Discrete Mathematics And Graph Theory

Syllabus

UNIT - 1 :- Mathematical logic

Introduction - statements and Notation, connectives, wellformed formulas, Tautology, duality laws, equivalence, implication, Normal forms, functionally complete set of connectives, inference theory of statement calculus, predicate calculus, inference theory of predict calculus.

UNIT - I

Mathematical Logic

Statement :- A statement is a collection of sentence which is either true or false, but not both.

Eg :- i) The sun rises in the east (True statement)

ii) $2+3=5$ (True)

iii) $12.5+9=16$ (False)

iv) $x+y=3$ (Not a statement)

Notation (Not a statement)

Connectives :-

The compound statements consists of two or more statements connected with connectives.

Notation	Logical operation
$\sim$	negation
$\wedge$	And
$\vee$	or
$\rightarrow$	Implies
$\leftrightarrow$	Biconditional

Examples :- i) Sameera is a little girl and she is playing.

ii) Raheem has ₹5 and he brought a chocolate.

Logical connectives

1) **Negation - ($\sim$)**

Let P be a statement, the negation of P is denoted by "$\sim P$" or "$\bar{P}$" or "$\neg P$".

The statement of $\sim P$ is true, then the negation is false.

P	$\sim P$
T	F
F	T

Conjunction - (And - $\wedge$)

let p, q are two statements then the conjunction of p & q is denoted by "$p \wedge q$" and read as "p and q".

If the conjunction $p \wedge q$ is True, then the statement is True otherwise it is false.

p	q	$p \wedge q$
T	T	T
T	F	F
F	T	F
F	F	F

Disjunction - (OR - $\vee$)

let p & q are two statements then the disjunction of p & q is denoted by "$p \vee q$" and read as "p or q".

If the disjunction is false when p & q are false. Otherwise it is true.

p	q	$p \vee q$
T	T	T
T	F	T
F	T	T
F	F	F

Conditional - ($\rightarrow$ - Implies)

let p & q are two statements. The conditional statement of p & q is denoted by "$p \rightarrow q$" and read as "p implies q".

If the conditional statement $p \rightarrow q$ is false when p is true and q is false. Otherwise it is true.

p	q	$p \rightarrow q$
T	T	T
T	F	F
F	T	T
F	F	T

Biconditional ($\leftrightarrow$)

Let p, q are two statements then the biconditional of p & q is denoted by "$p \leftrightarrow q$" and read as "p biconditional q"

If Both the statements are true then biconditional is true, if both the statements are false then the biconditional is true, otherwise it is false.

p	q	$p \leftrightarrow q$
T	T	T
T	F	F
F	T	F
F	F	T

Order for logical connectives:-

Connectives	order
$\sim$	1
$\wedge$	2
$\vee$	3
$\rightarrow$	4
$\leftrightarrow$	5

NAND:-

The word NAND is a combination of "NOT" & "AND", where NOT stands for negation & AND stands for Conjunction. NAND is denoted by "$\uparrow$"

i.e. $p \uparrow q \Leftrightarrow \sim(p \wedge q)$

Let p, q be the two statements then the truth table for $p \uparrow q$ is

p	q	$p \wedge q$	$p \uparrow q$
T	T	T	F
T	F	F	T
F	T	F	T
F	F	F	T

NOR :-

1, The word NOR is a combination of "NOT" & "OR"
NOT stands for negation and OR stands for disjunction($\vee$)
NOR is denoted by $\downarrow$
let p, q be the two statements then the NOR of p & q is

$p \downarrow q \Longleftrightarrow \sim(p \vee q)$

p	q	$p \vee q$	$p \downarrow q$
T	T	T	F
T	F	T	F
F	T	T	F
F	F	F	T

Exclusive OR ($\oplus$)

let p & q be the two statements then the exclusive OR is denoted by "$p \oplus q$". And it is defined as the negation of biconditional statement. i.e.

$p \oplus q \Longleftrightarrow \sim(p \longleftrightarrow q)$

p	q	$p \longleftrightarrow q$	$p \oplus q$
T	T	T	F
T	F	F	T
F	T	F	T
F	F	T	F

Truth table :-

A truth table is a table that shows a truth value of compound statement for all posible cases, i.e. The relation b/w truth values and statement

Compound statement

Combination of two or more statements

Problems

Construct a truth table for the following compound statements

1) $\sim(p \vee q) \vee (\sim p \wedge \sim q)$

p	q	$\sim p$	$\sim q$	$\sim p \wedge \sim q$ (b)	$p \vee q$	$\sim(p \vee q)$ (a)	$a \vee b$
T	T	F	F	F	T	F	F
T	F	F	T	F	T	F	F
F	T	T	F	F	T	F	F
F	F	T	T	T	F	T	T

Sol: Given two statements p & q. To find truth table for the compound statement is

2) $(\sim(p \wedge q) \vee r) \rightarrow \sim p$

Given 3 statements p, q, r. To find the truth table for the compound statement is

p	q	r	$\sim p$ (b)	$p \wedge q$	$\sim(p \wedge q)$	$\sim(p \wedge q) \vee r$ (a)	$a \rightarrow b$
T	T	T	F	T	F	T	F
T	T	F	F	T	F	F	T
T	F	T	F	F	T	T	F
T	F	F	F	F	T	T	F
F	T	T	T	F	T	T	T
F	T	F	T	F	T	T	T
F	F	T	T	F	T	T	T
F	F	F	T	F	T	T	T

Tautology:-

A statement is said to be tautology if its truth value is true (T) for any assignment of the truth values to its components.

Contradiction:-

A statement is said to be contradiction if its truth value is False (F) for any assignment of the truth values to its components.

3) Show that the statement
$(P \to (q \to r)) \to ((P \to q) \to (P \to r))$ is tautology

A) Given 3 statements P, q, r. To show that the given statement is Tautology

P	q	r	$q \to r$	(c) $P \to (q \to r)$	(a) $P \to q$	(b) $P \to r$	(d) $a \to b$	$c \to d$
T	T	T	T	T	T	T	T	T
T	T	F	F	F	T	F	F	T
T	F	T	T	T	F	T	T	T
T	F	F	T	T	F	F	T	T
F	T	T	T	T	T	T	T	T
F	T	F	F	T	T	T	T	T
F	F	T	T	T	T	T	T	T
F	F	F	T	T	T	T	T	T

4) Show that the statement is contradiction
$\sim(q \to r) \wedge r \wedge (P \to q)$

A) Given 3 statements P, q, r. To show that the given statement is contradiction

P	q	r	$q \to r$	(a) $\sim(q \to r)$	$P \to q$	(b) $r \wedge (P \to q)$	$a \wedge b$
T	T	T	T	F	T	T	F
T	T	F	F	T	T	F	F
T	F	T	T	F	F	F	F
T	F	F	T	F	F	F	F
F	T	T	T	F	T	F	F
F	T	F	F	T	T	T	F
F	F	T	T	F	T	F	F
F	F	F	T	F	T	T	F

5) $P \wedge (\sim q \vee q)$

A) Given two statements P, q. To find out the truth table

(a) P	q	$\sim q$	(b) $\sim q \vee q$	$a \wedge b$
T	T	F	T	T
T	F	T	T	T
F	T	F	T	F
F	F	T	T	F

6) $p \to (\sim q \vee r)$

A) Given 3 statements p, q, r. To find out the truth table

p	q	r	$\sim q$ (a)	$\sim q \vee r$ (b)	$a \to b$
T	T	T	F	T	T
T	T	F	F	F	F
T	F	T	T	T	T
T	F	F	T	T	T
F	T	T	F	T	T
F	T	F	F	F	T
F	F	T	T	T	T
F	F	F	T	T	T

7) $(p \leftrightarrow q) \leftrightarrow [(p \wedge q) \vee (\sim p \wedge \sim q)]$

A) Given 2 statements p, q. To find out the truth table

p	q	$\sim p$	$\sim q$	$p \wedge q$ (a)	$\sim p \wedge \sim q$ (b)	$a \vee b$ (d)	$p \leftrightarrow q$ (c)	$c \leftrightarrow d$
T	T	F	F	T	F	T	T	T
T	F	F	T	F	F	F	F	T
F	T	T	F	F	F	F	F	T
F	F	T	T	F	T	T	T	T

8) $\sim [p \vee (q \wedge r)] \leftrightarrow [(p \vee q) \wedge (p \to r)]$

A) Given 3 statements p, q, r. To find out truth table

p	q	r	$q \wedge r$	$p \vee (q \wedge r)$	$\sim[p \vee (q \wedge r)]$ (c)	$p \vee q$ (a)	$p \to r$ (b)	$a \wedge b$ (d)	$c \leftrightarrow d$
T	T	T	T	T	F	T	T	T	F
T	T	F	F	T	F	T	F	F	T
T	F	T	F	T	F	T	T	T	F
T	F	F	F	T	F	T	F	F	T
F	T	T	T	T	F	T	T	T	F
F	T	F	F	F	T	T	T	T	T
F	F	T	F	F	T	F	T	F	F
F	F	F	F	F	T	F	T	F	F

9) $(p \to q) \to (q \to p)$

p	q	$p \to q$ a	$q \to p$ b	$a \to b$
T	T	T	T	T
T	F	F	T	T
F	T	T	F	F
F	F	T	T	T

10) $p \to (\sim q \vee r)$

(a) p	q	r	$\sim q$	(b) $\sim q \vee r$	$a \to b$
T	T	T	F	T	T
T	T	F	F	F	F
T	F	T	T	T	T
T	F	F	T	T	T
F	T	T	F	T	T
F	T	F	F	F	T
F	F	T	T	T	T
F	F	F	T	T	T

11) $(q \to \sim p) \leftrightarrow (p \leftrightarrow q)$

p	q	$\sim p$	$q \to \sim p$ (a)	$p \leftrightarrow q$ (b)	$a \leftrightarrow b$
T	T	F	F	T	F
T	F	F	T	F	F
F	T	T	T	F	F
F	F	T	T	T	T

12) $(\sim p \leftrightarrow \sim q) \leftrightarrow (p \leftrightarrow q)$

p	q	$\sim p$	$\sim q$	$\sim p \leftrightarrow \sim q$ (a)	$p \leftrightarrow q$ (b)	$a \leftrightarrow b$
T	T	F	F	T	T	T
T	F	F	T	F	F	T
F	T	T	F	F	F	T
F	F	T	T	T	T	T

13) $[(p \vee q) \wedge \sim(\sim p \wedge (\sim q \vee \sim r))] \vee [(\sim p \wedge \sim q) \vee (\sim p \wedge \sim r)]$

p	q	r	(a) $\sim p$	$\sim q$	$\sim r$	(c) $p \vee q$	(b) $\sim q \vee \sim r$	$a \wedge b$	(d) $\sim(a \wedge b)$	(g) $c \wedge d$	(e) $\sim p \wedge \sim q$	(f) $\sim p \wedge \sim r$	(h) $e \vee f$	$g \vee h$
T	T	T	F	F	F	T	F	F	T	T	F	F	F	T
T	T	F	F	F	T	T	T	F	T	T	F	F	F	T
T	F	T	F	T	F	T	T	F	T	T	F	F	F	T
T	F	F	F	T	T	T	T	F	T	T	F	F	F	T
F	T	T	T	F	F	T	F	F	T	T	F	F	F	T
F	T	F	T	F	T	T	T	T	F	F	F	T	T	T
F	F	T	T	T	F	F	T	T	F	F	T	F	T	T
F	F	F	T	T	T	F	T	T	F	F	T	F	T	T

14) $p \wedge \sim(q \wedge r) \longleftrightarrow \sim[(p \to q) \vee (p \wedge \sim r)]$

(a) p	q	r	$\sim r$	$q \wedge r$	(b) $\sim(q \wedge r)$	(e) $a \wedge b$	(c) $p \to q$	(d) $p \wedge \sim r$	$c \vee d$	(f) $\sim c \vee d$	$e \leftrightarrow f$
T	T	T	F	T	F	F	T	F	T	F	T
T	T	F	T	F	T	T	T	T	T	F	F
T	F	T	F	F	T	T	F	F	F	T	T
T	F	F	T	F	T	T	F	T	T	F	F
F	T	T	F	T	F	F	T	F	T	F	T
F	T	F	T	F	T	F	T	F	T	F	T
F	F	T	F	F	T	F	T	F	T	F	T
F	F	F	T	F	T	F	T	F	T	F	T

14) $p \wedge \sim(q \wedge r) \longleftrightarrow \sim[(p \to q) \vee (p \wedge \sim r)]$

(a) p	q	r	$\sim r$	$q \wedge r$	(b) $\sim(q \wedge r)$	(e) $a \wedge b$	(c) $p \to q$	(d) $p \wedge \sim r$	$c \vee d$	$\sim(c \vee d)$	(f) $e \longleftrightarrow f$
T	T	T	F	T	F	F	T	F	T	F	T
T	T	F	T	F	T	T	T	T	T	F	F
T	F	T	F	F	T	T	F	F	F	T	T
T	F	F	T	F	T	T	F	T	T	F	F
F	T	T	F	T	F	F	T	F	T	F	T
F	T	F	T	F	T	F	T	F	T	F	T
F	F	T	F	F	T	F	T	F	T	F	T
F	F	F	T	F	T	F	T	F	T	F	T

Equivalence Formula :- (or) Logically equivalent

The two compound statements have the same truth values in which the truth value of the statement is called logically equivalent and it is denoted by "$\equiv$" or "$\Longleftrightarrow$"

15) Show that $[p \to (q \wedge r)] \equiv [(p \to q) \wedge (p \to r)]$ is equivalent

Given 3 statements p, q, r. To show that the given statement is equivalent

p	q	r	$q \wedge r$	(a) $p \to (q \wedge r)$	(b) $p \to q$	(c) $p \to r$	$b \wedge c$
T	T	T	T	T	T	T	T
T	T	F	F	F	T	F	F
T	F	T	F	F	F	T	F
T	F	F	F	F	F	F	F
F	T	T	T	T	T	T	T
F	T	F	F	T	T	T	T
F	F	T	F	T	T	T	T
F	F	F	F	T	T	T	T

$\longrightarrow \equiv \longleftarrow$

$\therefore$ The given statement is equivalent

Well-formed Formula (WFF)

A mathematical expressions are represented in well defined form using parenthesis (( ), brackets [ ], square brackets { } etc. An logic statement which is a collection of variables or statements is called a well defined formula or Well formed formula (WFF)

1, If 'P' is a statement or variable, then it is WFF.

2, If P is WFF then $\sim P$ is also WFF

3, If α & β are two statements which are WFF

$\alpha \wedge \beta$; $\alpha \vee \beta$; $\alpha \rightarrow \beta$; $\alpha \leftrightarrow \beta$ are also WFF.

Eg: i, $\sim(p \vee q)$ is WFF.

ii, $\sim(p \vee q) \Leftrightarrow \sim p$ is WFF.

iii, $[\sim p \wedge q] \rightarrow \sim p \vee q$ is not a WFF

Principle of duality or Duality law:-

Two statements are said to be duals of each other if either one can be obtained from other by replacing and ($\wedge$) by or ($\vee$) [by replacing "$\wedge$" by "$\vee$"] and "$\vee$" by "$\wedge$". The connectives $\wedge$ & $\vee$ is called duals of each other.

If the statement A contains a special variable "T or F" then its dual is obtained by replacing "T by F" & "F by T".

Eg: i, $(p \vee q) \wedge r$ then the dual is $(p \wedge q) \vee r$

ii, $(p \wedge q) \vee r$ then the dual is $(p \vee q) \wedge r$

iii, $\sim(p \vee q) \wedge (p \vee \sim(q \wedge \sim r))$ then dual is

$\sim(p \wedge q) \vee (p \wedge \sim(q \vee \sim r))$

iv, $(p \wedge q) \vee T$ then dual is $(p \vee q) \wedge F$

Converse, Inverse and contrapositive

1, The converse of the implication $p \rightarrow q$ is $q \rightarrow p$

2, The inverse of the implication $p \rightarrow q$ is $\sim p \rightarrow \sim q$

3, The contrapositive of the implication $p \rightarrow q$ is $\sim q \rightarrow \sim p$

16) Show that $p \leftrightarrow q \equiv (p \to q) \wedge (q \to p)$

A) Given two statements p & q. To show that given statement is equivalent

p	q	$p \leftrightarrow q$	(a) $p \to q$	(b) $q \to p$	$a \wedge b$
T	T	T	T	T	T
T	F	F	F	T	F
F	T	F	T	F	F
F	F	T	T	T	T

$\equiv$

∴ The given statement is equivalent

17) Show that $p \vee (q \wedge r) \equiv (p \vee q) \wedge (q \vee r)$

A) Given 3 statements p & q & r. To show that given statement is equivalent

p	q	r	$q \wedge r$	$p \vee (q \wedge r)$	(a) $p \vee q$	(b) $q \vee r$	$a \wedge b$
T	T	T	T	T	T	T	T
T	T	F	F	T	T	T	T
T	F	T	F	T	T	T	T
T	F	F	F	T	T	F	F
F	T	T	T	T	T	T	T
F	T	F	F	F	T	T	T
F	F	T	F	F	F	T	F
F	F	F	F	F	F	F	F

$\not\equiv$ Not equivalent

18) Show that

$p \leftrightarrow q \equiv (p \vee q) \to (p \wedge q)$

p	q	$p \leftrightarrow q$	(a) $p \vee q$	(b) $p \wedge q$	$a \to b$
T	T	T	T	T	T
T	F	F	T	F	F
F	T	F	T	F	F
F	F	T	F	F	T

$\equiv$ (equivalent)

19) Show that

$\sim(p \leftrightarrow q) \equiv p \leftrightarrow \sim q \equiv \sim p \leftrightarrow q$

p	q	$p \leftrightarrow q$	$\sim(p \leftrightarrow q)$	$\sim q$	$p \leftrightarrow \sim q$	$\sim p$	$\sim p \leftrightarrow q$
T	T	T	F	F	F	F	F
T	F	F	T	T	T	F	T
F	T	F	T	F	T	T	T
F	F	T	F	T	F	T	F

$\equiv$

20) Show that $\sim(p \to q) \equiv p \wedge \sim q$

p	q	$p \to q$	$\sim(p \to q)$	$\sim q$	$p \wedge \sim q$
T	T	T	F	F	F
T	F	F	T	T	T
F	T	T	F	F	F
F	F	T	F	T	F

$\therefore$ It is a equivalent $\equiv$

21) Show that $\sim(p \vee q) \equiv (\sim p) \wedge (\sim q)$

p	q	(a) $\sim p$	(b) $\sim q$	$p \vee q$	$\sim(p \vee q)$	$a \wedge b$
T	T	F	F	T	F	F
T	F	F	T	T	F	F
F	T	T	F	T	F	F
F	F	T	T	F	T	T

$\therefore$ The given statement is equivalent $\equiv$

20) Show that $\sim(p \to q) \equiv p \wedge \sim q$

p	q	$p \to q$	$\sim(p \to q)$	$\sim q$	$p \wedge \sim q$
T	T	T	F	F	F
T	F	F	T	T	T
F	T	T	F	F	F
F	F	T	F	T	F

$\therefore$ It is a equivalent $\equiv$

21) show that $\sim(p \vee q) \equiv (\sim p) \wedge (\sim q)$

p	q	(a) $\sim p$	(b) $\sim q$	$p \vee q$	$\sim(p \vee q)$	$a \wedge b$
T	T	F	F	T	F	F
T	F	F	T	T	F	F
F	T	T	F	T	F	F
F	F	T	T	F	T	T

$\therefore$ The given statement is equivalent $\equiv$

Statement formulas:-

let p, q are two statements then the following properties are used to define the truth tables.

1) Idempotent law:-

$p \wedge p = p$

$p \vee p = p$

2) Associative law:-

$(p \wedge q) \wedge r = p \wedge (q \wedge r)$

$(p \vee q) \vee r = p \vee (q \vee r)$

3) Commutative law:-

$p \wedge q = q \wedge p$

$p \vee q = q \vee p$

4) Demorgan's law:-

$\sim(p \wedge q) = (\sim p) \vee (\sim q)$

$\sim(p \vee q) = (\sim p) \wedge (\sim q)$

5) Absorption law:-

$p \wedge (p \vee q) = p$

$q \wedge (p \vee q) = q$

$p \vee (p \wedge q) = p$

6) Double Negation:-

$\sim(\sim p) = p$

$\sim(\sim q) = q$

Normal forms:-

The problem of determining whether the given statement formula is tautology or contradiction, is called a "decision problem".

The word product in place of conjuction and sum in the place of disjunction.

A product of variables and their negations in a formula is called an Elementary products.

Sum of variables and their negations in a formula is called Elementary sum.

$P \wedge q$, $q \vee \sim q$, $p \wedge \sim p$, $\sim q \wedge q$ are elementary products.

$p \vee q$, $p \vee \sim q$, $\sim p \vee q$, $\sim p \vee \sim q$ are elementary sums.

There are two types of normal forms

1, Disjunctive Normal form (DNF)

2, Conjunctive Normal form (CNF)

1, **Disjunctive Normal form (DNF):-**

A formula which is equivalent to a given statement and which consists of sum of elementary products is called DNF.

2, **Conjunctive Normal form (CNF):-**

A formula which is equivalent to a given statement and which consists of a product of elementary sum is called CNF.

Procedure:-

1, If the connectives $\rightarrow$ and $\leftrightarrow$ are present in the given formula, they are replaced by $\wedge, \vee, \sim$ ....

Eg: $P \rightarrow q$ is replaced by $\sim P \vee q$

(or)

$\sim P \wedge q$

2, If the given expression consists of '$\sim$' before the given formula; demorgans law is applies

3, If necessary the distributive law, Idempotent law, absorption laws are applied

Note:-

$p \wedge q = p$, $p \vee q = p$, $p \vee F = F$, $\sim p \wedge p = F$, $\sim q \wedge q = F$

$p \vee \sim p = T$, $q \vee \sim q = T$, $p \wedge T = p$, $p \vee T = T$, $p \wedge F = F$,

$p \vee F = p$.

22) Find the DNF of the statement $\sim(\sim(p \leftrightarrow q) \wedge r)$

A) Given statement is $\sim(\sim(p \leftrightarrow q) \wedge r)$

$\sim(\sim(p \leftrightarrow q) \wedge r)$

$\sim[\sim(p \wedge q) \vee (\sim p \wedge \sim q) \wedge r]$

$\sim[\sim(p \wedge q) \wedge \sim(\sim p \wedge \sim q) \wedge r]$ ($\because$ by using double negation)

$\sim[(\sim p \vee \sim q) \wedge (p \vee q) \wedge r]$

$[p \wedge q] \vee [\sim(p \vee q) \vee \sim r]$

$[p \wedge q] \vee [\sim(p \vee q) \vee \sim r]$

23) Find the DNF of the statement $p \vee (\sim p \rightarrow [q \vee (q \rightarrow \sim r)])$

Principle disjunctive Normal form (PDNF)

Min terms:- (True values) (all in table)

For a given number of variables the min terms consist of conjunctions, in which each statement variable (or) its negation but not both appears only once.

let p & q be the two statement variables then there will be 2^2 min terms given by $p \wedge q$, $p \wedge \sim q$, $\sim p \wedge q$, $\sim p \wedge \sim q$.

PDNF

For a given formula consisting of disjunctions of min terms only is called PDNF. PDNF is also called as Sum of products of canonical form.

Principle conjuctive Normal form (PCNF)

Max terms:- (F values) (all in table)

For a given number of variables, the max terms consist of disjunctions in each variable or its negation but not both appears only once.

let p & q be the two statement variables then there are 2^2 max terms given by $p \vee q$, $p \vee \sim q$, $\sim p \vee q$, $\sim p \vee \sim q$.

PCNF

For a given formula an equivalent formula consisting of conjunctions of max terms only is called PCNF. PCNF is also called as product of sum of canonical form

24) Find PDNF for $(p \wedge q) \vee (\sim p \wedge r) \vee (q \wedge r)$

To find PDNF for 3 statements p, q, r then truth table is

p	q	r	(a) $p \wedge q$	$\sim p$	(b) $\sim p \wedge r$	(d) $q \wedge r$	(c) $a \vee b$	$c \vee d$	Min terms
T	T	T	T	F	F	T	T	T	$p \wedge q \wedge r$ ✓
T	T	F	T	F	F	F	T	T	$p \wedge q \wedge \sim r$ ✓
T	F	T	F	F	F	F	F	F	$p \wedge \sim q \wedge r$
T	F	F	F	F	F	F	F	F	$p \wedge \sim q \wedge \sim r$
F	T	T	F	T	T	T	T	T	$\sim p \wedge q \wedge r$ ✓
F	T	F	F	T	F	F	F	F	$\sim p \wedge q \wedge \sim r$
F	F	T	F	T	T	F	T	T	$\sim p \wedge \sim q \wedge r$ ✓
F	F	F	F	T	F	F	F	F	$\sim p \wedge \sim q \wedge \sim r$

for PDNF consider only T values

The PDNF for the given statement is
$(p \wedge q \wedge r) \vee (p \wedge q \wedge \sim r) \vee (\sim p \wedge q \wedge r) \vee (\sim p \wedge \sim q \wedge r)$

25) Find PCNF for $(\sim p \vee \sim q) \rightarrow (p \leftrightarrow \sim q)$

To find the PCNF for the given statement p, q, then the truth table is

p	q	~p	~q	~p∨~q (a)	p↔~q (b)	a→b	max terms
T	T	F	F	F	F	T	~p∨~q
T	F	F	T	T	T	T	~p∨q
F	T	T	F	T	T	T	p∨~q
F	F	T	T	T	F	F ✓	p∨q

The PCNF for the given statement is p∨q

26) Find the PDNF for $p \rightarrow q$

To find PDNF for given statement p, q then truth table is

p	q	p→q	min terms
T	T	T	$p \wedge q$ ✓
T	F	F	$p \wedge \sim q$
F	T	T	$\sim p \wedge q$ ✓
F	F	T	$\sim p \wedge \sim q$ ✓

The PDNF for given statement is
$(p \wedge q) \vee (\sim p \wedge q) \vee (\sim p \wedge \sim q)$

27) Find the PCNF for $q \vee (p \vee \sim q)$ and also find PDNF

p	q (a)	~q	p∨~q (b)	a∨b	max terms	min terms
T	T	F	T	T	$p \wedge q$	$\sim p \vee \sim q$
T	F	T	T	T	$p \wedge \sim q$	$\sim p \vee q$
F	T	F	F	T	$\sim p \wedge q$	$p \vee \sim q$
F	F	T	T	T	$\sim p \wedge \sim q$	$p \vee q$

The PDNF for the given statement is
$(p \wedge q) \vee (p \wedge \sim q) \vee (\sim p \wedge q) \wedge (\sim p \wedge \sim q)$

The PCNF for the given statement is :

28) Find PDNF and PCNF for $(p \wedge \sim q \wedge \sim r) \vee (q \wedge r)$

p	q	r	~q	(b) ~r	(a) p∧~q	c a∧b	d q∧r	c∨d	min terms	max terms
T	T	T	F	F	F	F	T	T	p∧q∧r	~p∨~q∨~r
T	T	F	F	T	F	F	F	F	p∧q∧~r	~p∨~q∨r
T	F	T	T	F	T	F	F	F	p∧~q∧r	~p∨q∨~r
T	F	F	T	T	T	T	F	T	p∧~q∧~r	~p∨q∨r
F	T	T	F	F	F	F	T	T	~p∧q∧r	p∨~q∨~r
F	T	F	F	T	F	F	F	F	~p∧q∧~r	p∨~q∨r
F	F	T	T	F	F	F	F	F	~p∧~q∧r	p∨q∨~r
F	F	F	T	T	F	F	F	F	~p∧~q∧~r	p∨q∨r

The PDNF for the given statement is
$(p \wedge q \wedge r) \vee (p \wedge \sim q \wedge \sim r) \vee (\sim p \wedge q \wedge r)$

The PCNF for the given statement is
$(\sim p \vee \sim q \vee r) \wedge (\sim p \vee q \vee \sim r) \wedge (p \vee \sim q \vee r) \wedge (p \vee q \vee \sim r) \wedge (p \vee q \vee r)$

29) Find PDNF and PCNF for $p \to ((p \to q) \wedge \sim(\sim q \vee \sim p))$

(d) p	q	r	~q	~p	(a) p→q	(b) ~q∨~p	(c) ~b	(e) a∧c	d→e	min terms	max terms
T	T	T	F	F	T	F	T	T	T		
T	T	F	F	F	T	F	T	T	T		
T	F	T	T	F	F	T	F	F	F		
T	F	F	T	F	F	T	F	F	T		
F	T	T	F	T	T	T	F	F	T		
F	T	F	F	T	T	T	F	F	T		
F	F	T	T	T	T	T	F	F	T		
F	F	F	T	T	T	T	F	F	T		

29) Find PDNF and PCNF for $p \to ((p \to q) \wedge \sim(\sim q \vee \sim p))$

(d) p	q	~p	~q	(b) p→q	(a) ~q∨~p	(c) ~a	(e) b∧c	d→e	min terms	max terms
T	T	F	F	T	F	T	T	T	p∧q	~p∨~q
T	F	F	T	F	T	F	F	F	p∧~q	~p∨q
F	T	T	F	T	T	F	F	T	~p∧q	p∨~q
F	F	T	T	T	T	F	F	T	~p∧~q	p∨q

The PDNF for given statement is
$(p \wedge q) \vee (p \wedge \sim q) \vee (\sim p \wedge q) \vee (\sim p \wedge \sim q)$

The PCNF for given statement is $(\sim p \vee q)$

30) $(p \wedge q) \vee (\sim p \wedge r) \vee (q \wedge r)$

p	q	r	$\sim p$	(a) $p \wedge q$	(b) $\sim p \wedge r$	(d) $q \wedge r$	(c) $a \vee b$	$c \vee d$	min terms	max terms
T	T	T	F	T	F	T	T	T	$p \wedge q \wedge r$	$\sim p \vee \sim q \vee \sim r$
T	T	F	F	T	F	F	T	T	$p \wedge q \wedge \sim r$	$\sim p \vee \sim q \vee r$
T	F	T	F	F	F	F	F	F	$p \wedge \sim q \wedge r$	$\sim p \vee q \vee \sim r$
T	F	F	F	F	F	F	F	F	$p \wedge \sim q \wedge \sim r$	$\sim p \vee q \vee r$
F	T	T	T	F	T	T	T	T	$\sim p \wedge q \wedge r$	$p \vee \sim q \vee \sim r$
F	T	F	T	F	F	F	F	F	$\sim p \wedge q \wedge \sim r$	$p \vee q \vee \sim r$
F	F	T	T	F	T	F	T	T	$\sim p \wedge \sim q \wedge r$	$p \vee q \vee \sim r$
F	F	F	T	F	F	F	F	F	$\sim p \wedge \sim q \wedge \sim r$	$p \vee q \vee r$

The PDNF for given statement is
$(p \wedge q \wedge r) \vee (p \wedge q \wedge \sim r) \vee (\sim p \wedge q \wedge r) \vee (\sim p \wedge \sim q \wedge r)$

The PCNF for given statement is
$(\sim p \vee q \vee \sim r) \wedge (\sim p \vee q \vee r) \wedge (p \vee q \vee \sim r) \wedge (p \vee q \vee r)$

31) Find PDNF and PCNF for $(\sim p \vee \sim q) \rightarrow (p \leftrightarrow \sim p)$

p	q	$\sim p$	$\sim q$	(a) $\sim p \vee \sim q$	(b) $p \leftrightarrow \sim p$	$a \rightarrow b$	min terms	max terms
T	T	F	F	F	F	T	$p \wedge q$	$\sim p \vee \sim q$
T	F	F	T	T	F	F	$p \wedge \sim q$	$\sim p \vee q$
F	T	T	F	T	F	F	$\sim p \wedge q$	$p \vee \sim q$
F	F	T	T	T	F	F	$\sim p \wedge \sim q$	$p \vee q$

The PDNF $= (p \wedge q)$

PCNF $= (\sim p \vee q) \wedge (p \vee \sim q) \wedge (p \vee q)$

32) Find PDNF and PCNF for $(p \wedge q) \vee (\sim p \wedge q) \vee (q \wedge r)$

p	q	r	$\sim p$	(a) $p \wedge q$	(b) $\sim p \wedge q$	(d) $q \wedge r$	(c) $a \vee b$	$c \vee d$	min terms	max terms
T	T	T	F	T	F	T	T	T	$p \wedge q \wedge r$	$\sim p \vee \sim q \vee \sim r$
T	T	F	F	T	F	F	T	T	$p \wedge q \wedge \sim r$	$\sim p \vee \sim q \vee r$
T	F	T	F	F	F	F	F	F	$p \wedge \sim q \wedge r$	$\sim p \vee q \vee \sim r$
T	F	F	F	F	F	F	F	F	$p \wedge \sim q \wedge \sim r$	$\sim p \vee q \vee r$
F	T	T	T	F	T	T	T	T	$\sim p \wedge q \wedge r$	$p \vee \sim q \vee \sim r$
F	T	F	T	F	T	F	T	T	$\sim p \wedge q \wedge \sim r$	$p \vee q \vee \sim r$
F	F	T	T	F	F	F	F	F	$\sim p \wedge \sim q \wedge r$	$p \vee q \vee \sim r$
F	F	F	T	F	F	F	F	F	$\sim p \wedge \sim q \wedge \sim r$	$p \vee q \vee r$

PDNF $= (p \wedge q \wedge r) \vee (p \wedge q \wedge \sim r) \vee (\sim p \wedge q \wedge r) \vee (\sim p \wedge q \wedge \sim r)$

PCNF $= (\sim p \vee q \vee \sim r) \wedge (\sim p \vee q \vee r) \wedge (p \vee q \vee \sim r) \wedge (p \vee q \vee r)$

33) Find PDNF and PCNF for $p \wedge \sim(q \wedge r) \vee (p \to q)$

(a) p	q	r	$q \wedge r$	(b) $\sim(q \wedge r)$	(c) $a \wedge b$	(d) $p \to q$	$c \vee d$	min terms	max terms
T	T	T	T	F	F	T	T	$p \wedge q \wedge r$	$\sim p \vee \sim q \vee \sim r$
T	T	F	F	T	T	T	T	$p \wedge q \wedge \sim r$	$\sim p \vee \sim q \vee r$
T	F	T	F	T	T	F	T	$p \wedge \sim q \wedge r$	$\sim p \vee q \vee \sim r$
T	F	F	F	T	T	F	T	$p \wedge \sim q \wedge \sim r$	$\sim p \vee q \vee r$
F	T	T	T	F	F	T	T	$\sim p \wedge q \wedge r$	$p \vee \sim q \vee \sim r$
F	T	F	F	T	F	T	T	$\sim p \wedge q \wedge \sim r$	$p \vee q \vee \sim r$
F	F	T	F	T	F	T	T	$\sim p \wedge \sim q \wedge r$	$p \vee q \vee \sim r$
F	F	F	F	T	F	T	T	$\sim p \wedge \sim q \wedge \sim r$	$p \vee q \vee r$

The PDNF for the given statement is

$(p \wedge q \wedge r) \vee (p \wedge q \wedge \sim r) \vee (p \wedge \sim q \wedge r) \vee (p \wedge \sim q \wedge \sim r) \vee (\sim p \wedge q \wedge r) \vee$
$(\sim p \wedge q \wedge \sim r) \vee (\sim p \wedge \sim q \wedge r) \vee (\sim p \wedge \sim q \wedge \sim r)$

The PCNF is :-

34) Find PDNF and PCNF for $(p \to r) \wedge (q \leftrightarrow p)$

p	q	r	(a) $p \to r$	(b) $q \to p$	$a \wedge b$	min terms	max terms
T	T	T	T	T	T	$p \wedge q \wedge r$	$\sim p \vee \sim q \vee \sim r$
T	T	F	F	T	F	$p \wedge q \wedge \sim r$	$\sim p \vee \sim q \vee r$
T	F	T	T	F	F	$p \wedge \sim q \wedge r$	$\sim p \vee q \vee \sim r$
T	F	F	F	F	F	$p \wedge \sim q \wedge \sim r$	$\sim p \vee q \vee r$
F	T	T	T	F	F	$\sim p \wedge q \wedge r$	$p \vee \sim q \vee \sim r$
F	T	F	T	F	F	$\sim p \wedge q \wedge \sim r$	$p \vee q \vee \sim r$
F	F	T	T	T	T	$\sim p \wedge \sim q \wedge r$	$p \vee q \vee \sim r$
F	F	F	T	T	T	$\sim p \wedge \sim q \wedge \sim r$	$p \vee q \vee r$

PDNF $= (p \wedge q \wedge r) \vee (\sim p \wedge \sim q \wedge r) \vee (\sim p \wedge \sim q \wedge \sim r)$

PCNF $= (\sim p \vee \sim q \vee \sim r) \wedge (\sim p \vee q \vee \sim r) \wedge (\sim p \vee q \vee r) \wedge (p \vee \sim q \vee \sim r)$
$\wedge (p \vee q \vee \sim r)$

35) Find PDNF and PCNF for $P \vee (\neg P \leftrightarrow (q \vee (\neg q \rightarrow r)))$

(e) P	(a) q	r	(c) $\neg P$	$\neg q$	(b) $\neg q \rightarrow r$	(d) $a \vee b$	(f) $c \rightarrow d$	e∧f	min terms	max terms
T	T	T	F	F	T	T	T	T	$P \wedge q \wedge r$	$\neg P \vee \neg q \vee \neg r$
T	T	F	F	F	T	T	T	T	$P \wedge q \wedge \neg r$	$\neg P \vee \neg q \vee r$
T	F	T	F	T	T	T	T	T	$P \wedge \neg q \wedge r$	$\neg P \vee q \vee \neg r$
T	F	F	F	T	F	F	T	T	$P \wedge \neg q \wedge \neg r$	$\neg P \vee q \vee r$
F	T	T	T	F	T	T	T	T	$\neg P \wedge q \wedge r$	$P \vee \neg q \vee \neg r$
F	T	F	T	F	T	T	T	T	$\neg P \wedge q \wedge \neg r$	$P \vee \neg q \vee r$
F	F	T	T	T	T	T	T	T	$\neg P \wedge \neg q \wedge r$	$P \vee q \vee \neg r$
F	F	F	T	T	F	F	F	F	$\neg P \wedge \neg q \wedge \neg r$	$P \vee q \vee r$

PDNF $= (P \wedge q \wedge r) \vee (P \wedge q \wedge \neg r) \vee (P \wedge \neg q \wedge r) \vee (P \wedge \neg q \wedge \neg r) \vee$
$(\neg P \wedge q \wedge r) \vee (\neg P \wedge q \wedge \neg r) \vee (\neg P \wedge \neg q \wedge r)$

PCNF $= P \vee q \vee r$

36) Find PDNF and PCNF for $((\neg q \vee \neg r) \leftrightarrow \neg P) \wedge ((q \vee r) \rightarrow P)$

(d) P	q	r	(b) $\neg P$	$\neg q$	$\neg r$	(a) $\neg q \vee \neg r$	(e) $a \rightarrow b$	(c) $q \vee r$	(f) $c \rightarrow d$	e∧f	min terms	max terms
T	T	T	F	F	F	F	T	T	T	T	$P \wedge q \wedge r$	$\neg P \vee \neg q \vee \neg r$
T	T	F	F	F	T	T	F	T	T	F	$P \wedge q \wedge \neg r$	$\neg P \vee \neg q \vee r$
T	F	T	F	T	F	T	F	T	T	F	$P \wedge \neg q \wedge r$	$\neg P \vee q \vee \neg r$
T	F	F	F	T	T	T	F	F	T	F	$P \wedge \neg q \wedge \neg r$	$\neg P \vee q \vee r$
F	T	T	T	F	F	F	T	T	F	F	$\neg P \wedge q \wedge r$	$P \vee \neg q \vee \neg r$
F	T	F	T	F	T	T	T	T	F	F	$\neg P \wedge q \wedge \neg r$	$P \vee \neg q \vee r$
F	F	T	T	T	F	T	T	T	F	F	$\neg P \wedge \neg q \wedge r$	$P \vee q \vee \neg r$
F	F	F	T	T	T	T	T	F	T	T	$\neg P \wedge \neg q \wedge \neg r$	$P \vee q \vee r$

PDNF $= (P \wedge q \wedge r) \vee (\neg P \wedge \neg q \wedge \neg r)$

PCNF $=$ ~~[illegible]~~ $\wedge$
$= (\neg P \vee \neg q \vee r) \wedge (\neg P \vee q \vee \neg r) \wedge (\neg P \vee q \vee r) \wedge (P \vee \neg q \vee \neg r)$
$\wedge (P \vee \neg q \vee r) \wedge (P \vee q \vee \neg r)$

38) Find DNF for $\sim(p \to (q \wedge r))$

A) Given $\sim(\underbrace{p}_{p} \to \underbrace{(q \wedge r)}_{q})$ $\quad (\because p \to q = \sim p \vee q)$

$\sim(\sim p \vee q \wedge r)$

$p \wedge \sim(q \wedge r)$

$p \wedge (\sim q \vee \sim r)$ $\quad$ ($\because$ Apply demorgans law)

$(p \wedge \sim q) \vee (p \wedge \sim r)$

39) Find CNF for $\sim[(p \vee \sim q) \wedge \sim r]$

A) Given $\sim[(p \vee \sim q) \wedge \sim r]$

$\sim[(p \wedge \sim r) \vee (\sim q \wedge \sim r)]$

$\sim(p \wedge \sim r) \wedge \sim(\sim q \wedge \sim r)$

$(\sim p \vee r) \wedge (q \vee r)$

Find PDNF and PCNF without Truth table

PDNF :- ($\wedge$)

1) $\wedge \vee \wedge$ (It is in this form)

2) $p \to q = \sim p \vee q$

3) $p \leftrightarrow q = (\sim p \vee q) \wedge (p \vee \sim q)$

40) Find PDNF $(\sim p \vee \sim q) \to (p \leftrightarrow q)$

To find PDNF without truth table

Given statement is

$(\sim p \vee \sim q) \to (p \leftrightarrow q)$ $\quad [\because a \to b = \sim a \vee b]$

$\sim(\sim p \vee \sim q) \vee (p \leftrightarrow q)$

$\sim(\sim p \vee \sim q) \vee [(\sim p \vee q) \wedge (\sim q \vee p)]$

$(p \wedge q) \vee [(\sim p \vee q) \wedge (\sim q \vee p)]$

$\wedge \quad \vee \quad \wedge$

41) Find PDNF $(p \wedge q) \vee (\sim p \wedge q) \vee (q \wedge r)$

A) To find PDNF without truth table

Given statement is

$(p \wedge q) \vee (\sim p \wedge q) \vee (q \wedge r)$

$(p \wedge q) \vee (\sim p \wedge q) \vee (q \wedge r)$ p Value so T but processed:

$(p \wedge q \wedge T) \vee (\sim p \wedge q \wedge T) \vee (q \wedge r \wedge T)$

$(p \wedge q \wedge (r \vee \sim r)) \vee (\sim p \wedge q \wedge (r \vee \sim r)) \vee (q \wedge r \wedge (p \vee \sim p))$

$\Rightarrow (p \wedge q \wedge r) \vee (p \wedge q \wedge \sim r) \vee (\sim p \wedge q \wedge r) \vee (\sim p \wedge q \wedge \sim r)$

$\vee \underbrace{(q \wedge r \wedge p)}_{\text{(repeated)}} \vee (q \wedge r \wedge \sim p)$

$\Rightarrow (p \wedge q \wedge r) \vee (p \wedge q \wedge \sim r) \vee (\sim p \wedge q \wedge r) \vee (\sim p \wedge q \wedge \sim r)$

$\rightarrow \wedge \vee \wedge \vee \wedge \vee \wedge$

CNF and DNF H.W problms

42) Find CNF for $(p \wedge \sim(q \wedge r)) \vee (p \rightarrow q)$

A) Given $(p \wedge \sim(q \wedge r)) \vee (p \rightarrow q)$

$(p \wedge \sim(q \wedge r)) \vee (\sim p \vee q)$

$(p \wedge (\sim q \vee \sim r)) \vee (\sim p \vee q)$

$(p \vee (\sim p \vee q)) \wedge ((\sim q \vee \sim r) \vee (\sim p \vee q))$

$(p \vee q) \wedge ((\sim q \vee \sim r) \vee (\sim p \vee q))$

$\vee \wedge \vee \rightarrow$ CNF

43) Find DNF for $p \wedge (p \rightarrow q)$

A) Given $p \wedge (p \rightarrow q)$

$p \wedge (\sim p \vee q)$

$(p \wedge \sim p) \vee (p \wedge q)$

$\wedge \vee \wedge \rightarrow$ DNF

44) Find DNF for $p \wedge (q \rightarrow q)$

A) Given $p \wedge (q \rightarrow q)$

$p \wedge (\sim q \vee q)$

$(p \wedge \sim q) \vee (p \wedge q)$

$\wedge \vee \wedge \rightarrow$ DNF

45) Find DNF for $\sim(p \rightarrow (q \wedge r))$

A) Given $\sim(p \rightarrow (q \wedge r))$

$\sim(\sim p \vee (q \wedge r))$

$p \wedge \sim(q \wedge r)$

$p \wedge (\sim q \vee \sim r)$

$(p \wedge \sim q) \vee (p \wedge \sim r)$ $(\wedge \vee \wedge) \rightarrow$ DNF

46) Find CNF for $P \to ((P \to q) \wedge \sim(\sim q \vee \sim p))$

A) Given $P \to ((P \to q) \wedge \underline{\sim(\sim q \vee \sim p)})$

$P \to ((P \to q) \wedge (q \wedge p))$

$\sim P \vee ((P \to q) \wedge (q \wedge p))$

$(\sim P \vee (P \to q)) \wedge (\sim P \vee (q \wedge p))$

$(\underline{\sim P \vee (\sim P \vee q)}) \wedge ((\sim P \vee q) \wedge (\sim P \vee p))$

$(\sim P \vee q) \wedge ((\sim P \vee q) \wedge T)$

$(\sim P \vee q) \wedge (\sim P \vee q)$ [$\because \vee \wedge \vee \to$ CNF]

$\to \sim P \vee q$ → Same terms so write one

47) Find DNF for $\sim(P \to (q \wedge r))$

A) Given $\sim(P \to (q \wedge r))$

$\sim(\sim P \vee (q \wedge r))$

$P \wedge \sim(q \wedge r)$

$P \wedge (\sim q \vee \sim r)$

$(P \wedge \sim q) \vee (P \wedge \sim r)$

$\wedge \vee \wedge \to$ DNF

48) Find CNF for $\sim(P \to (q \wedge r))$

A) Given $\sim(P \to (q \wedge r))$

$\sim(\sim P \vee (q \wedge r))$

$P \wedge \sim(q \wedge r)$

$P \wedge (\sim q \vee \sim r) \to$ CNF

$\to (P \vee P) \wedge (\sim q \vee \sim r) \to \vee \wedge \vee$

49) Find DNF for $(\sim P \to r) \wedge (P \to q)$

A) Given $(\sim P \to r) \wedge (P \to q)$

$(\sim(\sim P) \vee r) \wedge (\sim P \vee q)$

$(P \vee r) \wedge (\sim P \vee q)$

$(P \wedge (\sim P \vee q)) \vee (r \wedge (\sim P \vee q))$

$(\underbrace{P \wedge \sim P}_{F}) \vee (P \wedge q) \vee (r \wedge \sim P) \vee (r \wedge q)$

$\to F \vee (P \wedge q) \vee (r \wedge \sim P) \vee (r \wedge q)$ [$\because$ Remove F]

$\to (P \wedge q) \vee (r \wedge \sim P) \vee (r \wedge q)$

$\to \wedge \vee \wedge \vee \wedge \to$ DNF

50) Find CNF for $(q \vee (p \wedge r)) \wedge \sim((p \vee r) \wedge q)$

A) Given $(q \vee (p \wedge r)) \wedge \sim((p \vee r) \wedge q)$

$\Rightarrow (q \vee (p \wedge r)) \wedge \underline{\sim((p \vee r) \wedge q)}$

$\Rightarrow (q \vee (p \wedge r)) \wedge (\underline{\sim(p \vee r)} \vee \sim q)$

$\Rightarrow (q \vee (p \wedge r)) \wedge (\underline{(\sim p \wedge \sim r) \vee \sim q})$

$\Rightarrow (\underline{q \vee (p \wedge r)}) \wedge ((\sim p \vee \sim q) \wedge (\sim r \vee \sim q))$

$\Rightarrow ((q \vee p) \wedge (q \vee r)) \wedge ((\sim p \vee \sim q) \wedge (\sim r \vee \sim q))$

$(q \vee p) \wedge (q \vee r) \wedge (\sim p \vee \sim q) \wedge (\sim r \vee \sim q)$

$\vee \wedge \vee \wedge \vee \wedge \vee \Rightarrow$ CNF

(51) doubt Find DNF for $p \vee (\sim p \rightarrow (q \vee (q \rightarrow \sim r)))$

A) Given $p \vee (\sim p \rightarrow (q \vee (\underline{q \rightarrow \sim r})))$

$p \vee (\underline{\sim p \rightarrow (q \vee (\sim q \vee \sim r))})$

$p \vee (\underline{\sim(\sim p)} \vee (q \vee (\sim q \vee \sim r)))$

$p \vee (p \vee ((q \vee \sim q) \vee (q \vee \sim r)))$

$p \vee (p \vee (q \vee (q \vee \sim r))$

$p \vee (p \vee (\underbrace{(q \vee q)}_{T} \vee (q \vee \sim r))$

$p \vee (p \vee (T \vee (q \vee \sim r)))$ ($\because$ Remove T)

$\underline{p \vee (p \vee (q \vee \sim r))}$

$(\underbrace{p \vee p}_{T}) \vee (p \vee (q \vee \sim r))$

$T \vee (p \vee (q \vee \sim r))$ ($\because$ Remove T)

$p \vee (q \vee \sim r)$

$(p \wedge p) \vee (q \vee \sim r)$

$(p \wedge p) \vee ((q \wedge q) \vee (\sim r \wedge \sim r))$

$(p \wedge p) \vee (q \wedge q) \vee \underbrace{(\sim r \wedge \sim r)}_{F}$

$\wedge \vee \wedge \vee \wedge \Rightarrow$ DNF

$\left[\begin{array}{l} \because p = p \wedge p \\ \sim r \wedge \sim r = \sim r \\ q \wedge q = q \end{array}\right.$

52) Find DNF for $p \to ((p \to q) \wedge \sim(\sim q \vee \sim p))$

A) Given $p \to ((p \to q) \wedge \sim(\sim q \vee \sim p))$

$p \to ((\underline{p \to q}) \wedge \sim(\sim q \vee \sim p))$

$p \to (\underline{(\sim p \vee q) \wedge (q \wedge p)})$

$p \to ((\sim p \wedge (q \wedge p)) \vee (q \wedge (q \wedge p)))$

$p \to ((\sim p \wedge q \wedge p) \vee (q \wedge q \wedge p))$

$p \to ((\sim p \wedge p \wedge q) \vee (q \wedge q \wedge p))$

$p \to ((F \wedge q) \vee (q \wedge p))$ $[\because F \wedge q = F]$

$p \to (F \vee (q \wedge p))$ $[\because F \vee (\text{anything}) = \text{anything}]$

$p \to (q \wedge p)$

$\sim p \vee (q \wedge p)$

$(\sim p \wedge \sim p) \vee (q \wedge p)$ $[\because \sim p$ can also be written as $\sim p \wedge \sim p]$

$\wedge \vee \wedge \Rightarrow$ DNF

53) Find DNF for $(q \vee (p \wedge r)) \wedge \sim((p \vee r) \wedge q)$

A) Given $(q \vee (p \wedge r)) \wedge \underline{\sim((p \vee r) \wedge q)}$

$\Rightarrow (q \vee (p \wedge r)) \wedge (\underline{\sim(p \vee r)} \vee \sim q)$

$\Rightarrow \underbrace{(q \vee (p \wedge r)) \wedge ((\sim p \wedge \sim r) \vee \sim q)}_{\text{multiplying each term}}$

$\Rightarrow [q \wedge (\sim p \wedge \sim r) \vee (\underline{q \wedge \sim q})] \vee [(p \wedge r) \wedge (\sim p \wedge \sim r) \vee (p \wedge r \wedge \sim$

$\Rightarrow (\underline{q \wedge (\sim p \wedge \sim r) \vee F}) \vee (p \wedge r \wedge (\sim p \wedge \sim r) \vee (p \wedge r \wedge \sim q))$

$\Rightarrow (q \wedge (\sim p \wedge \sim r) \vee (p \wedge r \wedge (\sim p \wedge \sim r) \vee (p \wedge \sim q \wedge r))$

$\Rightarrow (q \wedge \sim p \wedge \sim r) \vee (p \wedge r \wedge \sim p \wedge \sim r) \vee (p \wedge \sim q \wedge r)$

$\Rightarrow (\sim p \wedge q \wedge \sim r) \vee (\underline{p \wedge \sim p} \underline{\wedge r \wedge \sim r}) \vee (p \wedge \sim q \wedge r)$

$\Rightarrow (\sim p \wedge q \wedge \sim r) \vee (F \wedge F) \vee (p \wedge \sim q \wedge r)$

$\Rightarrow (\sim p \wedge q \wedge \sim r) \vee \underbrace{F \vee (p \wedge \sim q \wedge r)}$

$\Rightarrow (\sim p \wedge q \wedge \sim r) \vee (p \wedge \sim q \wedge r)$

$\wedge \vee \wedge$

$\Rightarrow$ DNF

5b) Find DNF for $\sim(p \vee q) \leftrightarrow (p \wedge q)$

A) Given $\underbrace{\sim(p \vee q)}_{r} \leftrightarrow \underbrace{(p \wedge q)}_{s}$

wkt: $r \leftrightarrow s = (r \to s) \wedge (s \to r)$

$\therefore$ above ques. becomes

$$(\underbrace{\sim(p \vee q)}_{p} \to \underbrace{(p \wedge q)}_{q}) \wedge (\underbrace{(p \wedge q)}_{p} \to \underbrace{\sim(p \vee q)}_{q})$$

$[\because p \to q = \sim p \vee q]$

$$(\sim(\sim(p \vee q) \vee (p \wedge q))) \wedge (\sim(p \wedge q) \vee \sim(p \vee q))$$

$$(\sim(\sim p \wedge \sim q) \vee (p \wedge q)) \wedge ((\sim p \vee \sim q) \vee (\sim p \wedge \sim q))$$

$$((p \vee q) \vee (p \wedge q)) \wedge ((\sim p \vee \sim q) \vee (\sim p \wedge \sim q))$$

$$\Rightarrow ((p \vee q) \wedge (\sim p \vee \sim q) \vee (p \vee q) \wedge (\sim p \wedge \sim q)) \vee ((p \wedge q) \wedge (\sim p \vee \sim q) \vee ((p \wedge q) \wedge (\sim p \wedge \sim q))$$

$$\Rightarrow ((\underline{p \vee q) \wedge \sim p}) \vee ((\underline{p \vee q \wedge \sim q}) \vee (\underline{p \vee q) \wedge \sim p}) \wedge (\underline{p \vee q) \wedge \sim q})) \vee ((\underline{p \wedge q) \wedge \sim p}) \vee ((\underline{p \wedge q) \wedge \sim q}) \vee ((\underline{p \wedge q) \wedge \sim p}) \wedge ((\underline{p \wedge q) \wedge \sim q})$$

($\because$ repetitions are there)

$$\Rightarrow ((p \vee q) \wedge \sim p) \vee ((p \vee q) \wedge \sim q) \vee ((p \wedge q) \wedge \sim p) \vee ((p \wedge q) \wedge \sim q)$$

$$\Rightarrow ((p \wedge \sim p) \vee (q \wedge \sim p)) \vee ((p \wedge \sim q) \vee (q \wedge \sim q)) \vee ((p \wedge \sim p) \wedge (q \wedge \sim p)) \vee ((p \wedge \sim q) \wedge (q \wedge \sim q))$$

$$\Rightarrow (F \vee (q \wedge \sim p)) \vee ((p \wedge \sim q) \vee F) \vee (F \wedge (q \wedge \sim p)) \vee ((p \wedge \sim q) \wedge F)$$

$$\Rightarrow (q \wedge \sim p) \vee (p \wedge \sim q) \vee F \vee F \vee F$$

$$\Rightarrow (q \wedge \sim p) \vee (p \wedge \sim q) \vee F$$

$$\Rightarrow (q \wedge \sim p) \vee (p \wedge \sim q)$$

$$\Rightarrow (\sim p \wedge q) \vee (p \wedge \sim q)$$

$\wedge \vee \wedge$

$\Rightarrow$ DNF

55) Find CNF for $\sim(p \vee q) \leftrightarrow (p \wedge q)$

A) Given $\sim(p \vee q) \leftrightarrow (p \wedge q)$

we already know that $p \leftrightarrow q = (p \to q) \wedge (q \to p)$.

let $p = \sim(p \vee q)$ and $q = p \wedge q$.

$\therefore \Rightarrow (\sim(p \vee q) \to (p \wedge q)) \wedge ((p \wedge q) \to \sim(p \vee q))$

$\Rightarrow ((\sim p \wedge \sim q) \to (p \wedge q)) \wedge ((p \wedge q) \to (\sim p \wedge \sim q))$

$\Rightarrow [\sim(\sim p \wedge \sim q) \vee (p \wedge q)] \wedge [(\sim(p \wedge q) \vee (\sim p \wedge \sim q))]$

$\Rightarrow ((p \vee q) \vee (p \wedge q)) \wedge ((\sim p \vee \sim q) \vee (\sim p \wedge \sim q))]$

$\Rightarrow (p \vee (p \vee q) \wedge q \vee (p \vee q)) \wedge ((\sim p \vee (\sim p \vee \sim q) \wedge (\sim q \vee (\sim p \vee \sim q))$

$\Rightarrow ((p \vee p \vee q) \wedge (q \vee p \vee q)) \wedge ((\sim p \vee \sim p \vee \sim q) \wedge (\sim q \vee \sim p \vee \sim q)$

$\Rightarrow ((p \vee q) \wedge (q \vee p) \wedge (\sim p \vee \sim q) \wedge (\sim q \vee \sim p)$

$\Rightarrow (p \vee q) \wedge (p \vee q) \wedge (\sim p \vee \sim q) \wedge (\sim p \vee \sim q)$

$\vee \wedge \vee \wedge \vee \wedge \vee \Rightarrow$ CNF

56) using CNF show $q \vee (p \wedge \sim q) \vee (\sim p \wedge \sim q)$ is Tautology

A) Given $q \vee (p \wedge \sim q) \vee (\sim p \wedge \sim q)$

$\Rightarrow q \vee ((p \vee \sim p) \wedge \sim q)$ [$\because \sim q$ is common term]

$\Rightarrow q \vee (p \vee \sim p) \wedge (q \vee \sim q)$ [$\because p \vee \sim p = T$, $q \vee \sim q = T$]

$\Rightarrow q \vee T \wedge T$ [$\because T \wedge T = T$]

$\Rightarrow q \vee T$ [$\because q \vee T = T$]

$\Rightarrow T$

$\therefore q \vee (p \wedge \sim q) \vee (\sim p \vee \sim q)$ is a tautology

57) Find CNF of $A \to (B \wedge C)$

A) Given $A \to (B \wedge C)$

$\sim A \vee (B \wedge C)$

$(\sim A \vee B) \wedge (\sim A \vee C)$

$\vee \wedge \vee \Rightarrow$ CNF

58) Find CNF of $\sim(p \to q) \vee (r \to p)$

A) Given $\sim(p \to q) \vee (r \to p)$

$\sim(\sim p \vee q) \vee (\sim r \vee p)$

$(p \wedge \sim q) \vee (\sim r \vee p)$

$\Rightarrow (p \vee (\sim r \vee p)) \wedge (\sim q \vee (\sim r \vee p)$

$\Rightarrow (p \vee p \vee \sim r) \wedge (\sim q \vee \sim r \vee p)$

$\vee \wedge \vee$ is a CNF.

59) Find CNF for $P \wedge (P \to q)$.

A) Given $P \wedge (P \to q)$

$\Rightarrow P \wedge (\sim P \vee q)$

$\Rightarrow \cancel{(P \wedge \sim P) \wedge (P \wedge q)}$

$\Rightarrow P \wedge (\sim P \vee q)$ $[\because P \vee P = P]$

$\Rightarrow (P \vee P) \wedge (\sim P \vee q)$

$\vee \wedge \vee \Rightarrow$ CNF

59. Find CNF for $P \wedge (P \to q)$.

A) Given $P \wedge (P \to q)$

$\Rightarrow P \wedge (\sim P \vee q)$

$\Rightarrow (\cancel{P} \wedge \cancel{\sim P}) \wedge (P \wedge q)$

$\Rightarrow P \wedge (\sim P \vee q)$ $[\because P \vee P = P]$

$\Rightarrow (P \vee P) \wedge (\sim P \vee q)$

$\vee \wedge \vee \Rightarrow$ CNF

Theory of inference for statement Calculus

The main aim of logic is to be provide rules of inference to a conclusion from certain premises. The theory associated with rules of inference is known as theory of inference for statement calculus.

Rules of inference:-

There are two types of rules of inference:

1) Rule P.

2) Rule T.

Rule P

The premise may be introduced at any point in the derivation.

Rule T

A formula may be introduced in a derivative, it is a tautology implied by one or more of the preceding formulas in the derivative.

Statement formulas or implication formulas

$I_1 = P \wedge q \Rightarrow P$
$I_2 = P \wedge q \Rightarrow q$ } simplification

$I_3 = P \Rightarrow P \vee q$
$I_4 = q \Rightarrow P \vee q$ } Addition

$I_5 = \sim P \Rightarrow P \to q$

$I_6 = q \Rightarrow P \to q$

$I_7 = \sim(P \to q) \Rightarrow P$

$I_8 = \sim(P \to q) \Rightarrow \sim q$

$I_9 = P, q \Rightarrow P \wedge q$ (Conjunction)

$I_{10} = P, \sim P \vee q \Rightarrow q$ (disjunctive syllogism)

$I_{11} = P, P \to q \Rightarrow q$ (modus ponens)

$I_{12} = \sim q, P \to q \Rightarrow \sim P$ (modus Tollens)

$I_{13} = P \to q, q \to r \Rightarrow P \to r$ (Hypothetices Syllogism)

60) Demonstrate that 'r' is a ~~valid~~ valid conclusion from the premises $p \to q$, $q \to r$, p

A) Given premises are $p \to q$, $q \to r$, p

To find the valid inference conclusion in terms of 'r' is as shown below.

Sl.No	Statement Calculus	Rule	Derivation
1)	P	Rule P	–
2)	$P \to q$	Rule P	–
3)	q	Rule T	From 1, 2 & $I_{11} = P, P \to q = q$
4)	$q \to r$	Rule P	–
5)	r	Rule T	From 3, 4 & $I_{11} = P, P \to q = q$

61) Show that $S \vee R$ is tautology implied by the premesis $P \vee Q$, $P \to R$, $Q \to S$.

$I_7 = \sim(P \to q) \Rightarrow P$

$I_8 = \sim(p \to q) \Rightarrow \sim q$

$I_9 = P, q \Rightarrow P \wedge q$ (Conjunction)

$I_{10} = \sim P, P \vee q \Rightarrow q$ (disjunctive Syllogism)

$I_{11} = P, P \to q \Rightarrow q$ (modus ponens)

$I_{12} = \sim q, P \to q \Rightarrow \sim P$ (modus Tollens)

$I_{13} = P \to q, q \to r \Rightarrow P \to r$ (Hypothetices Syllogism)

$I_{14} = P \vee Q, P \to R, Q \to R \Rightarrow R$ (dilamma)

6) Demonstrate that 'r' is a ~~valid~~ valid conclusion from the premises $p \to q$, $q \to r$, p

A) Given premises are $p \to q$, $q \to r$, p

To find the valid inference conclusion in terms of 'r' is as shown below.

Sl.No	Statement Calculus	Rule	Derivation
1)	P	Rule P	–
2)	$P \to q$	Rule P	–
3)	q	Rule T	From 1, 2 & $I_{11} = P, P \to q = q$
4)	$q \to r$	Rule P	–
5)	r	Rule T	From 3, 4 & $I_{11} = P, P \to q = q$

6) Show that $S \vee R$ is tautology implied by the premesis $P \vee Q$, $P \to R$, $Q \to S$.

Given premises are $P \vee Q$, $P \to R$, $Q \to S$

Step No.	Statement calculus	Rules	Derivation
1)	$P \vee Q$	Rule P	–
2)	$\sim P \to Q$	Rule T	(1) $(P \to q) \Leftrightarrow \sim p \vee q$
3)	$Q \to S$	Rule P	–
4)	$\sim P \to S$	Rule T	(2)(3), Hypothetices (I_{13})
5)	$\sim S \to P$	Rule T	(4), $p \to q \Rightarrow \sim q \to \sim p$
6)	$P \to R$	Rule P	–
7)	$\sim S \to R$	Rule P	(5), (6), $P \to q$, $q \to r \Rightarrow P \to r$
8)	$S \vee R$	Rule P	(7) $P \to q \Rightarrow \sim p \vee q$

62) Demonstrate that $P \to q$, $q \to r$, $r \to s$, $\sim s$, $P \vee T$ is a tautology

A) Given premesis are $P \to q$, $q \to r$, $r \to s$, $\sim s$, $P \vee T$

To show that the valid conclusion is a tautology

Step No.	Statement Calculus	Rules	Derivation
1)	$P \to q$	Rule P	—
2)	$q \to r$	Rule P	—
3)	$P \to r$	" T	from 1, 2 & $P \to q$, $q \to r = P \to r$ (Hypothesis)
4)	$r \to s$	" P	—
5)	$P \to s$	" T	from 3, 4 & Hypothesis
6)	$\sim s$	Rule P	—
7)	$\sim P$	Rule T	from 5, 6 & $P \to q$, $\sim q \Rightarrow \sim P$ from modules Tollens
8)	$P \vee T$	Rule P	—
9)	T	Rule T	$P \vee T = P \vee 1 = 1 = T$

63) Demonstrate that $p \to q$, $\sim p \to r$, $r \to s$, $\Rightarrow \sim q \to s$

A) Given premises are $p \to q$, $\sim p \to r$, $r \to s$

To find the valued conclusion $\Rightarrow \sim q \to s$

Step No.	Statement Calculus	Rules	Derivation
1)	$\sim P \to r$	Rule P	—
2)	$r \to s$	Rule P	—
3)	$\sim P \to s$	Rule T	From 1, 2 & $P \to q$, $q \to r = P \to r$ (hypothesis)
4)	$\sim s \to P$	Rule T	$P \to q = \sim q \to \sim p$
5)	$P \to q$	Rule P	—
6)	$\sim s \to q$	Rule T	from 4, 5
7)	$\sim q \to s$	Rule T	$P \to q = \sim q \to \sim P$

64) Demonstrate that 's' is a valid conclusion from the premises $p \to q$, $q \to r$, $\sim(q \wedge r)$, $s \vee p$

A) Given premises are $p \to q$, $q \to r$, $\sim(q \wedge r)$, $s \vee p$

To find the valid inference conclusion in terms of 's' is as shown below

step No.	statement Calculus	Rules	Derivation
1)	$p \to q$	Rule P	–
2)	$\sim(q \wedge r)$	Rule P	–
3)	r	Rule T	from 2, $p \wedge q \Rightarrow q$ (I_2) & so $\sim(p \wedge q)$

step No.	statement Calculus	Rules	Derivation
1)	$\sim(q \wedge r)$	P	–
2)	$p \to q$	P	–
3)	$q \to r$	P	–
4)	$\sim q$	T	(1) $(p \wedge q) \Rightarrow p$
5)	$\sim p$	T	(2)(4) $\sim q$, $p \to q \Rightarrow \sim p$ I_{12} (modus Tollens)
6)	$s \vee p$	P	–
7)	s	T	(5)(6) $\sim p$, $s \vee p \Rightarrow s$

65) Demonstrate that 'r' is a valid conclusion from the premises $p \to q$, $q \to r$, $\sim(p \wedge r)$, $p \vee r$

A) let $\sim r$ be the additional premisis ($\because$ Rule (P)

step No.	statement Calculus	Rules	Derivation
1)	$p \to q$	P	–
2)	$q \to r$	P	–
3)	$p \to r$	T	(1)(2), $p \to q$, $q \to r \Rightarrow p \to r$ I_{13} hypothesis syllolysm
4)	$\sim r$	T	(3) – $\sim(p \to q) \Rightarrow \sim q$ (but here no negation) I_8
5)	$\sim p$	T	(3)(4) $\sim q$, $p \to q \Rightarrow \sim p$ I_{12} modus Tollens
6)	$p \vee r$	P	–
7)	r	T	(5)(6) $\sim p$, $p \vee q \Rightarrow q$ I_{10} disjunctive syllolysm
8)	$\sim(p \wedge r)$	P	–
9)	$\sim p \vee \sim r$	T	(8) $\sim(p \wedge q) \Rightarrow \sim p \vee \sim q$
10)	$\sim p$	T	(5)(6) $\sim p$, $p \vee q \Rightarrow q$
11)	r	T	(10)(6) $\sim p$, $p \vee q \Rightarrow q$ I_{10} disjunctive syllolysm

66) Demonstrate that $r \to s$ is a valid conclusion

Premises $p \to (q \to s)$, $\sim r \vee p$, q.

A) Given premises :- $p \to (q \to s)$, $\sim r \vee p$, q

Conclusion : $R \to S$

The Antecedent part i.e. R can be taken as additional premises. This additional premise R and the given existing premises produces conclusion 'S'.

So, the premises are $P \to (Q \to S)$, $\sim R \vee P$, Q, r

Step	statement Calculus	Rules	Derivation
1)	$\sim R \vee P$	P	–
2)	$R \to P$	T	(1) $p \to q = \sim p \vee q$
3)	R	P	additional premises
4)	P	T	(2)(3) and I_{11}, $P, P \to q = q$ modus poneses
5)	$P \to (Q \to S)$	P	~~(4)(5) and I_{11}, P, P→q = q modus poneses~~
6)	$Q \to S$	T	(4)(5) and I_{11}, $P, P \to q = q$ modus poneses
7)	Q	P	–
8)	S	Rule T	(6)(7), and I_{11}, $P, P \to q = q$ modus poneses
9)	$R \to S$	Rule CP	(3),(8)

67) Demonstrate that $(p \to q) \to r$, $p \wedge s$, $q \wedge t \Rightarrow r$

Step	statement calculus	Rules	Derivation
1)	$p \wedge s$	P	–
2)	$q \wedge t$	P	–
3)	q	T	(2) $p \wedge q = p$ I_1
4)	t	T	(2) $p \wedge q = q$ I_2
5)	p	T	(1) $p \wedge q = p$ I_1
6)	s	T	(1) $p \wedge q = q$ I_2
7)	$p \to q$	T	(3). $q \Rightarrow p \to q$ (I_6)
8)	$(p \to q) \to r$	P	–
9)	r	T	$p, p \to q = q$ I_{11} modus poneses (7),(8)

assume p

68) Demonstrate that $p \to \sim q,\ r \to p,\ q \Rightarrow \sim r$

Step No.	Statement Calculus	Rules	Derivations
1)	$p \to \sim q$	P	–
2)	$q \to \sim p$	T	From 1, $p \to q = \sim q \to \sim p$
3)	q	P	–
4)	$\sim p$	T	(3), (2), $p, p \to q = q$ In modus ponenses
5)	$r \to p$	P	–
6)	$\sim p \to \sim r$	T	$p \to q = \sim q \to \sim p$
7)	$\sim r$	T	From 4, 6 $p, p \to q = q$ In modus ponenses.

PDNF problms without truthtables

69) PDNF for $p \to ((p \to q) \wedge \sim(\sim q \vee \sim p))$

A) Given

$p \to ((p \to q) \wedge \sim(\sim q \vee \sim p))$

$p \to ((p \to q) \wedge (q \wedge p))$

$p \to ((\sim p \vee q) \wedge (q \wedge p))$

$p \to (\sim p \wedge (q \wedge p) \vee (q \wedge q \wedge p))$

$p \to (\sim p \wedge q \wedge p \vee (q \wedge q \wedge p))$

$p \to ((\sim p \wedge p \wedge q) \vee (q \wedge p))$

$p \to ((F \wedge q) \vee (q \wedge p))$

$p \to (F \vee (q \wedge p))$

$p \to q \wedge p$

$\sim p \vee (q \wedge p)$

$\to (\sim p \wedge T) \vee (q \wedge p)$

$\to (\sim p \wedge (q \vee \sim q)) \vee (p \wedge q)$ (∵ distributive law)

$\to (\sim p \wedge q) \vee (\sim p \wedge \sim q) \vee (p \wedge q)$

$\to \wedge \vee \wedge \vee \wedge \to$ PDNF

70) PDNF for $(p \wedge q) \vee (p \wedge \sim q) \vee (\sim p \wedge q)$

A) Given

$(p \wedge q) \vee (p \wedge \sim q) \vee (\sim p \wedge q)$

$\Rightarrow (p \wedge q \wedge T) \vee (p \wedge \sim q \wedge T) \vee (\sim p \wedge q \wedge T)$

$\Rightarrow (p \wedge q \wedge (r \vee \sim r)) \vee (p \wedge \sim q \wedge (r \vee \sim r)) \vee (\sim p \wedge q \wedge (r \vee \sim r))$

$\Rightarrow (p \wedge q \wedge r) \vee (p \wedge q \wedge \sim r) \vee (p \wedge \sim q \wedge r) \vee (p \wedge \sim q \wedge \sim r)$
$\vee (\sim p \wedge q \wedge r) \vee (\sim p \wedge q \wedge \sim r)$

$\Rightarrow \wedge \vee \wedge \vee \wedge \vee \wedge \vee \wedge \vee \wedge \Rightarrow$ PDNF

PCNF problems without truth tables

71) $(p \wedge q) \vee (\sim p \wedge q)$

A) Given $(p \wedge q) \vee (\sim p \wedge q)$

(common)

$[q \wedge (p \vee \sim p)]$

$\Rightarrow (q \wedge T)$

$\Rightarrow q$

$\Rightarrow (q \vee q) \Rightarrow$ PCNF

7) PCNF for $p \vee (\sim p \rightarrow (q \vee (\sim q \rightarrow r)))$

A) Given $p \vee (\sim p \rightarrow (q \vee (\underline{\sim q \rightarrow r})))$

$\Rightarrow p \vee (\sim p \rightarrow (q \vee (q \vee r)))$

$\Rightarrow p \vee (\sim p \rightarrow (\underline{q \vee q} \vee r))$

$\Rightarrow p \vee (\sim p \rightarrow (q \vee r))$

$\Rightarrow p \vee (p \vee q \vee r)$

$\Rightarrow \underline{p \vee p} \vee q \vee r$

$\Rightarrow p \vee q \vee r \Rightarrow$ PCNF.

Unit-2

Set theory: Syllabus

The principle of inclusion- exclusion, Pigeon hole principle and its applications, functions - Composition of functions, inverse functions, recurssive functions, lattices and their properties.

Algebraic structures:

Algebraic's term, - examples and general properties, groups, semi groups, sub groups, monoids, homomorphism and isomorphism.

Basics of Set theory:-

A set is a well defined collection of objects called elements or members of the set

Elements of a set are usually denoted by lower-case letters while sets are denoted by capital letters.

Eg:- Natural numbers $N = \{1, 2, 3, 4 \ldots\}$

Vowels $O = \{A, E, I, O, U\}$

$O = \{a, e, i, o, u\}$

Formalisation of sets:-

There are two ways by which the sets can be described

1) Roster Notation:-

Roster Notation is a complete list of all the elements of the set. Eg: $B = \{2, 4, 6, 8, \ldots 40\}$

2) Set Builder Notation:-

It is used when the roster method is impossible.

Eg: $B = \{x / 2 \le x \le 40, x \text{ is even}\}$ $x/ \rightarrow x$ is such that

Subset:-

If every element in A is also the element of the set B then A is called subset of B., It is denoted by "$A \subseteq B$" Eg: $A = \{1, 3, 4, 5, 8, 9\}$ $\therefore A \subseteq B = \{1, 3, 5\}$

$B = \{1, 2, 3, 5, 7\}$

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Types of sets:-

1) **NULL set:-**
The set with no elements is called null set or empty set.
A null set is denoted by '$\emptyset$'
$\emptyset = \{ \}$

2) **Countable set:-**
A countable set is the set with the same cardinality as some subset of the set.
Eg: $R = \{ x / x$ is a natural number $\}$
R is a infinite set
Eg: $A = \{ x / x$ is a even number b/w 10 to 100$\}$
A is a finite set

3) **Universal sets:-**
All the sets are consider to be subsets of one particular set; this set is called universal set. and it is denoted by "μ or U"

4) **Power set:-**
The set of all subsets of a set 'A' is called the power set of 'A' and it is denoted by P(A).
If A has 'n' elements in it then P(A) has 2^n elements.
Eg: $A = \{1, 2, 3\}$
then $P(A) = \{\emptyset, \{1\}, \{2\}, \{3\}, \{1,2\}, \{1,3\}, \{2,3\}, \{1,2,3\}\}$
$P(A) = 2^n = 2^3 = 8$ terms.

5) **Equivalent sets:-**
Two sets are said to be equivalent if there is a one-one correspondance b/w the elements of the two sets.
Eg: $A = \{a, b, c, d\}$
$B = \{\alpha, \beta, \gamma, \delta\}$ ($\because$ same no of elements)
then A is equivalent to B. with 4 elements with each

6) **Disjoint sets:-**
Two sets are said to be disjoint if they have no common elements
Eg: $A = \{1, 2, 3\}$
$B = \{6, 7, 8, 9\}$
$\therefore$ A and B are disjoint sets i.e. $A \cap B = \emptyset$

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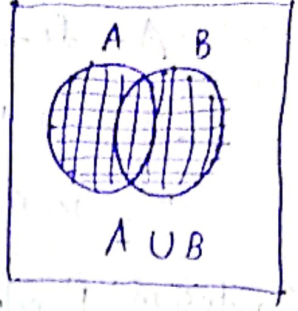
Union of two sets:-

The union of two sets A & B is the set of whose elements are in set A or set B or both. The union of sets A & B are denoted by "$A \cup B$"



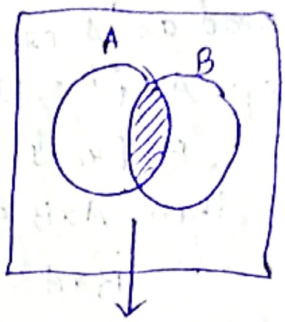
Eg: $A = \{1, 2, 3, 4\}$

$B = \{-6, -1\}$

$A \cup B = \{1, 2, 3, 4, -6, -1\}$

Intersection of two sets:-

The intersection of two sets A & B is the set of whose elements are all of the elements common to both A & B then Intersection of the sets of A & B is denoted by "$A \cap B$" and it is read as A intersection B.



Eg: $A = \{1, 2, 3, 4\}$

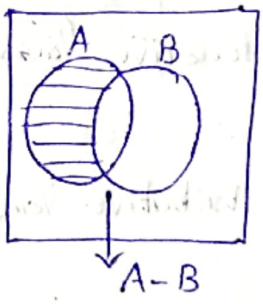
$B = \{3, 4, 5, 6, 7\}$

then $A \cap B = \{3, 4\}$

Difference of sets:-

If A & B are subsets of the universal set μ then the compliment of B in A is the set of all elements in A ~~with~~ ~~at~~ which are not in A and it is denoted by A−B

$A - B = \{x / x \in A \text{ and } x \notin B\}$



Ex: If $A = \{1, 2, 3, 4, 5\}$

$B = \{2, 3, 4\}$

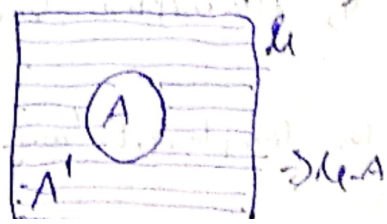
$A - B = \{1, \cancel{2}, \cancel{3}, \cancel{4}, 5\} - \{2, 3, 4\}$

$= \{1, 5\}$

Complement of a set:-

If μ is a universal set containing the set A then $\mu - A$ is called the complement of A & it is denoted by A'.

$\Rightarrow \mu - A = A' = \{x / x \notin A\}$



Ex: Let $\mu = \{1, 2, 3, 4, 5 \ldots\}$

$A = \{1, 2, 3, 4\}$

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$A' = \mu - A = \{5, 6, 7, 8, ---\}$

$B = \{2, 4, 6, 8\}$

$B' = \mu - B = \{1, 3, 5, 7, 9, 10, ------\}$.

Cartesian Products

Cartesian product is a set of direct products. Let us consider two sets A & B the set of all order pairs (a, b) where $a \in A$, $b \in B$ is called the cartesian product of A & B. and it is denoted by "$A \times B$" and read as "A cross B"

Eg:- $A = \{1, 2, 3\}$

$B = \{4, 5\}$

then $A \times B = \{(1,4)(1,5); (2,4)(2,5)(3,4)(3,5)\}$

$B \times A = \{(4,1)(4,2)(4,3)(5,1)(5,2)(5,3)\}$

Laws of set theory :-

1) Identity law - $A \cup \emptyset = A$, $A \cap \mu = A$
2) Domination law - $A \cup \mu = \mu$, $A \cap \emptyset = \emptyset$
3) Idempotent law - $A \cup A = A$, $A \cap A = A$
4) Complementary law - $(A')' = A$
5) Commutative law - $A \cup B = B \cup A$

   $A \cap B = B \cap A$
6) Associative laws - $A \cup (B \cup C) = (A \cup B) \cup C$

   $A \cap (B \cap C) = (A \cap B) \cap C$
7) Distributive law - $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

   $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$
8) Demorgan's law :- $(A \cup B)' = A' \cap B'$

   $(A \cap B)' = A' \cup B'$
9) Absorption law :- $A \cup (A \cap B) = A$

   $A \cap (A \cup B) = A$
10) Complement law :- $A \cup A' = \mu$

   $A \cap A' = \emptyset$

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AA 2M

Inclusion - Exclusion Principle :-

The inclusion - exclusion principle is a counting technique which generalizes the familiar method of obtaining the number of elements in the union of two finite sets symbolically expressed as

$$n(A \cup B) = n(A) + n(B) - n(A \cap B)$$

for 3 sets A, B, C is

$$n(A \cup B \cup C) = n(A) + n(B) + n(C) - n(A \cap B) - n(B \cap C) - n(C \cap A) + n(A \cap B \cap C)$$

where $n(A)$ = no. of elements in 'A'.

$n(B)$ = no. of elements in 'B'

$n(C)$ = no. of elements in 'C'

$n(A \cap B)$ = no. of elements in A & B which are common.

$n(B \cap C)$ = no. of elements in B & C which are common.

$n(C \cap A)$ = no. of elements in C & A which are common

$n(A \cap B \cap C)$ = no. of elements in A, B & C which are commos

Problems

1) Find the no. of mathematics student at a clg. taking at least one of the languages Telugu, Hindi and English given the following data with 65 study Telugu, 45 study Hindi, 42 study English, 20 study Telugu and hindi, 25 study Telugu and English, 15 study Hindi and English, 8 study all the 3 languages.

A) let T represents Telugu, H → Hindi and E → English

Given $n(T) = 65$, $n(H) = 45$, $n(E) = 42$

$n(T \cap H) = 20$, $n(T \cap E) = 25$, $n(H \cap E) = 15$, $n(T \cap H \cap E) = 8$

By inclusion and exclusion principle

$$n(T \cup H \cup E) = n(T) + n(H) + n(E) - n(T \cap H) - n(T \cap E) - n(H \cap E) + n(T \cap H \cap E)$$

$$= 65 + 45 + 42 - 20 - 25 - 15 + 8$$

$$n(T \cup H \cup E) = 100$$

∴ Total no. of mathematics student = 100

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2) A big software company develops a company employees of 60 computer programmers with 20 are in Java, 30 are in python and determine the no.of computer programmers that are not proficient in any of these two languages.

A) Given two programming languages JAVA and Python.
let J represents JAVA programming i.e. $n(J) = 20$
let P represents python programming i.e. $n(P) = 30$
to develop a company employees of 60 of two language is $n(J \cup P) = 60$.
By inclusion and exclusion principle

$$n(J \cup P) = n(J) + n(P) - n(J \cap P)$$

$$\Rightarrow n(J \cap P) = n(J) + n(P) - n(J \cup P)$$

$$= 20 + 30 - 60$$

$$= -10$$

3) Find the no.of mathematics student at a college taking at least one of the languages Telugu, Hindi and English. Given the following data, 30 study Telugu, 40 study Hindi, 35 study english and 20 study both Telugu & hindi, 10 study both English and telugu, no one study both hindi and english, 5 studies all 3 subjects.

A) let T represents Telugu, H → Hindi, E → English

Given $n(T) = 30$, $n(H) = 40$, $n(E) = 35$

$n(T \cap H) = 20$
$n(E \cap T) = 10$
$n(H \cap E) = 0$
$n(T \cap H \cap E) = 5$

By inclusion and exclusion principle

$$n(T \cup H \cup E) = n(T) + n(H) + n(E) - n(T \cap H) - n(H \cap E) - n(E \cap T) + n(T \cap H \cap E)$$

$$n(T \cup H \cup E) = 30 + 40 + 35 - 20 - 10 - 0 + 5$$

$$= 80$$

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Pigeon - hole principle and its application:-

If there are few boxes are available and also there are few objects that are greater than the total no. of boxes.

let 'm' represents the no. of boxes and 'n' represents the no. of objects then, $m < n \Rightarrow \boxed{n > m}$.

Each box one needs to place objects in the given boxes, then atleast one box must contains more than one such objects are called pigeon hole principle.

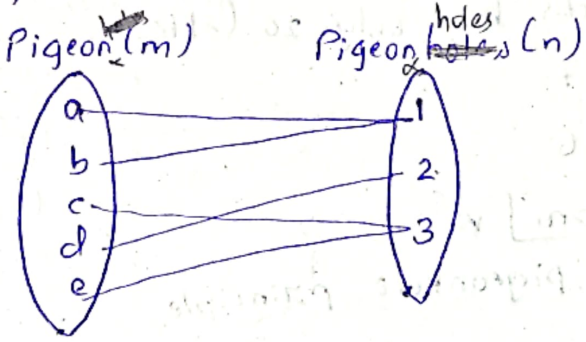
Applications:-

Suppose if 40 chairs are arranged in a rectangular array of 8 rows and 5 columns and 52 students are seated randomly in the chairs (one student on chair).

1) Some rows contains atleast 9 students
2) some ~~students~~ rows contains atmost 15 students
3) Some columns contains atleast 3 students
4) Some columns contains atleast 4 students.

Hence all the boxes in the column are not filled by the students. Hence pigeon hole principle does not exist.

Let us consider 'm' boxes with $\{a, b, c, d, e\}$ and with 'n' objects with $\{1, 2, 3\}$ then the pigeon hole principle is applicable as shown below



$m = 5$
$n = 3$
$n > m$
$3 \not> 5$

Therefore $m = 5$, $n = 3$

by pigeon hole principle we know that $n > m \Rightarrow 3 \not> 5$

hence pigeon hole principle does not apply.

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→ If 20 pigeons fly into 19 pigeon holes to roost at least one pigeon hole will have at least two pigeons. This is because of each pigeonholes only had one pigeon you could fit 19 pigeons.

here no. of objects (pigeons) $n = 20$

no. of boxes (pigeon holes) $m = 19$

$\boxed{n > m}$ ✓

∴ It obeys pigeonhole principle

→ If you have 13 people in a group atleast two of them will have the same birth moth because there are only 12 months in a year

here $n = 13$

$m = 12$

$\boxed{n > m}$ ✓

∴ It obeys pigeonhole principle.

→ In a group of 27 english words, at least two words must with the same letter. bcz alphabets has only 26 letters

here $n = 27$

$m = 26$

$\boxed{n > m}$ ✓

∴ It obeys pigeonhole principle

→ In any group of 8 people two were born on the same day of the week. bcz week has only 7 days

here $n = 8$

$m = 7$

$\boxed{n > m}$ ✓

∴ It obeys pigeon hole principle

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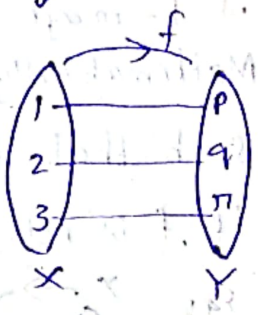
Functions :-

Let x and y be any two sets. A relation 'f' from x to y is called a function. If for every $x \in X$, there is a unique element $y \in Y$, such that $(x, y) \in f$

Eg :- let $X = \{1, 2, 3\}$

$Y = \{p, q, r\}$

and $f = \{(1, p)(2, q)(3, r)\}$



then $f(1) = p$, $f(2) = q$, $f(3) = r$

clearly f is a function from $X \to Y$.

(X to Y) i.e. $f : X \to Y$ (f is mapping from x to Y)

Domain and Range of a function :-

If $f : X \to Y$ is a function then X is called Domain of 'f' and Y is called the co-domain of 'f'.

The Range of 'f' is defined as the set of all images under 'f', it is denoted by

$f(x) = \{y / \text{for some } x \text{ in } X,\ f(x) = y\}$

Eg :- If the function f is defined by $f(x) = x^2 + 1$ on the set $\{-2, -1, 0, 1, 2\}$ then find the range of f.

Sol :- Given $f(x) = x^2 + 1$

Given set is $\{-2, -1, 0, 1, 2\}$

$f(-2) = (-2)^2 + 1 = 5$

$f(-1) = 1 + 1 = 2$

$f(0) = 1$

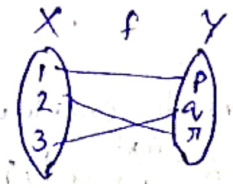
$f(1) = 2$ $\quad \therefore$ Range of $f = \{1, 2, 3\}$

$f(2) = 5$

Types of functions :-

1) One-to-one function :- (Injective functions)

A mapping $f : X \to Y$ is called one to one if distinct elements of X are mapped into distinct elements of Y. i.e. $f(x_1) = f(x_2) \Rightarrow x_1 = x_2$ for $x_1, x_2 \in X$



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3) **On-to function (surjection function)**

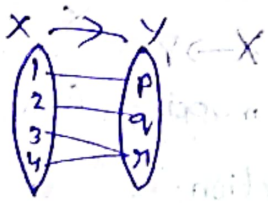
A mapping $f: X \to Y$ is called ON TO if the range set $R_f = Y$

like if $f: X \to Y$ is ON TO, then each element of Y is f-image of atleast one element of X.

Mathematically for all $\forall y \in b$ there exist $x \in A$ ~~such that $y \in b$~~ such that $f(x) = y$

if f is not ON TO then it is said to be into function

Eg:



Bijection function:-

A mapping $f: X \to Y$ is called a bijective function if it is both one-to one and ON TO functions, such mapping is called a bijection function.

Eg: Show that the mapping $f: R \to R$ defined by $f(x) = 2x+1$ for $x \in R$ is a bijective from R to R

let $f: R \to R$ defined by $f(x) = 2x+1$ for $x \in R$

1) **one-to one**

let x and y be the any two elements in R such that $f(x) = f(y)$

$\Rightarrow 2x+1 = 2y+1$

$2x = 2y$

$x = y$

$\Rightarrow f(x) = f(y) \Rightarrow x = y$

f is one to one function.

2) **ON TO :-**

let y be any element in the domain R. then $f(x) = y$

$\Rightarrow 2x+1 = y$

$x = \frac{y-1}{2}$

$x = \frac{y-1}{2} \in R$

$\therefore f$ is ON TO

$\therefore f$ is both one-to one & ON TO

$\therefore f: R \to R$ is a bijective

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4) **Composition of functions:-**

let $f: X \to Y$ and $g: Y \to Z$ these two function, then the composition of f and g denoted by "gof" is the function from X to Z defined as:

$(gof)(x) = g(f(x)), \forall x \in X$. ("gof" read as g circle f)

Eg: let $f(x) = x+2$, $g(x) = x-2$,

Ques 6) $h(x) = 3x$ for $x \in R$, where R is the set of real numbers then find gof, fog, fof, gog, foh, hof, hog, fogoh

A) Given $f(x) = x+2$; $g(x) = x-2$, $h(x) = 3x$

i) $gof(x) = g[f(x)] \Rightarrow g[x+2] \Rightarrow x+2-2 = x$

ii) $fog(x) = f[g(x)] \Rightarrow f(x-2) \Rightarrow x-2+2 = x$

iii) $fof = f[f(x)] \Rightarrow f(x+2) \Rightarrow x+2+2 \Rightarrow x+4$

iv) $gog \Rightarrow g[g(x)] \Rightarrow g(x-2) \Rightarrow x-2-2 \Rightarrow x-4$

v) $foh \Rightarrow f[h(x)] \Rightarrow f(3x) \Rightarrow 3x+2 \Rightarrow 3x+2$

vi) $hof \Rightarrow h[f(x)] \Rightarrow h[x+2] \Rightarrow 3(x+2) \Rightarrow 3x+6$

vii) $hog \Rightarrow h[g(x)] \Rightarrow h[x-2] \Rightarrow 3(x-2) \Rightarrow 3x-6$

viii) $fogoh \Rightarrow fog[h(x)] \Rightarrow fog[3x] \Rightarrow fo[3x-2]$

(or) $\Rightarrow f[3x-2] \Rightarrow 3x-2+2$

$\Rightarrow 3x$

viii) $fogoh \Rightarrow f[g(h(x))] \Rightarrow f[g(3x)]$

$\Rightarrow f[3x-2] \Rightarrow 3x-2+2$

$\Rightarrow 3x$

7) If $f(x) = x^2+3x+1$, $g(x) = 2x-3$ then find fof, gof, fog, gog,

[$\because (a+b+c)^2 = a^2+b^2+c^2+2(ab+bc+ca)$]

A) Given $f(x) = x^2+3x+1$, $g(x) = 2x-3$

i) $fof(x) = f[f(x)] \Rightarrow f[x^2+3x+1]$

$\Rightarrow (x^2+3x+1)^2 + 3(x^2+3x+1) + 1$

$\Rightarrow x^4+9x^2+1+2(3x^3+3x+x^2) + 3x^2+3x+3+1$

$\Rightarrow x^4+6x^3+14x^2+9x+5$

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ii) $gof(x) \Rightarrow g[f(x)] \Rightarrow g[x^2+3x+1]$

$\Rightarrow 2(x^2+3x+1)-3$

$\Rightarrow 2x^2+6x+2-3 \Rightarrow 2x^2+6x-1$

iii) $fog(x) \Rightarrow f[g(x)] \Rightarrow f[2x-3]$

$\Rightarrow (2x-3)^2+3(2x-3)+1$

$\Rightarrow 4x^2+9-12x+6x-9+1$

$\Rightarrow 4x^2-6x+1$

iv) $gog(x) \Rightarrow g[g(x)] \Rightarrow g[2x-3]$

$\Rightarrow 2(2x-3)-3$

$\Rightarrow 4x-6-3$

$\Rightarrow 4x-9$

Inverse functions:-

A function $f: X \to Y$ is said to be inverse of its inverse function f^{-1} is also a function from the range of f into X.

i.e. $f(x)=y$ then $x=f^{-1}(y)$ ($\because$ It should be onto function)

8) que. Example: let f is $f: R \to R$ given by $f(x) = x^3-2$ find f^{-1}.

A) 1) let $x_1, x_2 \in R$, such that $f(x_1) = f(x_2)$.

$\Rightarrow x_1^3-2 = x_2^3-2$

$\Rightarrow x_1^3 = x_2^3$

$\Rightarrow x_1 = x_2$

which is a one to one function.

2) If $f(x)=y$ then $x^3-2=y$

$x^3 = y+2$

$x = (y+2)^{1/3} \Rightarrow x = \sqrt[3]{y+2}$

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which is a on to function.

$\therefore f^{-1}$ is invertible and $f^{-1}(x) = \sqrt[3]{y+2}$ and $f^{-1}(y) = \sqrt[3]{x+}$

(∵ If the function is both one to one and on to then the function is "invertible")

9) If $f: R \to R$ is defined by $f(x) = ax+b$ for $a, b \in R$ then show that f is invertible and find inverse of f)

A) 1) Let $x_1, x_2 \in R$, such that $f(x_1) = f(x_2)$

$$ax_1 + \cancel{b} = ax_2 + \cancel{b}$$
$$\Rightarrow ax_1 = ax_2$$
$$\Rightarrow x_1 = x_2$$

which is a one to one function

2) If $f(x) = y$ then $ax + b = y$

$$ax = y - b$$
$$x = \frac{y-b}{a}$$

which is a on to function

$\therefore f^{-1}$ is invertible and $f^{-1}(x) = \frac{y-b}{a}$ and $f^{-1}(y) = \frac{x-b}{a}$

10) If $f: R \to R$ defined by $f(x) = \frac{5x-3}{2x+1}$ is invertible and find its inverse.

1) Let $x_1, x_2 \in R$ such that $f(x_1) = f(x_2)$

$$\frac{5x_1-3}{2x_1+1} = \frac{5x_2-3}{2x_2+1}$$
$$\Rightarrow (5x_1-3)(2x_2+1) = (5x_2-3)(2x_1+1)$$
$$\Rightarrow \cancel{10x_1x_2} + 5x_1 - 6x_2 - \cancel{3} = \cancel{10x_1x_2} - 6x_1 + 5x_2 - \cancel{3}$$
$$\Rightarrow 5x_1 + 6x_1 = 5x_2 + 6x_2$$
$$\Rightarrow 11x_1 = 11x_2 \Rightarrow x_1 = x_2$$ which is a one to one.

2) Let $f(x) = y$

$$\frac{5x-3}{2x+1} = y$$

$\Rightarrow 5x - 3 = y(2x+1)$
$\Rightarrow 5x - 3 = 2xy + y$
$\Rightarrow 5x - 2xy = y + 3$

$\Rightarrow x(5-2y) = y+3$

$x = \frac{y+3}{5-2y}$, which is a on to func.

$\therefore f$ is invertible and $f^{-1}(x) = \frac{y+3}{5-2y}$ and $f^{-1}(y) = \frac{x+3}{5-2x}$

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Ex) Let $f: R \to R$ given by $f(x) = ax^2 + b$. Show that f is invertible and find its f^{-1}.

1) Let $x_1, x_2 \in R$ such that $f(x_1) = f(x_2)$

$$ax_1^2 + \cancel{b} = ax_2^2 + \cancel{b}$$
$$ax_1^2 = ax_2^2$$
$$x_1^2 = x_2^2$$
$$x_1 = x_2$$

$\therefore$ which is a one to one function

2) If $f(x) = y$ then

$$ax^2 + b = y$$
$$ax^2 = y - b$$
$$x^2 = \frac{y-b}{a}$$
$$x = \sqrt{\frac{y-b}{a}}$$

$\therefore$ which is a onto function.

$\therefore f^{-1}$ is invertible and $f^{-1}(x) = \sqrt{\frac{y-b}{a}}$ and $f^{-1}(y) = \sqrt{\frac{x-b}{a}}$

Floor and ceiling functions:-

Let x be a real number then the least integer i.e. not less than x is called a ceiling of x and it is denoted by $\lceil x \rceil$

Eg: $\lceil 2.15 \rceil = 3$

$\lceil \sqrt{5} \rceil = 3$

$\lceil -7.4 \rceil = -7$

$\lceil -2 \rceil = -2$

Let x be a real number then the greatest integer that does not exceed x is called "floor" and it is denoted by "$\lfloor x \rfloor$"

Eg: $\lfloor 5.14 \rfloor = 5$

Eg: $\lfloor 7.15 \rfloor = 2$

Eg: $\lfloor -7.4 \rfloor = -8$

Eg: $\lfloor -2 \rfloor =$

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Recursive functions:-

A recursive function is a function which is defined in terms of it self. A recursive definition has two parts.

1) Definition of smallest argument i.e. $f(0)$ or $f(1)$

2) Definition of $f(n)$ given by $f(n-1), f(n-2), \ldots$

12) Example:- A recursive function is defined as $f(0) = 3$, $f(n+1) = 2$, $f(n) = 3$ then $f(1) = ?$

Given $f(0) = 3$, $f(n+1) = 2$, $f(n) = 3$

$\rightarrow f(1) = 2f(0) + 3 = 2(3)+3 \Rightarrow 6+3 = 9$

$f(2) = 2f(1) + 3 = 2(9)+3 = 21$

$f(3) = 2f(2) + 3 = 2(21)+3 = 45$

$f(4) = 2f(3) + 3 = 2(45)+3 = 93$

13) Calculate the value of the following recursive functions defined by $f(0) = 1$, $f(n) = nf(n-1)$

A) Given $f(0) = 1$, $f(n) = nf(n-1)$

If $n = 1$, then $f(1) = 1.f(1-1)$
$= 1.f(0)$
$= 1.1 = 1$

If $n = 2$ then $f(2) = 2f(2-1)$
$= 2f(1)$
$= 2(1) = 2$

If $n = 3$ then $f(3) = 3f(3-1)$
$= 3f(2)$
$= 3(2) = 6$

If $n = 4$ then $f(4) = 4f(4-1)$
$= 4f(3)$
$= 4(6) \rightarrow 24$

and so on .....

$\therefore$ The recursive function of given $f(n)$ is

$f(n) = 1!, 2!, 3!, 4!, \ldots\ldots n!$

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14) Consider the following and calculate the value of the following recurrsive function defined by $f(0)=0$, $f(n) = f(n-1)+2n+1$

A) Given $f(0)=0$, and $f(n) = f(n-1)+2n+1$

If $n=1$, then $f(1) = f(1-1)+2(1)+1$

$= f(0)+2+1$

$= 0+3$

$= 3$

If $n=2$ then $f(2) = f(2-1)+2(2)+1$

$= f(1)+4+1$

$= 3+5$

$= 8$

If $n=3$ then $f(3) = f(2)+2(3)+1$

$= 8+6+1$

$= 15$

If $n=4$ then $f(4) = f(3)+2(4)+1$

$= 15+8+1$

$= 24$

and so on ----

by observing the above values $f(n)$ generates a sequence of 3, 8, 15, 24 ------

~~Let us assume $f(k) = k(k+1)$~~

~~$f(k+1) = f(k)+2(k+1)+1$~~

~~$= k(k+1)+2(k+1)+1$~~

~~$= (k+1)(k+2)$ proving our induction step~~

$\therefore$ value of recurrsive func. $f(n) = (n+1)^2-1$ or n^2+2n

15) Calculate the value of the recurrsive function defined by $f(0)=5$, $f(n)=f(n-1)+2$

A) Given $f(0)=5$ and $f(n)=f(n-1)+2$

If $n=1$ then $f(1) = f(1-1)+2$

$= f(0)+2$

$= 5+2$

$= 7$

If $n=2$ then $f(2) = f(1)+2$

$= 7+2 \Rightarrow 9$

If $n=3$ then $f(3) = f(2)+2$

$= 9+2 \Rightarrow 11$

and so on ----

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by observing above pattern, $f(n)$ increases by 2 for each value of n.

$\therefore$ The function can be rewritten in closed form as

$f(n) = 5 + 2n$ (∵ here 5 add chesina bcz after 5 nunchi values vachinai)

$\Rightarrow f(n) = 2n + 5$.

16) Calculate the value of the recursive function defined by $f(0) = 3$, $f(n+1) = 2f(n) + 3$

A) Given $f(0) = 3$ and $f(n+1) = 2f(n) + 3$

If $n = 0 \Rightarrow f(0+1) = 2f(0)+3 \Rightarrow f(1) = 2(3)+3 \Rightarrow 9$ [∵ $f(1) = 9$]

If $n = 1$, then $f(1+1) = 2 \cdot f(1) + 3 \Rightarrow$ ~~2×9+3~~

$\Rightarrow f(2) = 2f(1) + 3 \Rightarrow 2 \times 9 + 3 = 21$

If $n = 2$, then $f(3) = 2f(2) + 3 = 2 \times 21 + 3 = 45$

If $n = 3$, then $f(4) = 2f(3) + 3 = 2 \times 45 + 3 = 93$

If $n = 4$, then $f(5) = 2f(4) + 3 = 2 \times 93 + 3 = 189$

$\therefore$ by observing above pattern

~~f(n+1) = 2f(n)+3~~, the recursive pattern is 9, 21, 45, 93, 189, ----

17) Consider the following recursive function for $x < y$ then $f(x,y) = 0$ if $y \le x$ then $f(x,y) = f(x-y, y) + 1$ find the value of $f(4,7)$ and $f(19,6)$

A) Given $f(x,y) = \begin{cases} 0, & x < y \\ f(x-y, y) + 1, & y \le x \end{cases}$

To find the values of $f(4,7)$ and $f(19,6)$

1) here $x = 4$ and $y = 7$ the condition $y \le x$ does not satisfies hence $f(4,7) = 0$

2) To find $f(19,6)$

Given $f(x-y, y) + 1$

here $x = 19$, $y = 6$

$f(19-6, 6) + 1$

$f(13, 6) + 1 \rightarrow (1)$

To find $f(13, 6)$

$f(13-6, 6) + 1$

$\Rightarrow f(7, 6) + 1 \rightarrow (2)$

To find $f(7, 6)$

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$\Rightarrow f(7-6, 6)+1$

$\Rightarrow f(1, 6)+1 \rightarrow (3)$

(here $x=1$, $y=6$ cond: $x<y$ $6<1$ false)

$\Rightarrow \because f(1,6)=0$

$\Rightarrow$ eq (3) $\Rightarrow 0+1 \Rightarrow 1$ i.e, $f(7,6)=1$

Since $f(13,6) = f(7,6)+1$

$= 1+1$

$= 2$

next $f(19,6) = f(13,6)+1$

$= 2+1 \Rightarrow 3$

18) Find the recursive function for $F_n = F_{n-1} + F_{n-2}$ where $F_0 = 0$, $F_1 = 1$

A) Given $F_n = F_{n-1} + F_{n-2}$

Initial conditions are $F_0 = 0$, $F_1 = 1$

If $n=2$, then $F_2 = F_{2-1} + F_{2-2}$

$= F_1 + F_0$

$= 1+0 \Rightarrow 1$

If $n=3$, then $F_3 = F_{3-1} + F_{3-2}$

$= F_2 + F_1 \Rightarrow 1+1 \Rightarrow 2$

If $n=4$, then $F_4 = F_{4-1} + F_{4-2}$

$= F_3 + F_2 \Rightarrow 2+1 \Rightarrow 3$

If $n=5$, then $F_5 = F_{5-1} + F_{5-2}$

$= F_4 + F_3 \Rightarrow 3+2 \Rightarrow 5$

$\vdots$

$\therefore F_n = F_{n-1} + F_{n-2}$ when $n \geq 2$

19) Find the recursive function for $T_n = 2T_{n-1}$ with $T_1 = 4$, $T_2 = 8$, $T_3 = 16$

A) Given $T_n = 2T_{n-1}$ and $T_1 = 4$, $T_2 = 8$, $T_3 = 16$

let

If $n=4$ then $T_4 = 2T_{4-1}$

$= 2T_3$

$= 2 \times 16$

$= 32$

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If $n=5$ then $T_5 = 2T_{5-1}$

$= 2T_4 \Rightarrow 2 \times 32$

$= 64$

If $n=6$ then $T_6 = 2T_{6-1}$

$= 2T_5 \Rightarrow 2 \times 64$

$= 128$.

by observing above pattern we can assume that the value is doubling each time

So the condition $T_{n+1} = 2T_n$ will satisfy

or another condition $T_n = 4 \times 2^{n-1}$ will also satisfy

$\therefore T_n = 2T_{n-1}$ if $T_{n+1} = 2T_n$ or $T_n = 4 \times 2^{n-1}$

or simply we can write $T_n = 2T_{n-1}$ when $n \geq 4$

20) Find the recurssive function for $T_n = 3T_{n-1} - 4$ with $T_1 = 3$

A) Given $T_n = 3T_{n-1} - 4$

Given initial conditions are $T_1 = 3$

now:

If $n=2$, then $T_2 = 3T_{2-1} - 4$

$= 3 \times T_1 - 4 \Rightarrow 3 \times 3 - 4 \Rightarrow 9 - 4 \Rightarrow 5$

If $n=3$, then $T_3 = 3T_{3-1} - 4$

$= 3T_2 - 4 \Rightarrow 3 \times 5 - 4 \Rightarrow 15 - 4 \Rightarrow 11$

If $n=4$, then $T_4 = 3T_{4-1} - 4$

$= 3T_3 - 4 \Rightarrow 3 \times 11 - 4 \Rightarrow 33 - 4 \Rightarrow 29$

If $n=5$, then $T_5 = 3T_{5-1} - 4$

$= 3T_4 - 4 \Rightarrow 3 \times 29 - 4 \Rightarrow 83$,

Sequence starts as: $(3, 5, 11, 29, \ldots)$

$\therefore T_n = 3T_{n-1} - 4$; $n \geq 2$

The recursive function for T_n is $T_n = 3T_{n-1} - 4$ with $T_1 = 3$

The sequence of T_n is 5, 11, 29, ----

Poset :- (Partially ordered set)

A set P together with a partial order R is called a poset. The relation R is often denoted by the symbol $\leq$ then the ordered pair $(P, \leq)$ is called a poset.

Hasse diagram :-

A partial order $\leq$ on a set P can be represented by means of diagram is known as hasse diagram of $(P, \leq)$.

1) Each element is represented by a small circle or dot.
2) The circle for $x \in P$ is drawn below the circle for $y \in P$ if $x < y$ and a line is drawn btwn x & y if y covers x.
3) If $x < y$ but y does not covers x then x and y are not connected directly by a single line.

Lattices :-

A Lattice is a partially ordered set $(l, \leq)$ in which every subset consisting of pair of elements has a lowest upper bound (LUB) and ~~greaste~~ greatest lower bond (GLB).

A lattice 'L' is an algebraic system with binary operations $\vee$ and $\wedge$, It is denoted by $(l, \wedge, \vee)$

l is called a lattice if it satisfies the following axioms.

1) Commutative law – $(a \wedge b) = (b \wedge a)$
$(a \vee b) = (b \vee a)$

2) Associative law – $a \wedge (b \wedge c) = (a \wedge b) \wedge c$
$a \vee (b \vee c) = (a \vee b) \vee c$

3) Absorption law :- $a \wedge (a \vee b) = a$
$a \vee (a \wedge b) = a$

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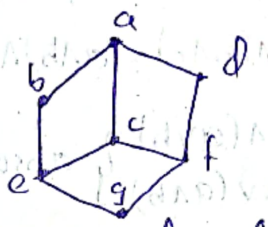
The LUB is denoted by $(a \vee b)$ and called as "joint"
The GLB is denoted by $(a \wedge b)$ and is called "meet".

21) Determine whether the following hasse diagram is a lattice or not.



A) To find the lattice to the diagram.

To find LUB and GLB for $A = \{a, b, c, d, e, f, g\}$

LUB:- The following table represents a lattice table for the given diagram

$\vee$	a	b	c	d	e	f	g
a	a	a	a	a	a	a	a
b	a	b	a	a	b	a	b
c	a	a	c	a	c	c	c
d	a	a	a	d	a	d	d
e	a	b	c	a	e	c	e
f	a	a	c	d	c	f	f
g	a	b	c	d	e	f	g

GLB: The following table represents a lattice table for the given diagram

$\wedge$	a	b	c	d	e	f	g
a	a	b	c	d	e	f	g
b	b	b	e	g	e	g	g
c	c	e	c	f	e	f	g
d	d	g	f	d	g	f	g
e	e	e	e	g	e	g	g
f	f	g	f	f	g	f	g
g	g	g	g	g	g	g	g

Each subset of two elements has a least upper bound (LUB) and a greatest lower bound (GLB) so it is a lattice

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Properties of Lattices

1) $a \wedge a = a$, $a \vee a = a$ → Idempotent law

2) $a \wedge b = b \wedge a$, $a \vee b = b \vee a$ → commutative law

3) $a \wedge (b \wedge c) = (a \wedge b) \wedge c$; $a \vee (b \vee c) = (a \vee b) \vee c$ → associative law

4) $a \wedge (a \vee b)$, $a \vee (a \wedge b)$ } absorption law.

5) $a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$, $a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$ } Distributive law

Theorem - 1 :-

Let $(l, \leq)$ be a lattice in which $\wedge$ and $\vee$ denotes the operation of meet and joint respectively for any $a \in l$, $a \leq b \Leftrightarrow a \wedge b = a \Leftrightarrow a \vee b = b$

Proof :-

we shall prove that $a \leq b \Leftrightarrow a \wedge b = a$

let us assume that $a \leq b$, since $a \leq a$ then

$$a \leq b = a \wedge b = a$$

next to assume that $a \wedge b = a$ but it is only possible if $a \leq b$ i.e $a \wedge b = a \Rightarrow a \leq b$

finally we assume that $a \leq b \Rightarrow a \wedge b = a$ and

$$a \wedge b = a \Rightarrow a \leq b$$

$$\Rightarrow a \leq b \Leftrightarrow a \wedge b = a$$

lly if $a \leq b \Leftrightarrow a \vee b = b$

Theorem - 2 :-

Let $(l, \leq)$ be a lattice then $b \subseteq c$ (subset) then $a \wedge b \leq a \wedge c$ $a \vee b \leq a \vee c$.

Proof :-

we know that

$a \in l$, $a \leq b \Leftrightarrow a \wedge b = a \Leftrightarrow a \vee b = b$

To show that

$b \subseteq c \Rightarrow a \wedge b \leq a \wedge b$

$\Rightarrow (a \wedge b) \wedge (a \wedge c) = a \wedge (b \wedge a) \wedge c$

$= (a \wedge a) \wedge (b \wedge c)$

$= a \wedge (b \wedge c) \Rightarrow a \wedge b$

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If $b \le c$ then $a \wedge b \le a \wedge c$

Theorem-3:

In a lattice $(L, \le)$, prove that $a \wedge (a \vee b) = a$, $\forall a, b \in L$.

Proof:

1, The statements follow from the definition of LUB and GLB

2, The definition of LUB and GLB treat a and b symmetrica so the result follow

3, a, from the definition of LUB, we have $a \le a \vee (b \vee c)$ and $b \vee c \le a \vee (b \vee c)$. Moreover, $b \le b \vee c$ and $c \le b \vee c$. So by transitivity $b \le a \vee (b \vee c)$ and $c \le a \vee (b \vee c)$. Thus $a \vee (b \vee c)$ is an upper bound of a and b. So by definition of least upper bound we have

$$a \vee b \le a \vee (b \vee c)$$

Since $a \vee (b \vee c)$ is an upper bound of $a \vee b$ and c we obtain $(a \vee b) \vee c \le a \vee (b \vee c)$

Similarly, $a \vee (b \vee c) \le (a \vee b) \vee c$. By the anti symmetry of $\le$ property 3(a) follows

3, b, The proof is analogous to the proof of part (a) and we omit it.

4, a, Since $a \wedge b \le a$ and $a \le a$, we see that a is an upper bound of $a \wedge b$ and a; so $a \vee (a \wedge b) \le a$. on the other hand, by the definitation of LUB we have $a \le a \vee (a \wedge b)$, so $a \vee (a \wedge b) = a$

b, The proof is analogous to the proof of part (a) and we omit it.

It follows from property 3 that we can write $a \vee (b \vee c)$ and $(a \vee b) \vee c$ mearly as $a \vee b \vee c$ and similarly for $a \wedge b \wedge c$ moreover, we can write.

LUB $(\{a_1, a_2, \ldots, a_n\})$ as $a_1 \vee a_2 \vee \ldots \vee a_n$

GLB $(\{a_1, a_2, \ldots a_n\})$ as $a_1 \wedge a_2 \wedge \ldots \wedge a_n$

Since we can show by induction that these joins and meets are independent of the grouping of terms.

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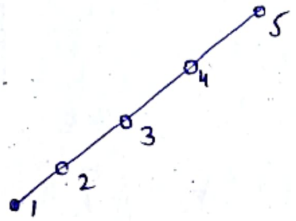
Hasse diagram Examples.

Example 1:- let $P = \{1,2,3,4,5\}$ and $\leq$ be the relation "less than or equal to" then the hasse diagram is shown below

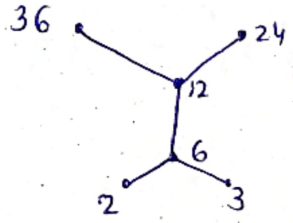
A)



It is totally order set

Example 2:- let $X = \{2,3,6,12,24,36\}$ and the relation $\leq$ be such that $x \leq y$ if x divides y. Draw the hasse diagram of $(X, \leq)$

A)



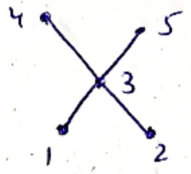
It is not a total order set

Example 3:- Draw the hasse diagram for the relation R on $A = \{1,2,3,4,5\}$ whose relation matrix given below

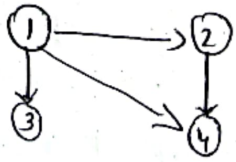
$$M_R = \begin{bmatrix} 1 & 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

A) $R = \{(1,1), (1,3), (1,4), (1,5), (2,2), (2,3), (2,4), (2,5), (3,3), (3,4), (3,5), (4,4), (5,5)\}$

Hasse diagram for M_R is

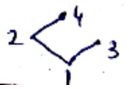


Example 4:- A partial order 'R' on the set $A = \{1,2,3,4\}$ is represented by the following digraph. Draw the hasse diagram for R



A) By examining the given digraph. we find that $R = \{(1,1), (1,2), (1,3), (1,4), (2,2), (2,4), (3,3), (4,4)\}$

Hasse diagram is



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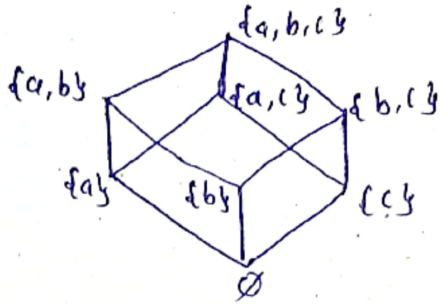
Example 5:- Draw the hasse diagram for the partial ordering $\subseteq$ on the power set $P(S)$ where $S = \{a, b, c\}$

A) Given $S = \{a, b, c\}$

$P(S) = \{\emptyset, \{a\}, \{b\}, \{c\}, \{a, b\}, \{b, c\}, \{a, c\}, \{a, b, c\}\}$

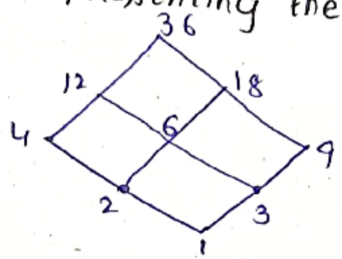
$\therefore$ Hasse diagram is



Example 6: Draw the hasse diagram representing the positive divisors of 36

A) $D_{36} = \{1, 2, 3, 4, 6, 9, 12, 18, 36\}$

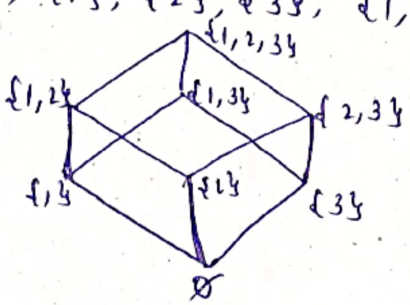
$\therefore$ The hasse diagram is



Lattice examples:

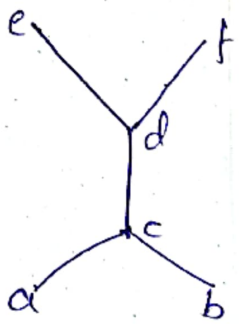
Example 1:- Let $x = \{1, 2, 3\}$ then the poset $(P(x), \subseteq)$ of all sub sets of x is a lattice or not

A) $P(x) = \{\emptyset, \{1\}, \{2\}, \{3\}, \{1, 2\}, \{1, 3\}, \{2, 3\}, \{1, 2, 3\}\}$



Each subset has a LUB and GLB. So it is lattice

NOTE: The following diagram is not a lattice, bcz, GLB (a,b) and LUB (e,f) does not exist



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Types of Lattices

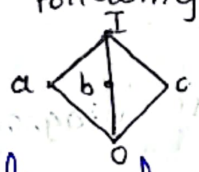
1) **Distributive lattice :-**

A lattice $(L, \wedge, \vee)$ is a distributive lattice if the following laws holds for $x, y, z \in L$ then if it satisfies

$$x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$$

$$x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$$

22) Show that the following diagram is a distributive lattice or not.



To the given diamond shape with vertices $\{a, b, c, O, I\}$

To find whether diamond is distributive lattice or not

consider $a \wedge (\underbrace{b \vee c}_{\text{upper bound}}) = a \wedge (\overset{\text{lower bound in diagram}}{I})$

$= a$

Now $(a \wedge b) \vee (a \wedge c)$

$= O \vee O$

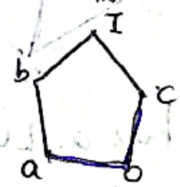
$= O$

$\therefore a \neq O$ which does not satisfies the distributive lattice

$\therefore a \wedge (b \vee c) \neq (a \wedge b) \vee (a \wedge c)$

$\therefore$ It is not a distributive lattice.

23) Show that the following diagram is distributive lattice or not



To the given pentagon shape with vertices $\{a, b, c, I, O\}$

To find whether pentagon is distributive lattice or not

Consider $a \wedge (b \vee c) = a \wedge I$

$= a$

Now $(a \wedge b) \vee (a \wedge c) = a \vee O$

$= a$

$\therefore a = a$

$\therefore a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$

$\therefore$ It is a distributive lattice

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24) Consider the poset $(L, \leq)$ where $L = \{1, 2, 3, 5, 30\}$ and partially ordered relation $\leq$ is defined as x & $y \in L$ then $x \leq y$ then show that the poset is a Lattice

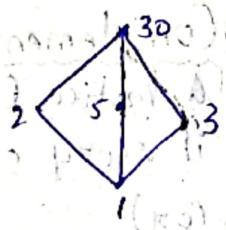
Given a poset $(L, \leq)$ with $L = \{1, 2, 3, 5, 30\}$

given condition is $x \leq y$

The poset of given set L is $\{(1,2), (1,3)\ (1,5), (1,30)$
$(2,3)\ (2,5), (2,30)\ (3,5)$
$(3,30)\ (5,30)\}$

The hasse diagram of the posets are

hasse diagram is a diamond shape



LUB:-

V	1	2	3	5	30
1	1	2	3	5	30
2	2	2	30	30	30
3	3	30	3	30	30
5	5	30	30	5	30
30	30	30	30	30	30

GLB:

Λ	1	2	3	5	30
1	1	1	1	1	1
2	1	2	1	1	2
3	1	1	3	1	3
5	1	1	1	5	5
30	1	2	3	5	30

let us consider $x = 2$, $y = 3$, $z = 5$ then

$x \wedge (y \vee z) \Rightarrow 2 \wedge (3 \vee 5)$
$= 2 \wedge (30) \Rightarrow 2$

now $(x \wedge y) \vee (x \wedge z) \Rightarrow (2 \wedge 3) \vee (2 \wedge 5)$
$= 1 \vee 1 \Rightarrow 1$

If $x = 1, y = 2, z = 3$ it satisfies the distributive property so check other value if any of it not satisfies then it is not distributive

$\therefore$ Given $(L, \leq)$ is not a distributive lattice for ~~[illegible]~~
$L = \{1, 2, 3, 5, 30\}$

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2) Bounded lattices:

A bounded lattice is an algebraic system with $(L, \wedge, \vee)$ if there exists a greatest element "1" and a least element "0" in the lattice

1) $\forall x \in L$; $x \wedge 1 = x$ & $x \vee 1 = 1$

2) $\forall$; $x \in L$, $x \wedge 0 = 0$ & $x \vee 0 = x$

The element '1' is called the upper bound

The element '0' is called the lower bound

3) Complemented Lattice:

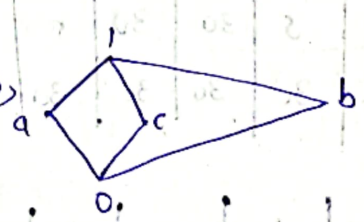
A lattice $(L, \wedge, \vee)$ is said to be a complemented lattice if every element in the lattice has a complement

(or)

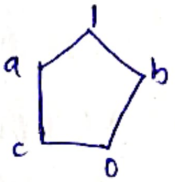
A bounded lattice with greatest element '1' and least element '0' then for the elements $a, b \in L$ element 'b' is the complement of 'a' if $a \vee b = 1$ & $a \wedge b = 0$

Ex: 1)

Eg: 3)



Eg: 2:



i) From Eg① $a \vee b = 1$ & $a \wedge b = 0$ which is a complement lattice

ii) From Eg② $a \vee b = 1$ & $a \wedge b = 0$ which is a complement lattice

iii) From Eg③ $a \vee b = 1$ & $a \vee b = 0$ which is a complement lattice

Note:

A complement lattice must contains '1' and '0' in the diagram.

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4) Modular lattice:

A modular lattice is a lattice that satisfies

$x \le b \Rightarrow x \vee (a \wedge b) = (x \vee a) \wedge b$

25) In a distributive law lattice $(L, \le)$, $a \wedge b = a \wedge c$ & $a \vee b = a \vee c$ implies that $b = c$. Prove this by using distributive lattice

A) Consider $(a \wedge b) \vee c = (a \vee c) \wedge (b \vee c)$ $\quad a \vee c = a \vee b$

$= (a \vee b) \wedge (b \vee c)$

commutative reverse

$= (b \vee a) \wedge (b \vee c)$

same so associative law

$= b \vee (a \wedge c) \rightarrow a \wedge c = a \wedge b$

$= b \vee (a \wedge b)$

$(a \wedge b) \vee c = b \longrightarrow (1)$

Consider $(a \wedge b) \vee c = (a \wedge c) \vee c \rightarrow a \wedge c = a \wedge b$

$= c \longrightarrow (2)$

$\therefore (1) = (2)$

i.e. $(a \wedge b) \vee c = b = c$

$\therefore \boxed{b = c}$

26) Prove In a distributive lattice if an element has a complement then this complement is unique.

A) let a is an element of lattice L

b & c are the two complement of a

let $a \vee b = 1 \rightarrow (1)$

$a \vee c = 1 \rightarrow (2)$

$a \wedge b = 0 \rightarrow (3)$

$a \wedge c = 0 \rightarrow (4)$

$a + a' = 1$

$a \cdot a' = 0$

like wise come

we have to prove $b = c$

now $b = b \wedge 1$

$= b \wedge (a \vee c)$ [$\because$ from eq (2)]

$= (b \wedge a) \vee (b \wedge c)$ (by using distributive law)

$= (a \wedge b) \vee (b \wedge c)$ (by commutative law)

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$= (a \wedge b) \vee (b \wedge c)$

$= 0 \vee (b \wedge c)$ ($\because$ from eq ③)

$= (a \wedge c) \vee (b \wedge c)$ ($\because$ from eq ④)

$= (a \vee b) \wedge c$ (by distributive law)

$= 1 \wedge c$ ($\because$ from eq ①)

$b = c$

$\therefore$ Hence the complement of an element in a distributive lattice is unique

Every distributive lattice is a modular lattice

Proof:-

Let $(L, \leq)$ be a distributive lattice

$\forall\, a, b, c \in L$

$a + (b * c) = \underbrace{(a+b) * (a+c)}_{\text{distributive law}}$

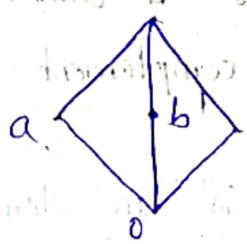
if $a \leq c$,

then $a + c = c$

$\therefore \boxed{a + (b * c) = (a+b) * c}$ → this is modular eq.

The modular equation is satisfied and L is modular

Eg: diamond lattice (M_5)



i, $a \vee (b \wedge c)$

$= a \vee 0$

$= a$

ii, $(a \vee b) \wedge (a \vee c)$

$= 1 \wedge 1$

$= 1$

$\wedge$ = lower bound

$\vee$ = upper bound

So, M_5 is not distributive | M_5 is modular

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Binary operations:

Let 'S' be a non empty set, if $f: S \to S$ is a mappi then the binary operation is $f: S \times S \to S$ is called a binary operation or binary composition in S.

The symbols $+, \cdot, *, \oplus$ etc are used to denote binary operations.

Eg: $a, b \in S \Rightarrow a+b \in S$,

here $+$ is called a binary operation

for $a, b \in S \Rightarrow a*b \in S$; here $*$ is a binary operati

Algebraic structure:

A non empty set G equipped with one or more binary operations is called an algebraic structure or algebraic system.

If '$*$' is a binary operation on G, then the algebrai structure can be written as $(G, *)$

Eg: $(N, +)$, $(Q, -1)$, $(R, +)$ etc are algebraic struc

Semi group:

An algebraic structure $(G, *)$ is called a semi group if the binary operation '$*$' is a binary operator then $(G, *)$ is said to be a semigroup if

1) $a, b \in G \Rightarrow a*b \in G \quad \forall a, b \in G$

2) $a*(b*c) = (a*b)*c, \quad \forall a, b, c \in G$

which is a closure and associative property

2B) The operator $*$ is defined by $a*b = a \ \forall a, b \in G$ then show that $(G, *)$ is a semi group

A) Given $a, b \in G$ defined by the operator $*$ given by $a*b = a$

i) Closure property:-

Since $a*b \in G$ such that $a*b = a \in G$

ii) Associative property:-

Let $a, b, c \in G$ such that $a*(b*c)$ (∵ here $b*c$ compar with $a*b = a$

$= a*b$

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$(a*b)*c \Rightarrow a*c$
$= a$

$\therefore a*(b*c) = (a*b)*c$

$*$ is a associative in G.

Hence $(G, *)$ is a semigroup.

29, The operator $*$ is defined by $a*b = a+b-ab$ $\forall a,b \in Z$ show that $(Z, *)$ is a semigroup.

A, let $a,b \in Z \Rightarrow a*b = a+b-ab \in Z$

here $*$ is a binary operation in Z.

To show that Z is a semi group which satisfies closure and associative properties

i, closure property:

let $a,b \in Z$ then $a*b = a+b-ab \in Z$

ii, Associative property:

let $a,b,c \in Z$ such that

$(a*b)*c = (a+b-ab)*c$
$= a+b-ab+c-(a+b-ab)c$
$= a+b-ab+c-ac-bc+abc$

$a*(b*c) = a*(b+c-bc)$
$= a+b+c-bc-a(b+c-bc)$
$= a+b+c-bc-ab-ac+abc$

$\therefore (a*b)*c = a*(b*c)$

$\therefore (Z, *)$ is a semigroup which satisfies both closure and associative property.

30, let Q be the set of rational numbers, $*$ is a binary operation defined on Q such that $a*b = a-b+ab$ for $a,b \in Q$. check whether $(Q,*)$ is a semigroup or not.

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ii) Associative property:

let $a, b, c, \in Q$ such that ~~$(a*b)*c$~~

$\Rightarrow (a*b)*c = (\underbrace{a-b+ab}_{a}) * \underbrace{c}_{b}$

$= a-b+ab-c+(a-b+ab)c$

$= a-b+ab-c+ac-bc+abc$

Now $a*(b*c) = \underbrace{a}_{a} * (\underbrace{b-c+bc}_{b})$

$= a-b+c-bc+a(b-c+bc)$

$= a-b+ab+c-ac+abc-bc$

$\therefore (a*b)*c \neq a*(b*c)$

$\therefore (Q, *)$ is not a semigroup which not satisfies associative property

3) let $(A, *)$ be a semigroup, show that a, b, c in A if $a*c = c*a$ and $b*c = c*b$ then $(a*b)*c = a*(b*c)$

A) let $a, b \in A \Rightarrow a*c = c*a$ and $b*c = c*b \in A$ where $*$ is a binary operation in A.

Given that A is a semigroup

To show that A is $(a*b)*c = a*(b*c)$

i) $a*c = c*a$

LHS: $(a*b)*c = (a*c)*b = (c*a)*b \rightarrow (1)$

ii) $b*c = c*b$

$a*(b*c) = (b*c)*a = c*(a*b)$

$= c*(a*b) \rightarrow (2)$

eq (1) = eq (2)

$(a*b)*c = a*(b*c)$

$\therefore$ Hence proved

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32) Show that the binary operation $*$ defined on $(R, *)$ where $x*y = x^y$ is not a semigroup.

A) let $x, y \in R \Rightarrow x*y = x^y \in R$; here $*$ is a binary operator in R.

To show that R is a semigroup which satisfies closure and associative property.

i) Closure property:

let $x, y \in R$ then $x*y = x^y \in R$

ii) Associative property:-

let $x, y, z \in R$ such that $x*(y*z) = (x*y)*z$

LHS: $(x*y)*z = x^y * z$ (∵ By the definition of $*$)

$= (x^y)^z \Rightarrow x^{yz}$

RHS: $x*(y*z) = x*y^z$

$\Rightarrow x^{y^z}$

LHS $\neq$ RHS

∴ It is not a semigroup

Homomorphism of Semigroup:

let $(S, *)$ and $(T, \circ)$ be any two semi groups then the mapping $f: S \to T$ such that for any two elements $a, b \in S$ such that $f(a*b) = f(a) \circ f(b)$ is called a semi group homomorphism.

(circle)

A homomorphism is a semigroup into itself is called a semigroup

Identity Element

let G be a non empty set and $*$ be any binary operation on G if there exists an element $e \in G$ such that $a*e = e*a = a$, $a \in G$ the 'e' is called an identity element on G

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Monoid :-

A semigroup $(G, *)$ with an identity element with respect to the binary operation $*$ is known as monoid.

$(Z, +)$ is a monoid and the identity is "zero".

$(Z, \cdot)$ is a monoid and the identity is "one".

Inverse element :

Let $(G, *)$ be an algebraic structure with the identity element 'e' in G with respect to $*$, an element $a \in G$ is said to be invertible if there exists an element $x \in G$ such that $a * x = x * a = e$

Group :-

If G is a non empty set and $*$ is a binary operation defined on G, such that the following conditions are satisfied then $(G, *)$ is a group

1) Closure law :

let $a, b \in G \Rightarrow a * b \in G$

2) Associative law:

For $a, b, c \in G \Rightarrow a * (b * c) = (a * b) * c$

3) Identity law:

for $a \in G$ then there exists $(\exists)$ $e \in G$ such that

$$a * e = e * a = a$$

4) Inverse law:

For $a \in G$ $\exists$ $b \in G$ such that

$$a * b = b * a = e$$

($\exists \rightarrow$ there exist)

where b is called inverse of a

33) Prove that $\{1, \omega, \omega^2\}$ is a group with respect to multiplication where $1, \omega, \omega^2$ are cube root of unity

A) Given multiplication is a operation defined on the set $G = \{1, \omega, \omega^2\}$. To show that $(G, \cdot)$ is a group

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$\cdot$	1	ω	ω^2
1	1	ω	ω^2
ω	ω	ω^2	$\omega^3 = 1$
ω^2	ω^2	$\omega^3 = 1$	$\omega^4 = \omega$

i) Closure law:-

From the table, $a, b \in G \Rightarrow a * b \in G$

let $a = 1$, $b = \omega$

now we are using dot product $\Rightarrow a.b = 1.\omega = \omega \in G$

ii) Associative law:

let $a, b, c \in G \Rightarrow a*(b*c) = (a*b)*c$

let $a = 1$, $b = \omega$, $c = \omega^2$

$\Rightarrow a.(b.c) \Rightarrow 1.(\omega.\omega^2)$

$\Rightarrow 1.(\omega^3) \Rightarrow 1.(1) \Rightarrow 1$

$\Rightarrow (a.b).c \Rightarrow (1.\omega).\omega^2$

$\Rightarrow \omega.\omega^2 \Rightarrow \omega^3 = 1$

$\therefore a.(b.c) = (a.b).c$

iii) Identity law:-

$a * e = e * a = a$

from the given data '1' is the identity element in G such that $a.e = e.a = a$

LHS: $a.e$

$\Rightarrow a.1 = a$

RHS: $e.a$

$= 1.a \Rightarrow a$

iv) Inverse law:-

$a * b = b * a = e$

let $a = \omega$, $b = \omega^2$

$a.b = \omega.\omega^2 = 1$

$b.a = \omega^2.\omega = 1$

Therefore all the conditions are satisfied.

Hence $(G, \cdot)$ is a group

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Abelian Group

If G is a non empty set and $*$ is a binary operation defined on G, such that the following conditions are satisfied then $(G, *)$ is an abelian group

1) Closure law:-

Let $a, b \in G \Rightarrow a*b = G$

2) Associative law:-

for $a, b, c \in G \Rightarrow a*(b*c) = (a*b)*c$

3) Identity law:-

for $a \in G$ $\exists$ $e \in G$ such that

$$a*e = e*a = 1$$

4) Inverse law:-

for $a \in G$ $\exists$ $b \in G$ such that

$$a*b = b*a = e$$

where 'b' is called the inverse of a.

5) Commutative law:-

Let $a, b \in G$ then $a*b = b*a \in G$

34) Prove that the set Z of all integers with binary operation $*$ defined by $a*b = a+b+1$ $\forall$ $a, b \in Z$ is abelian group

A) Given $a*b = a+b+1$ $\forall$ $a, b \in Z$ with binary operation $*$.

To show that $(Z, *)$ is a abelian group

i) closure law:-

Let $a, b \in Z \Rightarrow a*b \in Z$

Let $a = -1$, $b = 3$ ($\because$ we can take any integers, $b \in Z$ Z is given.)

Given $a*b = a+b+1$

$= -1+3+1 \Rightarrow 3 \in Z$

ii) Associative law

let $a, b, c \in Z$, then $a*(b*c) = (a*b)*c$

Let $a = 4$, $b = 5$, $c = 3$

LHS: $a*(b*c) \Rightarrow a*(b+c+1)$

$\Rightarrow a+b+c+1+1$

$= 4+5+3+2$

$14 \in Z$

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RHS: $(a*b)*c$

$\Rightarrow (a+b+1)*c$

$\Rightarrow a+b+1+c+1$

$\Rightarrow 4+5+1+3+1 \Rightarrow 14 \in Z$

LHS = RHS

$\therefore a*(b*c) = (a*b)*c$

iii) Identity Law:-

Let $a \in Z$ then there exist $e \in Z$ such that

$a*e = e*a = a$

LHS: $a*e$

$\Rightarrow a+e+1$

$\cancel{a}+e+1 = \cancel{a}$

$e+1 = 0$

$e = -1 \in Z$

RHS: $e*a$

$\Rightarrow e+a+1$

$e+\cancel{a}+1 = \cancel{a}$

$e+1 = 0$

$e = -1 \in Z$

$\therefore -1$ is the identity element in Z

iv) Inverse law:

Let $a \in Z$, $b \in Z$ such that

$a*b = b*a = e$

LHS: $a*b = e$

$a+b+1 = e$

$a+b+1 = -1$

$a+b = -2 \in Z$

$b = -2-a \in Z$

RHS: $b*a = e$

$b+a+1 = e$

$b+a+1 = -1$

$b+a = -2 \in Z$

$b = -2-a \in Z$

v) Commutative law:

Let $a, b \in Z$, such that $a*b = b*a$

let $a = -2$, $b = 0$

LHS: $a*b$

$\Rightarrow a+b+1$

$\Rightarrow -2+0+1$

$\Rightarrow -1$

RHS: $b*a$

$\Rightarrow b+a+1$

$\Rightarrow 0-2+1$

$\Rightarrow -1$

$\therefore a*b = b*a \in Z$

$\therefore (Z, *)$ is a abelian group.

$\therefore$ All 5 properties are satisfied.

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15) Show that the set Q_+ of all numbers forms abelian group under the composition defined by $*$ such that $a*b = \frac{ab}{3}$ $\forall$ $a,b \in Q_+$

A) Given Q_+ is a positive rational number with binary operation $*$.

To show that $(Q_+, *)$ is a abelian group.

i) closure law:-

let $a,b \in Q_+$ such that $a*b \in Q_+$

let $a=3$, $b=5$

now $a*b = \frac{ab}{3} = \frac{(3)(5)}{3} = 5 \Rightarrow \frac{5}{1} \in Q_+$

ii) Associative law:-

let $a,b,c \in Q_+$ such that $a*(b*c) = (a*b)*c$

let $a=2$, $b=3$, $c=4$

LHS: $a*(b*c)$

$\Rightarrow a*\left(\frac{bc}{3}\right)$

$\Rightarrow \frac{a \cdot \frac{bc}{3}}{3}$

$\Rightarrow \frac{abc}{9} \Rightarrow \frac{(2)(3)(4)}{9}$

$\Rightarrow \frac{24}{9}$

RHS: $(a*b)*c$

$\Rightarrow \left(\frac{ab}{3}\right)*c$

$\Rightarrow \frac{\frac{ab}{3}c}{3}$

$\Rightarrow \frac{abc}{9} \Rightarrow \frac{(2)(3)(4)}{9}$

$\Rightarrow \frac{24}{9}$

LHS = RHS,

$\therefore a*(b*c) = (a*b)*c$

iii) Identity law:

let $a \in Q_+$ then there exist $e \in Q_+$ such that

$a*e = e*a = a$

$a*e = a$	$e*a = a$
$\Rightarrow \frac{ae}{3} = a$	$\frac{ea}{3} = a$
$\Rightarrow e = 3 \in Q_+$	$e = 3 \in Q_+$

iv) Inverse law:-

let $a \in Q_+$, $b \in Q_+$ such that

$a*b = b*a = e$

now $a*b = e$

$\frac{ab}{3} = e$ $\Rightarrow ab = 3 \cdot 3$

$\Rightarrow ab = 9$ $\Rightarrow b = \frac{9}{a} \in Q_+$

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v) Commutative law

let $a, b \in Q_+$ such that $a * b = b * a$; let $a = 2, b = 3$

$$a * b = \frac{ab}{3} = \frac{(2)(3)}{3} = \frac{2}{1} \in Q_+$$

$$b * a = \frac{ba}{3} = \frac{(3)(2)}{3} \Rightarrow \frac{2}{1} \in Q_+$$

$\therefore (Q_+, *)$ is a abelian group

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$= \frac{(2)(3)}{3}$ | $\frac{(3)(2)}{3}$

$= \frac{2}{1} \in Q_1$ | $\Rightarrow \frac{2}{1} \in Q_1$

$\therefore (Q_+, *)$ is a abelian group

Homomorphism

(*mapping is same then it is homomorphism
*mapping is not same then it is isomorphism

let G & G' be any two groups with binary operations $*$ & Δ (delta) respectively.

Then a mapping $f: G \to G'$ is said to be Homomorphism if

$$f(a*b) = f(a) \Delta f(b) \ \forall a, b \in G$$

36) Eg: Let $(Z, +)$ and $(G, *)$ be any two groups defined as $G = \{2^n, \forall n \in Z\}$ then the function $f: Z \to G$ is $f(n) = 2^n$. then to show that $f: Z \to G$ is a homomorphishm

A) let $(Z, +)$; $(G, *)$ be the two groups then

$G = \{2^n, \forall n \in Z\}$

let $n_1, n_2 \in Z$, such that $f(n_1) = 2^{n_1}$; $f(n_2) = 2^{n_2}$

now, $f(n_1 + n_2) = 2^{n_1} + 2^{n_2}$

$= 2^{n_1 + n_2}$

$= 2^{n_1} * 2^{n_2}$

$$f(n_1 + n_2) = f(n_1) * f(n_2)$$

$\therefore f$ is homomorphism.

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Isomorphism:-

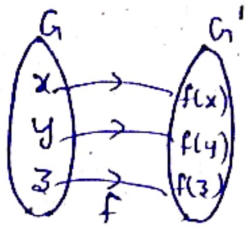
let $(G, +)$ and $(G', *)$ are two groups of a function $f: G \to G'$ is called isomorphism if and-only if the following three conditions are satisfied.

i) One-to-one:-

Mapping is one to correspondance.

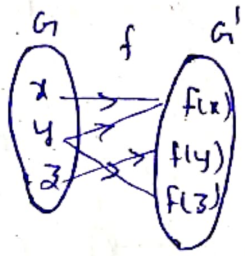
$f: G \to G'$ is one-to one in which Every element of G has a particular image of G' $\forall$ $x, y, z \in G$ &

$f(x), f(y), f(z) \in G'$



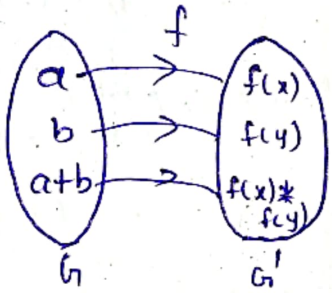
ii) One to:-

$f: G \to G'$ in which every element in G' has a preimage in G.



iii) Homomorphism:-

$f: G \to G'$ with binary operations $+$ & $*$ then $f(a+b) = f(a) * f(b)$



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Basic Counting principle:-

We have two basic counting principles

1. product rule.

2. Sum rule.

product rule:- In general, suppose that a procedure can be broken down into sequence of retasks namely $s_1, s_2, \ldots s_n$. If the first task s_2 can be alone in n_2 ways and the second task s_2 can done in n_2 ways. After the task has done and like wise rth task s_r. can be done in n_r ways. then they are $n_1, n_2, n_3 \ldots n_n$ ways of carrying out the procedure.

Sum rule: In general, suppose that task $s_1, s_2 \ldots s_n$ can be done in $n_1, n_2 \ldots n_n$ ways respectively and no two task can be done at the same time then the no. of ways of doing one of those task is $n_1 + n_2 \ldots n_n$.

Example:

1. There are 150 mathematic major student and 200 computer science major students at a college
(i) How many ways are there to select two representation so that one is mathematic major and other is computer science major
(ii) How many ways are there to pick one representatio who is either a mathematic major or computer science major.

Sol) No. of mathematic major students $n(M) = 150$

No. of computer science students $N(C) = 200$

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(i) product $250 \times 200 = 30000$ ways of selecting

(ii) sum $150 + 200 = 350$ ways

permutations:-

An orderd arrangment of objects from a set of distint objects is called a premutations.

An orderd arrangment of r elements of a set is called r permutations

The no of r permutations in a set with n elements is denoted by $p(n,r)$ or ^nP_r where $r \leq n$

$$\therefore p(n,r) = n(n-1)(n-2)\ldots(n-r+1) \text{ (or)}$$

$$= \frac{n!}{(n-r)!}$$

1. Find no. of 5-permutations of a set with 9 elements:

Sol) The no. of 5-permutations of a set with 9 elemen

$$P(\overset{n}{9},\overset{r}{5}) = 9 \times 8 \times 7 \times 6 \times 5 = 15120$$

(or)

$$P(9,5) = \frac{9!}{(9-5)!} = \frac{9 \times 8 \times 7 \times 6 \times 5 \times \cancel{4 \times 3 \times 2 \times 1}}{\cancel{4 \times 3 \times 2 \times 1}} = 15120$$

2. How many permutations of $\{a,b,c,d,e,f,g\}$ end with a.

Sol) The no. of given distinct objects in a given set without 'd' = 6.

$\therefore$ The required no. of permutations $= 6!$

$= 720$

3. How many permutations of letter A,B,C,D,E,F,G,H

(i) the string ED

(ii) The string ABC

(iii) The string ABC and CDE

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(i) considering ED as one object we have the following seven objects A, B, C, D, E, F, G, H. These seven objects can be permitted in $p(7,7) = 7! = 7\times6\times5\times4\times3\times2\times1$

$= 5040$ ways.

(ii) Considering ABC as one objects. These 6 objects can permitted in $p(6,6) = 6! = 720$ ways.

(iii) considering ABC and CDE contains common letters we include the string ABCDE in the permutations in the include both the string ABC & CDE. But it canot be used twice hence we have ABCDEFGH these four objects can be permitted as $4! = 4\times3\times2\times1 = 24$

4 In how many ways can 6 man and 6 women seat in a row.

(i) If any person seat not to anyother

(ii) If man & women must occupy alternative seats.

Sol:- First let man seats in a row

ie the no. of ways 6 man can seat on 6 seats $6P_6 = 6! = 720$ ways

similary women $6P_6 = 6!$

$= 720$

If any person seats any where =

720×720

$= 51400$ ways.

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The total no. of ways 6 man and 6 woman will seat in a row alternatively.

$$720 \times 720 + 720 \times 720$$

$$= 1036800.$$

5. How many ways are there to select to 1st prize winners & 2nd prize winner & third prize winner from hundred different people from a contest.

Sol: Since it is a matter of which prize person wins which prize, the no of ways of selecting these prize winners, the no. of permutations of a set of 100 elements.

$\therefore$ The required no. of possible is.

$$P(100,3) = {}^{100}P_3$$

$$= \frac{100!}{(100-3)!} = \frac{100 \times 99 \times 98 \times \cancel{97!}}{\cancel{97!}}$$

$$= 970,200 \text{ ways.}$$

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6. There are four bus lines between A and B, and three bus lines between B and C in how many ways can a man travel.

(i) By Bus from A to C via B

(ii) round trip from A to C via B.

(iii) round trip from A to C via B if he doesn't want to use a bus line more than once.

Sol: (i) There are four ways to go from A to B and three ways from B to C. Hence, there are

$4 \times 3 = 12$ ways to go from A to C via B.

(ii) There are 12 ways to go from A to C via B and 12 ways return. Hence, they are

$12 \times 12 = 144$ ways to travel round trip.

(iii) The man travel from A to B to C to B to A that is $A \to B \to C \to B \to A$

A man can travel four ways ~~A to B to C to B~~

A to B can travel 3 ways from B to C But he can travel in two ways from C to B and in three ways from B to A

Since he doesn't want to use a bus line more than once.

$$A \xrightarrow[4]{} B \xrightarrow[3]{} C \xrightarrow[2]{} B \xrightarrow[3]{} A$$

thus, there are 4, 3, 2, 3

$4 \times 3 \times 2 \times 3 = 72$ ways to can a man travel

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7. Suppose that there are eight runners in a race. the winner recives the gold medal and second winner reciuers the silver medal and the third place recives brown medal. how many different ways to award this medals. If all possible outcomes of the race can occure there are no ties.

Sol there are 8 choices for the Gold medal winner, 7 remaining choices for the silver medal winner and 6 for Bronz medal, so they are

$8 \times 7 \times 6 = 336$ ways to select 3 medals for 8 members.

permitation with repeation:

If there are n_1 indistinguisble objects of type 1, n_2 indistinguisble objects of type 2 and so on and n_k indistinguisble objects of type k and the no of different permitations n objects is

$$\frac{n!}{n_1! n_2! \cdots n_k!}$$

where $n = n_1 + n_2 \cdots n_k$.

1. find the no. of distinguisble permitations of letters from the following.

(i) BANANA..

(ii) CALCULUS

(iii) ~~ENGINERING~~ ENGINEERING

(iv) PEPPER.

i) In the given words, some of the letters are repeated. The word banana contains

B's - 1

A's - 3

N's = 2

Using all the letters of the words BANANA the no. of different strings

$$\frac{6!}{1\times 3!\times 2!}$$

$$\frac{6\times 5\times \cancel{4}^{2}\times \cancel{3\times 2\times 1}}{\cancel{3\times 2}\times 1\times \cancel{2}\times 1}$$

$= 60.$

(ii) CALCULUS

C's - 2

A's - 1

L's - 2

U's - 2

S's = 1

$$\frac{8!}{2!\times 2!\times 2!\times 1!\times 1!}$$

$$\frac{8\times 7\times \cancel{6}^{3}\times 5\times \cancel{4}^{2}\times 3\times \cancel{2\times 1}}{\cancel{2}\times 1\times \cancel{2}\times 1\times \cancel{2\times 1}}$$

$= 5040.$

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(iii) ENGINEERING

E's - 3

N's - 3

G's = 2

I's - 2

~~E's = 2~~

R's - 1

$$\frac{11!}{3! \times 3! \times 2! \times 2! \times 1!}$$

$$\frac{11 \times 10 \times \cancel{9}^{3} \times \cancel{8}^{4} \times 7 \times \cancel{6}^{3} \times 5 \times \cancel{4}^{2} \times \cancel{3 \times 2 \times 1}}{\cancel{3 \times 2 \times 1} \times \cancel{3} \times \cancel{2} \times 1 \times \cancel{2} \times 1 \times \cancel{2 \times 1} \times 1}$$

$= 277200.$

(iv) PEPPER

P's 3

E's - 2

R's - 1

$$\frac{6!}{3! \times 2! \times 1!}$$

$$\frac{6 \times 5 \times \cancel{4}^{2} \times \cancel{3 \times 2 \times 1}}{\cancel{3 \times 2 \times 1} \times \cancel{2} \times 1 \times 1}$$

$= 60$

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Find the no. of permutations of the word. MASSASAUGA ?

(i) In how many ways of there are 4 A's together?

(ii) In how many ways them begins with 's' letter?

In the given word, some of the letters are repeated. the word MASSASAUGA contains

M's - 1

A's - 4

S's - 3

U's - 1

G's - 1

(i) consider, replacing the 4 A's as 1 alphabet we need the arrangment of 7 alphabets

$$\frac{7!}{3!} = \frac{7\times6\times5\times4\times\cancel{3!}}{\cancel{3!}}$$

$= 840$ ways.

(ii) Consider, we are choosing 1 as before other alphabets 's'

we need the arrangment of 9 alphabets

$$\frac{9!}{4!\times2!} = \frac{9\times8\times7\times\cancel{6}^{3}\times5\times\cancel{4!}}{\cancel{4!}\times\cancel{2}\times1}$$

$= 7560$ ways

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Combinations:-

An unordered selection of ~~r elements~~ r-elements from a set is called r-combinations. Thus an r-combination is simply a subset with 'r' elements. The no. of r-combinations of a set with n-distinct elements is denoted by $C(n,r)$ (or) ${}^nC_r$

Formula:-

$$C(n,r) = \frac{n!}{r!(n-r)!} \quad \text{where } 0 \le r \le n$$

How many ways are there to select 5 players from a 10 members ~~tenis~~ tennis team. to make a trip for match at another school

The no. of ways selecting 5 members from 10 members of tennis group.

$$C(10,5) = \frac{10!}{5!(10-5)!}$$

$$\frac{10 \times 9 \times 8 \times 7 \times 6 \times 5!}{5!(5)!}$$

$$= \frac{\overset{2}{\cancel{10}} \times \overset{3}{\cancel{9}} \times \overset{2}{\cancel{8}} \times 7 \times \overset{3}{\cancel{6}} \times \cancel{5!}}{\cancel{5!} \times \cancel{5} \times \cancel{4} \times \cancel{3} \times \cancel{2} \times 1}$$

$$= 252.$$

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2. How many poker hands of 5 cards can be dealt from a standard deck of 52 cards? also how many ways are there to select 47 cards from a standard deck of 52 cards.

Sol: The no. of 5 cards hands is

$$C(52,5) = \frac{52!}{5!\,(52-5!)}$$

$$\frac{\overset{26}{\cancel{52}} \times \overset{17}{\cancel{51}} \times \overset{10}{\cancel{50}} \times 49 \times \overset{12}{\cancel{48}} \times \cancel{47!}}{\cancel{5} \times \cancel{4} \times \cancel{3} \times \cancel{2} \times 1 \times \cancel{47!}}$$

$$= 26 \times 17 \times 10 \times 49 \times 12$$

$$= 2598960.$$

3. Suppose that a department contains 10 men and 15 women. how many ways are there to form a comunity with 6 members if it must have more women than men.

Sol:- With the condition there must be more woreen that men in the community the options we get 4 women + 2 men, 5 women, 1 m 6 women + 0 men.

(i) 4 women + 2 men.

(ii) 5 women + 1 men.

(iii) 6 women + 0 men.

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(i) $C(15,4) + C(10,2)$

$$\frac{15!}{4! \times 11!} + \frac{10!}{2! \times 8!}$$

$= 1365 \times 45$

$= \cancel{1410}\ 61425$

(ii) $C(15,5) + C(10,1)$

$$\frac{15!}{5! \times 10!} + \frac{10!}{1! \times 9!}$$

3003×10

$= \cancel{3013}\ 30030$

(iii) $C(15,6) + C(10,0)$

$$\frac{15!}{6! \times 9!} + \frac{10!}{0! \times 10!}$$

$5005 + 0$

$61425 + 30030$

$\cancel{1410 + 3013 +}\ 5005$

$\cancel{= 9428} = 96460$

4. A women have 11 close relatives she wish to invite 5 of them to dinner. In how many ways she can invite them in the situation as

(i) There is no ristriction for the choice

(ii) Two particular persons will not attend seperatly.

(iii) Two particular person will not attend together.

Sol:- Given, that a women has 11 close friends she invites 5 of them to dinner in

$^{11}C_5$ ways.

case(i) when I don't attend the party

$^{9}C_4$ ways.

$$= \frac{9!}{4!(5!)} = \frac{\cancel{9}^{3} \times \cancel{8}^{2} \times 7 \times \cancel{6}^{3} \times \cancel{5!}}{\cancel{4} \times \cancel{3} \times \cancel{2} \times 1 \times \cancel{5!}} = 126$$

case(ii) when two don't attend the party

$^{9}C_4$ ways

$$= \frac{9!}{4! \times 5!} = 126$$

case(iii) when ~~both~~ of them don't attend the party

$^{9}C_5$

$$\frac{9!}{5! \times 4!} = 126$$

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BINOMIAL

STATEMENT:- let x, y be any two variables and let n be a non-negative integer. then

$$(x+y)^n = C_0^n x^n y^0 + {}^nC_1 x^{n-1} y^1 + \cdots + C_n^n x^{n-n} y^n.$$

$$= \sum_{i=0}^{n} C_i^n x^{n-i} \cdot y^i \quad \text{where } 0 \le i \le n.$$

proof:- we prove by using combinational concept.

when we expand $(x+y)^n$, the terms in the product are in the form $x^{n-i} \cdot y^i$ It is enough to choose $(n-i)$, x from sum n.

ie the coefficient of $x^{n-i} \cdot y^i$ is ${}^nC_{n-i}$

Thus $(x+y)^n = \sum_{i=0}^{n} C_i^n x^{n-i} \cdot y^i$

. find the coefficient of the following.

(i) $x^5 y^7$ in the expansion of $(x+3y)^{12}$

Sol:- from the binomal theroem. we know that

$$(x+y)^n = \sum_{i=0}^{n} C_i^n x^{n-i} \cdot y^i$$

form the given expansion $(x+3y)^{12}$

$$(x+3y)^{12} = \sum_{i=0}^{12} C_i^{12} x^{12-i} \cdot 3y^i$$

$$\sum_{i=0}^{12} C_i^{12} x^{12-i} \cdot x(3)^i \cdot y^i$$

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where $x^5 y^7$

$^{12}C_7 \cdot x^{12-7} \cdot (3)^7 \cdot y^7$

$= {}^{12}C_7\, x^5\, (3)^7 \cdot y^7 = {}^{12}C_7\, 3^7\, x^5\, y^7$

$^{12}C_7 = \dfrac{12!}{7!\times 5!}$

$\dfrac{12!}{7!\times 5!} \times (3)^7 \cdot x^5 y^7$

$x^5 y^7 \cdot 1732104.$

The coffecient of $x^5 y^7 = 1732104.$

<u>Multinomial theroem:</u>

Let $a_1, a_2, a_3 \ldots\ldots a_n$ belongs R (E) and n belongs to Z (E) with $n \geq 1$ then $(a_1 + a_2 + \ldots + a_n)^n$

$$\sum_{0 < k_1 k_2 \ldots k_m \leq n} \begin{bmatrix} n \\ k_1, k_2 \ldots k_m \end{bmatrix} a_1^{k_1} a_2^{k_2} \ldots a_m^{k_m}$$

$k_1 + k_2 + \ldots + k_m = n.$

Here, the sum is indexed, or, ordered m tuples, of Integers $k_1, k_2, \ldots\ldots k_m$ such that $0 \leq k_1, k_2 \ldots k_m \leq n$ and $k_1 + k_2 + \ldots + k_m = n$

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UNIT-III

Notes Continuation.....

Second Order Linear Homogeneous Recurrence Relations:

Method of Characteristic Roots:-

This method is used for evaluation of the solution of recurrence relation.

Consider an equation of the form

$$a_n + c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_m a_{n-m} = 0 \quad -①$$

is called linear homogeneous recurrence relation.

The characteristic equation of ① is obtained by taking $a_n = r^n$.

Then ① becomes.

$$r^n + c_1 r^{n-1} + c_2 r^{n-2} + \cdots + c_m r^{n-m} = 0. \quad -②$$

Divide throughout the equation ② by r^{n-m},

$$r^m + c_1 r^{m-1} + c_2 r^{m-2} + \cdots + c_m = 0. \quad -③$$

is called characteristic equation.

The General solution of ③ depends upon the nature of the roots.

Let the roots of eq ③ be $a_0, a_1, \ldots a_m$.

① All roots are real and distinct :-

Then G.S is.

$$a_n = c_1 a_1^n + c_2 a_2^n + \cdots + c_m a_m^n.$$

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② Two are equal and remaining are distinct:-

Then G.S is.

$$a_n = (c_1 + c_2 n) a_1^n + c_3 a_3^n + \cdots + c_m a_m^n .$$

③ Three roots are equal remaining are distinct:-

The G.S is.

$$a_n = (c_1 + c_2 n + c_3 n^2) a_1^n + c_4 a_4^n + \cdots + c_m a_m^n$$

④ Two roots are complex (ie) $\alpha \pm i\beta$.) roots and remaining are distinct:-

The G.S is

$$a_n = r^n (c_1 \cos n\theta + c_2 \sin n\theta) + c_3 a_3^n + \cdots + c_m a_m^n$$

where $r = \sqrt{\alpha^2 + \beta^2}$ and $\theta = \tan^{-1}\left(\frac{\beta}{\alpha}\right)$

Problems:-

1.) Solve the recurrence relation $a_n - 7a_{n-1} + 10a_{n-2} = 0$

Sol:- The given recurrence relation is.

$$a_n - 7a_{n-1} + 10a_{n-2} = 0 . \quad \text{--- (1)}$$

The characteristic equation of (1) is

Put $a_n = r^n$.

$$r^n - 7r^{n-1} + 10r^{n-2} = 0 .$$

Divide throughout the eqn by r^{n-2}, we get

$$r^2 - 7r + 10 = 0 .$$

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$\Rightarrow r^2 - 7r + 10 = 0$

$\Rightarrow r^2 - 5r - 2r + 10 = 0$

$\Rightarrow r(r-5) - 2(r-5) = 0$

$\Rightarrow (r-2)(r-5) = 0$

$\Rightarrow r = 2, 5.$

Soln. is $a_n = c_1 2^n + c_2 5^n.$

② Solve $a_n - 7a_{n-1} + 12a_{n-2} = 0$, $a_0 = 1$, $a_1 = 1$.

Sol:- Given $a_n - 7a_{n-1} + 12a_{n-2} = 0$, $a_0 = 1, a_1 = 1$ — ①

The ch. eqn of ① is

$r^2 - 7r + 12 = 0.$

$r^2 - 4r - 3r + 12 = 0$

$r(r-4) - 3(r-4) = 0$

$(r-3)(r-4) = 0.$

$r = 3, 4.$

The G.S is $\therefore a_n = c_1 3^n + c_2 4^n.$

At $n=0$, $a_0 = c_1 3^0 + c_2 4^0.$

$\Rightarrow 1 = c_1 + c_2.$ $(\because a_0 = 1)$

$\Rightarrow \boxed{c_1 + c_2 = 1}$ — ②

At $n=1$, $a_1 = c_1 3^1 + c_2 4^1$

$\Rightarrow 1 = 3c_1 + 4c_2$ $(\because a_1 = 1)$

$\Rightarrow \boxed{3c_1 + 4c_2 = 1}$ — ③.

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On solving ① & ③, we get.

$c_1 = 3, \quad c_2 = -2.$

$\therefore$ Solution is $a_n = 3(3^n) - 2(4^n)$

③ Solve $a_n - 8a_{n-1} + 21a_{n-2} - 18a_{n-3} = 0, n > 3$.

Sol: Given $a_n - 8a_{n-1} + 21a_{n-2} - 18a_{n-3} = 0$.

The ch. equation is.

$$r^3 - 8r^2 + 21r - 18 = 0.$$

To get the roots by synthetic division.

$$\begin{array}{r|rrrr} r=2 & 1 & -8 & 21 & -18 \\ & 0 & 2 & -12 & 18 \\ \hline & 1 & -6 & 9 & \boxed{0.} \end{array}$$

$r = 2, \quad r^2 - 6r + 9 = 0$

$(r-3)^2 = 0$

$r = 3, 3.$

$\therefore \quad r = 2, 3, 3.$

The G.S is $a_n = c_1 2^n + (c_2 + c_3 n) 3^n.$

④ Solve $a_n + a_{n-1} + a_{n-2} = 0$.

Sol: Given $a_n + a_{n-1} + a_{n-2} = 0$.

The Ch. $r^2 + r + 1 = 0$. $\quad a = 1, b = 1, c = 1$

$$r = \frac{-1 \pm \sqrt{1-4}}{2}. \quad \left(r = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \right)$$

$$= \frac{-1 \pm i\sqrt{3}}{2} = -\frac{1}{2} \pm i\frac{\sqrt{3}}{2}.$$

Here $\alpha = \frac{-1}{2}, \quad \beta = \frac{\sqrt{3}}{2}.$

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$\therefore$ The g.s for complex roots is.

$$a_n = r^n (c_1 \cos n\theta + c_2 \sin n\theta).$$

$$r = \sqrt{\alpha^2+\beta^2} = \sqrt{\frac{1}{4}+\frac{3}{4}} = \sqrt{\frac{4}{4}} = 1.$$

$$\theta = \tan^{-1}\left(\frac{\beta}{\alpha}\right) = \tan^{-1}\left(\frac{\frac{\sqrt{3}}{2}}{-\frac{1}{2}}\right) = \tan^{-1}(-\sqrt{3})$$

$$= \tan^{-1}\left(\tan\frac{2\pi}{3}\right) = \frac{2\pi}{3}.$$

$$\therefore a_n = 1^n \left(c_1 \cos n\left(\frac{2\pi}{3}\right) + c_2 \sin n\left(\frac{2\pi}{3}\right)\right)$$

* 5. Solve the recurrence relation $f_n = f_{n-1} + f_{n-2}$, $n \geq 2$ with the initial conditions $f_0 = 1$, $f_1 = 1$.

(or)

Solve the recurrence relation of Fibonacci Sequence.

Sol:- Fibonacci Sequence:-

The sequence $\{1, 1, 2, 3, 5, 8, 13, \ldots\}$ is called a fibonacci sequence. The recurrence relation for the Fibonacci sequence is.

$$f_n = f_{n-1} + f_{n-2}, \quad n \geq 2, \quad f_0 = 1, f_1 = 1.$$

Given. $f_n = f_{n-1} + f_{n-2}$.

$$\Rightarrow f_n - f_{n-1} - f_{n-2} = 0. \quad (1)$$

The ch. eq of (1) is.

$$r^2 - r - 1 = 0.$$

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$$r = \frac{1 \pm \sqrt{1+4}}{2} = \frac{1 \pm \sqrt{5}}{2} = \frac{1+\sqrt{5}}{2}, \frac{1-\sqrt{5}}{2}.$$

$\therefore$ Soln. is.

$$f_n = c_1\left(\frac{1+\sqrt{5}}{2}\right)^n + c_2\left(\frac{1-\sqrt{5}}{2}\right)^n \quad -(2)$$

Given $f_0 = 1$. $f_1 = 1$.

Put $n=0$, $f_0 = c_1\left(\frac{1+\sqrt{5}}{2}\right)^0 + c_2\left(\frac{1-\sqrt{5}}{2}\right)^0$.

$$1 = c_1 + c_2$$

$$\boxed{c_1 + c_2 = 1}, \quad -(3).$$

Put $n=1$, $f_1 = c_1\left(\frac{1+\sqrt{5}}{2}\right)^1 + c_2\left(\frac{1-\sqrt{5}}{2}\right)^1$

$$\Rightarrow \quad 1 = c_1\left(\frac{1+\sqrt{5}}{2}\right) + c_2\left(\frac{1-\sqrt{5}}{2}\right).$$

$$\Rightarrow \quad 1 = c_1\left(\frac{1+\sqrt{5}}{2}\right) + (1-c_1)\left(\frac{1-\sqrt{5}}{2}\right)$$

(from (3) $c_2 = 1 - c_1$)

$$\Rightarrow \quad 1 = c_1\left(\frac{1+\sqrt{5}}{2}\right) + \frac{1-\sqrt{5}}{2} - c_1\left(\frac{1-\sqrt{5}}{2}\right)$$

$$= c_1\left(\frac{1+\sqrt{5}}{2} - \frac{1-\sqrt{5}}{2}\right) + \frac{1-\sqrt{5}}{2}.$$

$$= c_1\left(\frac{1+\sqrt{5}-1+\sqrt{5}}{2}\right) + \frac{1-\sqrt{5}}{2}.$$

$$1 = c_1\left(\frac{2\sqrt{5}}{2}\right) + \frac{1-\sqrt{5}}{2}.$$

$$1 - \frac{1-\sqrt{5}}{2} = c_1\left(\frac{2\sqrt{5}}{2}\right).$$

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$$\Rightarrow \frac{2-1+\sqrt{5}}{2} = c_1\left(\frac{2\sqrt{5}}{2}\right)$$

$$\Rightarrow 1+\sqrt{5} = c_1(2\sqrt{5})$$

$$\Rightarrow c_1 = \frac{1+\sqrt{5}}{2\sqrt{5}}.$$

$$\because c_2 = 1-c_1 = 1-\frac{1+\sqrt{5}}{2\sqrt{5}} = \frac{\sqrt{5}-1}{2\sqrt{5}}.$$

Sub c_1 & c_2 values in (2).

$$f_n = \left(\frac{1+\sqrt{5}}{2\sqrt{5}}\right)\left(\frac{1+\sqrt{5}}{2}\right)^n + \left(\frac{\sqrt{5}-1}{2\sqrt{5}}\right)\left(\frac{1-\sqrt{5}}{2}\right)^n.$$

(6)* Solve $U_{n+2} - U_{n+1} - 12U_n = 0$, $U_0 = 0$, $U_1 = 13$.

Sol: Given $U_{n+2} - U_{n+1} - 12U_n = 0$, $U_0 = 0$, $U_1 = 13$.

The ch. eqn is.

$$r^2 - r - 12 = 0.$$

$$r^2 - 4r + 3r - 12 = 0$$

$$r(r-4) + 3(r-4) = 0.$$

$$(r+3)(r-4) = 0$$

$$r = -3, 4$$

$\therefore$ The general solution is

$$U_n = c_1(-3)^n + c_2(4)^n$$

$\because U_0 = 0$, $U_1 = 13$.

Put $n=0$, $U_0 = c_1(-3)^0 + c_2(4)^0$.

$$0 = c_1 + c_2.$$

$$\Rightarrow \boxed{c_2 = -c_1}.$$

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$U_1 = 13.$

Put $n=1$, $U_1 = c_1(-3)^1 + c_2(4)^1.$

$$13 = -3c_1 + 4c_2.$$

$$13 = -3c_1 + 4(-c_1)$$

$$13 = -3c_1 - 4c_1$$

$$13 = -7c_1.$$

$$\Rightarrow c_1 = -\frac{13}{7}.$$

$$\therefore c_2 = -c_1 = \frac{13}{7}.$$

- Sub. c_1 & c_2 values in the solution.

$$U_n = \left(-\frac{13}{7}\right)(-3)^n + \left(\frac{13}{7}\right)(4)^n.$$

(7)* Solve the recurrence relation $a_n = a_{n-1} + 2a_{n-2}$. with the initial conditions $a_0 = 2$, $a_1 = 7$.

Sol:- Given $a_n = a_{n-1} + 2a_{n-2}$, $a_0 = 2$, $a_1 = 7$.

$$\Rightarrow a_n - a_{n-1} - 2a_{n-2} = 0.$$

The ch. eqn is $r^2 - r - 2 = 0.$

$$\Rightarrow r^2 - 2r + r - 2 = 0$$

$$\Rightarrow r(r-2) + 1(r-2) = 0$$

$$\Rightarrow (r+1)(r-2) = 0$$

$$\Rightarrow r = -1, 2.$$

The G.S is $a_n = c_1(-1)^n + c_2(2)^n.$

Given $a_0 = 2$, $a_1 = 7.$

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Put $n=0$, $a_0 = c_1(-1)^0 + c_2(2)^0$.

$$2 = c_1 + c_2$$

$$c_1 + c_2 = 2$$

$$\boxed{c_2 = 2 - c_1}$$

Put $n=1$, $a_1 = c_1(-1)^1 + c_2(2)^1$.

$$7 = -c_1 + 2c_2.$$

$$7 = -c_1 + 2(2-c_1)$$

$$7 = -c_1 + 4 - 2c_1$$

$$7 = -3c_1 + 4.$$

$$-3c_1 = 7 - 4.$$

$$-3c_1 = 3.$$

$$c_1 = -1.$$

$$\therefore c_2 = 2 - c_1 = 2-(-1) = 3.$$

$\therefore c_1 = -1$, $c_2 = 3$.

Sub c_1 & c_2 values in the soln.

$$a_n = (-1)(-1)^n + 3(2)^n.$$

5) ★ Solve recurrence relation $a_n = 4a_{n-1} + 12a_{n-2}$. $n \geq 2$, $a_0 = 4$ & $a_1 = \frac{10}{3}$.

Sol: Given $a_n = 4a_{n-1} + 12a_{n-2}$.

$$a_n - 4a_{n-1} - 12a_{n-2} = 0.$$

The ch. eqn is

$$r^2 - 4r - 12 = 0.$$

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$\Rightarrow (r+2)(r-6)=0$

$\Rightarrow r=-2, 6.$

Soln is $a_n = c_1(-2)^n + c_2(6)^n$ — (1).

$a_0 = 4$, Put $n=0$ in (1)

$a_0 = c_1(-2)^0 + c_2(6)^0$.

$4 = c_1 + c_2$.

$c_1 + c_2 = 4 \Rightarrow c_2 = 4 - c_1$.

$a_1 = \frac{10}{3}$, Put $n=1$ in (1).

$a_1 = c_1(-2)^1 + c_2(6)^1$

$\frac{10}{3} = -2c_1 + 6c_2$

$\frac{10}{3} = -2c_1 + 6(4-c_1)$.

$\frac{10}{3} = -2c_1 + 24 - 6c_1$

$\Rightarrow -8c_1 = \frac{10}{3} - 24$.

$\Rightarrow -8c_1 = -\frac{62}{3}$

$\Rightarrow c_1 = \frac{\cancel{62}\,31}{3\times\cancel{8}\,4} = \frac{31}{12}$

$\therefore c_1 = \frac{31}{12}$.

$c_2 = 4 - c_1 = 4 - \frac{31}{12} = \frac{48-31}{12}$

$\therefore c_2 = \frac{17}{12}$.

Sub c_1 & c_2 in the solution.

$\therefore a_n = \frac{31}{12}(-2)^n + \frac{17}{12}(6)^n$.

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Solution Of Non-homogeneous Recurrence Relation:-

A linear inhomogeneous or non-homogeneous recurrence relation with constant coefficients is of the form.

$$a_n + c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_m a_{n-m} = f(n) \quad (1)$$

where $c_1, c_2, \ldots c_m$ are real constants and $f(n)$ is a given real valued function of n.

A general solution of the recurrence relation (1) is given by.

$$a_n = a_n^{(H)} + a_n^{(P)}$$

where $a_n^{(H)}$ is the general solution of the homogeneous part of the relation (1) (ie) relation (1) with $f(n)=0$, and $a_n^{(P)}$ is any particular solution of the relation (1).

To find the particular solution $a_n^{(P)}$ of (1). the following are some of the special cases.

(ie) Function $f(n)$ has any one of the following forms.

(i) Polynomial in n.
(ii) a constant
(iii) Powers of constant.

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then we may guess the forms of the particular solution and exactly find out by the method of Undetermined coefficients.

The table shows the particular solution for given $f(n)$.

S.No.	Function $f(n)$	Form of particular Solution. $a_n^{(P)}$.
1.)	A constant, c.	A constant, d
2.)	A linear function $c_0 + c_1 n$.	A linear function $d_0 + d_1 n$.
3)	$c_0 + c_1 n + c_2 n^2$.	$d_0 + d_1 n + d_2 n^2$.
4.)	An mth degree Polynomial. $c_0 + c_1 n + c_2 n^2 + \cdots + c_m n^m$	An mth degree Polynomial. $d_0 + d_1 n + d_2 n^2 + \cdots + d_m n^m$.
5)	r^n, $r \in R$.	$d r^n$.

If n initial conditions are given, then it will give the solution by solving the system of n equations with n unknowns.

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Problems

1. Solve the recurrence relation $a_n = 3a_{n-1} + 2^n$, $a_0 = 1$

Sol:- Given $a_n = 3a_{n-1} + 2^n$, $a_0 = 1$.

$$\Rightarrow a_n - 3a_{n-1} = 2^n \quad (1), \quad a_0 = 1.$$

is a non-homogeneous recurrence relation.

The associated homogeneous recurrence relation is. $a_n - 3a_{n-1} = 0$.

Its characteristic equation is

$$r - 3 = 0$$
$$r = 3.$$

$\therefore$ The homogeneous solution is.

$$a_n^{(H)} = c_1 3^n.$$

Since R.H.S of recurrence relation is 2^n.

Let the particular solution of the recurrence relation be. $a_n^{(P)} = d\,2^n$.

$$a_n = d\,2^n, \quad a_{n-1} = d\,2^{n-1}.$$

Sub. the above in (1), we get.

$$d\,2^n - 3 \cdot d \cdot 2^{n-1} = 2^n.$$

$$\Rightarrow 2^n [d - 3d \cdot 2^{-1}] = 2^n.$$

$$\Rightarrow d\left(1 - \frac{3}{2}\right) = 1.$$

$$\Rightarrow d\left(-\frac{1}{2}\right) = 1 \Rightarrow d = -2.$$

$$\therefore a_n^{(P)} = (-2)\,2^n = -(-2)^{n+1}$$

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Hence the general solution is.

$$a_n = a_n^{(H)} + a_n^{(P)}$$

$$\therefore \quad a_n = c_1(3)^n - (2)^{n+1} \quad -(2).$$

Given $a_0 = 1$.

Put $n=0$ in (2).

$$a_0 = c_1(3)^0 - (2)^{0+1}.$$

$$1 = c_1 - 2.$$

$$\Rightarrow \quad c_1 = 3.$$

Sub. c_1 value in (2).

$$\therefore a_n = 3(3)^n - 2^{n+1} = 3^{n+1} - 2^{n+1}.$$

*2.) Solve the recurrence relation

$$a_n - 7a_{n-1} + 10a_{n-2} = 6 + 8n.$$

Sol:- Given equation is

$$a_n - 7a_{n-1} + 10a_{n-2} = 6 + 8n \quad -(1).$$

ch. eqn. is $r^2 - 7r + 10 = 0$ of homogeneous equation of (1).

The roots are $r = 2, 5$.

$$a_n^{(H)} = c_1 2^n + c_2 5^n$$

Let $a_n^{(P)} = d_0 + d_1 n \quad -(2).$

Since the R.H.S is a linear polynomial.

$$a_n = d_0 + d_1 n$$

$$a_{n-1} = d_0 + d_1(n-1).$$

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$a_{n-2} = d_0 + d_1(n-2)$

Sub. the above values in (1)

$(d_0 + d_1 n) - 7(d_0 + d_1(n-1)) + 10(d_0 + d_1(n-2)) = 6+8n$

$\Rightarrow d_0 + d_1 n - 7d_0 - 7d_1(n-1) + 10d_0 + 10d_1(n-2) = 6+8n$

$\Rightarrow 4d_0 + d_1 n - 7d_1 n + 7d_1 + 10d_1 - 20d_1 = 6+8n.$

$\Rightarrow (4d_0 - 13d_1) + (d_1 - 7d_1 + 10d_1)n = 6+8n.$

$\Rightarrow (4d_0 - 13d_1) + 4d_1 n = 6+8n.$

Compare the constants and coeff's. on both sides,

we get. $4d_0 - 13d_1 = 6$ and $4d_1 = 8$

$\boxed{d_1 = 2.}$

$\Rightarrow 4d_0 - 13(2) = 6$

$\Rightarrow 4d_0 - 26 = 6$

$\Rightarrow 4d_0 = 32$

$\Rightarrow \boxed{d_0 = 8}$

Substitute d_0 & d_1 values in (2)

$\therefore a_n^{(P)} = 8 + 2n$

$\therefore$ General solution is

$a_n = a_n^{(H)} + a_n^{(P)}$

$a_n = c_1(2)^n + c_2(5)^n + 8 + 2n.$

(3). Solve the recurrence relation $a_n - 2a_{n-1} + a_{n-2} = 2$. with $a_0 = 25$, $a_1 = 16$.

Sol: Given $a_n - 2a_{n-1} + a_{n-2} = 2$, $a_0 = 25$, $a_1 = 16$.

The homogeneous recurrence relation is

$a_n - 2a_{n-1} + a_{n-2} = 0.$

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The ch. eqn is. $r^2 - 2r + 1 = 0$

$\Rightarrow (r-1)^2 = 0$

$\Rightarrow r = 1, 1.$

$\therefore$ The homogeneous solution is

$$a_n^{(H)} = (c_1 + c_2 n) 1^n.$$

Since R.H.S of recurrence relation is 2, a constant we assume the particular solution of the recurrence to be.

$$a_n^{(H)} = d, \text{ a constant.}$$

Using this solution in the given recurrence relation, we get $d - 2d + d = 2$.

(ie) $0 = 2$, which is impossible.

Thus our assumption is wrong.

Now, we assume that $a_n^{(P)} = nd$.

Using this solution in the given recurrence relation, we get.

$$nd - 2(n-1)d + (n-2)d = 2.$$

$\Rightarrow nd - 2nd + nd + 2d - 2d = 2.$

$\Rightarrow 0 = 2$, which is also impossible.

Thus, our assumption is wrong.

Now we assume that $a_n^{(H)} = n^2 d$.

Using this solution in the given recurrence relation, we get $n^2 d - 2(n-1)^2 d + (n-2)^2 d = 2$.

$\Rightarrow n^2 d - 2(n^2 - 2n + 1)d + (n^2 - 4n + 4)d = 2$

$\Rightarrow n^2 - 2n^2 d + 4nd - 2d + n^2 d - 4nd + 4d = 2$

$\Rightarrow d = 1.$

$\therefore a_n^{(P)} = n^2$

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Hence, the general solution is

$$a_n = a_n^{(H)} + a_n^{(P)}$$

$$\Rightarrow a_n = [c_1 + c_2 n]1^n + n^2.$$

Given $a_0 = 25$ and $a_1 = 16$.

Now, $a_0 = 25$

$$\Rightarrow (c_1 + 0) + 0 = 25$$

$$\Rightarrow c_1 = 25.$$

Also, $a_1 = 16$.

$$(c_1 + c_2) + 1^2 = 16$$

$$\Rightarrow c_1 + c_2 + 1 = 16$$

$$\Rightarrow c_2 = 16 - 1 - 25 = -10.$$

$\therefore$ The required solution is

$$a_n = (25 - 10n)1^n + n^2$$

$$\Rightarrow a_n = n^2 - 10n + 25.$$

4. Solve the recurrence relation $a_n + 4a_{n-1} + 4a_{n-2} = 8$ for $n \geq 2$ given that $a_0 = 1$ and $a_1 = 2$.

Sol: The given recurrence relation is

$$a_n + 4a_{n-1} + 4a_{n-2} = 8 \text{ and } a_0 = 1, a_1 = 2.$$

The ch. equation is $r^2 + 4r + 4 = 0$ for homogeneous equation $a_n + 4a_{n-1} + 4a_{n-2} = 0$.

$$r^2 + 4r + 4 = 0$$

$$(r+2)^2 = 0$$

$$r = -2, -2.$$

$\therefore$ The homogeneous solution.

$$a_n^{(H)} = (c_1 + c_2 n)(-2)^n.$$

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Since the R.H.S of the recurrence relation is 8. and -2 is the characteristic root of the equation repeated twice., we assume the particular solution of the recurrence relation to be.

$$a_n^{(P)} = d.$$

Putting $a_n^{(P)}$ for a_n in the given relation, we get,

$$d + 4d + 4d = 8.$$

$$9d = 8 \Rightarrow d = \frac{8}{9}.$$

$$\therefore a_n^{(P)} = \frac{8}{9}.$$

$\therefore$ The general solution of a_n is.

$$a_n = a_n^{(H)} + a_n^{(P)}.$$

$$a_n = (c_1 + c_2 n)(-2)^n + \frac{8}{9}.$$

Given $a_0 = 1$, $a_1 = 2$.

Put $n=0$ in the general solution, we get

$$a_0 = (c_1 + c_2(0))(-2)^0 + \frac{8}{9}$$

$$1 = c_1 + \frac{8}{9} \Rightarrow \boxed{c_1 = 1 - \frac{8}{9} = \frac{1}{9}}$$

Put $n=1$ in the general soln. we get

$$a_1 = (c_1 + c_2(1))(-2)^1 + \frac{8}{9}$$

$$2 = \left(\frac{1}{9} + c_2\right)(-2) + \frac{8}{9}$$

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$$2 = -\frac{2}{9} - 2c_2 + \frac{8}{9}$$

$$2 = \frac{6}{9} - 2c_2$$

$$2 = \frac{2}{3} - 2c_2 \Rightarrow -2c_2 = 2 - \frac{2}{3}$$

$$\Rightarrow -2c_2 = +\frac{4}{3} \Rightarrow \boxed{c_2 = -\frac{2}{3}}$$

Sub. c_1 & c_2 values in the general soln.

$$\therefore a_n = \left(\frac{1}{9} - \frac{2}{3}n\right)(-2)^n + \frac{8}{9}$$

(5.) Find the particular solution of the recurrence relation $a_{n+2} - 4a_{n+1} + 4a_n = 2^n$.

Sol: Given recurrence relation is

$$a_{n+2} - 4a_{n+1} + 4a_n = 2^n.$$

~~Since R.H.S is 2 and~~

The ch. eqn of $a_{n+2} - 4a_{n+1} + 4a_n = 0$ is. $r^2 - 4r + 4 = 0 \Rightarrow (r-2)^2 = 0$

$$\Rightarrow r = 2, 2.$$

$\therefore$ The homogeneous solution is.

$$a_n^{(H)} = (c_1 + c_2 n)\, 2^n.$$

Since R.H.S is 2^n and 2 is the double root of the ch. eqn., let us assume that the particular solution to be

$$a_n^{(P)} = d\, 2^n.$$

Using this solution in the given recurrence relation, we get.

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$$d2^{n+2} - 4d2^{n+1} + 4d2^n = 2^n.$$

$$\Rightarrow 2^n(d2^2 - 4d\cdot 2 + 4d) = 2^n$$

$$\Rightarrow 4d - 8d + 4d = 1.$$

$$\Rightarrow 0 = 1, \text{ which is impossible.}$$

$\Rightarrow$ Thus, our assumption is wrong.

Now, we assume that $a_n^{(P)} = dn2^n$.

Using this solution in the given recurrence relation, we get.

$$dn2^n - 4d(n-1)2^{n-1} + 4d(n-2)2^{n-2} = 2^n$$

$$\Rightarrow 2^n dn - 4dn2^{n-1} + 4d2^{n-1} + 4dn2^{n-2} - 8d2^{n-2} = 2^n.$$

$$\Rightarrow 2^n\left[dn - \frac{4dn}{2} + \frac{4d}{2} + \frac{4dn}{2^2} - \frac{8d}{2^2}\right] = 2^n$$

$$\Rightarrow dn - 2dn + 2d + dn - 2d = 1$$

$$\Rightarrow 0 = 1 \text{ which is impossible.}$$

$\Rightarrow$ Thus our assumption is wrong.

Now, we assume that $a_n^{(P)} = dn^2 2^n$.

Using this solution in the given recurrence relation, we get.

$$dn^2 2^n - 4d(n-1)^2 2^{n-1} + 4d(n-2)^2 2^{n-2} = 2^n.$$

$$\Rightarrow 4dn^2 - 8d(n-1)^2 + 4d(n-2)^2 = 4. \text{ (divide through by } 2^{n-2}).$$

$$\Rightarrow dn^2 - 2d(n-1)^2 + d(n-2)^2 = 1$$

$$\Rightarrow dn^2 - 2d(n^2 - 2n + 1) + d(n^2 - 4n + 4) = 1$$

$$\Rightarrow dn^2 - 2dn^2 + 4dn - 2d + dn^2 - 4dn + 4d = 1.$$

$$\Rightarrow 2d = 1$$

$$\Rightarrow d = \frac{1}{2}.$$

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$\therefore$ The particular solution ~~of~~ to 2^n is

$$a_n^{(P)} = \frac{1}{2} n^2 2^n = n^2 2^{n-1}$$

(6) Solve the recurrence relation $a_{n+2}^2 - 5a_{n+1}^2 + 6a_n^2 = 7n$ for $n \geq 0$. Given $a_0 = a_1 = 1$.

Sol:- Given recurrence relation is

$$a_{n+2}^2 - 5a_{n+1}^2 + 6a_n^2 = 7n, \quad a_0 = 1, a_1 = 1.$$

Let $b_n = a_n^2$. then the given relation becomes

$$b_{n+2} - 5b_{n+1} + 6b_n = 7n. \quad \text{--- (1)}$$

For the homogeneous part of this relation, the characteristic equation is.

$$r^2 - 5r + 6 = 0.$$
$$(r-2)(r-3) = 0$$
$$r = 3, 2 = 2, 3$$

$$\therefore \quad b_n^{(H)} = c_1 2^n + c_2 3^n.$$

Since the R.H.S of (1) is $7n$. ~~and~~ then we assume the particular solution of the recurrence relation (1) to be

$$b_n^{(P)} = d_0 + d_1 n.$$

Putting $b_n^{(P)}$ for b_n in the ~~given~~ relation (1), we get.

$$d_0 + d_1(n+2) - 5(d_0 + d_1(n+1)) + 6(d_0 + d_1 n) = 7n.$$

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$\Rightarrow d_0 + d_1 n + 2d_1 - 5d_0 - 5d_1 n - 5d_1 + 6d_0 + 6d_1 n = 7n$

$\Rightarrow (d_0 + 2d_1 - 5d_0 - 5d_1 + 6d_0) + d_1 n - 5d_1 n + 6d_1 n = 7n$

$\Rightarrow (2d_0 - 3d_1) + n(2d_1) = 7n.$

Comparing the coeff. of n and constants on both sides, we get.

$2d_0 - 3d_1 = 0$ $\qquad$ $2d_1 = 7$

$d_1 = \frac{7}{2}.$

$2d_0 = 3d_1$

$d_0 = \frac{3}{2} d_1$

$= \frac{3}{2} \times \frac{7}{2} = \frac{21}{4}.$

$\therefore d_0 = \frac{21}{4}$ and $d_1 = \frac{7}{2}.$

Putting these into $b_n^{(P)}$, then.

$b_n^{(P)} = \frac{21}{4} + \frac{7}{2} n.$

$\therefore b_n = b_n^{(H)} + b_n^{(P)}.$

$b_n = c_1 2^n + c_2 3^n + \frac{21}{4} + \frac{7}{2} n.$

Given that. $a_0 = 1$, $a_1 = 1.$

We supposed $b_n = a_n^2.$

for. $n = 0.$ $\qquad$ $b_0 = a_0^2$

$\Rightarrow b_0 = 1^2 = 1.$

for $n = 1$, $\qquad$ $b_1 = a_1^2$

$\Rightarrow b_1 = 1^2 = 1$

$\therefore b_0 = 1, b_1 = 1.$

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$\therefore$ for $b_0 = 1$, $\quad 1 = c_1 2^0 + c_2 3^0 + \frac{21}{4} + \frac{7}{2}(0)$

$$1 = c_1 + c_2 + \frac{21}{4}.$$

$$c_1 + c_2 = 1 - \frac{21}{4} = \frac{17}{4}.$$

for $b_1 = 1$, $\quad 1 = c_1 \cdot 2^1 + c_2 \cdot 3^1 + \frac{21}{4} + \frac{7}{2}(1)$

$$1 = 2c_1 + 3c_2 + \frac{21}{4} + \frac{7}{2}.$$

$$\Rightarrow 2c_1 + 3c_2 = 1 - \frac{21}{4} + \frac{7}{2}.$$

$$= \frac{4 - 21 + 14}{4} = \frac{18-21}{4} = \frac{3}{4}.$$

$$2c_1 + 3c_2 = \frac{3}{4}. \quad \text{—(a)}$$

$$c_1 + c_2 = \frac{17}{4} \quad \text{—(b)}$$

On solving (a) & (b) we get $c_2 = \frac{3}{4}$, $c_1 = -5$.

$\therefore$ ~~$b_n = \frac{3}{4}$~~

$$\therefore \quad b_n = (-5)\, 2^n + \frac{3}{4}(3^n) + \frac{21}{4} + \frac{7}{2} n.$$

$$\therefore \quad b_n = a_n^2 \Rightarrow a_n = \pm\sqrt{b_n} = \pm(b_n)^{1/2}.$$

$$a_n = \pm\left((-5)2^n + \frac{3}{4}(3^n) + \frac{21}{4} + \frac{7}{2} n\right)^{1/2}.$$

is the required solution.

(7). Solve the recurrence relation $a_n - 6a_{n-1} + 8a_{n-2} = 3^n$ where $a_0 = 3$, $a_1 = 7$.

Sol: Given recurrence relation is $a_n - 6a_{n-1} + 8a_{n-2} = 3^n$.
where $a_0 = 3$ and $a_1 = 7$.

The associated homogeneous eqn is $a_n - 6a_{n-1} + 8a_{n-2} = 0$

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It's characteristic equation is.

$$r^2 - 6r + 8 = 0.$$
$$(r-2)(r-4) = 0$$
$$r = 2, 4.$$

$\therefore$ Homogeneous solution is $a_n^{(H)} = c_1 2^n + c_2 4^n$.

We need to find the particular solution.

Let the particular solution be.

$$a_n^{(P)} = d(3^n)$$
$$a_n = d(3^n).$$

Using this equation in the given recurrence relation, we get.

$$d3^n - 6d3^{n-1} + 8d3^{n-2} = 3^n.$$
$$d3^n - 6d3^n \cdot 3^{-1} + 8d3^n \cdot 3^{-2} = 3^n.$$
$$d - \frac{6d}{3} + \frac{8d}{3^2} = 1$$
$$d - 2d + \frac{8d}{9} = 1$$
$$-d + \frac{8d}{9} = 1$$
$$-\frac{d}{9} = 1 \Rightarrow d = -9.$$

$$\therefore \quad a_n^{(P)} = (-9)3^n.$$

$\therefore$ The general solution is

$$a_n = a_n^{(H)} + a_n^{(P)}$$
$$a_n = c_1 2^n + c_2 4^n + (-9)3^n.$$

Given $a_0 = 3$, $a_1 = 7$.

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for $n=0$, $a_0 = c_1 2^0 + c_2 4^0 + (-9) 3^0$.

$$3 = c_1 + c_2 - 9 \Rightarrow c_1 + c_2 = 12 \quad \text{— (a)}$$

for $n=1$, $a_1 = c_1 2^1 + c_2 4^1 + (-9) 3^1$

$$7 = 2c_1 + 4c_2 - 27$$

$$\Rightarrow 2c_1 + 4c_2 = 34$$

$$\Rightarrow c_1 + 2c_2 = 17 \quad \text{— (b)}.$$

On solving (a) & (b), we get $c_1 = 7$, $c_2 = 5$

$$\therefore a_n = 7 \cdot 2^n + 5 \cdot 4^n + (-9) 3^n$$

(8) Solve the recurrence relation $a_n + 3a_{n-1} - 10a_{n-2} = n^2 + n + 1$.

Sol:- Given $a_n + 3a_{n-1} - 10a_{n-2} = n^2 + n + 1$.

The associated homogeneous eqn. is

$$a_n + 3a_{n-1} - 10a_{n-2} = 0.$$

Its characteristic equation is

$$r^2 + 3r - 10 = 0$$

$$(r-2)(r+5) = 0$$

$$r = 2, -5.$$

$\therefore$ Homogeneous soln. is $a_n^{(H)} = c_1(-5)^n + c_2(2^n)$

Since the R.H.S of (1). is $n^2 + n + 1$ then we assume the particular solution of (1) to be.

$$a_n^{(P)} = d_0 + d_1 n + d_2 n^2.$$

Using this solution in the recurrence relation, we get.

$$d_0 + d_1 n + d_2 n^2 + 3(d_0 + d_1(n-1) + d_2(n-1)^2) - 10(d_0 + d_1(n-2) + d_2(n-2)^2)$$

$$= 1 + n + n^2.$$

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$\Rightarrow d_0 + d_1 n + d_2 n^2 + 3d_0 + 3d_1 n - 3d_1 + 3d_2 n^2 - 6d_2 n$
$+ 3d_2 - 10d_0 - 10d_1 n + 20d_1 - 10d_2 n^2 + 40d_2 n - 40d_2$
$= n^2 + n + 1.$

$\Rightarrow (-6d_0 + 17d_1 - 37d_2) + (-6d_1 + 34d_2)n - 6d_2 n^2 = n^2 + n + 1$

Equating the coefficients of the terms on both sides, we get.

$-6d_0 + 17d_1 - 37d_2 = 1$ — (a)

$-6d_1 + 34d_2 = 1$ — (b)

$-6d_2 = 1$ — (c) $\Rightarrow \boxed{d_2 = \frac{-1}{6}}$

Sub. d_2 value in (b), we get $\boxed{d_1 = -\frac{10}{9}}$

Sub. d_1 & d_2 values in (a), we get.

$\boxed{d_0 = -\frac{247}{108}.}$

Sub. d_0, d_1 & d_2 values in $a_n^{(P)}$, we get.

$$a_n^{(P)} = -\frac{247}{108} - \frac{10}{9}n - \frac{1}{6}n^2.$$

$\therefore$ The general solution is.

$$a_n = a_n^{(H)} + a_n^{(P)}$$

$$\therefore a_n = c_1(-5)^n + c_2(2^n) - \frac{247}{108} - \frac{10n}{9} - \frac{n^2}{6}.$$

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(9) Solve the recurrence relation $a_n - 6a_{n-1} + 8a_{n-2} = n4^n$ $a_0 = 8$ and $a_1 = 22$.

Sol: Given recurrence relation is

$$a_n - 6a_{n-1} + 8a_{n-2} = n4^n \quad \text{--- (1)} \qquad a_0 = 8,\ a_1 = 22$$

Homogeneous equation of (1) is

$$a_n - 6a_{n-1} + 8a_{n-2} = 0.$$

It's characteristic equation is.

$$r^2 - 6r + 8 = 0$$

$$(r-2)(r-4) = 0$$

$$r = 2, 4$$

$\therefore$ Homogeneous solution is

$$a_n^{(H)} = c_1 2^n + c_2 4^n.$$

Since the R.H.S of (1) is $n \cdot 4^n$ then we assume the particular solution of (1) to be

$$a_n^{(P)} = (d_0 + d_1 n) 4^n.$$

Using this solution in the recurrence relation is

$$4^n(d_0 + d_1 n) - 6\,4^n(d_0 + d_1(n-1)) + 8\,4^n(d_0 + d_1(n-2)) = n \cdot 4^n.$$

$$d_0 + d_1 n - 6d_0 - 6d_1 n + 6d_1 + 8d_0 + 8d_1 n - 16d_1 = n$$

$$(d_0 - 6d_0 + 8d_0) + (6d_1 - 16d_1) + (d_1 - 6d_1 + 8d_1)n = n$$

$$(3d_0 - 10d_1) + 3d_1 n = n.$$

Equating the corresponding terms on both sides. We get.

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$$3d_0 - 10d_1 = 0$$

$$3d_1 = 1 \Rightarrow d_1 = \frac{1}{3}.$$

$$3d_0 - 10\left(\frac{1}{3}\right) = 0$$

$$\Rightarrow \quad 3d_0 = \frac{10}{3} \Rightarrow d_0 = \frac{10}{9}.$$

$$\therefore \quad d_0 = \frac{10}{9}, \quad d_1 = \frac{1}{3}.$$

Sub the values d_0 & d_1 in the solution.

$$a_n = a_n^{(H)} + a_n^{(P)}$$

$$a_n = c_1(2^n) + c_2(4^n) + \left(\frac{10}{9} + \frac{1}{3}n\right)4^n.$$

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UNIT-4

Graph Theory

Graph Theory:-

* Basic Concepts
* Graph Theory and its Applications
* Subgraphs
* Graph Representations: Adjacency and Incidence Matrices
* Isomorphic Graphs
* Paths and Circuits.
* Eulerian and Hamiltonian Graphs.

Basic Concepts:-

A graph G consists of two sets:

i) a non-empty set V whose elements are called vertices, nodes (or) points of G. Then the set ~~E(G)~~ V(G) is called the vertex set of G.

ii) a set E of edges such that each edge $e \in E$ associated with ordered (or) unordered pair of elements of V. The set E(G) is called the edge set of G

Mostly the graph G with vertices and edges E is written as $G = (V, E)$ (or) $G(V, E)$.

Graph:- A graph G has pair (V, E), where V is a non-empty finite set whose elements are called vertices (nodes (or) points). E is a another set whose elements are called edges (lines)

The graph G with vertices V and edges E is written as $G = (V, E)$ (or) $G(V, E)$

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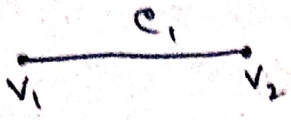
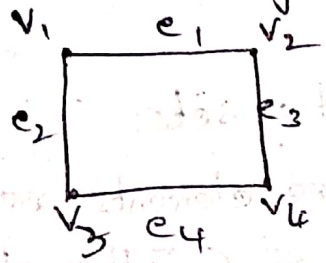
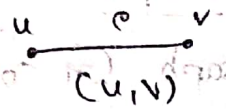
Ex:- 

Here $V = \{v_1, v_2\}$, $E = \{e_1\}$

Note:-

1. If an edge $e \in E$ is associated with an ordered pair (u, v) where $(u, v) \in V$.
2. e (edge) is connected to u and v are called end points of e.
3. Any two vertices connected by an edge in a graph is called adjacent vertices.
4. Any two edges e_1 and e_2 are incident with a common point (or) vertex then they are called adjacent edges.



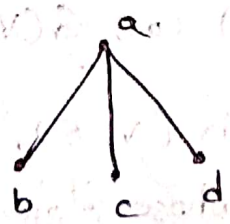


Here e has two adjacent vertices u and v

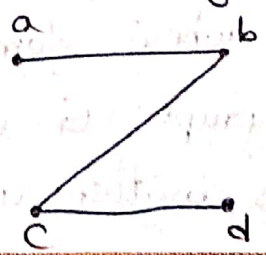
In the above graph v_1 has two adjacent edges e_1 and e_2 and v_1 denoted as incident vertex.

1. $V = \{a, b, c, d\}$ and $E = \{(a,b), (a,c), (a,d)\}$. draw the graph G.

Solution:-



2. Construct the vertices and edges from the given graph.



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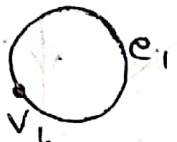
Solution:- The given graph $G=(V,E)$

$V=\{a,b,c,d\}$ $E=\{(a,b),(b,c),(c,d)\}$.

Types of graphs:-

1. Infinite graph: A graph with infinite vertex set.
2. finite graph: A graph with finite vertex set.
3. Simple graph: A graph in which each edge connects two different vertices & where no two edges connect the same pair of vertices is called a simple graph.
4. Multi graph:- Graphs that may have multiple edges connecting the same vertices are called multigraph

→ loop:- An edge of a graph G that join a node to itself is called a loop (or) self loop defined as $e_i=(v_1,v_1)$



5. Psuedo graph:- A graph in which loops and multi edges are allowed is called psuedo graph.
6. Undirected graph:- An undirected graph G has a set of vertices V and a set of edges E such that each edge $e \in E$ is associated with an unordered pair of vertices ($e \in (v_i,v_j)$ and (v_j,v_i))
7. Directed graph:- A directed graph G has a set of vertices V and a set of edges E such that each edge $e \in E$ is associated with an ordered pair of vertices, means directions on each edge $e \in (v_i,v_j)$.

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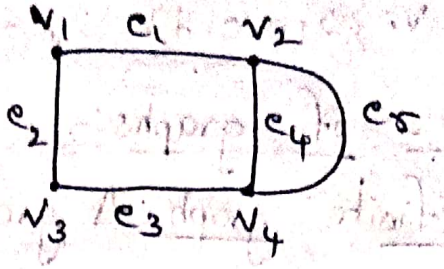
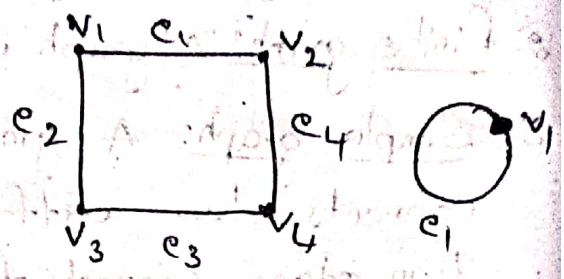
Examples :-

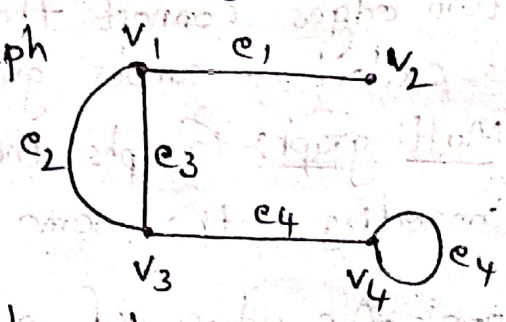
→ Example for Multi graph



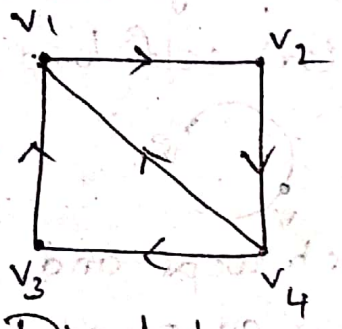
→ Example for simple graph



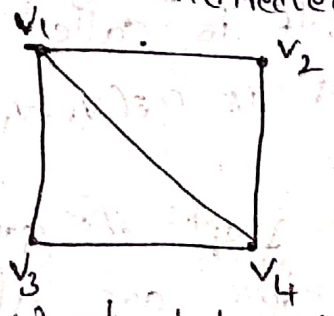
→ Example for psuedo graph



→ Examples for directed and undirected graph



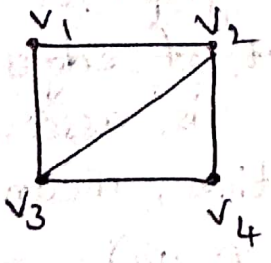
Directed graph



Undirected graph.

Degree of a vertex:- The degree of a vertex of an undirected graph G is the number of edges incident with it. The degree of the vertex denoted as deg(v) (or) d(v)

Ex:-



$deg(v_1) = 2$

$deg(v_2) = 3$

$deg(v_3) = 3$

$deg(v_4) = 2$

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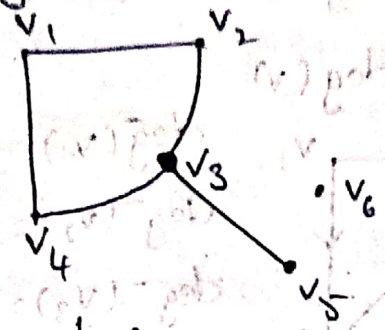
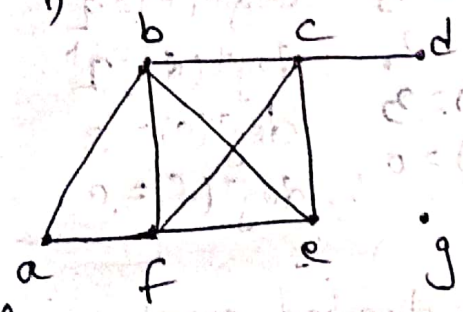
1. Construct degree of vertices from given diagram



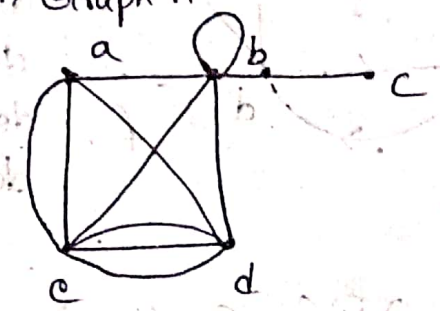
Solution:- $\deg(v_1) = 2$, $\deg(v_2) = 2$, $\deg(v_3) = 3$, $\deg(v_4) = 2$
$\deg(v_5) = 1$, $\deg(v_6) = 0$.

2. find the degrees of vertices

i) Graph G



ii) Graph H



Solution:- i) $\deg(a) = 2$, $\deg(b) = 4$, $\deg(c) = 4$,
$\deg(d) = 1$, $\deg(e) = 4$, $\deg(f) = 4$, $\deg(g) = 0$.

ii) $\deg(a) = 4$, $\deg(b) = 5$, $\deg(c) = 1$, $\deg(d) = 5$,
$\deg(e) = 6$

Note:- 1) A vertex of degree zero is called isolated
2) A vertex is pendent if and only if it has degree one.

Indegree and outdegree on directed graph:-

The Indegree of a vertex V of a directed graph G is the number of edges receiving (or) ending (or) coming at V and denoted as $\deg^-(V)$ (or) $\text{Indeg}(V)$.

The outdegree of a vertex V of a directed graph G is the number of edges going (or)

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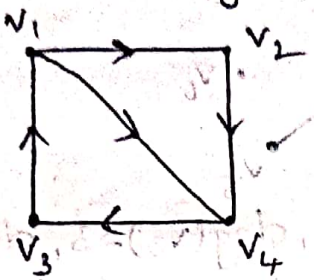
Starting (or) sending at v and denoted as $deg^+(v)$ (or) out deg (v)

Example:-

1)



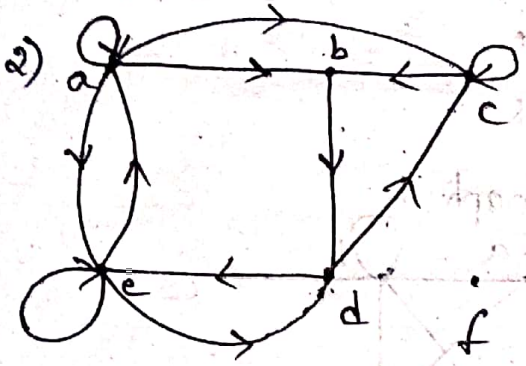
$deg^-(v_1) = 1$ $\quad deg^+(v_1) = 2$

$deg^-(v_2) = 1$ $\quad deg^+(v_2) = 1$

$deg^-(v_3) = 1$ $\quad deg^+(v_3) = 1$

$deg^-(v_4) = 2$ $\quad deg^+(v_4) = 1$

2)



$deg^-(a) = 2$ $\quad deg^+(a) = 4$

$deg^-(b) = 2$ $\quad deg^+(b) = 1$

$deg^-(c) = 3$ $\quad deg^+(c) = 2$

$deg^-(d) = 2$ $\quad deg^+(d) = 2$

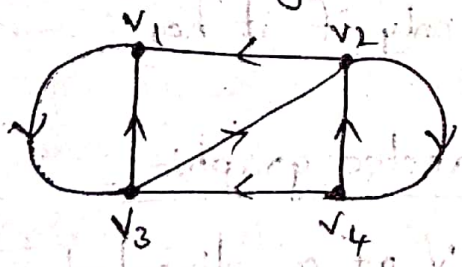
$deg^-(e) = 3$ $\quad deg^+(e) = 3$

$deg^-(f) = 0$ $\quad deg^+(f) = 0$

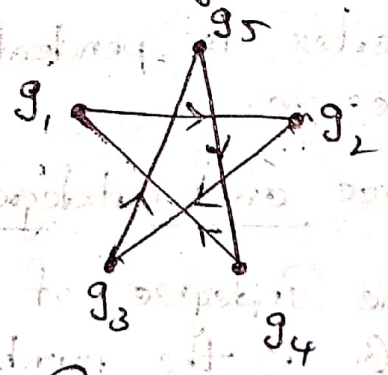
Note:- 1) If $G = (V, e)$ is a directed graph with edge e then $\sum_{v \in V} deg^-(v) = \sum_{v \in V} deg^-(v) = E$

2) $\sum_{i=1}^{n} deg(v_i) = 2E$ (Handshaking theorem)

1. Construct In degrees and out degrees from given graphs



Graph G_1



Graph G_2

Solution: Graph G_1

$deg^-(v_1) = 2$, $deg^-(v_2) = 2$, $deg^-(v_3) = 2$, $deg^-(v_4) = 1$

$deg^+(v_1) = 1$, $deg^+(v_2) = 2$, $deg^+(v_3) = 2$, $deg^+(v_4) = 2$

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Graph G_2

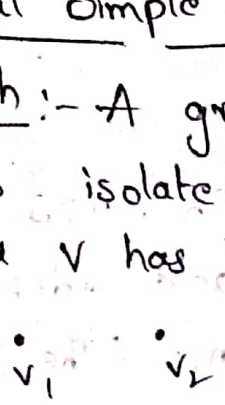
$\deg^-(G_1)=1$, $\deg^-(G_2)=1$, $\deg^-(G_3)=1$, $\deg^-(G_4)=1$, $\deg^-(G_5)=1$

$\deg^+(G_1)=1$, $\deg^+(G_2)=1$, $\deg^+(G_3)=1$, $\deg^+(G_4)=1$, $\deg^+(G_5)=1$

Some special Simple graphs:-

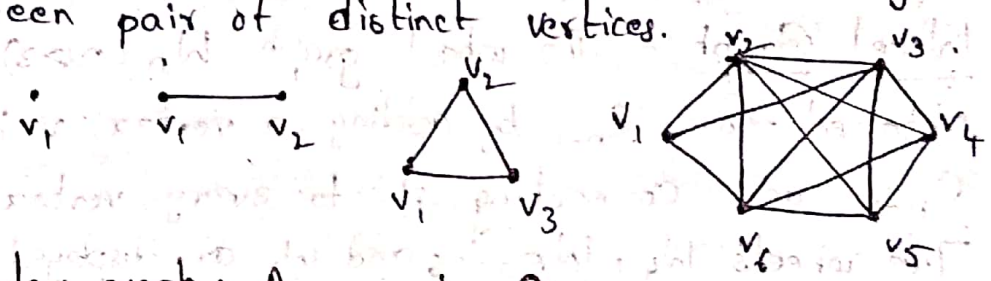
1. **Null graph**:- A graph G in which node (or) vertex is isolated node is called a nullgraph. The vertex v has '0' edeges.

Ex:-



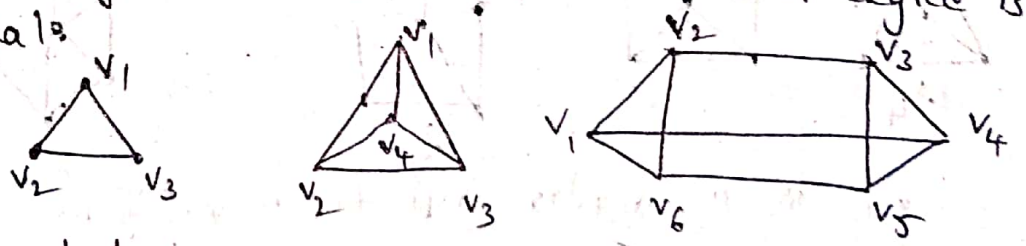
2. **Complete graph**:- A simple graph G is said to be complete graph if every vertex in G is connected with every other vertex. i.e., exactly one edge between pair of distinct vertices.

Ex:-



3. **Regular graph**:- A graph $G=(V,E)$ is said to be regular graph if it has all vertices of degree is equals

Ex:-



4. **Bipartiate graph**:- A graph $G=(V,E)$ is said to be bipartiate graph if the vertex V can be divided into 2 disjoint subsets V_1 and V_2 such that every edge e connects from V_1 to V_2. No edge is connects either two vertices in V_1 (or) V_2 of G.

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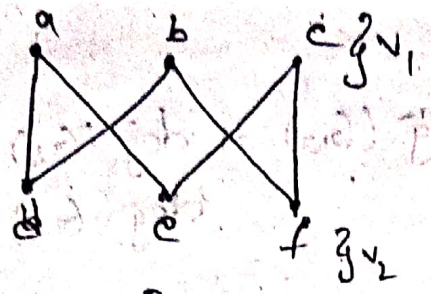
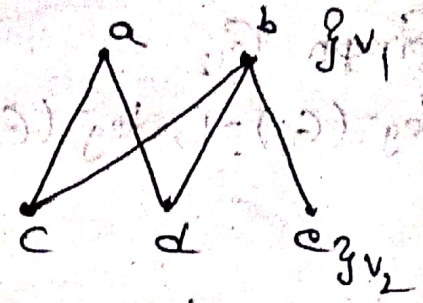
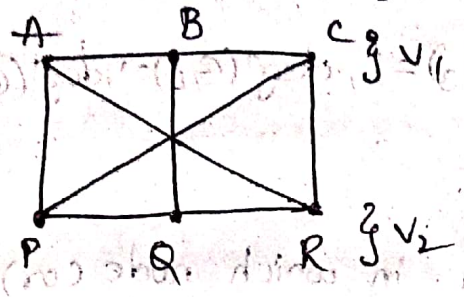
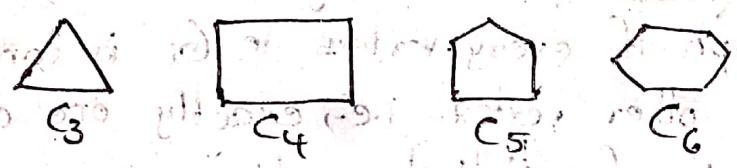
Ex:-



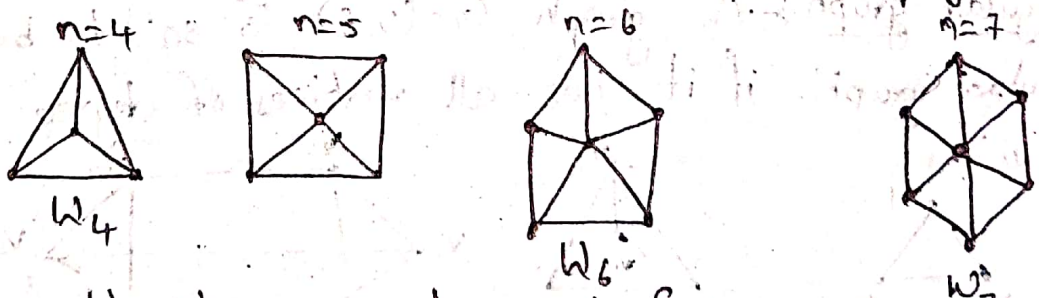




5. <u>Cycle graph</u>:- The cycle C_n, $n \geq 3$ consists of n vertices $1, 2, \ldots, n$ and edges $(1,2), (2,3), \ldots, (n-1, n)$ and $(n, 1)$. The cycles C_3, C_4, C_5 & C_6 are displayed.



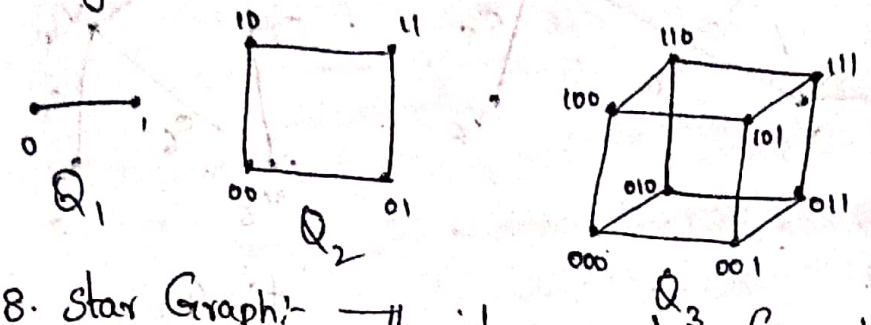
6. <u>Wheel Graph</u>:- The wheel graph W_n $(n > 3)$ is obtained from C_{n-1} by adding a vertex v inside C_{n-1} and connecting it to every vertex in C_{n-1}. The wheels W_4, W_5, W_6 and W_7 are displayed



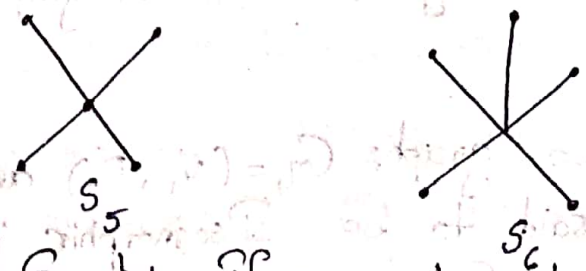
W_n is a regular graph for $n=4$. It has n vertices and $2n-2$ edges. The edges of a wheel graph which include the hub are called spokes.

7. <u>N-Cube graph</u>:- The N-cube denoted by Q_n, is the graph that has vertices representing the 2^n bit strings of length n. Two vertices are adjacent if and only if the bit strings that they

represent differ in exactly one bit position. The graphs Q_1, Q_2, Q_3 are displayed. Thus Q_n has 2^n vertices and $n.2^{n-1}$ edges, and is regular of degree n.



8. <u>Star Graph</u>:- The star graph of order n is a graph with n-1 edges, one vertex with degree n-1



9. <u>Mixed Graph</u>:- If a graph contains edges, some of which are directed and some are non-directed is called a mixed graph.

<u>SUB Graph</u>:- Given two graphs G and H, we say that H is a subgraph of G if the following conditions hold:

i) All the vertices and all the edges of H are in G

ii) Each edge of H has the same end points in H as in G.

Any subgraph of a graph G can be obtained by removing certain vertices and edges from G. It is understood that the removal of an edge leaves its points in place, whereas the removal of a vertex necessitates the removal of any edges with that vertex as an end point.

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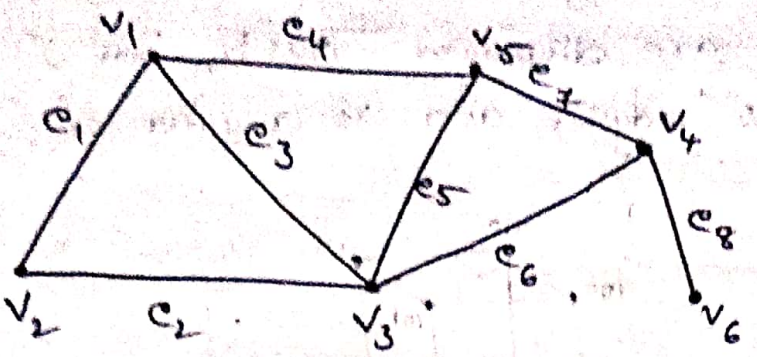


fig ⓐ

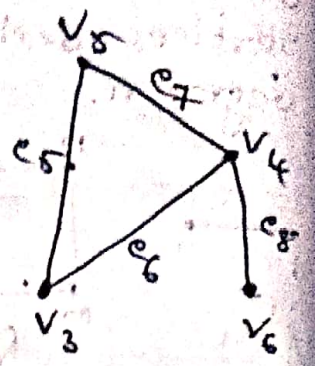


fig ⓑ

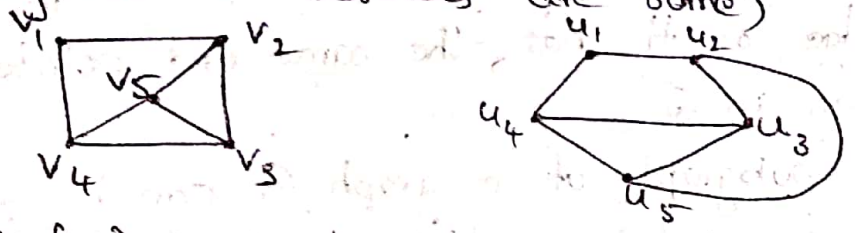
The graph in fig ⓑ is a subgraph of the fig ⓐ

<u>Isomorphism:-</u> Two graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ are said to be Isomorphic if there exists a Bijection $f: V_1 \to V_2$ such that $(u_1, v_1) \in E_1$ are ~~(or)~~ adjacent vertices in G_1 $(u_2, v_2) \in E_2$ are adjacent vertices in G_2.

degree of vertex in G_1 are equivalent to degree of vertex in G_2 if the adjacent vertices degrees are equal in G_1 and G_2 such that G_1 is isomorphic to G_2, then we write as $G_1 \cong G_2$.

(degrees of vertices are same)

Ex:-



$\deg(v_1) = 2 \Leftrightarrow \deg(u_1) = 2$

$\deg(v_2) = 3 \Leftrightarrow \deg(u_2) = 3$

$\deg(v_3) = 3 \Leftrightarrow \deg(u_3) = 3$

$\deg(v_4) = 3 \Leftrightarrow \deg(u_4) = 3$

$\deg(v_5) = 3 \Leftrightarrow \deg(u_5) = 3$

$\therefore$ The given subgraphs are in isomorphism.

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(OR)

Isomorphic Graphs:- Two graphs are called isomorphic if there is a one to one correspondence between their vertices and between their edges such that their incidence relationship is preserved.

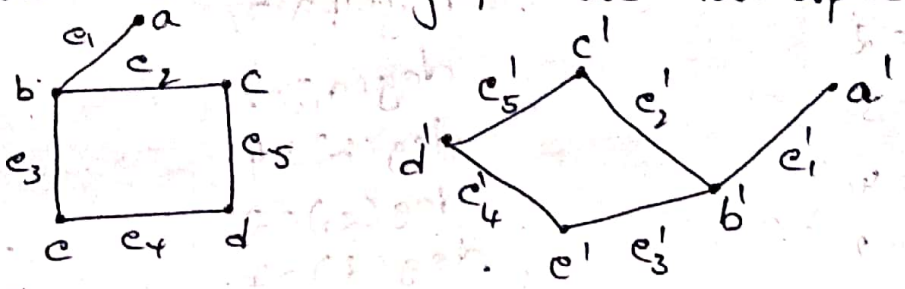
Thus, two graphs are isomorphic if

i) they have same number of edges

ii) they have same number of vertices

iii) they have equal number of vertices with a given degree.

1) Show that the two graphs are isomorphic



Solution: i) Both graphs have 5 edges.

ii) Both graphs have same number of vertices

iii) $\deg(a) = \deg(a') = 1$

$\deg(b) = \deg(b') = 3$

$\deg(c) = \deg(c') = 2$

$\deg(d) = \deg(d') = 2$

$\deg(e) = \deg(e') = 2$

Thus the vertices a, b, c, d, e correspond to a', b', c', d', e' respectively and the edges e_1, e_2, e_3, e_4, e_5 correspond to $e_1', e_2', e_3', e_4', e_5'$ respectively. Hence the two graphs are isomorphic

2) Example isomorphism of two graphs. Check whether the given graphs G_1 and G_2 are

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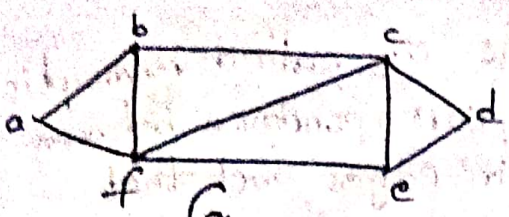
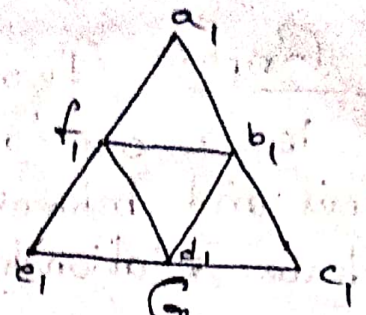
isomorphic (or) not? Give reasons.



G_1



G_2

Solution:- We observe that the given two graphs G_1 and G_2 have the same number of vertices (6) and same number of edges (9). But graph G_1 has 2 vertices of degree 4 whereas graph G_2 has 3 vertices of degree 4.

In graph G_1,

$\deg(a) = 2$

$\deg(b) = 3$

$\deg(c) = 4$

$\deg(d) = 2$

$\deg(e) = 3$

$\deg(f) = 4$

In graph G_2

$\deg(a_1) = 2$

$\deg(b_1) = 4$

$\deg(c_1) = 2$

$\deg(d_1) = 4$

$\deg(e_1) = 2$

$\deg(f_1) = 4$

$\therefore$ The two graphs G_1 and G_2 are not isomorphic.

Matrix representation of a graph:-

Matrix Representation of a graph has two types. They are 1. Adjacence matrix
2. Incidence matrix.

1. Adjacence matrix:-

Let $G = (V, E)$ be a simple graph with n vertices ordered from v_1 to v_2 then the adjacence matrix $A_m = [a_{ij}]_{n \times n}$ of G is an $n \times n$ symmetric matrix defined by

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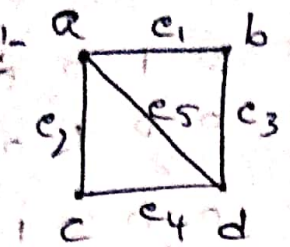
$$a_{ij} = \begin{cases} 1, & \text{when } v_i \text{ is adjacent to } v_j \\ 0, & \text{otherwise} \end{cases}$$

Ex:-



$$A_m = \begin{array}{c} \\ a \\ b \\ c \\ d \end{array} \begin{array}{c} \begin{array}{cccc} a & b & c & d \end{array} \\ \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix} \end{array}$$

2. Incidence matrix:-

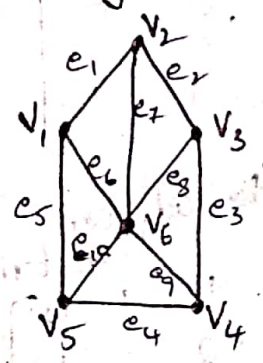
Let G be a graph with n vertices $V = \{v_1, v_2, \ldots, v_n\}$ and $E = \{e_1, e_2, \ldots, e_m\}$ define $n \times m$ matrix $I_m = [a_{ij}]_{n \times m}$,

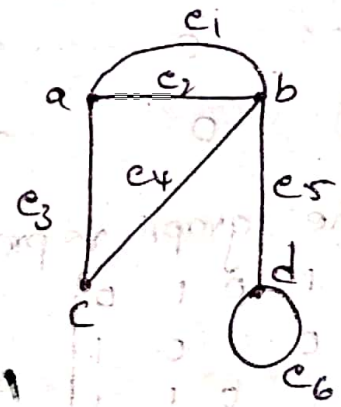
where $a_{ij} = \begin{cases} 1, & \text{when } v_i \text{ is incident with } e_j \\ 0, & \text{otherwise} \end{cases}$

Ex:-

$$I_m = \begin{array}{c} \\ a \\ b \\ c \\ d \end{array} \begin{array}{c} \begin{array}{ccccc} e_1 & e_2 & e_3 & e_4 & e_5 \end{array} \\ \begin{bmatrix} 1 & 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 \end{bmatrix} \end{array}$$

1. find the adjacence and Incidence matrix from given graphs.





Solution:- The adjacent & Incidence matrix from first graph is

$$A_m = \begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \\ v_6 \end{array} \begin{array}{c} \begin{array}{cccccc} v_1 & v_2 & v_3 & v_4 & v_5 & v_6 \end{array} \\ \begin{bmatrix} 0 & 1 & 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 & 1 & 0 \end{bmatrix} \end{array}$$

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and $I_m =$

$$\begin{array}{c|cccccccccc} & e_1 & e_2 & e_3 & e_4 & e_5 & e_6 & e_7 & e_8 & e_9 & e_{10} \\ \hline v_1 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ v_2 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ v_3 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ v_4 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 0 \\ v_5 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 1 \\ v_6 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 \end{array}$$

The adjacent and Incidence matrix to second graph is

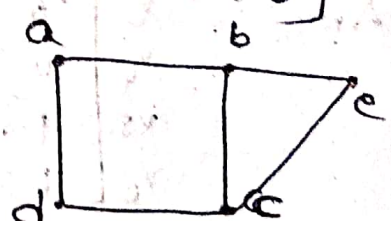
$$A_m = \begin{array}{c|cccc} & a & b & c & d \\ \hline a & 0 & 2 & 1 & 0 \\ b & 2 & 0 & 1 & 1 \\ c & 1 & 1 & 0 & 0 \\ d & 0 & 1 & 0 & 1 \end{array}$$

$$I_m = \begin{array}{c|cccccc} & e_1 & e_2 & e_3 & e_4 & e_5 & e_6 \\ \hline a & 1 & 1 & 1 & 0 & 0 & 0 \\ b & 1 & 1 & 0 & 1 & 1 & 0 \\ c & 0 & 0 & 1 & 1 & 0 & 0 \\ d & 0 & 0 & 0 & 0 & 1 & 1 \end{array}$$

2. Draw the graph represented by the adjacent matrix

$$A_{G_1} = \begin{bmatrix} 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \end{bmatrix} \quad A_{G_2} = \begin{bmatrix} 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 2 & 0 & 1 \\ 1 & 2 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 \end{bmatrix}$$

Solution: for A_{G_1} the graph is



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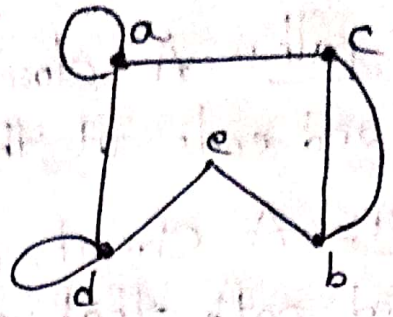
for A_{G_2} the graph is

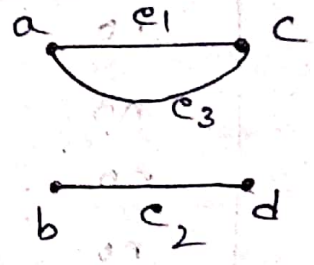


3. Draw the graph from given Incidence matrix.

$$I_G = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

Solution: for I_G the graph is

$$I_G = \begin{array}{c} \\ a \\ b \\ c \\ d \end{array} \begin{array}{c} \begin{array}{cccc} e_1 & e_2 & e_3 & e_4 \end{array} \\ \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \end{array}$$



Paths and Circuits:-

1. Walk:- A walk of a graph G is defined as an alternating sequence of vertices and edges $v_0 - e_0 - v_1 - e_1 - v_2 - e_2 - \ldots - e_n - v_n$. Starting and ending with vertices such that each line e_i is incident with v_i. A walk joining v_0 and v_n is called $v_0 - v_n$ walk.

   It contains only a single vertex such a walk is called trival walk.

2. Trail:- A walk is called a trail if all its edges are distinct.

3. Path:- A walk is called a path if all its vertices are distinct.

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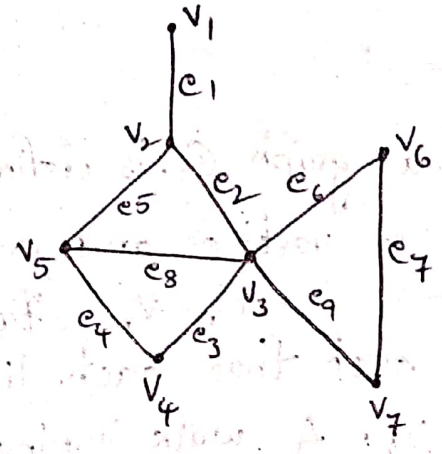
4. <u>Closed path</u> :- A closed path is a path that starts and ends at the same point (or) vertex.

5. <u>Circuit</u> :- A Circuit (or) Cycle is defined as a closed path that does not contain repeated edges (distinct edges).

	Repeated Edge	Repeated Vertex	Starts and Ends at same points?
Walk(open)	allowed	allowed	no
Walk(closed)	allowed	allowed	yes
Trail	no	allowed	no
Circuit	no	allowed	yes
Path	no	no	no
Cycle	no	first and last only	yes

<u>Example</u> :-

(1)



<u>Solution</u> :-

i) The sequence $v_1 - e_1 - v_2 - e_5 - v_5 - e_8 - v_3 - e_3 - v_4 - e_4 - v_5 - e_5 - v_2 - e_2 - v_3 - e_6 - v_6$ walk of length 8. It contains repeated vertices v_2, v_3 and v_5 and repeated edge e_5.

ii) The sequence $v_1 - e_1 - v_2 - e_5 - v_5 - e_3 - v_3 - e_3 - v_4 - e_4 - v_5$ is a trial. It contains repeated vertex v_5 but does not contain repeated edge

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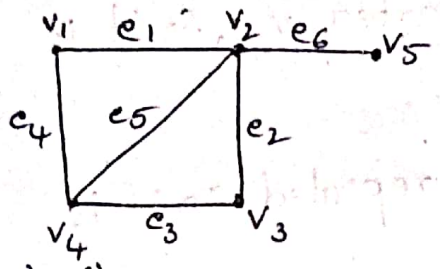
iii) The sequence $v_1 - e_1 - v_2 - e_5 - v_5 - e_8 - v_3 - e_3 - v_4$ is a path. It does not contain repeated vertex and repeated edge.

iv) The sequence $v_2 - e_2 - v_3 - e_3 - v_4 - e_4 - v_5 - e_5 - v_2$ is a cycle. It does not contain repeated vertex and repeated edge except the first and last vertex.

(2)



Solution:- i) $v_1 - e_1 - v_2 - e_2 - v_3 - e_3 - v_4 - e_5 - v_2 - e_2 - v_3$ are walks

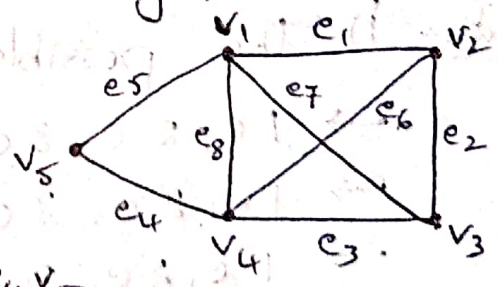
ii) $v_1 - e_1 - v_2 - e_5 - v_4 - e_3 - v_3 - e_2 - v_2 - e_6 - v_5$ is a trail.

iii) $v_1 - e_4 - v_4 - e_3 - v_3 - e_2 - v_2 - e_6 - v_5$ is a path

iv) $v_1 - e_1 - v_2 - e_5 - v_4 - e_3 - v_3 - e_2 - v_2 - e_1 - v_1$ is a closed path

v) $v_1 - e_4 - v_4 - e_3 - v_3 - e_2 - v_2 - e_1 - v_1$ are circuits.

(3) Determine of the following sequence are circuits & paths from below graph



a) $v_1 e_1 v_2 e_6 v_4 e_3 v_3 e_2 v_2$

b) $v_1 e_1 v_2 e_2 v_3 e_3 v_4 e_4 v_5$

c) $v_1 e_8 v_4 e_3 v_3 e_7 v_1 e_8 v_4$

d) $v_5 e_5 v_1 e_8 v_4 e_3 v_3 e_2 v_2 e_6 v_4 e_4 v_5$

e) $v_2 e_2 v_3 v_4 e_4 v_5 e_5 v_1 e_1 v_2$

Solution:-

a) Vertex v_2 is repeated twice, so it not a path. Starting vertex v_1 and ending vertex v_2, so it is not circuit.

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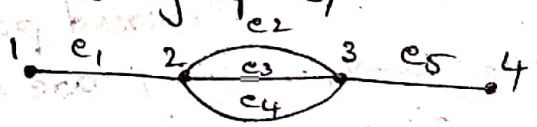
b) Here all vertices are distinct, so it is a path.
Starting vertex v_1 ending vertex v_5, so it is not a circuit.

c) Here vertex v_1, v_4 are repeated, so it is not a path.
Starting vertex v_1 ending vertex v_4, so it is not a circuit.

d) Here vertex v_5, v_4 are repeated, so it is not a path.
Starting vertex v_5 ending vertex v_5, so it is a circuit.

e) Here v_2 is repeated, so it is not a path
Starting vertex v_2, ending vertex v_2
so it is a circuit.

(4) let the graph G



i) How many paths are there from 1 to 4.
ii) How many trails are there from 1 to 4.

Solution: i) The possible paths are from 1 to 4

→ $1\ e_1\ 2\ e_3\ 3\ e_5\ 4$
→ $1\ e_1\ 2\ e_2\ 3\ e_5\ 4$
→ $1\ e_1\ 2\ e_4\ 3\ e_5\ 4$

ii) The possible trails are from 1 to 4.

→ $1-e_1-2-e_3-3-e_5-4$
→ $1-e_1-2-e_2-3-e_5-4$
→ $1-e_1-2-e_4-3-e_5-4$
→ $1-e_1-2-e_2-3-e_3-2-e_4-3-e_5-4$
→ $1-e_1-2-e_2-3-e_4-2-e_3-3-e_5-4$

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$\rightarrow 1 - e_1 - 2 - e_4 - 3 - e_2 - 2 - e_3 - 3 - e_5 - 4$

$\rightarrow 1 - e_1 - 2 - e_4 - 3 - e_3 - 2 - e_2 - 3 - e_5 - 4$

$\rightarrow 1 - e_1 - 2 - e_3 - 3 - e_4 - 2 - e_2 - 3 - e_5 - 4$

$\rightarrow 1 - e_1 - 2 - e_3 - 3 - e_2 - 2 - e_4 - 3 - e_5 - 4.$

Eulerian graph (or) Euler graph (or) Eulerian Circuit.

1. A trail in G is called an Eulerian trial (distinct edges).
2. It contains all vertices atleast once of G.
3. A closed Eulerian trail (starting and ending vertices same) is called Eulerian graph (or) Euler Circuit.

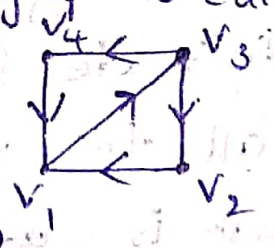
Euler path:-

1. A path in a Graph G is called an Euler path if it includes every edge exactly once (distinct edges).
2. Visit all vertices at least once.

Note:-

1. If G is a graph in which the degree of every vertex is even then it is possible to construct Euler circuit.
2. The graph G is a Euler path if atleast one degree of vertex is even.
3. If the given graph G is not a Euler circuit and path if and only if its vertices has odd degree.

(P) Determine whether the graph is Euler path (or) circuit.



Solution: from a given graph

Vertices $V = \{v_1, v_2, v_3, v_4\}$

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Indegree to every vertex is

$deg^-(v_1)=2$, $deg^-(v_2)=1$, $deg^-(v_3)=1$, $deg^-(v_4)=1$

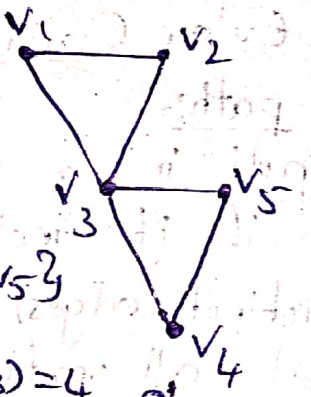
Outdegree to every vertex is

$deg^+(v_1)=1$, $deg^+(v_2)=1$, $deg^+(v_3)=2$, $deg^+(v_4)=1$

Here the vertices v_1 and v_3 has odd degree $=3$.

$\therefore$ The Euler path is $v_3-v_2-v_1-v_3-v_4-v_1$

2. from the given graph check whether Euler circuit (or) path



Solution: from the given graph

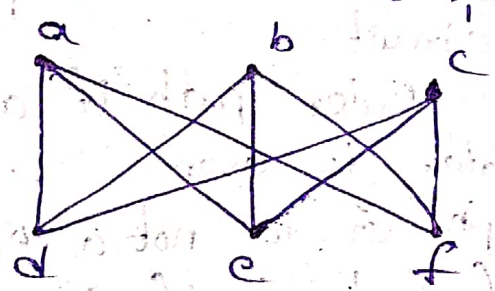
Vertices $V=\{v_1, v_2, v_3, v_4, v_5\}$

$deg(v_1)=2$, $deg(v_2)=2$, $deg(v_3)=4$, $deg(v_4)=2$, $deg(v_5)=2$

Here all the vertices have even degrees

$\therefore$ The Euler circuit is $v_1-v_2-v_3-v_4-v_5-v_3-v_1$

3.



Solution: from given graph vertices $=\{a, b, c, d, e, f\}$

$deg(a)=3$, $deg(b)=3$, $deg(c)=3$,

$deg(d)=3$ $deg(e)=3$, $deg(f)=3$

Here all the vertices have odd degrees $=3$

The path is ~~ad~~ $a-d-c-f-b-e-a-f$

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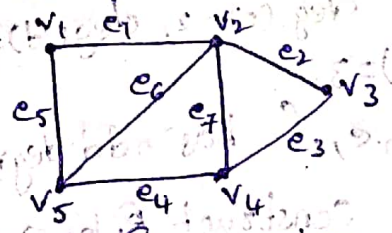
Here b – d, e – c edges are not covered.
So, it is not a path.

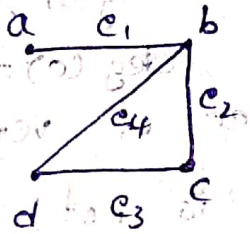
Hamiltonian graph :- A circuit in a graph G is called Hamiltonian circuit (or) Hamiltonian graph.

2. If it contains each vertex in G exactly once except for the starting and ending vertex that appears twice.

Hamiltonian path :- A Hamiltonian path is a path that contains all vertices of G where the end points (starting and ending vertices) may be distinct.

1. Determine which of the following graph is Hamaltonian circuit (or) path.

i)


ii)


Solution:- i) From given graph
vertices $V = \{v_1, v_2, v_3, v_4, v_5\}$

The path is $v_1 - v_2 - v_3 - v_4 - v_5 - v_1$
In this path all vertices visited exactly once except starting & ending vertices.
∴ It is Hamaltonian circuit.

ii) From given graph vertices $V = \{a, b, c, d\}$
The path is a–b–c–d
In this path all vertices visited exactly once and starting and ending vertices are distinct
∴ It is a Hamaltonian path.

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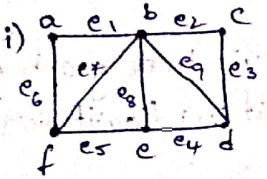
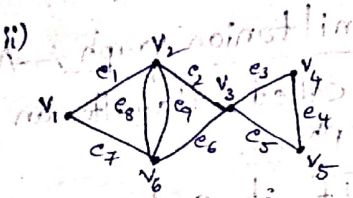
2. The given graph which is Hamaltonian circuit (or) Euler Circuit:

i)



ii)



Solution:

i) Case-i:- from the given graph

Vertices $V = \{a, b, c, d, e, f\}$.

The path is $a-b-c-d-e-f-a$

In this path all vertices visited exactly once except starting and ending vertex.

$\therefore$ It is a Hamaltonian circuit.

Case-ii:- for given graph degrees to all vertices $\deg(a)=2$, $\deg(b)=5$, $\deg(c)=2$, $\deg(d)=3$, $\deg(e)=3$, $\deg(f)=3$.

Degree of vertices b, d, e, f has odd degree.

It is not possible to construct Euler circuit.

ii) Case-i:- from the given graph

Vertices $V = \{v_1, v_2, v_3, v_4, v_5, v_6\}$

The path is $v_1-v_2-v_6-v_3-v_4-v_5$.

Here all vertices visited exactly once but starting and ending vertices are distinct.

$\therefore$ It is not a Hamiltonian circuit.

Case-ii:- from the given graph degrees to all vertices

$\deg(v_1)=2$, $\deg(v_2)=4$, $\deg(v_3)=4$, $\deg(v_4)=2$, $\deg(v_5)=2$, $\deg(v_6)=4$.

The path is $v_1-v_2-v_3-v_4-v_5-v_3-v_6-v_2-v_6-v_1$.

Here all edges exactly once and visited all vertices. starting and ending vertex is same.

$\therefore$ It is Euler circuit.

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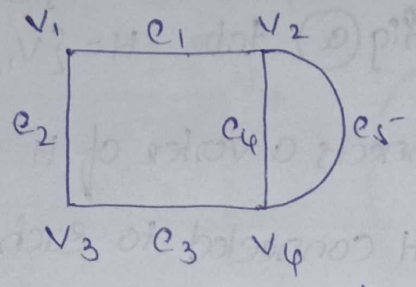
Unit - V

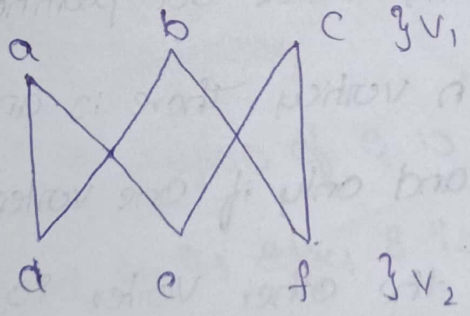
Multi Graphs

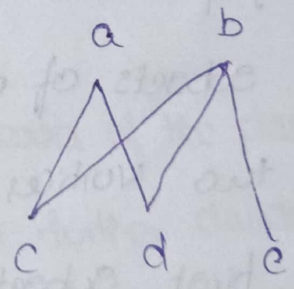
multi graphs: Graphs that may have multiple edges connecting the same vertices are called multi graphs.



Bipartite graph: A graph $G=(V,E)$ is said to be Bipartiate graph if the vertex V can be divided into 2 disjoint sub sets V_1 and V_2 such that every edge e connects from V_1 to V_2 NO edge is connects either two vertices in V_1 (or) V_2 of G.

Ex: $K_{3,3}$



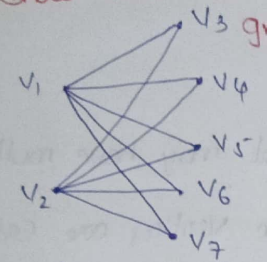


The set V can be partitioned in two sub sets V_1 & V_2 given by

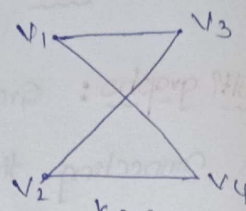
$V_1 = \{a, b, c\}$ $V_2 = \{d, e, f\}$

Hence $V = \{a, b, c, d, e, f\}$.

1) Show that the graph are complete bipartite graph



$K_{2,5}$ fig (a)



$K_{2,2}$ fig (b)

Sol^n:- for graph of fig (a) take $H = \{V_1, V_2\}$, $S = \{V_3, V_4, V_5, V_6, V_7\}$

∴ each edge of G connects a vertex of H to a vertex of S.

Also each vertex of H connected to each vertex of S.

Hence given graph is complete bipartite.

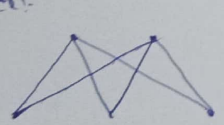
In fig (b) take $H = \{V_1, V_2\}$, $S = \{V_3, V_4\}$

Complete Bipartite Graphs

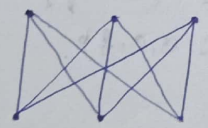
The complete bipartite graph $K_{m,n}$ is the graph that has its vertex set partitioned into two subsets of m & n vertices. There is an edge b/w two vertices if and only if one vertex is in the first subset & the other vertex is in the second subset.

the complete bipartite graphs $K_{2,3}$, $K_{3,3}$, $K_{3,5}$

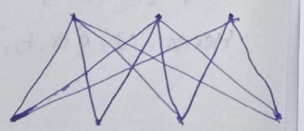
Ex:-



$K_{2,3}$



$K_{3,3}$

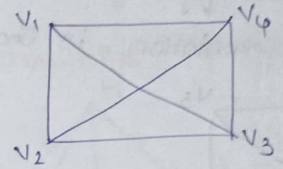


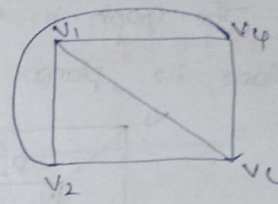
$K_{3,4}$.

Planar Graphs

A graph G is called planar if it can be drawn on a plane without intersecting edges. The point of intersection of edges is called crossover. Thus we say that a graph is called planar if it can be drawn on a plane without crossovers.

Ex:-

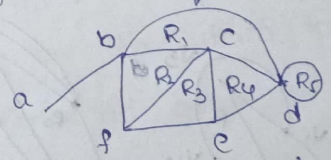




This shows that graph K_4 is planar.

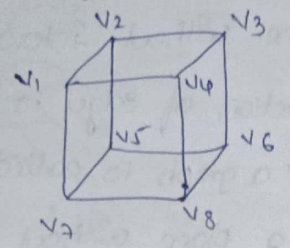
Regions: Let G be a planar graph. A plane representation of G divides the plane into regions. A region is characterized by the set of edges forming its boundary.

A region of G is an area of the plane which is enclosed by its edges & is not further divided into sub-area. If the enclosed area is finite then the region is called finite region otherwise infinite.

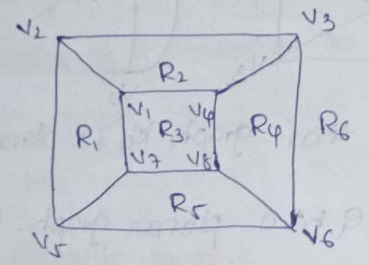


* the degree of region of planar graph is defined as the no. of edges in the boundary which encloses the regions.

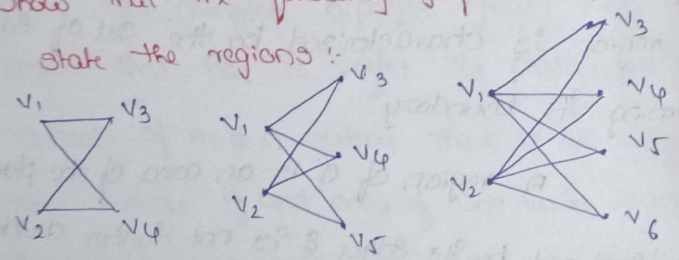
1) Show that the following graphs are planar & state the regions



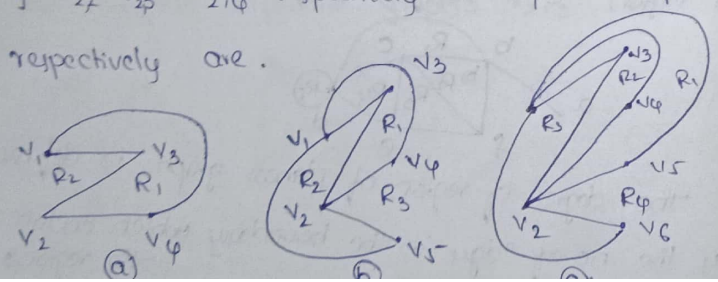
Soln:- The graph in above fig. can be embedded in plane i.e. planar representation is shown as



2) Show that the following graph are planar and state the regions:-



Soln:- The graphs in fig (a) (b) (c) are denoted by $K_{2,2}$, $K_{2,3}$, $K_{2,4}$ respectively. Their planar representation respectively are.

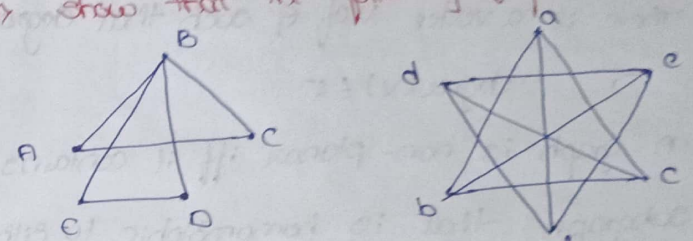


i) The graph (a) has: one finite region R_1 & an infinite region R_2.

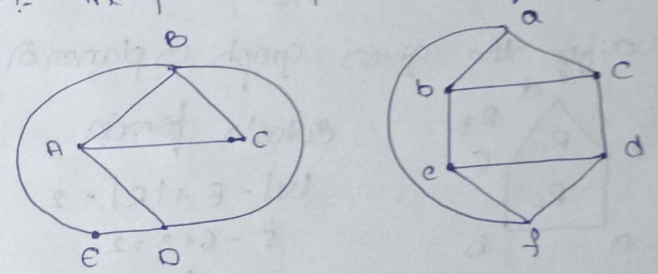
ii) The graph (b) has: R_1 & R_2 finite region & R_3 infinite region

iii) The graph (c) has: R_1, R_2 and R_3 finite regions and R_4 infinite region.

3) Show that the following graphs are planar:-



Soln:- The planar representation of graph is



Note:-

* A graph is planar if it has a drawing without crossings

following are the criteria to detect to planarity of a connected graph.

a) If G is connected planar graph then

$$|V| - |E| + |R| = 2$$

b) A complete graph K_n is planar iff $n \leq 4$; where n is the no. of vertices.

c) A complete bipartite graph $K_{m,n}$ is planar iff $m \leq 2$ or $n \leq 2$

d) A connected simple graph is planar with $|E| > 1$

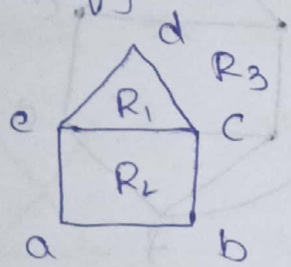
1) $|E| \leq 3|V| - 6$

2) There is a vertex v of G such that degree

$$\text{degree}(v) \leq 5$$

e) A graph is non-planar iff it contains a subgraph that is homomorphic to either K_5 (or) $K_{3,3}$

eg: verify the given graph is planar or not



Euler's formula.

$$|V| - E + |R| = 2$$

$$5 - 6 + 3 = 2$$

$$2 = 2.$$

The above graph is planar.

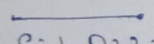
Corollary: If a connected planar simple graph has e edges and v vertices with $v \geq 3$ and no circuits of length three, then $e \leq 2v - 4$.

Euler's formula

A planar graph may have several different plane representations. A connected planar graph with n vertices & e edges have r regions given by $\boxed{r = e - n + 2}$ or $V - e + R = 2$

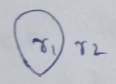
Let $e = 1$ then n may be equal to 1 or 2.

In case $e = 1$, $n = 2$, then no. of regions $r = 1 - 2 + 2 = 1$



$e = 1, n = 2 = V$ (Infinite Region)

In case $e = 1$, $n = 1$ then no. of regions $r = 1 - 1 + 2 = 2$



$e = 1, n = 1 = V$

In this graph has 2 regions r_1 & r_2.

Condition for a Graph To be planar.

If G is a connected simple planar graph with $n \geq 3$ vertices and e edges then to show that $e \leq 3n - 6$.

Proof:- Let G is a simple graph, i.e G has no loops & multiple edges, therefore, each region is bounded by atleast three edges and each edge belongs to exactly two regions so that we have, $2e \geq 3r$.

where r is the no. of regions in G.

We know that the Euler's formula is

$r = e - n + 2$

$\Rightarrow 3r = 3e - 3n + 6$ [multiplying both sides by 3]

$\Rightarrow 2e \geq 3e - 3n + 6$

$\Rightarrow 3n - 6 \geq e$

$e \leq 3n - 6$

1) Show that the graph K_5 is non-planar.

We know that K_5 is a simple graph & has 5 vertices i.e $n = 5 (= V)$

$\therefore e =$ no. of edges $= \frac{1}{2} n(n-1) = \frac{1}{2} \times 5(5-1) = 10$

Also $3n - 6 = 3 \times 5 - 6 = 15 - 6 = 9 < e$

$\therefore$ the inequality $e \leq 3n - 6$ is not satisfied

Hence the graph is non-planar.

2) Show that the graph K_6 is non-planar

K_6 is a simple graph with 6 vertices i.e $n = 6$

$\therefore e =$ no. of edges $= \frac{1}{2}(n)(n-1) = \frac{6 \times (6-1)}{2} = 15$

Also $3n - 6 = 3 \times 6 - 6 = 12 < e$

Hence the inequality $e \leq 3n - 6$ is not satisfied

Hence the graph is non-planar.

3) A connected plane graph has 10 vertices each of degree 3. Into how many regions does a representation of this planar graph split the plane?

Given, no. of vertices $n = 10$

Given degree of each vertex is 3.

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$\therefore \Sigma \deg(v) =$ degree of vertex × no. of vertices

$= 3 \times 10 = 30$

But, we know that

$\Sigma \deg(v) = 2e$

$30 = 2e \Rightarrow e = 15$

If r denotes the no. of regions, then by Euler's formula, we have

$r = e - n + 2$

$= 15 - 10 + 2 \Rightarrow r = 7$

4) A connected plane graph has 6 vertices. It has 2 vertices each of degree 4 and 4 vertices each of degree 2. In how many regions, does a representation of this planar graph split the plane?

Given $n = 6$

$\Sigma \deg(v) = 2 \times 4 + 4 \times 2 = 16$

But $\Sigma \deg(v) = 2e \Rightarrow 16 = 2e \Rightarrow e = 8$

If r denotes the no. of regions, then by Euler's formula, we have

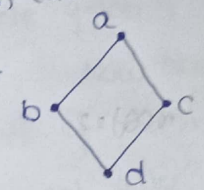
$r = e - n + 2$

$= 8 - 6 + 2 \Rightarrow r = 4$.

Graph colouring and covering

Colouring: An assignment of colours to the vertices of a graph and no two adjacent vertices get the same colour is called colouring of the graph (or) vertex colouring.

Ex:-



$f(a) =$ Red $\quad f(c) =$ pink

$f(b) =$ Blue $\quad f(d) =$ yellow.

Chromatic Number:-

The chromatic no. of a graph G is the minimum no. of colors required for proper coloring the vertices of G and is denoted by $\chi(G)$. Thus if a graph requires n different colors for its proper coloring then $\chi(G) = n$ and is n-chromatic graph.

Ex:-

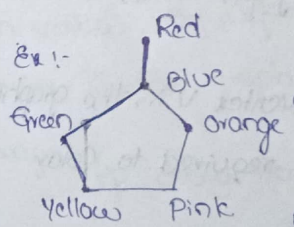


Fig6

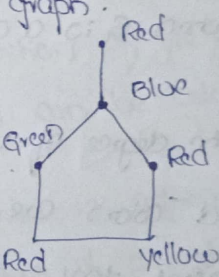


Fig-4

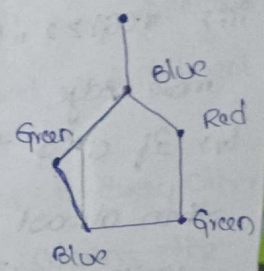


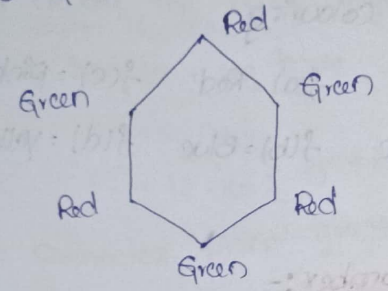
Fig-3

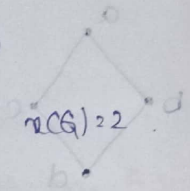
colors are used

three different proper coloring of G. With different no. of colors. fig ③ shows that three different colors are required for proper coloring of G.

Thus G is 3-chromatic $\chi(G) = 3$.

Ex: 2





The graph needs minimum two colors for proper coloring. Thus graph G is 2-chromatic.

Key points

i) If G is a null graph, then $\chi(G) = 1$ i.e chromatic

ii) A graph with only one edge is 2-chromatic.

iii) $\chi(G) \geq 2$, when G is a graph with one or more edge

iv) If d is the degree of vertex V in the graph G, then almost d colors are required to color vertices adjacent to V.

Theorem: A graph G with n vertices is a complete graph if and only if its chromatic polynomial in (λ) is given by.

$$P_n(\lambda) = \lambda(\lambda-1)(\lambda-2)\ldots(\lambda-n+1)$$

Proof: Let $V_1, V_2 \ldots V_n$ denote the n vertices of the complete graph G. Let λ numbers of colors be given.

Now any vertex V_1 can be colored by any one of the λ colors. i.e V_1 can be colored in λ different way

Similarly V_2 can be colored in $\lambda-1$ different way. from remaining $\lambda-1$ colors, V_3 can be colored in $\lambda-2$ ways and so on.

Hence $P_n(\lambda) = \lambda(\lambda-1)(\lambda-2)\ldots(\lambda-(n-1))$

$= \lambda(\lambda-1)(\lambda-2)\ldots(\lambda-n+1)$

Conversely, let $P_n(\lambda) = \lambda(\lambda-1)(\lambda-2)\ldots(\lambda-n+1)$ be the given chromatic polynomial of a graph having n vertices. By definition of $P_n(\lambda)$, any selected vertex of the graph can be colored in λ different ways. If every vertex is adjacent to every other vertex then the ways of properly coloring the second, third, fourth .... nth vertices are $\lambda-1$, $\lambda-2$, $\lambda-3$, ... $\lambda-(n-1)$ respectively. Hence the graph is a complete graph.

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Note :

A graph having n vertices is a tree if and only if its chromatic polynomial $P_n(\lambda)$ is given by

$$P_n(\lambda) = \lambda(\lambda-1)^{n-1}$$

1> write down the chromatic polynomial of a tree with 6 vertices.

Sol:- $P_n(\lambda) = \lambda(\lambda-1)^5$.

2> write down the chromatic polynomial of a complete graph with 7 vertices.

Sol: $P_n(\lambda) = \lambda(\lambda-1)(\lambda-2)(\lambda-3)(\lambda-4)(\lambda-5)(\lambda-6)$

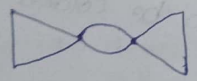
3> find the graph whose chromatic polynomial is $P_n(\lambda) = \lambda^5 - 4\lambda^4 + 6\lambda^3 - 4\lambda^2 + \lambda$

Sol:- we have $P_n(\lambda) = \lambda(\lambda^4 - 4\lambda^3 + 6\lambda^2 - 4\lambda + 1)$

$= \lambda(\lambda-1)^4$

Hence the graph is a tree with 5 vertices.

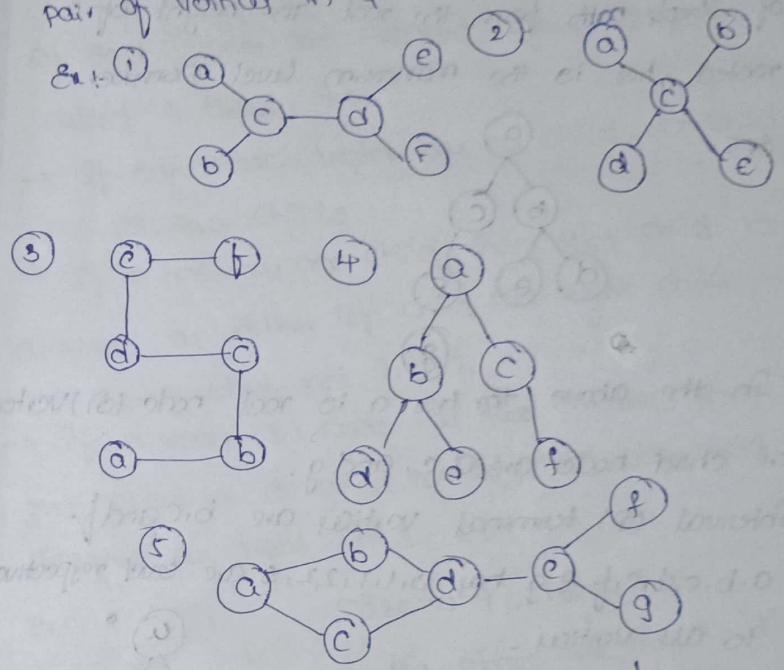
4> Draw a graph with 6 vertices containing an Eulerian Circuit but not hamiltonian Circuit.



Trees

A tree is a simple graph G such that there is a unique simple undirected path between each pair of vertices in G.

Ex:-



Here, 1, 2, 3 & 4 are trees, 5 is not a tree

∴ three tree denoted as "T".

rooted tree :-

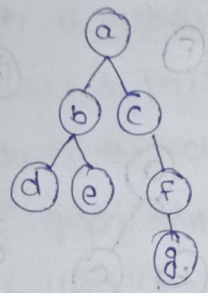
A rooted tree is a tree in which a particular vertex is designed as the root. (starting node or) vertex

→ If a vertex v of 'T' is a child vertex, if that vertex is a end vertex (or) exit vertex. such that V_0 as a root node $(V_0, e_1, V_1, e_2 \dots e_n V_n)$ and V_n is a child vertex in every path.

→ Except root & child vertices remaining all vertices are Internal (or) middle vertices.

→ The level of a vertex v in a tree is the length of simple path from the root. The height of a rooted tree is the maximum level number.

Ex:-



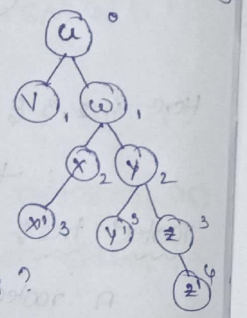
* In the above the tree a is root node (or) vertex.

Here child nodes are d, e, and g.

Internal (or) terminal vertices are b, c and f.

→ a, b, c, d, e, f & g has 0, 1, 1, 2, 2, 3 are level respectively to all vertices.

→ The height of tree is '3'.



i) From the given rooted tree 'T'

i) What is the root of T?

ii) find the levels & Internal vertices?

iii) What are the levels of w and z?

iv) find the childs of w and z?

Ans:- i) from the given tree root node is 'u'

ii) the levels of given tree u, v, w, x, y, z, x', y', z'
0 1 1 2 2 3 3 3 4

The Internal vertices are w, x, y, z.

iii) the level of w is 1 and level of z is 3

iv) the childs of w and z are x', y', z' and z'.

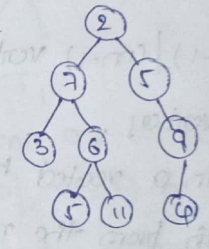
Binary tree :- A rooted tree in which the children of each vertex are assigned a fixed ordering is called a binary tree.

→ If either each vertex has no child, one child (or) two childs.

→ If a tree has one child then that child is designed as either left child (or) right child (but not both).

→ If a vertex (or) node has two children then the first child is designed as left child other child is designed as right child.

Ex:-



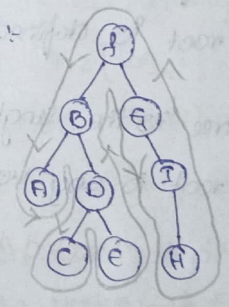
Size is 9, height h=3

→ the above tree size is '9' & height is 3'

the root node is '2'

the child nodes are 3, 5, 11, 4.

Ex:-

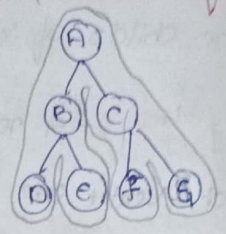


Preorder : FBADCEGIH

Inorder : ABCDEFGHI

Post order : ACEDBHIGF

1) construct preorder, Inorder & postorder from given graph.



Sol:- Preorder : ABDECFG

Post order : DEBFGCA

Properties of trees

1) A tree with n vertices has n-1 edges.

2) A full m-ary tree with i internal vertices contains n = mi+1 vertices.

3) A full m-ary tree with

i) n vertices has i = (n-1)/m Internal vertices & l = [(m-1)n+1]/m leaves.

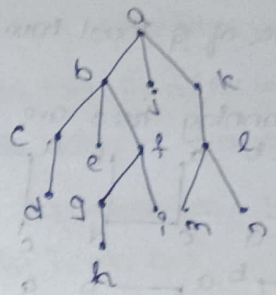
ii) i internal vertices has n = mi+1 vertices & l = (m-1)i+1 leaves.

iii) l leaves has n = (ml-1)/(m-1) vertices & i = (l-1)/(m-1) internal vertices.

4) The level of a vertex v in a rooted tree is the length of the unique path from the root to this vertex. The level of the root is defined to be zero.

5) The height of a rooted tree is the length of the longest path from the root to any vertex.

1) find the level of each vertex in the rooted tree. what is the height of this tree?



levels

a - 0

b, j, k - 1

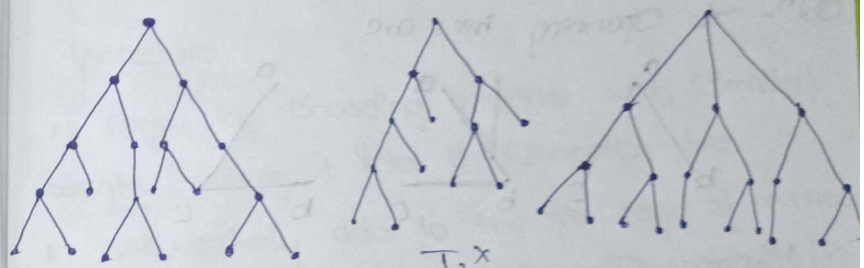
c, e, f, l - 2

d, g, i, m, n - 3

h - 4 → maximum hight is 4.

5) A rooted m-ary tree of hight h is balanced if all leaves are at levels h or h-1.

Eg which of the rooted trees are balanced.



✓ T_1 T_2 ✗ T_3 ✓

6) There are almost m^h leaves in an m-ary tree of height h.

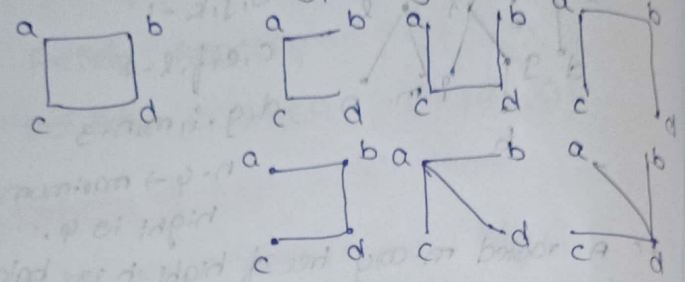
7) If an m-ary tree of height 'h' has l leaves, then $h \geq \lceil \log_m l \rceil$. If m-ary tree is full and balanced, then $h = \lceil \log_m l \rceil$

Spanning tree : A tree 'T' is a spanning tree of a graph G. T is a subgraph of G that contains all of the vertices of G.

If G is a connected graph with n vertices

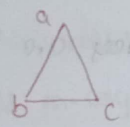
& m edges a spanning tree of G must have n-1 edges. m = n-1

Ex:

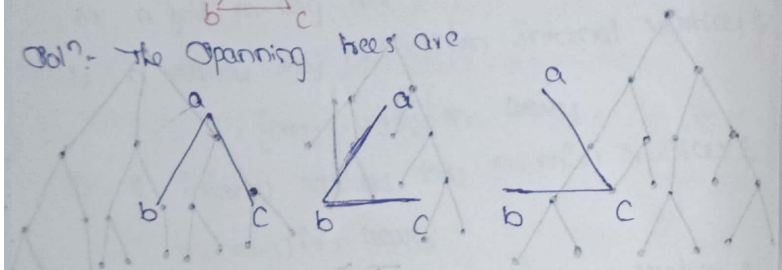


Spanning trees are

Q> Determine the spanning trees from given graph



Sol^n:- The spanning trees are



Minimal Spanning tree

Definition:- A minimal spanning tree in a connected weight graph is a spanning tree that has the smallest possible term of weights of its edges.

There are two methods for correct constructing minimum spanning tree (i.e 2 Algorithms)

1> Prim's Algorithm.

2> Krushkal's Algorithm.

Prim's Algorithm:

Procedure Prim(G: weighted connected undirected graph with 'n' vertices)

T: a minimum - weight edge.

for i = 1 to n-2

e := an edge of minimum weight incident to a vertex in T and not forming a simple circuit in T if added to T.

T := T with e added

return T {T is a minimum spanning tree of G}

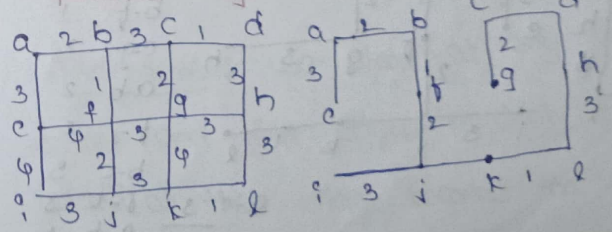
Procedure:

1> Begin by choosing an edge with smallest weight, putting it into the spanning tree.

2> recursively add to the tree edge of minimum weight that are incident to a vertex already in the tree, never forming a simple circuit with the tree, never forming a simple circuit with there edge's already in the tree.

3> Stop when n-1 edges have been added.

Eg: Produce minimum spanning tree using Prim's Algorithm



b-f-1
ab-2
ij-2
a.e-3
j-i-3
f-g 3
c-g 2
c-d 1
g-h-3
h-l 3
k-l 1
Total 26

Kruskal's Algorithm

Procedure :-

1) choose an edge in the graph with minimum weights.

2) successively add edges with minimum weights that do not form a simple circuit with the edges already choosen.

3) Stop after n-1 edges have been selected

Algorithm Kruskal

Procedure kruskal (G: weighted connected undirected graph with n vertices).

T := empty graph

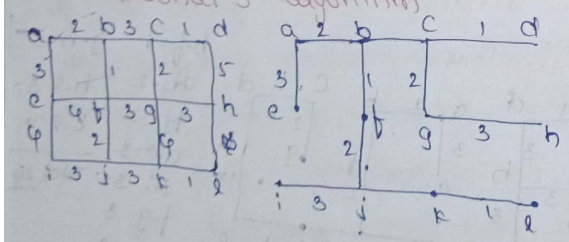
for i := 1 to n-1

e := any edge in G with smallest weight that does not form a simple circuit when added to T

T := T with e added.

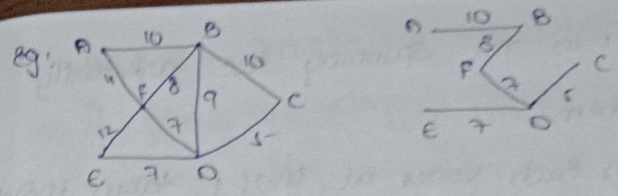
return T {T is a minimum spanning tree of G}.

eg:- Produce a minimum spanning tree using kruskal's algorithm



edge	weight
c-d	1
k-l	1
b-f	1
c-g	2
a-b	2
f-j	2
b-c	3
j-k	3
g-h	3
i-j	3
a-e	3
	29

eg:



Algorithm Breadth-First Search

Procedure BFS (G: connected graph with vertices $v_1, v_2 \ldots v_n$)

T := tree consisting only of vertex v_1

L := empty list

Put v_1 in the list L of unprocessed vertices

while L is not empty

begin

remove the first vertex, v from L

for each neighbour w of v

If w is not in L and not in T then

begin

add w to the end of the list L

add w and edge {v, w} to T

end

end.

BFS :

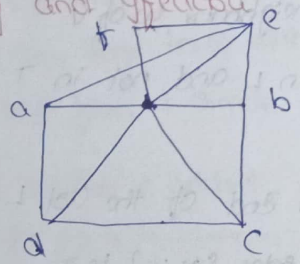
Procedure : we start at any vertex, taken as a root and find edges incident on it. The vertices added at this stage become the vertices at

level 1 in the spanning tree, arbitrarily order them.

Next for each vertex at level 1, taken in order use add edges incident in it so long as it does not produce a cycle.

This leads to the vertices at level 2, we repeat the procedure till no further edge can be added to any of the vertex at the last level reached.

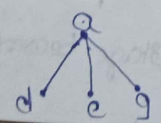
Eg: use BFS to find a spanning tree for the graph given below with the vertex ordering abcdefg and gfedcba



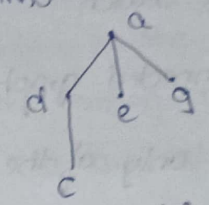
Let the vertex ordering be abcdefg.

1) Start with vertex 'a' as a root.

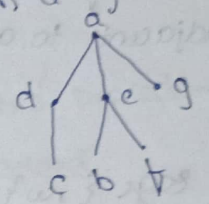
2) Add all the edges incident on 'a'. They are {a,b}, {a,e}, {a,g}. The vertices d,e,f,g are level 1 in the tree



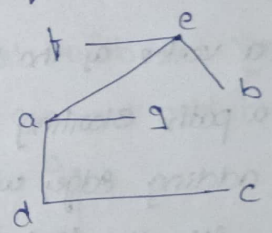
3) Add d to {d,c} but not {d,g} as it forms a cycle.



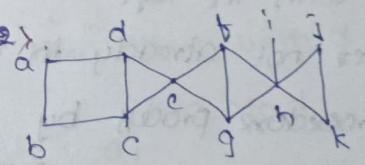
4) A e add {e,b} & {e,f} but not {e,g} as it form a cycle.

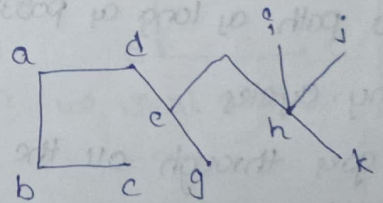


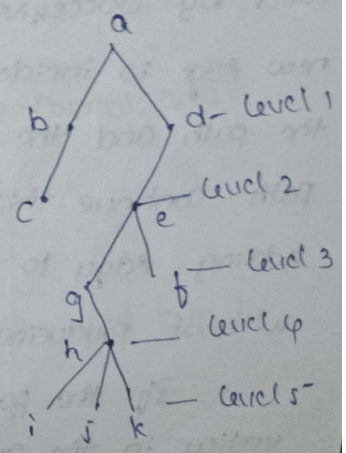
It contains all the vertices Hence it is a spanning tree. we can also write this spanning tree as follows.



Ex 2)







DFS (Also called Backtracking)

Algorithm Depth first search

Procedure DFS(G: connected graph with vertices $v_1, v_2, \ldots, v_n$)

T: = tree consisting only of the vertex v_1

visit(1)

procedure visit (v: vertex of G)

for each vertex w adjacent to and not yet in T

begin

add vertex w and edge {v,w} to T

visit(w)

end.

Procedure:

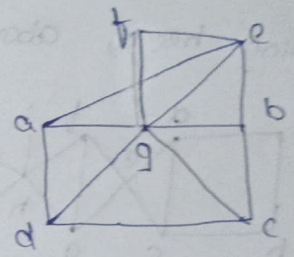
Arbitarily choose a vertex as the root of the spanning tree T. from a path starting at this vertex by successively adding edges where each new edge is incident with the last vertex in the path and the vertex not already in the path. continue this procedure process by adding edges to this path as long as possible without producing any cycles.

If the path goes through all the vertices in the graph, then the tree become a spanning tree. Otherwise more edges must be added, move back to next vertex to last vertex in the graph and it possible, form a new path starting at this vertex passing through that are not already visited.

If still all the vertices are not visited, move back to another vertex and try again.

Repeat this procedure until no more edges can be added.

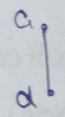
Q1) Apply DFS for



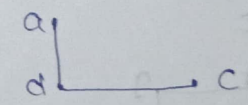
Let the vertex ordering be abcdefg.

i) Start with the vertex 'a' as a root.

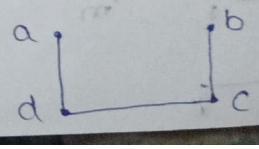
ii) we can not add b and c to a. Add d to a



iii) Now start with d. we cannot add b. Add c to d.



iv) we start with c. we can add b to c.



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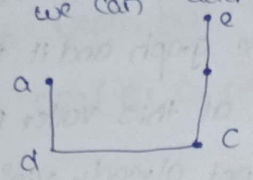
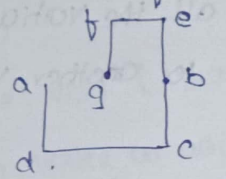
v) Start at b. we remain with vertices e, f, g we can add e to b.

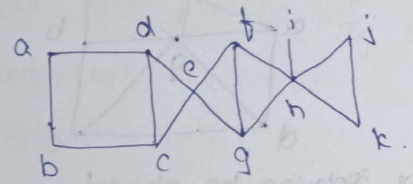


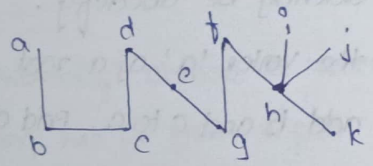
vi) we can add f to e and then g to f.



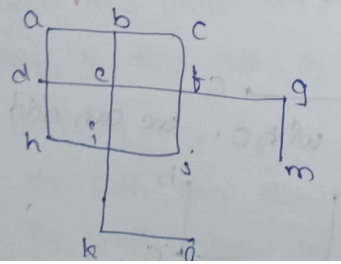
It contains all the vertices, hence it is the spanning tree as above.

2).

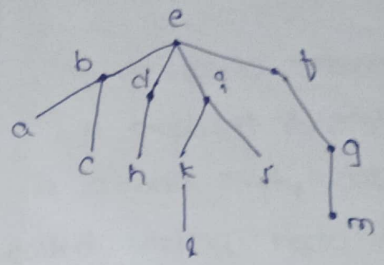




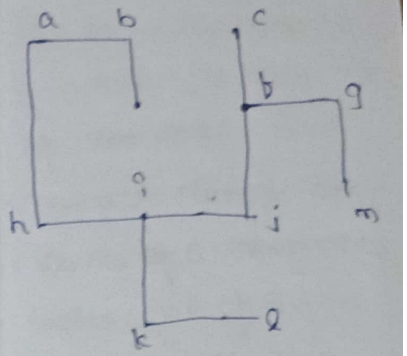
3) Using the BFS and DFS method, determine the spanning tree T for the graph G with 'e' as the root of T.



Sol: BFS



DFS



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