Markov Decision Theory

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Contents

1 Aim

Immediately after a patient arrives and begins waiting for service, want to decide when the next patient should be scheduled to arrive given the current state.

2 Assumptions

- Patients can be scheduled at any future (or present) time
- Patients are punctual and arrive when scheduled
- iid exponential service times with mean μ and single server

3 Parameters

μ	:	mean service time of each patient
c_W	:	cost of patient's waiting time (per unit time)
c_I	:	cost of doctor's idle time (per unit time)
n	:	number of patients remaining to be scheduled
k	:	current number of patients in queue (i.e., the current state)
j	:	number of patients in queue immediately after the next patient arrives (i.e., the next state)
a	:	time next patient is scheduled to arrive (relative to current time)
$C_n(k)$	:	expected cost of being in state k with n patients still to be scheduled
$p_a(k,j;\mu)$	:	probability of transitioning from state k to state j over the time interval a
$R_a(k,j;\mu,c_W,c_I)$	:	expected cost of transitioning from state k to state j over the time interval a

4 Optimal Policy

The optimal policy is the a^* given by the following form of Belmann's equation (for $n \geq 1$):

$$C_n(k) = \min_{a\geq 0}\left[\sum_{j=1}^{k+1} p_a(k,j;\mu)\Big(R_a(k,j;\mu,c_W,c_I) + C_{n-1}(j)\Big)\right] \tag{1}$$

5 None Remaining to be Scheduled

5.1 Expected Waiting Time

Let w_i be the expected waiting time of the patient that is currently in position i in the queue

$$w_i = \mu i$$

5.2 Expected Waiting Cost

$$\begin{aligned}
C_0(k) &= c_W \sum_{i=1}^{k} w_i \\
&= c_W \mu \sum_{i=1}^{k} i \\
C_0(k) &= \frac{c_W \mu k(k+1)}{2}
\end{aligned}$$

6 Erlang Distribution

- Let S_i be the service time of the patient that is currently in position i in the queue
- $S_1, \ldots, S_k$ are iid $\text{Exp}(\mu)$
- $X = \sum_{i=1}^{k} S_i$ is the total service time of the k patients
- $X \sim \text{Erlang}(k, \mu)$ such that for $x > 0$:

$$f(x; k, \mu) = \frac{1}{\mu \cdot (k-1)!} \left(\frac{x}{\mu}\right)^{k-1} \exp\left[\frac{-x}{\mu}\right]$$

6.1 Cumulative Distribution Function

- The number of patients served by the health care system from time 0 to time t is a poisson point process $W_t \sim \text{Poisson}\left(\frac{t}{\mu}\right)$ for $t > 0$
- $\{X > a\}$ and $\{W_a < k\}$ are both the events that not all of the k patients are served from time 0 to time a such that $\mathbb{P}(X > a) = \mathbb{P}(W_a < k)$

Where $a > 0$:

$$\begin{aligned}
F(a; k, \mu) &= \mathbb{P}(X \leq a) \\
&= 1 - \mathbb{P}(X > a) \\
&= 1 - \mathbb{P}(W_a < k) \\
&= 1 - \sum_{n=0}^{k-1} \mathbb{P}(W_a = n) \\
F(a; k, \mu) &= 1 - \sum_{n=0}^{k-1} \frac{1}{n!} \left(\frac{a}{\mu}\right)^n \exp\left[\frac{-a}{\mu}\right]
\end{aligned}$$

Where $a = 0$:

$$\begin{aligned}
F(0; k, \mu) &= \mathbb{P}(X \leq 0) \\
F(0; k, \mu) &= 0
\end{aligned}$$

In summary:

$$F(a;k,\mu) = \begin{cases} 1 - \sum_{n=0}^{k-1} \frac{1}{n!} \left(\frac{a}{\mu}\right)^n \exp\left[\frac{-a}{\mu}\right] & \text{where } a > 0 \\ 0 & \text{where } a = 0 \end{cases} \tag{2}$$

6.2 Conditional Expectation

Where $a > 0$:

$$\begin{aligned}
G(a;k,\mu) &= \mathbb{E}[X|X \le a] \\
&= \int x \cdot \mathbb{P}(X \in dx | X \le a) \\
&= \int x \cdot \frac{\mathbb{P}(X \in dx, X \le a)}{\mathbb{P}(X \le a)} \\
&= \frac{1}{F(a;k,\mu)} \int_0^a x \cdot f(x;k,\mu) dx \\
&= \frac{1}{F(a;k,\mu)} \int_0^a x \cdot \frac{1}{\mu \cdot (k-1)!} \left(\frac{x}{\mu}\right)^{k-1} \exp\left[\frac{-x}{\mu}\right] dx \\
&= \frac{\mu k}{F(a;k,\mu)} \int_0^a \frac{1}{\mu \cdot k!} \left(\frac{x}{\mu}\right)^{k} \exp\left[\frac{-x}{\mu}\right] dx \\
&= \frac{\mu k}{F(a;k,\mu)} \int_0^a f(x;k+1,\mu) dx \\
G(a;k,\mu) &= \mu k \cdot \frac{F(a;k+1,\mu)}{F(a;k,\mu)}
\end{aligned}$$

Where $a = 0$:

$$\begin{aligned}
G(0;k,\mu) &= \mathbb{E}[X|X \le 0] \\
G(0;k,\mu) &= 0
\end{aligned}$$

In summary:

$$G(a;k,\mu) = \begin{cases} \mu k \cdot \frac{F(a;k+1,\mu)}{F(a;k,\mu)} & \text{where } a > 0 \\ 0 & \text{where } a = 0 \end{cases} \tag{3}$$

7 Probability of Transition

7.1 Case 1 $a = 0$

$$p_a(k,j;\mu) = \mathbb{1}(j = k+1)$$

7.2 Case 2 $a > 0, k = 0$

$$p_a(k,j;\mu) = \mathbb{1}(j = 1)$$

7.3 Case 3 $a > 0, k \geq 1, j = 1$

$$\begin{aligned} p_a(k,j;\mu) &= \mathbb{P}\left(\sum_{i=1}^{k} S_i \leq a\right) \\ p_a(k,j;\mu) &= F(a;k,\mu) \end{aligned}$$

7.4 Case 4 $a > 0, k \geq 1, 2 \leq j \leq k$

$$\begin{aligned}
p_a(k,j;\mu) &= \mathbb{P}\left(\sum_{i=1}^{k-(j-1)} S_i \leq a, \sum_{i=1}^{k-(j-1)+1} S_i > a\right) \\
&= \mathbb{P}\left(\sum_{i=1}^{k-(j-1)} S_i \leq a, \sum_{i=1}^{k-(j-1)} S_i + S_{k-(j-1)+1} > a\right) \\
&= \mathbb{P}\left(\sum_{i=1}^{k-(j-1)} S_i \leq a, S_{k-(j-1)+1} > a - \sum_{i=1}^{k-(j-1)} S_i\right) \\
&= \int \mathbb{P}\left(\sum_{i=1}^{k-(j-1)} S_i \leq a, S_{k-(j-1)+1} > a - \sum_{i=1}^{k-(j-1)} S_i \,\middle|\, \sum_{i=1}^{k-(j-1)} S_i = z\right) \mathbb{P}\left(\sum_{i=1}^{k-(j-1)} S_i \in dz\right) \\
&= \int_0^\infty \mathbb{P}\left(z \leq a, S_{k-(j-1)+1} > a - z\right) f(z;k-(j-1),\mu)dz \\
&= \int_0^a \mathbb{P}\left(S_{k-(j-1)+1} > a - z\right) f(z;k-(j-1),\mu)dz \\
&= \int_0^a f(z;k-(j-1),\mu)\Big(1 - \mathbb{P}(S_{k-(j-1)+1} \leq a - z)\Big)dz \\
&= \int_0^a f(z;k-(j-1),\mu)\Big(1 - \mathbb{P}(S_1 \leq a - z)\Big)dz \\
&= \int_0^a f(z;k-(j-1),\mu)\Big(1 - F(a-z;1,\mu)\Big)dz \\
&= \int_0^a \frac{1}{\mu \cdot (k-(j-1)-1)!}\left(\frac{z}{\mu}\right)^{k-(j-1)-1} \exp\left[\frac{-z}{\mu}\right] \exp\left[\frac{-(a-z)}{\mu}\right] dz \\
&= \frac{1}{(k-(j-1)-1)!}\left(\frac{1}{\mu}\right)^{k-(j-1)} \exp\left[\frac{-a}{\mu}\right] \int_0^a z^{k-(j-1)-1} dz \\
&= \frac{1}{(k-(j-1)-1)!}\left(\frac{1}{\mu}\right)^{k-(j-1)} \exp\left[\frac{-a}{\mu}\right] \left[\frac{z^{k-(j-1)}}{k-(j-1)}\right]_{z=0}^{z=a} \\
&= \frac{1}{(k-(j-1))!}\left(\frac{a}{\mu}\right)^{k-(j-1)} \exp\left[\frac{-a}{\mu}\right] \\
&= \left[1 - \sum_{n=0}^{k-(j-1)-1} \frac{1}{n!}\left(\frac{a}{\mu}\right)^n \exp\left[\frac{-a}{\mu}\right]\right] - \left[1 - \sum_{n=0}^{k-(j-1)} \frac{1}{n!}\left(\frac{a}{\mu}\right)^n \exp\left[\frac{-a}{\mu}\right]\right] \\
p_a(k,j;\mu) &= F\Big(a;k-(j-1),\mu\Big) - F\Big(a;k-(j-1)+1,\mu\Big)
\end{aligned}$$

7.5 Case 5 $a > 0, k \geq 1, j = (k+1)$

$$\begin{aligned}
p_a(k,j;\mu) &= \mathbb{P}(S_1 > a) \\
&= 1 - \mathbb{P}(S_1 \leq a) \\
p_a(k,j;\mu) &= 1 - F(a;1,\mu)
\end{aligned}$$

7.6 All Other Cases

$$p_a(k,j;\mu) = 0$$

7.7 Summary

These results can be summarised as:

$$p_a(k,j;\mu) = \begin{cases} \mathbb{1}(j = k+1) & \text{where } a = 0 \\ \mathbb{1}(j = 1) & \text{where } a > 0,\ k = 0 \\ F(a;k,\mu) & \text{where } a > 0,\ k \geq 1,\ j = 1 \\ F\big(a; k-(j-1), \mu\big) - F\big(a; k-(j-1)+1, \mu\big) & \text{where } a > 0,\ k \geq 1,\ 2 \leq j \leq k \\ 1 - F(a;1,\mu) & \text{where } a > 0,\ k \geq 1,\ j = (k+1) \\ 0 & \text{otherwise} \end{cases} \tag{4}$$

8 Expected Cost of Transition

8.1 Case 1 $a = 0$

$$R_a(k,j;\mu,c_W,c_I) = 0$$

8.2 Case 2 $a > 0, k = 0$

$$R_a(k,j;\mu,c_W,c_I) = c_I a$$

8.3 Case 3 $a > 0, k \geq 1, j = 1$

$$
\begin{aligned}
R_a(k, j; \mu, c_W, c_I) &= \sum_{i=1}^{k} c_W \mathbb{E}\left[\sum_{l=1}^{i} S_l \;\middle|\; \sum_{n=1}^{k} S_n \leq a\right] + c_I \mathbb{E}\left[a - \sum_{n=1}^{k} S_n \;\middle|\; \sum_{n=1}^{k} S_n \leq a\right] \\
&= c_W \sum_{i=1}^{k} \sum_{l=1}^{i} \mathbb{E}\left[S_l \;\middle|\; \sum_{n=1}^{k} S_n \leq a\right] + c_I a - c_I \mathbb{E}\left[\sum_{n=1}^{k} S_n \;\middle|\; \sum_{n=1}^{k} S_n \leq a\right] \\
&= c_W \mathbb{E}\left[S_1 \;\middle|\; \sum_{n=1}^{k} S_n \leq a\right] \sum_{i=1}^{k} i + c_I a - c_I G(a; k, \mu) \\
&= \frac{c_W k(k+1)}{2} \mathbb{E}\left[S_1 \;\middle|\; \sum_{n=1}^{k} S_n \leq a\right] + c_I a - c_I G(a; k, \mu) \\
&= \frac{c_W (k+1)}{2} \mathbb{E}\left[\sum_{n=1}^{k} S_n \;\middle|\; \sum_{n=1}^{k} S_n \leq a\right] + c_I a - c_I G(a; k, \mu) \\
&= c_I a + \frac{c_W G(a; k, \mu)(k+1)}{2} - c_I G(a; k, \mu) \\
R_a(k, j; \mu, c_W, c_I) &= c_I a + \frac{G(a; k, \mu)\Big(c_W (k+1) - 2c_I\Big)}{2}
\end{aligned}
$$

8.4 Case 4 $a > 0, k \geq 1, 2 \leq j \leq k$

$$
\begin{aligned}
R_a(k, j; \mu, c_W, c_I) &= c_W a(j-1) + \sum_{i=1}^{k-(j-1)} c_W \mathbb{E}\left[\sum_{l=1}^{i} S_l \;\middle|\; \sum_{n=1}^{k-(j-1)} S_n \leq a\right] \\
&= c_W a(j-1) + c_W \sum_{i=1}^{k-(j-1)} \sum_{l=1}^{i} \mathbb{E}\left[S_l \;\middle|\; \sum_{n=1}^{k-(j-1)} S_n \leq a\right] \\
&= c_W a(j-1) + c_W \mathbb{E}\left[S_1 \;\middle|\; \sum_{n=1}^{k-(j-1)} S_n \leq a\right] \sum_{i=1}^{k-(j-1)} i \\
&= c_W a(j-1) + \frac{c_W \Big(k-(j-1)\Big)\Big(k-(j-1)+1\Big)}{2} \mathbb{E}\left[S_1 \;\middle|\; \sum_{n=1}^{k-(j-1)} S_n \leq a\right] \\
&= c_W a(j-1) + \frac{c_W \Big(k-(j-1)+1\Big)}{2} \mathbb{E}\left[\sum_{n=1}^{k-(j-1)} S_n \;\middle|\; \sum_{n=1}^{k-(j-1)} S_n \leq a\right] \\
R_a(k, j; \mu, c_W, c_I) &= c_W a(j-1) + \frac{c_W G\Big(a; k-(j-1), \mu\Big)\Big(k-(j-1)+1\Big)}{2}
\end{aligned}
$$

8.5 Case 5 $a > 0, k \geq 1, j = (k+1)$

$$
R_a(k, j; \mu, c_W, c_I) = c_W a k
$$

8.6 Summary

These results can be summarised as:

$$R_a(k,j;\mu,c_W,c_I) = \begin{cases} 0 & \text{where } a = 0 \\ c_I a + \frac{G(a;k,\mu)\left(c_w(k+1)-2c_I\right)}{2} & \text{where } a > 0,\ k \geq 0,\ j = 1 \\ c_W a(j-1) + \frac{c_W G\left(a;k-(j-1),\mu\right)\left(k-(j-1)+1\right)}{2} & \text{where } a > 0,\ k \geq 1,\ 2 \leq j \leq (k+1) \end{cases} \tag{5}$$