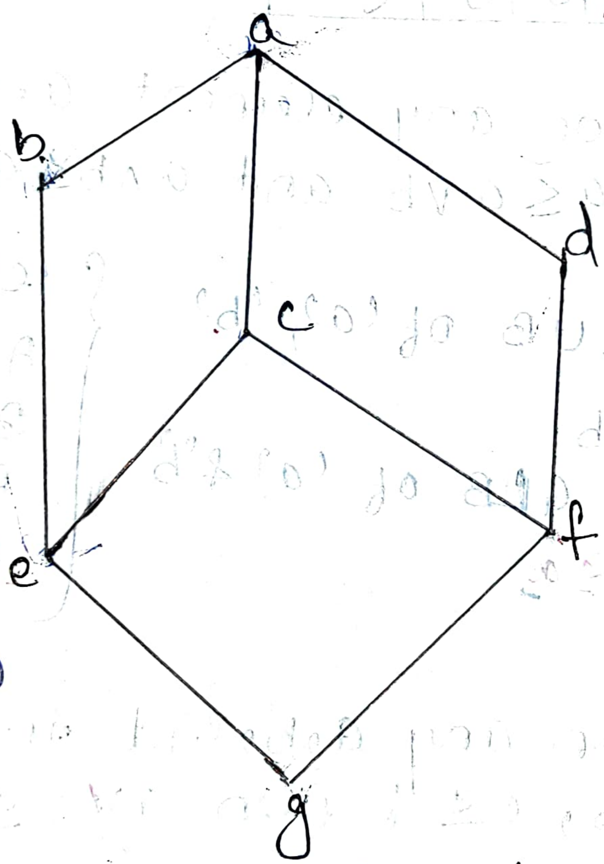
LATTICE AND BOOLEAN ALGEBRA :-

→ A set with relation i.e. $(A, \leq)$ is called lattice. Basically lattice has two operator

supremum i) least upper bound (LUB) or "JOIN". $\vee$

infimum ii) greatest lower bound (GUB) or "MEET". $\wedge$

Ex :-



LUB → least (minm) element of upper bound

GLB → greatest (maxm) element of lower bound

Join → उपर
meet → निचे

$c, f \to (a, c) \to c$

$e, a \to (a, c)$

Join operation — (LUB) or $(\vee)$ (least upper bound)

$\vee$	a	b	c	d	e	f	g
a	a	a	a	a	a	a	a
b	a	b	a	a	b	a	b
c	a	a	c	a	c	c	c
d	a	a	a	d	a	d	d
e	a	b	c	a	e	c	e
f	a	a	c	d	c	f	f
g	a	b	c	d	e	f	g

(b, f) → upper bound → a, d as b is not d

LUB = a.

MEET OPERATION (GLB) or ($\wedge$)

(move downward only)

$\wedge$	a	b	c	d	e	f	g
a	a	b	c	d	e	f	g
b	b	b	e	g	e	g	g
c	c	e	c	f	e	f	g
d	d	g	f	d	g	f	g
e	e	e	e	g	e	g	g
f	f	g	f	f	g	f	g
g	g	g	g	g	g	g	g

Join semi lattice + meets too

1) **Theorem :—** For any element a, b in a lattice $(A; \leq)$ then $a \leq a \vee b$ and $a \wedge b \leq a$

Proof :—

$a \vee b$ = LUB of 'a', 'b'

$a \leq a \vee b$

$a \wedge b$ = GLB of 'a' & 'b'

$a \wedge b \leq a$

Let A & B are subset

$A \subseteq A \cup B$

$B \subseteq A \cup B$

$A \cap B \subseteq A$

$A \cap B \subseteq B$.

2) **Theorem :—** For any a, b, c, d in a lattice $(A; \leq)$ if $a \leq b$, $c \leq d$ then $a \vee c \leq b \vee d$; $a \wedge c \leq b \wedge d$.

Proof :—

(1) $a \leq b$ (given). compare with $a \vee c \leq b \vee d$

$b \leq b \vee d$

$a \leq b \vee d$ (transitive) — (i)

$c \leq d$ (given)

$d \leq b \vee d$

$c \leq b \vee d$ (transitive) — (ii)

from (1) & (2)

$b \vee d$ is the upper bound of a & c

$\therefore a \vee c \leq b \vee d$.

ii) $a \wedge c \leq a$ (compare $a \wedge c \leq b \wedge d$).

$a \leq b$ (given)

$a \wedge c < b$ (transitively) — ③

$a \wedge c \leq c$

$c \leq d$ (given)

$a \wedge c \leq d$ (transitively) — ④

From ③ & ④

now, $a \wedge c$ is the lower bound of 'b' & 'd'

$\therefore a \wedge c \leq b \wedge d$.

Note:— Principle of duality:—

→ In a lattice of join operation (LUB) is valid than ~~meet~~ meet operation (GLB) holds good & vice versa

3) Theorem:— Join & meet operation are commutative :—

Proof:— a, b are two elements in a lattice

$(A; \leq)$ $a \vee b$ = LUB of 'a' & 'b'

join = LUB of 'b' & 'a'

$= b \vee a$.

$\Rightarrow \boxed{a \vee b = b \vee a}$

by principle of duality.

$\boxed{a \wedge b = b \wedge a}$

4) Theorem:— Join and meet operation are associative.

proof:— let a, b, c are 3 element in a lattice $(A; \leq)$, we have to show that join is associative that is

join: union
meet: intersect.

$(a \vee b) \vee c = a \vee (b \vee c)$ and meet is associative

i.e.

$(a \wedge b) \wedge c = a \wedge (b \wedge c)$

let

$(a \vee b) \vee c = a \vee (b \vee c)$.

let, $(a \vee b) \vee c = g$ and $a \vee (b \vee c) = h$

$\left\{ \begin{array}{l} A \cup B = X \\ A \subset A \cup B \\ A \subset X \\ B \subset X \end{array} \right.$

$a \vee (b \vee c) = h$

$a \leq h$ and $(b \vee c) \leq h$

$a \leq h$ and $b \leq h$, $c \leq h$

and

$a \vee b \leq h \vee h$

$\Rightarrow a \vee b \leq h$

$\Rightarrow (a \vee b) \vee c \leq h \vee h$

$\Rightarrow (a \vee b) \vee c \leq h$

$\Rightarrow g \leq h$. —— (1) $(\because (a \vee b) \vee c)$

Now,

$(a \vee b) \vee c = g$

$(a \vee b) \leq g$ and $c \leq g$

$a \leq g$, $b \leq g$ and $c \leq g$.

and $(b \vee c) \leq g \vee c$

$(b \vee c) \leq g$

Now

$$a \vee (b \vee c) \leq g \vee g$$

$\Rightarrow$ $a \vee (b \vee c) \leq g$.

$\Rightarrow$ $h \leq g$ —— (ii)

from (i) and (ii) we can get

$g = h$.

$(a \vee b) \vee c = a \vee (b \vee c)$.

and

By Principle of duality, we can get

$(a \wedge b) \wedge c = a \wedge (b \wedge c)$.

Q
5) In a lattice $(A; \leq)$, $a \in A$ then prove that
$a \vee a = a$
$a \wedge a = a$

$a \vee a = a$

$a \leq a \vee a$ —— (1)

$A \subseteq A \cup B$.
$A \subseteq A$.

$a \leq a$ and $a \leq a$.

$a \vee a \leq a \vee a$

$a \vee a \leq a$ —— (2)

from (1) and (2) $a \vee a = a$

by the Principle of duality

$\therefore$ $a \wedge a = a$

* Absorption Properties :-

* For any two element a and b in a lattice $(A, \leq)$ then $a \vee (a \wedge b) = a$ & $a \wedge (a \vee b) = a$.

* Proof :-

we have to proof $a \vee (a \wedge b) = a$

Now, $a \leq a \vee (a \wedge b)$ —— (i)

$a \leq a$ —— (ii)

$a \wedge b \leq a$ —— (iii) meet then

Now, join eqn (ii) and (iii) we get

$$a \vee (a \wedge b) \leq a \vee a$$

$$\Rightarrow a \vee (a \wedge b) \leq a \text{ —— (iv)}$$

Now, from eqn (i) and (iv) we get

$$a \vee (a \wedge b) = a.$$

By principle of duality

$$a \wedge (a \vee b) = a$$

proved

* Distributed Lattice :-

* A lattice is said to be distributed

If JOIN is distributed over MEET and vice-versa i.e both join and meet satisfies distributed properties

i.e

$$a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$$

&

$$a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c).$$

* Proof :- we have to show that $a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$.

RHS

$(a \vee b) \wedge (a \vee c)$

$= ((a \vee b) \wedge a) \vee ((a \vee b) \wedge c)$ (distributive property)

$= a \vee ((a \vee b) \wedge c)$ (absorption property)

$= a \vee ((a \wedge c) \vee (b \wedge c))$ (distributive)

$= \{a \vee (a \wedge c)\} \vee (b \wedge c)$ (associative)

$= a \vee (b \wedge c)$ (absorption property)

$=$ LHS

* By principale of duality :- $a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$ (proved)

* universal upper bound :- In a lattice $(A, \leq)$ an element 'a' is said to be universal upper

if there exist another any element $b \in A$ such that $a \geq b$,

* Basically universal upper bound is denoted as '1'.

* **Universal lower bound :-**

* In a lattice $(A, \leq)$ an element 'a' is said to be universal lower bound if there if there exist another any element $b \in A$ such that $a \leq b$.

* Basically universal lower bound is denoted as '0'.

* **Theorem :-**

* In a lattice $(A, \leq)$ with universal upper bound is 1 and universal lower bound is '0' zero, then for any element $a \in A$ prove the followings -

1) $a \vee 1 = 1$
2) $a \wedge 1 = a$
3) $a \vee 0 = a$
4) $a \wedge 0 = 0$.

$a \vee 1 = 1$	$a \wedge 1 = a$
$a \vee 0 = a$	$a \wedge 0 = 0$.

* Proof 1 :- we have to prove $a \vee 1 = 1$.

Now,

Less the join
$A \subseteq A \cup B$
$B \subseteq A \cup B$

$$1 \leq a \vee 1 \quad \text{--- (i)}$$

but

1 is the universal upper bound

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So,

$$a \vee 1 \leq 1 \quad \text{--- (ii)}$$

from (i) and (ii) we get

$$\boxed{a \vee 1 = 1.} \ *$$

proved

* Proof - 2 :- we have to to show that $a \wedge 1 = a$

$$a \wedge 1 \leq a. \quad \text{--- } (*)$$

$A \cap B \subseteq A$
$A \cap B \subseteq B$

as 1 is the universal upper bound.

$$a \leq 1 \quad \text{--- (1)}$$

&

$$a \leq a \quad \text{--- (2)}$$

Now, meet to both eqn - (1) and (2)

$$a \wedge a \leq 1 \wedge a.$$

$$a \leq 1 \wedge a.$$

$$a \leq a \wedge 1 \quad \text{commutative.} \longrightarrow ** .$$

from eqn $(*)$ & $(**)$ we get

$$\boxed{a \wedge 1 = a}$$

proved.

Proof - 3) we have to show $a \vee 0 = a$.

Now,

$$a \leq a \vee 0 \quad \text{——} \quad (*)$$

as

$x \vee 0$ is universal lower bound.

$\begin{cases} A \subseteq A \cup B \\ B \subseteq A \cup B \end{cases}$

$$0 \leq a \quad \text{——} \quad (i)$$

&

$$a \leq a \quad \text{——} \quad (ii)$$

taking join (i) and (ii) we get

$$0 \vee a \leq a \vee a$$

$$0 \vee a \leq a \quad \text{——} \quad (**)$$

$$a \vee 0 \leq a \quad \text{[commutative law]}$$

from $(*)$ & $(**)$, we get

$$\boxed{a \vee 0 = a}\ *.$$

Proof 4) we have to show $a \wedge 0 = 0$.

$\begin{cases} A \cap B \subseteq A \\ A \cap B \subseteq B \end{cases}$

Now

$$a \wedge 0 \leq 0 \quad \text{——} \quad * \quad \text{Theorem use.}$$

$\because$ 0 is the universal lower bound

$$0 \leq a \wedge 0 \quad \text{——} \quad **$$

from ~~[illegible]~~ $*$ & $**$ eqn

$$\boxed{a \wedge 0 = 0}\ *$$

Complimentary lattice :-

* Let $(A, \leq)$ be a lattice with universal upper bound is '1' and universal lower bound 0. then for any element $a \in A$ there exists another element $b \in A$ and b is called compliment of a if —

$a'/\bar{a} = b$

$$\boxed{a \vee b = 1}$$

and

$$\boxed{a \wedge b = 0}$$

* In a distributive lattice, if an element has compliment then it is "unique".

* Let $(A, \leq)$ be a lattice $a \in A$, has two compliment 'b' and 'c'.

So,

$a \vee c = 1$ $\quad$ $a \vee b = 1$

$a \wedge c = 0$ $\quad$ & $\quad$ $a \wedge b = 0$.

Now

$$b = b \wedge 1$$

$$= b \wedge (a \vee c)$$

$$= (b \wedge a) \vee (b \wedge c) \quad \text{(distributive)}$$

$$= 0 \vee (b \wedge c)$$

$$= (a \wedge c) \vee (b \wedge c)$$

$$= (c \wedge a) \vee (c \wedge b) \quad \text{commutati}$$

taking common c to out

$$= c \wedge (a \vee b) \quad \text{(distributive)}$$

take common

$$= c \wedge 1$$

$\Rightarrow$ $\boxed{b = c}$

$\therefore$ In a lattice every element has unique.

* **Boolean Algebra** → A lattice $(A, \leq)$ is called Boolean algebra when it defines boolean algebraic system i.e a set with atleast two binary operation.

eg:- $(A, \vee, \wedge, -)$

Imp
* **De-Morgan's Law** :- For any element 'a' & 'b' in a boolean algebra

* then prove that :-

(i) $\overline{a \vee b} = \bar{a} \wedge \bar{b}$

(ii) $\overline{a \wedge b} = \bar{a} \vee \bar{b}$

} This we simplify

proof :- we have to proof $\overline{a \vee b} = \bar{a} \wedge \bar{b}$

i.e (i) $(a \vee b) \vee (\bar{a} \wedge \bar{b}) = 1$.

(ii) $(a \vee b) \wedge (\bar{a} \wedge \bar{b}) = 0$.

$\quad\quad a \quad\quad \times \quad \bar{a}$

(i) LHS

$\quad a + \bar{a}$

$(a \vee b) \vee (\bar{a} \wedge \bar{b})$

$= ((a \vee b) \vee \bar{a}) \wedge ((a \vee b) \vee \bar{b})$ (distributive)

$= ((a \vee \bar{a}) \vee b) \wedge (a \vee (b \vee \bar{b}))$ (associative)

$= (1 \vee b) \wedge (a \vee 1)$

$= 1 \wedge 1$

$= 1 =$ RHS.

LHS = RHS.

$\boxed{(a \vee b) \vee (\bar{a} \wedge \bar{b}) = 1}$ *.

* $(a \vee b) \wedge (\bar{a} \wedge \bar{b}) = 0$

$\left\{ \begin{array}{l} \bar{a} = b \\ a \vee b = 1 \\ a \wedge b = 0 \end{array} \right\}$

LHS

$= (\bar{a} \wedge \bar{b}) \wedge (a \vee b)$ (commutative)

$= ((\bar{a} \wedge \bar{b}) \wedge a) \vee ((\bar{a} \wedge \bar{b}) \wedge b)$ (distributive)

$= ((\bar{a} \wedge a) \wedge \bar{b}) \vee (\bar{a} \wedge (b \wedge \bar{b}))$ (associative)

$= (0 \wedge \bar{b}) \vee (\bar{a} \wedge 0)$ $(\because \bar{a} \wedge a = 0.$

$= 0 \vee 0 = 0 =$ RHS. $\boxed{(a \vee b) \wedge (\bar{a} \wedge \bar{b}) = 0}$

Hence,

$\boxed{\overline{a \vee b} = \bar{a} \wedge \bar{b}}$ → from (1)

By Principle of duality

$\boxed{\overline{a \wedge b} = \bar{a} \vee \bar{b}}$ * → from (11)

* Minterms :-

→ let 'x' be a variable in boolean expression. Then $x_1, x_2, x_3, \ldots x_n$ is said to be minterms. If it is in the form of $x_1 \wedge x_2 \wedge x_3 \ldots \wedge x_n$

$\bar{x}_1 \wedge \bar{x}_2 \wedge \bar{x}_3 \wedge \ldots \wedge \bar{x}_n$

$\bar{x}_1 \wedge x_2 \wedge x_3 \wedge \ldots \wedge \bar{x}_n$

* Maxterms :- let x be a variable in boolean expression then $x_1, x_2, x_3 \ldots x_n$ is said to be maxterm. if it is in the form of $x_1 \vee x_2 \vee x_3 \ldots \vee x_n$

$\bar{x}_1 \vee \bar{x}_2 \vee \bar{x}_3$ ---- $\vee \bar{x}_n$

$\bar{x}_1 \vee x_2 \vee x_3$ ---- $\vee \bar{x}_n$

$\left(\begin{smallmatrix}\vee \to + \\ \wedge \to \times\end{smallmatrix}\right)$

$(A+\bar{B})(A+\bar{B}+C)$

★ **Disjunctive Normal form:—** (DNF) {POS}

→ It is the join of minterm. i.e

i.e. $(\bar{x}_1 \wedge x_2 \wedge \bar{x}_3) \vee (x_1 \wedge x_2 \wedge x_3) \vee (\bar{x}_1 \wedge \bar{x}_2 \wedge \bar{x}_3)$ - every variables should be Present

* **Conjuctive Normal form:—** (CNF) {SOP}

AB + BC̄

→ It is the meet of Maxterms.

i.e $(\bar{x}_1 \vee x_2 \vee \bar{x}_3) \wedge (x_1 \vee x_2 \vee x_3) \wedge (\bar{x}_1 \vee \bar{x}_2 \vee \bar{x}_3)$

Q) express the following Boolean expression in DNF & CNF form, where,

$$f(x_1, x_2, x_3) = (x_1 \wedge \bar{x}_2) \vee (\bar{x}_3 \wedge x_2)$$

Soln:—

$$f(x_1, x_2, x_3) = \{(x_1 \wedge \bar{x}_2) \wedge (x_3 \vee \bar{x}_3)\} \vee \{(\bar{x}_3 \wedge x_2) \wedge (x_1 \vee \bar{x}_1)\}$$

$$f(x_1, x_2, x_3) = \{(x_1 \wedge \bar{x}_2 \wedge x_3) \vee (x_1 \wedge \bar{x}_2 \wedge \bar{x}_3)\} \vee \{(\bar{x}_3 \wedge x_2 \wedge x_1) \vee (\bar{x}_3 \wedge x_2 \wedge \bar{x}_1)\}$$

$$f(x_1, x_2, x_3) = (x_1 \wedge \bar{x}_2 \wedge x_3) \vee (x_1 \wedge \bar{x}_2 \wedge \bar{x}_3) \vee (x_1 \wedge x_2 \wedge \bar{x}_3) \vee (\bar{x}_1 \wedge x_2 \wedge \bar{x}_3)$$

Table:-

x_1	x_2	x_3	$f(x_1,x_2,x_3)$
0	0	0	0
0	0	1	0
0	1	0	1
0	1	1	0
1	0	0	1
1	0	1	1
1	1	0	1
1	1	1	0

There, 1 represents CNF.
0 represents DNF.

CNF:-

$$(\bar{x}_1 \wedge x_2 \wedge \bar{x}_3) \vee (x_1 \wedge \bar{x}_2 \wedge \bar{x}_3) \vee (x_1 \wedge x_2 \wedge \bar{x}_3) \vee (x_1 \wedge \bar{x}_2 \wedge x_3)$$

DNF:-

$$(\bar{x}_1 \vee \bar{x}_2 \vee \bar{x}_3) \wedge (\bar{x}_1 \vee \bar{x}_2 \vee x_3) \wedge (\bar{x}_1 \vee x_2 \vee x_3) \wedge (x_1 \vee x_2 \vee x_3)$$

Q) $f(x_1,x_2,x_3) = \overline{\overline{(x_1 \vee x_2)} \vee (\bar{x}_1 \wedge x_3)}$

Soln:-

$$\overline{\overline{(x_1 \vee x_2)}} \wedge \overline{(\bar{x}_1 \wedge x_3)}$$

$$= (x_1 \vee x_2) \wedge (x_1 \vee \bar{x}_3)$$

$$f(x_1,x_2,x_3) = \{(x_1 \vee x_2) \vee (x_3 \wedge \bar{x}_3)\} \wedge \{(x_1 \vee \bar{x}_3) \vee (x_2 \wedge \bar{x}_2)\}$$

$= \{(x_1 \vee x_2) \vee (x_3 \vee \bar{x}_3)\} \wedge \{(x_1 \vee \bar{x}_3) \vee (x_2 \wedge \bar{x}_2)\}$

$= \{(x_1 \vee x_2 \vee x_3) \wedge (x_1 \vee x_2 \vee \bar{x}_3)\} \wedge \{(x_1 \vee \bar{x}_3 \vee x_2) \wedge (x_1 \vee \bar{x}_3 \vee \bar{x}_2)\}$

*

x_1	x_2	x_3	$f(x_1, x_2, x_3)$
0	0	0	0
0	0	1	0
0	1	0	1
0	1	1	0
1	0	0	1
1	0	1	1
1	1	0	1
1	1	1	1

CNF → $(\bar{x}_1 \wedge \bar{x}_2 \wedge \bar{x}_3) \vee (x_1 \wedge \bar{x}_2 \wedge \bar{x}_3) \vee (x_1 \wedge \bar{x}_2 \wedge x_3) \vee (x_1 \wedge x_2 \wedge \bar{x}_3) \vee (x_1 \wedge x_2 \wedge x_3)$

DNF → $(x_1 \vee x_2 \vee x_3) \wedge (x_1 \vee x_2 \vee \bar{x}_3) \wedge (x_1 \vee \bar{x}_2 \vee \bar{x}_3)$

Ch - Switch Circuit

* logical gates, K-Map, circuit design.