Queues with a Dynamic Schedule

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Outline

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Background

Queues with Scheduled Arrivals

- System where customers queue for service
- Instead of arriving randomly, customer arrival times are scheduled in advance
- Common example is appointments for doctor
- Scheduler determines arrival times to minimize expected cost
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Assumptions

- Single server
- Service times are independent exponential RVs with mean μ
- All customers arrive punctually
- Total number of customers is fixed

Static vs. Dynamic

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Literature Review

- Bailey (1952)
- Pegden and Rosenshine (1990)

Static Schedules

Objective Function

- Denote the expected waiting time of customer i by w_i

$$\mathbb{E}\Big[\text{total customers' waiting time}\Big] = \sum_{i=1}^{n} w_i$$

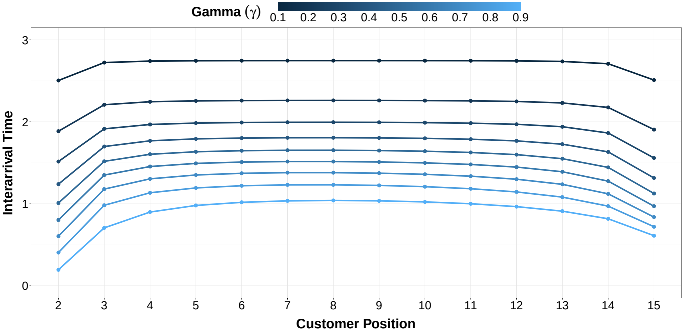
- Denote the interarrival time between customer i and customer $i+1$ by x_i

$$\mathbb{E}\Big[\text{server availability time}\Big] = \sum_{i=1}^{n-1} x_i + w_n + \mu$$

- Objective is to minimise a linear combination of these times

$$\phi(\mathbf{x}) = (1-\gamma)\sum_{i=1}^{n} w_i + \gamma\left(\sum_{i=1}^{n-1} x_i + w_n + \mu\right)$$

Model for 15 Customers



- Dome-shape: increase for first customers, remain constant, then decrease for last few customers
- As relative cost of server availability time (γ) increases, customers arrive earlier

Dynamic Schedules

Dynamic Schedules

- Static schedules are fixed for the duration of service
- Could be advantageous to allow schedule to vary during service
- Dynamic schedule is chosen progressively during service
- Arrival of customer i is scheduled on arrival of customer $i - 1$

Markov Decision Process

- Consider the problem of scheduling N customers
- Denote the number of customers in the system on arrival of customer i by k_i
- $\{k_1, \ldots, k_N\}$ is a discrete-time Markov chain
- On each customer's arrival, scheduler needs to schedule the arrival time of the next customer denoted by a
- Set of possible times is $\mathcal{A} = [0, \infty)$
- Naturally modelled as Markov decision process

Expected Cost of Schedule

- Denote the current state of n customers remaining to be scheduled and k customers currently in the system by (n, k)
- Initial state: $(N, 0)$
- Expected cost of state (n, k) by Bellman equation:

$$C_n^*(k) = \min_{a \geq 0} C_n(a, k) = \min_{a \geq 0} \left[\sum_{j=1}^{k+1} p_a(i, j) \Big(R_a(i, j) + C_{n-1}^*(j) \Big) \right]$$

Erlang Distribution

- Waiting time of customer in position $r+1$ is sum of r independent Exponential RVs with mean μ

$$X = \sum_{i=1}^{r} S_i \sim \text{Erlang}(r, \mu)$$

- Distribution function:

$$F(a; r) = \mathbb{P}(X \leq x) = \begin{cases} 0 & \text{where } x = 0 \\ 1 - \sum_{i=0}^{r-1} \frac{1}{i!} \left(\frac{x}{\mu}\right)^i \mathrm{e}^{\frac{-x}{\mu}} & \text{where } x > 0 \end{cases}$$

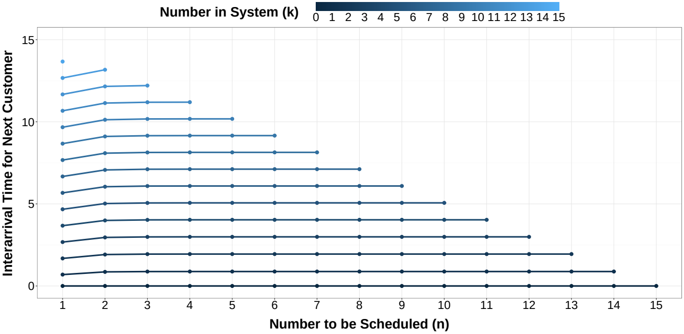
- Conditional expectation:

$$\mathbb{E}\Big[X \mid X \leq a\Big] = \mu r \times \frac{F(a; r+1)}{F(a; r)}$$

- Suppose $Y \sim \text{Exp}(\mu)$ independent of X

$$\mathbb{E}\Big[X \mid X \leq a, X + Y > a\Big] = \frac{ar}{r+1}$$

Model for 15 Customers



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Schedule Comparison

Expected Cost Comparison

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Simulation Studies

Simulation

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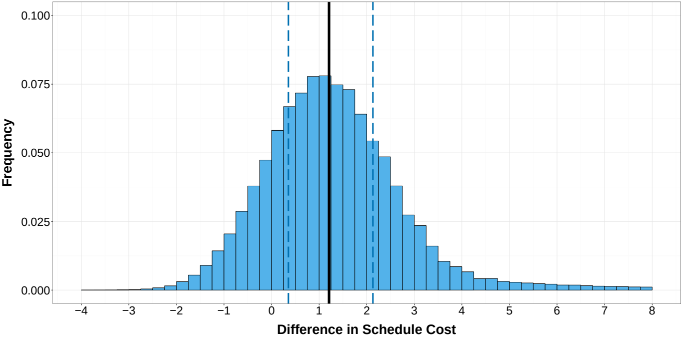
Schedule Cost



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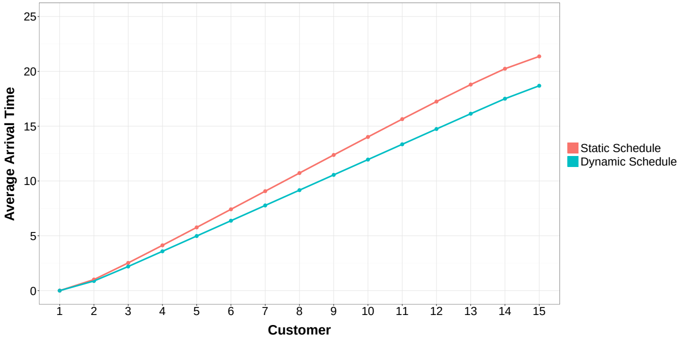
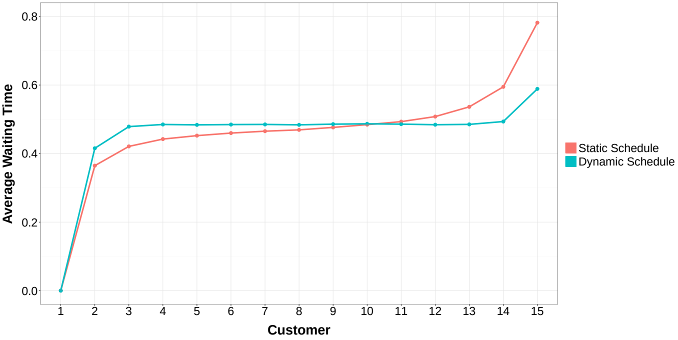
Customer Arrival Times



Customer Waiting Times



Conclusion

Conclusion

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References

References i

Bailey, Norman TJ (1952). "A study of queues and appointment systems in hospital out-patient departments, with special reference to waiting-times". *Journal of the Royal Statistical Society. Series B (Methodological)*, pp. 185–199.

Pegden, Claude Dennis and Matthew Rosenshine (1990). "Scheduling arrivals to queues". *Computers & Operations Research* 17.4, pp. 343–348.