i'': B \to B$ by letting $f_i''(x)$ be $f_i(x)$ if $x \in E$, $f_i'(x)$ otherwise. Then f_i'' is one-to-one and piecewise in G, and the pieces have the property of Baire as before. Furthermore, if A_i'' is the range of f_i'', then $A_i'' \cap E = A_i \cap E$ and $A_i'' \setminus E = A_i' \setminus E$ (since E is invariant under elements of G and hence under f_i and f_i'). Therefore, the sets A_i'' for $i \leq N$ are disjoint, so we are done. □

If the space $\mathscr{X}$ satisfies the hypotheses of both Theorem 3.1 and the Banach-Tarski results, then Lemma 2.4(b) combines the Banach-Tarski decomposition with the decomposition from 3.1 to give a Baire paradoxical decomposition of $\mathscr{X}$. In particular, we get the following two theorems:

Theorem 2.5. *Suppose G is a group of homeomorphisms acting freely on a Polish space $\mathscr{X}$, and G includes a subgroup which is free on more than one generator. Then, for any $N \geq 1$, $\mathscr{X}$ can be partitioned into N pieces with the property of Baire, each of which is Baire G-equidecomposable with $\mathscr{X}$.*

Proof. We may assume that G actually is a free group on finitely many generators, and hence countable. In fact, since a free group on 2 generators has subgroups which are free on any given finite number of generators, we may assume that G is a free group on $3N$ generators. We saw before that, since G acts freely on $\mathscr{X}$, there are N disjoint subsets of $\mathscr{X}$ each of which is G-equidecomposable with $\mathscr{X}$. This takes care of the first hypothesis of Lemma 2.4(b), while Theorem 3.1 takes care of the second hypothesis; now Lemma 2.4(b) and Corollary 2.3 give the desired result. □

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Theorem 2.6. *For any $n \geq 2$ and $N \geq 1$, the sphere S^n can be partitioned into N pieces with the property of Baire, each of which is Baire equidecomposable with S^n using rotations. The same holds for a ball in $\mathbf{R}^{n+1}$ or in S^{n+1}, using isometries of the space $\mathbf{R}^{n+1}$ or S^{n+1}.*

For balls in $\mathbf{R}^{n+1}$, this is equivalent to the following: any ball in $\mathbf{R}^{n+1}$ can be partitioned into finitely many pieces with the property of Baire which can be rearranged using isometries to form N disjoint balls identical to the original. The same applies to a ball in S^{n+1} unless the ball is so large that N disjoint copies of it will not fit in S^{n+1}.

Proof. Let G be a countable group of rotations of S^n which includes a subgroup which is free on $3N$ generators. (Such a subgroup exists for $n = 2$ by previously quoted results, and it can be extended to higher n by keeping the new coordinates fixed.) Clearly the set of fixed points of a nontrivial isometry of S^n or $\mathbf{R}^{n+1}$ is nowhere dense, so the union of the G-orbits of the fixed points of nonidentity elements of G is meager. Hence, Theorem 3.1 applies to G when $\mathscr{X}$ is either S^n or a ball in $\mathbf{R}^{n+1}$ or S^{n+1}. As described earlier in this section, the Banach-Tarski results also apply to S^n and to balls in $\mathbf{R}^{n+1}$ or S^{n+1} (although we use isometries of $\mathbf{R}^{n+1}$ or S^{n+1} rather than just rotations of the balls); hence, Lemma 2.4(b) and Corollary 2.3 give the desired result as before. □

Just as for unrestricted equidecomposability, one can use Theorem 2.6 and the Baire part of Proposition 2.2 to show

Corollary 2.7. *If A and B are bounded subsets of $\mathbf{R}^{n+1}$ or S^{n+1} $(n \geq 2)$ with the property of Baire and nonempty interior, then A and B are Baire equidecomposable using isometries.* □

(Of course, any subset of S^{n+1} is bounded.)

Unlike the other results about Baire equidecomposability, Corollary 2.7 was not proved directly from a corresponding result about open sets. This corresponding result is nonetheless true:

Corollary 2.8. *If A and B are nonempty bounded open subsets of $\mathbf{R}^{n+1}$ or S^{n+1} $(n \geq 2)$, then there are finitely many disjoint open sets $A_i \subseteq A$ $(1 \leq i \leq m)$ whose union is a dense open subset of A, and isometries σ_i of $\mathbf{R}^{n+1}$ or S^{n+1} such that the images $\sigma_i(A_i)$ are disjoint and their union is a dense open subset of B.*

Proof. We can prove this easily from Corollary 2.7, as follows: A can be partitioned into sets $A'_1, \ldots, A'_m$ with the property of Baire such that, for some isometries $\sigma_1, \ldots, \sigma_m$, the sets $\sigma_i(A'_i)$ are disjoint and their union is B. Let $\hat{A}_i$ be an open set such that the symmetric difference $\hat{A}_i \triangle A'_i$ is meager, and let $A_i = \hat{A}_i \cap A \cap \sigma_i^{-1}(B)$; since $A'_i \subseteq A \cap \sigma_i^{-1}(B)$, $A_i \triangle A'_i$ is also meager. It follows that the sets A_i are disjoint, since the open set $A_i \cap A_j$ is included in the meager set $(A_i \setminus A'_i) \cup (A_j \setminus A'_j)$ and no nonempty open set is meager. Also, $\bigcup_{i=1}^m A_i$ is an open subset of A which differs from A by a meager set, so it is a

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dense open subset of A. Similarly, the sets $\sigma_i(A_i)$ are disjoint and their union is dense in B, as desired. □

Analogues of Corollaries 2.7 and 2.8 for S^2 do hold, but the proofs are postponed until the beginning of §5, because they involve extensions of Theorem 3.1 and the Banach-Tarski results.

Of course, this proof of Corollary 2.8 depends on the Banach-Tarski theorem and hence uses the Axiom of Choice. One can show that Choice is not really necessary by appealing to standard metamathematical results (the Shoenfield absoluteness theorem [4, 8F.10]), or by proving Corollary 2.8 constructively directly from Theorem 3.1. The latter approach requires a version of the Banach-Schröder-Bernstein theorem for open sets. In order to state this, it is convenient to use the following definition: a function f is *openly piecewise in* G (where G is a group of homeomorphisms) if the domain of f can be partitioned into finitely many open sets, on each of which f agrees with some element of G. Hence, "there is a bijection from A to B which is openly piecewise in G" means that A can be partitioned into finitely many open sets which can be moved by elements of G and reassembled to form B.

Proposition 2.9 [9, pp. 118–119]. *Suppose G is a group of homeomorphisms on a Polish space $\mathscr{X}$, and A and B are open subsets of $\mathscr{X}$. Suppose that there are one-to-one functions $f\colon A' \to B$ and $g\colon B' \to A$ which are openly piecewise in G, where A' and B' are dense open subsets of A and B, respectively. Then there is a bijection $h\colon A'' \to B''$ which is openly piecewise in G, where A'' and B'' are dense open subsets of A and B, respectively.*

Proof. We follow the proof of Proposition 2.2. Let C_0 be the interior of $A \setminus g(B')$, and, for $n > 0$, let $C_n = g(f(C_{n-1} \cap A') \cap B')$. Now let $C = \bigcup_{n=0}^{\infty} C_n$, and let $A'' = (A' \cap C) \cup (g(B') \setminus \overline{C})$; define h on A'' by letting $h(x)$ be $f(x)$ if $x \in C$, $g^{-1}(x)$ otherwise.

Clearly C and $\mathscr{X} \setminus \overline{C}$ are open, so h is openly piecewise in G; we must see that h is one-to-one and that A'' and $B'' = h(A'')$ are dense in A and B, respectively. Since f and g^{-1} are one-to-one, checking that h is one-to-one reduces to showing that $f(A' \cap C)$ and $g^{-1}(g(B') \setminus \overline{C})$ are disjoint; but this is easy, because if $x \in f(A' \cap C_n)$ and $g(x)$ exists, then $g(x)$ is in C_{n+1} and hence not in $g(B') \setminus \overline{C}$.

To see that A'' is dense, note that any open subset of A which meets C must meet $A' \cap C$ (since C is open and A' is dense), while any nonempty open subset of A which does not meet C does not meet $\overline{C}$ but must meet $g(B')$ (if it did not, it would be included in C_0). To show the same for B'', suppose U is a nonempty open subset of B which does not meet the subset $f(A' \cap C)$ of B''. Then $U \cap B'$ is nonempty and open, and $g(U \cap B')$ does not meet C_0 (since it is included in $g(B')$) or C_n for $n > 0$ (since U does not meet $f(A' \cap C_{n-1})$). Therefore, $g(U \cap B')$ is included in $g(B') \setminus \overline{C}$, so $U \cap B'$ is included in B''. This shows that B'' is dense, so we are done. □

Given this result, Corollary 2.8 follows from the weaker version in which the sets A_i are not required to have union dense in A. To prove this weaker

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version, proceed as follows: Let C be an open ball included in A, and choose N so large that B can be covered by N translates of C. Applying Theorem 3.1, we get $m = 3N$ disjoint open subsets of C which can be rotated and reassembled in triples to form N dense open subsets of C; hence, these m sets (say A'_i for $i \leq m$) can be moved by isometries σ_i of $\mathbf{R}^{n+1}$ or S^{n+1} so that the resulting sets $B'_i = \sigma_i(A'_i)$ cover a dense open subset of B. Let $B_i = (B'_i \cap B) \setminus \bigcup_{j<i} \overline{B'_j}$; then the sets B_i are open and disjoint and their union is a dense subset of B, so the sets $A_i = \sigma_i^{-1}(B_i)$ satisfy the weakened conclusion of Corollary 2.8.

Note that Corollary 2.8 implies that, for any $n \geq 2$ and $R > 0$, the unit ball in $\mathbf{R}^{n+1}$ includes finitely many disjoint open sets which can be rearranged by isometries to form a dense subset of the ball of radius R. In fact, if C is any bounded subset of $\mathbf{R}^{n+1}$ which is the closure of its interior, then the unit ball includes finitely many disjoint open sets which can be rearranged by isometries so that the closure of their union is C.

Finally, we note that paradoxical decompositions using sets with the property of Baire imply the nonexistence of certain measures on the σ-algebra of sets with the property of Baire:

Proposition 2.10. *If $\mathscr{X}$ and G satisfy the hypotheses of Theorem* 3.1, *then there is no G-invariant finitely additive measure on the subsets of $\mathscr{X}$ with the property of Baire which gives $\mathscr{X}$ measure* 1 *and all meager sets measure* 0.

Proof. Suppose μ is such a measure, and let A_{ij} and f_{ij} be as in the conclusion of Theorem 3.1. Since $\bigcup_{i=1}^{3} f_{ij}(A_{ij})$ is comeager, it has measure 1; since μ is G-invariant, we have $\mu(A_{1j})+\mu(A_{2j})+\mu(A_{3j}) \geq 1$ for each j, so $\sum_{i,j} \mu(A_{ij}) \geq N$. This is impossible because the sets A_{ij} are disjoint and their union has measure at most 1. □

When the Banach-Tarski results are also available, Proposition 2.10 can be improved:

Proposition 2.11. *For $n \geq 2$, there is no isometry-invariant finitely additive measure on the subsets of S^n (or the bounded subsets of $\mathbf{R}^{n+1}$) with the property of Baire which gives S^n (the unit ball in $\mathbf{R}^{n+1}$) measure* 1.

Proof. Let μ be such a measure, and let X be S^n or the unit ball in $\mathbf{R}^{n+1}$, so that $\mu(X) = 1$. By Theorem 2.6, there are disjoint subsets A and B of X each of which is Baire equidecomposable with X. Since μ is isometry-invariant and the pieces in the decompositions are μ-measurable, we have $\mu(A) = \mu(B) = 1$; but A and B are disjoint subsets of X, so $\mu(X) \geq 2$, contradiction. □

3. The main theorem: decompositions using open sets

In this section we will prove the main result: given a sufficiently free group action by homeomorphisms on a Polish space (actually, all we need is a second countable Hausdorff space for which the Baire category theorem holds), there are finitely many disjoint open sets which can be rearranged so as to cover a dense open subset of the space more than once. As we saw in the preceding

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section, this implies that spaces such as S^n or $\mathbf{R}^{n+1}$ ($n \geq 2$) have paradoxical decompositions using pieces with the property of Baire. The important part of the following theorem is the case $N = 2$, but the proof of this case is no simpler than the proof for general N.

Theorem 3.1. *Suppose $\mathscr{X}$ is a Polish space, and G is a countable group of homeomorphisms of $\mathscr{X}$ which acts freely on a comeager subset of $\mathscr{X}$. Let $N \geq 1$ be given, and suppose that G has a free subgroup of rank $3N$ with free generators f_{ij} ($i = 1, 2, 3$; $1 \leq j \leq N$). Then there are disjoint open subsets A_{ij} of $\mathscr{X}$ such that, for each $j \leq N$, $f_{1j}(A_{1j}) \cup f_{2j}(A_{2j}) \cup f_{3j}(A_{3j})$ is dense in $\mathscr{X}$.*

Proof. Throughout this proof, we will use various forms of the letter i ($\bar{\imath}$, I, etc.) as variables ranging over $\{1, 2, 3\}$, while all sorts of j's will range over $\{1, 2, \ldots, N\}$.

For each i and j, we will construct increasing sequences $A_{ij}^0 \subseteq A_{ij}^1 \subseteq A_{ij}^2 \subseteq \cdots$ and $B_{ij}^0 \subseteq B_{ij}^1 \subseteq B_{ij}^2 \subseteq \cdots$ of open subsets of $\mathscr{X}$, in such a way that, if we let $A_{ij} = \bigcup_{n=0}^{\infty} A_{ij}^n$ and $B_{ij} = \bigcup_{n=0}^{\infty} B_{ij}^n$, then the sets A_{ij} will have the desired properties. The points in B_{ij}^n are to be excluded from A_{ij}^m for all m, and hence from A_{ij}. (In fact, it will turn out that the set B_{ij}^n resulting from this construction is just the union of the sets $A_{i'j'}^n$ for $(i', j') \neq (i, j)$; however, this fact is not used, and it is not true without modification for the variant proofs in the next section.) The initial sets are $A_{ij}^0 = B_{ij}^0 = \varnothing$. During the construction, the following properties (which are clearly satisfied for $n = 0$) will be maintained as induction hypotheses:

(1) Exclusion: $A_{ij}^n \cap B_{ij}^n = \varnothing$.

(2) Disjointness before moves: $A_{ij}^n \subseteq B_{i'j'}^n$ for $(i', j') \neq (i, j)$.

(3) Possible density after moves: If $\{i, i', i''\} = \{1, 2, 3\}$ and $z \in f_{ij}(B_{ij}^n) \cap f_{i''j}(B_{i''j}^n)$, then $z \in f_{i'j}(A_{i'j}^n)$.

We will also maintain one other induction hypothesis, to be stated later.

Properties (1) and (2) guarantee that the sets A_{ij}^n and $A_{i'j'}^n$ are disjoint for $(i', j') \neq (i, j)$; hence, A_{ij} and $A_{i'j'}$ will be disjoint. We must also ensure that, for each j, the set $\bigcup_{i=1}^{3} f_{ij}(A_{ij})$ is dense; that is, for every nonempty open set $Z \subseteq \mathscr{X}$, at least one of the three sets $f_{ij}(A_{ij})$ must meet Z. In fact, it will suffice to handle all nonempty sets Z in some base for the topology of $\mathscr{X}$. Fix a countable base $\mathscr{Z}$ for this topology, and let $\langle (\bar{\jmath}_n, Z_n) : n \geq 0 \rangle$ be an enumeration of $\{1, 2, \ldots, N\} \times (\mathscr{Z} \setminus \{\varnothing\})$. At stage n of the construction, we will construct A_{ij}^{n+1} and B_{ij}^{n+1} from A_{ij}^n and B_{ij}^n so as to ensure that at least one of the three sets $f_{\bar{\imath}\bar{\jmath}_n}(A_{\bar{\imath}\bar{\jmath}_n}^{n+1})$ ($\bar{\imath} = 1, 2, 3$) meets Z_n; once this is done for all n, the resulting sets A_{ij} will have all of the desired properties.

In order for this to work, we must ensure that an open set Z is never excluded from all three of $f_{1j}(A_{1j})$, $f_{2j}(A_{2j})$, and $f_{3j}(A_{3j})$ (i.e., included in all of $f_{1j}(B_{1j})$, $f_{2j}(B_{2j})$, and $f_{3j}(B_{3j})$), for any fixed j. If Z is excluded from two

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of the three sets $f_{ij}(A_{ij})$, then any nonempty open subset of Z must meet the third such set, so we might as well require that Z be included in this third set. Precisely, if i, i', i'' is a permutation of $1, 2, 3$ (i.e., $\{i, i', i''\} = \{1, 2, 3\}$) and the construction so far has given $Z \subseteq f_{ij}(B^n_{ij})$ and $Z \subseteq f_{i''j}(B^n_{i''j})$, then we require that $Z \subseteq f_{i'j}(A^n_{i'j})$. When stated in terms of individual points $z \in Z$, this is (3).

For any point z, if i, i', and j are as above, then (3) says that there is some relation between what we do at the point $x = f_{ij}^{-1}(z)$ and what we do at $x' = f_{i'j}^{-1}(z) = f_{i'j}^{-1}(f_{ij}(x))$. In fact, (3) can be restated in terms of x, as follows:

(3′) If $\{i, i', i''\} = \{1, 2, 3\}$, $x \in B^n_{ij}$, and $f_{i''j}^{-1}(f_{ij}(x)) \in B^n_{i''j}$, then $f_{i'j}^{-1}(f_{ij}(x)) \in A^n_{i'j}$.

From this we see that the points x, $x' = f_{i'j}^{-1}(f_{ij}(x))$, and $x'' = f_{i''j}^{-1}(f_{ij}(x))$ are related; what we do at x while constructing the new sets A^{n+1}_{IJ} and B^{n+1}_{IJ} may have effects at x' or x''. We therefore make the following definition:

Definition. The points x and x' are *linked*, or there is a *link* from x to x', if there are $i, i' \leq 3$ and $j \leq N$ with $i \neq i'$ such that $x' = f_{i'j}^{-1}(f_{ij}(x))$.

If we need to be more specific, we will say that x *is linked to* x' *via* i, i', j. Links are symmetric: if x is linked to x' via i, i', j, then x' is linked to x via i', i, j. Also, any point x is linked to only finitely many (in fact, at most $6N$) other points. (The links can be viewed as forming a graph. On an orbit on which G acts freely, this graph is made up of triangles, coming from the triples x, x', x''; since the elements f_{ij} of G are free generators of a free subgroup of G, it is not hard to show that the graph on this orbit has no simple cycles other than these triangles.)

At stage n in our construction, if we add x to B^n_{ij} and x is linked to x' via i, i', j, then (3) might force us to add x' to $A^n_{i'j}$. This will be forced only if the point $x'' = f_{i''j}^{-1}(f_{ij}(x))$ is in $B^n_{i''j}$ (where $\{i, i', i''\} = \{1, 2, 3\}$). We might say that the link is active in this case. Actually, for convenience at one point later in the proof, and for the sake of easy generalization, we will give a more inclusive definition:

Definition. A link from x to x' (via i, i', j) is *active* (for the sets A^n_{IJ} and B^n_{IJ}, $I \leq 3$, $J \leq N$) if at least one of the points x, x', or $x'' = f_{i''j}^{-1}(f_{ij}(x))$ is in $\bigcup_{I,J}(A^n_{IJ} \cup B^n_{IJ})$.

Thus, if one of the vertices in a linked triangle x, x', x'' is in some A^n_{IJ} or B^n_{IJ}, then the whole triangle is actively linked. The exact definition is relatively unimportant; what is important is that all links are inactive at stage 0, and that, if the link from x to x' is inactive, then adding x to B_{ij} does not directly force us to add x' to any set. We can now state the remaining induction

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hypothesis:

(4) Finite active linkage: For any $x \in \mathscr{X}$, the set of $y \in \mathscr{X}$ which are connected to x by a path of active links (i.e., there exist points $x_0, x_1, \ldots, x_m$ such that $x_0 = x$, $x_m = y$, and, for each k, x_k is linked to x_{k+1} via some i_k, i'_k, j_k and this link is active for the sets A^n_{IJ} and B^n_{IJ}) is finite.

(Equivalently, the graph of active links, which is a subgraph of the graph of links, has the property that all connected components are finite.) This is clearly true for $n = 0$.

We have now seen that the sets A^0_{ij} and B^0_{ij} satisfy hypotheses (1)–(4); we have also seen that, if we can maintain these hypotheses through an inductive construction of sets A^n_{ij} and B^n_{ij} which takes care of all of the pairs $(\bar{\jmath}, Z) = (\bar{\jmath}_n, Z_n)$, then the resulting sets A_{ij} will satisfy the conclusion of the theorem. Hence, it only remains to show

Lemma 3.2. *Assume the hypotheses of Theorem* 3.1. *If open sets* A^n_{ij} *and* B^n_{ij} $(i \leq 3,\ j \leq N)$ *satisfy hypotheses* (1)–(4), *and if* $\bar{\jmath} \leq N$ *and* Z *is a nonempty open subset of* $\mathscr{X}$, *then there are open sets* $A^{n+1}_{ij} \supseteq A^n_{ij}$ *and* $B^{n+1}_{ij} \supseteq B^n_{ij}$ *satisfying* (1)–(4) *such that, for some* $\bar{\imath} \leq 3$, $f_{\bar{\imath}\bar{\jmath}}(A^{n+1}_{\bar{\imath}\bar{\jmath}}) \cap Z \neq \varnothing$.

Proof. The construction will consist of two steps. In the first step, we will add finitely many points to the sets A^n_{ij} and B^n_{ij} to get sets $\hat{A}_{ij}$ and $\hat{B}_{ij}$ satisfying (1)–(4); one of these points will be a point $x_0 \in \hat{A}_{\bar{\imath}\bar{\jmath}}$ for some $\bar{\imath}$ such that $f_{\bar{\imath}\bar{\jmath}}(x_0) \in Z$. Of course, these sets probably will not be open. In the second step, we will "inflate" these finitely many points to get small open sets which, when added to A^n_{ij} and B^n_{ij}, give the desired sets A^{n+1}_{ij} and B^{n+1}_{ij}.

We start by finding a suitable point x_0 and index $\bar{\imath}$. Each of the open sets A^n_{ij} and B^n_{ij} has a nowhere dense boundary (any open set does); since G is countable, the set E of points in $\mathscr{X}$ which do not lie in any g-image ($g \in G$) of any of these boundaries is comeager. Therefore, we can choose a point $z_0 \in E$ which lies in Z and is also in the comeager set on which G acts freely. Properties (1) and (3) guarantee that, for at least one $\bar{\imath} \leq 3$, z_0 does not lie in $f_{\bar{\imath}\bar{\jmath}}(B^n_{\bar{\imath}\bar{\jmath}})$. Fix such an $\bar{\imath}$, and let $x_0 = f^{-1}_{\bar{\imath}\bar{\jmath}}(z_0)$; if we make sure that x_0 is in $A^{n+1}_{\bar{\imath}\bar{\jmath}}$, then $f_{\bar{\imath}\bar{\jmath}}(A^{n+1}_{\bar{\imath}\bar{\jmath}})$ will meet Z.

The ideas behind the construction of $\hat{A}_{ij}$ and $\hat{B}_{ij}$ are as follows: We will start by putting x_0 in $\hat{A}_{\bar{\imath}\bar{\jmath}}$. Once we do this, we will have to put x_0 in $\hat{B}_{ij}$ for all $(i, j) \neq (\bar{\imath}, \bar{\jmath})$ if we hope to satisfy (2). Then, if x_0 is linked to x' via i, i', j, we might have to add x' to $\hat{A}_{i'j}$ because of (3); then (2) will force us to add x' to the other $\hat{B}$'s, and this might bring (3) into play again, and so on. All of the points which come up in this process are connected to x_0 by a path of zero or more links (which are, in fact, active), so they all have the form $g(x_0)$ for some $g \in G$; we will keep track of the group elements g instead of the points $g(x_0)$.

We now give the construction precisely. Define subsets S_{ij} and T_{ij} of G

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simultaneously for all i and j by the following inductive clauses (these sets are to be the minimal sets meeting the following conditions):

(a) The identity element of G is in $S_{\bar{\imath}\bar{\jmath}}$.

(b) If $g \in S_{ij}$, then $g \in T_{i'j'}$ for $(i', j') \neq (i, j)$.

(c) If $\{i, i', i''\} = \{1, 2, 3\}$, $g \in T_{ij}$, and $f_{i''j}^{-1}(f_{ij}(g(x_0))) \in B_{i''j}^n$, then $f_{i'j}^{-1} \circ f_{ij} \circ g \in S_{i'j}$.

The set S_{ij} indicates which points are to be added to A_{ij}, while T_{ij} gives the points to be added to B_{ij}. Clause (a) ensures that the current density requirement (given by $\bar{\jmath}$ and Z) is met, while (b) and (c) enforce properties (2) and (3′), respectively. Thus, let $\hat{A}_{ij} = A_{ij}^n \cup \{g(x_0) \colon g \in S_{ij}\}$ and $\hat{B}_{ij} = B_{ij}^n \cup \{g(x_0) \colon g \in T_{ij}\}$; we will see that these sets satisfy properties (1)–(4). (It is not obvious that (c) as stated above suffices to prove (3) in all cases, but this turns out to be so.)

Claim 1. Every nonidentity element of S_{ij} or T_{ij} has the reduced form

$$(*) \qquad g = f_{i'_m j_m}^{-1} \circ f_{i_m j_m} \circ f_{i'_{m-1} j_{m-1}}^{-1} \circ f_{i_{m-1} j_{m-1}} \circ \cdots \circ f_{i'_1 j_1}^{-1} \circ f_{i_1 j_1}$$

for some numbers i_k, i'_k, and j_k; for $g \in S_{ij}$ we have $(i, j) = (i'_m, j_m)$, while for $g \in T_{ij}$ we have $(i, j) \neq (i'_m, j_m)$.

Proof. The fact that each element of S_{ij} or T_{ij} can be expressed in the form $(*)$ follows easily from the inductive definition of S_{ij} and T_{ij}, which also gives $i'_k \neq i_k$. We have $(i_k, j_k) \neq (i'_{k-1}, j_{k-1})$ for all k, because rules (b) and (c) are applied alternately. Therefore, the representation $(*)$ of g is in reduced form in the free group with generators f_{ij}. The last part of the claim follows immediately from the inductive definition. □

Since the generators f_{ij} are free generators in G, different sequences $\langle (i_k, i'_k, j_k) \colon k < m \rangle$ for $(*)$ give different elements g of G. Thus, given an element g of $\bigcup_{i,j}(S_{ij} \cup T_{ij})$, we can determine which of the sets S_{IJ} and T_{IJ} contain g: the unique expression of g as a reduced word in the generators f_{IJ} will have the form $f_{ij}^{-1} \circ w$ for some particular pair (i, j) (unless g is the identity, in which case let $(i, j) = (\bar{\imath}, \bar{\jmath})$), and g will be in S_{ij} and in $T_{i'j'}$ for all $(i', j') \neq (i, j)$. It follows that no g is in both S_{ij} and T_{ij}.

When written formally, the proof of Claim 1 is actually a proof by induction on the stage in the definition given by (a)–(c) at which a member g of G is added to the set S_{ij} or T_{ij} (in short, by induction on the definition of S_{ij} and T_{ij}). We use the same method to prove

Claim 2. For any g: if $g \in S_{ij}$, then $g(x_0) \notin B_{ij}^n$; if $g \in T_{ij}$, then $g(x_0) \notin A_{ij}^n$.

Proof. We prove this by induction; that is, we show that no application of one of the rules (a)–(c) can introduce a violation of one of these two statements unless

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such a violation was already present before the application of the rule. For rule (a), this is clear since x_0 was chosen to be a point not in $B^n_{\bar{i}\bar{j}}$. For (b), since $g \in S_{ij}$, we have $g(x_0) \notin B^n_{ij}$ (otherwise there was a previously existing violation), so $g(x_0) \notin A^n_{i'j'}$ by (2) (with (i', j') and (i, j) interchanged). Finally, for (c), since we have $g \in T_{ij}$, the induction hypothesis gives $g(x_0) \notin A^n_{ij}$; on the other hand, we have $f^{-1}_{i''j}(f_{ij}(g(x_0))) \in B^n_{i''j}$, and if we also had $f^{-1}_{i'j}(f_{ij}(g(x_0))) \in B^n_{i'j}$, then (3) (applied to $z = f_{ij}(g(x_0))$ with i and i' interchanged) would give $g(x_0) \in A^n_{ij}$, contradiction. Therefore, we have $f^{-1}_{i'j}(f_{ij}(g(x_0))) \notin B^n_{i'j}$. This completes the induction. □

Recall that z_0 was chosen to be in an orbit on which G acts freely, and that x_0 is also in this orbit. Hence, every x in this orbit has a unique representation as $g(x_0)$ with $g \in G$.

Another easy induction on the definition of S_{ij} and T_{ij} gives

Claim 3. If g is in one of the sets S_{ij} or T_{ij}, then x_0 is connected to $g(x_0)$ by a path of links that are active for the sets A^n_{ij} and B^n_{ij}. □

We are now ready to prove properties (1)–(4) for $\hat{A}_{ij}$ and $\hat{B}_{ij}$.

(1): If $x \in \hat{A}_{ij}$, then $x \in A^n_{ij}$ or $x = g(x_0)$ for some $g \in S_{ij}$; if $x \in \hat{B}_{ij}$, then $x \in B^n_{ij}$ or $x = h(x_0)$ for some $x \in T_{ij}$. We cannot have $x \in A^n_{ij}$ and $x \in B^n_{ij}$, by (1) for A^n_{ij} and B^n_{ij}. The combination $x \in A^n_{ij}$ and $x = h(x_0)$, $h \in T_{ij}$ is ruled out by Claim 2, as is the combination $x \in B^n_{ij}$ and $x = g(x_0)$, $g \in S_{ij}$. Finally, if $x = g(x_0) = h(x_0)$, then by freeness we must have $g = h$, so we cannot have both $g \in S_{ij}$ and $h \in T_{ij}$, by the remarks following Claim 1. Therefore, there is no way for x to be in both $\hat{A}_{ij}$ and $\hat{B}_{ij}$, so (1) holds.

(2): This is easy; the points in A^n_{ij} are handled by (2) for A^n_{ij}, while the points $g(x_0)$ for $g \in S_{ij}$ are handled by (b).

(3): We must show that, if $\{i, i', i''\} = \{1, 2, 3\}$, $z \in \mathscr{Z}$, $x = f^{-1}_{ij}(z)$ is in $\hat{B}_{ij}$, and $x'' = f^{-1}_{i''j}(z)$ is in $\hat{B}_{i''j}$, then $x' = f^{-1}_{i'j}(z)$ is in $\hat{A}_{i'j}$. Since x can be in either B^n_{ij} or $\{g(x_0): g \in T_{ij}\}$ and x'' can be in either $B^n_{i''j}$ or $\{g''(x_0): g'' \in T_{i''j}\}$, we have four cases to consider. If $x \in B^n_{ij}$ and $x'' \in B^n_{i''j}$, then (3) for the sets A^n_{IJ} and B^n_{IJ} gives $x' \in A^n_{i'j}$. If $x = g(x_0)$ for some $g \in T_{ij}$ and $x'' \in B^n_{i''j}$, then (c) gives $g' = f^{-1}_{i'j} \circ f_{ij} \circ g \in S_{i'j}$, so $x' = g'(x_0) \in \hat{A}_{i'j}$. The case $x \in B^n_{ij}$ and $x'' = g''(x_0)$ ($g'' \in T_{i''j}$) is handled the same way, with i and i'' (and x and x'') interchanged. This leaves the case $x = g(x_0)$ and $x'' = g''(x_0)$, which we will see cannot happen. By Claim 1, we can express g in the reduced form $(*)$, where the last pair (i'_m, j_m) (or $(\bar{\imath}, \bar{\jmath})$ if g is the identity) is different from (i, j) (otherwise g would be in S_{ij} rather than T_{ij}). We can also express g'' in this form, and here the last pair will be different from (i'', j). Therefore, $f_{ij} \circ g$ and

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$f_{i''j} \circ g''$ will be two different reduced words which send x_0 to the same point, contradicting the assumptions that the functions f_{ij} generate a free subgroup of G and G acts freely on the orbit of x_0. Therefore, (3) holds for the sets $\hat{A}_{ij}$ and $\hat{B}_{ij}$. (In terms of the graph of links mentioned earlier, note that, in building the sets S_{ij} and T_{ij}, one follows edges in this graph; for any g in S_{ij} or T_{ij}, there is a natural path from x_0 to $g(x_0)$ in the graph. If there were points $x = g(x_0)$ and $x'' = g''(x_0)$ with $g \in T_{ij}$ and $g'' \in T_{i''j}$ as above, then the link between x and x'' together with the paths from x_0 to $g(x_0)$ and to $g''(x_0)$ would form a cycle in the graph of links, and one can show that this cycle would include a simple cycle other than a triangle; we have already noted that this cannot occur.)

(4): Note that, by Claim 3 and (4) for A^n_{ij} and B^n_{ij}, there are only finitely many points of the form $g(x_0)$ where g is in one of the sets S_{ij} or T_{ij}. This means that only finitely many new links are activated when A^n_{ij} and B^n_{ij} are increased to $\hat{A}_{ij}$ and $\hat{B}_{ij}$, so (4) for the former implies (4) for the latter.

This completes the first step of the construction. In summary, we have found a point $x_0 \in \mathscr{X}$ and subsets S_{ij} and T_{ij} of G such that: x_0 is in a G-orbit on which G acts freely, and this orbit does not meet the boundary of any of the sets A^n_{ij} or B^n_{ij}; for each g in S_{ij} or T_{ij}, x_0 is connected to $g(x_0)$ by a path of active links (for the sets A^n_{IJ} and B^n_{IJ}); the sets $\hat{A}_{ij} = A^n_{ij} \cup \{g(x_0) \colon g \in S_{ij}\}$ and $\hat{B}_{ij} = B^n_{ij} \cup \{g(x_0) \colon g \in T_{ij}\}$ satisfy properties (1)–(4); and, for some $\bar{\imath} \leq 3$, $f_{\bar{\imath}\bar{\jmath}}(\hat{A}_{\bar{\imath}\bar{\jmath}})$ meets Z. These are all of the facts needed for the rest of the construction.

For the second step of the construction, we must find suitable open neighborhoods of the new points in $\hat{A}_{ij}$ and $\hat{B}_{ij}$ and add these neighborhoods to the sets A^n_{ij} and B^n_{ij} to get A^{n+1}_{ij} and B^{n+1}_{ij}. Since all of the new points are of the form $g(x_0)$ for some $g \in G$, it will suffice to find a suitable open neighborhood U_0 of x_0, and then define the new sets as follows: $A^{n+1}_{ij} = A^n_{ij} \cup \bigcup\{g(U_0) \colon g \in S_{ij}\}$ and $B^{n+1}_{ij} = B^n_{ij} \cup \bigcup\{g(U_0) \colon g \in T_{ij}\}$. These sets are clearly open sets which include the corresponding sets $\hat{A}_{ij}$ and $\hat{B}_{ij}$; hence, $f_{\bar{\imath}\bar{\jmath}}(A^{n+1}_{ij})$ meets Z. If we choose the neighborhood U_0 to be so small that the relevant images $g(U_0)$ are disjoint from each other and do not cross over the boundaries of the sets A^n_{ij} and B^n_{ij} (i.e., they are contained in or disjoint from these sets), then anything which happens at a point $g(x_0)$ will happen throughout the neighborhood $g(U_0)$; hence, (1)–(4) for $\hat{A}_{ij}$ and $\hat{B}_{ij}$ will imply (1)–(4) for A^{n+1}_{ij} and B^{n+1}_{ij}. We now give the proof of these statements (which has one unexpected twist).

Let S be the set of all $g \in G$ such that x_0 is connected to $g(x_0)$ by a path of links which are active for the sets $\hat{A}_{ij}$ and $\hat{B}_{ij}$ (we allow newly activated links as well as those which were active at stage n); clearly $S_{ij} \subseteq S$ and $T_{ij} \subseteq S$ for all i and j. Define S' to be the set of $g \in G$ such that either $g \in S$ or there is $g' \in S$ such that $g(x_0)$ is linked to $g'(x_0)$ (by a single inactive link).

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Equivalently, S' is the set of all elements of G of the form $f_{i'j}^{-1} \circ f_{ij} \circ g$ with $i, i' \leq 3$, $j \leq N$, and $g \in S$. Clearly (4) for $\hat{A}_{ij}$ and $\hat{B}_{ij}$ implies that S is finite, and it follows that S' is finite. Since the points $g(x_0)$ for $g \in S'$ are distinct, and none of them is on the boundary of any of the sets A_{ij}^n or B_{ij}^n, we can find an open set U_0 containing x_0 so small that the sets $g(U_0)$ for $g \in S'$ are disjoint and each of them is either included in or disjoint from each of the sets A_{ij}^n and B_{ij}^n. Now define A_{ij}^{n+1} and B_{ij}^{n+1} as above; once we show that these sets satisfy properties (1)–(4), we will be done.

From the definition of A_{ij}^{n+1} and B_{ij}^{n+1} and the disjointness of the sets $g(U_0)$ for $g \in S'$, the following two claims follow easily:

Claim 4. If $x \in g(U_0)$ for some $g \in S'$, then $x \in A_{ij}^{n+1}$ if and only if $g(x_0) \in \hat{A}_{ij}$, and similarly for B_{ij}^{n+1}. □

(Recall that the neighborhood U_0 was chosen to be so small that, for $g \in S'$, the points in $g(U_0)$ all behave exactly as $g(x_0)$ does with respect to the sets A_{ij}^n and B_{ij}^n.)

Claim 5. If $x \in \mathscr{X}$ is not in any of the sets $g(U_0)$ for $g \in S$, then $x \in A_{ij}^{n+1}$ if and only if $x \in A_{ij}^n$, and similarly for B_{ij}^{n+1}. □

At least one (in some cases both) of these claims applies to any point $x \in \mathscr{X}$.

We can now prove (1)–(4) for A_{ij}^{n+1} and B_{ij}^{n+1}.

(1): This follows easily from the preceding claims. If a point x is in one of the neighborhoods $g(U_0)$ where $g \in S$, then $g(x_0) \notin \hat{A}_{ij} \cap \hat{B}_{ij}$ by (1) for $\hat{A}_{ij}$ and $\hat{B}_{ij}$, so $x \notin A_{ij}^{n+1} \cap B_{ij}^{n+1}$ by Claim 4; if x is not in one of these neighborhoods, then $x \notin A_{ij}^n \cap B_{ij}^n$ by (1) for A_{ij}^n and B_{ij}^n, so $x \notin A_{ij}^{n+1} \cap B_{ij}^{n+1}$ by Claim 5.

(2): Consider the same two cases as for (1).

(3): Suppose z is such that $x = f_{ij}^{-1}(z) \in B_{ij}^{n+1}$ and $x'' = f_{i''j}^{-1}(z) \in B_{i''j}^{n+1}$. If neither x nor x'' is in $g(U_0)$ for any $g \in S$, then $x \in B_{ij}^n$ and $x'' \in B_{i''j}^n$, so $x' = f_{i'j}^{-1}(z) \in A_{i'j}^n$ by (3) for the sets A_{IJ}^n and B_{IJ}^n, so $x' \in A_{i'j}^{n+1}$. Now suppose $x \in g(U_0)$ and $g \in S$. (The case $x'' \in g(U_0)$ is symmetric; interchange x with x'' and i with i''.) Then $f_{i'j}^{-1} \circ f_{ij} \circ g$ and $f_{i''j}^{-1} \circ f_{ij} \circ g$ are in S', so, by Claim 4, $g(x_0) \in \hat{B}_{ij}$ and $(f_{i''j}^{-1} \circ f_{ij} \circ g)(x_0) \in \hat{B}_{i''j}$; now (3) for $\hat{A}_{ij}$ and $\hat{B}_{ij}$ gives $(f_{i'j}^{-1} \circ f_{ij} \circ g)(x_0) \in \hat{A}_{i'j}$, so $x' \in A_{i'j}^{n+1}$ by Claim 4 again.

(4): Let w be any point of $\mathscr{X}$, and consider the set of all points connected to w by a path of links which are active for the sets A_{ij}^{n+1} and B_{ij}^{n+1}. If this set contains no point which is in $g(U_0)$ for any $g \in S$, then all of the links connecting the set were in fact active for A_{ij}^n and B_{ij}^n. (Note: If the link from x to x' is activated by x'', so is the link from x to x''.) Hence, the set is finite by (4) for A_{ij}^n and B_{ij}^n. So suppose $y \in g(U_0)$ is connected to w, and $g \in S$.

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A point is connected to w if and only if it is connected to y, so it will suffice to show that only finitely many points are connected to y. If y is actively linked to y' via i, i', j, and $y'' = f_{i''j}^{-1}(f_{ij}(y))$ where $\{i, i', i''\} = \{1, 2, 3\}$, then at least one of the three points y, y', and y'' is in one of the sets A_{IJ}^{n+1} or B_{IJ}^{n+1}. Let $g' = f_{i'j}^{-1} \circ f_{ij} \circ g$ and $g'' = f_{i''j}^{-1} \circ f_{ij} \circ g$; both g' and g'' are in S'. Now, by Claim 4, $y \in A_{IJ}^{n+1}$ if and only if $g(x_0) \in \hat{A}_{IJ}$, and $y \in B_{IJ}^{n+1}$ if and only if $g(x_0) \in \hat{B}_{IJ}$; similar statements hold for y' and y''. Therefore, the link from $g(x_0)$ to $g'(x_0)$ is active for the sets $\hat{A}_{IJ}$ and $\hat{B}_{IJ}$; hence, it follows from the definition of S that $g' \in S$. Now this argument can be repeated starting at y', and so on; the result is that, for any path of active (for A_{IJ}^{n+1} and B_{IJ}^{n+1}) links starting at y, all of the links in the corresponding path starting at $g(x_0)$ are also active (for $\hat{A}_{IJ}$ and $\hat{B}_{IJ}$). Furthermore, if y is connected to two different points y' and y'' by such paths of links, this will give $y' = h'(y)$ and $y'' = h''(y)$ for some distinct elements h', h'' of G, and the corresponding points reached from $g(x_0)$ will be $h'(g(x_0))$ and $h''(g(x_0))$; since G acts freely on the orbit of x_0, these two points will also be different. Therefore, since $g(x_0)$ is connected to only finitely many points, y (and hence w) must be connected to only finitely many points.

This completes the proof that (1)–(4) hold for A_{ij}^{n+1} and B_{ij}^{n+1}, so Lemma 3.2 is proved. □

This completes the proof of Theorem 3.1 (and the results in §2). □

We stated earlier that the proof of Theorem 3.1 could be given constructively (without appeal to the Axiom of Choice); to do so requires some care at certain points in the proof, since one cannot rely on such seemingly innocuous statements as "a countable union of meager sets is meager." Since the group G was assumed to be countable, we can fix an enumeration of it in advance; hence, arbitrary choices from G pose no problem. We can also fix an enumeration of a countable dense subset of the separable metric space $\mathscr{X}$; this lets us explicitly construct a listing of a countable base for the topology of $\mathscr{X}$ (use balls with rational radii centered at points of the countable dense set). (Here we are just noting that the standard proof that separability implies second countability for metric spaces does not require the Axiom of Choice. The proof of Theorem 3.1 only requires second countability rather than separability.) From this we can give a constructive proof of the Baire category theorem (given an infinite sequence of dense open sets and a nonempty open set, one can find a point in the intersection of all of the sets): at each step in the standard proof [4, 2H.2] where we must choose an open set with certain properties, just choose the first suitable set from the countable base. This takes the arbitrariness out of the choice of z_0 at each stage. We must also check that the set E from which z_0 was chosen is really comeager; but this is no problem, since it was constructed as a countable (indexed by G) intersection of open dense sets. (Of course, finite intersections of open dense sets are open dense, and finite intersections of comeager sets are comeager, even without Choice. Actually, it is not hard to show that, if $E' \subseteq \mathscr{X}$ is meager, then so is $\bigcup_{g \in G} g(E')$; the list of nowhere dense sets composing E'

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can be copied uniformly to the images $g(E')$, and the resulting lists can be merged as usual.) The only other arbitrary choice made in the proof was that of $\bar{\imath}$, and here we can just choose the least possible value. Now the proof is completely constuctive.

4. Variations on a theorem

In this section we give extensions of Theorem 3.1 in several directions. Most of these extensions do not seem to follow directly from Theorem 3.1, so we will have to repeat the proof from §3 with modifications; instead of giving complete proofs, we indicate the changes which must be made in the proof of Theorem 3.1. Applications of these results are given in the next section.

Density and disjointness on both sides. The sets A_{ij} from Theorem 3.1 do not necessarily cover a dense subset of $\mathscr{X}$, and two image sets $f_{ij}(A_{ij})$ and $f_{i'j}(A_{i'j})$ are not necessarily disjoint. Either one of these properties can be arranged by modifying the sets A_{ij} after they are constructed. (To make the sets have dense union, simply add the interior of the complement of their union to A_{11}; to make different image sets with the same j disjoint, subtract the closure of $\bigcup_{i'<i} f_{ij}^{-1}(f_{i'j}(A_{i'j}))$ from A_{ij}.) Or one could apply a form of Proposition 2.9 (the open-sets version of the Banach-Schröder-Bernstein theorem) to get both properties, at the cost of increasing the number of open sets. With a little more work, however, we can get both of these properties to hold simultaneously without increasing the number of pieces:

Theorem 4.1. *Assume the hypotheses of Theorem* 3.1. *Then there are sets* A_{ij} *satisfying the conclusions of that theorem such that* $\bigcup_{i\le 3,\, j\le N} A_{ij}$ *is dense in* $\mathscr{X}$ *and, for each* j, *the sets* $f_{1j}(A_{1j})$, $f_{2j}(A_{2j})$, *and* $f_{3j}(A_{3j})$ *are disjoint.*

Proof. We follow the proof of Theorem 3.1; as before, we build increasing sequences of open sets A_{ij}^n and B_{ij}^n such that the unions $A_{ij} = \bigcup_{n=0}^{\infty} A_{ij}^n$ have the required properties. In order to make $\bigcup_{i\le 3,\, j\le N} A_{ij}$ dense in $\mathscr{X}$, there are additional requirements to be met: each nonempty Z in the countable base for the topology must meet at least one of the sets A_{ij}. We therefore revise the list of pairs $(\bar{\jmath}_n, Z_n)$ so that it also includes all pairs $(0, Z)$; if $(\bar{\jmath}_n, Z_n) = (0, Z)$, then we ensure that, for some i, j, $Z \cap A_{ij}^{n+1} \neq \varnothing$.

We maintain the same induction hypotheses (1)–(4) as before, but add two new hypotheses:

(1) Exclusion: $A_{ij}^n \cap B_{ij}^n = \varnothing$.

(2) Disjointness before moves: $A_{ij}^n \subseteq B_{i'j'}^n$ for $(i', j') \neq (i, j)$.

(3) Possible density after moves: If $\{i, i', i''\} = \{1, 2, 3\}$ and $z \in f_{ij}(B_{ij}^n) \cap f_{i''j}(B_{i''j}^n)$, then $z \in f_{i'j}(A_{i'j}^n)$.

(4) Finite active linkage: For any $x \in \mathscr{X}$, the set of $y \in \mathscr{X}$ which are connected to x by a path of active links is finite.

(5) Possible density before moves: If $x \in B_{i'j'}^n$ for all $(i', j') \neq (i, j)$, then $x \in A_{ij}^n$.

(6) Disjointness after moves: If $z \in f_{ij}(A_{ij}^n)$ and $i' \neq i$, then $z \in f_{i'j}(B_{i'j}^n)$.

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These are again satisfied at stage 0 by the sets $A_{ij}^0 = B_{ij}^0 = \varnothing$. It now suffices to prove a new version of Lemma 3.2:

Lemma 4.2. *Assume the hypotheses of Theorem* 4.1. *If open sets* A_{ij}^n *and* B_{ij}^n $(1 \le i \le 3,\ 1 \le j \le N)$ *satisfy hypotheses* (1)–(6), *and if* $0 \le \hat{\jmath} \le N$ *and* Z *is a nonempty open subset of* $\mathscr{X}$, *then there are open sets* $A_{ij}^{n+1} \supseteq A_{ij}^n$ *and* $B_{ij}^{n+1} \supseteq B_{ij}^n$ *satisfying* (1)–(6) *such that*: *if* $\hat{\jmath} = 0$, *then, for some* $i \le 3$ *and* $j \le n$, $A_{ij}^{n+1} \cap Z \ne \varnothing$; *if* $\hat{\jmath} > 0$, *then, for some* $\bar{\imath} \le 3$, $f_{\bar{\imath}\hat{\jmath}}(A_{\bar{\imath}\hat{\jmath}}^{n+1}) \cap Z \ne \varnothing$.

Proof. As before, find a point $z_0 \in Z$ whose G-orbit is acted on freely by G and does not meet the boundaries of the sets A_{ij}^n and B_{ij}^n. The definition of x_0 now depends on $\hat{\jmath}$. If $\hat{\jmath} > 0$, then let $\bar{\jmath} = \hat{\jmath}$. By (3), we can choose $\bar{\imath} \le 3$ such that $z_0 \notin f_{\bar{\imath}\bar{\jmath}}(B_{\bar{\imath}\bar{\jmath}}^n)$; let $x_0 = f_{\bar{\imath}\bar{\jmath}}^{-1}(z_0)$. If $\hat{\jmath} = 0$, then (5) guarantees the existence of $\bar{\imath} \le 3$ and $\bar{\jmath} \le N$ such that $z_0 \notin B_{\bar{\imath}\bar{\jmath}}^n$; now let $x_0 = z_0$. In either case, we have $x_0 \notin B_{\bar{\imath}\bar{\jmath}}^n$, and we will be done once we construct open sets $A_{ij}^{n+1} \supseteq A_{ij}^n$ and $B_{ij}^{n+1} \supseteq B_{ij}^n$ satisfying (1)–(6) such that $x_0 \in A_{\bar{\imath}\bar{\jmath}}^{n+1}$.

Define the sets S_{ij} and T_{ij} by the same inductive clauses as before:

(a) The identity element of G is in $S_{\bar{\imath}\bar{\jmath}}$.

(b) If $g \in S_{ij}$, then $g \in T_{i'j'}$ for $(i', j') \ne (i, j)$.

(c) If $\{i, i', i''\} = \{1, 2, 3\}$, $g \in T_{ij}$, and $f_{i''j}^{-1}(f_{ij}(g(x_0))) \in B_{i''j}^n$, then $f_{i'j}^{-1} \circ f_{ij} \circ g \in S_{i'j}$.

Also, define sets S'_{ij} and T'_{ij} inductively as follows:

(a′) The identity element of G is in $S'_{\bar{\imath}\bar{\jmath}}$.

(b′) If $g \in S'_{ij}$ and $i' \ne i$, then $f_{i'j}^{-1} \circ f_{ij} \circ g \in T'_{i'j}$.

(c′) If $(i, j) \ne (i', j')$, $g \in T'_{ij}$, and $g(x_0) \in B_{IJ}^n$ for all (I, J) other than (i, j) and (i', j'), then $g \in S'_{i'j'}$.

Then let $\hat{A}_{ij} = A_{ij}^n \cup \{g(x_0) \colon g \in S_{ij} \cup S'_{ij}\}$ and $\hat{B}_{ij} = B_{ij}^n \cup \{g(x_0) \colon g \in T_{ij} \cup T'_{ij}\}$. (The sets S_{ij} and T_{ij} are constructed so that $\hat{A}_{ij}$ and $\hat{B}_{ij}$ will satisfy (2) and (3), while the sets S'_{ij} and T'_{ij} enforce (5) and (6). As we will see, there is no need to mix these two parts of the construction in a single inductive definition.)

Every element of S'_{ij} or T'_{ij} (other than the identity element in $S'_{\bar{\imath}\bar{\jmath}}$) has the reduced form

$$(*) \qquad g = f_{i'_m j_m}^{-1} \circ f_{i_m j_m} \circ f_{i'_{m-1} j_{m-1}}^{-1} \circ f_{i_{m-1} j_{m-1}} \circ \cdots \circ f_{i'_1 j_1}^{-1} \circ f_{i_1 j_1}$$

(the same as before) with $(i_1, j_1) = (\bar{\imath}, \bar{\jmath})$, while no element of S_{ij} or T_{ij} has this form. The reduced word for an element of T'_{ij} has f_{ij}^{-1} as its leftmost component, while the reduced word for an element of S'_{ij} does not. Also, if $g \in S'_{ij}$, then $g(x_0) \notin B_{ij}^n$; if $g \in T'_{ij}$, then $g(x_0) \notin A_{ij}^n$. (These facts are proved by induction on the inductive definition of S'_{ij} and T'_{ij}.) Hence, the proof that (1) holds for $\hat{A}_{ij}$ and $\hat{B}_{ij}$ goes through as before.

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In the proofs of (2) and (3), we have some extra cases to consider. For (2), if $x \in \hat{A}_{ij}$, then either $x \in A^n_{ij}$, $x = g(x_0)$ for some $g \in S_{ij}$, or $x = g(x_0)$ for some $g \in S'_{ij}$. In the first of these cases, we have $x \in B^n_{i'j'} \subseteq \hat{B}_{i'j'}$; in the second, (b) gives $x \in \hat{B}_{i'j'}$. Finally, if $x = g(x_0)$ where $g \in S'_{ij}$ is not the identity, then g was added to S'_{ij} by clause (c′), and the hypotheses of (c′) imply that $g(x_0) \in \hat{B}_{i'j'}$ for all $(i', j') \neq (i, j)$.

To verify (3), suppose $\{i, i', i''\} = \{1, 2, 3\}$; we must show that, if $x = f^{-1}_{ij}(z)$ is in $\hat{B}_{ij}$ and $x'' = f^{-1}_{i''j}(z)$ is in $\hat{B}_{i''j}$, then $x' = f^{-1}_{i'j}(z)$ is in $\hat{A}_{i'j}$. If $x = g(x_0)$ for some $g \in T'_{ij}$, then g has the reduced form $f^{-1}_{ij} \circ f_{Ij} \circ h$ for some word $h \in S'_{Ij}$ and some $I \neq i$. Hence, $h(x_0) \in \hat{A}_{Ij}$. If $I = i''$, then $h(x_0) = x'' \notin \hat{B}_{Ij}$ by (1), contradiction; if $I = i'$, then $h(x_0) = x' = f^{-1}_{i'j}(z) \in \hat{A}_{Ij}$, as desired. If $x'' = g(x_0)$ for some $g \in T'_{i''j}$, the same argument applies with x and x'' (and i and i'') interchanged. The remaining cases are the four from the proof of Lemma 3.2, which are handled exactly as before.

For (4), note that, if g is in S'_{ij} or T'_{ij}, then x_0 is connected to $g(x_0)$ by a path of links, all of which are active (for A^n_{IJ} and B^n_{IJ}) except possibly for the first; since there are only two possibilities for this first link, (4) also holds for $\hat{A}_{ij}$ and $\hat{B}_{ij}$.

Now suppose x satisfies the hypothesis of (5). For each (i', j') other than (i, j), x is in $\hat{B}_{i'j'}$ and hence is either in $B^n_{i'j'}$, of the form $g(x_0)$ for some $g \in T_{i'j'}$, or of the form $g'(x_0)$ for some $g' \in T'_{i'j'}$. If the second of these possibilities holds for some (i', j'), then the definition of $T_{i'j'}$ implies that $g \in S_{i''j''}$ for some (i'', j''); then (i'', j'') must be (i, j) (since we already verified (1)), so $x \in \hat{A}_{ij}$. So suppose the second possibility does not arise. If the third possibility holds for some (i', j'), then this (i', j') is unique (by freeness, we can determine g' from x, and then (i', j') can be determined from the reduced form of g'), so x must be in B^n_{IJ} for all other (I, J), and now (c′) with (i, j) and (i', j') interchanged gives $x \in \hat{A}_{ij}$. Finally, if $x \in B^n_{i'j'}$ for all $(i', j') \neq (i, j)$, then (5) for A^n_{IJ} and B^n_{IJ} gives $x \in A^n_{ij}$.

The verification of (6) proceeds by a similar consideration of three cases. This completes the proof that $\hat{A}_{ij}$ and $\hat{B}_{ij}$ satisfy (1)–(6).

We can now define S, S', and U_0 as before, and let

$$A^{n+1}_{ij} = A^n_{ij} \cup \bigcup \{g(U_0) \colon g \in S_{ij} \cup S'_{ij}\}$$

and

$$B^{n+1}_{ij} = B^n_{ij} \cup \bigcup \{g(U_0) \colon g \in T_{ij} \cup T'_{ij}\};$$

then properties (1)–(6) hold at stage $n+1$ by the old argument. This completes the proof of the lemma. □

Clearly the resulting sets A_{ij} have the desired properties, so the theorem is proved. □

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Equidecomposability with small open sets. Now suppose that, instead of finding disjoint open subsets of $\mathscr{X}$ which can be rearranged to cover $\mathscr{X}$ densely more than once, we are given an open subset O of $\mathscr{X}$ and want to find disjoint open subsets of O which can be rearranged to cover $\mathscr{X}$ densely (once or more than once). An obvious necessary condition is that finitely many images of O under elements of G must suffice to cover a dense subset of $\mathscr{X}$. If G contains a suitable free subgroup, then this necessary condition is also sufficient.

Theorem 4.3. *Assume the hypotheses of Theorem* 3.1. *Suppose that R is a finite subset of G such that the elements f_{ij} of G are independent over R (i.e., the elements f_{ij} freely generate a group G' such that $\langle R, G'\rangle$ is the free product of $\langle R\rangle$ and G'). Then there are sets A_{ij} satisfying the conclusions of Theorem* 3.1 *such that, for $r, s \in R$, $i, i' \leq 3$, and $j, j' \leq N$, $r^{-1}(A_{ij})$ and $s^{-1}(A_{i'j'})$ are disjoint unless $(r, i, j) = (s, i', j')$.*

Proof. We may assume $R \neq \varnothing$. We proceed as in the proof of Theorem 3.1, constructing sets A^n_{ij} and B^n_{ij} satisfying four inductive hypotheses. Properties (1) and (3) are unchanged. Property (2) is modified to state:

(2) If $x \in A^n_{ij}$, $r, s \in R$, and $(r, i, j) \neq (s, i', j')$, then $s(r^{-1}(x)) \in B^n_{i'j'}$.

The modification of property (4) must be done carefully; we have to consider connections by active links and by combinations $s \circ r^{-1}$ for $r, s \in R$, but we cannot allow arbitrary combinations of the latter if we want (4) to hold at stage 0. So we use the following definition:

Definition. The point x is *actively R-connected* (for the sets A^n_{IJ} and B^n_{IJ}) to y if there are sequences $x_1, x_2, \ldots, x_m$ and $y_1, y_2, \ldots, y_m$ such that $x_1 = x$, $y_m = y$, $y_k = s_k(r_k^{-1}(x_k))$ for some $r_k, s_k \in R$, y_k is linked to x_{k+1} via some i_k, i'_k, j_k and this link is active for A^n_{IJ} and B^n_{IJ}, and for no k do we have $(r_{k+1}, i'_k, j_k) = (s_{k+1}, i_{k+1}, j_{k+1})$.

This relation is easily seen to be symmetric, but it is not necessarily transitive. The revised (4) is:

(4) Any point is actively R-connected to only finitely many other points.

This clearly holds at stage 0, since R is finite.

As before, we define sets S_{ij} and T_{ij} by three inductive clauses. Clauses (a) and (c) are not modified; clause (b) is now:

(b) If $g \in S_{ij}$, then $s \circ r^{-1} \circ g \in T_{i'j'}$ for $(s, i', j') \neq (r, i, j)$.

Now all elements of S_{ij} have the (not necessarily reduced) form

$$(**) \qquad g = f^{-1}_{i'_m j_m} \circ f_{i_m j_m} \circ s_m \circ r_m^{-1} \circ \cdots \circ f^{-1}_{i'_1 j_1} \circ f_{i_1 j_1} \circ s_1 \circ r_1^{-1}$$

for some $r_k, s_k \in R$, $i_k, i'_k \leq 3$ (with $i_k \neq i'_k$), and $j_k \leq N$, while elements of T_{ij} have the form $s_{m+1} \circ r^{-1}_{m+1} \circ g$ with g as in $(**)$. The excluded case in (b) ensures that $(s_k, i_k, j_k) \neq (r_k, i'_{k-1}, j_{k-1})$. Therefore, the assumed properties of R and f_{ij} imply as before that properties (1)–(3) hold for the sets $\hat{A}_{ij}$ and

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$\hat{B}_{ij}$. For (4), note that, if x is actively R-connected to y for the sets $\hat{A}_{IJ}$ and $\hat{B}_{IJ}$, then either no newly activated link was used in the path from x to y, in which case x was actively R-connected to y for A^n_{IJ} and B^n_{IJ}, or some newly activated link was used, in which case z is actively R-connected to y for A^n_{IJ} and B^n_{IJ}, where z is one of the points involved in a newly activated link (namely the last such point in the path from x to y). Since there are only finitely many such z's, and (4) for A^n_{IJ} and B^n_{IJ} can be applied to each such z and to x, we see that only finitely many y's can be actively R-connected to x for $\hat{A}_{IJ}$ and $\hat{B}_{IJ}$, so (4) holds for these sets.

Now define S to be the set of $g \in G$ such that x_0 and $g(x_0)$ are actively R-connected (for the sets $\hat{A}_{ij}$ and $\hat{B}_{ij}$), and let S' be the set of elements of G which are of either of the forms $s \circ r^{-1} \circ g$ or $f_{i'j}^{-1} \circ f_{ij} \circ g$ for some $g \in S$, along with all $g' \in G$ such that, for some $g \in S$, $g(x_0)$ and $g'(x_0)$ are actively R-connected for the sets $\hat{A}_{ij}$ and $\hat{B}_{ij}$. Then define U_0, A^{n+1}_{ij}, and B^{n+1}_{ij} as before. The rest of the proof goes as before, except for property (4). Here we consider a given point $w \in \mathscr{X}$, and note that the set W of all points actively R-connected to w at stage n is finite by the induction hypothesis. For any x which is actively R-connected to w at stage $n+1$ but not at stage n, look at the first link in the path from w to x which is active at stage $n+1$ but not at stage n, and let y be the point which activates this link; then y is either in W or linked to an element of W, $y \in g(U_0)$ for some $g \in S$, and y is actively R-connected to x at stage $n+1$. But, for any active R-connection starting at y, there is a corresponding active R-connection starting at $g(x_0)$ (for the sets $\hat{A}_{ij}$ and $\hat{B}_{ij}$), so the set of points actively R-connected to y is finite; since W is finite, there are only finitely many possible points y, so the set of points actively R-connected to w at stage $n+1$ is finite. □

Corollary 4.4. *Assume the hypotheses of Theorem* 4.3. *If O is an open subset of $\mathscr{X}$ such that $\bigcup_{r\in R} r(O)$ is dense in $\mathscr{X}$, then there are disjoint open subsets C_{ijr} of O and functions $h_{ijr} \in G$ $(i \le 3,\ j \le N,\ r \in R)$ such that, for each j, $\bigcup_{i\le 3,\, r\in R} h_{ijr}(C_{ijr})$ is dense in $\mathscr{X}$.*

Proof. Construct A_{ij} as in Theorem 4.3. The sets $A_{ij} \cap \bigcup_{r\in R} r(O)$ also satisfy the conclusion of this theorem, since the set of points being removed from A_{ij} is nowhere dense and this property is preserved by the homeomorphisms f_{ij}. Therefore, the sets $C_{ijr} = r^{-1}(A_{ij} \cap r(O))$ and the functions $h_{ijr} = f_{ij} \circ r$ have the desired properties. □

Incomplete spaces and locally commutative actions. We will now see that the hypotheses of Theorem 3.1 can be weakened in two ways: the space $\mathscr{X}$ need not be complete, and the action of G need only be locally commutative, rather than free, on a dense subset of $\mathscr{X}$. (An action of G is *locally commutative* if any two elements of G with a common fixed point commute.) Most of the results in this subsection were suggested to us by J. Mycielski.

First, consider a countable group G with a free subgroup on $3N$ generators such that G acts freely by homeomorphisms on a *countable* metric space (e.g.,

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G itself with a suitable metric). The proof of Theorem 3.1 goes through in this situation, except for one point: we cannot use the Baire category theorem to choose a point z_0 avoiding all images of boundaries of the sets A_{ij}^n and B_{ij}^n. Instead, we make sure that these sets have no boundary (i.e., they are clopen). To do this, we merely have to make sure that the neighborhood U_0 of x_0 chosen in the last part of the proof is clopen; this is easy in a countable space (let U_0 be a ball with center x_0 whose radius is sufficiently small and is not one of the countably many distances between pairs of points in the space).

Once one has Theorem 3.1 for countable metric spaces, one can prove it for an arbitrary separable metric space $\mathscr{X}$ on which a countable group G (with a subgroup free on $3N$ generators) acts by homeomorphisms, as long as there is a dense G-invariant subset of $\mathscr{X}$ on which G acts freely. (In a Polish space, this is equivalent to acting freely on a comeager set, since the set of fixed points of a homeomorphism is closed.) This is a consequence of the following proposition:

Proposition 4.5. *Suppose $\mathscr{Y}$ is a topological space, G is a group of homeomorphisms of $\mathscr{Y}$, $N \geq 1$, elements f_{ij} $(i \leq 3,\ j \leq N)$ of G are given, and $\mathscr{X}$ is a dense G-invariant subspace of $\mathscr{Y}$. Then the conclusion of Theorem* 3.1 *holds for $\mathscr{X}$ if and only if it holds for $\mathscr{Y}$.*

Proof. First, suppose we have open subsets A_{ij} of $\mathscr{X}$ as in the conclusion of Theorem 3.1. In $\mathscr{Y}$, let B_{ij} be the interior of the closure of A_{ij}. Then $A_{ij} \subseteq B_{ij}$, because if $U \subseteq \mathscr{Y}$ is an open set such that $U \cap \mathscr{X} = A_{ij}$, then A_{ij} is dense in U, so $U \subseteq B_{ij}$. Since the set $\bigcup_{i=1}^3 f_{ij}(A_{ij})$ is dense in $\mathscr{X}$ for each j and $\mathscr{X}$ is dense in $\mathscr{Y}$, $\bigcup_{i=1}^3 f_{ij}(B_{ij})$ is dense in $\mathscr{Y}$ for each j. Also, if $x \in B_{ij}$, then there is a neighborhood of x in which A_{ij} is dense; this cannot be true simultaneously for (i, j) and for (i', j'), because, if it were, there would be a nonempty open set U in which both A_{ij} and $A_{i'j'}$ were dense, and then $A_{ij} \cap U$ and $A_{i'j'} \cap U$ would be dense open subsets of $\mathscr{X} \cap U$, which is impossible because they are disjoint. Therefore, the open sets B_{ij} are disjoint, as required.

On the other hand, suppose we have the required subsets B_{ij} of $\mathscr{Y}$, and let $A_{ij} = B_{ij} \cap \mathscr{X}$. Since the sets B_{ij} are disjoint, the sets A_{ij} are disjoint. Also, since $\mathscr{X}$ is dense in $\mathscr{Y}$, any dense open subset of $\mathscr{Y}$ has an intersection with $\mathscr{X}$ which is dense in $\mathscr{X}$. In particular, the sets $\bigcup_{i=1}^3 f_{ij}(A_{ij})$ are dense in $\mathscr{X}$, as desired. □

Now, if G acts freely on a dense subset of a given separable metric space $\mathscr{X}$, then one can easily find a countable dense subset of $\mathscr{X}$ on which G acts freely, apply the version of Theorem 3.1 for countable spaces to this subspace, and use Proposition 4.5 to get the desired result for $\mathscr{X}$. Alternatively, instead of working "from below" like this, one can work "from above" by embedding $\mathscr{X}$ in a complete space $\mathscr{Y}$ on which G still acts by homeomorphisms, and then applying (the original form of) Theorem 3.1 and Proposition 4.5. This cannot always be done using the ordinary metric completion of $\mathscr{X}$, but the following proposition shows that a suitable space $\mathscr{Y}$ can be found (although the embedding of $\mathscr{X}$ into $\mathscr{Y}$ may be only a homeomorphism rather than an isometry).

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Proposition 4.6. *Suppose $\mathscr{X}$ is a topological space with a separable metric, and G is a countable group acting on $\mathscr{X}$ by homeomorphisms. Then there is a topological space $\mathscr{Y}$ such that $\mathscr{X}$ is a dense subspace of $\mathscr{Y}$, $\mathscr{Y}$ has a complete metric, and the action of G on $\mathscr{X}$ extends to an action of G on $\mathscr{Y}$ by homeomorphisms.*

Proof. Fix a separable metric on $\mathscr{X}$, and let $\mathscr{Y}_0$ be the completion of $\mathscr{X}$ under this metric; then $\mathscr{X}$ is a dense subspace of the Polish space $\mathscr{Y}_0$. By Lavrentiev's theorem [2, §35 II], any homeomorphism g from $\mathscr{X}$ to $\mathscr{X}$ can be extended to a homeomorphism $\hat{g}$ from A_g to B_g, where A_g and B_g are G_δ subsets of $\mathscr{Y}_0$ including $\mathscr{X}$. Note that $\hat{g} \circ \hat{h}$ and $\widehat{g \circ h}$ must agree wherever they are both defined, because they are continuous and agree on the dense subset $\mathscr{X}$. Similarly, $\widehat{g^{-1}}$ agrees with $\hat{g}^{-1}$ on $A_{g^{-1}} \cap B_g$; furthermore, if e is the identity of G, then $\hat{e}$ is the identity function on $A_e = B_e$. Let D_0 be the intersection of the sets A_g and B_g for $g \in G$. Given D_n, let D_{n+1} be the intersection of $\hat{g}(D_n) \cap \hat{g}^{-1}(D_n)$ for $g \in G$ (so $D_{n+1} \subseteq D_n$); let $\mathscr{Y} = \bigcap_{n=0}^{\infty} D_n$. Then the sets D_n and $\mathscr{Y}$ are all G_δ sets including $\mathscr{X}$, and the functions $\hat{g} \restriction \mathscr{Y}$ for $g \in G$ are homeomorphisms from $\mathscr{Y}$ to $\mathscr{Y}$. (For example, to see that $\mathscr{Y}$ is closed under $\hat{g}$ for $g \in G$, note that, if $x \in D_{n+1}$, then $\hat{g}(x) \in D_n$, since $D_{n+1} \subseteq \hat{g}^{-1}(D_n)$.) Since $\mathscr{Y}$ is G_δ, it is itself a Polish space (Alexandrov's theorem [2, §33 VI]) extending $\mathscr{X}$ on which G acts by homeomorphisms. □

Until now, we have assumed that the group G acts freely, at least on a dense subset of the given space. R. Robinson and T. Dekker showed that, for the Banach-Tarski results, the assumption that G acts freely could be weakened so as to require only local commutativity. We will now see that the results for free actions by homeomorphisms can be combined with those for locally commutative actions without a topology to get a version of Theorem 3.1 for locally commutative actions on separable metric spaces.

Theorem 4.7. *Suppose $\mathscr{X}$ is a separable metric space, and G is a countable group of homeomorphisms of $\mathscr{X}$ which acts locally commutatively on a dense (G-invariant) subset of $\mathscr{X}$. Let $N \geq 1$ be given, and suppose that G has a free subgroup of rank $3N$ with free generators f_{ij} $(i = 1, 2, 3;\ 1 \leq j \leq N)$. Then there are disjoint open subsets A_{ij} of $\mathscr{X}$ such that, for each $j \leq N$, $f_{1j}(A_{1j}) \cup f_{2j}(A_{2j}) \cup f_{3j}(A_{3j})$ is dense in $\mathscr{X}$.*

Proof. By Proposition 4.5, if the conclusion holds on the dense subspace on which G acts locally commutatively, then it holds on $\mathscr{X}$; therefore, we may assume that G acts locally commutatively on all of $\mathscr{X}$. We may also assume that G is the free group generated by the elements f_{ij}.

For each $g \in G$, let U_g be the interior of the set of fixed points of g. Let $\mathscr{X}' = \bigcup_{g \in G \setminus \{e\}} U_g$ (where e is the identity in G) and $\mathscr{X}'' = \mathscr{X} \setminus \overline{\mathscr{X}'}$. Then $\mathscr{X}'$ and $\mathscr{X}''$ are disjoint G-invariant open subsets of $\mathscr{X}$ (since $h(U_g) \subseteq U_{h \circ g \circ h^{-1}}$), and $\mathscr{X}' \cup \mathscr{X}''$ is dense in $\mathscr{X}$. Therefore, if we find disjoint open sets $A'_{ij} \subseteq \mathscr{X}'$

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and $A''_{ij} \subseteq \mathscr{X}''$ such that $f_{1j}(A'_{1j}) \cup f_{2j}(A'_{2j}) \cup f_{3j}(A'_{3j})$ is dense in $\mathscr{X}'$ for each j and similarly for A''_{ij}, then the sets $A_{ij} = A'_{ij} \cup A''_{ij}$ will have the desired properties.

First, consider $\mathscr{X}''$. The definition of $\mathscr{X}''$ ensures that, for each nonidentity $g \in G$, the set of fixed points of G is nowhere dense in $\mathscr{X}''$ (since this set is closed). Apply Proposition 4.6 to get a Polish space $\mathscr{Y}''$ with $\mathscr{X}''$ as a dense subspace such that the action of G on $\mathscr{X}''$ can be extended to an action of G on $\mathscr{Y}''$ by homeomorphisms. Since the set of fixed points of a nonidentity element of G is closed in $\mathscr{Y}''$ and does not include a nonempty open subset of $\mathscr{X}''$, it must be nowhere dense in $\mathscr{Y}''$. Therefore, G acts freely on a comeager subset of $\mathscr{Y}''$, so Theorem 3.1 and Proposition 4.5 give the desired open subsets of $\mathscr{X}''$.

It remains to handle $\mathscr{X}'$. We will use the standard fact that any subgroup of a free group is free [3, Corollary 2.9]. In particular, if two elements g and h of G commute, then the subgroup of G generated by g and h is both Abelian and free, so it must be cyclic, so g and h are powers of a common element of G, and there exist integers m, n (not both 0) such that $g^m = h^n$. We will also use the fact that, if g and h generate a free non-Abelian group, then they are free generators for this group [3, Corollary 2.13.1]. This applies to any two noncommuting elements of a free group (the subgroup they generate is free and non-Abelian), so, if g and h are noncommuting elements of G, then g^m and h^n do not commute for nonzero m, n, so in particular $g^m \neq h^n$.

For $g \in G$, let $S_g = \bigcup_{m=1}^{\infty} U_{g^m}$. Clearly $U_g \subseteq U_{g^k}$ and in fact $U_{g^m} \subseteq U_{g^{mk}}$ for all m and k, so $S_g = S_{g^k}$ for $k > 0$. The same applies for $k < 0$, since $U_g = U_{g^{-1}}$. Therefore, if g and g' are commuting nonidentity elements of G, then they are common powers of some $h \in G$, so $S_g = S_h = S_{g'}$. On the other hand, if g and h do not commute, then g^m and h^n do not commute for $m, n > 0$, so $U_{g^m} \cap U_{h^n} = \varnothing$ since G acts locally commutatively on $\mathscr{X}$, so $S_g \cap S_h = \varnothing$. Therefore, the sets S_g for $g \in G \setminus \{e\}$ form a partition of $\mathscr{X}'$ into open subsets. This partition is clearly G-invariant, since if x and y are in S_g, then $h(x)$ and $h(y)$ are in $S_{h \circ g \circ h^{-1}}$. Therefore, if we let $Y' = \{S_g : g \in G \setminus \{e\}\}$, then the action of G on $\mathscr{X}'$ induces an action of G on Y'.

We will now show that the action of G on Y' is locally commutative. It will suffice to show that, if $h(S_g) = S_g$, then $S_h = S_g$, because we saw above that $S_h = S_{h'}$ if and only if h and h' commute. Fix an element x of S_g; then $h(x)$ is also in S_g, so we can find $m, n > 0$ such that g^m fixes x and g^n fixes $h(x)$. Then g^m and $h^{-1} \circ g^n \circ h$ both fix the point x, so they must commute because g acts locally commutatively on $\mathscr{X}$. Hence, g and h cannot be free generators for a subgroup of G, so g and h must commute in G, so $S_h = S_g$, as desired.

Now we can apply the results of R. Robinson and T. Dekker [9, pp. 45–46] to partition Y' into $2N$ sets B_{ij} ($i \leq 2$, $j \leq N$) such that

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$(f_{2j}^{-1} \circ f_{1j})(B_{1j}) = Y' \setminus B_{2j}$ for each j. Let $B_{3j} = \varnothing$ for all j, and then let $A'_{ij} = \bigcup\{S_g : S_g \in B_{ij}\}$; clearly the sets A'_{ij} are disjoint open subsets of $\mathscr{X}'$ such that $f_{1j}(A'_{1j}) \cup f_{2j}(A'_{2j}) \cup f_{3j}(A'_{3j}) = \mathscr{X}'$ for each j, so we are done. □

Congruent open sets. The construction in Theorem 4.1 gives a decomposition of a dense open subset of $\mathscr{X}$ into $3N$ different open subsets of $\mathscr{X}$ which can be rearranged to form N dense subsets of $\mathscr{X}$. It turns out that one can in fact construct these pieces so as to be identical (up to the action of G); this follows from the following result:

Theorem 4.8. *Suppose $\mathscr{X}$ is a Polish space and G is a countable group of homeomorphisms of $\mathscr{X}$ which acts freely on a comeager subset of $\mathscr{X}$. Then, for any $N \geq 3$, if elements f_i $(1 \leq i \leq N)$ of G are free generators for a free subgroup of G, then there is an open subset A of $\mathscr{X}$ such that the sets $f_i(A)$ are disjoint and their union is dense in $\mathscr{X}$. In fact, if $f_{ij} \in G$ for $1 \leq i \leq j$ and $3 \leq j \leq N$ are free generators for a free subgroup of G, then there is an open set A such that, for each j, the sets $f_{ij}(A)$ for $i \leq j$ are disjoint and have dense union.*

Proof. We will just give the proof of the first part; the proof of the second part is the same except for extra bookkeeping (to keep track of the disjointness and density requirements for each j). Again we define A to be the union of an increasing sequence $A^0 \subseteq A^1 \subseteq A^2 \subseteq \cdots$; we also define open sets $B^0 \subseteq B^1 \subseteq B^2 \subseteq \cdots$ to be excluded from A. The induction hypotheses to be maintained are:

(1) Exclusion: $A^n \cap B^n = \varnothing$.
(2) Disjointness after moves: If $z \in f_i(A^n)$ and $i' \neq i$, then $z \in f_{i'}(B^n)$.
(3) Possible density after moves: If $i \leq N$, and $z \in f_{i'}(B^n)$ for all $i' \neq i$, then $z \in f_i(A^n)$.
(4) Finite active linkage: For any $x \in \mathscr{X}$, the set of $y \in \mathscr{X}$ which are connected to x by a path of active links is finite.

The definition of "links" is as before except for differences in indexing, but the definition of "active links" must be modified:

Definition. The points x and x' are *linked*, or there is a *link* from x to x', if there are $i, i' \leq N$ with $i \neq i'$ such that $x' = f_{i'}^{-1}(f_i(x))$. A link from x to x' is *active* (for A^n and B^n) if any point $f_{I'}^{-1}(f_I(x))$ or $f_{I'}^{-1}(f_I(x'))$ $(I, I' \leq N)$ is in A^n or B^n.

It is still true that adding one new point to A^{n+1} or B^{n+1} activates only finitely many links.

We must show how to get from A^n and B^n to A^{n+1} and B^{n+1} so that, for a given nonempty open set $Z = Z_n$, one of the sets $f_i(A^{n+1})$ will meet Z.

As usual, we find a point $z_0 \in Z$ outside all of the (countably many) relevant meager sets, choose $\bar{\imath} \leq N$ such that $z_0 \notin f_{\bar{\imath}}(B^n)$, and let $x_0 = f_{\bar{\imath}}^{-1}(z_0)$. Then define sets $S^n, T^n \subseteq G$ by:

(a) The identity element of G is in S^n.

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(b) If $g \in S^n$, then $f_{i'}^{-1} \circ f_i \circ g \in T^n$ for all i and i' such that $i \neq i'$ and $f_{i'}^{-1}(f_i(g(x_0))) \notin B^n$.

(c) If $g \in T^n$ and $i \neq i'$, and if $f_{i''}^{-1}(f_i(g(x_0))) \in B^n$ for all i'' other than i or i', then $f_{i'}^{-1} \circ f_i \circ g \in S^n$.

Now let $\hat{A} = A^n \cup \{g(x_0) \colon g \in S^n\}$ and $\hat{B} = B^n \cup \{g(x_0) \colon g \in T^n\}$.

First, induction on clauses (a)–(c) shows that: if $g \in S^n$, then $g(x_0) \notin B^n$; if $g \in T^n$, then $g(x_0) \notin A^n$. Using this, we now show

Claim. Any nonidentity h in S^n, and any h in T^n, has the form

$$(\dagger) \qquad f_{i'_m}^{-1} \circ f_{i_m} \circ f_{i'_{m-1}}^{-1} \circ f_{i_{m-1}} \circ \cdots \circ f_{i'_1}^{-1} \circ f_{i_1}$$

where $i_k \neq i'_k \neq i_{k+1}$ for all k; here m is even for $h \in S^n$ and odd for $h \in T^n$. Furthermore, h was added to S^n or T^n by applying clause (c) or clause (b) to $g = f_{i'_{m-1}}^{-1} \circ f_{i_{m-1}} \circ \cdots \circ f_{i'_1}^{-1} \circ f_{i_1}$, with $i = i_m$ and $i' = i'_m$.

Proof. Again we induct on clauses (a)–(c). The basis case (a) is vacuous. Suppose h was added to T^n by an application of (b), so $h = f_{I'}^{-1} \circ f_I \circ h'$ for some $h' \in S^n$ and $I' \neq I$ such that $f_{I'}^{-1}(f_I(h'(x_0))) = h(x_0)$ is not in B^n. If h' is the identity, we can just let $i_1 = I$ and $i'_1 = I'$ and be done; so assume h' is not the identity. By the induction hypothesis, h' has the form ($\dagger$) with m even. Let $i_{m+1} = I$ and $i'_{m+1} = I'$; then we will have the desired form for h if we show that $i_{m+1} \neq i'_m$. Let $g = f_{i'_{m-1}}^{-1} \circ f_{i_{m-1}} \circ \cdots \circ f_{i'_1}^{-1} \circ f_{i_1}$; then $g \in T^n$ and h' was added to S^n by applying (c) to g. We may assume $(i_{m+1}, i'_{m+1}) \neq (i'_m, i_m)$, since otherwise $h = g$, so h was already in S^n and has the desired form by the induction hypothesis. If $i_{m+1} = i'_m$ but $i'_{m+1} \neq i_m$, then the fact that (c) could be applied to g implies that $f_{i'_{m+1}}^{-1}(f_{i_m}(g(x_0))) \in B^n$, which is impossible because $f_{i'_{m+1}}^{-1}(f_{i_m}(g(x_0))) = h(x_0)$, which, as we saw earlier, is not in B^n. Therefore, we must have $i_{m+1} \neq i'_m$, so h has the desired form.

Finally, suppose h was added to S^n by an application of (c). Again we have $h = f_{I'}^{-1} \circ f_I \circ h'$, where now $h' \in T^n$, $I' \neq I$, and $f_{I''}^{-1}(f_I(h'(x_0))) \in B^n$ for all I'' other than I or I'. The induction hypothesis lets us express h' in the form ($\dagger$) with m odd. Let $i_{m+1} = I$ and $i'_{m+1} = I'$, and let $g = f_{i'_{m-1}}^{-1} \circ f_{i_{m-1}} \circ \cdots \circ f_{i'_1}^{-1} \circ f_{i_1} \in S^n$. Again it will suffice to show $i_{m+1} \neq i'_m$, and again we may assume $(i_{m+1}, i'_{m+1}) \neq (i'_m, i_m)$, since otherwise h would be equal to g and the induction hypothesis would give the desired results. We know that h' was added to T^n by an application of (b) to g. Now, if $i_{m+1} = i'_m$ but $i'_{m+1} \neq i_m$, then the applicability of (c) to h' implies that $f_{i'_m}^{-1}(f_{i_{m+1}}(h'(x_0))) \in B^n$; but $f_{i'_m}^{-1} \circ f_{i_{m+1}} \circ h' = g$, so we have $g \in S^n$ and $g(x_0) \in B^n$, contradicting the fact immediately preceding the claim. Therefore, we must have $i_{m+1} \neq i'_m$, so we are done.

The proof of (1)–(4) for the sets $\hat{A}$ and $\hat{B}$ goes pretty much as before,

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except for some changes for (3). (The extra restriction at the end of (b) causes no problems with (2); when (b) is being applied to verify that a point is in $f_{i'}(\hat{B})$, the restriction does not apply unless the point is in $f_{i'}(B^n) \subseteq f_{i'}(\hat{B})$ anyway.) For (3), all of the points $f_{i'}^{-1}(z)$ for $i' \neq i$ are in $\hat{B}$ and hence either in B^n or in $\{g(x_0)\colon g \in T^n\}$. If all of them are in B^n, we can apply (3) for A^n and B^n; if exactly one of them is of the form $g(x_0)$ for $g \in T^n$, then we can fix that particular i' and apply (c) with i and i' interchanged. Now suppose that there is more than one $i' \neq i$ such that $f_{i'}^{-1}(z)$ has the form $g(x_0)$ with $g \in T^n$. For each such i', we can express the corresponding g in the form (†). If the resulting i'_m is different from i', then $f_{i'} \circ g$ is a reduced word with leftmost component $f_{i'}$ which sends x_0 to z; this cannot happen for more than one i', since different reduced words send x_0 to different points. Therefore, there must be an i' such that $f_{i'}^{-1}(z) = g(x_0)$ where $g \in T^n$ has the form (†) with $i'_m = i'$. Hence, $h = f_{i_m}^{-1} \circ f_{i'_m} \circ g$ (which is (†) with the two leftmost functions removed) must be in S^n, so h is not in T^n and $h(x_0)$ is not in B^n. This gives $h(x_0) = f_{i_m}^{-1}(z) \notin \hat{B}$, so i_m must be equal to i; since $h \in S^n$, this gives $f_i^{-1}(z) \in \hat{A}$, as desired.

The rest of the proof (the definitions of S, S', A^{n+1}, and B^{n+1}) works out as before. □

This theorem implies that the decomposition in Theorem 4.1 can be accomplished with identical pieces: apply the second part of Theorem 4.8 with N replaced by $3N$, and note that any three of the pieces $f_{I,3N}(A)$ (say for $I = i, i', i''$) can be moved by maps $f_{13} \circ f_{i,3N}^{-1}$, $f_{23} \circ f_{i',3N}^{-1}$, and $f_{33} \circ f_{i'',3N}^{-1}$, respectively, to form a dense subset of $\mathscr{X}$.

Note that, in general, one cannot hope to get an open set A and functions f_{ij} for $i \leq j$ as above simultaneously for $2 \leq j \leq N$, or for $3 \leq j < \infty$. These can both be seen by looking at the sphere with the group of rotations. A dense open subset of the sphere can be covered with two disjoint congruent open sets (namely two hemispheres), but there is no open set A such that a dense open subset can be covered both by two copies of A and by three or more disjoint copies of A, as we will see in the next section (Proposition 5.6 and the paragraph that follows it). Also, the measure of the open set A is positive and preserved under rotations; if this measure is greater than $1/N$, then the sphere cannot include N disjoint copies of A.

Partial homeomorphisms. Our final modification of Theorem 3.1 will weaken the requirement that G be a group of homeomorphisms from $\mathscr{X}$ to $\mathscr{X}$. The proof works just as well (although the result is harder to state) for homeomorphisms from a space $\mathscr{X}$ to a space $\mathscr{Y}$, since all of the compositions involved in the proof are alternations of given functions and inverses of these functions. Also, the requirement that these functions map all of $\mathscr{X}$ to all of $\mathscr{Y}$ is not necessary.

If $\mathscr{X}$ and $\mathscr{Y}$ are Polish spaces, a *partial homeomorphism* from $\mathscr{X}$ to $\mathscr{Y}$ is a homeomorphism from some open subset of $\mathscr{X}$ to some open subset of

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$\mathscr{Y}$. A partial homeomorphism $f: \mathscr{X} \to \mathscr{Y}$ and a partial homeomorphism $g: \mathscr{Y} \to \mathscr{Z}$ can be composed to give a partial homeomorphism $g \circ f: \mathscr{X} \to \mathscr{Z}$ (but the domain of $g \circ f$ may be smaller than that of f; it might even be empty). Also, if f is a partial homeomorphism from $\mathscr{X}$ to $\mathscr{Y}$, then f^{-1} is a partial homeomorphism from $\mathscr{Y}$ to $\mathscr{X}$ (but composing f and f^{-1} gives only a partial identity function). Note that the image of a nowhere dense set under a partial homeomorphism is nowhere dense.

Given a set, whose elements will be referred to as "generators" in analogy with preceding arguments, an expression of the form

$$b_m^{-1} \circ a_m \circ b_{m-1}^{-1} \circ a_{m-1} \circ \cdots \circ b_1^{-1} \circ a_1,$$

where a_k and b_k are in the given set, is called an *alternating word* on these generators. If each generator is (or represents) a partial homeomorphism from $\mathscr{X}$ to $\mathscr{Y}$, then each alternating word represents a partial homeomorphism from $\mathscr{X}$ to $\mathscr{X}$. The alternating word of length 0 represents the identity function from $\mathscr{X}$ to $\mathscr{X}$; an alternating word is *nontrivial* if it has length greater than 0. An alternating word is *reduced* if it does not contain any occurrences of pairs $a^{-1} \circ a$ or $a \circ a^{-1}$ for any generator a.

In addition to replacing homeomorphisms with partial homeomorphisms, we will weaken the hypotheses of Theorem 3.1 in one more way. In the proof of Theorem 3.1, inductive hypothesis (4) held at stage 0 because there were no active links at all at stage 0, and this was because, for a fixed j, every element of the space was initially eligible for membership in at least three different sets $f_{ij}(A_{ij})$. We now want to consider situations where some points are eligible for membership in only two of the sets; in such cases, some links will be active at stage 0. These links form an initial active link graph $\mathscr{G}$ on $\mathscr{X}$ defined as follows: there is an edge from x to x' iff there is $y \in \mathscr{Y}$ which is in the ranges of exactly two of the given partial homeomorphisms and the preimages of y under these functions are x and x'. This is no problem as long as there are few enough such links for (4) to hold, at least on a comeager set. The resulting theorem can be stated as follows:

Theorem 4.9. *Suppose $\mathscr{X}$ and $\mathscr{Y}$ are Polish spaces, and P is a finite set of partial homeomorphisms from $\mathscr{X}$ to $\mathscr{Y}$ such that the following statements are true:*

(A) *For every nontrivial reduced alternating word on the set of generators P, the set of fixed points of the corresponding partial homeomorphism from $\mathscr{X}$ to $\mathscr{X}$ is nowhere dense in $\mathscr{X}$.*

(B) *The set of $y \in \mathscr{Y}$ which are in the ranges of at least two elements of P is dense.*

(C) *If $\mathscr{G}$ is the initial active link graph defined above, then there is a comeager subset D of $\mathscr{X}$ such that all connected components of $\mathscr{G} \restriction D$ are finite.*

Then there are disjoint open sets $A_p \subseteq \mathrm{domain}(p)$ for $p \in P$ such that the union of the images $p(A_p)$ is dense in $\mathscr{Y}$.

Now assume further that:

(D) *The set of $x \in \mathscr{X}$ which are in the domains of at least three of the functions in P is dense.*

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Then we can construct disjoint open sets $A_p \subseteq \mathrm{domain}(p)$ *with union dense in* $\mathscr{X}$ *such that the images* $p(A_p)$ *are disjoint and have union dense in* $\mathscr{Y}$.

Proof. We will prove the first part first; as usual, this will be a variant of the proof of Theorem 3.1. Let Q be the (countable) set of partial homeomorphisms from $\mathscr{X}$ to $\mathscr{X}$ coming from alternating words on P. There is a comeager subset D_0 of $\mathscr{X}$ such that: D_0 is included in some set D as in (C); if D' is the set defined in (B), then $p(D_0) \subseteq D'$ for all $p \in P$; D_0 is disjoint from the boundaries of the domains of the functions in Q and from the sets of fixed points of those elements of Q which come from nontrivial alternating reduced words on P; and, for any $q \in Q$, $q(D_0) \subseteq D_0$. (The complement of D, the inverse images of the complement of D' under all $p \in P$, the sets of fixed points of relevant elements of Q, and the boundaries of the domains of the elements of Q are all meager, so when put together they form a meager subset of $\mathscr{X}$; the union of this meager subset and all of its images under members of Q is also meager, and we can let D_0 be the complement of this union.)

We will construct open sets $A_p^0 \subseteq A_p^1 \subseteq \cdots$ and $B_p^0 \subseteq B_p^1 \subseteq \cdots$ for $p \in P$ so that the sets $A_p = \bigcup_{n=0}^{\infty} A_p^n$ have the desired properties (in particular, $A_p^n \subseteq \mathrm{domain}(p)$). The inductive hypotheses to be maintained are:

(1) Exclusion: $A_p^n \cap B_p^n = \varnothing$.
(2) Disjointness before moves: $A_p^n \subseteq B_{p'}^n$ for $p' \neq p$.
(3) Possible density after moves: If $p \in P$, $z \in p(D_0)$, and there is no $p' \neq p$ in P such that $z \in p'(\mathscr{X} \setminus B_{p'}^n)$, then $z \in p(A_p^n)$.
(4) Finite active linkage: For any $x \in D_0$, the set of points connected to x by a path of active links is finite.

Here we define "link" and "active link" as follows:

Definition. Points x and x' in $\mathscr{X}$ are *linked* if there are $p, p' \in P$ such that $p(x) = p'(x')$; the link is *active* if the point $p(x)$ is in at most two of the sets $p''(\mathscr{X} \setminus B_{p''}^n)$ for $p'' \in P$ (including p and p').

Note that if two points are linked and one of them is in D_0, then the other one is also in D_0.

The sets A_p^0 and B_p^0 are defined to be empty as usual. It is easy to verify that (1)–(4) hold for these sets; hypotheses (B) and (C) (as used in the definition of D_0) imply (3) and (4), respectively. Given open sets A_p^n and B_p^n satisfying (1)–(4), we must see how to enlarge them to open sets A_p^{n+1} and B_p^{n+1} satisfying (1)–(4) so that one of the images $p(A_p^{n+1})$ meets a given nonempty open set $Z \subseteq \mathscr{Y}$.

The union of the boundaries of the sets A_p^n and B_p^n for $p \in P$ is meager, so all of its images under functions $q \in Q$ are also meager; let E be the complement of the union of these images. Then E is comeager and $q(E) \subseteq E$ for $q \in Q$. Hypothesis (B) implies that at least one $p \in P$ has a range which meets Z; in fact, since D_0 and E are comeager, we can choose a point $z_0 \in Z$

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such that $z_0 \in p(D_0 \cap E)$ for some $p \in P$. Now (1) and (3) imply that, for some $\bar{p} \in P$, $\bar{p}^{-1}(z_0)$ exists and is not in $B^n_{\bar{p}}$. (If this is not true for $\bar{p} = p$, then $z_0 \notin p(A_p)$ by (1), so $z_0 \in \bar{p}(\mathscr{X} \setminus B^n_{\bar{p}})$ for some $\bar{p} \neq p$ by (3).) Fix this $\bar{p}$, and let $x_0 = \bar{p}^{-1}(z_0)$; since D_0 and E are closed under all $q \in Q$, $x_0 \in D_0 \cap E$.

Now construct sets S_p and T_p ($p \in P$) of alternating words on P according to the following inductive definition:

(a) The trivial word is in $S_{\bar{p}}$.

(b) If $g \in S_p$, then $g \in T_{p'}$ for $p' \neq p$.

(c) If p and p' are distinct members of P, $g \in T_p$, $p'^{-1}(p(g(x_0)))$ exists, and there is no $p'' \in P$ other than p or p' such that $p(g(x_0)) \in p''(\mathscr{X} \setminus B^n_{p''})$, then $p'^{-1} \circ p \circ g \in S_{p'}$.

Then every element of S_p (other than the trivial word in $S_{\bar{p}}$) is a reduced word starting with p^{-1}, while every nontrivial element of T_p is a reduced word starting with p'^{-1} for some $p' \neq p$. Hence, no word is in both S_p and T_p. Also, if $g \in S_p$, then $g(x_0)$ exists and is not in B^n_p, but is in the domain of p; if $g \in T_p$, then $g(x_0)$ exists and is not in A^n_p. (This is proved by induction on g as before; for the part involving (c), we must keep track of which $p'' \in P$ have the point $p(g(x_0))$ in their ranges, and the number of such p'' may vary, but this is no problem. Of course, since $x_0 \in D_0$, all of the points $g(x_0)$ are in D_0.)

If g and g' are distinct reduced alternating words on P, then, once the largest common initial segment of g and g' is cancelled, $g'^{-1} \circ g$ becomes a nontrivial reduced alternating word; since $x_0 \in D_0$, this word cannot send x_0 to itself. Therefore, g and g' cannot map x_0 to the same point.

Now, if we let $\hat{A}_p = A^n_p \cup \{g(x_0) \colon g \in S_p\}$ and $\hat{B}_p = B^n_p \cup \{g(x_0) \colon g \in T_p\}$, then the sets $\hat{A}_p$ and $\hat{B}_p$ will satisfy properties (1)–(4); the proof goes as before with minor differences. (For example, for (3), we are assuming that, for all p' other than p such that z is in the range of p', the point $p'^{-1}(z)$ is in $\hat{B}_{p'}$, and hence either (i) $p'^{-1}(z) \in B^n_{p'}$ or (ii) $p'^{-1}(z) \in \{g(x_0) \colon g \in T_{p'}\}$. If (i) holds for all such p', then we can apply (3) for the sets A^n_p and B^n_p; if (ii) holds for exactly one such p', then we can use (c); if (ii) holds for more than one p', we get a contradiction to the preceding paragraph.)

Next, define S to be the set of all alternating reduced words g such that $g(x_0)$ exists and the links in the path of links from x_0 to $g(x_0)$ (one can read off this path from g) are all active for the sets $\hat{A}_p$ and $\hat{B}_p$; since these sets satisfy (4), S is finite. Let S' be the set of all words obtained by taking a word of the form $p'^{-1} \circ p \circ g$ ($p, p' \in P$, $g \in S$) and performing any necessary cancellations to obtain a reduced word. Since $x_0 \in D_0 \cap E$, we can

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find a neighborhood U_0 of x_0 so small that: for any partial homeomorphism q corresponding to an element of S', if x_0 is not in the domain of q, then U_0 is disjoint from this domain, while if x_0 is in this domain, then U_0 is included in the domain and $q(U_0)$ is either included in or disjoint from each of the sets A_p^n and B_p^n and each of the domains of the elements of P; and, for any two such partial homeomorphisms q and q', $q(U_0)$ and $q'(U_0)$ are disjoint. Let $A_p^{n+1} = A_p^n \cup \bigcup\{g(U_0) \colon g \in S_p\}$ and $B_p^{n+1} = B_p^n \cup \bigcup\{g(U_0) \colon g \in T_p\}$.

One can now verify the following facts: if $x \notin g(U_0)$ for all $g \in S$, then $x \in A_p^{n+1}$ if and only if $x \in A_p^n$, and similarly for B_p^{n+1}; if $x \in g(U_0)$ for some $g \in S'$, then $x \in A_p^{n+1}$ if and only if $g(x_0) \in \hat{A}_p$, and similarly for B_p^{n+1}; if $x \in g(U_0)$ and $g \in S$, then $x \in \operatorname{domain}(p'^{-1} \circ p)$ if and only if $g(x_0) \in \operatorname{domain}(p'^{-1} \circ p)$ for any $p, p' \in P$. These facts allow us to prove (1)–(4) for A_p^{n+1} and B_p^{n+1} as before; one simply has to do a little extra work to keep track of the domains of the relevant partial homeomorphisms. In particular, if $g \in S_p$, then $g(x_0)$ is in the domain of p, so $g(U_0)$ is included in this domain; this shows that A_p^{n+1} is included in the domain of p, since A_p^n was.

This completes the proof of the first part of the theorem. For the second part, we will follow the modifications used to get the proof of Theorem 4.1 from that of Theorem 3.1, and make corresponding modifications in the above proof.

First, we add another property to the list of properties of D_0: D_0 is included in the set defined in hypothesis (D). (This set is open and dense, so we can still make D_0 comeager.) Also, we make a minor change in the definition of "active link":

Definition. Points x and x' in $\mathscr{X}$ are *linked* if there are $p, p' \in P$ such that $p(x) = p'(x')$; the link is *active* if the point $p(x)$ is in at most two of the sets $p''(\mathscr{X} \setminus B_{p''}^n)$ for $p'' \in P$ (including p and p'), or if there is $p'' \in P$ such that $p''^{-1}(p(x))$ exists and is in all but at most two of the sets $B_{\hat{p}}$, $\hat{p} \in P$.

The extra clause is needed so that the analogue of the following statement from the proof of Lemma 4.2 will hold here: if g is in S'_{ij} or T'_{ij}, then x_0 is connected to $g(x_0)$ by a path of links, all of which are active (for A_{IJ}^n and B_{IJ}^n) except possibly for the first. This held in the proof of 4.2 because we used a rather inclusive definition of "active link" to begin with; here, when some points are in more of the ranges of the functions $p \in P$ than others, we have to be more careful. (Actually, to get the above statement, it would suffice to state the new clause for p'' equal to p or p'; however, we want to preserve the property that, in a cluster of links formed by several points in $\mathscr{X}$ being mapped by various p's to the same point in $\mathscr{Y}$, the links are either all active or all inactive.)

We now add three new inductive hypotheses:

(5) Possible density before moves: If $p \in P$ and $x \in B_{p'}^n$ for all $p' \neq p$ in P, then $x \in A_p^n$.

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(6) Disjointness after moves: If p and p' are distinct members of P and $z \in p(A_p^n)$ is in the range of p', then $z \in p'(B_{p'}^n)$.

(7) Inclusion within domain: If $p \in P$ and $x \in D_0 \setminus B_p^n$, then x is in the domain of p.

Actually, (7) is not really an inductive hypothesis, because we will take care of it once and for all at stage 0. To do this, we define B_p^0 to be the interior of the complement of the domain of p; A_p^0 is defined to be empty as before. Now, if $x \in D_0$, then x is not on the boundary of the domain of p, since $p^{-1} \circ p \in Q$ has the same domain as p; hence, if x is not in the domain of p, then x is in the interior of the complement of this domain, so $x \in B_p^0 \subseteq B_p^n$. The parts of B_p^n outside the domain of p have little effect on properties (1)–(6); it is easy to prove that these properties hold for the sets A_p^0 and B_p^0. (For (5), if a point were in all but two of the sets B_p^0, some neighborhood of this point would be included in all but two of these sets, and this would contradict (D). For the same reason, the new part of the definition of "active link" is vacuous for the sets A_p^0 and B_p^0, so (C) implies (4) as before.)

To get from A_p^n and B_p^n to A_p^{n+1} and B_p^{n+1}, so as to meet a given nonempty open set $Z \subseteq \mathscr{Y}$, we choose z_0 and $\bar{p}$ as we did in the first part. The sets S_p and T_p are defined as above, but we also define sets S_p' and T_p' of alternating words by the following conditions:

(a′) The trivial word is in $S_{\bar{p}}'$.

(b′) If $g \in S_p'$ and $p'^{-1}(p(g(x_0)))$ exists, then $p'^{-1} \circ p \circ g \in T_{p'}'$.

(c′) If p and p' are distinct members of P, $g \in T_p'$, and $g(x_0) \in B_{\hat{p}}^n$ for all $\hat{p}$ other than p and p', then $g \in S_{p'}'$.

Then let $\hat{A}_p = A_p^n \cup \{g(x_0) \colon g \in S_p \cup S_p'\}$ and $\hat{B}_p = B_p^n \cup \{g(x_0) \colon g \in T_p \cup T_p'\}$. Every element of S_p' or T_p' (other than the trivial word in $S_{\bar{p}}'$) is an alternating reduced word with rightmost component $\bar{p}$, while no element of S_p or T_p has as its rightmost component $\bar{p}$. Also, if $g \in S_p'$, then $g(x_0) \notin B_p^n$ (and hence $g(x_0)$ is in the domain of p, by (7)); if $g \in T_p'$, then $g(x_0) \notin A_p^n$. Now the proof that $\hat{A}_p$ and $\hat{B}_p$ satisfy (1)–(6) (and (7), since $B_p^0 \subseteq \hat{B}_p$) is as it was for Theorem 4.1.

Now define U_0, S, and S' as in the first part, and let $A_p^{n+1} = A_p^n \cup \bigcup\{g(U_0) \colon g \in S_p \cup S_p'\}$ and $B_p^{n+1} = B_p^n \cup \bigcup\{g(U_0) \colon g \in T_p \cup T_p'\}$ We again get $A_p^{n+1} \subseteq \text{domain}(p)$, and the proof of properties (1)–(6) for these new sets goes through as before.

We must also consider requirements on the $\mathscr{X}$ side: for any nonempty open set Z in some countable base for $\mathscr{X}$, we have to be able to enlarge A_p^n and B_p^n to A_p^{n+1} and B_p^{n+1} so that at least one of the sets A_p^{n+1} meets Z. Here we just find $z_0 \in D_0 \cap E \cap Z$, choose $\bar{p} \in P$ so that $z_0 \notin B_{\bar{p}}^n$ (such a $\bar{p}$ exists by

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(5) and (1), and z_0 is in the domain of $\bar{p}$ by (7)), let $x_0 = z_0$, and proceed as above. This completes the proof of the second part of the theorem. □

One can also consider a situation where there are Polish spaces $\mathscr{Y}_1, \mathscr{Y}_2, \ldots, \mathscr{Y}_N$ to be covered, and for each $j \leq N$ there is a finite set P_j of partial homeomorphisms from $\mathscr{X}$ to $\mathscr{Y}_j$. (The spaces Y_j may or may not be the same space, but the sets P_j are required to be disjoint.) It is easy to write out the analogues of hypotheses (A)–(D) for this situation. (For (A), we just consider alternating reduced words of the form $b_m^{-1} \circ a_m \circ \cdots \circ b_1^{-1} \circ a_1$ where, for each k, a_k and b_k come from the same set P_j; for (D), we require that the set of $x \in \mathscr{X}$ which are in at least three of the domains of the partial homeomorphisms in $\bigcup_{j=1}^N P_j$ be dense.) Then the result is that, if the analogues of (A)–(C) hold, then there are disjoint open sets $A_p \subseteq \text{domain}(p)$ for $p \in \bigcup_{j=1}^N P_j$ such that, for each j, $\bigcup_{p \in P_j} p(A_p)$ is dense in $\mathscr{Y}_j$; if the analogue of (D) also holds, then the sets A_p can also be made to have union dense in $\mathscr{X}$ while, for each j, the images $p(A_p)$ for $p \in P_j$ are disjoint in $\mathscr{Y}_j$. We do not need to repeat the proof of Theorem 4.9 to get this result; simply apply Theorem 4.9 with $\mathscr{Y}$ being defined to be the sum (disjoint union) of the spaces Y_j, and with P defined to be all of the sets P_j together (with each map in P_j composed with the canonical embedding from $\mathscr{Y}_j$ to $\mathscr{Y}$). (For (A), we do not have to worry about the alternating reduced words not satisfying the condition that each pair a_k, b_k come from a single P_j, since any such word will be evaluated as the empty partial homeomorphism.)

If one really wanted to, one could weaken the conditions in Theorem 4.9 a little further (e.g., by changing (D) to allow more points to be in only two domains, and then giving a more complicated formulation of (C) so that inductive hypothesis (4) will hold at stage 0), but we see little point in doing so; we only went so far as to state the theorem as above because it is needed for a result in the next section.

5. Equidecomposability in S^2, congruent sets, and minimal decompositions

The variants of Theorem 3.1 can be combined with corresponding results about decompositions using arbitrary pieces to get more results about decompositions using pieces with the property of Baire. In this section, we give several examples of this: analogues of Corollaries 2.7 and 2.8 for the sphere S^2; decompositions of S^n or related spaces into congruent pieces with the property of Baire; and Baire paradoxical decompositions with as few pieces as possible.

Equidecomposability of subsets of S^2 or H^2. In order to show that bounded subsets of $\mathbf{R}^{n+1}$ $(n \geq 2)$ or S^n $(n \geq 3)$ with the property of Baire and nonempty interior are Baire equidecomposable, we used the facts that any bounded set can be covered by finitely many translates of a given ball (symmetrical neighborhood of a point) and that any ball is Baire paradoxical; the latter fact was true because the ball had a sufficiently free group of isometries.

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This does not work within the sphere S^2, because a ball in S^2 does not have enough isometries; however, Corollary 4.4 provides a suitable substitute.

Theorem 5.1. *If A and B are nonempty open subsets of S^2, then there are finitely many disjoint open sets $A_i \subseteq A$ $(1 \leq i \leq m)$ whose union is a dense open subset of A, and rotations σ_i of S^2 such that the images $\sigma_i(A_i)$ are disjoint and their union is a dense open subset of B.*

Proof. Let f_1, f_2, f_3, g_1, g_2 be free generators for a free group of rotations of S^2. Clearly the rotations g_1 and g_2 are of infinite order, and they are rotations around different axes l_1 and l_2; it is easy to see that any nonempty closed subset of S^2 which is invariant under g_1 and g_2 must be invariant under any rotation around axis l_1 or around l_2, and hence must be the entire sphere. Therefore, if x is any point in A, then the orbit of x under the subgroup $H = \langle g_1, g_2 \rangle$ is dense in S^2, which means that the images $h(A)$ for $h \in H$ cover S^2. Since S^2 is compact, there is a finite set $R \subseteq H$ such that the images $r(A)$ for $r \in R$ cover S^2. The hypotheses of Corollary 4.4 have now been met (with $N = 1$), so, by that corollary, there exist disjoint open sets C_i and rotations h_i ($i \leq m = 3|R|$) such that the images $D_i = h_i(C_i)$ have union dense in S^2. Let $B_i = (D_i \cap B) \setminus \bigcup_{i' < i} \overline{B_{i'}}$; then the open sets B_i are disjoint with union a dense subset of B, and they are the images of the disjoint subsets $A_i = h_i^{-1}(B_i)$ of A under the rotations h_i.

The same argument shows that there are finitely many disjoint open subsets of B which can be moved by rotations to form a dense open subset of A. The desired result now follows from Proposition 2.9. □

As usual, this must be combined with a result about decompositions into arbitrary sets: Any two subsets of S^2 with nonempty interior are equidecomposable. For the proof of this fact (which is more involved than that of the Banach-Tarski theorem), we refer to Wagon [9, p. 113]. Now Lemma 2.4(a) and Proposition 2.2 give

Corollary 5.2. *Any two subsets of S^2 with the property of Baire and nonempty interior are Baire equidecomposable using rotations.* □

In almost the same way, we can give another proof of the corresponding results for S^n ($n \geq 3$). The only difference is in the construction of the set R, because it is no longer true that, under any two free rotations, all orbits must be dense. Instead, we find a finite set $\{\rho_1, \ldots, \rho_m\}$ of rotations such that the sets $\rho_i(A)$ cover S^n, without worrying about freeness. Now, using the general fact that the set of sequences $g_1, \ldots, g_m, f_1, f_2, f_3$ of rotations which are free generators for a free subgroup of the rotation group SO_{n+1} is dense (in fact, comeager) in SO_{n+1}^{m+3} [9, p. 83], we can find such a sequence where the rotations g_i are so close to the rotations ρ_i that the images $g_i(A)$ for $i \leq m$ also cover S^n. Now, if $R = \{g_1, \ldots, g_m\}$, then the rotations f_i generate a free subgroup which is independent of R, so Corollary 4.4 can be applied as before.

Similarly, we can prove that any two bounded subsets of the hyperbolic plane H^2 with the property of Baire and nonempty interior are Baire equidecompos-

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able. Of course, we need the corresponding result for general equidecomposability, that any two bounded subsets of H^2 with nonempty interior are equidecomposable; this is a result of Mycielski [5]. To get a version of this for open sets, first note that the methods from chapter 6 of Wagon [9] can be used to prove the following: the set of infinite sequences $f_1, f_2, \ldots$ of isometries of H^2 such that the isometries f_i are independent (i.e., they are free generators for a free group of isometries) is comeager (in the infinite-dimensional product of the group of isometries of H^2, topologized in the standard way). In particular, we can find such a sequence with the property that the points $f_m(0)$, $m = 1, 2, \ldots$, are dense in H^2, where 0 is the origin in H^2. Now, suppose A is a nonempty open subset of H^2 and B is a bounded open subset of H^2; we may assume $0 \in A$. Since the points $f_m(0)$ are dense and B is bounded, we can find a finite subset R of $\{f_m : m \geq 4\}$ such that the images $r(A)$ for $r \in R$ cover $\bigcup_{i=1}^{3} f_i^{-1}(\overline{B})$. Now apply Theorem 4.3 to the space H^2 with $N = 1$ (so we can forget the subscript j) to get open subsets A_i ($i = 1, 2, 3$) of H^2 such that the images $f_i(A_i)$ cover a dense subset of H^2 but the sets $r^{-1}(A_i)$ ($r \in R$, $i \leq 3$) are disjoint. Then the sets $A \cap r^{-1}(A_i \cap f_i^{-1}(B))$ ($r \in R$, $i \leq 3$) are disjoint open subsets of A whose images under the corresponding maps $f_i \circ r$ cover a dense open subset of B. Now, as usual, we can reduce these sets so that their images are disjoint subsets of B (but their union is still dense) and apply Lemma 2.4(a) and the relevant forms of the Banach-Schröder-Bernstein theorem (Propositions 2.9 and 2.2) to get

Theorem 5.3. (a) *If A and B are nonempty bounded open subsets of the hyperbolic plane H^2, then there are dense open subsets A' and B' of A and B, respectively, such that A' can be partitioned into finitely many open sets which can be rearranged by isometries of H^2 to form B'.*

(b) *Any two bounded subsets of H^2 with the property of Baire and nonempty interior are Baire equidecomposable.* □

Another method for proving the results in this subsection uses Theorem 4.9 instead of Theorem 4.3. Let A and B be nonempty bounded open sets in S^2 or in H^2. Using the comeagerness of sequences of independent isometries, one can find a finite set R of independent isometries of S^2 or of H^2 such that the images $r(A)$ for $r \in R$ cover every point of B at least three times. Let P be the set of restrictions $r \restriction (A \cap r^{-1}(B))$ for $r \in R$, and let $\mathscr{X} = A$ and $\mathscr{Y} = B$; then the hypotheses of the first part of Theorem 4.9 are satisfied, and the conclusion of that theorem gives finitely many disjoint open subsets of A which can be rearranged to cover a dense open subset of B.

Partitions into congruent sets. As we did with Theorem 5.1, we are going to combine Theorem 4.8 with a corresponding result using arbitrary sets to get partitions of S^n into congruent sets with the property of Baire. Here, however, we must be more careful, because instead of having separate pieces for the comeager part (coming from the open sets) and the meager part (coming from the arbitrary sets), we must combine corresponding pieces.

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First, we review some definitions concerning congruences [9, p. 44]. Fix a positive integer r. A *congruence* is specified by two subsets L and R of $\{1, 2, \dots, r\}$, and is written formally as $\bigcup_{k\in L} A_k \cong \bigcup_{k\in R} A_k$, where A_k ($k \leq r$) are variables. The congruence is *proper* if both L and R are nonempty proper subsets of $\{1, \dots, r\}$. Now suppose G is a group acting on a set X, and a system of congruences is given by pairs $L_i, R_i \subseteq \{1, \dots, r\}$ for $i \leq m$; a *solution* to the system of congruences in X is a sequence of sets $A_k \subseteq X$ ($k \leq r$) which are disjoint and have union X, such that, for each $i \leq m$, there is $\sigma_i \in G$ such that $\sigma_i(\bigcup_{k\in L_i} A_k) = \bigcup_{k\in R_i} A_k$ (i.e., σ_i witnesses congruence number i). It is clear that only proper congruences are useful here, and that, if σ_i witnesses the congruence given by L_i and R_i, then σ_i also witnesses the complementary congruence given by L_i^c and R_i^c, where $S^c = \{1, \dots, r\} \setminus S$. A system of congruences is called *weak* if, among all congruences which can be deduced from the system by taking complements and applying transitivity, there is no congruence of the form $\bigcup_{k\in L} A_k \cong \bigcup_{k\in L^c} A_k$ which requires some set to be congruent to its complement.

The main result about solutions to systems of congruences using arbitrary sets is the following: Suppose the pairs L_i, R_i for $i \leq m$ specify a proper system of congruences. Suppose G is a group acting on a set X, and elements σ_i ($1 \leq i \leq m$) of G are free generators for a free subgroup of G. Finally, suppose that either G acts freely on X, or the system of congruences is weak and G acts locally commutatively on X (i.e., any two elements of G with a common fixed point commute). Then there is a solution to the system of congruences in X such that σ_i witnesses congruence number i [9, p. 46].

Clearly the action of the group of rotations on the sphere S^2 is locally commutative. This is not true of higher-dimensional rotation groups, but it is known that the rotation group on the sphere S^n ($n \geq 2$) has subgroups which are free on more than one generator (and hence subgroups which are free on any given finite number of generators) which act locally commutatively (even freely if n is odd) on S^n [9, p. 57]. It follows easily that S^n can be partitioned into N congruent pieces if $N \geq 3$; one merely has to write down the congruences $A_1 \cong A_i$ for $2 \leq i \leq N$ and note that they form a weak system if $N \geq 3$. If one adds the additional congruences $A_1 \cong \bigcup_{k=i}^N A_k$ for $3 \leq i < N$, then the new system is still weak, and any solution to it in S^n gives a partition of S^n into N congruent sets, any j of which (for $3 \leq j \leq N$) can be rearranged to form a partition of S^2 into j pieces [9, p. 88]. This is the result we combine with Theorem 4.8 to get

Theorem 5.4. *For any $N \geq 3$ and $n \geq 2$, the sphere S^n can be partitioned into N congruent pieces with the property of Baire; in fact, there is a set $A \subseteq S^n$ with the property of Baire such that, for $3 \leq j \leq N$, S^n can be partitioned into j pieces congruent to A.*

Proof. Clearly the first part is a special case of the second part. To prove the latter, fix rotations f_{ij} ($1 \leq i \leq j$, $3 \leq j \leq N$) which are free generators for a locally commutative group of rotations on S^n. As we have noted before, any nontrivial isometry of S^n has a nowhere dense set of fixed points, so the

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hypotheses of Theorem 4.8 are satisfied; let A' be the open set obtained from this theorem.

Note that the rotations $\sigma_{ij} = f_{ij} \circ f_{13}^{-1}$ for $(i, j) \neq (1, 3)$ are independent (i.e., they are free generators for the subgroup they generate). Now define a system of congruences with N variables (i.e., $r = N$), one for each pair (i, j) as above other than $(1, 3)$, by: $L_{ij} = \{1\}$ for all i, j; if $i < j$, then $R_{ij} = \{i\}$; if $i = j$, then $R_{ij} = \{j, j+1, \ldots, N\}$. This is just the weak system of congruences given before this theorem, listed redundantly and with some trivial congruences $A_1 \cong A_1$. Therefore, there are sets B_k $(k \leq N)$ which are a solution to this system of congruences, with σ_{ij} witnessing the congruence for pair (i, j).

We now use the method of Lemma 2.4 to combine the open sets with the arbitrary sets. For each pair (i, j) with $1 \leq i \leq j$ and $3 \leq j \leq N$, let $A'_{ij} = f_{ij}(A')$, and let B'_{ij} be B_i if $i < j$, $\bigcup_{k=j}^{N} B_k$ if $i = j$. Then we have $\sigma_{ij}(A'_{13}) = A'_{ij}$ and $\sigma_{ij}(B'_{13}) = B'_{ij}$ for $(i, j) \neq (1, 3)$; also, for each j, the sets B'_{ij} $(i \leq j)$ form a partition of S^n, while the open sets A'_{ij} form a partition of a dense open subset E_j of S^n. Let D' be the union of all of the images of the meager set $\bigcup_{j=3}^{N}(S^n \setminus E_j)$ under the elements of the group G generated by the rotations f_{ij}; then D' is meager. Now let $A_{ij} = (A'_{ij} \setminus D') \cup (B'_{ij} \cap D')$. These sets have the property of Baire (since A'_{ij} is open and D' and $B'_{ij} \cap D'$ are meager); using the G-invariance of D', it is easy to verify that $\sigma_{ij}(A_{13}) = A_{ij}$ for all (i, j) other than $(1, 3)$ (so the sets are all congruent to $A = A_{13}$), and that, for each j, the sets A_{ij} for $i \leq j$ form a partition of S^n (into j sets congruent to A). □

As in the notes following Theorem 4.8, it follows that there are paradoxical decompositions of S^n using congruent pieces with the property of Baire. Also note that, if A is as in the conclusion of Theorem 5.4, and B is an open set such that $A \triangle B$ is meager, then for any j $(3 \leq j \leq N)$ there are j disjoint sets congruent to B whose union is dense in S^2. It follows just as it did for Theorem 4.8 that one cannot strengthen Theorem 5.4 to work for $2 \leq j \leq N$, or for $3 \leq j < \infty$.

It is now natural to ask which systems of congruences have solutions using subsets of S^2 with the property of Baire, and which systems have solutions using open sets which are only required to cover a dense open subset of S^2. A complete answer to the first question, and a partial answer to the second, will appear in a forthcoming paper [1]. Here we will just show that not all weak systems have solutions. To see this, we use the following fact:

Proposition 5.5. *If B and C are disjoint subsets of S^2 with the property of Baire such that $B \cong B \cup C$, then C is meager.*

Proof. Let σ be an isometry witnessing the congruence, and let B' and C' be the unique regular-open sets such that $B \triangle B'$ and $C \triangle C'$ are meager. (Those who are not familiar with regular-open sets should see Oxtoby [8, Chapter 4]. The relevant fact here is that B' is the largest open set such that $B \triangle B'$ is

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meager, and similarly for C'.) Since $B \cup C \subseteq \sigma(B)$, we have $B' \cup C' \subseteq \sigma(B')$. Also, since B and C are disjoint, $B' \cap C'$ is a meager open set and hence empty. If λ is the standard rotation-invariant probability measure on S^2, then $\lambda(B') = \lambda(\sigma(B')) \geq \lambda(B') + \lambda(C')$, so $\lambda(C') \leq 0$, so C' must be empty, so C is meager. □

One can use this fact to show that, for example, the weak system of congruences defined in the second paragraph of the proof of Theorem 5.4 has no solutions using subsets of S^2 with the property of Baire, or using open subsets whose union is just required to be dense, if $N \geq 4$. Recall that this system included the congruences $A_1 \cong A_3$ and $A_1 \cong A_3 \cup A_4$, so $A_3 \cong A_3 \cup A_4$, so the set A_4 must be meager; but all of the sets are required to be congruent, so they are all meager, so their union must be meager and hence not all of S^2 or even a dense open subset of S^2.

Let G be a countable free group of rotations of S^2. Note that, since each element of G other than the identity has only two fixed points, G acts freely on $S^2 \setminus D$ for some countable set D. But $S^2 \setminus D$ is a G_δ set in S^2 and is therefore a Polish space itself [2, §33 VI], and the preceding paragraphs hold for this new space as well. Hence, even in a Polish space on which a free group of homeomorphisms acts freely, it may be that not all weak systems of congruences are solvable.

Decompositions with a minimal number of pieces. In §2 we were not very careful about how many pieces were used in our decompositions. Now we will see what the minimum number of pieces is for a paradoxical decomposition of S^2 or the unit ball in $\mathbf{R}^3$ using pieces with the property of Baire; here we cannot use results like the Banach-Schröder-Bernstein theorem which increase the number of pieces.

We first obtain a lower bound: S^2 is not Baire equidecomposable with two copies of S^2 using fewer than six pieces, and the same applies to the unit ball in $\mathbf{R}^3$. This was first proved by F. Wehrung by an argument about Borel-measurable partitions of unity [10]; we will give a different proof here.

Proposition 5.6. *Suppose G is a group of homeomorphisms of a Polish space $\mathscr{X}$, such that there is a G-invariant probability measure on $\mathscr{X}$ which gives any nonempty open set positive measure. Let A and B be disjoint open subsets of $\mathscr{X}$, and assume that there are elements σ and τ of G such that $\sigma(A) \cup \tau(B)$ is dense in $\mathscr{X}$. Then $A \cup B$ is dense.*

Proof. Let μ be the invariant probability measure. Since $\sigma(A) \cup \tau(B)$ is dense, $\overline{\sigma(A)} \cup \overline{\tau(B)} = \mathscr{X}$, so any point not in $\overline{\sigma(A)}$ is in $\overline{\tau(B)}^\circ$, where C° is the interior of C. Therefore, $\mu(\overline{\sigma(A)}) + \mu(\overline{\tau(B)}^\circ) \geq 1$. Since σ and τ are homeomorphisms, $\sigma(\overline{A}) = \overline{\sigma(A)}$ and $\tau(\overline{B}^\circ) = \overline{\tau(B)}^\circ$; since μ is G-invariant, $\mu(\overline{A}) + \mu(\overline{B}^\circ) \geq 1$. But any point in $\overline{B}^\circ$ has a neighborhood included in $\overline{B}$, and B is dense in this neighborhood; such a neighborhood cannot meet A, since A and B are disjoint open sets. Therefore, $\overline{A}$ and $\overline{B}^\circ$ are disjoint, so $\mu(\overline{A} \cup \overline{B}^\circ) \geq 1$. It now follows that $A \cup B$ is dense, because if E were a nonempty open set which did not meet $A \cup B$, then E would be disjoint from

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$\overline{A} \cup \overline{B}$, and this would give $\mu(\overline{A} \cup \overline{B}^{\circ}) \leq 1 - \mu(E) < 1$, contradiction. □

Note that such a probability measure exists for the group of rotations of S^n for any n. Also note that, if G and $\mathscr{X}$ are as in Proposition 5.6 and A is an open set such that the union of two G-congruent copies of A is dense, then $\mathscr{X}$ cannot include three disjoint G-congruent copies of A, since any two of them would have to be dense; this is the result mentioned in the notes following Theorem 4.8.

Theorem 5.7. *Suppose G is a group of homeomorphisms of a Polish space $\mathscr{X}$, such that there is a G-invariant probability measure on $\mathscr{X}$ which gives any nonempty open set positive measure. Suppose there are disjoint subsets A_i $(1 \leq i \leq m)$ and B_j $(1 \leq j \leq m')$ of $\mathscr{X}$ with the property of Baire, and elements σ_i and τ_j of G, such that $\bigcup_{i=1}^{m} \sigma_i(A_i)$ and $\bigcup_{j=1}^{m'} \tau_j(B_j)$ are both comeager in $\mathscr{X}$. Then $m \geq 3$ and $m' \geq 3$.*

Proof. We prove that $m > 2$; the proof that $m' > 2$ is identical. Suppose $m \leq 2$; we may assume $m = 2$, since we can just let $A_2 = \varnothing$ if necessary. Let $\hat{A}_1$ and $\hat{A}_2$ be open sets such that $A_i \triangle \hat{A}_i$ is meager for $i = 1, 2$. Then $\sigma_1(\hat{A}_1) \cup \sigma_2(\hat{A}_2)$ is comeager and hence dense in $\mathscr{X}$. Also, $\hat{A}_1$ and $\hat{A}_2$ are disjoint, since otherwise their intersection would be nonmeager and $A_1 \cap A_2$ would also have to be nonmeager. Therefore, by Proposition 5.6, $\hat{A}_1 \cup \hat{A}_2$ is dense and hence comeager, so $A_1 \cup A_2$ is comeager, so the sets B_j are all meager, so $\bigcup_{j=1}^{m'} \tau_j(B_j)$ is meager, contradiction. □

So, for spaces such as S^2, paradoxical decompositions using sets with the property of Baire, or using open sets to cover a dense open set, must use at least six pieces. We will now see that six pieces are also sufficient for S^2; to do this we need a version of Lemma 2.4 which does not waste pieces (by using the Banach-Schröder-Bernstein theorem, for example).

Lemma 5.8. (a) *Suppose $f_1, f_2, \ldots, f_m$ are homeomorphisms from a topological space $\mathscr{X}$ to itself, and A and B are subsets of $\mathscr{X}$ with the property of Baire. Suppose that there are disjoint sets $A'_1, \ldots, A'_m$ with union A such that the sets $f_i(A'_i)$ are disjoint and have union B. Also suppose that there are disjoint sets $A''_1, \ldots, A''_m$ with the property of Baire whose union is $A \setminus D$ for some meager D, such that the sets $f_i(A''_i)$ are disjoint and their union is $B \setminus D'$ for some meager set D'. Then there are disjoint sets $A_1, \ldots, A_m$ with the property of Baire with union A such that the sets $f_i(A_i)$ are disjoint and have union B.*

(b) *Suppose $f_1, f_2, \ldots, f_m$ and $g_1, g_2, \ldots, g_n$ are homeomorphisms from a topological space $\mathscr{X}$ to itself, and B is a subset of $\mathscr{X}$ with the property of Baire. Suppose that there are disjoint sets $A'_1, \ldots, A'_m$ and $B'_1, \ldots, B'_n$ with union B such that the sets $f_i(A'_i)$ are disjoint and have union B, and the sets $g_i(B'_i)$ are also disjoint with union B. Also suppose that there are disjoint sets $A''_1, \ldots, A''_m$ and $B''_1, \ldots, B''_n$ with the property of Baire whose union is $B \setminus D$ for some meager D, such that the sets $f_i(A''_i)$ are disjoint and their union is*

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$B \setminus D'$ for some meager set D', and the sets $g_i(B_i'')$ are disjoint and their union is $B \setminus D''$ for some meager set D''. Then there are disjoint sets $A_1, \dots, A_m$ and $B_1, \dots, B_n$ with the property of Baire with union B such that the sets $f_i(A_i)$ are disjoint and have union B, and the sets $g_i(B_i)$ are disjoint and have union B.

Proof. We just prove (a); the proof of (b) is the same. Let G be the (countable) group generated by $f_1, \dots, f_m$, and let E by the union of all of the sets $h(D)$ and $h(D')$ for $h \in G$; then E is meager. Now define A_i to be $(A_i' \cap E) \cup (A_i'' \setminus E)$. As usual, A_i has the property of Baire since A_i'' has the property of Baire and E and $A_i' \cap E$ are meager; the G-invariance of E shows that $f_i(A_i) = (f_i(A_i') \cap E) \cup (f_i(A_i'') \setminus E)$, so the sets A_i form a partition of A and the sets $f_i(A_i)$ form a partition of B, as desired. □

Theorem 5.9. *Suppose G is a group of homeomorphisms acting on a Polish space $\mathscr{X}$, and this action is free on a comeager set and locally commutative (i.e., any two elements of G with a common fixed point commute). Also suppose that G has a subgroup which is free on more than one generator. Then $\mathscr{X}$ can be partitioned into six pieces $A_1, A_2, A_3, B_1, B_2, B_3$ with the property of Baire such that, for some elements $f_1, f_2, f_3, g_1, g_2, g_3$ of G, the sets $f_1(A_1), f_2(A_2), f_3(A_3)$ form a partition of $\mathscr{X}$, and the sets $g_1(B_1), g_2(B_2), g_3(B_3)$ form a partition of $\mathscr{X}$. (In other words, $\mathscr{X}$ is Baire equidecomposable with two copies of $\mathscr{X}$ using six pieces.)*

Proof. Choose the functions $f_i, g_i \in G$ so that they are free generators for a free subgroup of G; let $\sigma = f_1^{-1} \circ f_2$ and $\tau = g_1^{-1} \circ g_2$. Then σ and τ are free generators for the subgroup $\langle \sigma, \tau \rangle$, so $\mathscr{X}$ can be partitioned into four pieces C_1, C_2, C_3, C_4 such that C_1 and $\sigma(C_2)$ are disjoint with union $\mathscr{X}$, and C_3 and $\tau(C_4)$ are disjoint with union $\mathscr{X}$ [9, p. 37]. (Just find a solution to the congruences $C_2 \cong C_2 \cup C_3 \cup C_4$ and $C_4 \cong C_1 \cup C_2 \cup C_4$ witnessed by σ and τ; this system of congruences is weak.) Now let $A_1' = C_1$, $A_2' = C_2$, $A_3' = \varnothing$, $B_1' = C_3$, $B_2' = C_4$, and $B_3' = \varnothing$. Then each of $\{A_1', A_2', B_1', B_2'\}$, $\{f_1(A_1'), f_2(A_2')\}$, and $\{g_1(B_1'), g_2(B_2')\}$ is a partition of $\mathscr{X}$. Let A_1'', A_2'', A_3'', B_1'', B_2'', and B_3'' be the open sets produced by Theorem 4.1 for the functions f_i and g_i. Lemma 5.8(b) now gives the desired result. □

This shows that the minimal number of pieces in a paradoxical decomposition from one n-sphere to two ($n \geq 2$) using pieces with the property of Baire is six; in fact, the same method shows that the minimal number of pieces to go from one n-sphere to N n-spheres is $3N$.

There appears to be some inefficiency in the way we are using the free rotations to get open sets, as compared to the way they are used to get arbitrary sets via congruences. When obtaining the four arbitrary pieces, only two rotations are used, because the other two pieces are not moved; in the construction of open sets, we applied a different rotation to each piece. However, if one has rotations f_{ij} and open sets A_{ij} such that $\bigcup_{i=1}^{3} f_{ij}(A_{ij})$ is dense for each j, then one gets the same result using the rotations $\sigma_{ij} = f_{1j}^{-1} \circ f_{ij}$, since this just

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rotates the entire dense set after it is built; since σ_{1j} is the identity, we are only using the free rotations σ_{2j} and σ_{3j}. Of course, the proof of Theorem 3.1 and its variants uses the rotations f_{ij}; one could give the proof in terms of the rotations σ_{ij}, but it would be much more tedious to keep track of which expressions are reduced words in the free group.

For the unit ball B in $\mathbf{R}^3$, the proofs of Proposition 5.6 and Theorem 5.7 go through using ordinary Lebesgue measure (normalized to give B measure 1, if desired); B is not closed under isometries of $\mathbf{R}^3$, but this does not matter because the measure is still invariant. One therefore finds that at least six pieces are needed to get two balls from one. On the other hand, the standard method for getting a minimal decomposition of a ball uses the four pieces needed to decompose a sphere by rotations, together with a fifth piece which is just a single point that will become the center of the second sphere [9, p. 40]. This point is "stolen" from one of the spheres making up $B \setminus \{0\}$ (where 0 is the center of B), and the decomposition of that sphere is revised to make up for it. If one now repeats the proof of Lemma 5.8, making sure that the meager set E includes the "stolen" point, then there is no problem combining this decomposition of B with the decomposition obtained from Theorem 4.1 (for which only rotations are needed) to get a decomposition from B to two copies of B using seven pieces with the property of Baire (one of which is just a single point). In fact, the same argument applies to higher-dimensional balls, so we get

Proposition 5.10. *A ball of radius* 1 *in* $\mathbf{R}^{n+1}$ $(n \geq 2)$ *is Baire equidecomposable with two such balls, using seven pieces.* □

The same method shows that one ball and N balls are Baire equidecomposable, using $4N-1$ pieces. However, unlike the situation for arbitrary decompositions, this method for getting decompositions of a ball from decompositions of a sphere does not necessarily produce optimal results here:

Theorem 5.11. *A ball of radius* 1 *in* $\mathbf{R}^3$ *is Baire equidecomposable with two such balls, using six pieces.*

Proof. Let B be the closed unit ball in $\mathbf{R}^3$ (the same proof works for the open ball). As usual, getting a decomposition using sets with the property of Baire requires a decomposition using arbitrary sets and a decomposition using open sets. Let f_1, f_2, f_3, g_1, g_2, and g_3 be free generators for a free group of rotations of $\mathbf{R}^3$ around the origin (the center of B). This list of mappings cannot be used for the decomposition (the centers of both new balls would have to come from the center of the old ball); however, we will see that, if $\mathbf{w}$ is a suitable vector and $T_{\mathbf{w}}$ is the translation of $\mathbf{R}^3$ sending the origin to $\mathbf{w}$, then we can get the desired decomposition using the isometries f_1, f_2, f_3, g_1, g_2, and $T_{\mathbf{w}} \circ g_3$. By Lemma 5.8(b), it will suffice to get an arbitrary-sets decomposition using these mappings and an open-sets decomposition (of dense open subsets of B) using these mappings; we will use R. Robinson's refinement of the Banach-Tarski theorem for the former and Theorem 4.9 for the latter.

Let 1 be the identity map on $\mathbf{R}^3$, and let $\sigma = f_1^{-1} \circ f_2$, $\sigma' = f_1^{-1} \circ f_3$,

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$\tau = g_1^{-1} \circ g_2$, and $\tau' = g_1^{-1} \circ g_3$. It is easy to check that σ, σ', τ, and τ' are free generators for a free group of rotations. As we have seen before, getting the first copy of B via maps f_1, f_2, f_3 is equivalent to getting it via $1, \sigma, \sigma'$, since we can just rotate the entire ball by f_1^{-1} after it is assembled. Similarly, using g_1, g_2, and $T_{\mathbf{w}} \circ g_3$ for the second copy is equivalent to using 1, τ, and $g_1^{-1} \circ T_{\mathbf{w}} \circ g_3$. Note that $g_1^{-1} \circ T_{\mathbf{w}} \circ g_3 = T_{\mathbf{v}} \circ \tau'$, where $\mathbf{v} = g_1^{-1}(\mathbf{w})$.

Instead of specifying the requirements on $\mathbf{w}$ directly, it is more convenient to choose a suitable vector $\mathbf{v}$ and let $\mathbf{w} = g_1(\mathbf{v})$. Let $\mathbf{v}_1$ be a unit vector perpendicular to the axis of the rotation τ which is in a free orbit under the action of the group generated by σ, τ, and τ' (the latter condition only excludes countably many unit vectors, so such a vector exists). The vector $\mathbf{v}$ will be a small positive multiple of $\mathbf{v}_1$; the magnitude $|\mathbf{v}|$ will be specified later.

For the arbitrary-sets decomposition, we use R. Robinson's method for getting such a decomposition of B using five pieces [9, p. 41]. Here two pieces are not moved (i.e., moved by the identity map 1), one is moved by σ, one is moved by τ, and the fifth piece is a single point moved to the origin. This single point is usually taken to be on the boundary of B, but it can actually be any point of B in an orbit on which the group $\langle \sigma, \tau \rangle$ acts freely; in particular, it can be the point $\tau'^{-1}(-\mathbf{v})$, so we can take the map moving it to the origin to be $T_{\mathbf{v}} \circ \tau'$. We now have a decomposition using the desired isometries (the isometry σ' is not needed, so we can take the corresponding piece to be the empty set).

It now remains to find a decomposition between one dense open subset of B and two such subsets, using open sets (call them A_1, A_2, A_3, B_1, B_2, and B_3) and the isometries f_1, f_2, f_3, g_1, g_2, and $T_{\mathbf{w}} \circ g_3$. For this, we will use Theorem 4.9, in the form given immediately after the proof of that theorem. The partial homeomorphisms will be the six isometries above, restricted so that their domains and ranges are included in B. However, the space $\mathscr{X}$ will not be all of B; as we will soon see, hypothesis (C) of Theorem 4.9 is not satisfied by B.

Let ℓ be the axis of the rotation τ, and let $\pm\mathbf{u}$ be the unit vectors in the direction of ℓ. (Recall that $\mathbf{v}$ is perpendicular to ℓ.) When continuously rotated around the line ℓ, the ball $T_{\mathbf{v}}(B)$ sweeps out a shape which includes all of B except for two small "caps," one containing $\mathbf{u}$ (call it C) and one containing $-\mathbf{u}$ ($-C$); see Figure 5.1. In particular, the sets $g_1(C)$ and $g_1(-C)$ lie outside $(T_{\mathbf{w}} \circ g_3)(B)$, so they will have to be taken care of by the sets B_1 and B_2. But any point in C or $-C$ which is not on ℓ is in an infinite orbit of $\tau = g_1^{-1} \circ g_2$ which is included in C or $-C$; this shows that hypothesis (C) of Theorem 4.9 cannot hold for the entire ball B. To get around this, we will handle C and $-C$ separately, and just work with the rest of B. Let $\mathscr{X} = B \setminus (C \cup -C)$; then $\mathscr{X}$ is a closed set in $\mathbf{R}^3$ (an open set if B is an open ball) and hence a Polish space in its own right. Let $\mathscr{Y}_1 = B$ and $\mathscr{Y}_2 = g_1(\mathscr{X})$. Define the partial homeomorphisms p_1, p_2, p_3 from $\mathscr{X}$ to $\mathscr{Y}_1$ by letting p_i be the restriction of f_i to the interior of $\mathscr{X}$; similarly, define the partial homeomorphisms q_1, q_2, q_3 from $\mathscr{X}$ to $\mathscr{Y}_2$ to be the maps g_1, g_2, $T_{\mathbf{w}} \circ g_3$, with each map h restricted to

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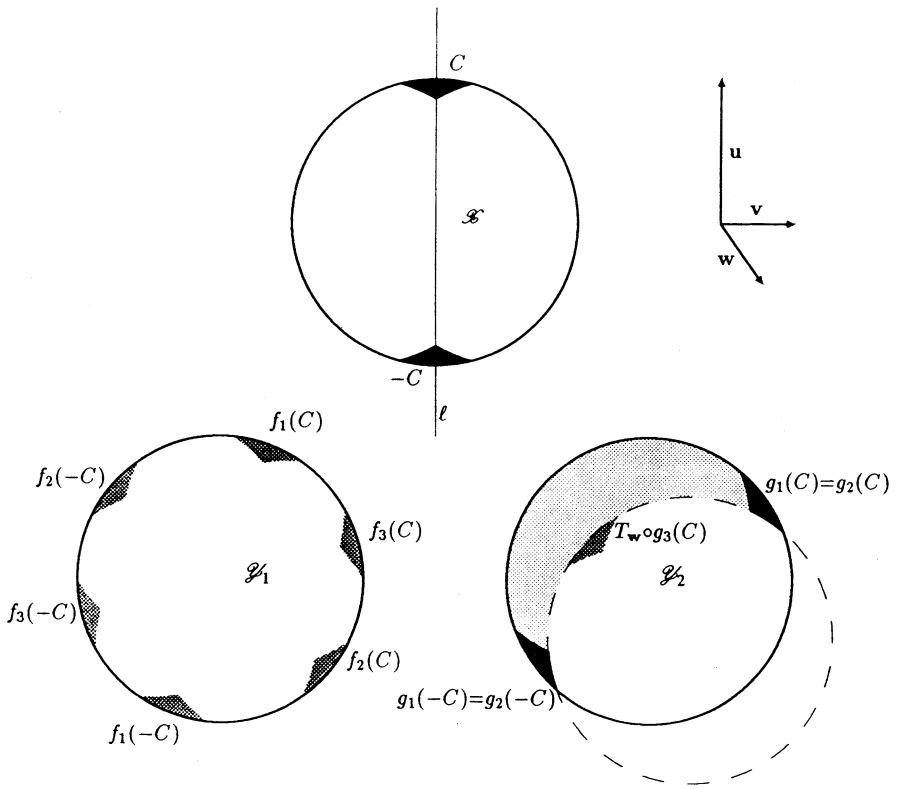

FIGURE 5.1. Domain and range spaces for application of Theorem 4.9.

the interior of $\mathscr{X} \cap h^{-1}(\mathscr{Y}_2)$. (Here and throughout this proof, the word "interior" will refer to the interior in the space $\mathbf{R}^3$, not the interior in $\mathscr{X}$.) We will show that, if $|\mathbf{v}|$ is small enough, then these spaces and partial homeomorphisms satisfy hypotheses (A)–(D) of the modified Theorem 4.9. Given this, apply the theorem to get open sets $A_1, A_2, A_3, \hat{B}_1, B_2, B_3$, and let B_1 be the union of $\hat{B}_1$ and the interiors of C and $-C$; then the sets $A_1, A_2, A_3, B_1, B_2, B_3$ are as desired.

We first consider (A). Let w be a nontrivial reduced alternating word on the generators p_1, p_2, p_3, q_1, q_2, and q_3; then there are corresponding words w' and w'' on the lists of generators $f_1, f_2, f_3, g_1, g_2, g_3$ and $f_1, f_2, f_3, g_1, g_2, T_{\mathbf{w}}\circ g_3$, respectively. Since the rotations were chosen as free generators, w' gives a nontrivial rotation ρ of $\mathbf{R}^3$. Using the fact that the translations form a normal subgroup of the isometry group of $\mathbf{R}^3$ (in particular, if h is an isometry fixing the origin and $\mathbf{x} \in \mathbf{R}^3$, then $h \circ T_{\mathbf{x}} = T_{h(\mathbf{x})} \circ h$), one sees that w'' gives an isometry of the form $T_{\mathbf{y}} \circ \rho$ for some $\mathbf{y}$. Since ρ is not the identity, $T_{\mathbf{y}} \circ \rho$ cannot be the identity, so its set of fixed points is nowhere dense in $\mathbf{R}^3$. The partial homeomorphism given by w is just the restriction of $T_{\mathbf{y}} \circ \rho$ to some

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domain, so its set of fixed points is also nowhere dense (in $\mathbf{R}^3$ and hence in $\mathscr{X}$).

For (B) and (C), we will use the fact that the caps $\pm C$ can be made as small in diameter as we like (i.e., can be required to lie within arbitrarily small neighborhoods of $\pm\mathbf{u}$), simply by requiring $|\mathbf{v}|$ to be sufficiently small; this follows from the fact that $\mathbf{v}$ and ℓ are perpendicular. Now, for (B), we need to see that the ranges of p_1, p_2, and p_3 cover a dense subset of $\mathscr{Y}_1 = B$ twice, while the ranges of q_1, q_2, and q_3 cover a dense subset of $\mathscr{Y}_2$ twice. The only points in the interior of $\mathscr{Y}_1$ not in the range of p_i ($i = 1, 2, 3$) are those in $f_i(\pm\overline{C})$, so we must see that these six cap images do not overlap, if $|\mathbf{v}|$ is sufficiently small. Here it will suffice to show that the six points $f_i(\pm\mathbf{u})$ are distinct, because we can require the caps to lie within very small neighborhoods of the points. But if we had, say, $f_1(\mathbf{u}) = f_2(-\mathbf{u})$, then $f_2^{-1} \circ f_1 \circ f_2^{-1} \circ f_1$ would be a rotation fixing the origin and the point $\mathbf{u}$, so it would be a rotation around the axis ℓ, so it would commute with $g_1^{-1} \circ g_2$, contradicting the fact that the six given rotations are free generators. Therefore, the six points are distinct, so the interior of $\mathscr{Y}_1$ is covered twice. As for $\mathscr{Y}_2$, since $\mathscr{X}$ is fixed under $\tau = g_1^{-1} \circ g_2$, we have $\mathscr{Y}_2 = g_1(\mathscr{X}) = g_2(\mathscr{X})$, so the ranges of q_1 and q_2 both cover the entire interior of $\mathscr{Y}_2$. This completes the proof of (B).

Next, for (C), let $\mathscr{G}$ be the initial active link graph; we will see that $\mathscr{G}$ has finite connected components, if $|\mathbf{v}|$ is small enough. Note that only points in the interior of $\mathscr{X}$ can be endpoints of edges of $\mathscr{G}$, since the domains of the partial homeomorphisms were limited to this interior. The graph $\mathscr{G}$ consists of two parts. The first part comes from the fact that $T_{\mathbf{w}}(B)$ does not cover B; for any point $\mathbf{y}$ in the interior of $\mathscr{Y}_2$ which is not in the interior of $T_{\mathbf{w}}(B)$ (the light gray part of Figure 5.1), the points $\mathbf{x} = g_2^{-1}(\mathbf{y})$ and $\mathbf{x}' = g_1^{-1}(\mathbf{y}) = \tau(\mathbf{x})$ are initially actively linked. Let $\mathscr{G}_1$ be the subgraph of $\mathscr{G}$ consisting of these edges. The second part comes from the fact that we are using $\mathscr{X}$ instead of B, and hence there are initially active links between images of the caps $\pm C$. If $\{i, i', i''\} = \{1, 2, 3\}$ and $\mathbf{y} \in f_{i''}(\pm\overline{C})$ is in the ranges of p_i and $p_{i'}$, then the points $\mathbf{x} = p_i^{-1}(\mathbf{y})$ and $\mathbf{x}' = p_{i'}^{-1}(\mathbf{y}) = f_{i'}^{-1}(f_i(\mathbf{x}))$ are initially actively linked. The same applies to those points in $T_{\mathbf{w}}(g_3(\pm\overline{C}))$ which are in the ranges of both q_1 and q_2. (These are the dark gray parts of Figure 5.1.) Let $\mathscr{G}_2$ be the subgraph of $\mathscr{G}$ comprising these edges; then $\mathscr{G} = \mathscr{G}_1 \cup \mathscr{G}_2$.

We will now show that every connected component of $\mathscr{G}_1$ is finite, and that, if $|\mathbf{v}|$ is sufficiently small, then each connected component of $\mathscr{G}$ contains at most one edge from $\mathscr{G}_2$. It follows that every component of $\mathscr{G}$ is made up of one or two components of $\mathscr{G}_1$ and is therefore finite, as desired.

Let $\mathbf{x}$ be a point in the interior of $\mathscr{X}_1$; we must see that the component of $\mathbf{x}$ in $\mathscr{G}_1$ is finite. The only points which could be connected to $\mathbf{x}$ by an edge of $\mathscr{G}_1$ are $\tau(\mathbf{x})$ and $\tau^{-1}(\mathbf{x})$; by iteration, we see that the component of $\mathbf{x}$ in $\mathscr{G}_1$ is included in the set $\{\tau^n(\mathbf{x}) : n \in \mathbf{Z}\}$, and the only possible edges of $\mathscr{G}_1$ between points in this set are those between $\tau^n(\mathbf{x})$ and $\tau^{n+1}(\mathbf{x})$. If $\mathbf{x}$ is on ℓ, then these points all coincide, so $\mathbf{x}$ is the only point in its $\mathscr{G}_1$-component; so assume $\mathbf{x}$ is not on ℓ. Since τ is a member of a free group of rotations, it must be

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a rotation about ℓ through an angle $2\pi\theta$ where θ is irrational. Hence, the points $\tau^n(\mathbf{x})$ for $n \in \mathbf{Z}$ are distinct. To prove that the $\mathscr{G}_1$-component of $\mathbf{x}$ is finite, it will suffice to show that there exist a positive k and a nonpositive k such that there is no edge in $\mathscr{G}_1$ from $\tau^{k-1}(\mathbf{x})$ to $\tau^k(\mathbf{x})$; if k_1 is the least such positive k and k_2 is the greatest such nonpositive k, then the $\mathscr{G}_1$-component of $\mathbf{x}$ is $\{\tau^n(\mathbf{x}): k_2 \leq n < k_1\}$.

Since $\mathbf{x}$ is in the interior of $\mathscr{X}$ (and hence not in the closure of C or of $-C$), the circle obtained by rotating $\mathbf{x}$ continuously around ℓ must meet the interior of $T_{\mathbf{v}}(B)$, so an entire arc of that circle lies within the interior of $T_{\mathbf{v}}(B)$. We now use the well-known fact that the multiples $\theta, 2\theta, 3\theta, \ldots$ of an irrational number θ have fractional parts which are dense in the interval $(0, 1)$ [7, §6.4]. This implies that the points $\tau^n(\mathbf{x})$, $n = 1, 2, \ldots$, are dense in the above circle, so there must be a positive k such that $\tau^k(\mathbf{x})$ is in the interior of $T_{\mathbf{v}}(B)$. Then $g_1(\tau^k(\mathbf{x})) = g_2(\tau^{k-1}(\mathbf{x}))$ is in the interior of $T_{\mathbf{w}}(B)$ as well as the interior of B, so there is no edge of $\mathscr{G}_1$ from $\tau^{k-1}(\mathbf{x})$ to $\tau^k(\mathbf{x})$. The argument for negative k is similar, so $\mathscr{G}_1$ has finite connected components.

We must now see how to ensure that each connected component of $\mathscr{G}$ contains at most one edge from $\mathscr{G}_2$. Let S be the set consisting of the fourteen points $f_{i'}^{-1}(f_i(\pm\mathbf{u}))$ $(i \neq i')$ and $g_1^{-1}(g_3(\pm\mathbf{u}))$. As in the proof of (B), one can see that these points are distinct; in fact, the points $\tau^k(\mathbf{x})$ for $\mathbf{x} \in S$ and $k \in \mathbf{Z}$ are all distinct. (Otherwise we would have $\tau^k(h(\mathbf{u})) = h'(\pm\mathbf{u})$ with $k \in \mathbf{Z}$ and h, h' among the rotations $f_{i'}^{-1} \circ f_i$ and $g_1^{-1} \circ g_3$, with either $k \neq 0$ or $h \neq h'$. Then $(h'^{-1} \circ \tau^k \circ h)^2(\mathbf{u})$ would be $\mathbf{u}$, so $(h'^{-1} \circ \tau^k \circ h)^2$ would commute with τ, which is impossible because the reduced form of $(h'^{-1} \circ \tau^k \circ h)^2$ involves at least one of f_1, f_2, f_3, and g_3.) In particular, the points in S are different from $\mathbf{u}$ and $-\mathbf{u}$ (which are fixed under τ). The argument of the preceding paragraph shows that, for each $\mathbf{x} \in S$, there exist a positive k and a negative k such that $\tau^k(\mathbf{x}) \cdot \mathbf{v}_1 > 0$. Fix $K > 0$ such that, for each $\mathbf{x} \in S$, there is a positive $k < K$ such that $\tau^k(\mathbf{x}) \cdot \mathbf{v}_1 > 0$, and there is a negative $k > -K$ such that $\tau^k(\mathbf{x}) \cdot \mathbf{v}_1 > 0$. Now choose $\varepsilon > 0$ so small that, for each $\mathbf{x} \in S$, there exist a positive $k < K$ and a negative $k > -K$ such that $\tau^k(\mathbf{x}) \cdot \mathbf{v}_1 > \varepsilon$ (i.e., $\tau^k(\mathbf{x})$ is on the same side of the plane $\mathbf{v}_1^\perp$ perpendicular to $\mathbf{v}_1$ through the origin as $\mathbf{v}_1$ is, and is at distance greater than ε from that plane). Let U be a neighborhood of $\mathbf{u}$ of diameter less than $\varepsilon/2$ such that the open sets $\tau^k(h(\pm U))$, where $-K \leq k \leq K$ and h is one of the rotations $f_{i'}^{-1} \circ f_i$ or $g_1^{-1} \circ g_3$, are disjoint from each other and from $\pm U$. We are finally ready to specify the magnitude of $\mathbf{v}$: let $|\mathbf{v}|$ be a positive number less than ε which is so small that the six caps $f_i(\pm\overline{C})$ are disjoint, $\overline{C} \subseteq U$, $T_{\mathbf{w}}(g_3(\overline{C})) \subseteq g_3(U)$, and $T_{\mathbf{w}}(g_3(-\overline{C})) \subseteq g_3(-U)$, where $\mathbf{w} = g_1(\mathbf{v})$.

We will show that, if $|\mathbf{v}|$ is as above, then a component of $\mathscr{G}$ which contains an edge of $\mathscr{G}_2$ cannot contain another such edge. First, consider a component of $\mathscr{G}$ which contains an edge of $\mathscr{G}_2$ arising from a point $\mathbf{y} \in T_{\mathbf{w}}(g_3(\pm\overline{C}))$. For

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definiteness, suppose $\mathbf{y} \in T_{\mathbf{w}}(g_3(\overline{C}))$; the case $\mathbf{y} \in T_{\mathbf{w}}(g_3(-\overline{C}))$ is the same. This edge connects the points $\mathbf{x} = g_1^{-1}(\mathbf{y})$ and $\tau^{-1}(\mathbf{x}) = g_2^{-1}(\mathbf{y})$. Since $\mathbf{y} \in T_{\mathbf{w}}(g_3(\overline{C}))$, we have $\mathbf{y} \in g_3(U)$, so $\mathbf{x} \in g_1^{-1}(g_3(U))$. Fix a positive $k_1 < K$ and a negative $k_2 > -K$ such that $\tau^{k_1}(g_1^{-1}(g_3(\mathbf{u}))) \cdot \mathbf{v}_1 > \varepsilon$ and $\tau^{k_2}(g_1^{-1}(g_3(\mathbf{u}))) \cdot \mathbf{v}_1 > \varepsilon$. We will show that the component of $\mathbf{x}$ is included in the finite set $\{\tau^k(\mathbf{x}) \colon k_2 \le k < k_1\}$.

Let H be the half-space $\{\mathbf{z} \colon \mathbf{z} \cdot \mathbf{v}_1 > |\mathbf{v}|/2)\}$; then it is easy to see that $B \cap H$ is included in the interior of $T_{\mathbf{v}}(B)$. Since U has diameter less than $\varepsilon/2$ and $\tau^{k_1}(g_1^{-1}(g_3(\mathbf{u}))) \cdot \mathbf{v}_1 > \varepsilon$, we have $\tau^{k_1}(g_1^{-1}(g_3(U))) \subseteq H$, so $\tau^{k_1}(\mathbf{x})$ is in the interior of $T_{\mathbf{v}}(B)$. Hence, there is no edge in $\mathscr{G}_1$ from $\tau^{k_1-1}(\mathbf{x})$ to $\tau^{k_1}(\mathbf{x})$. Similarly, there is no edge in $\mathscr{G}_1$ from $\tau^{k_2-1}(\mathbf{x})$ to $\tau^{k_2}(\mathbf{x})$. Now note that, by the choice of U, the sets $\tau^k(g_1^{-1}(g_3(U)))$ for $k_2 \le k < k_1$ are disjoint from each other and from the caps $f_{i'}^{-1}(f_i(\pm\overline{C}))$ $(i \ne i')$ and $g_i^{-1}(T_{\mathbf{w}}(g_3(\pm\overline{C})))$ $(i = 1, 2)$, except that $g_1^{-1}(T_{\mathbf{w}}(g_3(\overline{C}))) \subseteq g_1^{-1}(g_3(U))$ and $g_2^{-1}(T_{\mathbf{w}}(g_3(\overline{C}))) \subseteq \tau^{-1}(g_1^{-1}(g_3(U)))$. Therefore, the only edge of $\mathscr{G}_2$ which has an endpoint in the set $\{\tau^k(\mathbf{x}) \colon k_2 \le k < k_1\}$ is the edge from $\tau^{-1}(\mathbf{x})$ to $\mathbf{x}$. This completes the proof that no edge of $\mathscr{G}$ connects a point in this set to a point not in the set, so the $\mathscr{G}$-component of $\mathbf{x}$ is included in this set and therefore contains no edge of $\mathscr{G}_2$ other than the given one.

Second, consider a component of $\mathscr{G}$ which contains an edge of $\mathscr{G}_2$ arising from a point $\mathbf{y} \in f_i(\pm\overline{C})$ for some $i \le 3$; again we assume for definiteness that $\mathbf{y} \in f_i(\overline{C})$. Then the edge in question connects the point $\mathbf{x} = f_{i'}^{-1}(\mathbf{y}) \in f_{i'}^{-1}(f_i(U))$ to the point $\mathbf{x}' = f_{i''}^{-1}(\mathbf{y}) \in f_{i''}^{-1}(f_i(U))$, where $\{i, i', i''\} = \{1, 2, 3\}$. As in the first case, find a positive $k_1 < K$ and a negative $k_2 > -K$ such that, for $m = 1, 2$, $\tau^{k_m}(\mathbf{x})$ is the interior of $T_{\mathbf{v}}(B)$, and hence there is no edge of $\mathscr{G}_1$ from $\tau^{k_m-1}(\mathbf{x})$ to $\tau^{k_m}(\mathbf{x})$. Similarly, find a positive $k_3 < K$ and a negative $k_4 > -K$ such that there is no edge in $\mathscr{G}_1$ from $\tau^{k_m-1}(\mathbf{x}')$ to $\tau^{k_m}(\mathbf{x}')$ $(m = 3, 4)$. Then, by the choice of U, the sets $\tau^k(f_{i'}^{-1}(f_i(U)))$ $(k_2 \le k < k_1)$ and $\tau^k(f_{i''}^{-1}(f_i(U)))$ $(k_4 \le k < k_3)$ are disjoint from each other and from the caps $f_{I'}^{-1}(f_I(\pm\overline{C}))$ $(I \ne I')$ and $g_I^{-1}(T_{\mathbf{w}}(g_3(\pm\overline{C})))$ $(I = 1, 2)$, except that $f_{i'}^{-1}(f_i(\overline{C})) \subseteq f_{i'}^{-1}(f_i(U))$ and $f_{i''}^{-1}(f_i(\overline{C})) \subseteq f_{i''}^{-1}(f_i(U))$. This means that the only edge of $\mathscr{G}_2$ with an endpoint in the finite set

$$\{\tau^k(\mathbf{x}) \colon k_2 \le k < k_1\} \cup \{\tau^k(\mathbf{x}') \colon k_4 \le k < k_3\}$$

is the edge from $\mathbf{x}$ to $\mathbf{x}'$. By the choice of k_m $(m = 1, 2, 3, 4)$, no edge of $\mathscr{G}_1$ connects a point in this finite set to a point outside the set. Therefore, the $\mathscr{G}$-component of $\mathbf{x}$ is included in this set and hence contains no edge of $\mathscr{G}_2$ other than the edge from $\mathbf{x}$ to $\mathbf{x}'$. This completes the proof of (C).

Finally, (D) is trivial, because each of p_1, p_2, and p_3 has as domain the entire interior of $\mathscr{X}$. Therefore, Theorem 4.9 gives the desired decomposition using open sets, so we are done. □

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One can probably generalize this to get decompositions from one unit ball in $\mathbf{R}^{n+1}$ ($n \geq 2$) to N such balls using $3N$ pieces, but we have not worked out the details.

6. Conclusion and open problems

We have seen that problems about decompositions into sets with the property of Baire are quite closely related to problems about decompositions using open sets. The method presented here works quite well for the latter type of problem, but it is very specific to open sets; hence, it cannot handle problems about properties even slightly different from the property of Baire. For example, M. Talagrand has asked us whether S^2 is equidecomposable with two copies of S^2 using pieces A_i which have the property of Baire universally (i.e., for each closed subspace $\mathscr{Y}$ of S^2, $A_i \cap \mathscr{Y}$ has the property of Baire in $\mathscr{Y}$); we cannot answer this. Also, it is open whether S^2 can be decomposed into three congruent sets which are Borel, or Lebesgue measurable, or even (as far as we know) of positive inner measure. For a longer list of related problems, most of which are still open, see Wagon [9, Appendix C].

Mycielski asks whether, for the case of the unit ball in $\mathbf{R}^3$, one can find *connected* open sets satisfying the conclusion of Theorem 3.1. It seems reasonable to expect that one can achieve this using a modified version of the proof of Theorem 3.1, but we have not yet succeeded in doing so. On the other hand, we have not yet ruled out the possibility that there are connected open subsets of S^2 satisfying the conclusion of Theorem 3.1, although this seems highly implausible (but then, so do most of the main results of this paper).

Another problem is to determine the minimal number of pieces for specific instances of Baire equidecomposability: between the unit ball in $\mathbf{R}^4$ and two such balls, or between the unit ball in $\mathbf{R}^3$ and three such balls, or between a ball of radius 1 and a ball of radius r, etc. The last of these seems likely to be totally intractable, because it is quite difficult just to determine how many translates of the unit ball are required to cover a larger ball.

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DEPARTMENT OF MATHEMATICS, OHIO STATE UNIVERSITY, COLUMBUS, OHIO 43210
Current address, M. Foreman: Department of Mathematics, University of California, Irvine, CA 92717
E-mail address, R. Dougherty: rld@math.ohio-state.edu
E-mail address, M. Foreman: mforeman@math.uci.edu

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