

# CIS 410 Midterm Report on Hidden Subgroup Problem

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## 1 Motivation

The hidden subgroup problem (HSP) is an computational algebra problem which has been shown to have a lot of interesting consequences and motivations. For example, the *Shor's quantum algorithm* of factoring integers and solving the discrete logarithm problem can be reduced to solving the HSP on finite abelian groups.

**Definition 1.** Given a group  $G$ , a subgroup  $H \leq G$ , and a set  $X$ , a function  $f : G \rightarrow X$  **hides**  $H$  if  $\forall g_1, g_2 \in G, f(g_1) = f(g_2)$  iff  $g_1H = g_2H$ , that is,  $g_1, g_2$  are in the same coset of  $H$ .

**Definition 2.** Now, the **Hidden Subgroup Problem (HSP)** is a problem with inputs: a group  $G$ , a set  $X$ , and a function oracle  $f : G \rightarrow X$  hiding a subgroup  $H$ . The function oracle uses  $\log(|G| + |X|)$  bits. The desired output is a generating set of  $H$ .

It is known that there exists a quantum algorithm which solves with certainty a hidden subgroup problem of an arbitrary finite group in a polynomial (in  $\log|G|$ ) number of calls to the oracle. In addition, quantum computers have been shown to have very good speedups for some instances of the problem. In fact, because quantum computers can solve the HSP on finite abelian groups in polynomial time, it is possible for quantum computers to factor integers much faster than classical computers can.

Two unknowns regarding the HSP are whether the symmetric group and the dihedral group have efficient quantum algorithms for solving HSP. If an efficient quantum algorithm were to be found for the symmetric group HSP, we would have an efficient algorithm for *graph isomorphism*, a very important problem in theoretical computer science and for Eugene Luks. A polynomial time dihedral group HSP algorithm would give a polynomial time algorithm for solving the *shortest vector problem on lattices*, a problem which is...(line truncated)...

Our group has some background in abstract algebra and algebraic number theory, so this is an attractive topic for us to explore. Also, one of us is studying the shortest vector problem for his undergraduate thesis, so this is of increased interest.

## 2 Midterm Report

### 2.1 Quantum Query Complexity of HSP is polynomial

Our motivation is to find an efficient quantum algorithm which can solve the HSP for any arbitrary finite group  $G$  in a polynomial calls to the given oracle. Given  $r$  many distinct subgroups of  $G$ , we are looking for a generating set for one of the subgroups. We can assume that any algorithms for the HSP always output a subset of a subgroup  $H$ ; if an algorithm outputs some subset  $X \not\subseteq H$ , we simply find the intersection of  $X$  with  $H$  by keeping  $x \in X$  only if  $f(x) = f(1_G)$ .

Let  $f$  be a function satisfying the conditions of the HSP. Fix an ordering of the distinct subgroups  $H_1, H_2, \dots, H_r$  such that  $|H_i| \geq |H_{i+1}|$  for all  $1 \leq i \leq r$ . Also let  $N = |G|$  and consider  $n = \log|G|$  to be the input size.<sup>2</sup>

**Theorem 3.** *There exists a quantum algorithm that solves the HSP for any finite group  $G$  in  $O(\log^4|G|)$  calls to the oracle. The algorithm has exponentially small error probability in  $\log|G|$ .*

The algorithm considers  $2 + 2s$  registers, where  $s$  is a positive integer chosen to lower the error probability: 1st contains a subgroup index  $1 \leq v \leq r$ , 2nd contains a counter  $1 \leq l \leq r$ , remaining  $2s$  are pairs of couplets so that in each couplet the first contains an element of  $G$  and the second some image of  $f$ . We call the first register in a couplet as a "subgroup" register and the second as a "function" register.

We say that a function  $f$  is  $H$ -periodic if  $f$  is constant of the left cosets of a subgroup  $H$  of  $G$ .  $H$  being a *hidden subgroup* of  $f$  is an instance of  $f$  taking distinct values on distinct cosets of  $H$ .

A left *translation* of a subgroup  $H$  is a subset  $T \subseteq G$  such that for any  $g \in G$ ,  $g = th$  for some  $t \in T$  and  $h \in H$ .

We define an operator **Test** so that  $\text{Test} = \text{Test}_r \cdots \text{Test}_2 \cdot \text{Test}_1$ , where each unitary operator  $\text{Test}_i$  tests whether  $f$  is  $H_i$ -periodic. Each  $\text{Test}_i$  is defined by  $\text{Test}_i = Q_i \otimes P_{s,i} + I \otimes P_{s,i}^\perp$  where

$$Q_i : \quad \begin{cases} |0\rangle |0\rangle & \mapsto |i\rangle |1\rangle \\ |v\rangle |l\rangle & \mapsto |v\rangle |l+1\rangle, \end{cases} \quad \text{if } l > 0 \quad \text{and } P_{s,i} = \left( \sum_{t \in T_i} |tH_i\rangle \langle tH_i| \otimes I \right)^{\otimes s}$$

Here  $Q_i$  acts on the first two registers so that once the second register is increased from 0 to 1, the first register stays the same, and  $P_{s,i}$  is the projector of the  $s$  couplets. The effect of  $\text{Test}_i$  is that  $Q_i$  is applied on the first two registers if  $s$  subgroup registers are in coset states of  $H_i$ .

The initial state is defined as

$$|\Psi_{init}\rangle = |0\rangle |0\rangle \otimes \left( \frac{1}{\sqrt{N}} \sum_{g \in G} |g\rangle |f(g)\rangle \right)^{\otimes s}$$

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<sup>1</sup>If a function is  $H$ -periodic then it is also  $H'$ -periodic for a proper subgroup  $H'$  of  $H$ . And we want to test for bigger subgroups first

<sup>2</sup>We know that the number of  $r$  is  $2^{O(n^2)}$  since any  $H_i$  is generated by a set of at most  $n$  elements of  $G$

**Lemma 1.** *If  $f$  is  $H_i$ -periodic, then*

$$\text{Test}_i |\Psi_{init}\rangle = |i\rangle |1\rangle \otimes \left( \frac{1}{\sqrt{N}} \sum_{g \in G} |g\rangle |f(g)\rangle \right)^{\otimes s}$$

*Proof.* First, we realize that if  $f$  is  $H_i$ -periodic then  $s$  subgroup registers are in superposition of coset states  $|tH_i\rangle = \frac{1}{\sqrt{|H_i|}} \sum_{h \in H_i} |th\rangle$  for  $t \in T_i$ , a translation of  $H_i$ . Also,  $f$  begin  $H_i$ -periodic implies that  $f(t) = f(th)$  for all  $t \in T_i$  and  $h \in H_i$ . So the state  $\frac{1}{\sqrt{N}} \sum_{g \in G} |g\rangle |f(g)\rangle = \frac{1}{\sqrt{N}} \sum_{t \in T_i} |tH_i\rangle |f(g)\rangle$  lives in  $+1$ -eigenspace of  $P_{1,i}$ , and hence  $P_{s,i}$  leaves the  $s$  coulets untouched. Thus, the lemma follows.  $\square$

At the end of the day, what we want to do is to apply the unitary **Test** to the initial state  $|\Psi_{init}\rangle$  to get  $|\Psi_{final}\rangle$ . Then, we measure the first register of  $|\Psi_{final}\rangle$  to get the subgroup index  $v$ , and if  $1 \leq v \leq r$  we output a generating set for  $H_i$ , otherwise we output  $1_G$ . But our output may be a wrong answer.

As we iterate through  $r$  tests for the subgroups, we would wish to only alter the state marginally so that it is safe to continue to test for the next  $H_{i+1}$  subgroup.

**Lemma 2.** *If  $f$  is not  $H_i$ -periodic, then the distance  $|\langle \text{Test}_i |\Psi_{init}\rangle \rangle - |\Psi_{init}\rangle|$  is at most  $\frac{2}{2^{s/2}}$ .*

The next lemma follows since distances add up linearly.

**Lemma 3.** *If  $f$  is not  $H_i$ -periodic for any  $1 \leq i \leq j$ , then the distance  $|\langle \Psi_j \rangle - |\Psi_{init}\rangle|$  is at most  $\frac{2j}{2^{s+2}}$ .*

At the beggining in the Theorem, we stated that the error probability is exponential. Great news is that we can make the algorithm exact by the use of amplitude amlification. This is something we want to show in the final paper.

## 2.2 Dihedral Group HSP

In order to solve Dihedral Group HSP, we first need to characterize the subgroups of the Dihedral Group  $D_N = \{r, s | \text{ord}(r) = N, \text{ord}(s) = 2, srs = r^{-1}\}$ . First, there are the cyclic subgroups generated by the set  $\{r^k | k \in \mathbb{Z}_N\}$ , which are normal in  $D_N$ . There are also dihedral subgroups, which are of the form  $D_m$ , where  $m$  divides  $N$ . And the other subgroups are generated by  $sr^k$  and are of order 2.

In 1998, Ettinger and Hoyer showed that it is possible to reduce the problem of solving the hidden subgroup problem on  $D_N$  with hiding function  $f$  to calculating  $k$  assuming the hidden subgroup is generated  $sr^k$ . The reason is fairly simple. If the hidden subgroup is a cyclic subgroup of  $D_N$ , then it is normal in  $D_N$  and can be calculated in polynomial time using previous results. If it is a dihedral subgroup  $D_m$ , then the hidden subgroup itself has a cyclic subgroup - it hides a cyclic subgroup  $\langle r^k \rangle$ . We take the factor group  $D_N / \langle r^k \rangle$  after calculating  $k$ , and we find the hidden subgroup that remains  $D_m / \langle r^k \rangle$ , which is just an

order 2 subgroup. To recap, first we try to find if there is a cyclic subgroup that is hidden. If  $\langle r^a \rangle$  is hidden, we calculate  $a$  using previously known results about finding hidden normal subgroups. After that, we take the factor group  $D_N / \langle r^a \rangle$  and see if the function  $f$  hides an order 2 subgroup generated by  $sr^k$  or if only the trivial subgroup remains. If both a cyclic and an order 2 subgroup was detected, we have a hidden dihedral subgroup. Otherwise, it is either a cyclic or an order 2 subgroup that is hidden and we have detected it. Now we move to the algorithm for detecting an order 2 subgroup.

## References

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