② Find the inverse Z-transform of

$$X(z) = \frac{z^{-1}}{3-4z^{-1}+z^{-2}} \; ; \; ROC; \; |z|>1$$

Solution: $X(z)$ can be written as,

$$X(z) = \frac{z^{-1}}{3-4z^{-1}+z^{-2}} = \frac{z}{3z^2-4z+1}$$

$$= \frac{z}{3\left[z^2 - \frac{4}{3}z + \frac{1}{3}\right]} = \frac{1}{3} \cdot \frac{z}{(z-1)(z-\frac{1}{3})}$$

$$\therefore \frac{X(z)}{z} = \frac{1}{3} \cdot \frac{1}{(z-1)(z-\frac{1}{3})} = \frac{A}{z-1} + \frac{B}{z-\frac{1}{3}}$$

Now, $A = (z-1)\dfrac{X(z)}{z}\Big|_{z=1} = (z-1)\cdot\dfrac{1}{3}\cdot\dfrac{1}{(z-1)(z-\frac{1}{3})}\Big|_{z=1} = \dfrac{1}{3}\cdot\dfrac{1}{[1-\frac{1}{3}]}$

$$A = +\frac{1}{2}$$

$$B = \left(z-\frac{1}{3}\right)\frac{X(z)}{z}\Big|_{z=\frac{1}{3}} = \left(z-\frac{1}{3}\right)\cdot\frac{1}{3}\cdot\frac{1}{(z-1)(z-\frac{1}{3})}\Big|_{z=\frac{1}{3}}$$

$$B = \frac{1}{3}\cdot\frac{1}{[\frac{1}{3}-1]} = -\frac{1}{2}$$

$$\therefore \quad \frac{X(z)}{z} = \frac{1/2}{z-1} - \frac{1/2}{z-\frac{1}{3}}$$

$$\Rightarrow \quad X(z) = \frac{\frac{1}{2}z}{z-1} - \frac{\frac{1}{2}z}{z-\frac{1}{3}}$$

$$\Rightarrow \quad X(z) = \frac{\frac{1}{2}z}{z-1} - \frac{\frac{1}{2}z}{z-\frac{1}{3}}$$

Apply, $x(n) = Z^{-1}\left[\frac{z}{z-P_K}\right] = (P_K)^n u(n)$; ROC; $|z| > |P_K|$

$$\therefore \quad x(n) = \frac{1}{2}\cdot(1)^n u(n) - \frac{1}{2}\left(\frac{1}{3}\right)^n u(n)$$

(3). Find the inverse Z-transform of

$$X(z) = \frac{(1/6)z^{-1}}{\left[1-\frac{1}{2}z^{-1}\right]\left[1-\frac{1}{3}z^{-1}\right]};\ \text{ROC}: |z| > \frac{1}{2}$$

Solution: The $X(z)$ can be written as,

$$X(z) = \frac{(1/6)z}{\left[z-\frac{1}{2}\right]\left[z-\frac{1}{3}\right]}$$

On applying partial fraction expansion,

$$\frac{X(z)}{z} = \frac{1/6}{\left(z-\frac{1}{2}\right)\left(z-\frac{1}{3}\right)} = \frac{C_1}{z-\frac{1}{2}} + \frac{C_2}{z-\frac{1}{3}}$$

Now, $C_1 = \left(z - \frac{1}{2}\right) \frac{X(z)}{z} \Bigg|_{z=\frac{1}{2}} = \frac{1}{6} \cdot \frac{1}{\left(z - \frac{1}{3}\right)} \Bigg|_{z=\frac{1}{2}}$

$$C_1 = \frac{1}{6} \cdot \frac{1}{\left(\frac{1}{2} - \frac{1}{3}\right)} = 1$$

$$C_2 = \left(z - \frac{1}{3}\right) \frac{X(z)}{z} \Bigg|_{z=\frac{1}{3}} = \frac{1}{6} \cdot \frac{1}{\left(z - \frac{1}{2}\right)} \Bigg|_{z=\frac{1}{3}}$$

$$C_2 = \frac{1}{6} \cdot \frac{1}{\left(\frac{1}{3} - \frac{1}{2}\right)} = -1$$

$$\therefore \quad X(z) = \frac{z}{z - \frac{1}{2}} - \frac{z}{z - \frac{1}{3}}$$

$$\therefore \quad X(z) = \frac{z}{z-\frac{1}{2}} - \frac{z}{z-\frac{1}{3}}$$

Apply $x(n) = z^{-1}\left[\frac{z}{z-P_k}\right] = (P_k)^n \cdot u(n)$; ROC ; $|z| > |P_k|$

$$\therefore \quad x(n) = \left(\frac{1}{2}\right)^n u(n) - \left(\frac{1}{3}\right)^n u(n)$$

① Using the power series expansion technique, find the inverse Z-transform of the following $X(z)$:

(a) $X(z) = \dfrac{z}{2z^2 - 3z + 1}$; ROC; $|z| < \frac{1}{2}$

(b) $X(z) = \dfrac{z}{2z^2 - 3z + 1}$; ROC; $|z| > 1$

Solution: (a). Since ROC is $|z| < \frac{1}{2}$, therefore $x(n)$ will be a non-causal sequence. So, $N(z)$ and $D(z)$ must be arranged in ascending powers of Z or in descending powers of z^{-1} before long division.

$$\begin{array}{r l}
 & z+3z^2+7z^3+15z^4+31z^5+63z^6+\cdots \\
1-3z+2z^2 & \overline{\big)\, z} \\
 & \underline{-z \mp 3z^2 \pm 2z^3} \\
 & \quad 3z^2-2z^3 \\
 & \quad \underline{-3z^2 \mathop{-}\limits_{+} 9z^3 \mathop{+}\limits_{-} 6z^4} \\
 & \qquad 7z^3-6z^4 \\
 & \qquad \underline{-7z^3 \mp 21z^4 \pm 14z^5} \\
 & \qquad\quad 15z^4-14z^5 \\
 & \qquad\quad \underline{-15z^4 \mathop{-}\limits_{+} 45z^5 \mathop{+}\limits_{-} 30z^6} \\
 & \qquad\qquad 31z^5-30z^6 \\
 & \qquad\qquad \underline{-31z^5 \mathop{-}\limits_{+} 93z^6 \mathop{+}\limits_{-} 62z^7} \\
 & \qquad\qquad\quad 63z^6-62z^7
\end{array}$$

$$\therefore \quad X(z) = z+3z^2+7z^3+15z^4+31z^5+63z^6+\cdots$$

$\therefore \quad X(z) = z + 3z^2 + 7z^3 + 15z^4 + 31z^5 + 63z^6 + \cdots$

since ROC: $|z| < \frac{1}{2}$, so $x(n)$ will be non-causal sequence.

$\therefore \quad x(n) = \{\ldots, 63, 31, 15, 7, 3, 1, \underset{\uparrow}{0}\}$

(b). Since ROC is $|z| > 1$, therefore $x(n)$ will be a causal signal. So, $N(z)$ and $D(z)$ must be arranged in descending powers of z or in ascending powers of z^{-1} before long division.

$$\frac{1}{2}z^{-1}+\frac{3}{4}z^{-2}+\frac{7}{8}z^{-3}+\frac{15}{16}z^{-4}+\frac{31}{32}z^{-5}+\cdots$$

$$2z^2-3z+1 \;\overline{\big)\; z \qquad\qquad\qquad\qquad}$$

$$z-\frac{3}{2}+\frac{1}{2}z^{-1}$$

$$-\quad+\quad-$$

$$\frac{3}{2}-\frac{1}{2}z^{-1}$$

$$\frac{3}{2}-\frac{9}{4}z^{-1}+\frac{3}{4}z^{-2}$$

$$-\quad+\quad-$$

$$\frac{7}{4}z^{-1}-\frac{3}{4}z^{-2}$$

$$\frac{7}{4}z^{-1}-\frac{21}{8}z^{-2}+\frac{7}{8}z^{-3}$$

$$-\quad+\quad-$$

$$\frac{15}{8}z^{-2}-\frac{7}{8}z^{-3}$$

$$\frac{15}{8}z^{-2}-\frac{45}{16}z^{-3}+\frac{15}{16}z^{-4}$$

$$-\quad+\quad-$$

$$\frac{31}{16}z^{-3}-\frac{15}{16}z^{-4}$$

$$\therefore \; X(z) = \frac{1}{2}z^{-1} + \frac{3}{4}z^{-2} + \frac{7}{8}z^{-3} + \frac{15}{16}z^{-4} + \frac{31}{32}z^{-5} + \cdots$$

Since ROC is $|z| > 1$, $x(n)$ will be causal sequence.

Hence $x(n) = \left\{ \underset{\uparrow}{0}, \frac{1}{2}, \frac{3}{4}, \frac{7}{8}, \frac{15}{16}, \frac{31}{32}, \cdots \right\}$