Tutorial Sheet-1

Ex-1

Given point $P(-2, 6, 3)$ and vector $\mathbf{A} = y\mathbf{a}_x + (x + z)\mathbf{a}_y$, express P and $\mathbf{A}$ in cylindrical and spherical coordinates. Evaluate $\mathbf{A}$ at P in the Cartesian, cylindrical, and spherical systems.

Solution:

At point P: $x = -2$, $y = 6$, $z = 3$. Hence,

$$\rho = \sqrt{x^2 + y^2} = \sqrt{4 + 36} = 6.32$$

$$\phi = \tan^{-1}\frac{y}{x} = \tan^{-1}\frac{6}{-2} = 108.43^\circ$$

$$z = 3$$

$$r = \sqrt{x^2 + y^2 + z^2} = \sqrt{4 + 36 + 9} = 7$$

$$\theta = \tan^{-1}\frac{\sqrt{x^2 + y^2}}{z} = \tan^{-1}\frac{\sqrt{40}}{3} = 64.62^\circ$$

Thus,

$$P(-2, 6, 3) = P(6.32, 108.43°, 3) = P(7, 64.62°, 108.43°)$$

In the Cartesian system, **A** at P is

$$\mathbf{A} = 6\mathbf{a}_x + \mathbf{a}_y$$

For vector **A**, $A_x = y$, $A_y = x + z$, $A_z = 0$. Hence, in the cylindrical system

$$\begin{bmatrix} A_\rho \\ A_\phi \\ A_z \end{bmatrix} = \begin{bmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} y \\ x + z \\ 0 \end{bmatrix}$$

or

$$A_\rho = y\cos\phi + (x + z)\sin\phi$$

$$A_\phi = -y\sin\phi + (x + z)\cos\phi$$

$$A_z = 0$$

But $x = \rho\cos\phi$, $y = \rho\sin\phi$, and substituting these yields

$$\begin{aligned} \mathbf{A} = (A_\rho, A_\phi, A_z) = {} & [\rho\cos\phi\sin\phi + (\rho\cos\phi + z)\sin\phi]\mathbf{a}_\rho \\ & + [-\rho\sin^2\phi + (\rho\cos\phi + z)\cos\phi]\mathbf{a}_\phi \end{aligned}$$

At P

$$\rho = \sqrt{40}, \qquad \tan\phi = \frac{6}{-2}$$

Hence,

$$\cos\phi = \frac{-2}{\sqrt{40}}, \qquad \sin\phi = \frac{6}{\sqrt{40}}$$

$$\begin{aligned}\mathbf{A} &= \left[\sqrt{40}\cdot\frac{-2}{\sqrt{40}}\cdot\frac{6}{\sqrt{40}} + \left(\sqrt{40}\cdot\frac{-2}{\sqrt{40}} + 3\right)\cdot\frac{6}{\sqrt{40}}\right]\mathbf{a}_\rho \\ &\quad + \left[-\sqrt{40}\cdot\frac{36}{40} + \left(\sqrt{40}\cdot\frac{-2}{\sqrt{40}} + 3\right)\cdot\frac{-2}{\sqrt{40}}\right]\mathbf{a}_\phi \\ &= \frac{-6}{\sqrt{40}}\mathbf{a}_\rho - \frac{38}{\sqrt{40}}\mathbf{a}_\phi = -0.9487\mathbf{a}_\rho - 6.008\mathbf{a}_\phi\end{aligned}$$

Similarly, in the spherical system

$$\begin{bmatrix} A_r \\ A_\theta \\ A_\phi \end{bmatrix} = \begin{bmatrix} \sin\theta\cos\phi & \sin\theta\sin\phi & \cos\theta \\ \cos\theta\cos\phi & \cos\theta\sin\phi & -\sin\theta \\ -\sin\phi & \cos\phi & 0 \end{bmatrix} \begin{bmatrix} y \\ x+z \\ 0 \end{bmatrix}$$

or

$$A_r = y\sin\theta\cos\phi + (x+z)\sin\theta\sin\phi$$

$$A_\theta = y\cos\theta\cos\phi + (x+z)\cos\theta\sin\phi$$

$$A_\phi = -y\sin\phi + (x+z)\cos\phi$$

But $x = r \sin\theta \cos\phi$, $y = r \sin\theta \sin\phi$, and $z = r\cos\theta$. Substituting these yields

$$\begin{aligned}
\mathbf{A} &= (A_r, A_\theta, A_\phi) \\
&= r[\sin^2\theta \cos\phi \sin\phi + (\sin\theta\cos\phi + \cos\theta)\sin\theta\sin\phi]\mathbf{a}_r \\
&\quad + r[\sin\theta\cos\theta\sin\phi\cos\phi + (\sin\theta\cos\phi + \cos\theta)\cos\theta\sin\phi]\mathbf{a}_\theta \\
&\quad + r[-\sin\theta\sin^2\phi + (\sin\theta\cos\phi + \cos\theta)\cos\phi]\mathbf{a}_\phi
\end{aligned}$$

At P

$$r = 7, \qquad \tan\phi = \frac{6}{-2}, \qquad \tan\theta = \frac{\sqrt{40}}{3}$$

Hence,

$$\cos\phi = \frac{-2}{\sqrt{40}}, \qquad \sin\phi = \frac{6}{\sqrt{40}}, \qquad \cos\theta = \frac{3}{7}, \qquad \sin\theta = \frac{\sqrt{40}}{7}$$

$$\begin{aligned}
\mathbf{A} &= 7 \cdot \left[\frac{40}{49} \cdot \frac{-2}{\sqrt{40}} \cdot \frac{6}{\sqrt{40}} + \left(\frac{\sqrt{40}}{7} \cdot \frac{-2}{\sqrt{40}} + \frac{3}{7}\right) \cdot \frac{\sqrt{40}}{7} \cdot \frac{6}{\sqrt{40}}\right]\mathbf{a}_r \\
&\quad + 7 \cdot \left[\frac{\sqrt{40}}{7} \cdot \frac{3}{7} \cdot \frac{6}{\sqrt{40}} \cdot \frac{-2}{\sqrt{40}} + \left(\frac{\sqrt{40}}{7} \cdot \frac{-2}{\sqrt{40}} + \frac{3}{7}\right) \cdot \frac{3}{7} \cdot \frac{6}{\sqrt{40}}\right]\mathbf{a}_\theta \\
&\quad + 7 \cdot \left[\frac{-\sqrt{40}}{7} \cdot \frac{36}{40} + \left(\frac{\sqrt{40}}{7} \cdot \frac{-2}{\sqrt{40}} + \frac{3}{7}\right) \cdot \frac{-2}{\sqrt{40}}\right]\mathbf{a}_\phi \\
&= \frac{-6}{7}\mathbf{a}_r - \frac{18}{7\sqrt{40}}\mathbf{a}_\theta - \frac{38}{\sqrt{40}}\mathbf{a}_\phi \\
&= -0.8571\mathbf{a}_r - 0.4066\mathbf{a}_\theta - 6.008\mathbf{a}_\phi
\end{aligned}$$

Note that $|\mathbf{A}|$ is the same in the three systems; that is,

$$|\mathbf{A}(x, y, z)| = |\mathbf{A}(\rho, \phi, z)| = |\mathbf{A}(r, \theta, \phi)| = 6.083$$

PRACTICE EXERCISE 2.1

(a) Convert points $P(1, 3, 5)$, $T(0, -4, 3)$, and $S(-3, -4, -10)$ from Cartesian to cylindrical and spherical coordinates.

(b) Transform vector

$$\mathbf{Q} = \frac{\sqrt{x^2 + y^2}\,\mathbf{a}_x}{\sqrt{x^2 + y^2 + z^2}} - \frac{yz\,\mathbf{a}_z}{\sqrt{x^2 + y^2 + z^2}}$$

to cylindrical and spherical coordinates.

(c) Evaluate $\mathbf{Q}$ at T in the three coordinate systems.

Answer: (a) $P(3.162, 71.56°, 5)$, $P(5.916, 32.31°, 71.56°)$, $T(4, 270°, 3)$, $T(5, 53.13°, 270°)$, $S(5, 233.1°, -10)$, $S(11.18, 153.43°, 233.1°)$

(b) $\dfrac{\rho}{\sqrt{\rho^2 + z^2}}(\cos\phi\,\mathbf{a}_\rho - \sin\phi\,\mathbf{a}_\phi - z\sin\phi\,\mathbf{a}_z)$, $\sin\theta\,(\sin\theta\cos\phi - r\cos^2\theta\sin\phi)\mathbf{a}_r + \sin\theta\cos\theta\,(\cos\phi + r\sin\theta\sin\phi)\mathbf{a}_\theta - \sin\theta\sin\phi\,\mathbf{a}_\phi$

(c) $0.8\mathbf{a}_x + 2.4\mathbf{a}_z$, $0.8\mathbf{a}_\phi + 2.4\mathbf{a}_z$, $1.44\mathbf{a}_r - 1.92\mathbf{a}_\theta + 0.8\mathbf{a}_\phi$.

(a) At P(1,3,5), x = 1, y = 3, z = 5,

$\rho = \sqrt{x^2 + y^2} = \sqrt{10}$, z = 5, $\phi = \tan^{-1} y/x = \tan^{-1} 3 = 71.6$

$P(\rho,\phi,z) = P(\sqrt{10}, \tan^{-1} 3, 5) = \underline{\underline{P(3.162, 71.6°, 5)}}$

Spherical system:

$r = \sqrt{x^2 + y^2 + z^2} = \sqrt{35} = 5.916$

$\theta = \tan^{-1} \sqrt{x^2 + y^2}/z = \tan^{-1} \sqrt{10}/5 = \tan^{-1} 0.6325 = 32.31°$

$P(r,\theta,\varphi) = \underline{\underline{P(5.916, 32.31°, 71.56°)}}$

At T(0,-4,3), x = 0 y = -4, z = 3;

$\rho = \sqrt{x^2 + y^2} = 4, z = 3, \varphi = \tan^{-1} y/x = \tan^{-1} - 4/0 = 270°$

$\underline{\underline{T(\rho,\varphi,z) = T(4, 270°, 3)}}$.

Spherical system:

$r = \sqrt{x^2 + y^2 + z^2} = 5, \theta = \tan^{-1} \rho/z = \tan^{-1} 4/3 = 53.13°$.

$\underline{\underline{T(r,\theta,\varphi) = T(5, 53.13°, 270°)}}$.

At S(-3-4-10), x = -3, y = -4, z = -10;

$\rho = \sqrt{x^2 + y^2} = 5,\ \phi = \tan^{-1} - 4/-3 = 233.1°$

$\underline{\underline{S(\rho,\phi,z) = S(5, 233.1, -10)}}$.

Spherical system:

$r = \sqrt{x^2 + y^2 + z^2} = 5\sqrt{5} = 11.18.$

$\theta = \tan^{-1} \rho/z = \tan^{-1} \dfrac{5}{-10} = 153.43°;$

$\underline{\underline{S(r,\theta,\phi) = S(11.18, 153.43°, 233.1°)}}$.

(b) In Cylindrical system, $\rho = \sqrt{x^2 + y^2}$; $yz = z\rho\sin\phi$,

$$Q_x = \frac{\rho}{\sqrt{\rho^2 + z^2}}; \quad Q_y = 0; \quad Q_z = -\frac{z\rho\sin\phi}{\sqrt{\rho^2 + z^2}};$$

$$\begin{bmatrix} Q_\rho \\ Q_\phi \\ Q_z \end{bmatrix} = \begin{bmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} Q_x \\ 0 \\ Q_z \end{bmatrix};$$

$$Q_\rho = Q_x\cos\phi = \frac{\rho\cos\phi}{\sqrt{\rho^2 + z^2}}, \qquad Q_\phi = -Q_x\sin\phi = \frac{-\rho\sin\phi}{\sqrt{\rho^2 + z^2}}$$

Hence,

$$\boldsymbol{Q} = \frac{\rho}{\sqrt{\rho^2 + z^2}}(\cos\phi\,\boldsymbol{a}_\rho - \sin\phi\,\boldsymbol{a}_\phi - z\sin\phi\,\boldsymbol{a}_z).$$

In Spherical coordinates:

$$Q_x = \frac{r\sin\theta}{r} = \sin\theta;$$

$$Q_z = -r\sin\phi\sin\theta\, r\cos\theta\,\frac{1}{r} = -r\sin\theta\cos\theta\sin\phi.$$

$$\begin{bmatrix} Q_r \\ Q_\theta \\ Q_\phi \end{bmatrix} = \begin{bmatrix} \sin\theta\cos\phi & \sin\theta\sin\phi & \cos\theta \\ \cos\theta\cos\phi & \cos\theta\sin\phi & -\sin\theta \\ -\sin\phi & \cos\phi & 0 \end{bmatrix} \begin{bmatrix} Q_x \\ 0 \\ Q_z \end{bmatrix};$$

$$Q_r = Q_x \sin\theta\cos\phi + Q_z \cos\theta = \sin^2\theta\cos\phi - r\sin\theta\cos^2\theta\sin\phi.$$
$$Q_\theta = Q_x \cos\theta\cos\phi - Q_z \sin\theta = \sin\theta\cos\theta\cos\phi + r\sin^2\theta\cos\theta\sin\phi.$$
$$Q_\phi = -Q_x \sin\phi = -\sin\theta\sin\phi.$$

$$\therefore\ \boldsymbol{Q} = \sin\theta\left(\sin\theta\cos\phi - r\cos^2\theta\sin\phi\right)\boldsymbol{a}_r + \sin\theta\cos\theta(\cos\phi + r\sin\theta\sin\phi)\boldsymbol{a}_\theta - \sin\theta\sin\phi\,\boldsymbol{a}_\phi\ .$$

At T :

$$\bar{Q}(x,y,z) = \frac{4}{5}\bar{a}_x + \frac{12}{5}\bar{a}_z = 0.8\bar{a}_x + 2.4\bar{a}_z\ ;$$

$$\bar{Q}(\rho,\phi,z) = \frac{4}{5}(\cos 270^\circ\,\bar{a}_\rho - \sin 270^\circ\,\bar{a}_\phi - 3\sin 270^\circ\,\bar{a}_z$$

$$= 0.8\bar{a}_\phi + 2.4\bar{a}_z\ ;$$

$$\bar{Q}(r,\theta,\phi) = \frac{4}{5}(0 - \frac{45}{25}(-1))\bar{a}_r + \frac{4}{5}(\frac{3}{5})(0 + \frac{20}{5}(-1))\bar{a}_\theta - \frac{4}{5}(-1)\bar{a}_\phi$$

$$= \frac{36}{25}\bar{a}_r - \frac{48}{25}\bar{a}_\theta + \frac{4}{5}\bar{a}_\phi = \underline{\underline{1.44\bar{a}_r - 1.92\bar{a}_\theta + 0.8\bar{a}_\phi}}\ ;$$

Note, that the magnitude of vector Q = 2.53 in all 3 cases above.

2.1 Express the following points in Cartesian coordinates:

(a) $P(1, 60^\circ, 2)$

(b) $Q(2, 90^\circ, -4)$

(c) $R(, 45^\circ, 210^\circ)$

(d) $T(4, \pi/2, \pi/6)$

2.2 Express the following points in cylindrical and spherical coordinates:

(a) $P(1, -4, -3)$

(b) $Q(3, 0, 5)$

(c) $R(-2, 6, 0)$

2.3 (a) If $V = xz - xy + yz$, express V in cylindrical coordinates.

(b) If $U = x^2 + 2y^2 + 3z^2$, express U in spherical coordinates.

2.4 Transform the following vectors to cylindrical and spherical coordinates:

(a) $\mathbf{D} = (x + z)\mathbf{a}_y$

(b) $\mathbf{E} = (y^2 - x^2)\mathbf{a}_x + xyz\mathbf{a}_y + (x^2 - z^2)\mathbf{a}_z$

2.5 Convert the following vectors to cylindrical and spherical systems:

(a) $\mathbf{F} = \dfrac{x\mathbf{a}_x + y\mathbf{a}_y + 4\mathbf{a}_z}{\sqrt{x^2 + y^2 + z^2}}$

(b) $\mathbf{G} = (x^2 + y^2)\left[\dfrac{x\mathbf{a}_x}{\sqrt{x^2 + y^2 + z^2}} + \dfrac{y\mathbf{a}_y}{\sqrt{x^2 + y^2 + z^2}} + \dfrac{z\mathbf{a}_z}{\sqrt{x^2 + y^2 + z^2}}\right]$

2.6 Express the following vectors in Cartesian coordinates:

(a) $\mathbf{A} = \rho(z^2 + 1)\mathbf{a}_\rho - \rho z \cos\phi\, \mathbf{a}_\phi$

(b) $\mathbf{B} = 2r \sin\theta \cos\phi\, \mathbf{a}_r + r \cos\theta \cos\theta\, \mathbf{a}_\theta - r \sin\phi\, \mathbf{a}_\phi$

2.7 Convert the following vectors to Cartesian coordinates:

(a) $\mathbf{C} = z \sin\phi\, \mathbf{a}_\rho - \rho \cos\phi\, \mathbf{a}_\phi + 2\rho z \mathbf{a}_z$

(b) $\mathbf{D} = \dfrac{\sin\theta}{r^2}\mathbf{a}_r + \dfrac{\cos\theta}{r^2}\mathbf{a}_\theta$

2.8 Prove the following:

(a)
$$\begin{aligned}
\mathbf{a}_x \cdot \mathbf{a}_\rho &= \cos\phi \\
\mathbf{a}_x \cdot \mathbf{a}_\phi &= -\sin\phi \\
\mathbf{a}_y \cdot \mathbf{a}_\rho &= \sin\phi \\
\mathbf{a}_y \cdot \mathbf{a}_\phi &= \cos\phi
\end{aligned}$$

(b)
$$\begin{aligned}
\mathbf{a}_x \cdot \mathbf{a}_r &= \sin\theta\cos\phi \\
\mathbf{a}_x \cdot \mathbf{a}_\theta &= \cos\theta\cos\phi \\
\mathbf{a}_y \cdot \mathbf{a}_r &= \sin\theta\sin\phi
\end{aligned}$$

$$\begin{aligned}
\mathbf{a}_y \cdot \mathbf{a}_\theta &= \cos\theta\sin\phi \\
\mathbf{a}_z \cdot \mathbf{a}_r &= \cos\theta \\
\mathbf{a}_z \cdot \mathbf{a}_\theta &= -\sin\theta
\end{aligned}$$

Solutions

2.2

(a) Given P(1,-4,-3), convert to cylindrical and spherical values;

$$\rho = \sqrt{x^2 + y^2} = \sqrt{1^2 + (-4)^2} = \sqrt{17} = 4.123.$$

$$\phi = \tan^{-1}\frac{y}{x} = \tan^{-1}\frac{-4}{1} = 284.04°.$$

$$\therefore P(\rho, \phi, z) = \underline{\underline{(4.123,\ 284.04°,\ -3)}}.$$

Spherical:

$$r = \sqrt{x^2 + y^2 + z^2} = \sqrt{1 + 16 + 9} = 5.099.$$

$$\theta = \tan^{-1}\frac{\rho}{z} = \tan^{-1}\frac{4.123}{-3} = 126.04°.$$

$$P(r, \theta, \phi) = \underline{\underline{P(5.099, 126.04°,\ 284.04°)}}.$$

(b) $\rho = 3, \quad \phi = \tan^{-1}\frac{y}{x} = \tan^{-1}\frac{0}{3} = 0^o$

$$Q(\rho, \phi, z) = \underline{\underline{Q(3, 0^o, 5)}}$$

$$r = \sqrt{9 + 0 + 25} = 5.831, \quad \theta = \tan^{-1}\frac{\rho}{z} = \tan^{-1}\frac{3}{5} = 30.96^o$$

$$Q(r, \theta, \phi) = \underline{\underline{Q(5.831, 30.96^o, 0^o)}}$$

(c) $\rho = \sqrt{4 + 36} = 6.325, \quad \phi = \tan^{-1}\frac{6}{-2} = 108.4^o$

$$R(\rho, \phi, z) = \underline{\underline{R(6.325, 108.4^o, 0)}}$$

$$r = \rho = 6.325, \quad \theta = \tan^{-1}\frac{\rho}{z} = \tan^{-1}\frac{6.325}{0} = 90^o$$

$$R(r, \theta, \phi) = \underline{\underline{R(6.325, 90^o, 108.4^o)}}$$

2.3

(a)

$$x = \rho\cos\phi, \qquad y = \rho\sin\phi,$$

$$V = \underline{\underline{\rho z\cos\phi - \rho^2\sin\phi\cos\phi + \rho z\sin\phi}}$$

(b)

$$\begin{aligned} U &= x^2 + y^2 + z^2 + y^2 + 2z^2 \\ &= r^2 + r^2\sin^2\theta\sin^2\phi + 2r^2\cos^2\theta \\ &= \underline{\underline{r^2[1 + \sin^2\theta\sin^2\phi + 2\cos^2\theta]}} \end{aligned}$$

2.4

(a)

$$\begin{bmatrix} D_\rho \\ D_\phi \\ D_z \end{bmatrix} = \begin{bmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ x+z \\ 0 \end{bmatrix}$$

$$D_\rho = (x+z)\sin\phi = (\rho\cos\phi + z)\sin\phi$$

$$D_\phi = (x+z)\cos\phi = (\rho\cos\phi + z)\cos\phi$$

$$\bar{D} = \underline{\underline{(\rho\cos\phi + z)[\sin\phi\,\bar{a}_\rho + \cos\phi\,\bar{a}_\phi]}}$$

Spherical: Since $D_x = D_z = 0$, we may leave out the first and third column of the transformation matrix. Thus,

$$\begin{bmatrix} D_r \\ D_\theta \\ D_\phi \end{bmatrix} = \begin{bmatrix} \ldots & \sin\theta\sin\phi & \ldots \\ \ldots & \cos\theta\sin\phi & \ldots \\ \ldots & \cos\phi & \ldots \end{bmatrix} \begin{bmatrix} 0 \\ x+z \\ 0 \end{bmatrix}$$

$$D_r = (x+z)\sin\theta\sin\phi = r(\sin\theta\cos\phi + \cos\theta)\sin\theta\sin\phi.$$

$$D_\theta = (x+z)\cos\theta\sin\phi = r(\sin\theta\cos\phi + \cos\theta)\cos\theta\sin\phi.$$

$$D_\phi = (x+z)\cos\phi \qquad = r(\sin\theta\cos\phi + \cos\theta)\cos\phi.$$

$$\bar{D} = \underline{\underline{r(\sin\theta\cos\phi + \cos\theta)[\sin\theta\sin\phi\,\bar{a}_r + \cos\theta\sin\phi\,\bar{a}_\theta + \cos\phi\,\bar{a}_\phi]}}.$$

(b) Cylindrical:

$$\begin{bmatrix} E_\rho \\ E_\phi \\ E_z \end{bmatrix} = \begin{bmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} y^2 - x^2 \\ xyz \\ x^2 - z^2 \end{bmatrix}$$

$$\begin{aligned} E_\rho &= (y^2 - x^2)\cos\phi + xyz\sin\phi \\ &= \rho^2(\sin^2\phi - \cos^2\phi)\cos\phi + \rho^2 z\cos\phi\sin^2\phi \\ &= -\rho^2\cos 2\phi\cos\phi + \rho^2 z\sin^2\phi\cos\phi. \end{aligned}$$

$$\begin{aligned} E_\phi &= -(y^2 - x^2)\sin\phi + xyz\cos\phi \\ &= \rho^2\cos 2\phi\sin\phi + \rho^2 z\sin\phi\cos^2\phi. \end{aligned}$$

$$E_z = x^2 - z^2 = \rho^2\cos^2\phi - z^2.$$

$$\bar{E} = \underline{\underline{\rho^2\cos\phi(z\sin^2\phi - \cos 2\phi)\,\bar{a}_\rho + \rho^2\sin\phi(z\cos^2\phi + \cos 2\phi)\,\bar{a}_\phi + (\rho^2\cos^2\phi - z^2)\,\bar{a}_z}}.$$

In spherical:

$$\begin{bmatrix} E_r \\ E_\theta \\ E_\phi \end{bmatrix} = \begin{bmatrix} \sin\theta\cos\phi & \sin\theta\sin\phi & \cos\theta \\ \cos\theta\cos\phi & \cos\theta\sin\phi & -\sin\theta \\ -\sin\phi & \cos\phi & 0 \end{bmatrix} \begin{bmatrix} y^2 - x^2 \\ xyz \\ x^2 - z^2 \end{bmatrix}$$

$$E_r = (y^2 - x^2)\sin\theta\cos\phi + xyz\sin\theta\sin\phi + (x^2 - z^2)\cos\theta;$$

but $x = r\sin\theta\cos\phi, \quad y = r\sin\theta\sin\phi, \quad z = r\cos\theta;$

$$\begin{aligned} E_r = {} & r^2\sin^3\theta(\sin^2\phi - \cos^2\phi)\cos\phi + r^3\sin^3\theta\cos\theta\sin^2\phi\cos\phi \\ & + r^2(\sin^2\theta\cos^2\phi - \cos^2\theta)\cos\theta; \end{aligned}$$

$$E_\theta = (y^2 - x^2)\cos\theta\cos\phi + xyz\cos\theta\sin\phi - (x^2 - z^2)\sin\theta;$$
$$= -r^2\sin^2\theta\cos 2\phi\cos\theta\cos\phi + r^3\sin^2\theta\cos^2\theta\sin^2\phi\cos\phi - r^2(\sin^2\theta\cos^2\phi - \cos^2\theta)\sin\theta;$$
$$E_\phi = (x^2 - y^2)\sin\phi + xyz\cos\phi$$
$$= r^2\sin^2\theta\cos 2\phi\sin\phi + r^3\sin^2\theta\cos^2\phi\sin\phi\cos\theta;$$
$$\bar{E} = [-r^2\sin^3\theta\cos 2\phi\cos\phi + r^3\sin^3\theta\cos\theta\sin^2\phi\cos\phi + r^2(\sin^2\theta\cos^2\phi - \cos^2\theta)\cos\theta]\bar{a}_r +$$
$$[-r^2\sin^2\theta\cos 2\phi\cos\theta\cos\phi + r^3\sin^2\theta\cos^2\theta\sin^2\phi\cos\phi - r^2\sin\theta(\sin^2\theta\cos^2\phi - \cos^2\theta)]\bar{a}_\theta +$$
$$+ \quad \underline{\underline{[r^2\sin^2\theta\cos 2\phi\sin\phi + r^3\sin^2\theta\cos^2\phi\sin\phi\cos\theta]\bar{a}_\phi}}$$

2.8

(a) $\vec{a}_x \cdot \vec{a}_\rho = (\cos\phi\, \vec{a}_\rho - \sin\phi\, \vec{a}_\phi) \cdot \vec{a}_\rho = \cos\phi$

$\vec{a}_x \cdot \vec{a}_\phi = (\cos\phi\, \vec{a}_\rho - \sin\phi\, \vec{a}_\phi) \cdot \vec{a}_\phi = -\sin\phi$

$\vec{a}_y \cdot \vec{a}_\rho = (\sin\phi\, \vec{a}_\rho + \cos\phi\, \vec{a}_\phi) \cdot \vec{a}_\rho = \sin\phi$

$\vec{a}_y \cdot \vec{a}_\phi = (\sin\phi\, \vec{a}_\rho + \cos\phi\, \vec{a}_\phi) \cdot \vec{a}_\phi = \cos\phi$

(b) Since $\vec{a}_\rho$, $\vec{a}_\phi$, and $\vec{a}_z$ are mutually orthogonal,

$\vec{a}_z \cdot \vec{a}_z = 1$, $\vec{a}_z \cdot \vec{a}_\rho = 0 = \vec{a}_z \cdot \vec{a}_\phi$.

Also, $\vec{a}_x \cdot \vec{a}_z = 0 = \vec{a}_y \cdot \vec{a}_z$. Hence

$$\begin{bmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \vec{a}_x \cdot \vec{a}_\rho & \vec{a}_x \cdot \vec{a}_\phi & \vec{a}_x \cdot \vec{a}_z \\ \vec{a}_y \cdot \vec{a}_\rho & \vec{a}_y \cdot \vec{a}_\phi & \vec{a}_y \cdot \vec{a}_z \\ \vec{a}_z \cdot \vec{a}_\rho & \vec{a}_z \cdot \vec{a}_\phi & \vec{a}_z \cdot \vec{a}_z \end{bmatrix}$$

(c) In spherical system,

$$\vec{a}_x = \sin\theta\cos\phi\,\vec{a}_r + \cos\theta\cos\phi\,\vec{a}_\theta - \sin\phi\,\vec{a}_\phi$$

$$\vec{a}_y = \sin\theta\sin\phi\,\vec{a}_r + \cos\theta\sin\phi\,\vec{a}_\theta + \cos\phi\,\vec{a}_\phi$$

$$\vec{a}_z = \cos\theta\,\vec{a}_r - \sin\theta\,\vec{a}_\theta.$$

Hence $\vec{a}_x \cdot \vec{a}_r = \sin\theta\cos\phi$,

$\vec{a}_x \cdot \vec{a}_\theta = \cos\theta\cos\phi$,

$\vec{a}_y \cdot \vec{a}_r = \sin\theta\sin\phi$,

$\vec{a}_y \cdot \vec{a}_\theta = \cos\theta\sin\phi$,

$\vec{a}_z \cdot \vec{a}_r = \cos\theta$,

$\vec{a}_z \cdot \vec{a}_\theta = -\sin\theta$.