Steady State Error

ECE305

Steady State Error

- If the output of a control system at steady state does not exactly match with the input, the system is said to have steady state error

- Any physical control system inherently suffers steady-state error in response to certain types of inputs.

- A system may have no steady-state error to a step input, but the same system may exhibit nonzero steady-state error to a ramp input.

Derivation of Steady State Error

Consider a simple closed loop system using negative feedback as shown in the Fig.

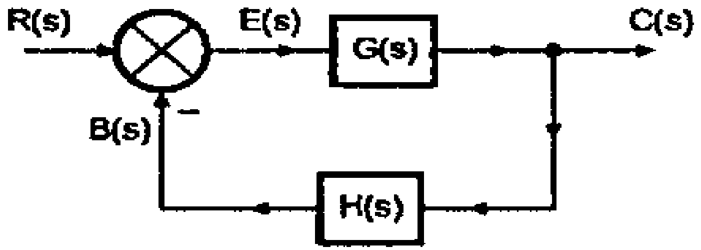


Fig.

where $E(s)$ = Error signal, and $B(s)$ = Feedback signal

Now, $E(s) = R(s) - B(s)$

But $B(s) = C(s)H(s)$

$$\therefore \quad E(s) = R(s) - C(s)H(s)$$

and $$C(s) = E(s)G(s)$$

$$\therefore \quad E(s) = R(s) - E(s)\,G(s)H(s)$$

$$\therefore \quad E(s) + E(s)G(s)H(s) = R(s)$$

$$\therefore \quad \boxed{\begin{aligned} E(s) &= \frac{R(s)}{1+G(s)H(s)} \quad \text{for nonunity feedback} \\ E(s) &= \frac{R(s)}{1+G(s)} \quad \text{for unity feedback} \end{aligned}}$$

This E(s) is the error in Laplace domain and is expression in `s'. We want to calculate the error value. In time domain, corresponding error will be e(t). Now steady state of the system is that state which remains as $t \to \infty$.

$$\therefore \quad \boxed{\text{Steady state error, } e_{ss} = \lim_{t \to \infty} e(t)}$$

Now we can relate this in Laplace domain by using **final value theorem** which states that,

$$\lim_{t \to \infty} F(t) = \lim_{s \to 0} sF(s) \qquad \text{where } F(s) = L\{ F(t) \}$$

$$\text{Therefore,} \quad \boxed{e_{ss} = \lim_{t \to \infty} e(t) = \lim_{s \to 0} sE(s) \quad \text{where } E(s) \text{ is } L\{ e(t) \}.}$$

Substituting E(s) from the expression derived, we can write

$$\boxed{e_{ss} = \lim_{s \to 0} \frac{sR(s)}{1 + G(s)H(s)}}$$

Classification of Control Systems

- Control systems may be classified according to their ability to follow step inputs, ramp inputs, parabolic inputs, and so on.

- The magnitudes of the steady-state errors due to these individual inputs are indicative of the goodness of the system.

Classification of Control Systems

- Consider the unity-feedback control system with the following open-loop transfer function

$$G(s) = \frac{K(T_a s + 1)(T_b s + 1)\cdots(T_m s + 1)}{s^N(T_1 s + 1)(T_2 s + 1)\cdots(T_p s + 1)}$$

- It involves the term s^N in the denominator, representing N poles at the origin.

- A system is called type 0, type 1, type 2, ... , if N=0, N=1, N=2, ... , respectively.

Classification of Control Systems

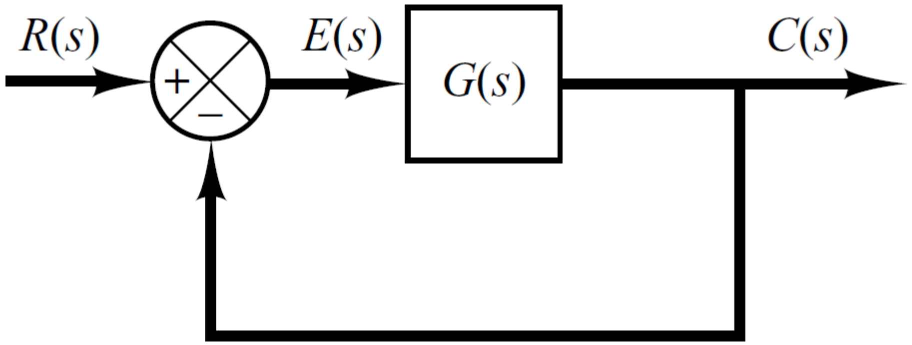
- As the type number is increased, accuracy is improved.

- However, increasing the type number aggravates the stability problem.

- A compromise between steady-state accuracy and relative stability is always necessary.

Steady State Error of Unity Feedback Systems

- Consider the system shown in following figure.



- The closed-loop transfer function is

$$\frac{C(s)}{R(s)} = \frac{G(s)}{1 + G(s)} \qquad G(s) = \frac{K(T_a s + 1)(T_b s + 1)\cdots(T_m s + 1)}{s^N(T_1 s + 1)(T_2 s + 1)\cdots(T_p s + 1)}$$

Steady State Error of Unity Feedback Systems

- Steady state error is defined as the error between the input signal and the output signal when $t \to \infty$.

- The transfer function between the error signal E(s) and the input signal R(s) is $\frac{E(s)}{R(s)} = \frac{1}{1+G(s)}$
- The final-value theorem provides a convenient way to find the steady-state performance of a stable system.
- Since E(s) is $E(s) = \frac{1}{1 + G(s)} R(s)$
- The steady state error is

$$e_{ss} = \lim_{t \to \infty} e(t) = \lim_{s \to 0} sE(s) = \lim_{s \to 0} \frac{sR(s)}{1 + G(s)}$$

Static Error Constants

- The static error constants are figures of merit of control systems. The higher the constants, the smaller the steady-state error.

- In a given system, the output may be the position, velocity, pressure, temperature, or the like.

- Therefore, in what follows, we shall call the output “position,” the rate of change of the output “velocity,” and so on.

- This means that in a temperature control system “position” represents the output temperature, “velocity” represents the rate of change of the output temperature, and so on.

Static Position Error Constant (K_p)

- The steady-state error of the system for a unit-step input is

$$e_{ss} = \lim_{s \to 0} \frac{\cancel{s}}{1 + G(s)} \frac{1}{\cancel{s}}$$

$$= \frac{1}{1 + G(0)}$$

- The static position error constant K_p is defined by

$$K_p = \lim_{s \to 0} G(s) = G(0)$$

- Thus, the steady-state error in terms of the static position error constant K_p is given by

$$e_{ss} = \frac{1}{1 + K_p}$$

Static Position Error Constant (K_p)

- For a Type 0 system

$$K_p = \lim_{s\to 0} \frac{K(T_a s + 1)(T_b s + 1)\cdots}{(T_1 s + 1)(T_2 s + 1)\cdots} = K$$

- For Type 1 or higher order systems

$$K_p = \lim_{s\to 0} \frac{K(T_a s + 1)(T_b s + 1)\cdots}{s^N(T_1 s + 1)(T_2 s + 1)\cdots} = \infty, \qquad \text{for } N \geq 1$$

- For a unit step input the steady state error **e_{ss}** is

$$e_{ss} = \frac{1}{1 + K}, \qquad \text{for type 0 systems}$$

$$e_{ss} = 0, \qquad \text{for type 1 or higher systems}$$

Static Velocity Error Constant (K_v)

- The steady-state error of the system for a unit-ramp input is

$$e_{ss} = \lim_{s \to 0} \frac{s}{1 + G(s)} \frac{1}{s^2}$$

$$= \lim_{s \to 0} \frac{1}{sG(s)}$$

- The static velocity error constant K_v is defined by

$$K_v = \lim_{s \to 0} sG(s)$$

- Thus, the steady-state error in terms of the static velocity error constant K_v is given by

$$e_{ss} = \frac{1}{K_v}$$

Static Velocity Error Constant (K_v)

- For a Type 0 system

$$K_v = \lim_{s \to 0} \frac{sK(T_a s + 1)(T_b s + 1)\cdots}{(T_1 s + 1)(T_2 s + 1)\cdots} = 0$$

- For Type 1 systems

$$K_v = \lim_{s \to 0} \frac{sK(T_a s + 1)(T_b s + 1)\cdots}{s(T_1 s + 1)(T_2 s + 1)\cdots} = K$$

- For type 2 or higher order systems

$$K_v = \lim_{s \to 0} \frac{sK(T_a s + 1)(T_b s + 1)\cdots}{s^N(T_1 s + 1)(T_2 s + 1)\cdots} = \infty, \qquad \text{for } N \geq 2$$

Static Velocity Error Constant (K_v)

- For a ramp input the steady state error e_{ss} is

$$e_{ss} = \frac{1}{K_v} = \infty, \qquad \text{for type 0 systems}$$

$$e_{ss} = \frac{1}{K_v} = \frac{1}{K}, \qquad \text{for type 1 systems}$$

$$e_{ss} = \frac{1}{K_v} = 0, \qquad \text{for type 2 or higher systems}$$

Static Acceleration Error Constant (K_a)

- The steady-state error of the system for parabolic input is

$$e_{ss} = \lim_{s \to 0} \frac{s}{1 + G(s)} \frac{1}{s^3}$$

$$= \frac{1}{\lim_{s \to 0} s^2 G(s)}$$

- The static acceleration error constant K_a is defined by

$$K_a = \lim_{s \to 0} s^2 G(s)$$

- Thus, the steady-state error in terms of the static acceleration error constant K_a is given by

$$e_{ss} = \frac{1}{K_a}$$

Static Acceleration Error Constant (K_a)

- For a Type 0 system

$$K_a = \lim_{s\to 0} \frac{s^2 K(T_a s + 1)(T_b s + 1)\cdots}{(T_1 s + 1)(T_2 s + 1)\cdots} = 0$$

- For Type 1 systems

$$K_a = \lim_{s\to 0} \frac{s^2 K(T_a s + 1)(T_b s + 1)\cdots}{s(T_1 s + 1)(T_2 s + 1)\cdots} = 0$$

- For type 2 systems

$$K_a = \lim_{s\to 0} \frac{s^2 K(T_a s + 1)(T_b s + 1)\cdots}{s^2(T_1 s + 1)(T_2 s + 1)\cdots} = K$$

- For type 3 or higher order systems

$$K_a = \lim_{s\to 0} \frac{s^2 K(T_a s + 1)(T_b s + 1)\cdots}{s^N(T_1 s + 1)(T_2 s + 1)\cdots} = \infty, \qquad \text{for } N \geq 3$$

Static Acceleration Error Constant (K_a)

- For a parabolic input the steady state error $\mathbf{e_{ss}}$ is

$$e_{ss} = \infty, \qquad \text{for type 0 and type 1 systems}$$

$$e_{ss} = \frac{1}{K}, \qquad \text{for type 2 systems}$$

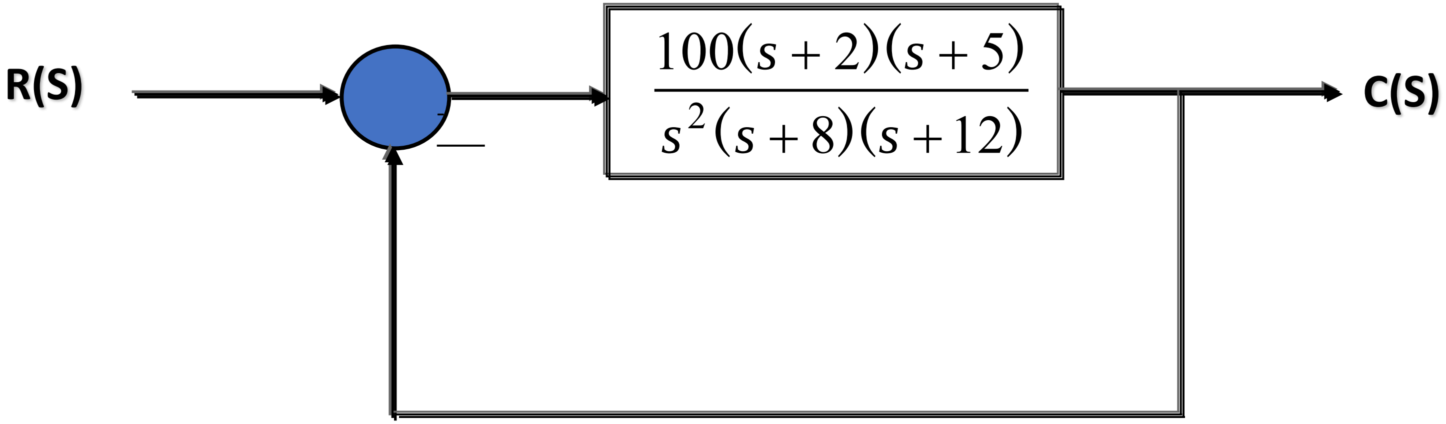
$$e_{ss} = 0, \qquad \text{for type 3 or higher systems}$$

Summary

	Step Input $r(t) = 1$	Ramp Input $r(t) = t$	Acceleration Input $r(t) = \frac{1}{2}t^2$
Type 0 system	$\frac{1}{1 + K}$	∞	∞
Type 1 system	0	$\frac{1}{K}$	∞
Type 2 system	0	0	$\frac{1}{K}$

Example 2

- For the system shown in figure below evaluate the static error constants and find the expected steady state errors for the standard step, ramp and parabolic inputs.



Example 2

$$G(s)=\frac{100(s+2)(s+5)}{s^2(s+8)(s+12)}$$

$$K_p=\lim_{s\to 0}G(s)$$

$$K_p=\lim_{s\to 0}\left(\frac{100(s+2)(s+5)}{s^2(s+8)(s+12)}\right)$$

$$K_p=\infty$$

$$K_v=\lim_{s\to 0}sG(s)$$

$$K_v=\lim_{s\to 0}\left(\frac{100s(s+2)(s+5)}{s^2(s+8)(s+12)}\right)$$

$$K_v=\infty$$

$$K_a=\lim_{s\to 0}s^2G(s)$$

$$K_a=\lim_{s\to 0}\left(\frac{100s^2(s+2)(s+5)}{s^2(s+8)(s+12)}\right)$$

$$K_a=\left(\frac{100(0+2)(0+5)}{(0+8)(0+12)}\right)=10.4$$

Example 2

$K_p = \infty$ $\quad K_v = \infty$ $\quad K_a = 10.4$

$$e_{ss} = \frac{1}{1 + K_p} = 0$$

$$e_{ss} = \frac{1}{K_v} = 0$$

$$e_{ss} = \frac{1}{K_a} = 0.09$$

Example *Find the steady state error for various types of standard test inputs for a unity feedback system with*

$$G(s) = \frac{K}{s(s+5)\,(s+10)}$$

(a) $K = 10$ (b) $K = 200$

Solution :

$$G(s)H(s) = \frac{K}{s(s+5)\ (s+10)} = \frac{K}{s \times 5 \times \left(1+\frac{s}{5}\right) \times 10 \times \left(1+\frac{s}{10}\right)}$$?

$$= \frac{\left(\frac{K}{50}\right)}{s(1+0.2s)\ (1+0.1s)}$$

$$K_p = \lim_{s \to 0} G(s)H(s) = \lim_{s \to 0} \frac{\left(\frac{K}{50}\right)}{s(1+0.2s)\ (1+0.1s)} = \infty$$

$$K_v = \lim_{s \to 0} sG(s)H(s) = \lim_{s \to 0} \frac{s\left(\frac{K}{50}\right)}{s(1+0.2s)\ (1+0.1s)} = \frac{K}{50}$$

$$K_a = \lim_{s \to 0} s^2G(s)H(s) = \lim_{s \to 0} \frac{s^2\left(\frac{K}{50}\right)}{s(1+0.2s)\ (1+0.1s)} = 0.$$

$\therefore$ For step input of magnitude 1,

$$e_{ss} = \frac{1}{1+K_p} = \frac{1}{1+\infty} = 0 \qquad \text{... for any value of K.}$$

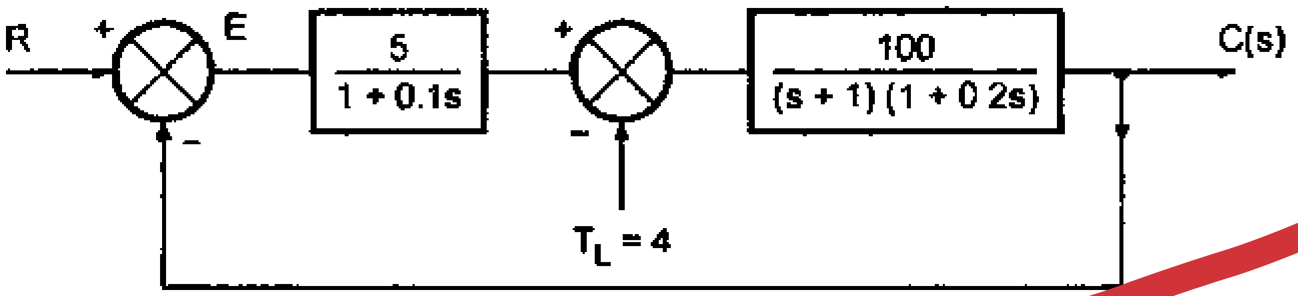
For ramp input of magnitude 1,

$$e_{ss} = \frac{1}{K_v} = \frac{50}{K}$$

a) For K = 10, $e_{ss} = 5$

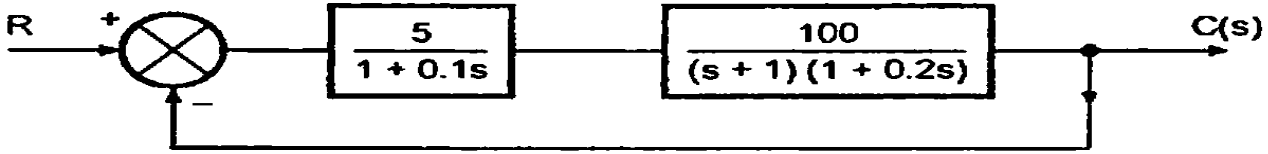
b) For K = 200, $e_{ss} = 0.25$

Example *In the system given, the command input is R = 10 and disturbance signal is $T_L = 4$, what is the steady state error ?*



Solution : Using superposition principle, consider inputs separately.

a) R acting, $\quad T_L = 0$



$$\therefore \qquad G(s)H(s) = \frac{500}{(1+0.1s)\ (s+1)\ (1+0.2s)}$$

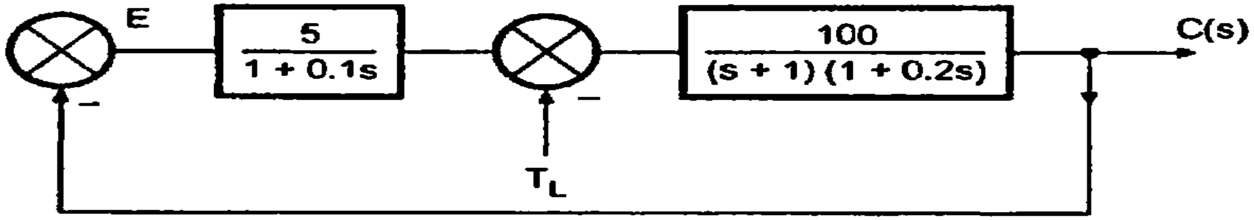
For step input

$$K_p = \lim_{s \to 0} G(s)H(s) = 500$$

?

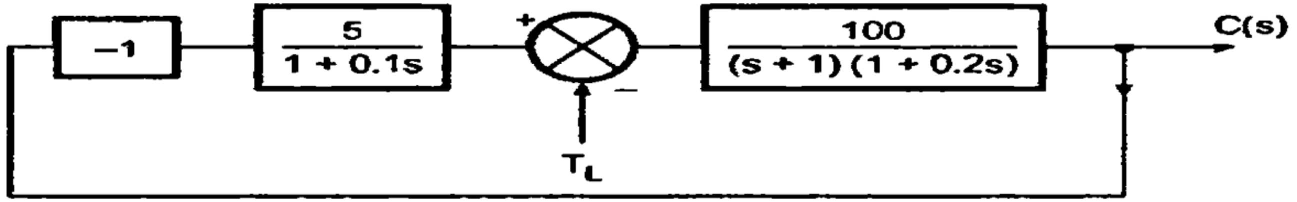
$$\therefore \qquad e_{ss1} = \frac{A}{1+K_p} \quad \text{where } A = \text{magnitude of step} = \frac{10}{1+500} = \frac{10}{501}$$

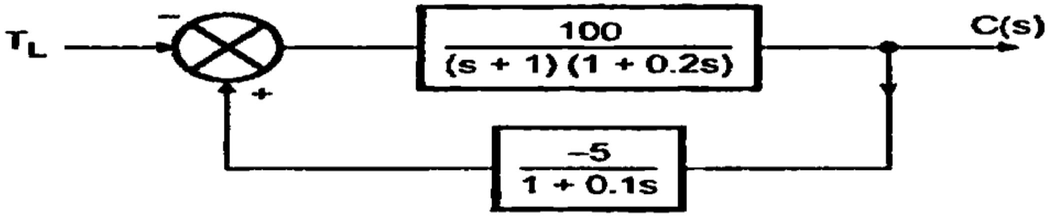
b) T_L acting, $\quad R = 0$



$$E(s) = -C(s)$$

As system is not in standard form, error coefficient method cannot be used.





$$\frac{C(s)}{T(s)} = \frac{\dfrac{100}{(1+s)\ (1+0.2s)}}{1 - \dfrac{100}{(1+s)\ (1+0.2s)} \times \left(\dfrac{-5}{1+0.1s}\right)}$$

$$\frac{C(s)}{T(s)} = \frac{100(1+0.1s)}{(1+s)\ (1+0.2s) \times (1+0.1s) + 500}$$

Now $\quad T(s) = \dfrac{-4}{s}$ −ve sign as T_L applied with −ve sign.

$\therefore \quad C(s) = \dfrac{-400(1+0.1s)}{s[(1+s)\ (1+0.2s) \times (1+0.1s) + 500]}$

but $\quad E(s) = -\,C(s) = \dfrac{+400(1+0.1s)}{s[(1+s)\ (1+0.2s) \times (1+0.1s) + 500]}$

$\therefore \quad e_{ss2} = \lim\limits_{s \to 0} sE(s) = \lim\limits_{s \to 0} \dfrac{400(1+0.1s)}{[(1+s)\ (1+0.2s) \times (1+0.1s) + 500]}$

$$= \left[\frac{400}{1+500}\right] = \frac{+400}{501}$$

$\therefore$ Total error $\quad e_{ss} = e_{ss1} + e_{ss2} = \dfrac{10}{501} + \dfrac{400}{501} = \dfrac{410}{501} = 0.8183$