Time Domain Analysis

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Introduction

- In time-domain analysis the response of a dynamic system to an input is expressed as a function of time.

- It is possible to compute the time response of a system if the nature of input and the mathematical model of the system are known.

Introduction

The time response of a control system consists of two parts:

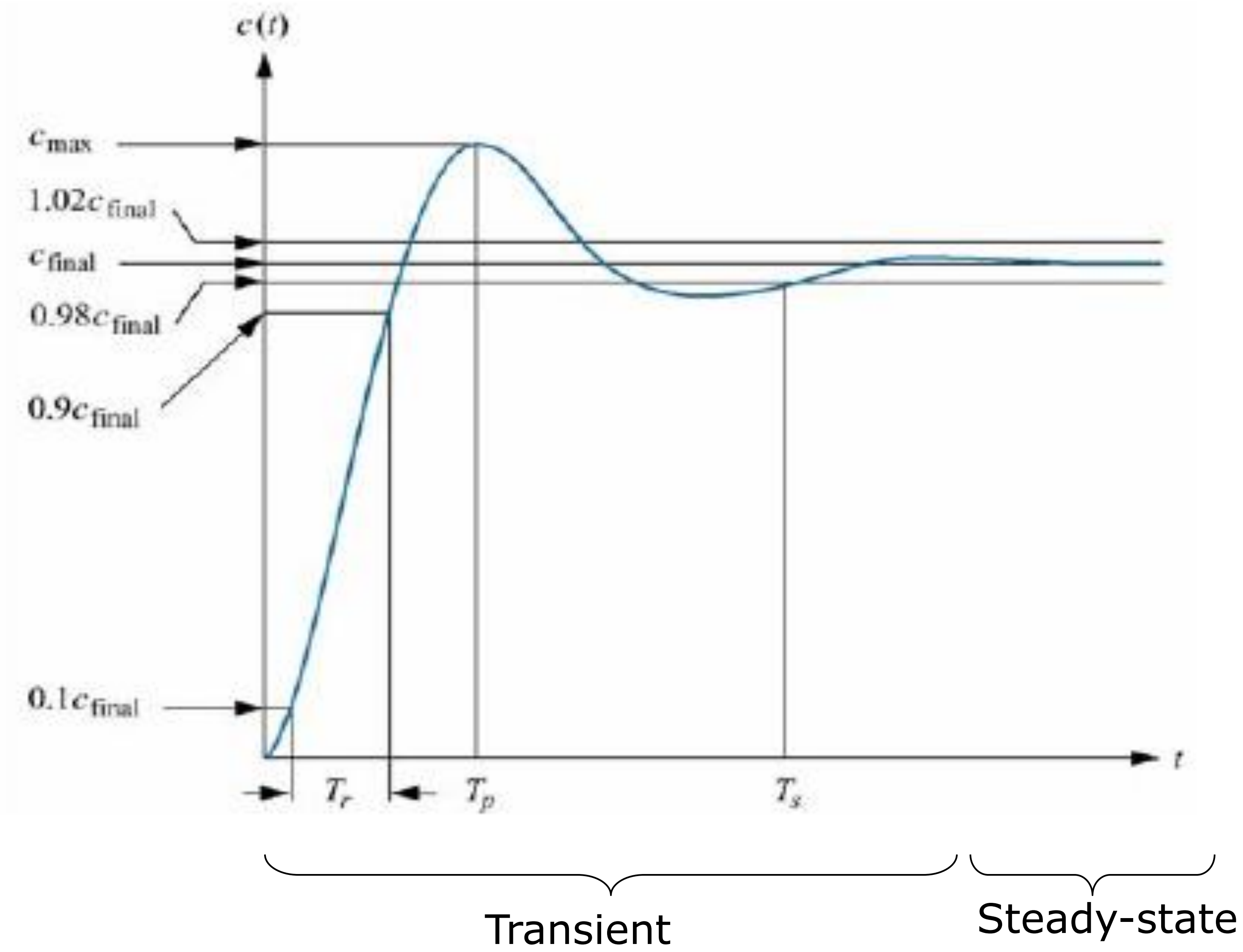
1. Transient response

   - from initial state to the final state – purpose of control systems is to provide a desired response.

2. Steady-state response

   - the manner in which the system output behaves as *t* approaches infinity – the error after the transient response has decayed, leaving only the continuous response.

Introduction



Standard Test Signals

- The characteristics of actual input signals are a sudden shock, a sudden change, a constant velocity, and constant acceleration.

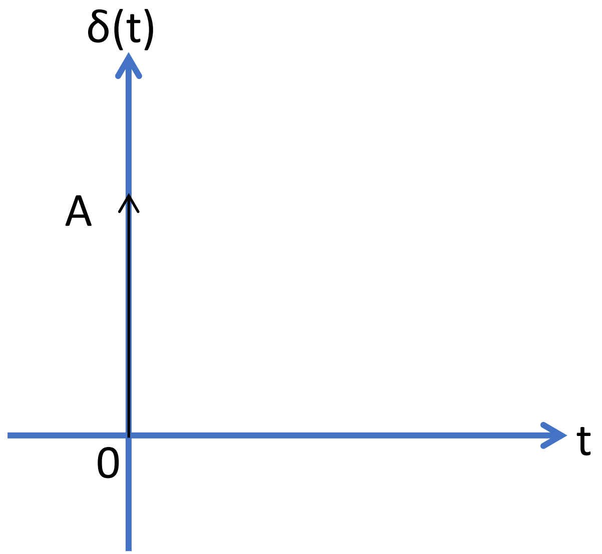
- The dynamic behavior of a system is therefore judged and compared under application of standard test signals – an impulse, a step, a constant velocity, and constant acceleration.

Standard Test Signals

- Impulse signal
  - The impulse signal imitate the sudden shock characteristic of actual input signal.

$$\delta(t) = \begin{cases} A & t = 0 \\ 0 & t \neq 0 \end{cases}$$

  - If A=1, the impulse signal is called unit impulse signal.

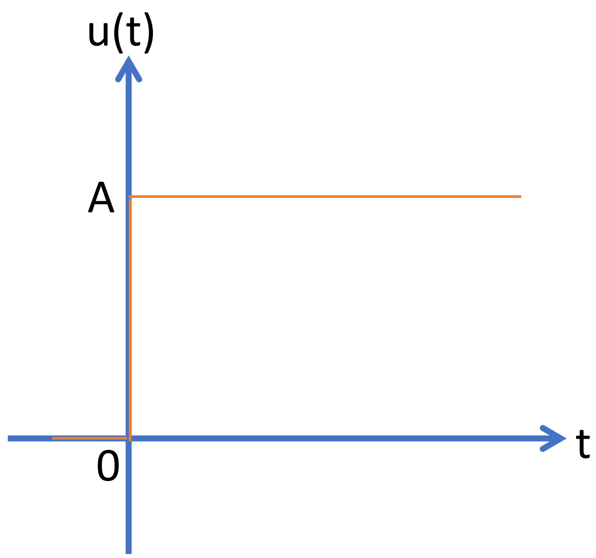


Standard Test Signals

- Step signal
  - The step signal imitate the sudden change characteristic of actual input signal.

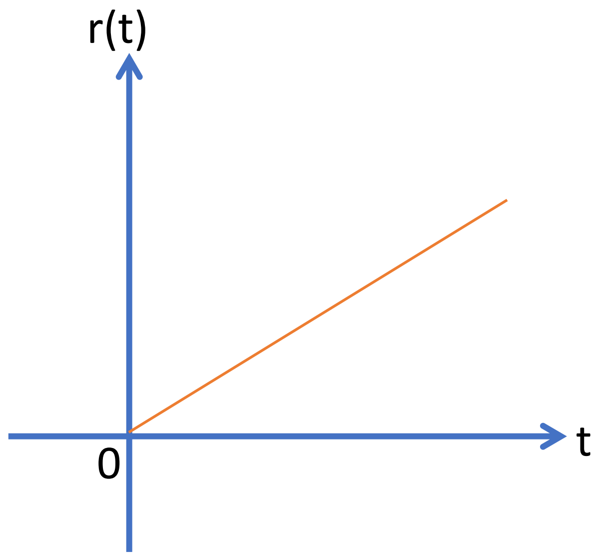
$$u(t) = \begin{cases} A & t \geq 0 \\ 0 & t < 0 \end{cases}$$

  - If A=1, the step signal is called unit step signal



Standard Test Signals

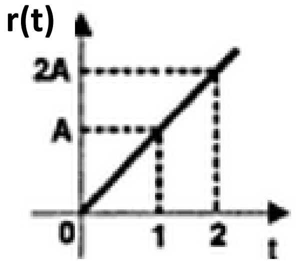
- Ramp signal
  - The ramp signal imitate the constant velocity characteristic of actual input signal.



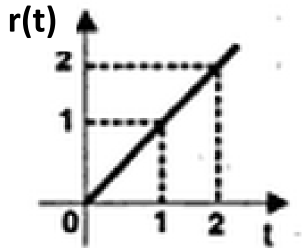
$$r(t) = \begin{cases} At & t \geq 0 \\ 0 & t < 0 \end{cases}$$

  - If *A=1*, the ramp signal is called unit ramp signal

ramp signal with slope A



unit ramp signal

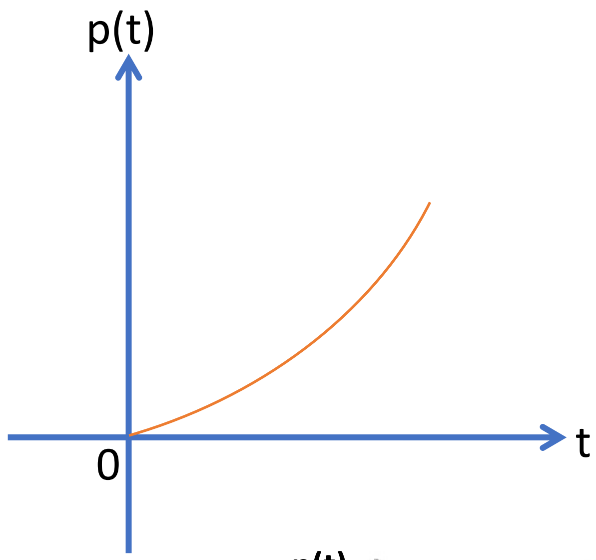


Standard Test Signals

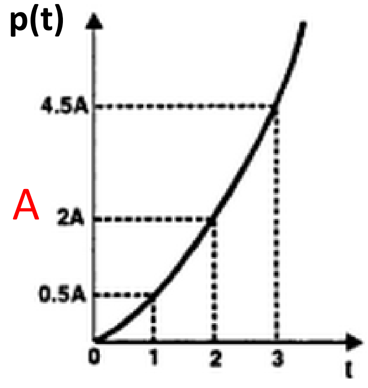
- Parabolic signal
  - The parabolic signal imitate the constant acceleration characteristic of actual input signal.

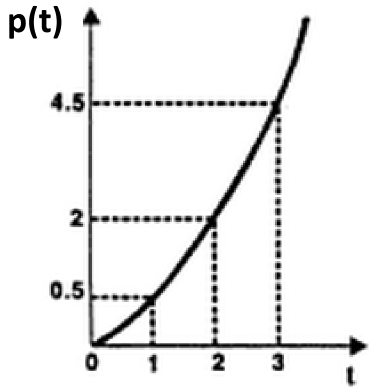
$$p(t) = \begin{cases} \dfrac{At^2}{2} & t \geq 0 \\ 0 & t < 0 \end{cases}$$

  - If $A=1$, the parabolic signal is called unit parabolic signal.



parabolic signal with slope A





Unit parabolic signal

Relation between standard Test Signals

- Impulse
- Step
- Ramp
- Parabolic

$$\delta(t) = \begin{cases} A & t = 0 \\ 0 & t \neq 0 \end{cases}$$

$$\int \qquad \frac{d}{dt}$$

$$u(t) = \begin{cases} A & t \geq 0 \\ 0 & t < 0 \end{cases}$$

$$\int \qquad \frac{d}{dt}$$

$$r(t) = \begin{cases} At & t \geq 0 \\ 0 & t < 0 \end{cases}$$

$$\int \qquad \frac{d}{dt}$$

$$p(t) = \begin{cases} \dfrac{At^2}{2} & t \geq 0 \\ 0 & t < 0 \end{cases}$$

Laplace Transform of Test Signals

- Impulse

$$\delta(t) = \begin{cases} A & t = 0 \\ 0 & t \neq 0 \end{cases}$$

$$L\{\delta(t)\} = \delta(s) = A$$

- Step

$$u(t) = \begin{cases} A & t \geq 0 \\ 0 & t < 0 \end{cases}$$

$$L\{u(t)\} = U(s) = \frac{A}{S}$$

Laplace Transform of Test Signals

- Ramp

$$r(t) = \begin{cases} At & t \geq 0 \\ 0 & t < 0 \end{cases}$$

$$L\{r(t)\} = R(s) = \frac{A}{s^2}$$

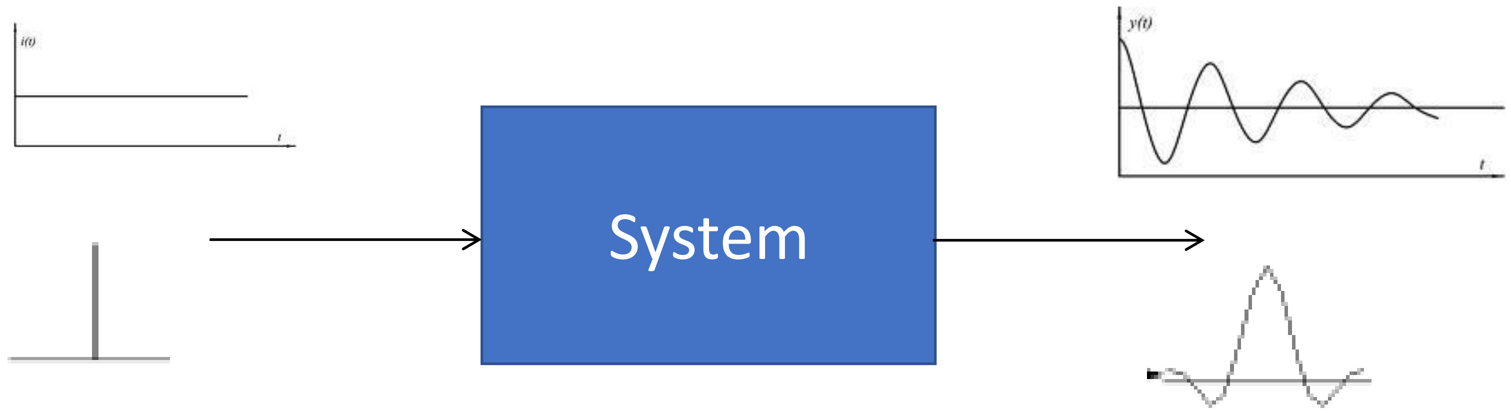
- Parabolic

$$p(t) = \begin{cases} \dfrac{At^2}{2} & t \geq 0 \\ 0 & t < 0 \end{cases}$$

$$L\{p(t)\} = P(s) = \frac{A}{S^3}$$

Time Response of Control Systems

- Time response of a dynamic system response to an input expressed as a function of time.



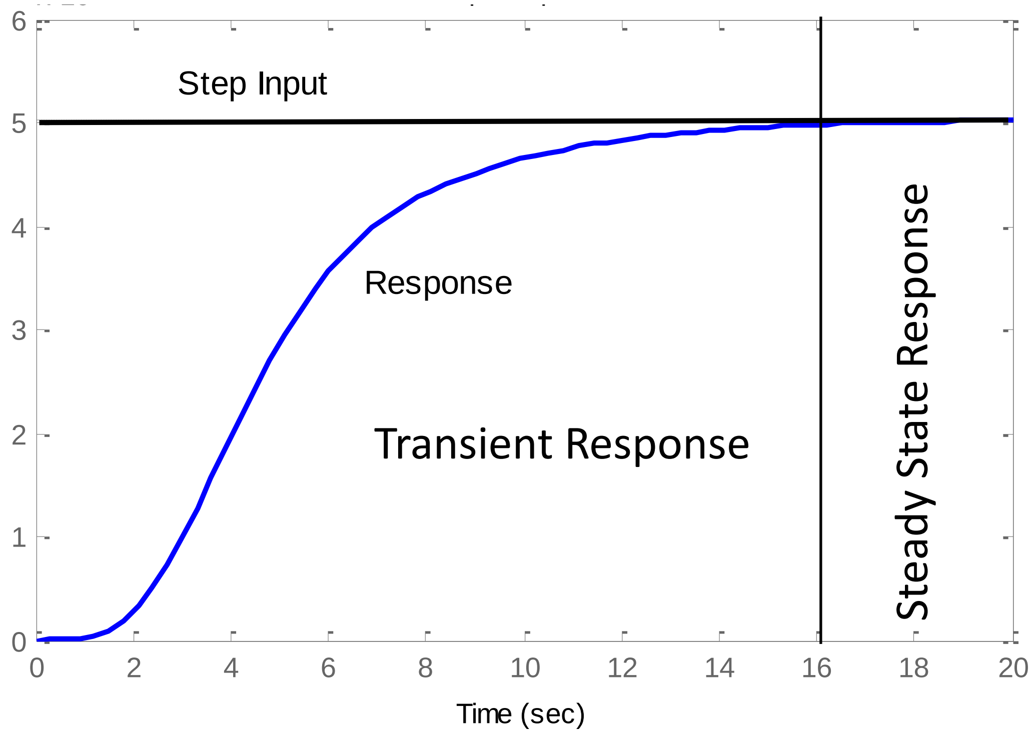
- The time response of any system has two components
    - Transient response
    - Steady-state response.

Time Response of Control Systems

- When the response of the system is changed from equilibrium it takes some time to settle down.

- This is called transient response.

- The response of the system after the transient response is called steady state response.



Time Response of Control Systems

- Transient response depend upon the system poles only and not on the type of input.

- It is therefore sufficient to analyze the transient response using a step input.

- The steady-state response depends on system dynamics and the input quantity.

- It is then examined using different test signals by final value theorem.

Introduction

- The first order system has only one pole.

$$\frac{C(s)}{R(s)} = \frac{K}{Ts+1}$$

- Where ***K*** is the D.C gain and ***T*** is the time constant of the system.

- Time constant is a measure of how quickly a 1^{st} order system responds to a unit step input.

- D.C Gain of the system is ratio between the input signal and the steady state value of output.

Introduction

- The first order system given below.

$$G(s) = \frac{10}{3s+1}$$

- D.C gain is 10 and time constant is 3 seconds.

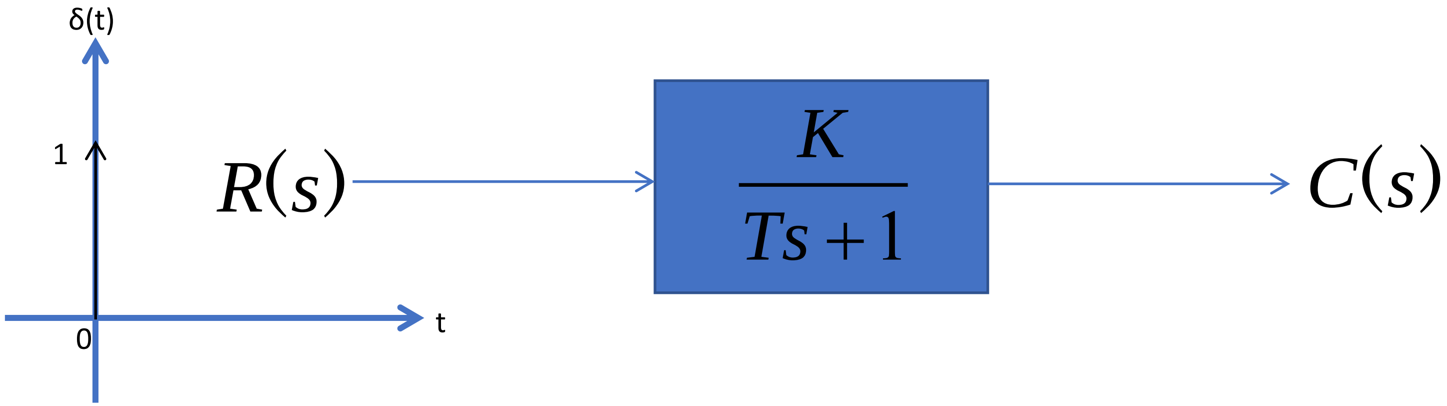
- For the following system

$$G(s) = \frac{3}{s+5} = \frac{3/5}{1/5s+1}$$

- D.C Gain of the system is 3/5 and time constant is 1/5 seconds.

Impulse Response of 1st Order System

- Consider the following 1st order system



$$R(s) = \delta(s) = 1$$

$$C(s) = \frac{K}{Ts+1}$$

Impulse Response of 1st Order System

$$C(s) = \frac{K}{Ts+1}$$

- Re-arrange following equation as

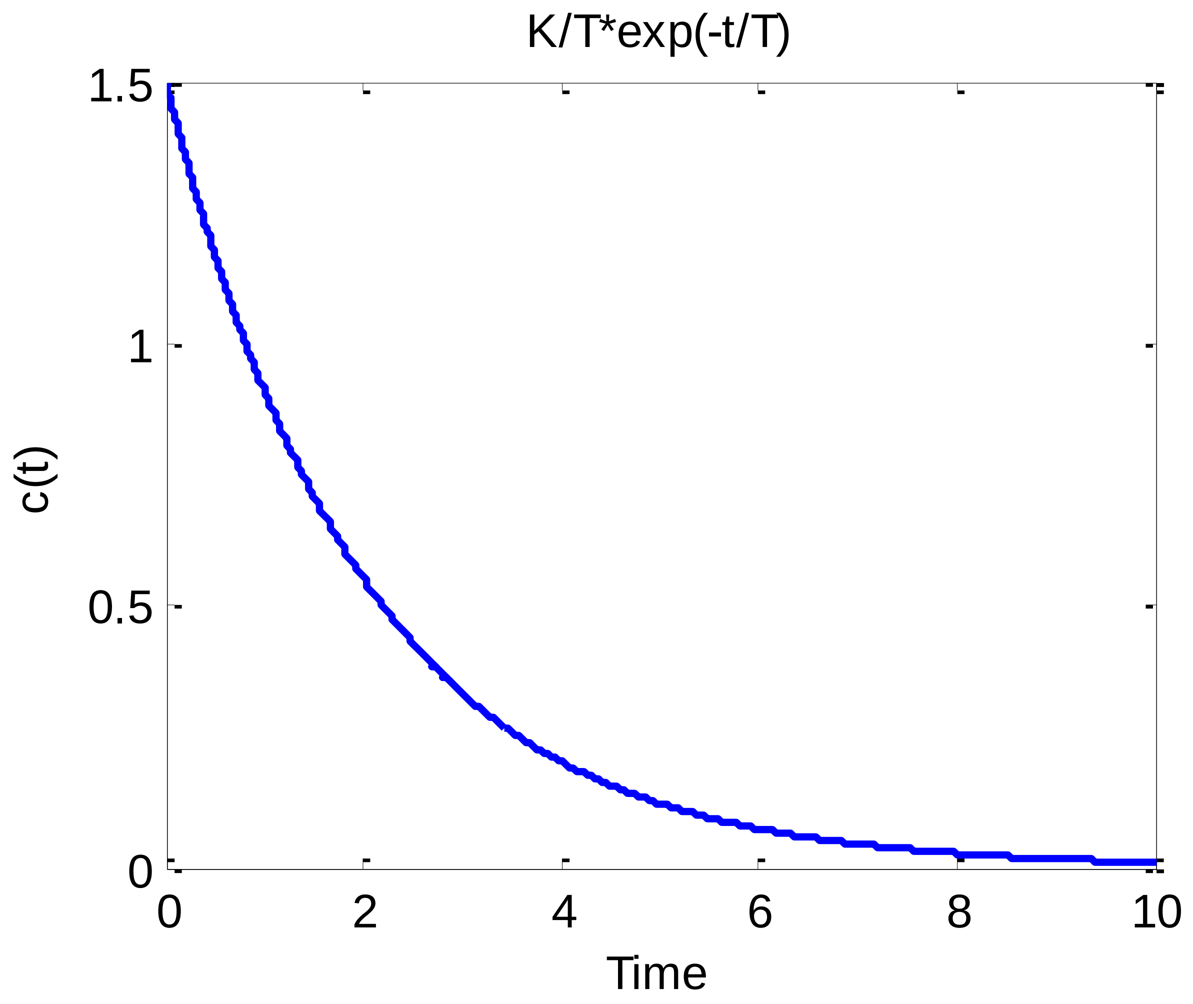
$$C(s) = \frac{K/T}{s+1/T}$$

- In order to compute the response of the system in time domain we need to compute inverse Laplace transform of the above equation.

$$L^{-1}\left(\frac{C}{s+a}\right) = Ce^{-at} \qquad c(t) = \frac{K}{T}e^{-t/T}$$

Impulse Response of 1st Order System

- If K=3 and T=2s then $c(t) = \frac{K}{T} e^{-t/T}$



Step Response of 1st Order System

- Consider the following 1st order system

$$R(s) \longrightarrow \boxed{\frac{K}{Ts+1}} \longrightarrow C(s)$$

$$R(s) = U(s) = \frac{1}{s}$$

$$C(s) = \frac{K}{s(Ts+1)}$$

- In order to find out the inverse Laplace of the above equation, we need to break it into partial fraction expansion (page 867 in the Textbook)

$$C(s) = \frac{K}{s} - \frac{KT}{Ts+1}$$

Step Response of 1st Order System

$$C(s) = K\left(\frac{1}{s} - \frac{T}{Ts+1}\right)$$

- Taking Inverse Laplace of above equation

$$c(t) = K\left(u(t) - e^{-t/T}\right)$$

- Where u(t)=1

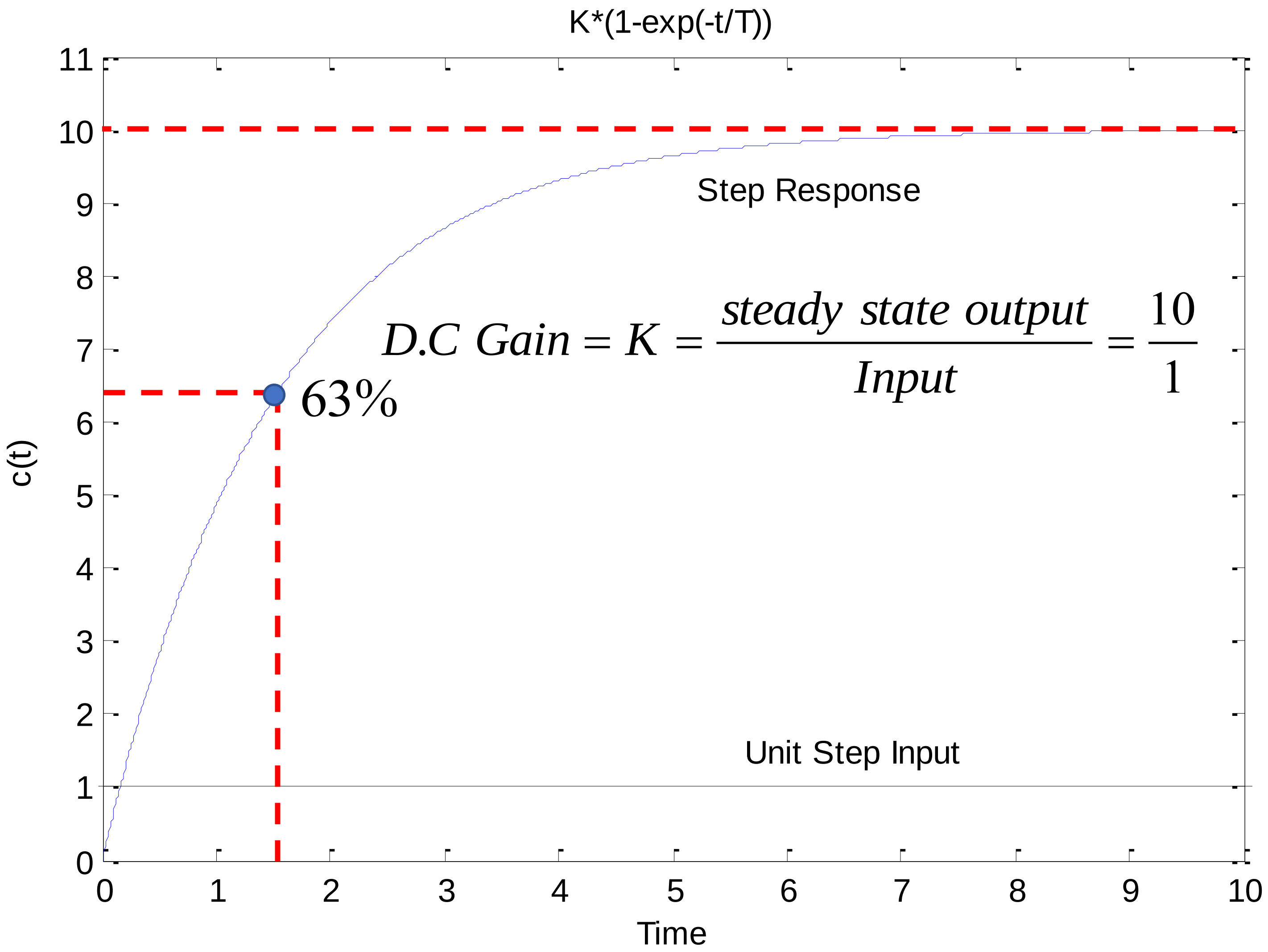
$$c(t) = K\left(1 - e^{-t/T}\right)$$

- When t=T (time constant)

$$c(t) = K\left(1 - e^{-1}\right) = 0.632K$$

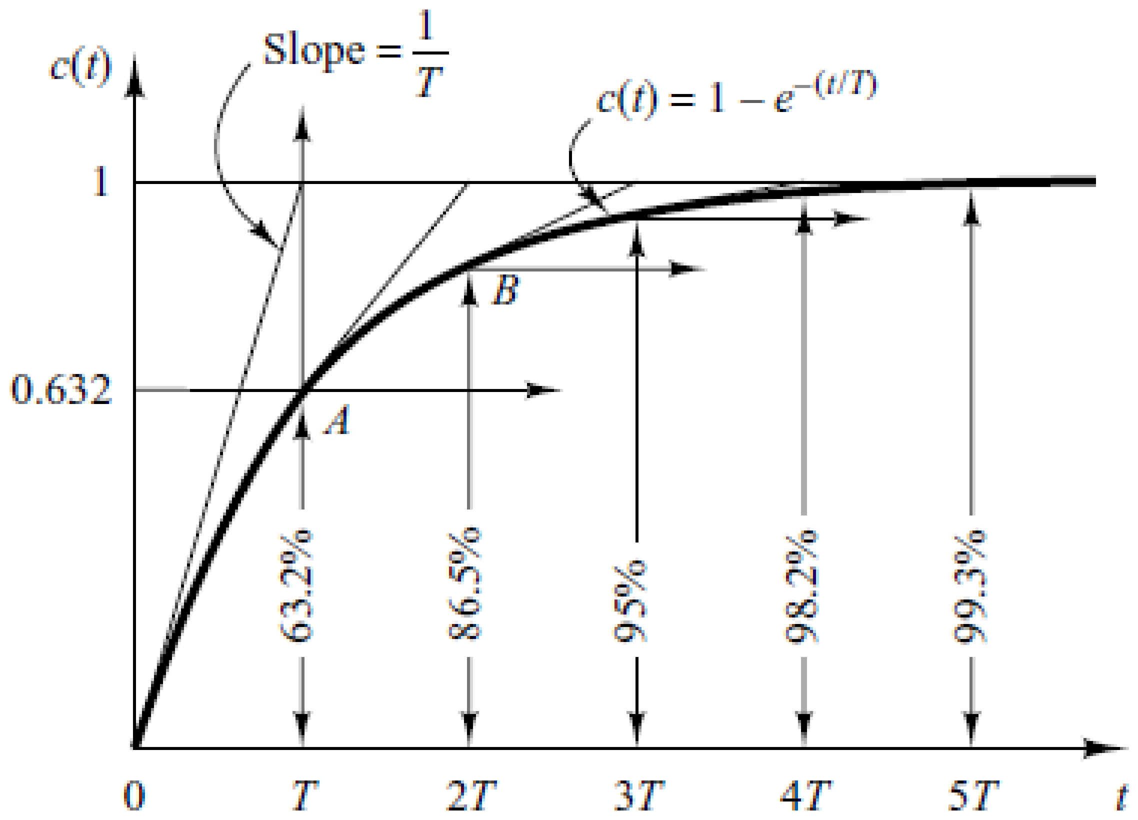
Step Response of 1st Order System

- If K=10 and T=1.5s then $c(t) = K\left(1 - e^{-t/T}\right)$



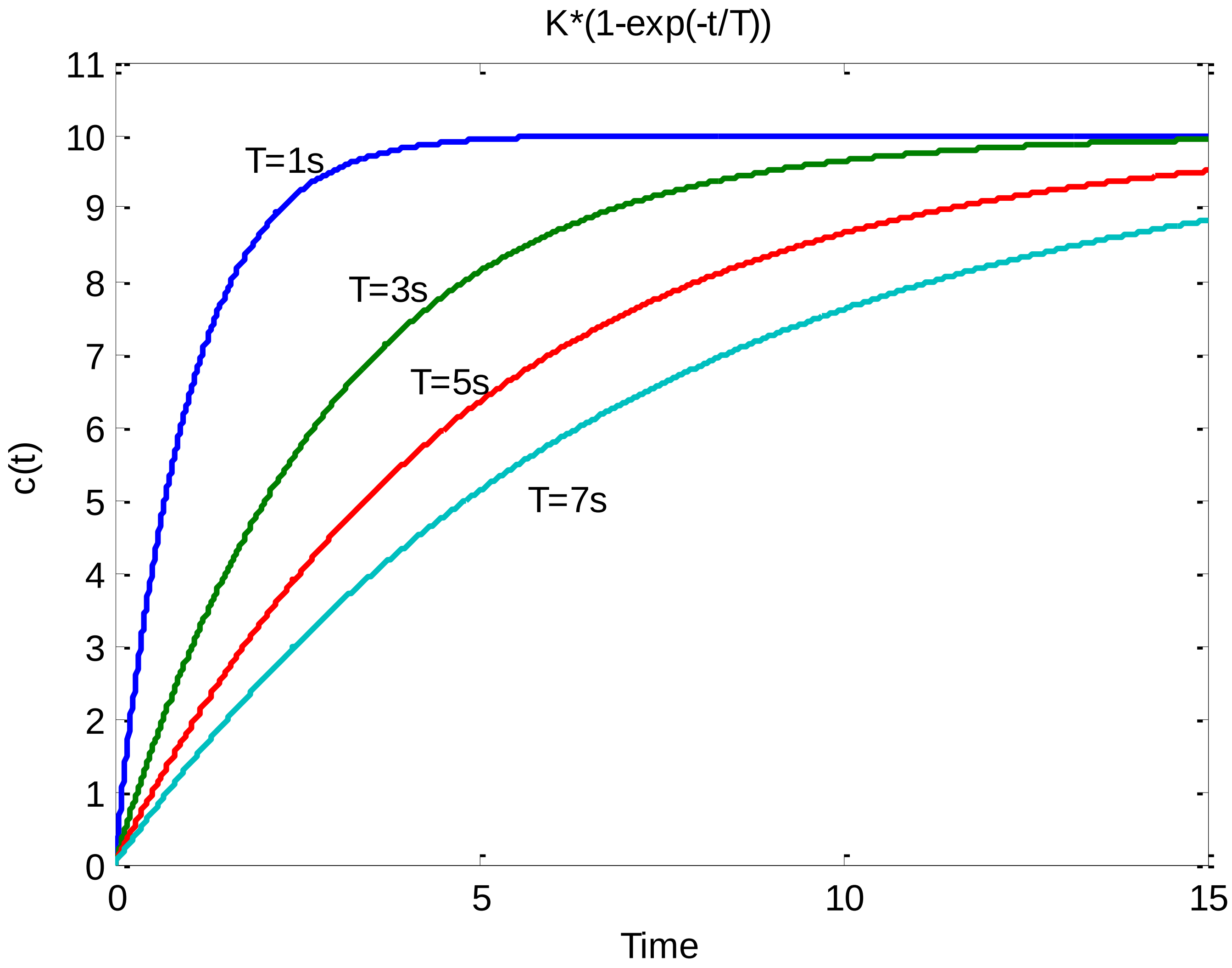
Step Response of 1st order System

- System takes five time constants to reach its final value.



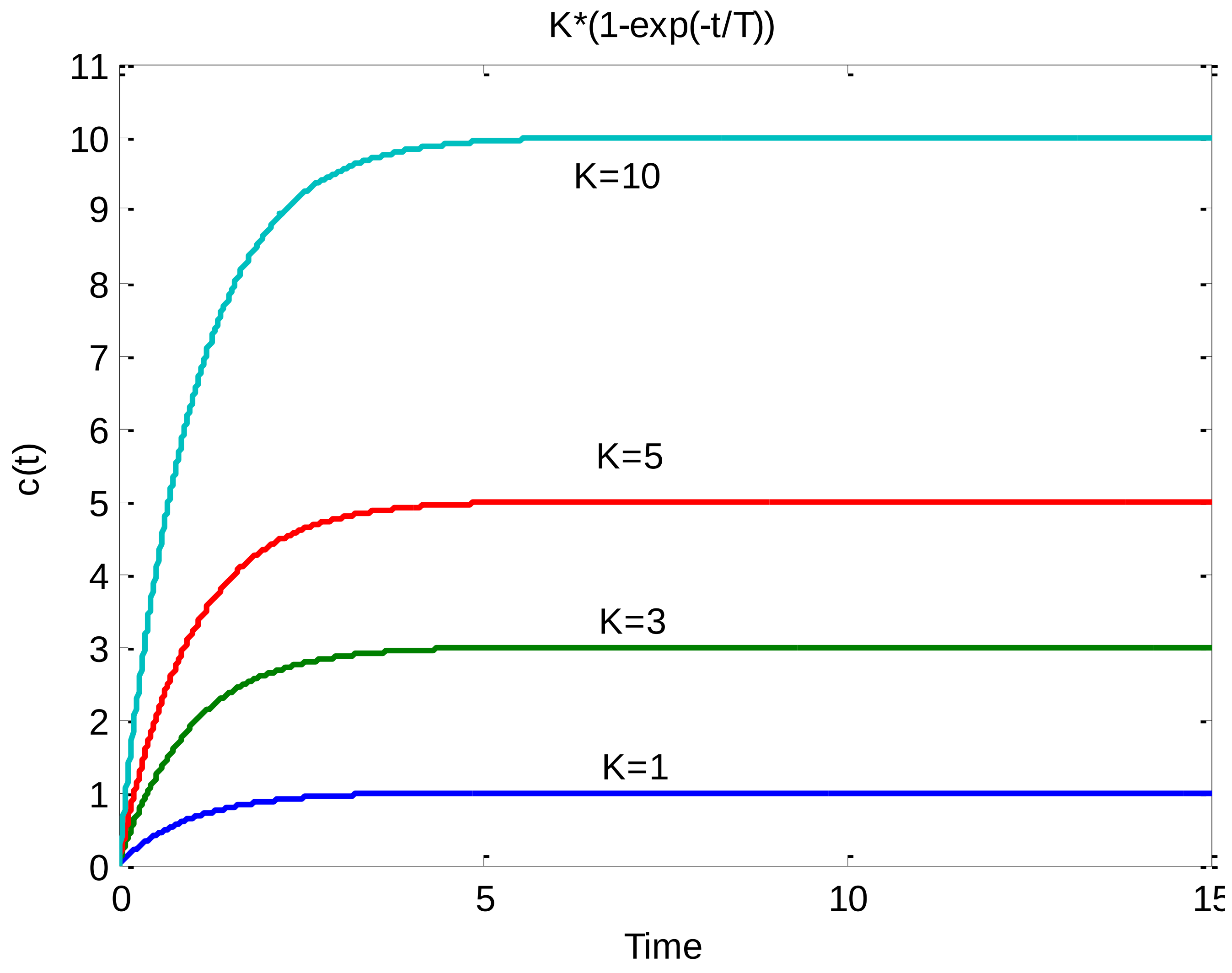
Step Response of 1st Order System

- If K=10 and T=1, 3, 5, 7 $$c(t) = K\left(1 - e^{-t/T}\right)$$



Step Response of 1st Order System

- If K=1, 3, 5, 10 and T=1 $$c(t) = K\left(1 - e^{-t/T}\right)$$



Relation Between Step and impulse response

- The step response of the first order system is

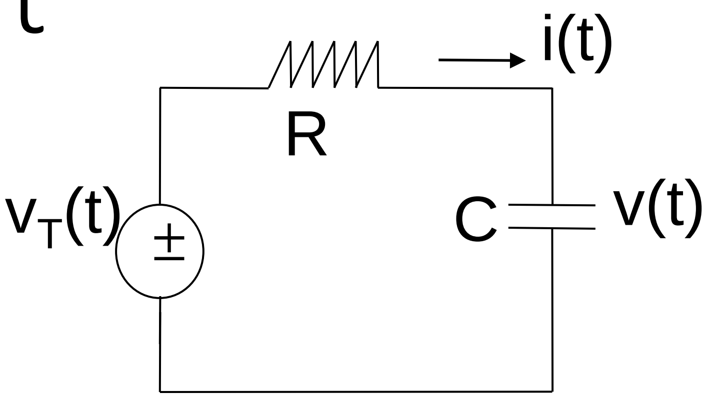
$$c(t) = K\left(1 - e^{-t/T}\right) = K - Ke^{-t/T}$$

- Differentiating c(t) with respect to t yields

$$\frac{dc(t)}{dt} = \frac{d}{dt}\left(K - Ke^{-t/T}\right)$$

$$\frac{dc(t)}{dt} = \frac{K}{T}e^{-t/T}$$

Analysis of Simple RC Circuit



$$R \cdot i(t) + v(t) = v_T(t)$$

$$i(t) = \frac{d(Cv(t))}{dt} = C\frac{dv(t)}{dt}$$

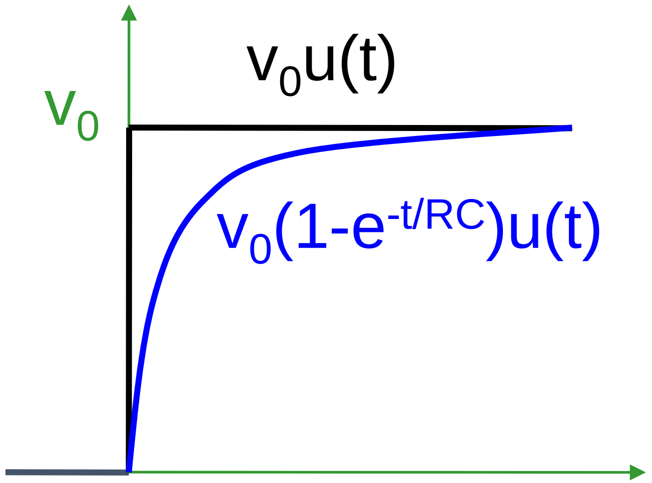
$$\Rightarrow RC\frac{dv(t)}{dt} + v(t) = v_T(t)$$

state variable

Input waveform

Analysis of Simple RC Circuit

Step-input response:



$$RC\frac{dv(t)}{dt} + v(t) = v_0 u(t)$$

$$v(t) = Ke^{-t/RC} + v_0 u(t)$$

match initial state:

$$v(0) = 0 \quad \Rightarrow \ K + v_0 u(t) = 0 \ \Rightarrow \ K + v_0 = 0$$

output response for step-input:

$$v(t) = v_0 (1 - e^{-t/RC})u(t)$$

RC Circuit

- $v(t) = v_0(1 - e^{-t/RC})$ -- waveform

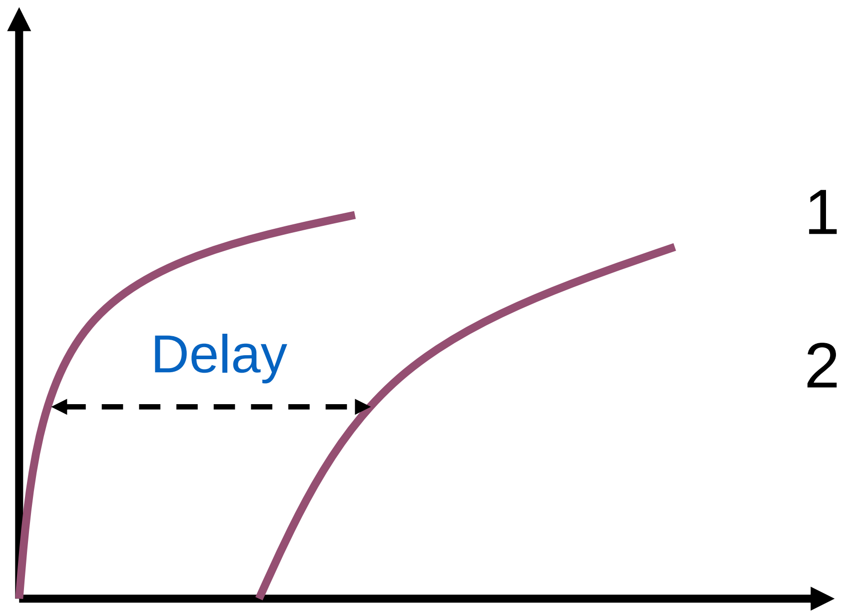
  under step input $v_0 u(t)$

- $v(t)=0.5v_0 \Rightarrow t = 0.69RC$
  - i.e., delay = 0.69RC (50% delay)

  $v(t)=0.1v_0 \Rightarrow t = 0.1RC$

  $v(t)=0.9v_0 \Rightarrow t = 2.3RC$
  - i.e., rise time = 2.2RC (if defined as time from 10% to 90% of Vdd)
- For simplicity, industry uses $T_D = RC$ (= Elmore delay)

Elmore Delay



1. 50%-50% point delay
2. Delay=0.69 RC

Example 1

- Impulse response of a 1st order system is given below.

$$c(t) = 3e^{-0.5t}$$

- Find out
  - Time constant T
  - D.C Gain K
  - Transfer Function
  - Step Response

Example 1

- The Laplace Transform of Impulse response of a system is actually the transfer function of the system.
- Therefore taking Laplace Transform of the impulse response given by following equation.

$$c(t) = 3e^{-0.5t}$$

$$C(s) = \frac{3}{S+0.5} \times 1 = \frac{3}{S+0.5} \times \delta(s)$$

$$\frac{C(s)}{\delta(s)} = \frac{C(s)}{R(s)} = \frac{3}{S+0.5}$$

$$\frac{C(s)}{R(s)} = \frac{6}{2S+1}$$

Example 1

- Impulse response of a 1st order system is given below.

$$c(t) = 3e^{-0.5t}$$

- Find out
  - Time constant T=2
  - D.C Gain K=6
  - Transfer Function $\dfrac{C(s)}{R(s)} = \dfrac{6}{2S+1}$
  - Step Response

Example 1

- For step response integrate impulse response

$$c(t) = 3e^{-0.5t}$$

$$\int c(t)dt = 3\int e^{-0.5t}dt$$

$$c_s(t) = -6e^{-0.5t} + C$$

- We can find out C if initial condition is known e.g. $c_s(0)=0$

$$0 = -6e^{-0.5\times 0} + C$$
$$C = 6$$

$$c_s(t) = 6 - 6e^{-0.5t}$$

Example 1

- If initial conditions are not known then partial fraction expansion is a better choice

$$\frac{C(s)}{R(s)} = \frac{6}{2S+1}$$

since $R(s)$ is a step input, $R(s) = \frac{1}{s}$

$$C(s) = \frac{6}{s(2S+1)}$$

$$\frac{6}{s(2S+1)} = \frac{A}{s} + \frac{B}{2s+1}$$

$$\frac{6}{s(2S+1)} = \frac{6}{s} - \frac{6}{s+0.5}$$

$$c(t) = 6 - 6e^{-0.5t}$$

Ramp Response of 1st Order System

- Consider the following 1st order system

$$R(s) \longrightarrow \boxed{\frac{K}{Ts+1}} \longrightarrow C(s)$$

$$R(s) = \frac{1}{s^2}$$

$$C(s) = \frac{K}{s^2(Ts+1)}$$

- The ramp response is given as

$$c(t) = K\left(t - T + Te^{-t/T}\right)$$

Parabolic Response of 1st Order System

- Consider the following 1st order system

$$R(s) \longrightarrow \boxed{\frac{K}{Ts+1}} \longrightarrow C(s)$$

$$R(s) = \frac{1}{s^3} \quad \text{Therefore,} \quad C(s) = \frac{K}{s^3(Ts+1)}$$

Practical Determination of Transfer Function of 1st Order Systems

- Often it is not possible or practical to obtain a system's transfer function analytically.
- Perhaps the system is closed, and the component parts are not easily identifiable.
- The system's step response can lead to a representation even though the inner construction is not known.
- With a step input, we can measure the time constant and the steady-state value, from which the transfer function can be calculated.

Practical Determination of Transfer Function of 1st Order Systems

- If we can identify *T* and *K* empirically we can obtain the transfer function of the system.

$$\frac{C(s)}{R(s)} = \frac{K}{Ts+1}$$

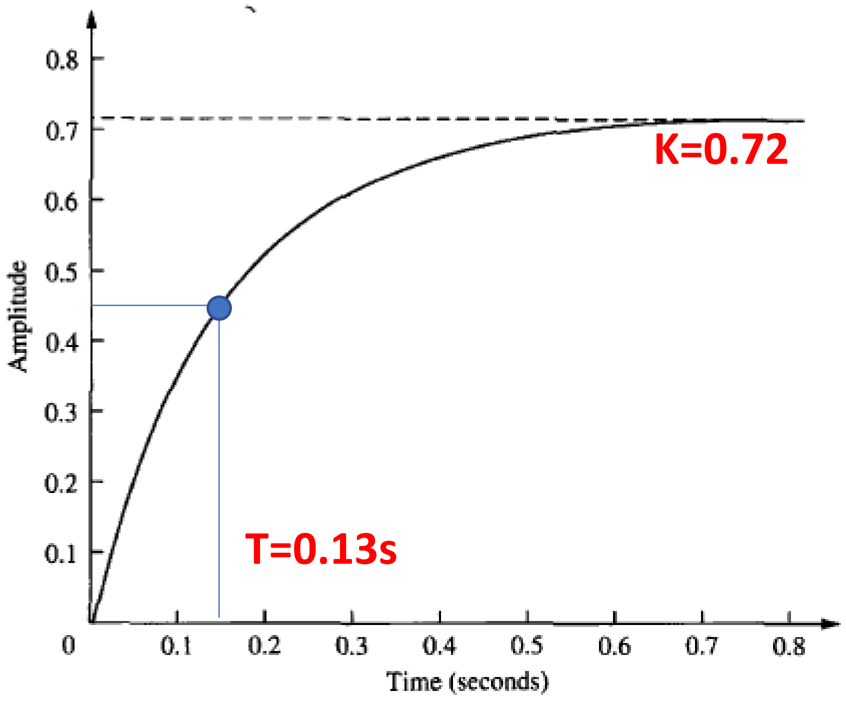
Practical Determination of Transfer Function of 1st Order Systems

- For example, assume the unit step response given in figure.

- From the response, we can measure the time constant, that is, the time for the amplitude to reach 63% of its final value.

- Since the final value is about 0.72 the time constant is evaluated where the curve reaches 0.63 x 0.72 = 0.45, or about 0.13 second.

- K is simply steady state value.



- Thus transfer function is obtained as:

$$\frac{C(s)}{R(s)} = \frac{0.72}{0.13s+1} = \frac{5.5}{s+7.7}$$

First Order System with a Zero

$$\frac{C(s)}{R(s)} = \frac{K(1+\alpha s)}{Ts+1}$$

- Zero of the system lie at *-1/α* and pole at *-1/T*.

- Step response of the system would be:

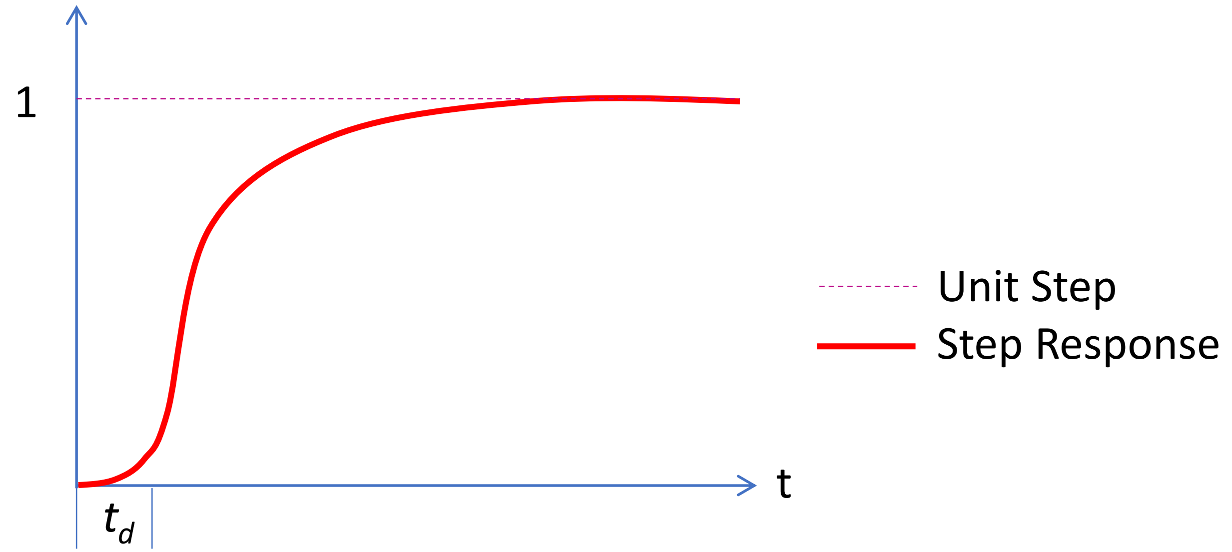
$$C(s) = \frac{K(1+\alpha s)}{s(Ts+1)}$$

$$C(s) = \frac{K}{s} + \frac{K(\alpha - T)}{(Ts+1)}$$

$$c(t) = K + \frac{K}{T}(\alpha - T)e^{-t/T}$$

First Order System With Delays

$$\frac{C(s)}{R(s)} = \frac{K}{Ts+1}e^{-st_d}$$



First Order System With Delays

$$\frac{C(s)}{R(s)} = \frac{10}{3s+1}e^{-2s}$$

$$C(s) = \frac{10}{s(3s+1)}e^{-2s}$$

$$L^{-1}[e^{-\partial s}F(s)] = f(t-\partial)u(t-\partial)$$

$$L^{-1}[(\frac{10}{s} + \frac{-10}{s+1/3})e^{-2s}] =$$

$$[10(t-2) - 10e^{-1/3(t-2)}]u(t-2)$$