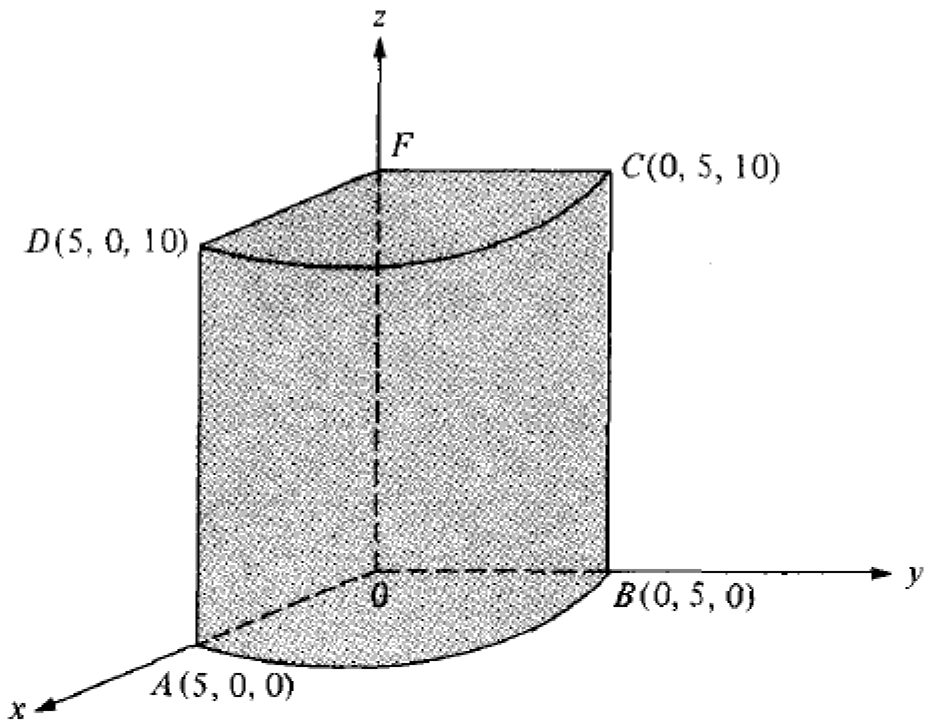
Tutorial Sheet-2

Ex-1



Consider the object shown in Figure 3.7. Calculate

(a) The distance BC

(b) The distance CD

(c) The surface area $ABCD$

(d) The surface area ABO

(e) The surface area $AOFD$

(f) The volume $ABDCFO$

Solution:

Although points A, B, C, and D are given in Cartesian coordinates, it is obvious that the object has cylindrical symmetry. Hence, we solve the problem in cylindrical coordinates. The points are transformed from Cartesian to cylindrical coordinates as follows:

$$A(5, 0, 0) \rightarrow A(5, 0°, 0)$$

$$B(0, 5, 0) \rightarrow B\left(5, \frac{\pi}{2}, 0\right)$$

$$C(0, 5, 10) \rightarrow C\left(5, \frac{\pi}{2}, 10\right)$$

$$D(5, 0, 10) \rightarrow D(5, 0°, 10)$$

(a) Along BC, $dl = dz$; hence,

$$BC = \int dl = \int_0^{10} dz = 10$$

(b) Along CD, $dl = \rho\, d\phi$ and $\rho = 5$, so

$$CD = \int_0^{\pi/2} \rho\, d\phi = 5\, \phi \Big|_0^{\pi/2} = 2.5\pi$$

(c) For $ABCD$, $dS = \rho\, d\phi\, dz$, $\rho = 5$. Hence,

$$\text{area } ABCD = \int dS = \int_{\phi=0}^{\pi/2} \int_{z=0}^{10} \rho\, d\phi\, dz = 5 \int_0^{\pi/2} d\phi \int_0^{10} dz \Big|_{\rho=5} = 25\pi$$

(d) For ABO, $dS = \rho\, d\phi\, d\rho$ and $z = 0$, so

$$\text{area } ABO = \int_{\phi=0}^{\pi/2} \int_{\rho=0}^{5} \rho\, d\phi\, d\rho = \int_0^{\pi/2} d\phi \int_0^{5} \rho\, d\rho = 6.25\pi$$

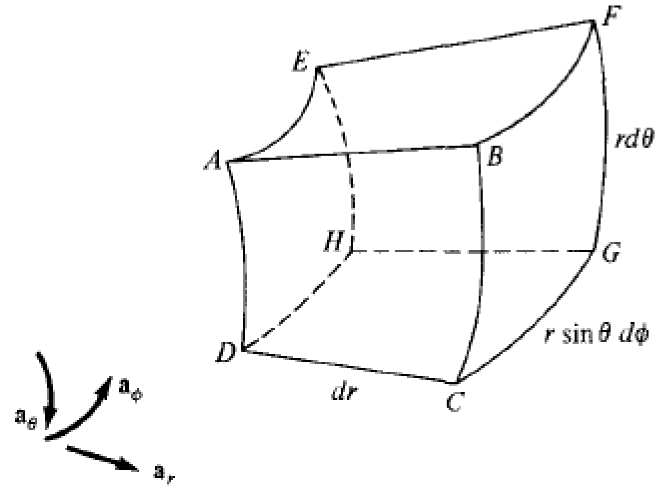
(e) For $AOFD$, $dS = d\rho\, dz$ and $\phi = 0°$, so

$$\text{area } AOFD = \int_{\rho=0}^{5} \int_{z=0}^{10} dp\, dz = 50$$

(f) For volume $ABDCFO$, $dv = \rho\, d\phi\, dz\, d\rho$. Hence,

$$v = \int dv = \int_{\rho=0}^{5} \int_{\phi=0}^{\pi/2} \int_{z=0}^{10} \rho\, d\phi\, dz\, d\rho = \int_0^{10} dz \int_0^{\pi/2} d\phi \int_0^{5} \rho\, d\rho = 62.5\pi$$

EX-2



E
F
A
B
$rd\theta$
H
G
D
$r \sin\theta \, d\phi$
dr
C
$\mathbf{a}_\theta$
$\mathbf{a}_\phi$
$\mathbf{a}_r$

Refer to Figure 3.26; disregard the differential lengths and imagine that the object is part of a spherical shell. It may be described as $3 \le r \le 5$, $60° \le \theta \le 90°$, $45° \le \phi \le 60°$ where surface $r = 3$ is the same as $AEHD$, surface $\theta = 60°$ is $AEFB$, and surface $\phi = 45°$ is $ABCD$. Calculate

(a) The distance DH

(b) The distance FG

(c) The surface area $AEHD$

(d) The surface area $ABDC$

(e) The volume of the object

(a) $$DH = \int_{\phi=45°}^{\phi=60°} r \sin\theta \, d\phi \Big|_{r=3,\theta=90°} = 3(1)\left[\frac{\pi}{3} - \frac{\pi}{4}\right] = \frac{\pi}{4} = \underline{\underline{0.7854.}}$$

(b) $$FG = \int_{\theta=60°}^{\theta=90°} r d\theta \Big|_{r=5} = 5\left(\frac{\pi}{2} - \frac{\pi}{3}\right) = \frac{5\pi}{6} = \underline{\underline{2.618.}}$$

(c)

$$AEHD = \int_{\theta=60°}^{\theta=90°} \int_{\phi=45°}^{\phi=60°} r^2 \sin\theta \, d\theta \, d\phi \Big|_{r=3} = 9(-\cos\theta)\Big|_{\theta=60°}^{\theta=90°} \phi\Big|_{\phi=45°}^{\phi=60°}$$

$$= 9\left(\frac{1}{2}\right)\left(\frac{\pi}{12}\right) = \frac{3\pi}{8} = \underline{\underline{1.178.}}$$

(d)

$$ABCD = \int_{r=3}^{r=5} \int_{\theta=60}^{\theta=90} r d\theta \, dr = \frac{r^2}{2}\Big|_{r=3}^{r=5} \left(\frac{\pi}{2} - \frac{\pi}{3}\right) = \frac{4\pi}{3} = \underline{\underline{4.189.}}$$

(e)

$$\text{Volume} = \int_{r=3}^{r=5} \int_{\phi=45°}^{\phi=60°} \int_{\theta=60}^{\theta=90} r^2 \sin\theta dr \, d\theta \, d\phi = \frac{r^3}{3}\Big|_{r=3}^{r=5} (-\cos\theta)\Big|_{\theta=60°}^{\theta=90°} \phi\Big|_{\phi=45°}^{\phi=60°} = \frac{1}{3}(98)\left(\frac{1}{2}\right)\frac{\pi}{12}$$

$$= \underline{\underline{\frac{49\pi}{36} = 4.276.}}$$

Ex-3

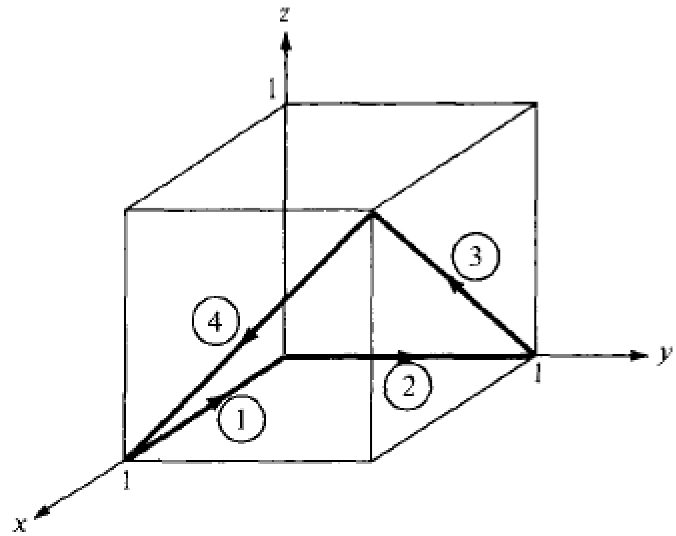


Figure 3.10

Given that $\mathbf{F} = x^2\mathbf{a}_x - xz\mathbf{a}_y - y^2\mathbf{a}_z$, calculate the circulation of $\mathbf{F}$ around the (closed) path shown in Figure 3.10.

Solution:

The circulation of $\mathbf{F}$ around path L is given by

$$\oint_L \mathbf{F}\cdot d\mathbf{l} = \left(\int_1 + \int_2 + \int_3 + \int_4\right)\mathbf{F}\cdot d\mathbf{l}$$

where the path is broken into segments numbered 1 to 4 as shown in Figure 3.10.

For segment 1, $y = 0 = z$

$$\mathbf{F} = x^2\mathbf{a}_x, \qquad d\mathbf{l} = dx\,\mathbf{a}_x$$

Notice that $d\mathbf{l}$ is always taken as along $+\mathbf{a}_x$ so that the direction on segment 1 is taken care of by the limits of integration. Thus,

$$\int_1 \mathbf{F}\cdot d\mathbf{l} = \int_1^0 x^2 dx = \frac{x^3}{3}\bigg|_1^0 = -\frac{1}{3}$$

For segment 2, $x = 0 = z$, $\mathbf{F} = -y^2\mathbf{a}_z$, $d\mathbf{l} = dy\,\mathbf{a}_y$, $\mathbf{F} \cdot d\mathbf{l} = 0$. Hence,

$$\int_2 \mathbf{F} \cdot d\mathbf{l} = 0$$

For segment 3, $y = 1$, $\mathbf{F} = x^2\mathbf{a}_x - xz\mathbf{a}_y - \mathbf{a}_z$, and $d\mathbf{l} = dx\,\mathbf{a}_x + dz\,\mathbf{a}_z$, so

$$\int_3 \mathbf{F} \cdot d\mathbf{l} = \int (x^2 dx - dz)$$

But on 3, $z = x$; that is, $dx = dz$. Hence,

$$\int_3 \mathbf{F} \cdot d\mathbf{l} = \int_0^1 (x^2 - 1)\, dx = \frac{x^3}{3} - x \bigg|_0^1 = -\frac{2}{3}$$

For segment 4, $x = 1$, so $\mathbf{F} = \mathbf{a}_x - z\mathbf{a}_y - y^2\mathbf{a}_z$, and $d\mathbf{l} = dy\,\mathbf{a}_y + dz\,\mathbf{a}_z$. Hence,

$$\int_4 \mathbf{F} \cdot d\mathbf{l} = \int (-z\,dy - y^2 dz)$$

But on 4, $z = y$; that is, $dz = dy$, so

$$\int_4 \mathbf{F} \cdot d\mathbf{l} = \int_1^0 (-y - y^2)\, dy = -\frac{y^2}{2} - \frac{y^3}{3} \bigg|_1^0 = \frac{5}{6}$$

By putting all these together, we obtain

$$\oint_L \mathbf{F} \cdot d\mathbf{l} = -\frac{1}{3} + 0 - \frac{2}{3} + \frac{5}{6} = -\frac{1}{6}$$

EX-4

Calculate the circulation of

$$\mathbf{A} = \rho \cos \phi \, \mathbf{a}_\rho + z \sin \phi \, \mathbf{a}_z$$

around the edge L of the wedge defined by $0 \leq \rho \leq 2$, $0 \leq \phi \leq 60°$, $z = 0$ and shown in Figure 3.11.

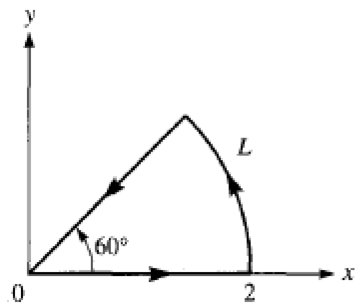
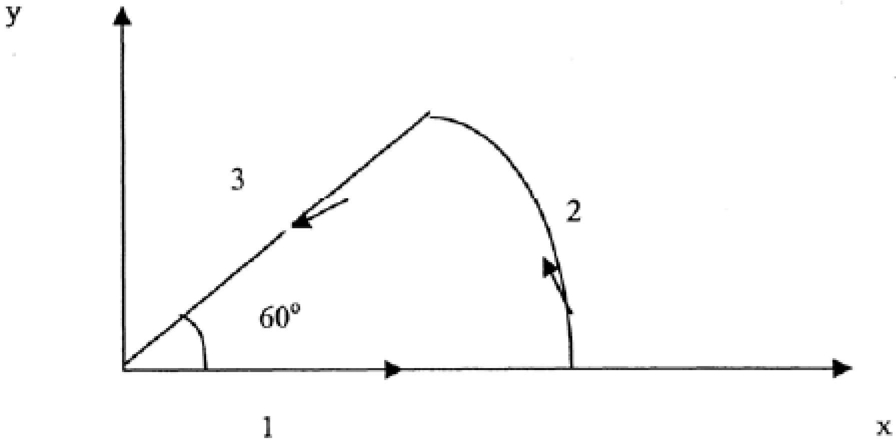


Figure 3.11

Sol:



$$\oint_L A \bullet dl = \left(\int_1 + \int_2 + \int_3\right) \bar{A} \bullet \bar{dl} = C_1 + C_2 + C_3$$

Along (1), $C_1 = \int \bar{A} \bullet \bar{dl} = \int_0^2 \rho \cos\phi \, d\rho |_{\phi=0} = \frac{\rho^2}{2}\Big|_0^2 = 2.$

Along (2), $\bar{dl} = \rho d\phi \, \bar{a}_\phi$. $\bar{A} \bullet \bar{dl} = 0$, $C_2 = 0$

Along (3), $C_3 = \int_2^0 \rho \cos\phi \, d\rho_{\phi=60^\circ} = -\frac{\rho^2}{2}\Big|_0^2 \left(\frac{1}{2}\right) = -1$

$$\oint_l \bar{A} \bullet \bar{dl} = C_1 + C_2 + C_3 = 2 + 0 - 1 = \underline{\underline{1}}$$

DEL OPERATOR

The del operator, written ∇, is the vector differential operator. In Cartesian coordinates,

$$\boxed{\nabla = \frac{\partial}{\partial x}\mathbf{a}_x + \frac{\partial}{\partial y}\mathbf{a}_y + \frac{\partial}{\partial z}\mathbf{a}_z} \qquad (3.16)$$

This vector differential operator, otherwise known as the *gradient operator,* is not a vector in itself, but when it operates on a scalar function, for example, a vector ensues. The operator is useful in defining

1. The gradient of a scalar V, written as ∇V
2. The divergence of a vector $\mathbf{A}$, written as $\nabla \cdot \mathbf{A}$
3. The curl of a vector $\mathbf{A}$, written as $\nabla \times \mathbf{A}$
4. The Laplacian of a scalar V, written as $\nabla^2 V$

In Cylindrical coordinate

$$\boxed{\nabla = \mathbf{a}_\rho \frac{\partial}{\partial \rho} + \mathbf{a}_\phi \frac{1}{\rho}\frac{\partial}{\partial \phi} + \mathbf{a}_z \frac{\partial}{\partial z}}$$

In Spherical coordinate

$$\boxed{\nabla = \mathbf{a}_r \frac{\partial}{\partial r} + \mathbf{a}_\theta \frac{1}{r}\frac{\partial}{\partial \theta} + \mathbf{a}_\phi \frac{1}{r \sin\theta}\frac{\partial}{\partial \phi}}$$

GRADIENT OF A SCALAR

The **gradient** of a scalar field *V* is a vector that represents both the magnitude and the direction of the maximum space rate of increase of V.

$$\boxed{\nabla V = \frac{\partial V}{\partial x}\mathbf{a}_x + \frac{\partial V}{\partial y}\mathbf{a}_y + \frac{\partial V}{\partial z}\mathbf{a}_z}$$

for cylindrical coordinates,

$$\boxed{\nabla V = \frac{\partial V}{\partial \rho}\mathbf{a}_\rho + \frac{1}{\rho}\frac{\partial V}{\partial \phi}\mathbf{a}_\phi + \frac{\partial V}{\partial z}\mathbf{a}_z}$$

and for spherical coordinates,

$$\boxed{\nabla V = \frac{\partial V}{\partial r}\mathbf{a}_r + \frac{1}{r}\frac{\partial V}{\partial \theta}\mathbf{a}_\theta + \frac{1}{r\sin\theta}\frac{\partial V}{\partial \phi}\mathbf{a}_\phi}$$

EX-5

Find the gradient of the following scalar fields:

(a) $V = e^{-z}\sin 2x \cosh y$

(b) $U = \rho^2 z \cos 2\phi$

(c) $W = 10r\sin^2\theta\cos\phi$

Solution:

(a) $$\nabla V = \frac{\partial V}{\partial x}\mathbf{a}_x + \frac{\partial V}{\partial y}\mathbf{a}_y + \frac{\partial V}{\partial z}\mathbf{a}_z$$
$$= 2e^{-z}\cos 2x \cosh y\,\mathbf{a}_x + e^{-z}\sin 2x \sinh y\,\mathbf{a}_y - e^{-z}\sin 2x\cosh y\,\mathbf{a}_z$$

(b) $$\nabla U = \frac{\partial U}{\partial \rho}\mathbf{a}_\rho + \frac{1}{\rho}\frac{\partial U}{\partial \phi}\mathbf{a}_\phi + \frac{\partial U}{\partial z}\mathbf{a}_z$$
$$= 2\rho z\cos 2\phi\,\mathbf{a}_\rho - 2\rho z\sin 2\phi\,\mathbf{a}_\phi + \rho^2\cos 2\phi\,\mathbf{a}_z$$

(c) $$\nabla W = \frac{\partial W}{\partial r}\mathbf{a}_r + \frac{1}{r}\frac{\partial W}{\partial \theta}\mathbf{a}_\theta + \frac{1}{r\sin\theta}\frac{\partial W}{\partial \phi}\mathbf{a}_\phi$$
$$= 10\sin^2\theta\cos\phi\,\mathbf{a}_r + 10\sin 2\theta\cos\phi\,\mathbf{a}_\theta - 10\sin\theta\sin\phi\,\mathbf{a}_\phi$$

Ex-6

Determine the gradient of the following scalar fields:

(a) $U = x^2y + xyz$

(b) $V = \rho z\sin\phi + z^2\cos^2\phi + \rho^2$

(c) $f = \cos\theta\sin\phi\ln r + r^2\phi$

Answer: (a) $y(2x + z)\mathbf{a}_x + x(x + z)\mathbf{a}_y + xy\mathbf{a}_z$

(b) $(z \sin \phi + 2\rho)\mathbf{a}_\rho + (z \cos \phi - \frac{z^2}{\rho} \sin 2\phi)\mathbf{a}_\phi +$
$(\rho \sin \phi + 2z \cos^2 \phi)\mathbf{a}_z$

(c) $\left(\frac{\cos \theta \sin \phi}{r} + 2r\phi\right)\mathbf{a}_r - \frac{\sin \theta \sin \phi}{r} \ln r\, \mathbf{a}_\theta +$
$\left(\frac{\cot \theta}{r} \cos \phi \ln r + r \operatorname{cosec} \theta\right)\mathbf{a}_\phi$

Ex-7

Given $W = x^2y^2 + xyz$, compute ∇W and the direction derivative dW/dl in the direction $3\mathbf{a}_x + 4\mathbf{a}_y + 12\mathbf{a}_z$ at $(2, -1, 0)$.

Solution:

$$\nabla W = \frac{\partial W}{\partial x}\mathbf{a}_x + \frac{\partial W}{\partial y}\mathbf{a}_y + \frac{\partial W}{\partial z}\mathbf{a}_z$$
$$= (2xy^2 + yz)\mathbf{a}_x + (2x^2y + xz)\mathbf{a}_y + (xy)\mathbf{a}_z$$

At $(2, -1, 0)$: $\nabla W = 4\mathbf{a}_x - 8\mathbf{a}_y - 2\mathbf{a}_z$

Hence,

$$\frac{dW}{dl} = \nabla W \cdot \mathbf{a}_l = (4, -8, -2) \cdot \frac{(3, 4, 12)}{13} = -\frac{44}{13}$$

Ex-8

Given $\Phi = xy + yz + xz$, find gradient Φ at point (1, 2, 3) and the directional derivative of Φ at the same point in the direction toward point (3, 4, 4).

Answer: $5\mathbf{a}_x + 4\mathbf{a}_y + 3\mathbf{a}_z$, 7.

$$\nabla\Phi = (y+z)\bar{a}_x + (x+z)\bar{a}_y + (x+y)\bar{a}_z$$

$$\text{At } (1,2,3) \quad \nabla\Phi = \underline{\underline{(5,4,3)}}$$

$$\nabla\Phi \bullet \bar{a}_l = (5,4,3) \bullet \frac{(2,2,1)}{3} = \frac{21}{3} = \underline{\underline{7}},$$

$$\text{where } (2,2,1) = (3,4,4) - (1,2,3)$$

Ex-9

Find the angle at which line $x = y = 2z$ intersects the ellipsoid $x^2 + y^2 + 2z^2 = 10$.

Solution:

Let the line and the ellipsoid meet at angle ψ as shown in Figure 3.13. The line $x = y = 2z$ can be represented by

$$\mathbf{r}(\lambda) = 2\lambda\mathbf{a}_x + 2\lambda\mathbf{a}_y + \lambda\mathbf{a}_z$$

where λ is a parameter. Where the line and the ellipsoid meet,

$$(2\lambda)^2 + (2\lambda)^2 + 2\lambda^2 = 10 \rightarrow \lambda = \pm 1$$

Taking $\lambda = 1$ (for the moment), the point of intersection is $(x, y, z) = (2, 2, 1)$. At this point, $\mathbf{r} = 2\mathbf{a}_x + 2\mathbf{a}_y + \mathbf{a}_z$.

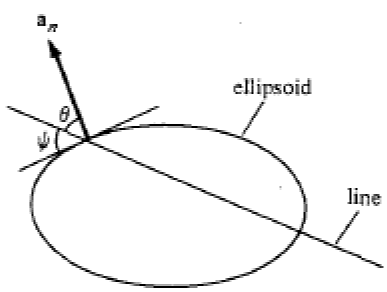


Figure 3.13 For Example 3.5; plane of intersection of a line with an ellipsoid.

The surface of the ellipsoid is defined by

$$f(x, y, z) = x^2 + y^2 + 2z^2 - 10$$

The gradient of f is

$$\nabla f = 2x\,\mathbf{a}_x + 2y\,\mathbf{a}_y + 4z\,\mathbf{a}_z$$

At (2, 2, 1), $\nabla f = 4\mathbf{a}_x + 4\mathbf{a}_y + 4\mathbf{a}_z$. Hence, a unit vector normal to the ellipsoid at the point of intersection is

$$\mathbf{a}_n = \pm\frac{\nabla f}{|\nabla f|} = \pm\frac{\mathbf{a}_x + \mathbf{a}_y + \mathbf{a}_z}{\sqrt{3}}$$

Taking the positive sign (for the moment), the angle between $\mathbf{a}_n$ and $\mathbf{r}$ is given by

$$\cos\theta = \frac{\mathbf{a}_n \cdot \mathbf{r}}{|\mathbf{a}_n \cdot \mathbf{r}|} = \frac{2 + 2 + 1}{\sqrt{3}\sqrt{9}} = \frac{5}{3\sqrt{3}} = sin\,\psi$$

Hence, $\psi = 74.21°$. Because we had choices of + or − for λ and $\mathbf{a}_n$, there are actually four possible angles, given by $\sin\psi = \pm 5/(3\sqrt{3})$.

Ex-10

The **divergence** of **A** at a given point P is the *outward* flux per unit volume as the volume shrinks about P.

Hence,

$$\operatorname{div} \mathbf{A} = \nabla \cdot \mathbf{A} = \lim_{\Delta v \to 0} \frac{\oint_S \mathbf{A} \cdot d\mathbf{S}}{\Delta v} \tag{3.32}$$

For Cartesian

$$\boxed{\nabla \cdot \mathbf{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}}$$

For Cyllendrical

$$\boxed{\nabla \cdot \mathbf{A} = \frac{1}{\rho} \frac{\partial}{\partial \rho} (\rho A_\rho) + \frac{1}{\rho} \frac{\partial A_\phi}{\partial \phi} + \frac{\partial A_z}{\partial z}}$$

For Spherical

$$\boxed{\nabla \cdot \mathbf{A} = \frac{1}{r^2} \frac{\partial}{\partial r} (r^2 A_r) + \frac{1}{r \sin \theta} \frac{\partial}{\partial \theta} (A_\theta \sin \theta) + \frac{1}{r \sin \theta} \frac{\partial A_\phi}{\partial \phi}}$$

EX-10

Determine the divergence of these vector fields:

(a) $\mathbf{P} = x^2yz\,\mathbf{a}_x + xz\,\mathbf{a}_z$

(b) $\mathbf{Q} = \rho \sin\phi\,\mathbf{a}_\rho + \rho^2 z\,\mathbf{a}_\phi + z\cos\phi\,\mathbf{a}_z$

(c) $\mathbf{T} = \dfrac{1}{r^2}\cos\theta\,\mathbf{a}_r + r\sin\theta\cos\phi\,\mathbf{a}_\theta + \cos\theta\,\mathbf{a}_\phi$

Solution:

(a)
$$
\begin{aligned}
\nabla \cdot \mathbf{P} &= \frac{\partial}{\partial x}P_x + \frac{\partial}{\partial y}P_y + \frac{\partial}{\partial z}P_z \\
&= \frac{\partial}{\partial x}(x^2yz) + \frac{\partial}{\partial y}(0) + \frac{\partial}{\partial z}(xz) \\
&= 2xyz + x
\end{aligned}
$$

(b)
$$
\begin{aligned}
\nabla \cdot \mathbf{Q} &= \frac{1}{\rho}\frac{\partial}{\partial \rho}(\rho Q_\rho) + \frac{1}{\rho}\frac{\partial}{\partial \phi}Q_\phi + \frac{\partial}{\partial z}Q_z \\
&= \frac{1}{\rho}\frac{\partial}{\partial \rho}(\rho^2\sin\phi) + \frac{1}{\rho}\frac{\partial}{\partial \phi}(\rho^2 z) + \frac{\partial}{\partial z}(z\cos\phi) \\
&= 2\sin\phi + \cos\phi
\end{aligned}
$$

(c)
$$
\begin{aligned}
\nabla \cdot \mathbf{T} &= \frac{1}{r^2}\frac{\partial}{\partial r}(r^2T_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial \theta}(T_\theta\sin\theta) + \frac{1}{r\sin\theta}\frac{\partial}{\partial \phi}(T_\phi) \\
&= \frac{1}{r^2}\frac{\partial}{\partial r}(\cos\theta) + \frac{1}{r\sin\theta}\frac{\partial}{\partial \theta}(r\sin^2\theta\cos\phi) + \frac{1}{r\sin\theta}\frac{\partial}{\partial \phi}(\cos\theta) \\
&= 0 + \frac{1}{r\sin\theta}2r\sin\theta\cos\theta\cos\phi + 0 \\
&= 2\cos\theta\cos\phi
\end{aligned}
$$

Ex-11

Determine the divergence of the following vector fields and evaluate them at the specified points.

(a) $\mathbf{A} = yz\mathbf{a}_x + 4xy\mathbf{a}_y + y\mathbf{a}_z$ at $(1, -2, 3)$

(b) $\mathbf{B} = \rho z \sin\phi\, \mathbf{a}_\rho + 3\rho z^2 \cos\phi\, \mathbf{a}_\phi$ at $(5, \pi/2, 1)$

(c) $\mathbf{C} = 2r\cos\theta\cos\phi\, \mathbf{a}_r + r^{1/2}\mathbf{a}_\phi$ at $(1, \pi/6, \pi/3)$

Answer: (a) $4x$, 4, (b) $(2 - 3z)z\sin\phi$, -1, (c) $6\cos\theta\cos\phi$, 2.598.