Queues with a Dynamic Schedule

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Declaration

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Signed

John Gilbertson

Abstract

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Chapter 1

Introduction

Queues with scheduled arrivals occur frequently in society. These are queues where instead of customers arriving randomly, their arrival times are scheduled in advance. A common example is a doctor's surgery where patients are given appointment times. These queues also occur in shipping when ship docking times are scheduled in advance.

In this thesis, we make several assumptions about the underlying system. First, we assume a single server queue with exponential service times. We assume that customers arrive punctually at their scheduled arrival times. In addition, all customers have the same mean service time μ.

The objective is to find a schedule of customer arrival times that minimises a linear combination of the expected total waiting time of all the customers and the expected server availability time (i.e., the expected time between the start and end of service). This linear combination is referred to as the expected cost of a schedule.

A customer's waiting time is the time from the customer's arrival until the server begins serving that customer. Throughout this thesis, we refer to the number of customers in the system at a given point in time. This number includes both any customers currently being served and any customers who have arrived and are currently waiting for service.

Scheduling customers involves a trade off between customer waiting time and server availability time. Scheduling customers to arrive further apart decreases the expected waiting time of the customers as there are less customers in the system at the time of each customer's arrival, but increases the expected server availability time. On the other hand, scheduling customers to arrive closer together decreases the expected server availability time, but increases the expecter waiting time of the customers.

Bailey (1952) was the first to study such queues. Bailey proposed a rule whereby the first two customers should be scheduled to arrive at the start of service and the remaining customers should be scheduled to arrive at fixed intervals. Bailey found that this rule reduced time wasted by customers without signficiantly increasing the server's idle time.

Many authors have extended on Bailey's ideas. Pegden and Rosenshine (1990) propose a method for determing the optimal scheduled arrival times. Mendel (2006) extends the model to allow for no-shows, whereby a customer fails to arrive for service. Fiems, Koole, and Nain (2007) include emergency requests that immediately halt the server.

Unfortunately, few authors have considered the problem of adjusting a given schedule. Most authors assume that a schedule is fixed at the start of service and cannot be altered during service. This lack of freedom to alter a schedule could be a restrictive assumption. In this thesis, we examine the potential advantage of being able to adjust some scheduled arrival times during service.

In Chapter 3, we derive a method for determing the optimal sequence of arrival times for a given number of customers. This method assumes that the schedule is fixed for the duration of service. We call such a schedule a 'static schedule.' The results in this chapter largely follow the work of Pegden and Rosenshine (1990).

In Chapter 4, we consider a special case where customer arrivals are scheduled iteratively (i.e., one by one). A customer's arrival time is only decided on the arrival of the customer to be served immediately before them. Such a schedule is called a 'dynamic schedule.' This schedule is equivalent to the scheduler being able to freely adjust each customer's scheduled arrival time up until the arrival of the customer to be served immediately before them.

Towards the end of both Chapters 3 and 4, a number of example models are considered. These models help to understand the properties of each of the schedules. We examine the simple case of a small number of customers to be scheduled, and the more interesting case of a larger number of customers to be scheduled.

The dynamic schedule is simply a more general case of the static schedule. In Chapter 5, we compare the expected cost of the two schedules and confirm this intuitive result. Moreover, we investigate the percentage difference between the expected cost of each schedule as the number of customers to be scheduled increases.

In Chapter 6, we seek a broader understanding of the differences between the two schedules. We simulate a million runs of each schedule and compare the two cost distributions. The mean customer arrival time and customer waiting time under each schedule is examined. This chapter gives us a fuller understanding of the properties of each schedule that are difficult to derive analytically.

Chapter 2

Literature Review

Queues with scheduled arrivals are widely studied. There is a large body of literature studying the potential of appointment systems to reduce customer waiting times and waiting room congestion. This research is essential as health care providers in particular are under a great deal of pressure to improve service quality and efficiency (Goldsmith, 1989). Before we explore the detail of this thesis, it is important to review some of the papers that study this problem.

Fomundam and Herrmann (2007), and Cayirli and Veral (2003) provide comprehensive surveys of research on appointment scheduling. Most of the papers on scheduled arrivals in health care can be classed into two categories. Those that design algorithms to determine schedules, and those that evaluate schedules using simulation. While simulation studies can easily model complicated customer flows, queuing models often provide more generic results than simulation (Green, 2006).

The foundation paper on modeling queues with scheduled arrivals is Bailey (1952). Bailey proposes that customers' waiting times can be reduced without a significant increase in the server's idle time. The Bailey rule, which is commonly referenced in literature, is that customers should be scheduled to arrive at fixed intervals with two customers scheduled to arrive at the start of service. Bailey found that a great deal of time wasted by customers could be reduced without a signficant increase in the server's idle time. Under the Bailey rule, customers with late appointments will wait longer than those with early appointments. This lack of uniformity might be perceived as unfair and thus an undesirable property of a schedule.

Pegden and Rosenshine (1990) extend on Bailey's paper. They present an algorithm to iteratively determine the optimal arrival times for n customers that need to be scheduled. The optimal arrival times are those that minimise a weighted sum of the expected customers' waiting time and the expected server's total availability time. Pegden and Rosenshine prove that their objective function is convex for $n \leq 4$, thus their algorithm finds the optimal schedule. While they conjecture that the objective function is convex for all n, it has not been proven.

Stein and Côté (1994) apply Pegden and Roshenshine's model to obtain numerical results for situations with more than three customers. The optimal times between successive customers become near constant as n grows. This is the often observed dome-shape. Optimal appointment intervals exhibit a common pattern where they initially increase towards the middle of a session, and then decrease. Stein and Côte simplify their model by requiring the intervals between arriving customers to be constant. This commonly studied restriction makes the model more easily applicable in practice without significant altering the results.

Stein and Côte apply queuing theory results to solve the model for the optimal arrival interval assuming the queue reaches its steady state distribution. This assumption greatly reduces the computation required. However, in practice, it is common to find services that never reach steady state. Babes and Sarma (1991) attempted to apply steady state queuing theory, but found their results tended to be very different from those observed in real operation.

These key papers by Bailey, Pegden and Rosenshine, and Stein and Côte provide the basis for a more realistic exploration of health care systems. DeLaurentis et al. (2006) point out that customer no-shows can lead to a waste of resources. Mendel (2006) incorporates the probability of a customer not showing up into the model presented by Pegden and Rosenshine. Unsurprisingly, no-shows lead to lower expected waiting times for customers who do show up.

The presence of walk-ins (regular and emergency) can disrupt a schedule. Gupta, Zoreda, and Kramer (1971) propose a system where non-routine requests are superimposed on top of routine scheduled requests. Fiems, Koole, and Nain (2007) investigate the effect of emergency requests on the waiting times of scheduled customers. Fiems et al. model a system with deterministic service times and discrete time. Despite this research, Cayirli and Veral suggest that walk-ins are neglected in most studies. Further research could investigate their effect on optimal arrival times.

Mondschein and Weintraub (2003) observe that the majority of the literature assumes that demand is exogenous and independent of customers' waiting times. These papers assume the total number of customers is fixed and independent of waiting times. The vast majority of servers are now private (including medical servers), so face competitive environments. Mondschein and Weintraub thus present a model where demand depends on the customers' expected waiting time.

Few authors have attempted to model a dynamic schedule. Wang (1993) considers the problem of scheduling a new customer when there are already a number of customers with fixed scheduled arrival times. The aim is to determine the optimal arrival time for the new customer such that the objective function remains optimal. However, Wang has been criticised for not having a truly dynamic model (Erdogan and Denton, 2011). The initial scheduling of the customers ignores the possibility of an additional customer needing to be scheduled.

Simulation is a useful tool to analyse the effectiveness of appointment policies. Kao and Tung (1981) use simulation to compliment their results obtained from queuing theory. Ho and Lau (1992) study the performance of eight different appointment rules under different scenarios. They find that no rule will perform well under all circumstances.

Case studies can test the real world applications of an appointment system. While they lack generalisation, they are necessary to compliment the theoretical research. Rockart and Hofmann (1969) show individual block systems lead to more punctual doctors and patients, and less no-shows. Walter (1973) indicates that the simple grouping of inpatients and outpatients results in a substantial reduction in the doctor's idle time.

Unfortunately, Cayirli and Veral (2003) lament that despite much published work, the impact of appointment systems on outpatient clinics has been limited. Doctors are often unwilling to change their old habits. O'Keefe (1985) had their proposed appointment system of classifying patients rejected by staff. Huarng and Hou Lee (1996) were unable to implement their system due to staff resistance. Bennett and Worthington (1998) found their recommendations were not implemented successfully. Future research must attempt to develop models that will be accepted and implemented in real health care services.

Chapter 3

Static Schedule

This chapter largely follows the results of Pegden and Rosenshine (1990). The aim is to derive a method for choosing an optimal static schedule. A static schedule is a sequence of n customer arrival times $t_1, \ldots, t_n$ chosen at the start of service and fixed for the duration of service.

For simplicity, these results assume the customer service times are independent and identically distributed (iid) exponential random variables with mean μ. There is a single server. All customers are punctual and arrive at their scheduled arrival time.

3.1 Objective Function

The optimal schedule minimises the expected cost, which is a linear combination of the expected total customers' waiting time and total expected server availability time.

Denote the expected waiting time of customer i as w_i. The expected total customers' waiting time is the sum of the individual customer's expected waiting times.

$$\mathbb{E}\Big[\text{total customer's waiting time}\Big] = \sum_{i=1}^{n} w_i \tag{3.1}$$

Instead of solving for the customer arrival times, it is easier to solve for the arrival time of the first customer t_1 and the customer interarrival times $\mathbf{x} = (x_1, \ldots, x_{n-1})$ where $x_i = t_{i+1} - t_i$. The expected total server availability time is the expected time from the start of service until the end of service. This the sum of the last customer's scheduled arrival time, the last customer's expected waiting time and the last customer's expected service time.

$$\mathbb{E}\Big[\text{total server availability time}\Big] = \left(t_1 + \sum_{i=1}^{n-1} x_i\right) + w_n + \mu \tag{3.2}$$

Denote c_W and c_I as the per unit time cost of the expected total customer's waiting time and the per unit time cost of the expected total server availability time respectively. The objective function to be minimised is thus,

$$\phi(t_1, \mathbf{x}_n) = c_W \sum_{i=1}^{n} w_i + c_S \left[t_1 + \sum_{i=1}^{n-1} x_i + w_n + \mu\right] \tag{3.3}$$

The first customer should obviously be scheduled for the start of service so $t_1 = 0$. Moreover, can scale the objective function by dividing by $(c_W + c_S)$ and defining $\gamma = \frac{c_S}{c_W + c_S}$.

$$\phi(\mathbf{x}_n) \coloneqq (1-\gamma) \sum_{i=1}^{n} w_i + \gamma \left[\sum_{i=1}^{n-1} x_i + w_n + \mu\right] \tag{3.4}$$

The optimal static schedule is thus the interarrival times that minimise Equation 3.4

3.2 Expected Waiting Time

We want to express w_i (i.e., the expected waiting time of customer i) as a function of the interarrival times $\mathbf{x}_n$. If there are j customers in the system just prior to the arrival of customer i, then $w_i = j\mu$ by the memoryless property of the exponential distribution. The number of customers in the system refers to both customers being served and customers waiting for service.

Denote the number of customers in the system just prior to the arrival of customer i as N_i. Thus, the expected waiting time of customer i is given by

$$w_i = \sum_{j=0}^{i-1} (j\mu)\mathbb{P}(N_i = j) \tag{3.5}$$

The probability of a given number of customers in the system can be expressed recursively. This probability depends on the number of departures from the system between the arrival of customer $(i-1)$ and customer i. The number of departures is the minimum of $(N_{i-1} + 1)$ and a Poisson random variable with mean $\frac{x_{i-1}}{\mu}$.

The full recursive expression for the probability of j customers in the system immediately before the arrival of customer i is

$$\mathbb{P}(N_i = j) = \begin{cases} 1 & \text{for } i = 1,\ j = 0 \\ \sum_{k=1}^{i-1} \mathbb{P}(N_{i-1} = k-1) \left[1 - \sum_{l=0}^{k-1} \frac{x_{i-1}^l}{\mu^l l!} e^{\frac{-x_{i-1}}{\mu}}\right] & \text{for } i \geq 2,\ j = 0 \\ \sum_{k=0}^{i-j-1} \mathbb{P}(N_{i-1} = j+k-1) \left[\frac{x_{i-1}^k}{\mu^k k!} e^{\frac{-x_{i-1}}{\mu}}\right] & \text{for } i \geq 2,\ j \geq 1 \end{cases} \tag{3.6}$$

3.3 Computing the Objective Function

Pegden and Rosenshine (1990) suggest a similar algorithm to Algorithm 1 for computing the value of the objective function given by Equation 3.4 for a given vector $\mathbf{x}_n$ and parameters γ and μ.

Algorithm 1 Return $\phi(\mathbf{x}_n)$ for a given vector $\mathbf{x}_n$, γ and μ

```
function ObjectiveFunction(x_n, γ, μ)
    for i = 1, 2, ..., n do
        for j = 0, 1, ..., (i − 1) do
            compute P(N_i = j) by Equation 3.6
    for i = 1, 2, ..., n do
        compute w_i by Equation 3.5
    return φ(x_n) computed by Equation 3.4
```

3.4 Example Models

3.4.1 Model for 2 Customers

We consider the simplest case of this model where there are two customers to be scheduled (i.e., $n = 2$). As the first customer is scheduled to arrive at the start of service, the only unknown variable is the optimal interarrival time between the first and second customer (i.e., x_1).

By Equation 3.5, the expected waiting times of the two customers are

$$\begin{aligned} w_1 &= 0 \\ w_2 &= \mu e^{\frac{-x_1}{\mu}} \end{aligned} \tag{3.7}$$

By Equation 3.4, the objective function to be minimised is given by

$$\phi(x_1) = \mu\left[\gamma + \exp\left(\frac{-x_1}{\mu}\right)\right] + \gamma x_1 \tag{3.8}$$

This objective function is convex as

$$\forall x_1 \quad \phi''(x_1) = \frac{1}{\mu}\exp\left(\frac{-x_1}{\mu}\right) > 0 \tag{3.9}$$

Due to the convexity of the objective function, the optimal policy that minmises $\phi(x_1)$ can be found by solving:

$$\phi'(x_1) = 0 \implies -\exp\left(\frac{-x_1}{\mu}\right) + \gamma = 0 \tag{3.10}$$

Therefore, the optimal policy is:

$$x_1^* = \arg\min_{x_1} \phi(x_1) = -\mu \ln \gamma \tag{3.11}$$

As the server availability cost increases relative to the customer waiting cost (i.e., γ increases), the second customer is scheduled to arrive earlier (i.e., x_1^* decreases).

Unfortunately, as Pegden and Rosenshine (1990) found, no general algebraic solution exists for more than two customers (i.e., $n \geq 3$). All other cases need to be solve numerically. Pegden and Rosenshine (1990) proved that the objective function $\phi(\mathbf{x}_n)$ is convex for $n = 1, 2, 3, 4$. Moreover, they conjecture that it appears to be convex for general n, but are unable to prove it.

3.4.2 Model for 15 Customers

The more interesting cases concern scheduling a larger number of customers. We consider here the case of $n = 15$. Without loss of generality, can let $\mu = 1$. For $\mu \neq 1$, the solutions found are the optimal values of $\mu\mathbf{x}_n = (\mu x_1, \ldots, \mu x_{n-1})$.

Minimising the objective function is a constrained optimization problem. A numeric solution can be found using `scipy.optimize.minimize` in Python.

Figure 3.1 plots the optimal interarrival times $\mathbf{x}_n^* = (x_1^*, \ldots, x_{14}^*)$ for scheduling 15 customers with a mean service time $\mu = 1$ for various values of γ.

The optimal interarrival time increases for the initial few customers, remains constant for the majority of customers, and then it decreases for the last few customers. This is the dome-shape that was observed by Stein and Côté (1994)

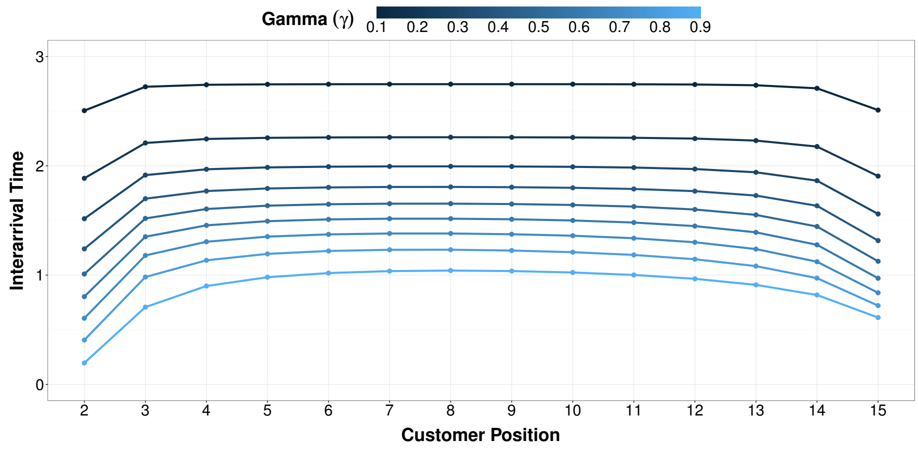


Figure 3.1: Optimal interarrival time for each of the 14 customers after the first customer. Figure generated assuming $\mu = 1$. Each line is plotted for a fixed value of $\gamma = \frac{c_S}{c_S + c_W}$. The darkest line is for $\gamma = 0.1$. As γ increases, the lines become lighter.

and Mendel (2006). The observation that the first customers arrive close together obeys Bailey's Low that was first recommended by Bailey (1952). As γ increases (i.e., the relative cost of server availability), the optimal interarrival times decrease while obeying the same general shape.

The common approach to efficiently finding the optimal static schedule involves simplifying the model by assuming a constant interarrival time (i.e., $x_1^* = \ldots = x_{n-1}^*$). Stein and Côté (1994) justifies this simplification by Theorem 1.1 of Hajek (1983). The theorem states that the average customer waiting time for an exponential server queue will be at a minimum for constant interarrival times. This guarantees that constant interarrival times is optimal for $\gamma = 0$ (i.e., $c_S = 0$), but does not imply that it's optimal for $\gamma > 0$.

The constant interarrival simplification significantly reduces computation cost, but Figure 3.1 suggests that it is clearly not optimal for the first few and last few customers. In addition, it is not possible to include this simplification in the dynamic schedule model presented next. Thus, this simplification is not pursued here.

Chapter 4

Dynamic Schedule

The static schedule presented in Chapter 3 and commonly throughout the scheduled arrivals literature is fixed for the duration of service. It was intuitively be advantageous to allow the schedule to vary during service. For example, if the service times of the first few customers are longer than expected, then it would be beneficial to schedule the remaining customers to arrive later.

This chapter approaches the problem of choosing a schedule in a similar way to Chapter 3. The aim is to find the arrival time of the first customer and the customer interarrival times that minimise the expected cost. The assumptions on iid service times and punctual customers are the same as Chapter 3.

This dynamic schedule is chosen progressively during service. Immediately after a customer arrives and begins waiting for service, the scheduler chooses the arrival time of the next customer. Most real-world situations are obviously more restrictive than this. It is often not possible to schedule the customer arrivals one-by-one.

Real-world situations would sometimes allow a degree of flexibility in regards to rescheduling customers. The theory is that if the (probably unrealistic) dynamic schedule presented here is significantly better than the static schedule, than the rescheduling ability should be included in real-world models for scheduled arrivals. However, if the dynamic schedule performs similarly to the static schedule than it is likely reasonable to ignore the rescheduling in the model formulation.

4.1 Objective Function

The state (n, k) refers to n customers remaining to be scheduled and k customers currently in the system (i.e., either waiting or being served). The time the next customer is scheduled to arrive relative to the current time is a.

The expected cost of a schedule is a linear combination of the expected total customers' waiting time and the total expected server availability time. The expected cost of the current state is a function of the expected cost involved in transitioning to the next state, the expected cost of the next state and the probability of transitioning to the next state (over all possible next states).

For any $n \geq 1$, the probability of transitioning from state (n,k) to state $(n-1,j)$ over time interval a if the next customer is scheduled to arrive in a time units is denoted by $p_a(k,j)$. The expected cost over this transition is denoted by $R_a(k,j)$. Note that j is the total number of customers in the system immediately after the next customer's arrival.

For $n \geq 1$, the expected cost of state (n,k) can be computed by the following form of Bellman's equation:

$$C_n^*(k) := \min_{a\geq 0} C_n(a,k) = \min_{a\geq 0}\left[\sum_{j=1}^{k+1} p_a(k,j)\Big(R_a(k,j) + C_{n-1}^*(j)\Big)\right] \tag{4.1}$$

Equation 4.1 is a recursive equation involving C^*. The optimal dynamic schedule is found by solving for each customer interarrival time a iteratively. The optimal policy a^* is the interarrival time that attains the minimum cost whereby

$$C_n^*(k) = C_n(a^*,k) = \min_{a\geq 0} C_n(a,k) \tag{4.2}$$

4.2 Base Case

Solving Equation 4.1 requires a solution for the base case where $n = 0$. The state $(0,k)$ is the state where there are no customers remaining to be scheduled and k customers currently in the system.

Denote the expected waiting time of the customer that is currently in position i as w_i. This expected waiting time is the summation of the expected service times of all the customers in positions $\{1,\ldots,i-1\}$. The expected total customers' waiting time for the k remaining customers is the sum of their individual waiting times.

$$\begin{aligned}\mathbb{E}\Big[\text{total customer's waiting time at state } (0,k)\Big] &= \sum_{i=1}^{k} w_i = \sum_{i=1}^{k}\mu(i-1) \\ &= \frac{\mu k(k-1)}{2}\end{aligned} \tag{4.3}$$

If there are no customers remaining to be scheduled, then the expected total server availability time is the expected time until the customer currently in the last position in the queue finishes service. This time is the summation of the expected service times of all the customers currently in the system.

$$\mathbb{E}\Big[\text{total server availability time at state } (0,k)\Big] = \sum_{i=1}^{k} \mu = k\mu \tag{4.4}$$

The per unit costs c_W and c_S are defined as before in Chapter 3. The cost of the base case is thus given by

$$C_0^*(k) = c_W \frac{\mu k(k-1)}{2} + c_S k\mu \tag{4.5}$$

In order to compare $C_0^*(k)$ with the cost of the static schedule, need to scale it by dividing by $(c_S + c_W)$ and defining $\gamma = \frac{c_S}{c_S+c_W}$.

$$C_0^*(k) := (1-\gamma)\frac{\mu k(k-1)}{2} + \gamma k\mu \tag{4.6}$$

4.3 Erlang Distribution

The transition probability and expected transition cost for the dynamic schedule depend on the cdf and conditional expectation of the Erlang distribution, which are shown here.

Denote the service time of the customer that is currently in position i in the queue as S_i. For n customers, the service times $S_1, \ldots, S_n$ are iid exponential random variables with mean μ.

For $r \geq 1$, the waiting time of the customer in position $(r+1)$ is given by

$$X = \sum_{i=1}^{r} S_i \sim \text{Erlang}(r, \mu) \tag{4.7}$$

which has the pdf

$$f(x;r) := \frac{1}{\mu(r-1)!}\left(\frac{x}{\mu}\right)^{r-1} \mathrm{e}^{\frac{-x}{\mu}} \tag{4.8}$$

The cdf of the Erlang distribution is

$$F(a;r) \coloneqq \mathbb{P}(X \le a) = \begin{cases} 0 & \text{for } a = 0 \\ 1 - \sum_{i=0}^{r-1} \frac{1}{i!} \left(\frac{a}{\mu}\right)^i \mathrm{e}^{\frac{-a}{\mu}} & \text{for } a > 0 \end{cases} \tag{4.9}$$

For $r \ge 1$, $F(a;r)$ is continuous for $a \ge 0$ as

$$\lim_{a \to 0^+} F(a;r) = 0 = F(0;r) \tag{4.10}$$

The conditional expectation of X given $X \le a$ is

$$G(a;r) \coloneqq \mathbb{E}[X|X \le a] = \begin{cases} 0 & \text{for } a = 0 \\ \frac{\mu r F(a;r+1)}{F(a;r)} & \text{for } a > 0 \end{cases} \tag{4.11}$$

This expression makes intutive sense. For $a > 0$ and $r \ge 1$, $\frac{F(a;r+1)}{F(a;r)} < 1$. In addition, the mean of the Erlang distribution is $\mathbb{E}[X] = \mu r$. Thus, for $a > 0$ and $r \ge 1$, $\mathbb{E}[X|X \le a] < \mathbb{E}[X]$ as expected.

Moreover,

$$\lim_{a \to \infty} \frac{F(a;r+1)}{F(a;r)} = \frac{\lim_{a \to \infty} F(a;r+1)}{\lim_{a \to \infty} F(a;r)} = \frac{1}{1} = 1 \tag{4.12}$$

Thus, $\lim_{a \to \infty} \mathbb{E}[X|X \le a] = \mathbb{E}[X]$ as expected.

Finally, suppose there is an exponential random variable Y with mean μ that is independent of X. The conditional expectation of X given $X \le a$ and $X + Y > a$ is

$$H(a;r) \coloneqq \mathbb{E}[X|X \le a, X + Y > a] = \frac{ar}{r+1} \tag{4.13}$$

The conditional expectation $H(a;r)$ doesn't depend on μ.

4.4 Transition Probability

The transition probability $p_a(k,j)$ is the probability that the queue length changes from k customers initially to j customers on the arrival of the next customer after a time units. In other words, it is the probability that there are $k - (j-1)$ departures from the queue over a time interval of length a.

The transition probability is most easily derived on a case by case basis. The full derivation is included in the appendix and only the final equation is presented here.

$$p_a(k,j) \coloneqq \begin{cases} \mathbb{1}(j=1) & \text{for } k=0 \\ F(a;k) & \text{for } k \geq 1, j=1 \\ F(a;k-j+1) - F(a;k-j+2) & \text{for } k \geq 2, 2 \leq j \leq k \\ 1-F(a;1) & \text{for } k \geq 1, j=(k+1) \\ 0 & \text{otherwise} \end{cases} \tag{4.14}$$

Given a current state (n,k) and a time interval a, the total probability over all possible next states $(n-1,j)$ is one.

$$\forall k \in \mathbb{N}_0, a \geq 0 \quad \sum_{j=1}^{k+1} p_a(k,j) = 1 \tag{4.15}$$

4.5 Expected Transition Cost

The expected transition cost is the expected cost of transitioning from state (n,k) to state $(n-1,j)$ where the next customer is scheduled to arrive in a time units.

In a similar way to the transition probability, the expected transition cost is derived on a case by case basis. To compare the dynamic schedule with the static schedule, the cost is scaled by dividing by $(c_S + c_W)$ and defining $\gamma = \frac{c_S}{c_S+c_W}$. The full derivation is included in the appendix and the final equation is presented here.

$$R_a(k,j) \coloneqq \begin{cases} \gamma a & \text{for } k \in \{0,1\} \\ (1-\gamma)\frac{G(a;k)(k-1)}{2} + \gamma a & \text{for } k \geq 2, j=1 \\ (1-\gamma)\frac{a(k+j-3)}{2} + \gamma a & \text{for } k \geq 2, 2 \leq j \leq k+1 \end{cases} \tag{4.16}$$

4.6 Example Models

4.6.1 Model for 2 Customers

Assume there are two customers that need to be scheduled for service. The initial state is $(2,0)$ (i.e., two customers to schedule and no customers currently in the system). The possible states during service are all the states in the set

$$\left\{(n,k) \in \{0,1,2\}^2 : n+k \leq 2\right\} \tag{4.17}$$

For $n \geq 1$, the expected cost of state $(n, 0)$ (i.e., no customers currently in the system) as a function of the time interval a until the next customer arrival is given by

$$C_n(a, 0) = C^*_{n-1}(1) + \gamma a \tag{4.18}$$

As $\gamma \in (0, 1)$, the optimal policy a^* where $C_n(a, 0)$ attains a minimum is

$$a^* = \arg\min_{a \geq 0} C_n(a, 0) = 0 \tag{4.19}$$

Thus, if there are no customers currently waiting, then the next customer should be scheduled to arrive immediately. This agrees with the result of the static schedule that the first customer should be schedule to arrive immediately at the state of service.

As the transition occurs immediately, the expected cost of state $(n, 0)$ equals the expected cost of state $(n - 1, 1)$ for $n \geq 1$.

$$C^*_n(0) = C^*_{n-1}(1) \tag{4.20}$$

For $n \geq 1$, the expected cost of state $(n, 1)$ (i.e., one customer currently in the system) as a function of the time interval a until the next customer arrival is given by

$$C_n(a, 1) = C^*_{n-1}(1) + \mathrm{e}^{\frac{-a}{\mu}} \left[C^*_{n-1}(2) - C^*_{n-1}(1) \right] + \gamma a \tag{4.21}$$

The optimal policy a^* where $C_n(a, 1)$ attains a minimum is

$$a^* = \arg\min_{a \geq 0} C_n(a, 0) = \mu \ln \left[\frac{C^*_{n-1}(2) - C^*_{n-1}(1)}{\gamma\mu} \right] \tag{4.22}$$

Returning to the case of scheduling two customers using this dynamic schedule where initial state is $(2, 0)$. By Equation 4.19, the first customer should be scheduled to arrive immediately. On the first customer's arrival, the system is at state $(1, 1)$. By Equation 4.22, the second customer should be scheduled to arrive a^* after the arrival of the first customer where

$$a^* = \mu \left[\frac{C^*_0(2) - C^*_0(1)}{\gamma\mu} \right] = -\mu \ln \gamma \tag{4.23}$$

This dynamic schedule for two customers is exactly the same as the static schedule for two customers. As the first customer is scheduled to arrive at the start of service, all arrival times are decided at the start of service. Therefore, the optimal policy is the same for both schedules.

Unfortunately, for $k \geq 2$, no algebraic solution exists for the optimal policy for state (n, k). Thus, cannot analytically derive the dynamic schedule for more than two customers. All other cases need to be solved analytically.

4.6.2 Model for 15 Customers

The more interesting cases concern scheduling a larger number of customers. We consider here the case of $n = 15$. Without loss of generality, can let $\mu = 1$. For $\mu \neq 1$, the optimal policies a^* found are the optimal interarrival times μa^*. For simplicity, assume that the per unit time costs c_W and c_S are both equal such that $\gamma = 0.5$.

Finding the optimal dynamic schedule for 15 customers involves solving several contrained optimisation problems. Need to solve for the optimal policy a^* at each possible state in the set

$$\left\{(n, k) \in \{0, \ldots, 15\}^2 : n + k \leq 15\right\} \tag{4.24}$$

In a similar way to Chapter 3, the expected cost of each state (n, k) is found using `scipy.optimize.minimize` in Python. The results are plotted in Figure 4.1.

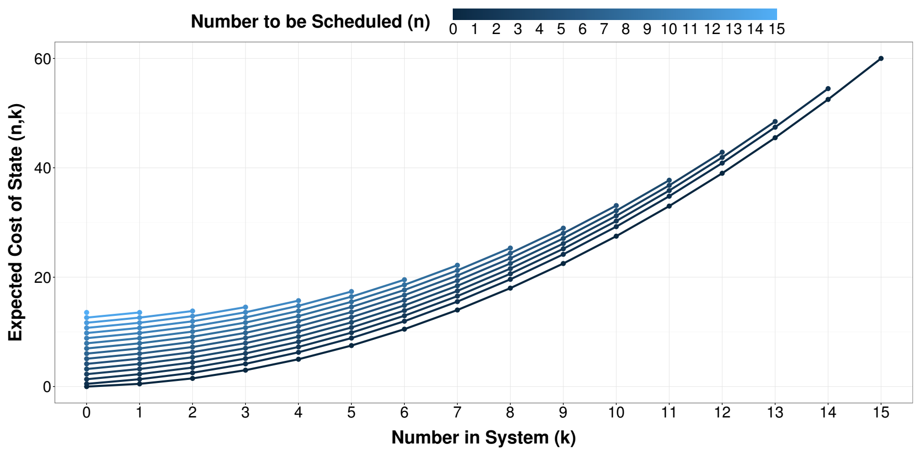


Figure 4.1: Expected cost of possible states with 15 total customers. Figure generated assuming $\gamma = \frac{c_S}{c_S + c_W} = 0.5$ and $\mu = 1$. Each line is plotted for a fixed value of the number of customers still to be scheduled (n). The darkest line is for $n = 0$. As n increases, the lines become lighter.

The cost of the initial state $(15, 0)$ is 13.55, thus the expected cost of serving 15 customers with a dynamic schedule is 13.55. The worst state in Figure 4.1 is the state $(0, 15)$, which is all 15 customers in the system. This can occur if all 15 customers are scheduled to arrive immediately at the start of service (to ensure minimal server availability time) or if the first customer has an extremely long service time.

The lines plotted on Figure 4.1 are the expected costs with fixed n and varying k. As the number of customers in the system (k) increases, the expected cost increases exponentially. By contrast, as the number of customers to be scheduled (n) increases with k fixed, the expected cost of the state increases approximately linearly at a significantly smaller rate.

It's difficult to observe, but for $n \geq 1$, Figure 4.1 shows that $C_n^*(0) = C_{n-1}^*(1)$. This confirms the previous result that if there are no customers currently in the system (e.g., initially), then it is always optimal to schedule the next arrival immediately. The expected cost does not change as the next arrival occurs immediately.

Figure 4.2 plots the corresponding interarrival times a^* for the states plotted in Figure 4.1. This figure does not include any optimal times for $n = 0$ as there is no next customer to schedule in those states.

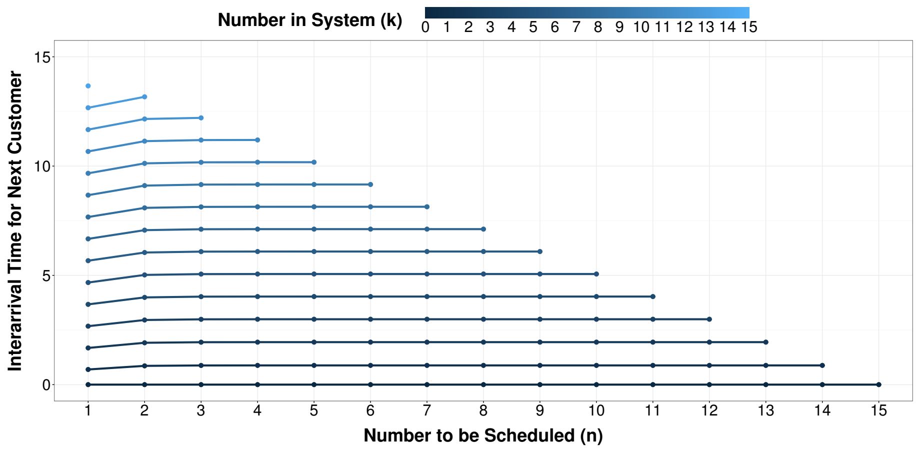


Figure 4.2: Optimal interarrival time for each possible state with 15 total customers. Figure generated assuming $\gamma = \frac{c_S}{c_S + c_W} = 0.5$ and $\mu = 1$. Each line is plotted for a fixed value of the number of customers in the system (k). The darkest line is for $k = 0$. As k increases, the lines become lighter.

Understanding Figure 4.2 is helped by looking at some examples. The optimal interarrival time for the initial state $(15, 0)$ is 0. The first customer should be scheduled to arrive immediately. In addition, the optimal interarrival time for the state $(3, 10)$ is 10.17. If (on arrival of a customer), there are 10 customers in the system and 3 customers remaining to be scheduled, then the next customer should be scheduled to arrive in 10.17 time units. This is slightly above the expected service time of the 10 customers currently in the system to account for the customer waiting cost.

The first pattern to notice is if there are no customers currently waiting (i.e., $k = 0$), then (as discussed earlier) the optimal policy is to schedule the next arrival immediately. This makes intutive sense as scheduling the next arrival immediately minimises the expected total server availability time without affecting the expected total customers' waiting times.

As k increases for fixed n, the optimal interarrival time a^* appears to increase at a slightly decreasing rate. The optimal a^* increases by approximately μ for each additional k. In contrast, for $n \geq 2$ and fixed k, a^* appears to be constant at a value similar to μk (i.e., the expected time for the system to be empty). The optimal a^* appears to be independent of the the number of customers still to be scheduled provided that there are at least two customers still to be scheduled.

Chapter 5

Schedule Comparison

5.1 Expected Cost Comparison

Suppose that there are N customers to be scheduled. In the case of the static schedule, the optimal policy is given by $\mathbf{x}_N^* = (x_1^*, \ldots, x_{N-1}^*)$ and the expected cost of the optimal policy is given by $\phi(\mathbf{x}_N^*)$. In the case of the dynamic schedule, the initial state is $(N, 0)$, so the expected cost of the optimal policy is given by $C_N^*(0)$.

Clearly, as the dynamic schedule has the ability to match the optimal policy for the static schedule, $C_N^*(0) \leq \phi(\mathbf{x}_N^*)$ (i.e., the dynamic schedule cannot have greater expected cost) regardless of the number of customers to be scheduled. As found in Chapter 4, equality is attained for the cases where $N \in \{0, 1, 2\}$ as the schedules are identical in those cases.

Figure 5.1 plots the expected costs of both the static and dynamic schedules against the number of customers to be scheduled (N) assuming $\gamma = 0.5$ and $\mu = 1$.

As expected, the costs are identical for $N \in \{0, 1, 2\}$. For all other N values, the cost of the static schedule is greater than the cost of the dynamic schedule. The cost difference appears to be minimal for $N \geq 5$, but as N increases the cost difference increases.

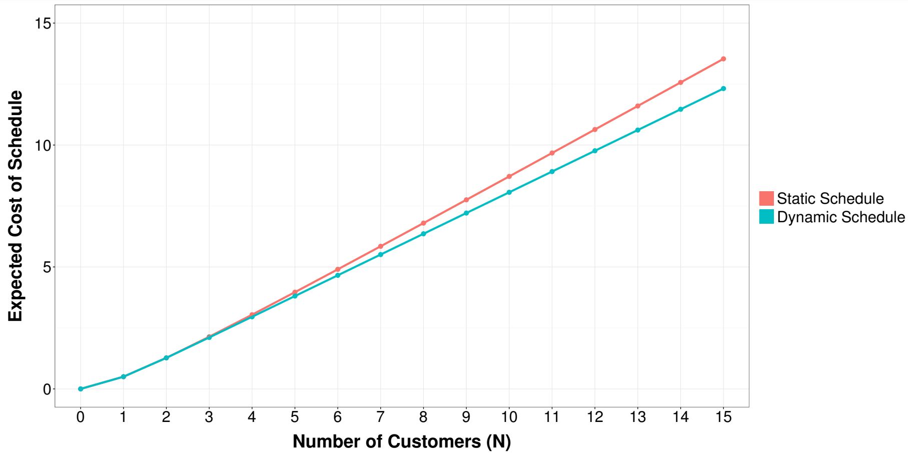


Figure 5.1: Plot of the expected cost of each schedule against the number of customers to be scheduled (N) for both the static and dynamic schedules where $N \in \{0, \ldots, 15\}$, $\gamma = \frac{c_S}{c_S + c_W} = 0.5$ and $\mu = 1$.

5.2 Expected Percentage Cost Saving

Define the expected percentage cost saving ΔC (i.e., the expected percentage difference between the cost of the static schedule and the dynamic schedule).

$$\Delta C := 100 \times \frac{\phi(\mathbf{x}_N^*) - C_N^*(0)}{\phi(\mathbf{x}_N^*)} \tag{5.1}$$

Figure 5.2 plots the percentage cost saving by using the dynamic schedule as opposed to the static schedule (ΔC) against γ for various values of the number of customers (N) to be scheduled.

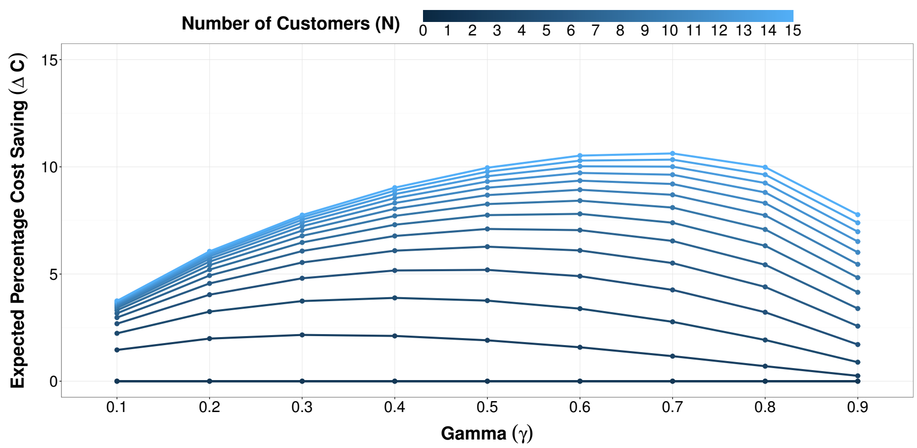


Figure 5.2: Plot of the percentage cost saving (ΔC) against $\gamma = \frac{c_S}{c_S + c_W}$ where $N = \{0, \ldots, 15\}$ and $\mu = 1$.

For $N \in \{0, 1, 2\}$, $\Delta C = 0$ for all values of γ as the schedules are identical.

As N increases with γ held constant, ΔC increases as the dynamic schedule begins to outperform the static schedule. ΔC increases at a decreasing rate (i.e., the curves become closer together as N increases). The maximal value of ΔC is 10.6%. Even if N were to increase further beyond 15, it doesn't appear that the expected percentage cost saving would exceed 15%.

For the extreme values of γ (i.e., $\gamma = 0.1$ and $\gamma = 0.9$), ΔC is at a minimum for each value of N. An extreme value of γ indicates that the either the customer waiting cost or the server availability cost is significantly priorities (i.e., $c_W \gg c_S$ or $c_S \gg c_W$). If one of the costs is heavily prioritised, there is little difference between the static and dynamic schedules, thus ΔC is small.

For each value of $N \geq 3$, the peak value of ΔC occurs at a middle value of γ. As N increases, the peak occurs at a larger value of γ. For $N = 3$ the peak occurs at $\gamma = 0.3$, whereas for $N = 15$ the peak occurs at $\gamma = 0.7$.

The difference between the dynamic and static schedule is most significant for middle values of γ (e.g., $\gamma \in [0.4, 0.7]$). For $\gamma = 0.1$, it doesn't appear that the expected percentage cost saving would exceed 5% indicating very little difference between the schedules.

Chapter 6

Simulation Studies

The previous chapter compared the static and dynamic schedules in terms of the expected cost of each schedule. We found for 15 customers, the dynamic schedule is about $5 - 10\%$ better than the static schedule depending on the value of γ. However, this does not provide any information about the other properties of each schedule. For example, one of the scheduled could have a greater variability in customer waiting times.

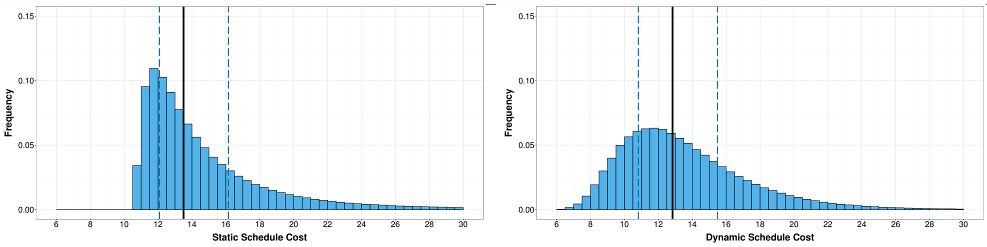
To understand each schedule more fully, we'll use a simulation study. We simulated a million runs and measured the performance of both the static and dynamic schedules on each run. Each run involved simulating the service times of the 15 customers assuming $\mu = 1$. All the simulations were completed assuming $\gamma = 0.5$.

6.1 Schedule Cost

The mean costs of the static and dynamic schedule were 15.05 and 13.55 respectively. As expected, these mean costs match closely the expected costs of the schedules. Figure 6.1 plots the histograms of the costs of the static and dynamic schedules for each run of the simulation.

The distribution of the cost of the static schedule is right skewed. The peak of the distribution occurs at around 12, but the median is 13.5. In contrast, the distribution of the dynamic schedule is more symmetric with the peak ocurring much closer to the median of 12.8. The dynamic schedule is more flexible and thus less prone to runs with high cost.

Figure 6.2 plots the histogram of the difference in costs between the two schedules. This is a plot of the cost of the static schedule minus the cost of the dynamic schedule for the same run.



(a) (b)

Figure 6.1: Histogram plot of the costs of the static (a) and dynamic (b) schedules for each simulation run where $\mu = 1$ and $\gamma = \frac{c_S}{c_S + c_W} = 0.5$. The black vertical line indicates the median and the blue dashes lines indicate the upper and lower quartiles.

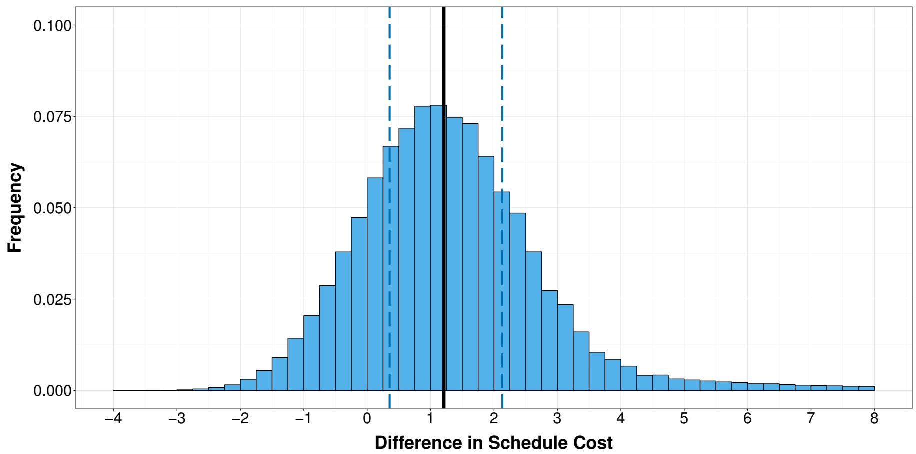


Figure 6.2: Histogram plot of the cost of the static schedule minus the cost of the dynamic schedule for each simulation run where $\mu = 1$ and $\gamma = \frac{c_S}{c_S + c_W} = 0.5$. The black vertical line indicates the median and the blue dashes lines indicate the upper and lower quartiles.

While the expected cost of the static schedule is greater than the expected cost of the dynamic schedule, the static schedule outperforms the dynamic schedule for a proportion of the simulation runs. This is indicated by a negative cost difference in the plotted histogram. The 25-th percentile of the difference in cost is 0.35. For more than 75% of runs, the static schedule has a larger cost than the dynamic schedule. The 75-th percentile is 2.13 (i.e., more than twice the mean service time). For a large proportion of the runs, the dynamic schedule significantly outperforms the static schedule.

6.2 Customer Arrival Times

We have now observed that the dynamic schedule performs better than the static schedule both in expectation and for the majority of the simulation runs. The next step is to attempt to understand how the dynamic schedule is able to outperform the static schedule. Figure 6.3 plots the average arrival time for each of the 15 customers for each schedule.

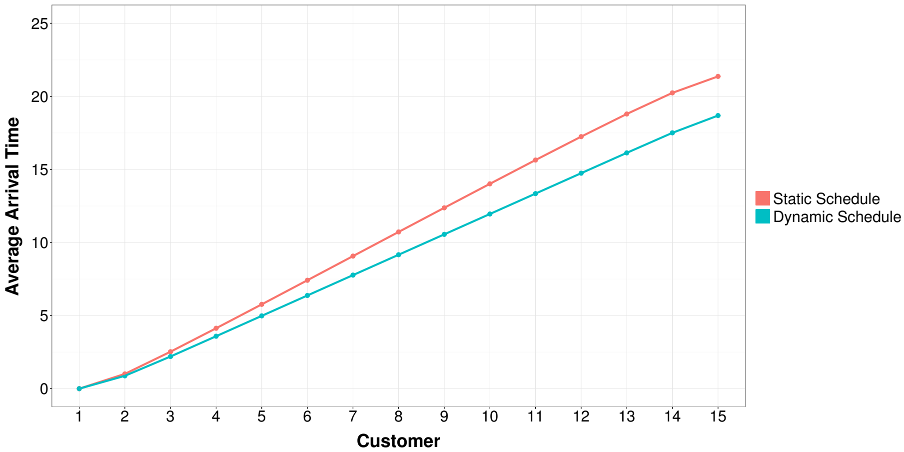


Figure 6.3: Plot of the average arrival time for each of the 15 customers for each schedule where $\mu = 1$ and $\gamma = \frac{c_S}{c_S + c_W} = 0.5$.

The static schedule has fixed arrival times, whereas the arrival times are chosen progressively for the dynamic schedule. For all 15 customers, the mean arrival time for the dynamic schedule is never earlier than the arrival time of the static schedule. The mean arrival times appear to be very similar for the first four or five customers. However, the later customers in the dynamic schedule appear to arrive significantly earlier than the corresponding customers in the static schedule.

6.3 Customer Waiting Times

We would expect that the earlier arrival of the customers in the dynamic schedule would lead to those customers waiting longer. This hypothesis is examined by plotting the histogram of the difference in average customer waiting time between the two schedule in Figure 6.4. As before, this is a plot of the average waiting time of the static schedule minus the average waiting time of the dynamic schedule.

The 75-th percentile of the difference in average waiting time is 0. As expected, for the majority of the runs, the customers in the static schedule have a shorter average waiting time than the customers in the dynamic schedule.

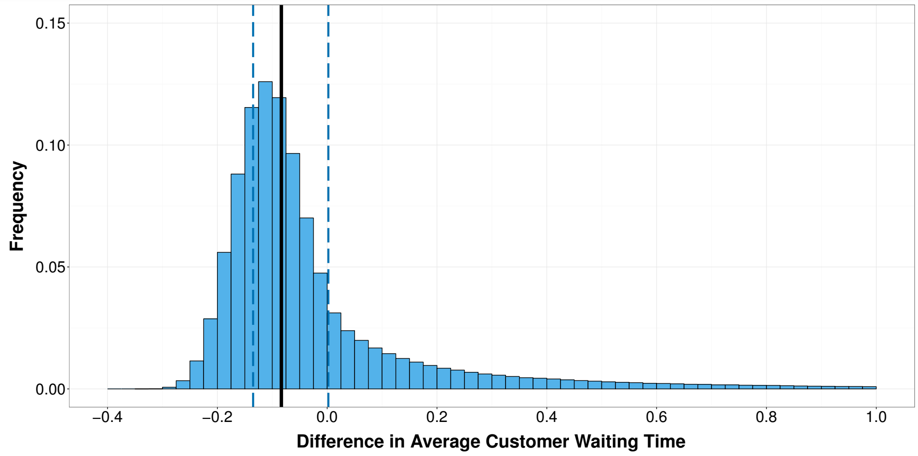


Figure 6.4: Histogram plot of the average waiting time for the static schedule minus the average waiting time for the dynamic schedule where $\mu = 1$ and $\gamma = \frac{c_S}{c_S + c_W} = 0.5$. The black vertical line indicates the median and the blue dashes lines indicate the upper and lower quartiles.

However, the minimum difference in average waiting time is only -0.33, whereas the maximum is 8.57. It appears that when the dynamic schedule has a shorter waiting time than the static schedule, it tends to have a significantly shorter waiting time. The dynamic schedule is less prone to extremely long waiting times for the customers. This is expected behaviour as the dynamic schedule is able to react to customers having longer than expected service times and reschedule the later customers.

Another important consideration is the treatment of all the customers individually. Figure 6.5 plots the average waiting time of each customer individually.

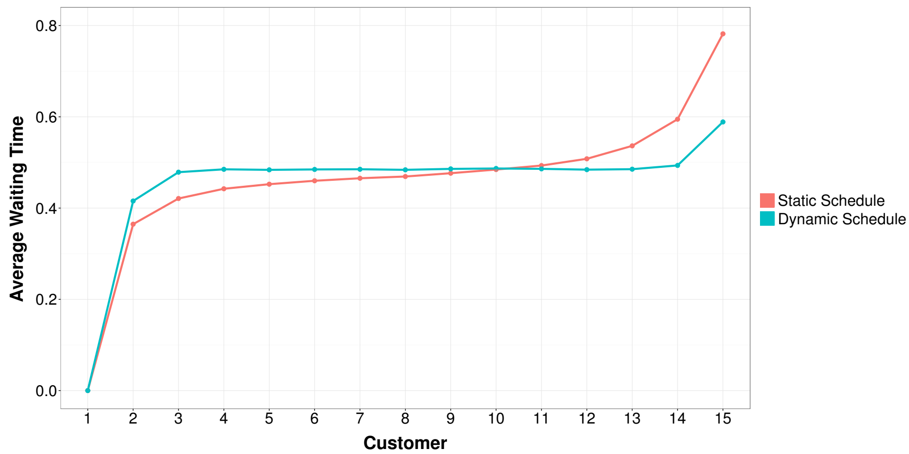


Figure 6.5: Plot of the average waiting time for each of the 15 customers for each schedule where $\mu = 1$ and $\gamma = \frac{c_S}{c_S + c_W} = 0.5$.

In both schedules, the average waiting time of the first customer is 0 as they are served immediately. The first few customers wait longer on average in the dynamic schedule than the static schedule. In contrast, the later few customers tend to wait significantly longer in the static schedule. From customer 3 up to customer 14, their average waiting times are increases for the static schedule, but approximately constant in the dynamic schedule. It appears that the dynamic schedule is significantly 'fairer' and able to spread the waiting time more evenly amongst the customers. The static schedule tends to favour the earlier customers.

Intuitively, customers prefer schedules where they are served immediately and do not have to wait. Figure 6.6 plots the proportion of runs where each customer does not have to wait.

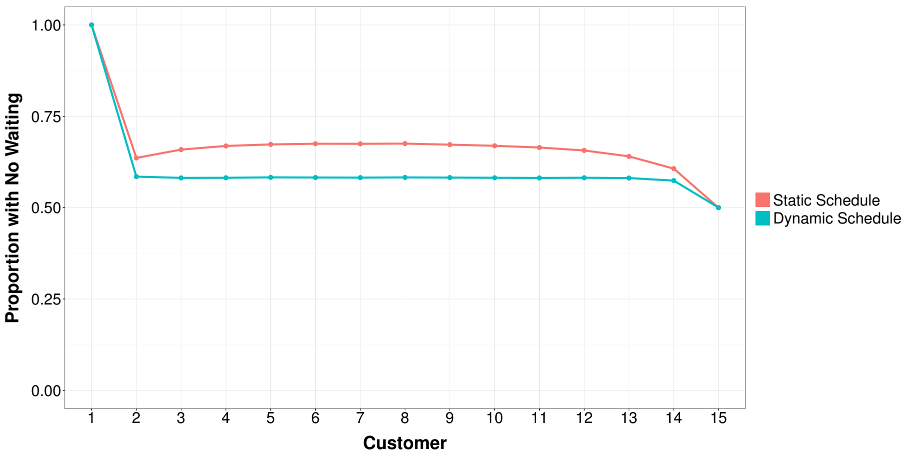
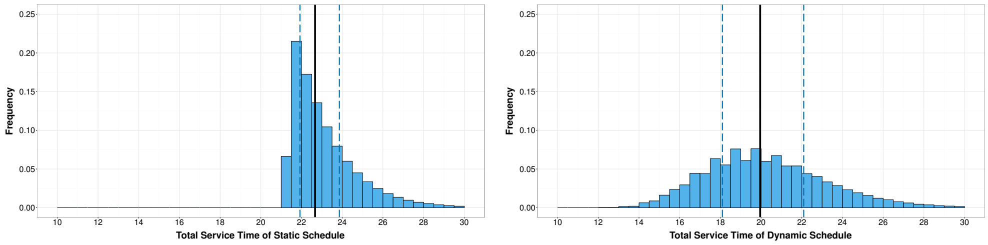


Figure 6.6: Plot of the proportion of all runs where the customer does not wait for each of the 15 customers for each schedule where $\mu = 1$ and $\gamma = \frac{c_S}{c_S + c_W} = 0.5$.

In both schedules, the first customer never waits and is served immediately in 100% of the runs. The dynamic schedule again appears to be fairer. The proportion of runs with no waiting appears to be constant for all except the first and last customers. All customers (except the first and last) are served immediately a greater proportion of the time in the static schedule than the dynamic schedule. While this is advantageous for the customers, it leads to a longer idle time of the server and thus a longer server availability time.

6.4 Server Availability Time

The schedules do not only attempt to minimise customer waiting time, but they also attempt to minimise the total server availability time. Figure 6.7 plots the histograms of the total server availability times of the static and dynamic schedules for each run of the simulation.



(a) (b)

Figure 6.7: Histogram plot of the total server availability times for the static (a) and dynamic (b) schedules for each simulation run where $\mu = 1$ and $\gamma = \frac{c_S}{c_S+c_W} = 0.5$. The black vertical line indicates the median and the blue dashes lines indicate the upper and lower quartiles.

The median total server availability time is 22.66 for the static schedule and 19.95 for the dynamic schedule. While the customer waiting time appears to be generally similar for both schedules, the dynamic schedule appears to attain its lower expected cost due to its lower total server availability time. This agrees with the idea from Chapter 5 that the expected percentage cost saving is lowest where $\gamma = \frac{c_S}{c_S+c_W}$ is small. If the per unit time of the server availability time is small then the dynamic schedule's ability to reduce the total server availability time is less effective at reduce the cost of the schedule.

Moreover, the minimum total server availability time is significantly lower in the case of the dynamic schedule. The minimum total server availability time is 21.37 for the static schedule and 12.11 for the dynamic schedule. This is especially surprising as both minimums occur in the same run of the simulation. The static schedule is significantly limited by the fixed arrival of the last customer at 21.36 time units. The dynamic schedule is able to adjust the customer arrival times and attain a broader range of total server availability times that are lower on average.

Chapter 7

Conclusion

Conclusion goes here.

Appendix A

Dynamic Schedule Derivation

A.1 Transition Probability

This is a derivation of the probability $p_a(k, j)$. This is the probability that there are $k - (j - 1)$ departures from a queue over a time units assuming the queue initially has k customers and no new arrivals. The customer service times are iid exponential random variables with mean μ.

The probabililty is most easily derived on a case by case basis.

A.1.1 Case 1 $k = 0$

The first case is that the queue initially has no customers. For any $a \geq 0$, there will be no departures from the queue over a time units.

$$p_a(k, j) = \mathbb{1}(j = 1) \tag{A.1}$$

A.1.2 Case 2 $k \geq 1, j = 1$

Denote the service times of the k customers as $S_1, \ldots, S_k$. If $j = 1$, then $p_a(k, j)$ is the probability of k departures from the queue over a time units. The sum $\sum_{i=1}^{k} S_i$ has an Erlang distribution with cdf $F(a; k)$.

$$p_a(k, j) = \mathbb{P}\left(\sum_{i=1}^{k} S_i \leq a\right) = F(a; k) \tag{A.2}$$

A.1.3 Case 3 $k \geq 2, 2 \leq j \leq k$

For $2 \leq j \leq k$, $p_a(k, j)$ is the probability that the total service time of the first $k - (j - 1)$ customers is less that a, and the total service time of the first

$k-(j-1)+1$ customers is greater than a.

$$\begin{aligned} p_a(k,j) &= \mathbb{P}\left(\sum_{i=1}^{k-(j-1)} S_i \leq a, \sum_{i=1}^{k-(j-1)+1} S_i > a\right) \\ &= \mathbb{P}\left(\sum_{i=1}^{k-(j-1)} S_i \leq a, S_{k-(j-1)+1} > a - \sum_{i=1}^{k-(j-1)} S_i\right) \end{aligned} \tag{A.3}$$

Condition the probability on $\sum_{i=1}^{k-(j-1)} S_i = z$, which has pdf $f(z; k-(j-1))$ and integrate over all possible values of z.

$$\begin{aligned} p_a(k,j) &= \int_0^\infty \mathbb{P}(z \leq a, S_{k-(j-1)+1} > a-z) f\big(z; k-(j-1)\big)\ dz \\ &= \int_0^a \mathbb{P}(S_{k-(j-1)+1} > a-z) f\big(z; k-(j-1)\big)\ dz \\ &= \frac{1}{(k-j)!}\left(\frac{1}{\mu}\right)^{k-j+1} \mathrm{e}^{\frac{-a}{\mu}} \int_0^a z^{k-j}\ dz \\ &= \frac{1}{(k-j+1)!}\left(\frac{a}{\mu}\right)^{k-j+1} \mathrm{e}^{\frac{-a}{\mu}} \\ &= F(a; k-j+1) - F(a; k-j+2) \end{aligned} \tag{A.4}$$

A.1.4 Case 4 $k \geq 1, j = k+1$

For $j = k+1$, $p_a(k,j)$ is the probability that the service time of the first customer is longer than a time units.

$$p_a(k,j) = \mathbb{P}(S_1 > a) = 1 - \mathbb{P}(S_1 \leq a) = 1 - F(a;1) \tag{A.5}$$

A.1.5 All Other Cases

All other cases have zero probability.

$$p_a(k,j) = 0 \tag{A.6}$$

A.1.6 Summary

These results can be summarised as:

$$p_a(k,j) = \begin{cases} \mathbb{1}(j=1) & \text{for } k=0 \\ F(a;k) & \text{for } k \geq 1, j=1 \\ F(a;k-j+1) - F(a;k-j+2) & \text{for } k \geq 2, 2 \leq j \leq k \\ 1 - F(a;1) & \text{for } k \geq 1, j=(k+1) \\ 0 & \text{otherwise} \end{cases} \tag{A.7}$$

A.2 Expected Transition Cost

This is a derivation of the expected cost $R_a(k,j)$. This is the expected cost of transitioning from the state (n,k) to the state $(n-1,j)$ over a time units if the next customer is scheduled to arrive in a time units. The expected cost is a linear combination of the expected total customers' waiting times and the expected server availability time. The per unit time costs c_W and c_S are defined as in Chapter 3.

In a similar way to the transition probability, the cost is most easily derived on a case by case basis.

A.2.1 Case 1 $k \in \{0,1\}$

The first case is that the system initially has either no customers or only a single customer. In this case, no customers are waiting during the transition, so the total customers waiting time is zero. The only cost is the cost of expected server availability time during the transition. The server is available for the entire transition, so the server availability time is a.

$$R_a(k,j) = c_S a \tag{A.8}$$

A.2.2 Case 2 $k \geq 2, j=1$

If $j=1$, then all k customers finish service during the transition. This implies that $\sum_{n=1}^{k} S_n \leq a$. The total customers' waiting time is a linear combination of the service times given that the sum of all k is smaller than k. The server is still available for the entire transition, so the server availability time is a.

$$\begin{aligned} R_a(k,j) &= c_W \sum_{i=2}^{k} \mathbb{E}\left[\sum_{l=1}^{i-1} S_l \,\middle|\, \sum_{n=1}^{k} S_n \le a\right] + c_S a \\ &= c_W \mathbb{E}\left[S_1 \,\middle|\, \sum_{n=1}^{k} S_n \le a\right] \sum_{i=2}^{k} (i-1) + c_S a \\ &= c_W \frac{k(k-1)}{2} \mathbb{E}\left[S_1 \,\middle|\, \sum_{n=1}^{k} S_n \le a\right] + c_S a \\ &= c_W \frac{(k-1)}{2} \mathbb{E}\left[\sum_{n=1}^{k} S_n \,\middle|\, \sum_{n=1}^{k} S_n \le a\right] + c_S a \end{aligned} \tag{A.9}$$

The term $\mathbb{E}\left[\sum_{n=1}^{k} S_n \,\middle|\, \sum_{n=1}^{k} S_n \le a\right]$ is one of the conditional expectations defined in Chapter 4.

$$R_a(k,j) = c_W \frac{G(a;k)(k-1)}{2} + c_S a \tag{A.10}$$

A.2.3 Case 3 $k \ge 2, 2 \le j \le k$

For $2 \le j \le k$, then the first $k-(j-1)$ customers finish service during the transition, but customer $k-(j-1)+1$ does not. This implies that $\sum_{n=1}^{k-(j-1)} S_n \le a$ and $\sum_{n=1}^{k-(j-1)+1} S_n > a$. The last $j-2$ customers wait for the entire transition. In addition, the server is available for the entire transition.

$$
\begin{aligned}
R_a(k,j) &= c_W \sum_{i=2}^{k-(j-2)} \mathbb{E}\left[\sum_{l=1}^{i-1} S_l \,\middle|\, \sum_{n=1}^{k-(j-1)} S_n \le a, \sum_{n=1}^{k-(j-1)+1} S_n > a\right] \\
&\quad + c_W \sum_{i=1}^{j-2} a + c_S a \\
&= c_W \mathbb{E}\left[S_1 \,\middle|\, \sum_{n=1}^{k-(j-1)} S_n \le a, \sum_{n=1}^{k-(j-1)+1} S_n > a\right] \sum_{i=2}^{k-(j-2)} (i-1) \\
&\quad + c_W a(j-2) + c_S a \\
&= c_W \frac{\big[k-(j-1)\big]\big[k-(j-2)\big]}{2} \mathbb{E}\left[S_1 \,\middle|\, \sum_{n=1}^{k-(j-1)} S_n \le a, \sum_{n=1}^{k-(j-1)+1} S_n > a\right] \\
&\quad + c_W a(j-2) + c_S a \\
&= c_W \frac{\big[k-(j-2)\big]}{2} \mathbb{E}\left[\sum_{n=1}^{k-(j-1)} S_n \,\middle|\, \sum_{n=1}^{k-(j-1)} S_n \le a, \sum_{n=1}^{k-(j-1)+1} S_n > a\right] \\
&\quad + c_W a(j-2) + c_S a
\end{aligned} \tag{A.11}
$$

The term $\mathbb{E}\left[\sum_{n=1}^{k-(j-1)} S_n \,\middle|\, \sum_{n=1}^{k-(j-1)} S_n \le a, \sum_{n=1}^{k-(j-1)+1} S_n > a\right]$ is another of the conditional expectations defined in Chapter 4.

$$
\begin{aligned}
R_a(k,j) &= c_W \left[\frac{H\big(a; k-(j-1)\big)\big[k-(j-2)\big]}{2} + a(j-2)\right] + c_S a \\
&= c_W \left[\frac{a\big[k-(j-1)\big]}{2} + a(j-2)\right] + c_S a \\
&= c_W \frac{a(k+j-3)}{2} + c_S a
\end{aligned} \tag{A.12}
$$

A.2.4 Case 4 $k \geq 2, j = (k+1)$

For $j = k+1$, no customers finish service during the transition. All customers except the first customer wait for the entire transition. The server is available for the entire transition.

$$
R_a(k,j) = c_W \sum_{i=1}^{k-1} a + c_S a = c_W a(k-1) + c_S a \tag{A.13}
$$

A.2.5 Summary

In order to compare the dynamic schedule with the static schedule, need to scale these costs by dividing by $(c_S + c_W)$ and defining $\gamma = \frac{c_S}{c_S + c_W}$. These results can be summarised as:

$$R_a(k, j) = \begin{cases} \gamma a & \text{for } k \in \{0, 1\} \\ (1-\gamma)\frac{G(a;k)(k-1)}{2} + \gamma a & \text{for } k \geq 2, j = 1 \\ (1-\gamma)\frac{a(k+j-3)}{2} + \gamma a & \text{for } k \geq 2, 2 \leq j \leq k+1 \end{cases} \tag{A.14}$$

Bibliography

Bailey, Norman TJ (1952). "A study of queues and appointment systems in hospital out-patient departments, with special reference to waiting-times". In: *Journal of the Royal Statistical Society. Series B (Methodological)*, pp. 185–199.

Rockart, John F and Paul B Hofmann (1969). "Physician and patient behavior under different scheduling systems in a hospital outpatient department". In: *Medical Care* 7.6, pp. 463–470.

Gupta, Ishwar, Juan Zoreda, and Nathan Kramer (1971). "Hospital manpower planning by use of queueing theory". In: *Health Services Research* 6.1, pp. 76–82.

Walter, SD (1973). "A comparison of appointment schedules in a hospital radiology department". In: *British Journal of Preventive & Social Medicine* 27.3, pp. 160–167.

Kao, Edward PC and Grace G Tung (1981). "Bed allocation in a public health care delivery system". In: *Management Science* 27.5, pp. 507–520.

Hajek, Bruce (1983). "The proof of a folk theorem on queuing delay with applications to routing in networks". In: *Journal of the ACM (JACM)* 30.4, pp. 834–851.

O'Keefe, Robert M (1985). "Investigating outpatient departments: Implementable policies and qualitative approaches". In: *Journal of the Operational Research Society* 36.8, pp. 705–712.

Goldsmith, Jeff (1989). "A radical prescription for hospitals". In: *Harvard Business Review.*

Pegden, Claude Dennis and Matthew Rosenshine (1990). "Scheduling arrivals to queues". In: *Computers & Operations Research* 17.4, pp. 343–348.

Babes, Malika and GV Sarma (1991). "Out-patient queues at the Ibn-Rochd health centre". In: *Journal of the Operational Research Society* 42.10, pp. 845–855.

Ho, Chrwan-Jyh and Hon-Shiang Lau (1992). "Minimizing total cost in scheduling outpatient appointments". In: *Management Science* 38.12, pp. 1750–1764.

Wang, P Patrick (1993). "Static and dynamic scheduling of customer arrivals to a single-server system". In: *Naval Research Logistics (NRL)* 40.3, pp. 345–360.

Stein, William E and Murray J Côté (1994). "Scheduling arrivals to a queue". In: *Computers & Operations Research* 21.6, pp. 607–614.

Huarng, Fenghueih and Mong Hou Lee (1996). "Using simulation in out-patient queues: A case study". In: *International Journal of Health Care Quality Assurance* 9.6, pp. 21–25.

Bennett, Joanne C and DJ Worthington (1998). "An example of a good but partially successful OR engagement: Improving outpatient clinic operations". In: *Interfaces* 28.5, pp. 56–69.

Cayirli, Tugba and Emre Veral (2003). "Outpatient scheduling in health care: A review of literature". In: *Production and Operations Management* 12.4, pp. 519–549.

Mondschein, Susana V and Gabriel Y Weintraub (2003). "Appointment policies in service operations: A critical analysis of the economic framework". In: *Production and Operations Management* 12.2, pp. 266–286.

DeLaurentis, Po-Ching et al. (2006). "Open access appointment scheduling - An experience at a community clinic". In: *IIE Annual Conference*. Institute of Industrial Engineers.

Green, Linda (2006). "Queueing analysis in healthcare". In: *Patient flow: reducing delay in healthcare delivery*. Springer, pp. 281–307.

Mendel, Sharon (2006). "Scheduling arrivals to queues: A model with no-shows". MA thesis. Tel-Aviv University.

Fiems, Dieter, Ger Koole, and Philippe Nain (2007). "Waiting times of scheduled patients in the presence of emergency requests". In: *Technisch Rapport*.

Fomundam, Samuel and Jeffrey W Herrmann (2007). *A survey of queuing theory applications in healthcare*. University of Maryland, The Insitute for Systems Research.

Erdogan, S Ayca and Brian Denton (2011). "Dynamic appointment scheduling of a stochastic server with uncertain demand". In: *INFORMS Journal on Computing* 25.1, pp. 116–132.