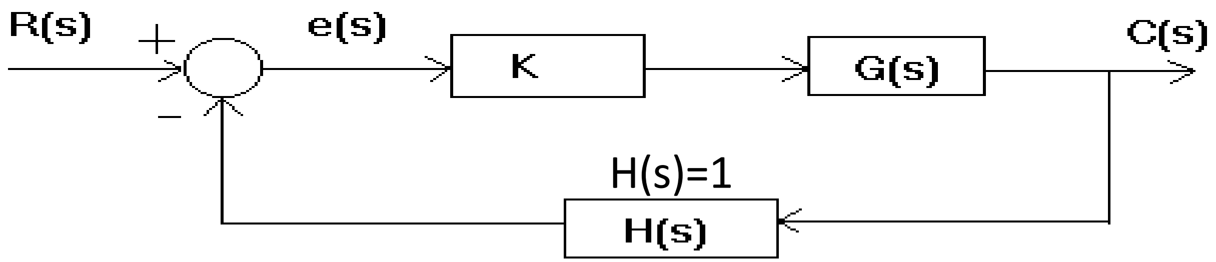
What is root locus……?

- Simple definition : A curve or other figure formed by all the points satisfying a particular equation of the relation between coordinates, or by a point, line, or surface moving according to mathematically defined conditions.
- Root locus : The root locus is the path of the roots of the characteristic equation traced out in the *s*-plane as a system parameter (K) is changed.($0<K<\infty$)
- It can be used to describe qualitatively the performance of a system as various parameters are changed
- It gives graphic representation of a system's transient response and also stability
- We can see the range of stability, instability, and the conditions that cause a system to break into oscillation

Root locus concept…



$$T.F.= \frac{K\,G(s)}{1+K\,G(s)}$$

Characteristic equation: $1+KG(s)=0$

$$1+K\frac{N(s)}{D(s)}=0$$

$$D(s)+KN(s)=0$$

When *K=0*, this collapses to $D(s)=0$.
Since the roots of $D(s)=0$ are the **poles of G(s),**
those are the closed-loop poles for $K=0$.

When $K=\infty$, $D(s)+KN(s)=\frac{1}{K}+\frac{N(s)}{D(s)}=0$ $\frac{N(s)}{D(s)}=0$

thus the roots of N(s)=0 are the **zeros of G(s).**

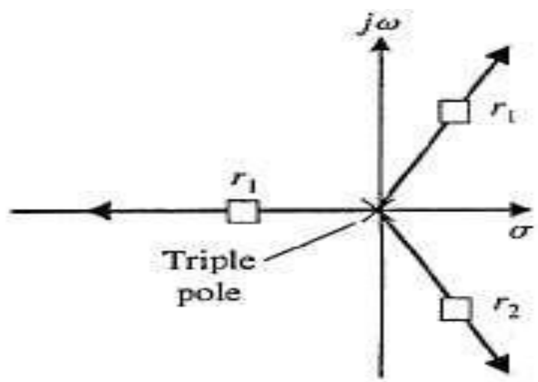
Construction of Root locus…

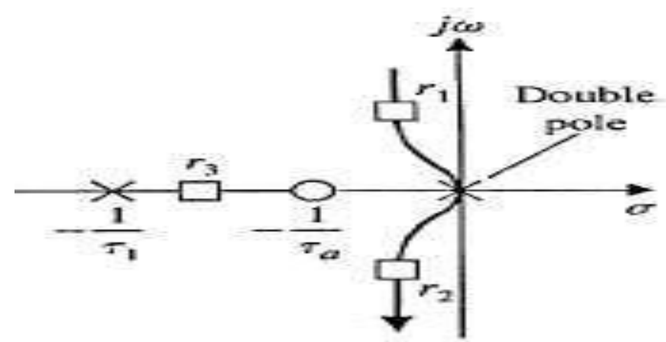
❖ There are many RULES which are used to construct Root locus are described below…

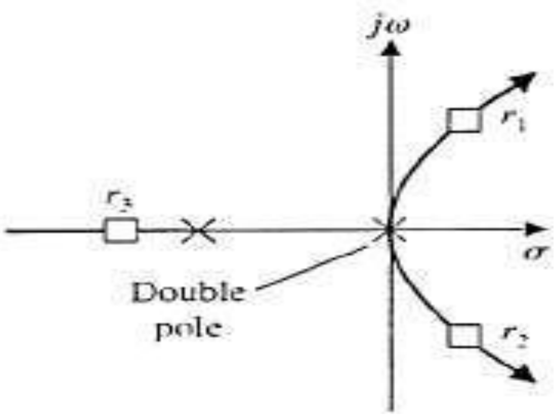
- Rule-1

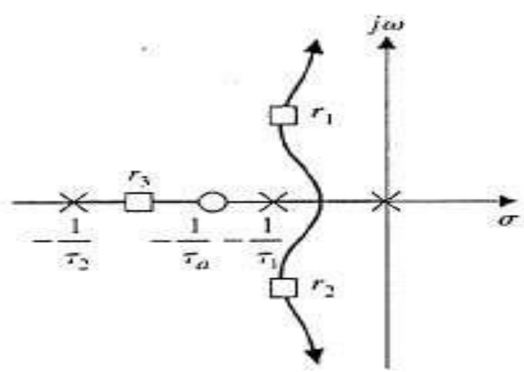
  The Root locus is symmetrical about the real axis.

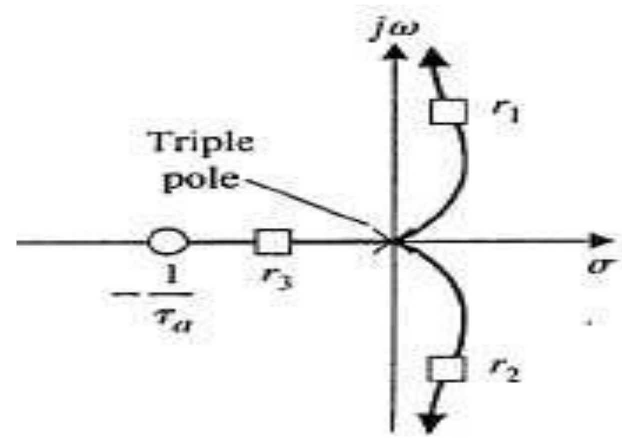
  e.g.











Draw the roo...

$G(s)H(s) = \dfrac{k}{s(s+3)(s+6)}$. **Obtain value of k when $\xi = 0.6$ from root Lou**

...1:

Here, Number of poles = n = 3

Number of zeros = m = 0

$\therefore$ Number of Loci = n = 3

and Number of Loci ending at infinity = n − m = 3

Thus there are three root loci. An p

Step 2 :

Draw the poles and zeros to suitable scale. Here there are no zeros.

Step 3 :

Real axis Loci is found as follows :

Moving from origin on negative X - axis, root locus is present whene poles & zeros to out right is odd. Accordingly real axis Loci is :

(a) Present Between $-3 < \sigma < 0$

(b) Absent Between $-\sigma < < -3$

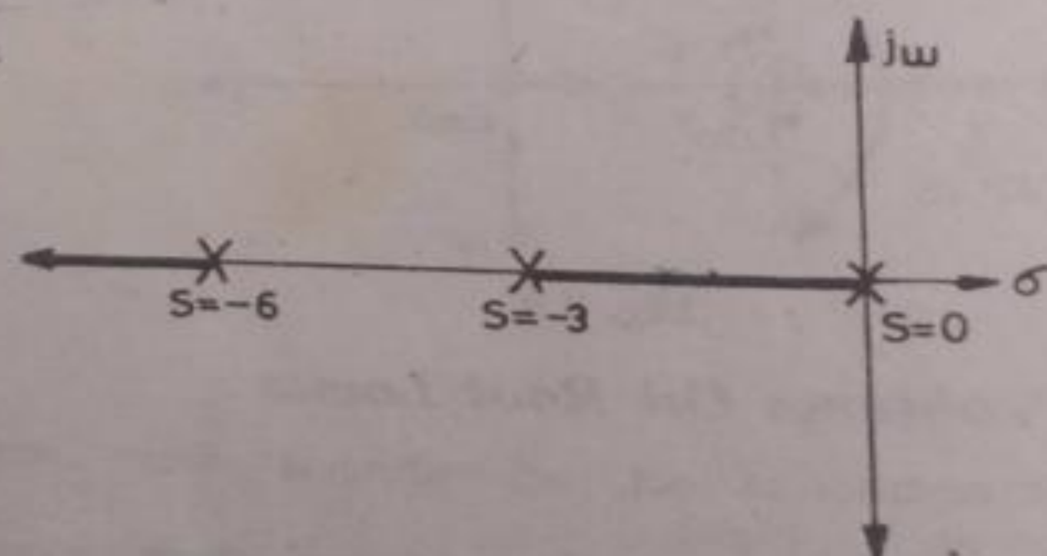
(c) Present Between $-\infty < \sigma < -6$

Refer Fig. 8.10 (a).

Rough Nature :

(i) There are 3 Loci. They start from poles at s = 0, s = − 3, s zeros in this example all will proceed to infinity along (n − m

(ii) Between s = 0 and s = − 3 real axis, two Loci start towrd must breakaway between 0 & − 3 (as root Loci begins on a pc This is seen from Fig. 8.10. (a).



Step 4 :   Fig. 8.10

Fig. 8.10 (a)

Step 4 :

(a) Number of asymptotes = $n - m = 3$

(b) Angle of Asymptotes = (odd multiple of) $\left(\frac{180}{n-m}\right)$

$$\beta = (2x+1)\left(\frac{180}{3}\right) \quad x = 0, 1, 2$$

$$\text{i.e. } \beta = 60^0, 180^0, 300^0.$$

Step 5 :

Centroid :

As per centroid formula,

$$\sigma_c = \frac{(0+(-3)+(-6))-\text{NIL}}{3-0} = -3 \qquad \sigma_c = -3$$

Step 6 :

$3-0$

Step 6 :

Breakaway points :

$$\text{Now } k = -\frac{D(s)}{N(s)} \qquad k = \frac{s(s+3)(s+6)}{1}$$

$$k = -\left(s^3 + 9s^2 + 18s\right)$$

Now, $\frac{dk}{ds} = 0$ $\qquad 0 = (3s^2 + 18s + 18)$

i.e. $0 = s^2 + 6s + 6$

Solving $s = -4.73$ and -1.27

Select the values of s where real axis Loci from step 3 is present. Here As re
between 0 & -3 $s = -1.27$ is accepted. Similarly $s = -4.73$ is dropped fo
xis loci is absent)

:Not needed, as no complex poles or zeros is present.

:Characteristic equation is $D + KN = 0$

$s(s+3)(s+6) + k(1) = 0$ $\qquad \therefore s^3 + 9s^2 + 18s + k = 0$

Routh - Array is drawn below.

outh - Array is drawn below.

$$\begin{array}{c|cc} s^3 & 1 & 18 \\ s^2 & 9 & k \\ s^1 & \frac{162-k}{9} & - \\ s^0 & k & \end{array}$$

for stability, column 1 should be positive.

Maximum value of k for stable system is obtained by equating column 1 s^0

$\therefore k > 0$ & $\frac{162-k}{9} > 0$ i.e. $k < 162$ i.e. $k_{mar} = 162$

Combining $0 < k < 162$ is stable range of k.

For s^2 row, equation is

$$9s^2 + k = 0$$

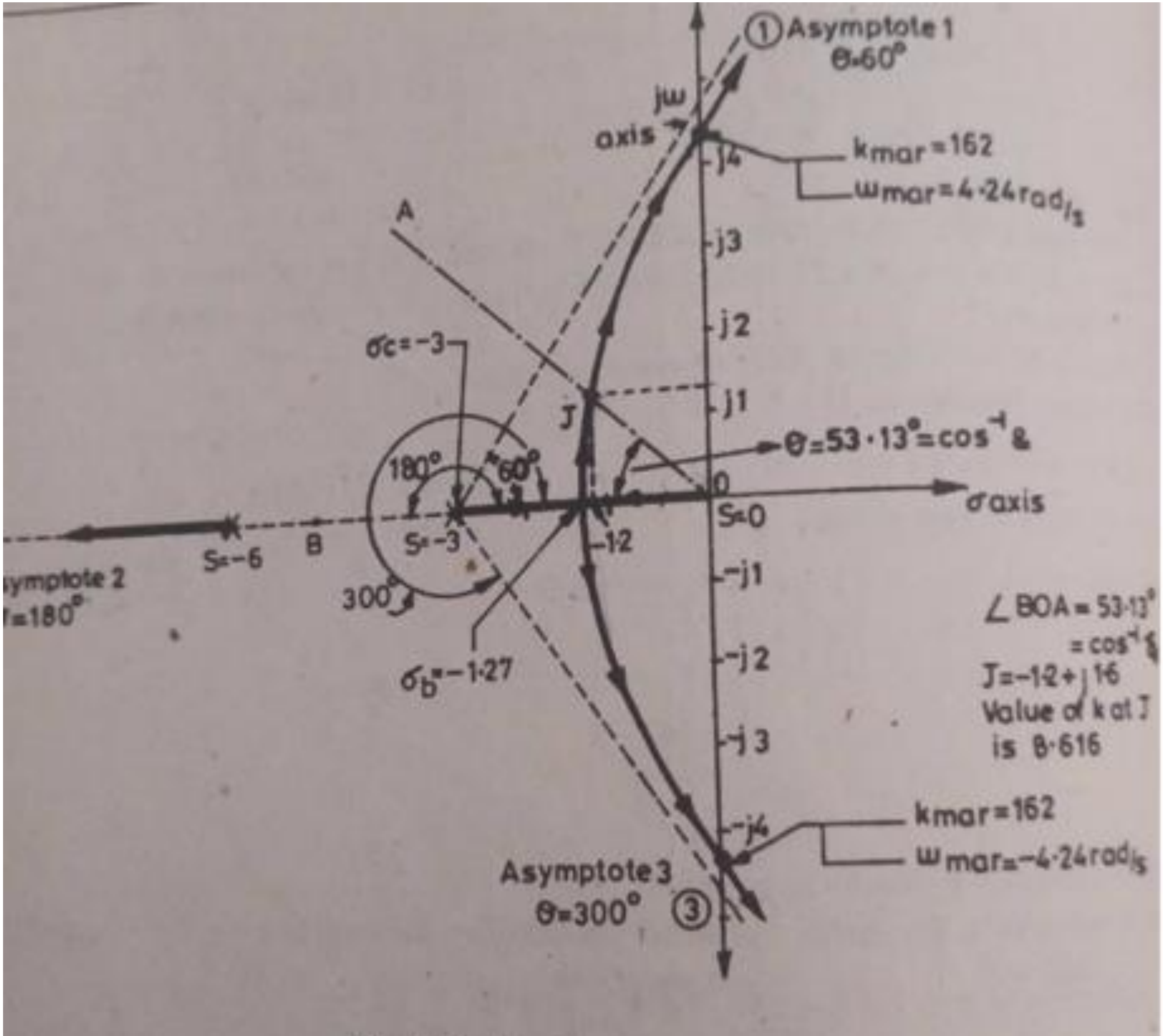
$$\therefore 9s^2 = -162$$

$$\therefore s = \pm j\,4.24 = \pm j\,\omega_{mar}$$

i.e. $9s^2 = -$

$\therefore s^2 = -$

$\therefore \omega_{mar} = 4.$



① Asymptote 1
θ=60°
jω
axis
kmar=162
ωmar=4·24rad/s
A
j4
j3
j2
j1
σc=−3
J
θ=53·13°=cos⁻¹ξ
180°
60°
0
σ axis
S=0
S=−6
B
S=−3
−1·2
symptote 2
=180°
300°
−j1
∠BOA=53·13°
=cos⁻¹ξ
J=−1·2+j1·6
Value of k at J
is 8·616
σb=−1·27
−j2
−j3
−j4
kmar=162
ωmar=−4·24rad/s
Asymptote 3
θ=300°
③

- Rule-2

  Each branch of the root locus originates from an open loop pole at K=0 and terminates at

  K=∞, either on open loop zero or on infinity.

  The numbers of branches of the root locus terminating on infinity is equal to (n-m).

  Where, n= numbers of poles

  m=numbers of zeros

  e.g.

$$G(s) \cdot H(s) = \frac{k}{s(s+2+2j)(s+2-2j)}$$

3 poles at $s = 0$,
$s = -2 \pm 2j$

Step 1 : Obtain total number of loci :

Here, n = Number of poles
$= 3$

m = Number of zeros
$= 0$

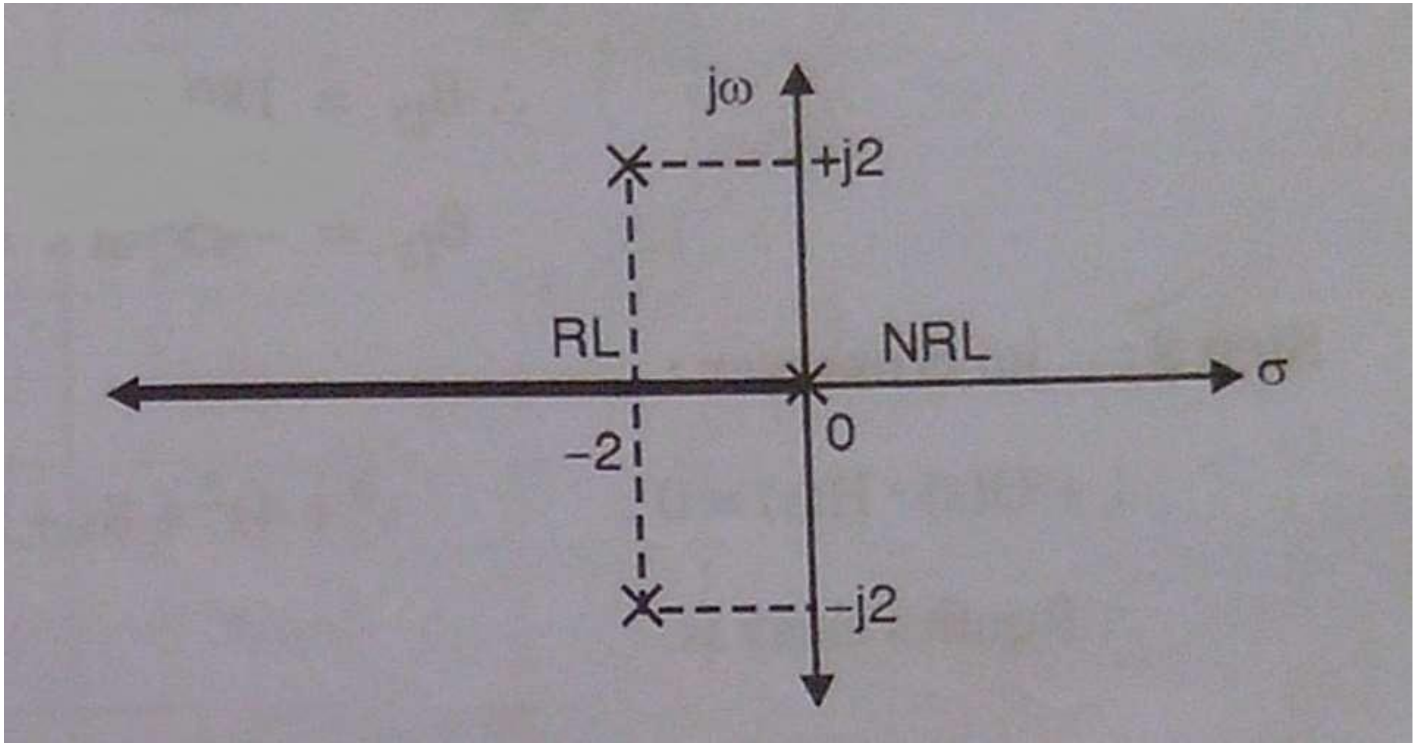
$\therefore$ Number of loci $= 3$

- Rule-3

  Segments of the real axis having an odd number of real axis open loop poles plus open loop zeros to their right are parts of the root locus.

  If numbers of (poles + zeros) at right side to the segment = ODD, then it is a part of Root locus, else it is not.

  e.g.



- Rule-4

  (n-m) is equals to numbers of asymptotes

  For to find asymptotes making angles with real axis…

  X=0,1,…(n-m-1)

  β =180(2X+1)/(n-m)

  e.g.

Step 4 : Calculate number of asymptotes and angle of asymptotes :

(a) Number of asymptotes $= n - m = 3$

(b) Angle of asymptotes, $\beta = \dfrac{(2x+1)\,180^\circ}{n-m}$, $x = 0, 1, 2$

$\therefore \quad \beta_1 = 60^\circ, \beta_2 = 180^\circ, \beta_3 = 300^\circ.$

- Rule-5

  The point of intersection of the asymptotes with the real axis is at

$$O_a = \frac{\sum_{i=1}^{n} p_i - \sum_{j=1}^{m} z_j}{n - m}$$

e.g.

Step 5 : **Centroid :**

$$\sigma_c = \frac{\sum \text{Real part of poles} - \sum \text{Real part of zeros}}{\text{Poles} - \text{zeros}} = \frac{0 - 2 - 2 - 0}{3} = -1.33$$

- Rule-6

  The breakaway point of the root locus are determined from the roots of the equation

$$\frac{dK}{dS} = 0$$

S should be real for breakaway point.

If it is not real, then no breakaway point.

e.g.

Step 6 : **Breakaway point :**

$$1 + G(s) \cdot H(s) = 0$$

$$\therefore \quad 1 + \frac{k}{s(s+2+2j)(s+(2-2j))} = 0$$

$$\therefore \quad s^3 + 4s^2 + 8s + k = 0$$

$$\therefore \quad k = \frac{-D(s)}{N(s)} = -s^3 - 4s^2 - 8s$$

$$\therefore \quad \frac{dk}{ds} = -3s^2 - 8s - 8$$

$$s = -1.33 \pm j\,0.94$$

$\therefore$ No Breakaway point.

- Rule-7

  The intersection of the root locus branches with the imaginary axis can be determined by use of the routh criterion.

  e.g.

Step 8 : $j\omega$ crossover :

$1 + G(s) \cdot H(s) = 0$; $s^3 + 4s^2 + 8s + k = 0$

$\therefore$ Routh's array is

s^3	1	8
s^2	4	k
s^1	$\frac{32-k}{4}$	0
s^0	k	

$32 - k = 0$ $\quad\therefore\quad k_{mar} = 32$

$\therefore\quad 4s^2 + 32 = 0$ $\quad\therefore\quad s = \pm j2.82$

$\therefore\quad \omega_{mar} = \pm 2.82$ rad/sec.

❖ Rule 8 is only applicable when either complex poles or complex zeros are there

- Rule-8

  The angle of departure

  suppose…

$$GH = \frac{K}{S(S+2+j2)(S+2-j2)} \qquad GH = \frac{K}{S(S+2+j2)}\Bigg|_{s=-2+j2}$$

$$GH = \frac{K}{(-2+j2)(-2+j2+2+j2)} \qquad AngGH' = \frac{\angle 0^0}{\angle 135^0 \angle 90^0}$$

$$AngGH' = -225^0 \qquad \theta_D = 180 + AngGH'$$

$$\theta_D = -45\left(at\_s = -2+2j\right)$$

$$\theta_D = 45\left(at\_s = -2-2j\right)$$

Example 9.13 : *For* $G(s)H(s) = \dfrac{K(s+2)}{s(s+4)(s^2+2s+2)}$ *, calculate angles of departures at complex conjugate poles.*

Solution : P = 4 , Z = 1

Poles are at s = 0 , −4 , − 1 ± j
Zero at s = −2.

Draw Pole-Zero plot.

Let us calculate ϕ_d at the pole s = −1 + j.

Join all other poles to this pole and measure or calculate the angles ϕ_{P1}, ϕ_{P2}, ϕ_{P3} as shown in the Fig. 9.17.

Join all zeros to this pole and calculate ϕ_{Z1}.

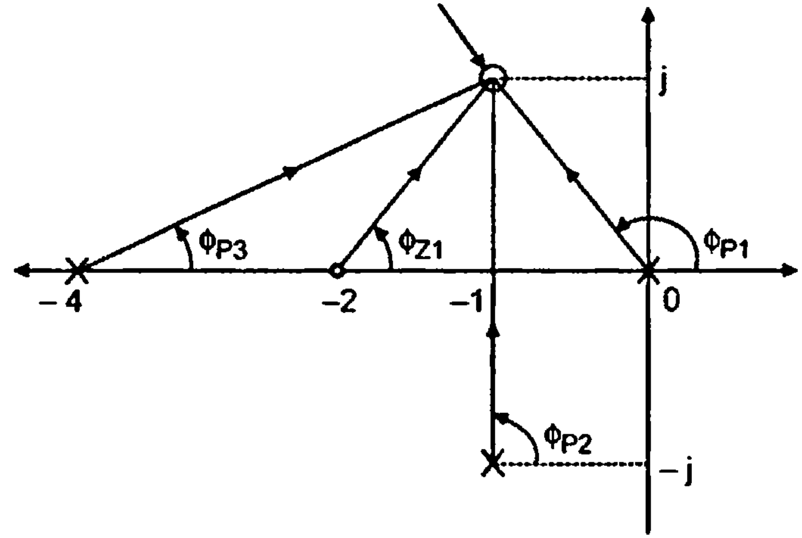


Fig. 9.17

Then ,
$$\sum \phi_P = \phi_{P1} + \phi_{P2} + \phi_{P3} \text{ while}$$
$$\sum \phi_Z = \phi_{Z1}$$

From geometry of the Fig. 9.17 we can calculate,

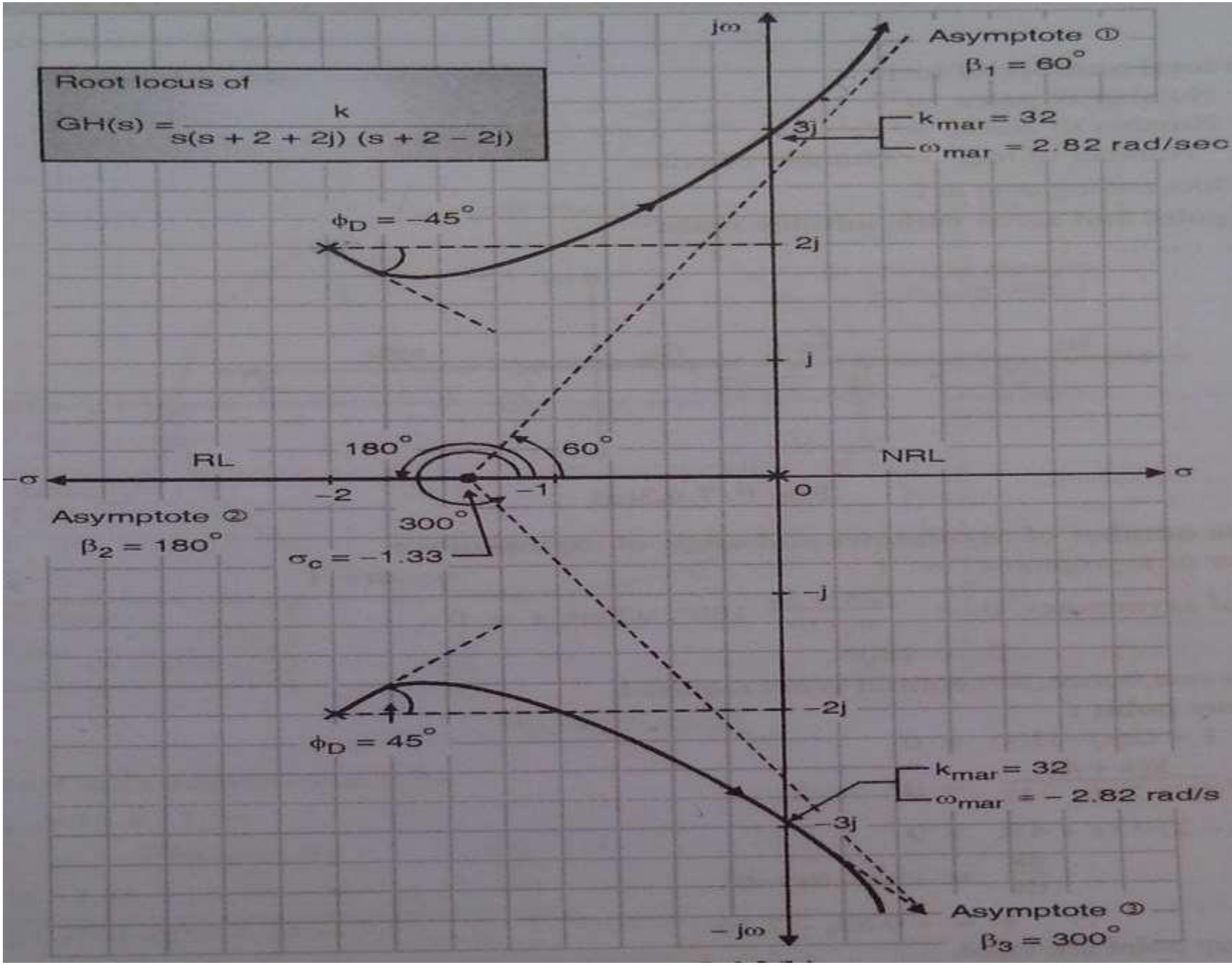
$$\phi_{P1} = 135^\circ, \quad \phi_{P2} = 90^\circ, \quad \phi_{P3} = 18.43^\circ$$

$$\therefore \quad \sum \phi_P = 135^\circ + 90^\circ + 18.43^\circ = 243.43^\circ$$

$$\sum \phi_Z = \phi_{Z1} = 45^\circ$$

$$\therefore \quad \phi = \sum \phi_P - \sum \phi_Z = 243.43^\circ - 45^\circ = 198.43^\circ$$

$$\phi_d = 180^\circ - \phi = 180^\circ - 198.43^\circ = -18.43^\circ$$



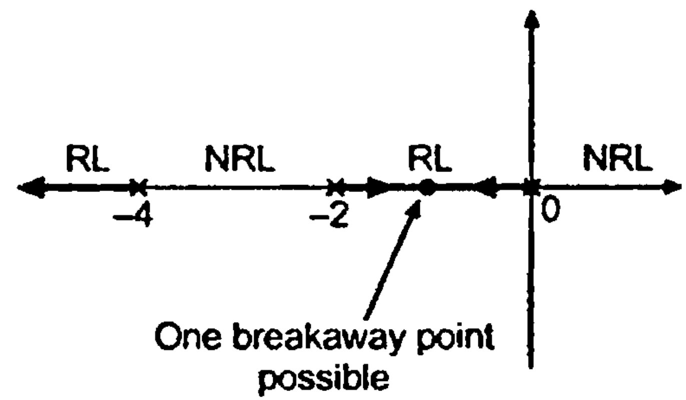
Root locus of
GH(s) = k / s(s + 2 + 2j) (s + 2 − 2j)
jω
Asymptote ①
β1 = 60°
kmar = 32
ωmar = 2.82 rad/sec
3j
φD = −45°
2j
j
180°
60°
RL
NRL
σ
−σ
−2
−1
0
300°
Asymptote ②
β2 = 180°
σc = −1.33
−j
−2j
φD = 45°
kmar = 32
ωmar = − 2.82 rad/s
−3j
Asymptote ③
β3 = 300°
− jω

Example 9.14 : *For a unity feedback system,* $G(s) = \dfrac{K}{s(s+4)(s+2)}$. *Sketch the rough nature of the root locus showing all details on it. Comment on the stability of the system.*

Solution : Step 1 : General information from $G(s)H(s) = \dfrac{K}{s(s+2)(s+4)}$

P = 3, Z = 0, number of branches N = P = 3. No finite zero so all P − Z = 3 branches will terminate at infinity. Starting points are locations of open loop poles i.e. 0, − 2, − 4.

Step 2 : Pole-Zero plot and sections of real axis.



Directions of branches away from poles. One breakaway point exists between 0 and − 2 according to general prediction.

Sections of real axis identified as a part of the root locus as to right side sum of poles and zeros is odd for those sections.

Step 3 : Angles of asymptotes.

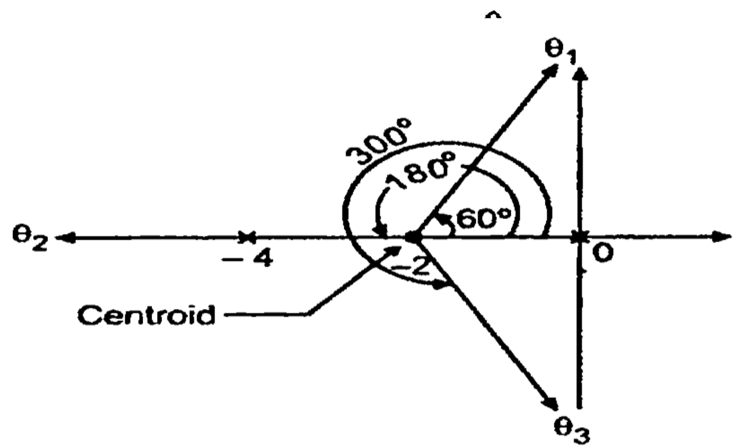
3 branches are approaching to ∞ , 3 asymptotes are required.

$$\theta = \frac{(2q+1)\,180^\circ}{P-Z}, \qquad q = 0, 1, 2$$

$$\therefore\ \theta_1 = \frac{180^\circ}{3} = 60^\circ,\ \theta_2 = \frac{(2+1)\,180^\circ}{3} = 180^\circ,\ \theta_3 = \frac{(2\times 2+1)\,180^\circ}{3} = 300^\circ$$

Step 4 : Centroid

$$\sigma = \frac{\sum \text{R.P. of poles} - \sum \text{R.P. of zeros}}{P-Z} = \frac{0-2-4}{3} = -2$$



Branches will approach to ∞ along these lines which are asymptotes.

Step 5 : To find breakaway point (Refer Rule No. 6). Characteristic equation is

$$1 + G(s)H(s) = 0$$

$$1 + \frac{K}{s(s+2)(s+4)} = 0$$

$$\therefore \quad s^3 + 6s^2 + 8s + K = 0$$

$$\therefore \quad K = -s^3 - 6s^2 - 8s \qquad \ldots (1)$$

$$\frac{dK}{ds} = -3s^2 - 12s - 8 = 0$$

$$\text{i.e.} \quad 3s^2 + 12s + 8 = 0$$

Roots i.e. breakaway points $= \dfrac{-12 \pm \sqrt{144 - 4 \times 3 \times 8}}{2 \times 3} = -0.845 \, , \, -3.15$

As there is no root locus between – 2 to – 4 , – 3.15 cannot be a breakaway point. It also can be confirmed by calculating `K' for s = – 3.15. It will be negative that confirms s = – 3.15 is not a breakaway point.

For $s = -3.15$, $K = -3.079$ (Substituting in equation for K)

But as there has to be breakaway point between `0' and `– 2' , s = – 0.845 is vaild breakaway point.

For $s = -0.845$ $K = +3.079$

As K is positive **$s = -0.845$ is vaild breakaway point.**

Step 6 : Intersection point with imaginary axis.

Characteristic equation

$$s^3 + 6s^2 + 8s + K = 0$$

Routh's array

$$\begin{array}{c|cc} s^3 & 1 & 8 \\ s^2 & 6 & K \\ s^1 & \frac{48-4}{6} & 0 \\ s^0 & K & \end{array}$$

$K_{marginal} = 48$ which makes row of s^1 as row of zeros.

$$A(s) = 6s^2 + K = 0$$

$$K_{mar} = 48$$

$$\therefore\ 6s^2 + 48 = 0$$

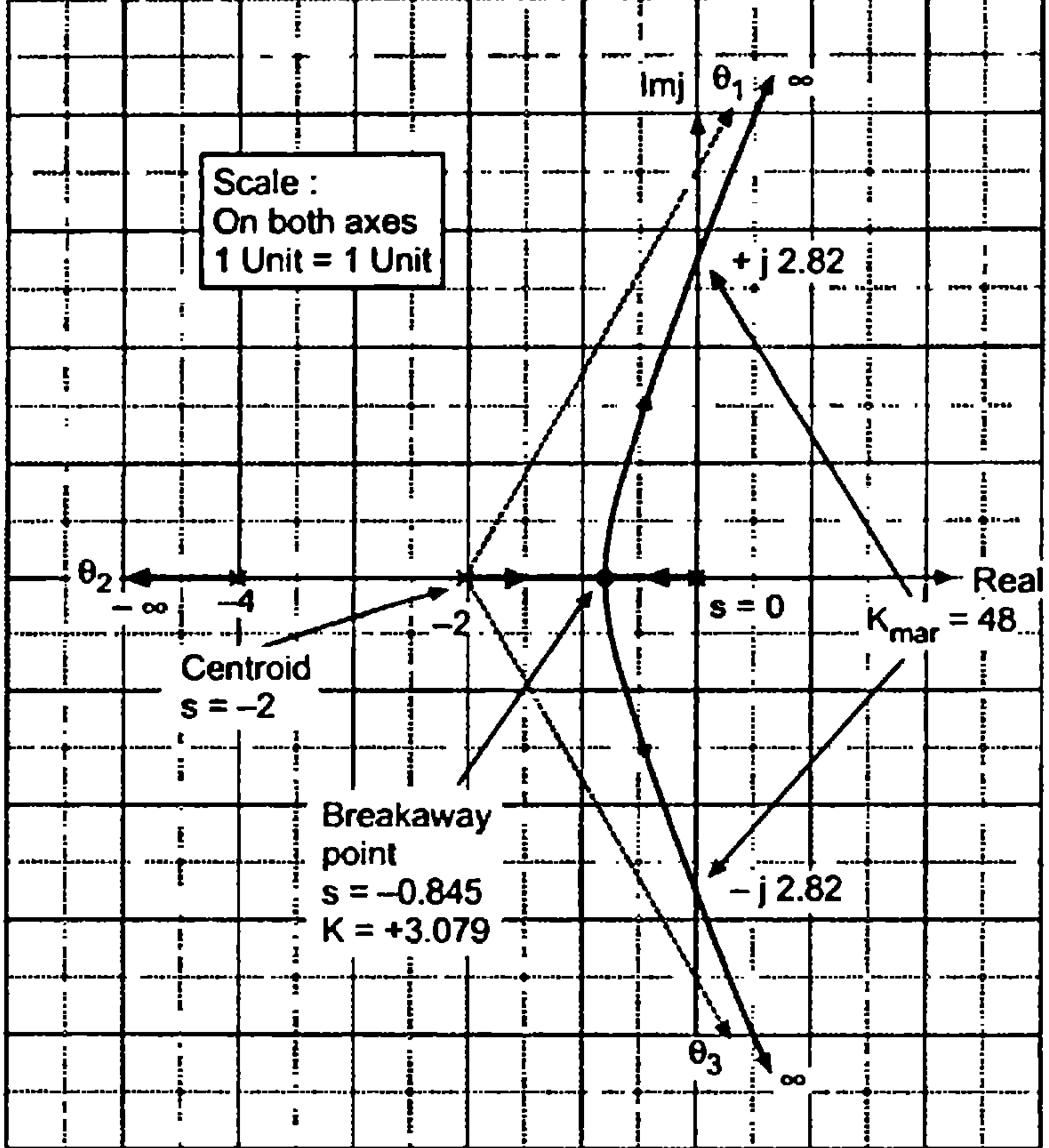
Intersection of root locus with imaginary axis is at ± j 2.828 and corresponding value of K_{mar} = 48.

Step 7 : As there are no complex conjugate poles or zeros, no angles of departures or arrivals are required to be calculated.

Step 8 : The complete root locus is as shown below.

Step 9 : Prediction about stability :

For $0 < K < 48$, all the roots are in left half of s–plane hence system is absolutely stable. For $K_{mar} = +48$, a pair of dominant roots on imaginary axis with remaining root in left half. So system is marginally stable oscillating at 2.82 rad/sec. For $48 < K < \infty$, dominant roots are located in right half of s-plane hence system is unstable.


Imj
θ_1
∞
Scale :
On both axes
1 Unit = 1 Unit
+ j 2.82
θ_2
– ∞
–4
–2
s = 0
Real
K_{mar} = 48
Centroid
s = –2
Breakaway
point
s = –0.845
K = +3.079
– j 2.82
θ_3
∞

Ex. 1.

Ex. 7.6.4 : A feedback control system has open loop trasfer function

$$G(s)\,H(s) = \frac{k}{s(s+4)\,(s^2+4s+20)}$$

Plot the root locus for K = 0 to ∞. Indicate all the points on it.

Soln. :

Step 1 : Obtain total number of loci :

Number of poles $= 4$,

Poles at s $= 0, -4, -2 \pm j4$

Number of zeros $= 0$

Number of loci ending at ∞ $= 4$

Step 2 : Draw the poles and zeros with suitable scale.

Step 3 : Real axis loci is :

Root locus is present between 0 to – 4 and it is absent from – 4 to ∞ as total number of poles and zeros are even number.

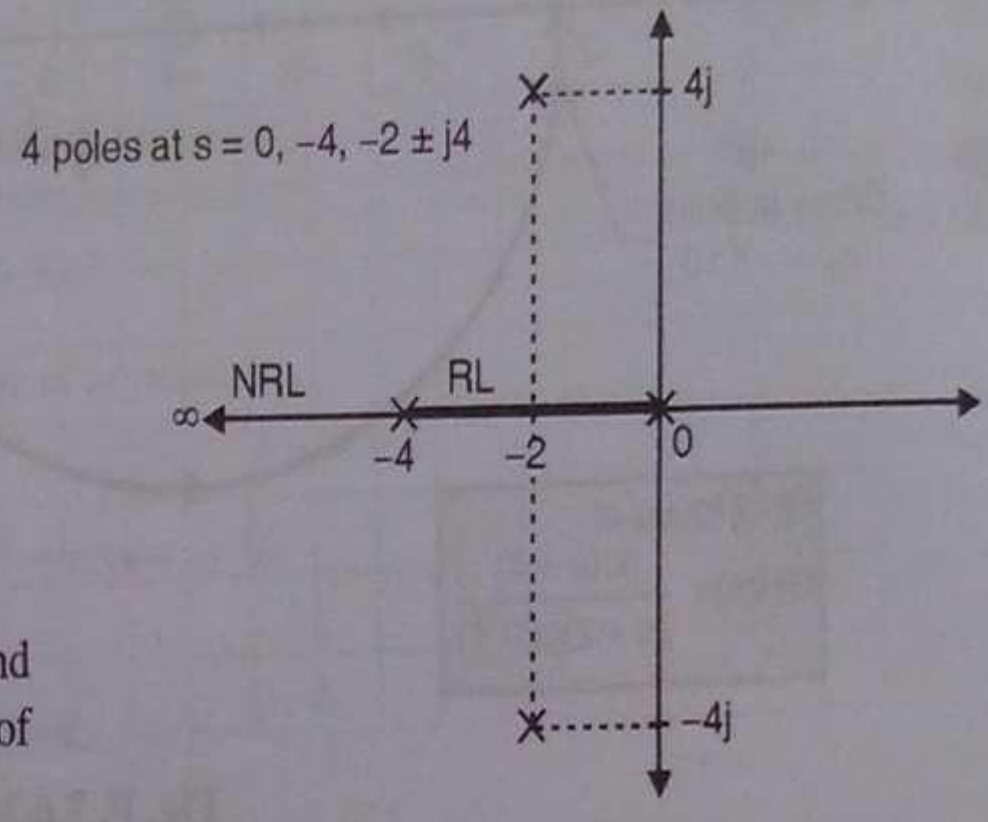


Fig. P. 7.6.4(a)

Step 4 : **Calculate number of asymptotes and angle of asymptotes :**

$$\beta = \frac{(2x+1)180^\circ}{n-m} \quad x = 0, 1, 2, 3 \qquad n-m = 4.$$

$$\beta = 45^\circ, 135^\circ, 225^\circ, 315^\circ$$

Step 5 : **Centroid :**

$$\sigma_c = \frac{\sum \text{Real poles} - \sum \text{Real zeros}}{n-m}$$

$$\sigma_c = \frac{0-4-2-2}{4} = -2$$

Step 6 : **Breakaway point :**

$$1 + G(s)\,H(s) = 0$$

$$1 + \frac{k}{s\,(s+4)\,(s^2+4s+20)} = 0$$

$$\therefore s^4 + 8\,s^3 + 36s^2 + 80s + k = 0$$

$$\therefore k = -(s^4 + 8s^3 + 36s^2 + 80s)$$

$$\frac{dk}{ds} = -(4s^3 + 24s^2 + 72s + 80)$$

$$\therefore s^3 + 6s^2 + 18s + 20 = 0$$

As root locus is present between 0 to – 4. So choose s = – 2 and obtain root using trial and error method.

– 2	1	6	18	20
		– 2	– 8	– 2
	1	4	10	0

$$\therefore \quad (s+2)(s^2+4s+10) = 0$$

$$\therefore \quad s = -2 \text{ and } -2 \pm j2.45$$

All these are valid breakaway point

Step 7 : Angle of departure :

$$GH = \frac{K}{s(s+4)(s+2+j4)(s+2-j4)}$$

$$\therefore GH' = \left.\frac{k}{s(s+4)(s+2+j4)}\right|_{s=-2+j4}$$

$$= \frac{k}{(-2+j4)(-2+j4+4)(-2+j4+2+j4)}$$

$$= \frac{k}{(-2+j4)(2+j4)(j8)} = \frac{k\angle 0^\circ}{\angle 116.56 \angle 63.43 \angle 90^\circ} = \frac{k}{270^\circ} = -270^\circ$$

$$GH' = 180 + \arg GH' = 180 - 270 = -90^\circ$$

$$\phi_D \text{ at } -2+j4 = -90^\circ$$

$$\phi_D \text{ at } -2-j4 = +90^\circ$$

The Routh's array is,

$$\begin{array}{c|ccc}
s^4 & 1 & 36 & k \\
s^3 & 8 & 80 & 0 \\
s^2 & 26 & k & \\
s^1 & \dfrac{2080-8k}{26} & 0 & \\
s^0 & k & &
\end{array}$$

For stability there should be no sign change in the first column. Maximum value of k for stable system is obtained by equating column s^0 and s^1 rows to zero.

$$\therefore \quad k > 0 \text{ and } \frac{2080-8k}{26} > 0$$

$$\therefore \quad \frac{2080-8k}{26} = 0$$

$$\therefore \quad K_{mar} = 260$$

The Auxiliary equation is,

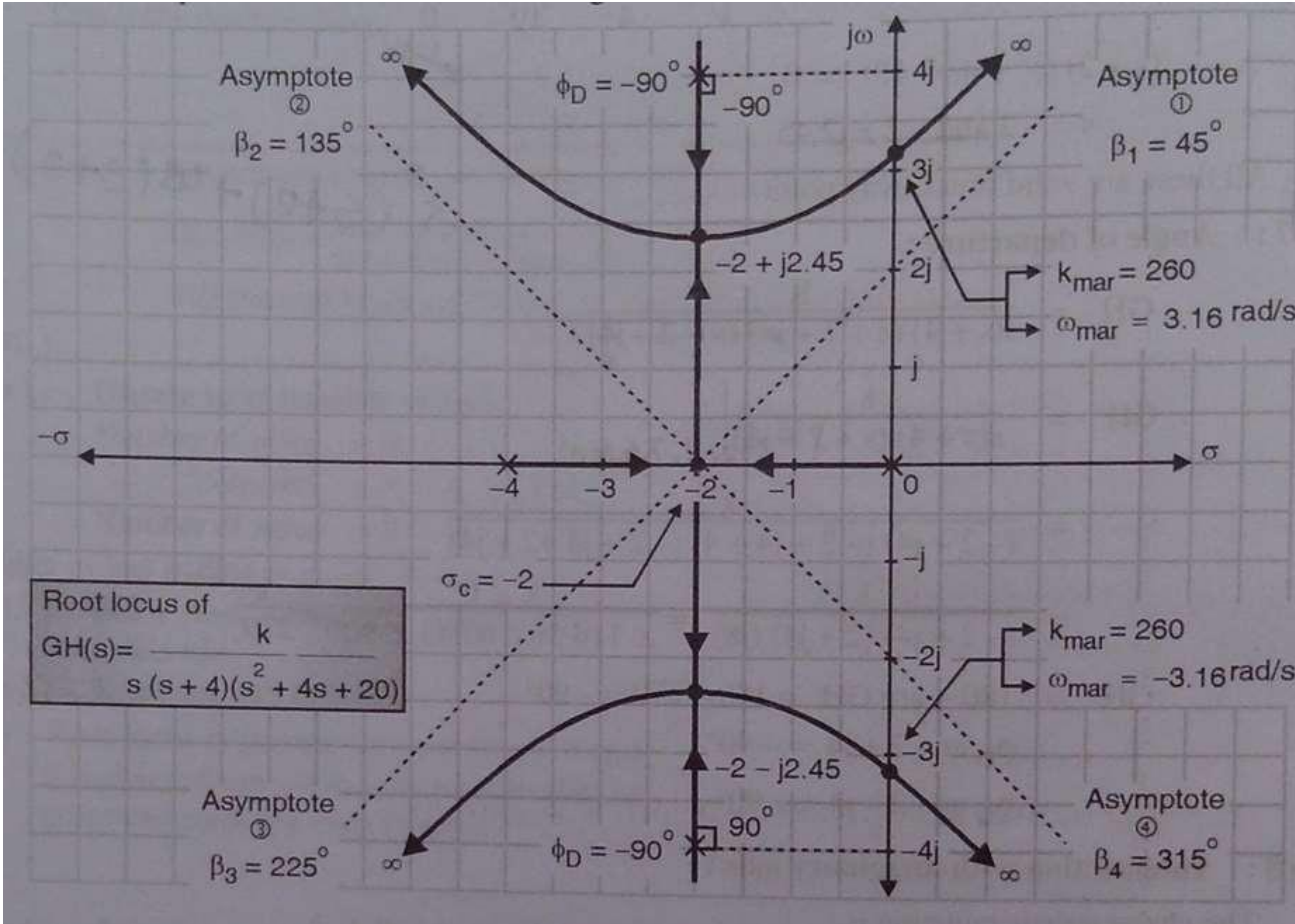
$$26s^2 + k = 0$$

$$\therefore \quad 26s^2 + 260 = 0$$

$$s^2 = -10$$

$$\therefore \quad s = \pm\sqrt{10} = \pm 3.162\,j$$

$$\therefore \quad \omega_{mar} = \pm 3.162 \text{ rad/sec}$$



Asymptote ②
$\beta_2 = 135^\circ$
Asymptote ①
$\beta_1 = 45^\circ$
Asymptote ③
$\beta_3 = 225^\circ$
Asymptote ④
$\beta_4 = 315^\circ$
$j\omega$
∞
$\phi_D = -90^\circ$
-90°
90°
4j
3j
2j
j
–j
–2j
–3j
–4j
$-2 + j2.45$
$-2 - j2.45$
$k_{mar} = 260$
$\omega_{mar} = 3.16$ rad/s
$k_{mar} = 260$
$\omega_{mar} = -3.16$ rad/s
$-\sigma$
σ
–4
–3
–2
–1
0
$\sigma_c = -2$
Root locus of
$GH(s) = \dfrac{k}{s(s+4)(s^2+4s+20)}$

Ex. 2.

EXAMPLE 7.10 : A feedback control system has open–loop transfer function

$$G(s)H(s) = \frac{K(s+2)(s+3)}{(s+1)(s-1)}$$

Find the root locus as K is varied from 0 to ∞.

The open–loop poles are located at s = –1 and 1 while zeros are located at s = –2 and –3.

Therefore n = 2, m = 2.

The pole zero configuration is shown in Fig. E.7.10.

Rule 2 : Accoding to this rule, two branches of root locus originate at K = 0 from the open–loop poles and terminate at K = ∞ on open–loop zeros. As n = m, no root locus branch is terminating on infinity.

Rule 3 : According to this rule, root locus exists on the real axis between the segments $-1 \le s \le 1$ and $-3 \le s \le -2$.

Rule 4 : As no root locus branch tends to infinity, no asymptotes are required and hence centroid is also not required to be found out.

Rule 6 : Break away points are determined from the condition

$$\frac{dK}{ds} = 0$$

From the characteristic equation,

$$K = -\frac{(s+1)(s-1)}{(s+2)(s+3)}$$

$$= -\frac{s^2-1}{s^2+5s+6}$$

Therefore,

$$\frac{dK}{ds} = \left[\frac{(s^2+5s+6)\cdot 2s - (s^2-1)(2s+5)}{(s^2+5s+6)^2}\right] = 0$$

$$2s\,(s^2+5s+6) - (s^2-1)\,(2s+5) = 0$$

$$2s^3 + 10s^2 + 12s - (2s^3 + 5s^2 - 2s - 5) = 0$$

$$5s^2 + 14s + 5 = 0$$

or $$s^2 + 2.8s + 1 = 0$$

Roots are $$= \frac{-2.8 \pm \sqrt{(2.8)^2 - 4}}{2}$$

$$= \frac{-2.8 \pm \sqrt{3.84}}{2}$$

$$= \frac{-2.8 \pm 1.96}{2}$$

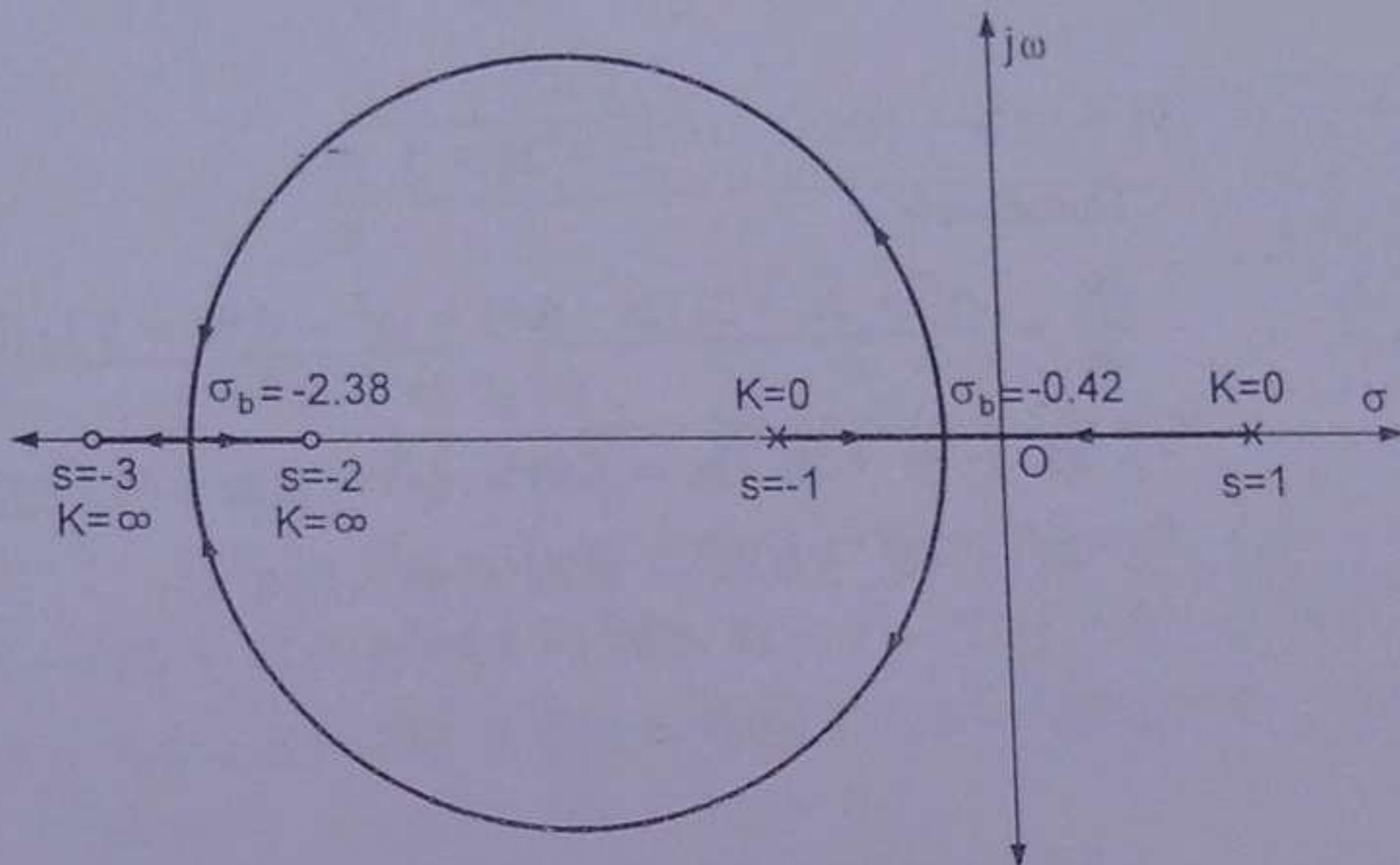
$$= -1.4 \pm 0.98$$

$$= -0.42,\ -2.38$$

Rule 7 : There is no complex pole or zero. Hence the angle of departure/arrival is not required to be found out.

Rule 8 : There are not asymptotes and crossing the imaginary axis. So intersection with imaginary axis is not required to be found.

The root locus plot is shown in Fig. E.7.10.



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