

Queues with a Dynamic Schedule

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Background

Static Schedules

Dynamic Schedules

Schedule Comparison

Simulation Studies

Background

Queues with Scheduled Arrivals

- System where customers queue for service
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Assumptions

- Single server
- Service times are independent exponential RVs with mean μ
- All customers arrive punctually
- Total number of customers is fixed

Static vs. Dynamic

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Static Schedules

Objective Function

- Denote the expected waiting time of customer i by w_i

$$\mathbb{E}[\text{total customers' waiting time}] = \sum_{i=1}^n w_i$$

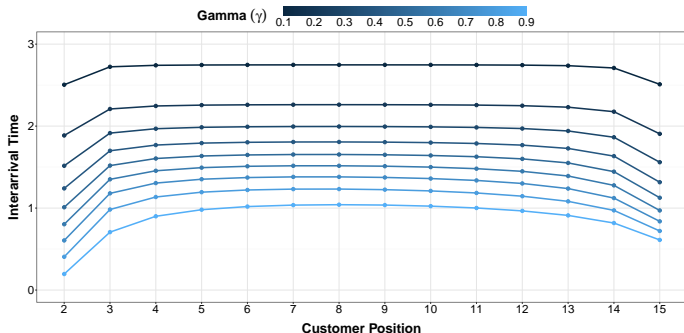
- Denote the interarrival time between customer i and customer $i + 1$ by x_i

$$\mathbb{E}[\text{server availability time}] = \sum_{i=1}^{n-1} x_i + w_n + \mu$$

- Objective is to minimise a linear combination of these times

$$\phi(\mathbf{x}) = (1 - \gamma) \sum_{i=1}^n w_i + \gamma \left(\sum_{i=1}^{n-1} x_i + w_n + \mu \right)$$

Model for 15 Customers



- **Dome-shape:** increase for first customers, remain constant, then decrease for last few customers
- As relative cost of server availability time (γ) increases, customers arrive earlier

Dynamic Schedules

Markov Decision Process

- Consider the problem of scheduling N customers
- Denote the number of customers in the system on arrival of customer i by k_i
- $\{k_1, \dots, k_N\}$ is a discrete-time Markov chain
- On each customer's arrival, scheduler needs to schedule the arrival time of the next customer denoted by a
- Set of possible times is $\mathcal{A} = [0, \infty)$
- Naturally modelled as Markov decision process

Expected Cost of Schedule

- Denote the current state of n customers remaining to be scheduled and k customers currently in the system by (n, k)
- Expected cost of state (n, k) for $n \geq 1$:

$$C_n^*(k) = \min_{a \geq 0} C_n(a, k) = \min_{a \geq 0} \left[\sum_{j=1}^{k+1} p_a(i, j) \left(R_a(i, j) + C_{n-1}^*(j) \right) \right]$$

Erlang Distribution

- Waiting time of customer in position $r + 1$ is sum of r independent Exponential RVs with mean μ

$$X = \sum_{i=1}^r S_i \sim \text{Erlang}(r, \mu)$$

- Distribution function:

$$F(a; r) = \mathbb{P}(X \leq x) = \begin{cases} 0 & \text{where } x = 0 \\ 1 - \sum_{i=0}^{r-1} \frac{1}{i!} \left(\frac{x}{\mu}\right)^i e^{-\frac{x}{\mu}} & \text{where } x > 0 \end{cases}$$

- Conditional expectation:

$$\mathbb{E}[X \mid X \leq a] = \mu r \times \frac{F(a; r+1)}{F(a; r)}$$

- Suppose $Y \sim \text{Exp}(\mu)$ independent of X

$$\mathbb{E}[X \mid X \leq a, X + Y > a] = \frac{ar}{r+1}$$

Schedule Comparison

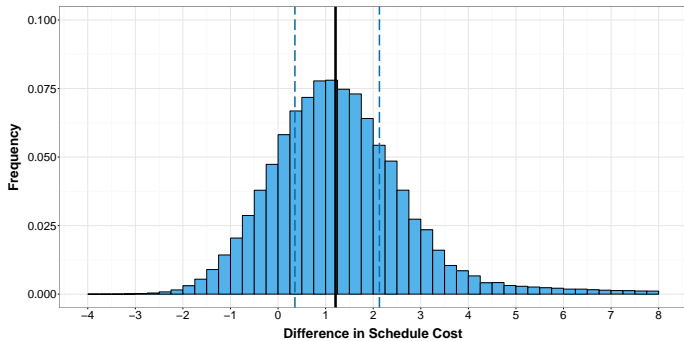
Expected Cost Comparison



Simulation Studies

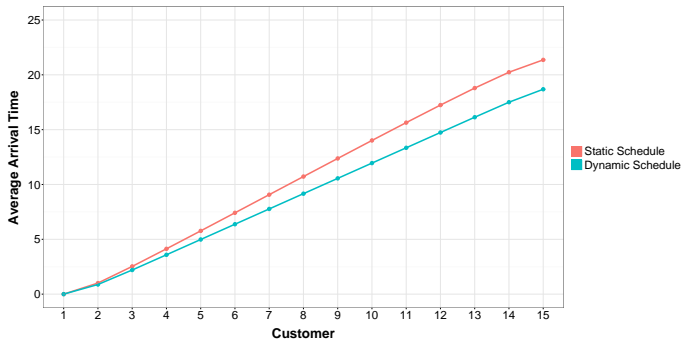


Schedule Cost

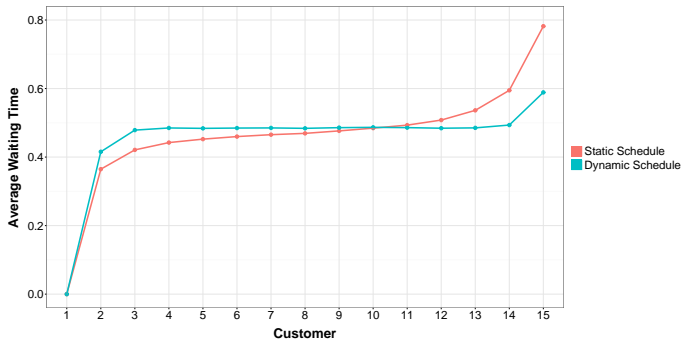


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Customer Arrival Times



Customer Waiting Times



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