

Tutorial Sheet-1

Ex-1

Given point $P(-2, 6, 3)$ and vector $\mathbf{A} = y\mathbf{a}_x + (x + z)\mathbf{a}_y$, express P and \mathbf{A} in cylindrical and spherical coordinates. Evaluate \mathbf{A} at P in the Cartesian, cylindrical, and spherical systems.

Solution:

At point P : $x = -2, y = 6, z = 3$. Hence,

$$\rho = \sqrt{x^2 + y^2} = \sqrt{4 + 36} = 6.32$$

$$\phi = \tan^{-1} \frac{y}{x} = \tan^{-1} \frac{6}{-2} = 108.43^\circ$$

$$z = 3$$

$$r = \sqrt{x^2 + y^2 + z^2} = \sqrt{4 + 36 + 9} = 7$$

$$\theta = \tan^{-1} \frac{\sqrt{x^2 + y^2}}{z} = \tan^{-1} \frac{\sqrt{40}}{3} = 64.62^\circ$$

Thus,

$$P(-2, 6, 3) = P(6.32, 108.43^\circ, 3) = P(7, 64.62^\circ, 108.43^\circ)$$

In the Cartesian system, \mathbf{A} at P is

$$\mathbf{A} = 6\mathbf{a}_x + \mathbf{a}_y$$

For vector \mathbf{A} , $A_x = y$, $A_y = x + z$, $A_z = 0$. Hence, in the cylindrical system

$$\begin{bmatrix} A_\rho \\ A_\phi \\ A_z \end{bmatrix} = \begin{bmatrix} \cos \phi & \sin \phi & 0 \\ -\sin \phi & \cos \phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} y \\ x + z \\ 0 \end{bmatrix}$$

or

$$A_\rho = y \cos \phi + (x + z) \sin \phi$$

$$A_\phi = -y \sin \phi + (x + z) \cos \phi$$

$$A_z = 0$$

But $x = \rho \cos \phi$, $y = \rho \sin \phi$, and substituting these yields

$$\begin{aligned} \mathbf{A} = (A_\rho, A_\phi, A_z) &= [\rho \cos \phi \sin \phi + (\rho \cos \phi + z) \sin \phi] \mathbf{a}_\rho \\ &+ [-\rho \sin^2 \phi + (\rho \cos \phi + z) \cos \phi] \mathbf{a}_\phi \end{aligned}$$

At P

$$\rho = \sqrt{40}, \quad \tan \phi = \frac{6}{-2}$$

Hence,

$$\begin{aligned}\cos \phi &= \frac{-2}{\sqrt{40}}, \quad \sin \phi = \frac{6}{\sqrt{40}} \\ \mathbf{A} &= \left[\sqrt{40} \cdot \frac{-2}{\sqrt{40}} \cdot \frac{6}{\sqrt{40}} + \left(\sqrt{40} \cdot \frac{-2}{\sqrt{40}} + 3 \right) \cdot \frac{6}{\sqrt{40}} \right] \mathbf{a}_\rho \\ &\quad + \left[-\sqrt{40} \cdot \frac{36}{40} + \left(\sqrt{40} \cdot \frac{-2}{\sqrt{40}} + 3 \right) \cdot \frac{-2}{\sqrt{40}} \right] \mathbf{a}_\phi \\ &= \frac{-6}{\sqrt{40}} \mathbf{a}_\rho - \frac{38}{\sqrt{40}} \mathbf{a}_\phi = -0.9487 \mathbf{a}_\rho - 6.008 \mathbf{a}_\phi\end{aligned}$$

Similarly, in the spherical system

$$\begin{bmatrix} A_r \\ A_\theta \\ A_\phi \end{bmatrix} = \begin{bmatrix} \sin \theta \cos \phi & \sin \theta \sin \phi & \cos \theta \\ \cos \theta \cos \phi & \cos \theta \sin \phi & -\sin \theta \\ -\sin \phi & \cos \phi & 0 \end{bmatrix} \begin{bmatrix} y \\ x + z \\ 0 \end{bmatrix}$$

or

$$A_r = y \sin \theta \cos \phi + (x + z) \sin \theta \sin \phi$$

$$A_\theta = y \cos \theta \cos \phi + (x + z) \cos \theta \sin \phi$$

$$A_\phi = -y \sin \phi + (x + z) \cos \phi$$

But $x = r \sin \theta \cos \phi$, $y = r \sin \theta \sin \phi$, and $z = r \cos \theta$. Substituting these yields

$$\begin{aligned}\mathbf{A} &= (A_r, A_\theta, A_\phi) \\ &= r[\sin^2 \theta \cos \phi \sin \phi + (\sin \theta \cos \phi + \cos \theta) \sin \theta \sin \phi] \mathbf{a}_r \\ &\quad + r[\sin \theta \cos \theta \sin \phi \cos \phi + (\sin \theta \cos \phi + \cos \theta) \cos \theta \sin \phi] \mathbf{a}_\theta \\ &\quad + r[-\sin \theta \sin^2 \phi + (\sin \theta \cos \phi + \cos \theta) \cos \phi] \mathbf{a}_\phi\end{aligned}$$

At P

$$r = 7, \quad \tan \phi = \frac{6}{-2}, \quad \tan \theta = \frac{\sqrt{40}}{3}$$

Hence,

$$\begin{aligned}\cos \phi &= \frac{-2}{\sqrt{40}}, \quad \sin \phi = \frac{6}{\sqrt{40}}, \quad \cos \theta = \frac{3}{7}, \quad \sin \theta = \frac{\sqrt{40}}{7} \\ \mathbf{A} &= 7 \cdot \left[\frac{40}{49} \cdot \frac{-2}{\sqrt{40}} \cdot \frac{6}{\sqrt{40}} + \left(\frac{\sqrt{40}}{7} \cdot \frac{-2}{\sqrt{40}} + \frac{3}{7} \right) \cdot \frac{\sqrt{40}}{7} \cdot \frac{6}{\sqrt{40}} \right] \mathbf{a}_r \\ &\quad + 7 \cdot \left[\frac{\sqrt{40}}{7} \cdot \frac{3}{7} \cdot \frac{6}{\sqrt{40}} \cdot \frac{-2}{\sqrt{40}} + \left(\frac{\sqrt{40}}{7} \cdot \frac{-2}{\sqrt{40}} + \frac{3}{7} \right) \cdot \frac{3}{7} \cdot \frac{6}{\sqrt{40}} \right] \mathbf{a}_\theta \\ &\quad + 7 \cdot \left[\frac{-\sqrt{40}}{7} \cdot \frac{36}{40} + \left(\frac{\sqrt{40}}{7} \cdot \frac{-2}{\sqrt{40}} + \frac{3}{7} \right) \cdot \frac{-2}{\sqrt{40}} \right] \mathbf{a}_\phi \\ &= \frac{-6}{7} \mathbf{a}_r - \frac{18}{7\sqrt{40}} \mathbf{a}_\theta - \frac{38}{\sqrt{40}} \mathbf{a}_\phi \\ &= -0.8571 \mathbf{a}_r - 0.4066 \mathbf{a}_\theta - 6.008 \mathbf{a}_\phi\end{aligned}$$

Note that $|\mathbf{A}|$ is the same in the three systems; that is,

$$|\mathbf{A}(x, y, z)| = |\mathbf{A}(\rho, \phi, z)| = |\mathbf{A}(r, \theta, \phi)| = 6.083$$

PRACTICE EXERCISE 2.1

(a) Convert points $P(1, 3, 5)$, $T(0, -4, 3)$, and $S(-3, -4, -10)$ from Cartesian to cylindrical and spherical coordinates.

(b) Transform vector

$$\mathbf{Q} = \frac{\sqrt{x^2 + y^2} \mathbf{a}_x}{\sqrt{x^2 + y^2 + z^2}} - \frac{yz \mathbf{a}_z}{\sqrt{x^2 + y^2 + z^2}}$$

to cylindrical and spherical coordinates.

(c) Evaluate \mathbf{Q} at T in the three coordinate systems.

Answer: (a) $P(3.162, 71.56^\circ, 5)$, $P(5.916, 32.31^\circ, 71.56^\circ)$, $T(4, 270^\circ, 3)$,
 $T(5, 53.13^\circ, 270^\circ)$, $S(5, 233.1^\circ, -10)$, $S(11.18, 153.43^\circ, 233.1^\circ)$

(b) $\frac{\rho}{\sqrt{\rho^2 + z^2}} (\cos \phi \mathbf{a}_\rho - \sin \phi \mathbf{a}_\phi - z \sin \phi \mathbf{a}_z)$, $\sin \theta (\sin \theta \cos \phi -$
 $r \cos^2 \theta \sin \phi) \mathbf{a}_r + \sin \theta \cos \theta (\cos \phi + r \sin \theta \sin \phi) \mathbf{a}_\theta - \sin \theta \sin \phi \mathbf{a}_\phi$

(c) $0.8\mathbf{a}_x + 2.4\mathbf{a}_z$, $0.8\mathbf{a}_\phi + 2.4\mathbf{a}_z$, $1.44\mathbf{a}_r - 1.92\mathbf{a}_\theta + 0.8\mathbf{a}_\phi$.

(a) At P(1,3,5), $x = 1$, $y = 3$, $z = 5$,
 $\rho = \sqrt{x^2 + y^2} = \sqrt{10}$, $z = 5$, $\phi = \tan^{-1} y/x = \tan^{-1} 3 = 71.6$
 $P(\rho, \phi, z) = P(\sqrt{10}, \tan^{-1} 3, 5) = \underline{P(3.162, 71.6^\circ, 5)}$

Spherical system:

$$r = \sqrt{x^2 + y^2 + z^2} = \sqrt{35} = 5.916$$

$$\theta = \tan^{-1} \sqrt{x^2 + y^2} / z = \tan^{-1} \sqrt{10} / 5 = \tan^{-1} 0.6325 = 32.31^\circ$$

$$P(r, \theta, \phi) = \underline{P(5.916, 32.31^\circ, 71.56^\circ)}$$

At T(0,-4,3), $x = 0$, $y = -4$, $z = 3$;
 $\rho = \sqrt{x^2 + y^2} = 4$, $z = 3$, $\phi = \tan^{-1} y/x = \tan^{-1} -4/0 = 270^\circ$
 $T(\rho, \phi, z) = \underline{T(4, 270^\circ, 3)}$.

Spherical system:

$$r = \sqrt{x^2 + y^2 + z^2} = 5$$

$$\theta = \tan^{-1} \rho / z = \tan^{-1} 4/3 = 53.13^\circ$$

$$T(r, \theta, \phi) = \underline{T(5, 53.13^\circ, 270^\circ)}$$

At S(-3-4-10), $x = -3$, $y = -4$, $z = -10$;
 $\rho = \sqrt{x^2 + y^2} = 5$, $\phi = \tan^{-1} -4/-3 = 233.1^\circ$
 $S(\rho, \phi, z) = \underline{S(5, 233.1^\circ, -10)}$.

Spherical system:

$$r = \sqrt{x^2 + y^2 + z^2} = 5\sqrt{5} = 11.18$$

$$\theta = \tan^{-1} \rho / z = \tan^{-1} \frac{5}{-10} = 153.43^\circ$$

$$S(r, \theta, \phi) = \underline{S(11.18, 153.43^\circ, 233.1^\circ)}$$

(b) In Cylindrical system, $\rho = \sqrt{x^2 + y^2}$; $yz = z\rho \sin \phi$,
 $Q_x = \frac{\rho}{\sqrt{\rho^2 + z^2}}$; $Q_y = 0$; $Q_z = -\frac{z\rho \sin \phi}{\sqrt{\rho^2 + z^2}}$;

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$$\begin{bmatrix} Q_\rho \\ Q_\phi \\ Q_z \end{bmatrix} = \begin{bmatrix} \cos \phi & \sin \phi & 0 \\ -\sin \phi & \cos \phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} Q_x \\ 0 \\ Q_z \end{bmatrix};$$

$$Q_\rho = Q_x \cos \phi = \frac{\rho \cos \phi}{\sqrt{\rho^2 + z^2}}, \quad Q_\phi = -Q_x \sin \phi = \frac{-\rho \sin \phi}{\sqrt{\rho^2 + z^2}}$$

Hence,

$$\underline{Q} = \frac{\rho}{\sqrt{\rho^2 + z^2}} (\cos \phi \underline{a}_\rho - \sin \phi \underline{a}_\phi - z \sin \phi \underline{a}_z).$$

In Spherical coordinates:

$$Q_x = \frac{r \sin \theta}{r} = \sin \theta;$$

$$Q_z = -r \sin \phi \sin \theta r \cos \theta \frac{1}{r} = -r \sin \theta \cos \theta \sin \phi.$$

$$\begin{bmatrix} Q_r \\ Q_\theta \\ Q_\phi \end{bmatrix} = \begin{bmatrix} \sin \theta \cos \phi & \sin \theta \sin \phi & \cos \theta \\ \cos \theta \cos \phi & \cos \theta \sin \phi & -\sin \theta \\ -\sin \phi & \cos \phi & 0 \end{bmatrix} \begin{bmatrix} Q_x \\ 0 \\ Q_z \end{bmatrix};$$

$$Q_r = Q_x \sin \theta \cos \phi + Q_z \cos \theta = \sin^2 \theta \cos \phi - r \sin \theta \cos^2 \theta \sin \phi.$$

$$Q_\theta = Q_x \cos \theta \cos \phi - Q_z \sin \theta = \sin \theta \cos \theta \cos \phi + r \sin^2 \theta \cos \theta \sin \phi.$$

$$Q_\phi = -Q_x \sin \phi = -\sin \theta \sin \phi.$$

$$\therefore \mathbf{Q} = \sin \theta (\sin \theta \cos \phi - r \cos^2 \theta \sin \phi) \mathbf{a}_r + \sin \theta \cos \theta (\cos \phi + r \sin \theta \sin \phi) \mathbf{a}_\theta - \sin \theta \sin \phi \mathbf{a}_\phi.$$

At T :

$$\bar{Q}(x, y, z) = \frac{4}{5} \bar{a}_x + \frac{12}{5} \bar{a}_z = 0.8 \bar{a}_x + 2.4 \bar{a}_z;$$

$$\bar{Q}(\rho, \phi, z) = \frac{4}{5} (\cos 270^\circ \bar{a}_\rho - \sin 270^\circ \bar{a}_\phi - 3 \sin 270^\circ \bar{a}_z)$$

$$= 0.8 \bar{a}_\theta + 2.4 \bar{a}_z;$$

$$\begin{aligned} \bar{Q}(r, \theta, \phi) &= \frac{4}{5} (0 - \frac{45}{25} (-1)) \bar{a}_r + \frac{4}{5} (\frac{3}{5}) (0 + \frac{20}{5} (-1)) \bar{a}_\theta - \frac{4}{5} (-1) \bar{a}_\phi \\ &= \frac{36}{25} \bar{a}_r - \frac{48}{25} \bar{a}_\theta + \frac{4}{5} \bar{a}_\phi = \underline{\underline{1.44 \bar{a}_r - 1.92 \bar{a}_\theta + 0.8 \bar{a}_\phi}}; \end{aligned}$$

Note, that the magnitude of vector $\mathbf{Q} = 2.53$ in all 3 cases above.

2.1 Express the following points in Cartesian coordinates:

- (a) $P(1, 60^\circ, 2)$
- (b) $Q(2, 90^\circ, -4)$
- (c) $R(, 45^\circ, 210^\circ)$
- (d) $T(4, \pi/2, \pi/6)$

2.2 Express the following points in cylindrical and spherical coordinates:

- (a) $P(1, -4, -3)$
- (b) $Q(3, 0, 5)$
- (c) $R(-2, 6, 0)$

2.3 (a) If $V = xz - xy + yz$, express V in cylindrical coordinates.

(b) If $U = x^2 + 2y^2 + 3z^2$, express U in spherical coordinates.

2.4 Transform the following vectors to cylindrical and spherical coordinates:

- (a) $\mathbf{D} = (x + z) \mathbf{a}_y$
- (b) $\mathbf{E} = (y^2 - x^2) \mathbf{a}_x + xyz \mathbf{a}_y + (x^2 - z^2) \mathbf{a}_z$

2.5 Convert the following vectors to cylindrical and spherical systems:

$$(a) \mathbf{F} = \frac{x\mathbf{a}_x + y\mathbf{a}_y + 4\mathbf{a}_z}{\sqrt{x^2 + y^2 + z^2}}$$

$$(b) \mathbf{G} = (x^2 + y^2) \left[\frac{x\mathbf{a}_x}{\sqrt{x^2 + y^2 + z^2}} + \frac{y\mathbf{a}_y}{\sqrt{x^2 + y^2 + z^2}} + \frac{z\mathbf{a}_z}{\sqrt{x^2 + y^2 + z^2}} \right]$$

2.6 Express the following vectors in Cartesian coordinates:

$$(a) \mathbf{A} = \rho(z^2 + 1)\mathbf{a}_\rho - \rho z \cos \phi \mathbf{a}_\phi$$

$$(b) \mathbf{B} = 2r \sin \theta \cos \phi \mathbf{a}_r + r \cos \theta \cos \theta \mathbf{a}_\theta - r \sin \phi \mathbf{a}_\phi$$

2.7 Convert the following vectors to Cartesian coordinates:

$$(a) \mathbf{C} = z \sin \phi \mathbf{a}_\rho - \rho \cos \phi \mathbf{a}_\phi + 2\rho z \mathbf{a}_z$$

$$(b) \mathbf{D} = \frac{\sin \theta}{r^2} \mathbf{a}_r + \frac{\cos \theta}{r^2} \mathbf{a}_\theta$$

2.8 Prove the following:

$$(a) \mathbf{a}_x \cdot \mathbf{a}_\rho = \cos \phi$$

$$\mathbf{a}_x \cdot \mathbf{a}_\phi = -\sin \phi$$

$$\mathbf{a}_y \cdot \mathbf{a}_\rho = \sin \phi$$

$$\mathbf{a}_y \cdot \mathbf{a}_\phi = \cos \phi$$

$$(b) \mathbf{a}_x \cdot \mathbf{a}_r = \sin \theta \cos \phi$$

$$\mathbf{a}_x \cdot \mathbf{a}_\theta = \cos \theta \cos \phi$$

$$\mathbf{a}_y \cdot \mathbf{a}_r = \sin \theta \sin \phi$$

$$\mathbf{a}_y \cdot \mathbf{a}_\theta = \cos \theta \sin \phi$$

$$\mathbf{a}_z \cdot \mathbf{a}_r = \cos \theta$$

$$\mathbf{a}_z \cdot \mathbf{a}_\theta = -\sin \theta$$

Solutions

2.2

- (a) Given $P(1, -4, -3)$, convert to cylindrical and spherical values;

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$$\rho = \sqrt{x^2 + y^2} = \sqrt{1^2 + (-4)^2} = \sqrt{17} = 4.123.$$

$$\phi = \tan^{-1} \frac{y}{x} = \tan^{-1} \frac{-4}{1} = 284.04^\circ.$$

$$\therefore P(\rho, \phi, z) = \underline{\underline{(4.123, 284.04^\circ, -3)}}.$$

Spherical:

$$r = \sqrt{x^2 + y^2 + z^2} = \sqrt{1 + 16 + 9} = 5.099.$$

$$\theta = \tan^{-1} \frac{\rho}{z} = \tan^{-1} \frac{4.123}{-3} = 126.04^\circ.$$

$$P(r, \theta, \phi) = \underline{\underline{P(5.099, 126.04^\circ, 284.04^\circ)}}.$$

$$(b) \quad \rho = 3, \quad \phi = \tan^{-1} \frac{y}{x} = \tan^{-1} \frac{0}{3} = 0^\circ$$

$$Q(\rho, \phi, z) = \underline{\underline{Q(3, 0^\circ, 5)}}$$

$$r = \sqrt{9 + 0 + 25} = 5.831, \quad \theta = \tan^{-1} \frac{\rho}{z} = \tan^{-1} \frac{3}{5} = 30.96^\circ$$

$$Q(r, \theta, \phi) = \underline{\underline{Q(5.831, 30.96^\circ, 0^\circ)}}$$

$$(c) \quad \rho = \sqrt{4 + 36} = 6.325, \quad \phi = \tan^{-1} \frac{6}{-2} = 108.4^\circ$$

$$R(\rho, \phi, z) = \underline{\underline{R(6.325, 108.4^\circ, 0)}}$$

$$r = \rho = 6.325, \quad \theta = \tan^{-1} \frac{\rho}{z} = \tan^{-1} \frac{6.325}{0} = 90^\circ$$

$$R(r, \theta, \phi) = \underline{\underline{R(6.325, 90^\circ, 108.4^\circ)}}$$

2.3

(a)

$$x = \rho \cos \phi, \quad y = \rho \sin \phi,$$

$$V = \underline{\underline{\rho z \cos \phi - \rho^2 \sin \phi \cos \phi + \rho z \sin \phi}}$$

(b)

$$\begin{aligned} U &= x^2 + y^2 + z^2 + y^2 + 2z^2 \\ &= r^2 + r^2 \sin^2 \theta \sin^2 \phi + 2r^2 \cos^2 \theta \\ &= \underline{\underline{r^2 [1 + \sin^2 \theta \sin^2 \phi + 2 \cos^2 \theta]}} \end{aligned}$$

2.4

(a)

$$\begin{bmatrix} D_\rho \\ D_\phi \\ D_z \end{bmatrix} = \begin{bmatrix} \cos \phi & \sin \phi & 0 \\ -\sin \phi & \cos \phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ x+z \\ 0 \end{bmatrix}$$

$$D_\rho = (x+z) \sin \phi = (\rho \cos \phi + z) \sin \phi$$

$$D_\phi = (x+z) \cos \phi = (\rho \cos \phi + z) \cos \phi$$

$$\bar{D} = \underline{(\rho \cos \phi + z)[\sin \phi \bar{a}_\rho + \cos \phi \bar{a}_\phi]}$$

Spherical: Since $D_x = D_z = 0$, we may leave out the first and third column of the transformation matrix. Thus,

$$\begin{bmatrix} D_r \\ D_\theta \\ D_\phi \end{bmatrix} = \begin{bmatrix} \dots & \sin \theta \sin \phi & \dots \\ \dots & \cos \theta \sin \phi & \dots \\ \dots & \cos \phi & \dots \end{bmatrix} \begin{bmatrix} 0 \\ x+z \\ 0 \end{bmatrix}$$

$$D_r = (x+z) \sin \theta \sin \phi = r(\sin \theta \cos \phi + \cos \theta) \sin \theta \sin \phi.$$

$$D_\theta = (x+z) \cos \theta \sin \phi = r(\sin \theta \cos \phi + \cos \theta) \cos \theta \sin \phi.$$

$$D_\phi = (x+z) \cos \phi = r(\sin \theta \cos \phi + \cos \theta) \cos \phi.$$

$$\bar{D} = \underline{r(\sin \theta \cos \phi + \cos \theta)[\sin \theta \sin \phi \bar{a}_r + \cos \theta \sin \phi \bar{a}_\theta + \cos \phi \bar{a}_\phi]}.$$

(b) Cylindrical:

$$\begin{bmatrix} E_\rho \\ E_\phi \\ E_z \end{bmatrix} = \begin{bmatrix} \cos \phi & \sin \phi & 0 \\ -\sin \phi & \cos \phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} y^2 - x^2 \\ xyz \\ x^2 - z^2 \end{bmatrix}$$

$$\begin{aligned} E_\rho &= (y^2 - x^2) \cos \phi + xyz \sin \phi \\ &= \rho^2 (\sin^2 \phi - \cos^2 \phi) \cos \phi + \rho^2 z \cos \phi \sin^2 \phi \\ &= -\rho^2 \cos 2\phi \cos \phi + \rho^2 z \sin^2 \phi \cos \phi. \end{aligned}$$

$$\begin{aligned} E_\phi &= -(y^2 - x^2) \sin \phi + xyz \cos \phi \\ &= \rho^2 \cos 2\phi \sin \phi + \rho^2 z \sin \phi \cos^2 \phi. \end{aligned}$$

$$E_z = x^2 - z^2 = \rho^2 \cos^2 \phi - z^2.$$

$$\underline{\underline{\vec{E} = \rho^2 \cos \phi (z \sin^2 \phi - \cos 2\phi) \bar{a}_\rho + \rho^2 \sin \phi (z \cos^2 \phi + \cos 2\phi) \bar{a}_\phi + (\rho^2 \cos^2 \phi - z^2) \bar{a}_z.}}$$

In spherical:

$$\begin{bmatrix} E_r \\ E_\theta \\ E_\phi \end{bmatrix} = \begin{bmatrix} \sin \theta \cos \phi & \sin \theta \sin \phi & \cos \theta \\ \cos \theta \cos \phi & \cos \theta \sin \phi & -\sin \theta \\ -\sin \phi & \cos \phi & 0 \end{bmatrix} \begin{bmatrix} y^2 - x^2 \\ xyz \\ x^2 - z^2 \end{bmatrix}$$

$$E_r = (y^2 - x^2) \sin \theta \cos \phi + xyz \sin \theta \sin \phi + (x^2 - z^2) \cos \theta;$$

$$\text{but } x = r \sin \theta \cos \phi, \quad y = r \sin \theta \sin \phi, \quad z = r \cos \theta;$$

$$\begin{aligned} E_r &= r^2 \sin^3 \theta (\sin^2 \phi - \cos^2 \phi) \cos \phi + r^3 \sin^3 \theta \cos \theta \sin^2 \phi \cos \phi \\ &\quad + r^2 (\sin^2 \theta \cos^2 \phi - \cos^2 \theta) \cos \theta; \end{aligned}$$

$$\begin{aligned}
E_\phi &= (y^2 - x^2) \cos \theta \cos \phi + xyz \cos \theta \sin \phi - (x^2 - z^2) \sin \theta; \\
&= -r^2 \sin^2 \theta \cos 2\phi \cos \theta \cos \phi + r^3 \sin^2 \theta \cos^2 \theta \sin^2 \phi \cos \phi - r^2 (\sin^2 \theta \cos^2 \phi - \cos^2 \theta) \sin \theta; \\
E_\psi &= (x^2 - y^2) \sin \phi + xyz \cos \phi \\
&= r^2 \sin^2 \theta \cos 2\phi \sin \phi + r^3 \sin^2 \theta \cos^2 \phi \sin \phi \cos \theta; \\
\vec{E} &= [-r^2 \sin^3 \theta \cos 2\phi \cos \phi + r^3 \sin^3 \theta \cos \theta \sin^2 \phi \cos \phi + r^2 (\sin^2 \theta \cos^2 \phi - \cos^2 \theta) \cos \theta] \bar{a}_r + \\
&[-r^2 \sin^2 \theta \cos 2\phi \cos \theta \cos \phi + r^3 \sin^2 \theta \cos^2 \theta \sin^2 \phi \cos \phi - r^2 \sin \theta (\sin^2 \theta \cos^2 \phi - \cos^2 \theta)] \bar{a}_\theta + \\
&+ \underline{\underline{[r^2 \sin^2 \theta \cos 2\phi \sin \phi + r^3 \sin^2 \theta \cos^2 \phi \sin \phi \cos \theta] \bar{a}_\phi}}
\end{aligned}$$

2.8

$$\begin{aligned} \text{(a)} \quad \vec{a}_x \cdot \vec{a}_\rho &= (\cos\phi \vec{a}_\rho - \sin\phi \vec{a}_\phi) \cdot \vec{a}_\rho = \cos\phi \\ \vec{a}_x \cdot \vec{a}_\phi &= (\cos\phi \vec{a}_\rho - \sin\phi \vec{a}_\phi) \cdot \vec{a}_\phi = -\sin\phi \end{aligned}$$

$$\begin{aligned} \vec{a}_y \cdot \vec{a}_\rho &= (\sin\phi \vec{a}_\rho + \cos\phi \vec{a}_\phi) \cdot \vec{a}_\rho = \sin\phi \\ \vec{a}_y \cdot \vec{a}_\phi &= (\sin\phi \vec{a}_\rho + \cos\phi \vec{a}_\phi) \cdot \vec{a}_\phi = \cos\phi \end{aligned}$$

(b) Since \vec{a}_r , \vec{a}_ϕ , and \vec{a}_z are mutually orthogonal,

$$\vec{a}_z \cdot \vec{a}_z = 1, \quad \vec{a}_z \cdot \vec{a}_r = 0 = \vec{a}_z \cdot \vec{a}_\phi.$$

Also, $\vec{a}_x \cdot \vec{a}_z = 0 = \vec{a}_y \cdot \vec{a}_z$. Hence

$$\begin{bmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \vec{a}_x \cdot \vec{a}_r & \vec{a}_x \cdot \vec{a}_\phi & \vec{a}_x \cdot \vec{a}_z \\ \vec{a}_y \cdot \vec{a}_r & \vec{a}_y \cdot \vec{a}_\phi & \vec{a}_y \cdot \vec{a}_z \\ \vec{a}_z \cdot \vec{a}_r & \vec{a}_z \cdot \vec{a}_\phi & \vec{a}_z \cdot \vec{a}_z \end{bmatrix}$$

(c) In spherical system,

$$\vec{a}_x = \sin\theta \cos\phi \vec{a}_r + \cos\theta \cos\phi \vec{a}_\theta - \sin\phi \vec{a}_\phi$$

$$\vec{a}_y = \sin\theta \sin\phi \vec{a}_r + \cos\theta \sin\phi \vec{a}_\theta + \cos\phi \vec{a}_\phi$$

$$\vec{a}_z = \cos\theta \vec{a}_r - \sin\theta \vec{a}_\theta.$$

$$\text{Hence } \vec{a}_x \cdot \vec{a}_r = \sin\theta \cos\phi,$$

$$\vec{a}_x \cdot \vec{a}_\theta = \cos\theta \cos\phi,$$

$$\vec{a}_y \cdot \vec{a}_r = \sin\theta \sin\phi,$$

$$\vec{a}_y \cdot \vec{a}_\theta = \cos\theta \sin\phi,$$

$$\vec{a}_z \cdot \vec{a}_r = \cos\theta,$$

$$\vec{a}_z \cdot \vec{a}_\theta = -\sin\theta.$$