Properties of Z-transform

(1). Linearity Property

if $x_1(n) \xleftrightarrow{ZT} X_1(z)$ with $ROC = R_1$

and $x_2(n) \xleftrightarrow{ZT} X_2(z)$ with $ROC = R_2$

Then $a x_1(n) + b x_2(n) \xleftrightarrow{ZT} a X_1(z) + b X_2(z)$ with $ROC = R_1 \cap R_2$

Proof:

$$Z\left[a x_1(n)+b x_2(n)\right] = \sum_{n=-\infty}^{\infty}\left[a x_1(n)+b x_2(n)\right] z^{-n}$$

$$= \sum_{n=-\infty}^{\infty} a x_1(n) z^{-n} + \sum_{n=-\infty}^{\infty} b x_2(n) z^{-n}$$

$$= a\sum_{n=-\infty}^{\infty} x_1(n) z^{-n} + b\sum_{n=-\infty}^{\infty} x_2(n) z^{-n}$$

$$= a X_1(z) + b X_2(z) \; ; \quad \text{ROC}; R_1 \cap R_2$$

$$\therefore a x_1(n) + b x_2(n) \xleftrightarrow{ZT} a X_1(z) + b X_2(z)$$

(2). Time shifting Property

if $x(n) \xleftrightarrow{ZT} X(Z)$ with ROC = R [with zero initial condition]

then $x(n-m) \xleftrightarrow{ZT} Z^{-m} X(Z)$ with ROC = R except for the possible addition or deletion of the origin or infinity.

Proof:

$$Z[x(n-m)] = \sum_{n=-\infty}^{\infty} x(n-m) Z^{-n}.$$

Let $n-m = p$, then $n = m+p$.

$$\therefore \quad Z[x(n-m)] = \sum_{p=-\infty}^{\infty} x(p) Z^{-(m+p)}$$

$$= Z^{-m} \sum_{p=-\infty}^{\infty} x(p) Z^{-p}$$

$$= Z^{-m} . X(z)$$

$$\therefore \quad x(n-m) \xleftrightarrow{ZT} Z^{-m} X(z)$$

$$\& \quad x(n+m) \xleftrightarrow{ZT} Z^{m} X(z)$$

(3) Multiplication by an Exponential sequence Property

if $x(n) \xleftrightarrow{ZT} X(z)$ with $ROC = R$

then $a^n x(n) \xleftrightarrow{ZT} X\left(\frac{z}{a}\right)$ with $ROC = |a|R$, where a is a complex number.

Proof: $$Z\left[a^n x(n)\right] = \sum_{n=-\infty}^{\infty} a^n . x(n) . Z^{-n}$$

$$= \sum_{n=-\infty}^{\infty} x(n) . (aZ^{-1})^n$$

$$= \sum_{n=-\infty}^{\infty} x(n) \left(\frac{Z}{a}\right)^{-n} = X\left(\frac{Z}{a}\right)$$

$$\therefore a^n x(n) \xleftrightarrow{ZT} X\left(\frac{Z}{a}\right)$$

(4). Time Reversal Property

if $x(n) \xleftrightarrow{ZT} X(z)$ with ROC = R

then $x(-n) \xleftrightarrow{ZT} X\left(\frac{1}{z}\right)$ with ROC = $\frac{1}{R}$

Proof:

$$Z[x(-n)] = \sum_{n=-\infty}^{\infty} x(-n) z^{-n}$$

Let $p = -n$, then

$$Z[x(-n)] = \sum_{p=-\infty}^{\infty} x(p) z^{p}$$

$$= \sum_{p=-\infty}^{\infty} x(p) (z^{-1})^{-p}$$

$$= X(z^{-1}) = X\left(\frac{1}{z}\right)$$

$$\therefore\ x(-n) \xleftrightarrow{ZT} X(z^{-1}).$$

(5). Multiplication by n or Differentiation in Z-domain Property

if $x(n) \xleftrightarrow{ZT} X(z)$ with ROC = R

then $nx(n) \xleftrightarrow{ZT} -z \frac{d}{dz} X(z)$ with ROC = R

Proof: $$Z[n\,x(n)] = \sum_{n=-\infty}^{\infty} n\,x(n) z^{-n}$$

Now $$Z[x(n)] = \sum_{n=-\infty}^{\infty} x(n) z^{-n} = X(z)$$

$$\therefore \frac{d}{dz} X(z) = \sum_{n=-\infty}^{\infty} x(n) . \frac{d}{dz} . z^{-n}$$

$$= \sum_{n=-\infty}^{\infty} x(n) . (-n) . z^{-n-1}$$

$$\frac{d}{dz}X(z) = -z^{-1}\sum_{n=-\infty}^{\infty}[n.x(n)].z^{-n}$$

$$\Rightarrow \frac{d}{dz}X(z) = -z^{-1}\,Z[nx(n)]$$

$$\Rightarrow Z[nx(n)] = -z\frac{d}{dz}X(z).$$ Hence proved.

(6). <u>Conjugation Property</u>

if $x(n) \xleftrightarrow{ZT} X(z)$ with ROC = R

then $x^*(n) \xleftrightarrow{ZT} X^*(z^*)$ with ROC = R

Proof:

$$Z[x^*(n)] = \sum_{n=-\infty}^{\infty} x^*(n) z^{-n}$$

$$= \left[\sum_{n=-\infty}^{\infty} x(n) (z^*)^{-n}\right]^*$$

$$= [X(z^*)]^* = X^*(z^*)$$

$$\therefore \quad x^*(n) \xleftrightarrow{ZT} X^*(z^*)$$

(7). Convolution Property.

if $\quad x_1(n) \xleftrightarrow{ZT} X_1(z)$, with $ROC = R_1$

and $\quad x_2(n) \xleftrightarrow{ZT} X_2(z)$, with $ROC = R_2$

then $\quad x_1(n) * x_2(n) \xleftrightarrow{ZT} X_1(z)\, X_2(z)$, with $ROC = R_1 \cap R_2$

Proof: The linear convolution of $x_1(n)$ and $x_2(n)$ is given by,

$$x_1(n) * x_2(n) = \sum_{K=-\infty}^{\infty} x_1(K)\, x_2(n-K)$$

Let $x(n) = x_1(n) * x_2(n)$

then $$Z[x(n)] = X(z) = \sum_{n=-\infty}^{\infty} x(n) Z^{-n}$$

$$\therefore\ Z[x_1(n) * x_2(n)] = \sum_{n=-\infty}^{\infty} [x_1(n) * x_2(n)] Z^{-n}$$

$$\therefore\ Z[x_1(n) * x_2(n)] = \sum_{n=-\infty}^{\infty} [x_1(n) * x_2(n)] Z^{-n}$$

$$= \sum_{n=-\infty}^{\infty} \left[\sum_{K=-\infty}^{\infty} x_1(K) x_2(n-K)\right] Z^{-n}$$

$$= \sum_{n=-\infty}^{\infty} \sum_{K=-\infty}^{\infty} x_1(K) x_2(n-K) Z^{-(n-K)} Z^{-K}$$

On interchanging the order of summations,

$$X(Z) = \sum_{K=-\infty}^{\infty} x_1(K) Z^{-K} \sum_{n=-\infty}^{\infty} x_2(n-K) Z^{-(n-K)}$$

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$$X(z) = \sum_{k=-\infty}^{\infty} x_1(k) z^{-k} \sum_{n=-\infty}^{\infty} x_2(n-k) z^{-(n-k)}$$

Replacing $(n-k)$ by p in the second summation,

$$X(z) = \sum_{k=-\infty}^{\infty} x_1(k) z^{-k} \sum_{p=-\infty}^{\infty} x_2(p) z^{-p}$$

$$= X_1(z) X_2(z)$$

$$x_1(n) * x_2(n) \xleftrightarrow{ZT} X_1(z) X_2(z) \; ; \; \text{ROC} \; ; \; R_1 \cap R_2$$

8. Correlation Property

if $x_1(n) \xleftrightarrow{ZT} X_1(z)$ and $x_2(n) \xleftrightarrow{ZT} X_2(z)$

then $$R_{x_1 x_2}(n) = x_1(n) * x_2(-n) \xleftrightarrow{ZT} X_1(z)\, X_2(z^{-1})$$

Proof: We know that cross correlation between $x_1(n)$ and $x_2(n)$ is given by,

$$R_{x_1 x_2}(n) = \sum_{l=-\infty}^{\infty} x_1(l)\, x_2(l-n)$$

then $$Z\left[x_1(n) * x_2(-n)\right] = \sum_{n=-\infty}^{\infty}\left[\sum_{l=-\infty}^{\infty} x_1(l)\, x_2(l-n)\right] z^{-n}$$

Interchanging the order of summation,

$$Z\left[x_1(n) * x_2(-n)\right] = \sum_{l=-\infty}^{\infty} x_1(l)\left[\sum_{n=-\infty}^{\infty} x_2(l-n)\, z^{-n}\right]$$

Let $l - n = m$ in the second summation,

$$Z[x_1(n) * x_2(-n)] = \sum_{l=-\infty}^{\infty} x_1(l) \left[\sum_{m=-\infty}^{\infty} x_2(m) Z^{-(l-m)} \right]$$

$$= \left[\sum_{l=-\infty}^{\infty} x_1(l) Z^{-l} \right] \left[\sum_{m=-\infty}^{\infty} x_2(m) Z^{m} \right]$$

$$= \left[\sum_{l=-\infty} x_1(l) Z^{-l} \right] \left[\sum_{m=-\infty}^{\infty} x_2(m) (Z^{-1})^{-m} \right]$$

$$= X_1(z)\, X_2(z^{-1})$$

$$\therefore \quad R_{x_1 x_2}(n) = x_1(n) * x_2(-n) \xleftrightarrow{ZT} X_1(z)\, X_2(z^{-1}).$$

(9) Initial Value Theorem

if $x(n) \xleftrightarrow{ZT} X(z)$, where $x(n)$ is causal signal,

then $\underset{n \to 0}{Lt}\, x(n) = x(0) = \underset{z \to \infty}{Lt}\, X(z)$

Proof: Z-transform of a causal signal is given by

$$Z[x(n)] = X(z) = \sum_{n=0}^{\infty} x(n) z^{-n}$$

$$= x(0) + x(1) z^{-1} + x(2) z^{-2} + \cdots$$

$$= x(0) + \frac{x(1)}{z} + \frac{x(2)}{z^2} + \cdots$$

Taking the limit $z \to \infty$ on both sides,

$$\underset{z \to \infty}{Lt} X(z) = \underset{z \to \infty}{Lt} \left[x(0) + \frac{x(1)}{z} + \frac{x(2)}{z^2} + \cdots \right]$$

$$= x(0) + 0 + 0 + \cdots = x(0)$$