2WB05 Simulation Lecture 8: Generating random variables

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Discrete distributions

Discrete version of Inverse Transform Method

Let X be a discrete random variable with probabilities

$$P(X = x_i) = p_i, \qquad i = 0, 1, \dots, \qquad \sum_{i=0}^{\infty} p_i = 1.$$

To generate a realization of X, we first generate U from $U(0, 1)$ and then set $X = x_i$ if

$$\sum_{j=0}^{i-1} p_j \leq U < \sum_{j=0}^{i} p_j.$$

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Discrete distributions

So the algorithm is as follows:

- Generate U from $U(0, 1)$;
- Determine the index I such that

$$\sum_{j=0}^{I-1} p_j \leq U < \sum_{j=0}^{I} p_j$$

  and return $X = x_I$.

The second step requires a *search*; for example, starting with $I = 0$ we keep adding 1 to I until we have found the (smallest) I such that

$$U < \sum_{j=0}^{I} p_j$$

Note: The algorithm needs exactly one uniform random variable U to generate X; this is a nice feature if you use variance reduction techniques.

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Discrete distributions

Array method: when X has a finite support

Suppose $p_i = k_i/100$, $i = 1, \ldots, m$,
where k_i's are integers with $0 \leq k_i \leq 100$

Construct array $A[i]$, $i = 1, \ldots, 100$ as follows:
set $A[i] = x_1$ for $i = 1, \ldots, k_1$
set $A[i] = x_2$ for $i = k_1 + 1, \ldots, k_1 + k_2$, etc.

Then, first sample a random index I from $1, \ldots, 100$:
$I = 1 + \lfloor 100U \rfloor$ and set $X = A[I]$

Discrete distributions

Bernoulli

Two possible outcomes of X (success or failure):

$$P(X = 1) = 1 - P(X = 0) = p.$$

Algorithm:

- Generate U from $U(0, 1)$;
- If $U \leq p$, then $X = 1$; else $X = 0$.

Discrete distributions

Discrete uniform

The possible outcomes of X are $m, m+1, \ldots, n$ and they are all equally likely, so

$$P(X = i) = \frac{1}{n - m + 1}, \qquad i = m, m+1, \ldots, n.$$

Algorithm:

- Generate U from $U(0, 1)$;
- Set $X = m + \lfloor (n - m + 1)U \rfloor$.

Note: No search is required, and compute $(n - m + 1)$ ahead.

Discrete distributions

Geometric

A random variable X has a geometric distribution with parameter p if

$$P(X = i) = p(1-p)^i, \qquad i = 0, 1, 2, \ldots;$$

X is the number of failures till the first success in a sequence of Bernoulli trials with success probability p.

Algorithm:

- Generate independent Bernoulli(p) random variables $Y_1, Y_2, \ldots$; let I be the index of the first successful one, so $Y_I = 1$;
- Set $X = I - 1$.

Alternative algorithm:

- Generate U from $U(0, 1)$;
- Set $X = \lfloor \ln(U)/\ln(1-p) \rfloor$.

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Discrete distributions

Binomial

A random variable X has a binomial distribution with parameters n and p if

$$P(X = i) = \binom{n}{i} p^i (1-p)^{n-i}, \qquad i = 0, 1, \ldots, n;$$

X is the number of successes in n independent Bernoulli trials, each with success probability p.

Algorithm:

- Generate n Bernoulli(p) random variables $Y_1, \ldots, Y_n$;
- Set $X = Y_1 + Y_2 + \cdots + Y_n$.

Alternative algorithms can be derived by using the following results.

Discrete distributions

Let $Y_1, Y_2, \ldots$ be independent geometric(p) random variables, and I the smallest index such that

$$\sum_{i=1}^{I+1} (Y_i + 1) > n.$$

Then the index I has a binomial distribution with parameters n and p.

Let $Y_1, Y_2, \ldots$ be independent exponential random variables with mean 1, and I the smallest index such that

$$\sum_{i=1}^{I+1} \frac{Y_i}{n - i + 1} > -\ln(1 - p).$$

Then the index I has a binomial distribution with parameters n and p.

Discrete distributions

Negative Binomial

A random variable X has a negative-binomial distribution with parameters n and p if

$$P(X = i) = \binom{n+i-1}{i} p^n (1-p)^i, \qquad i = 0, 1, 2, \ldots;$$

X is the number of failures before the n-th success in a sequence of independent Bernoulli trials with success probability p.

Algorithm:

- Generate n geometric(p) random variables $Y_1, \ldots, Y_n$;
- Set $X = Y_1 + Y_2 + \cdots + Y_n$.

Discrete distributions

Poisson

A random variable X has a Poisson distribution with parameter λ if

$$P(X = i) = \frac{\lambda^i}{i!} e^{-\lambda}, \qquad i = 0, 1, 2, \ldots;$$

X is the number of events in a time interval of length 1 if the inter-event times are independent and exponentially distributed with parameter λ.

Algorithm:

- Generate exponential inter-event times $Y_1, Y_2, \ldots$ with mean 1; let I be the smallest index such that

$$\sum_{i=1}^{I+1} Y_i > \lambda;$$

- Set $X = I$.

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Discrete distributions

Poisson (alternative)

- Generate U(0,1) random variables $U_1, U_2, \ldots$;
  let I be the smallest index such that
  $$\prod_{i=1}^{I+1} U_i < e^{-\lambda};$$
- Set $X = I$.