ELECTROMAGNETIC FIELD THEORY

Amp Circuit Law – Maxwell’s Equation

By: Shakti Raj Chopra

Books

Text Book:

- 1. PRINCIPLES OF ELECTROMAGNETICS
- By - MATTHEW N.O. SADIKU, 4th 2009 OXFORD UNIVERSITY
- PRESS, INDIA

Reference Book:

- 1. ELECTROMAGNETIC WAVES AND RADIATING SYSTEMS
- By - EDWARD C. JORDAN 5th PRENTICE HALL

AMPERE'S CIRCUIT LAW

- **Ampere's circuit law** states that the line integral of the tangential component of H around a *closed* path is the same as the net current Ienc. enclosed by the path.
- In other words, the circulation of H equals Ienc; that is,

$$\boxed{\oint \mathbf{H} \cdot d\mathbf{l} = I_{\text{enc}}}$$

- By applying Stoke's theorem to the left-hand side of eq., we obtain

$$I_{\text{enc}} = \oint_L \mathbf{H} \cdot d\mathbf{l} = \int_S (\nabla \times \mathbf{H}) \cdot d\mathbf{S}$$

- But

$$I_{\text{enc}} = \int_S \mathbf{J} \cdot d\mathbf{S}$$

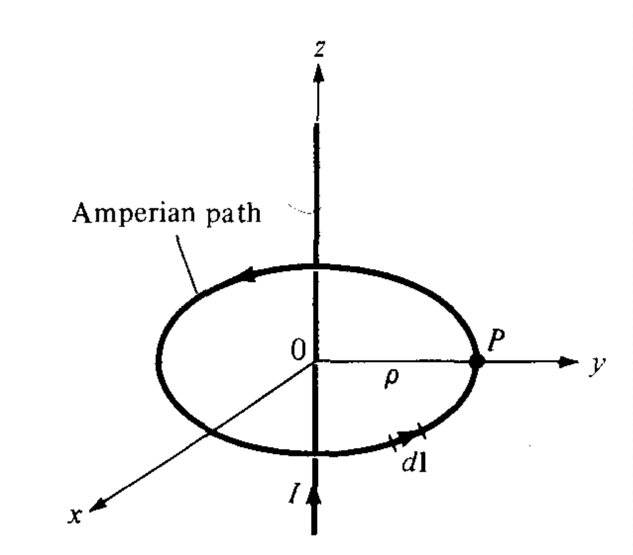
- Comparing the surface integrals in eqs

$$\boxed{\nabla \times \mathbf{H} = \mathbf{J}}$$

- This is the third Maxwell's equation to be derived; it is essentially Ampere's law in differential (or point) form.
- we should observe that $\nabla \times \mathbf{H} = \mathbf{J} \neq 0$; that is, magnetostatic field is not conservative.

APPLICATIONS OF AMPERE'S LAW

- A. **Infinite Line Current**:
- Consider an infinitely long filamentary current I along the z-axis as in Figure.



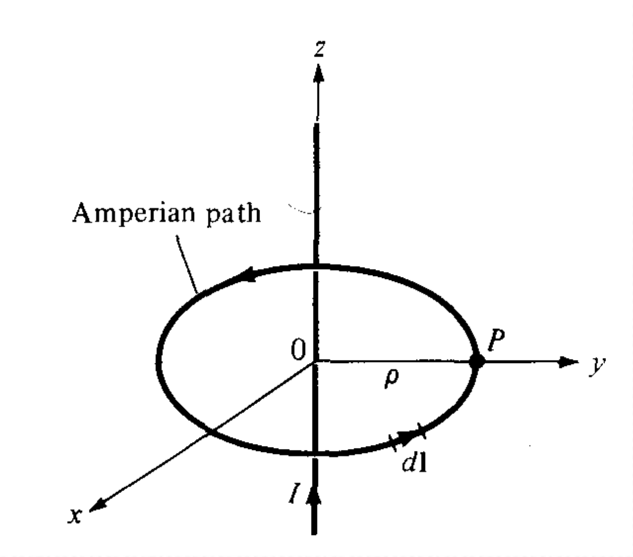
To determine H at an observation point *P*, we allow a closed path pass through *P*. This path, on which Ampere's law is to be applied, is known as an *Amperian path*.

$$I = \int H_\phi \mathbf{a}_\phi \cdot \rho \, d\phi \, \mathbf{a}_\phi = H_\phi \int \rho \, d\phi = H_\phi \cdot 2\pi\rho$$

$$\boxed{\mathbf{H} = \frac{I}{2\pi\rho} \mathbf{a}_\phi}$$

APPLICATIONS OF AMPERE'S LAW

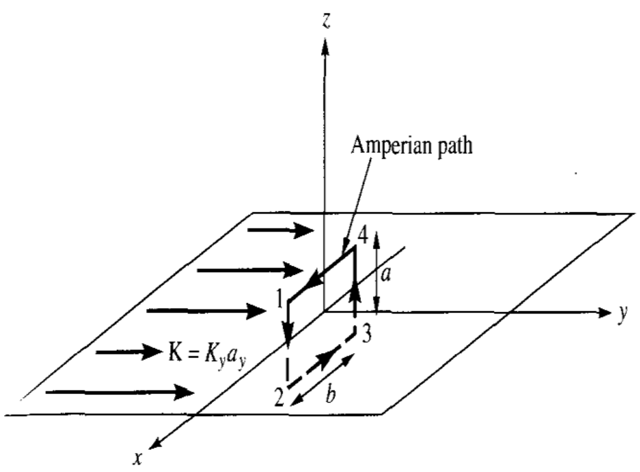
- Consider an infinitely long filamentary current I along the z-axis as in Figure.



To determine H at an observation point *P*, we allow a closed path pass through *P*. This path, on which Ampere's law is to be applied, is known as an *Amperian path*.

APPLICATIONS OF AMPERE'S LAW

- **B. Infinite Sheet of Current**:
- Consider an infinite current sheet in the $z = 0$ plane. If the sheet has a uniform current density $\mathbf{K} = K_y \mathbf{a}_y$ A/m as shown in Figure, applying Ampere's law to the rectangular closed path (Amperian path) gives



$$\oint \mathbf{H} \cdot d\mathbf{l} = I_{enc} = K_y b$$

$$\mathbf{H} = \begin{cases} H_o \mathbf{a}_x & z > 0 \\ -H_o \mathbf{a}_x & z < 0 \end{cases}$$

$$\begin{aligned} \oint \mathbf{H} \cdot d\mathbf{l} &= \left(\int_1^2 + \int_2^3 + \int_3^4 + \int_4^1 \right) \mathbf{H} \cdot d\mathbf{l} \\ &= 0(-a) + (-H_o)(-b) + 0(a) + H_o(b) \\ &= 2H_o b \end{aligned}$$

- But $\oint \mathbf{H} \cdot d\mathbf{l} = I_{enc} = K_y b$ hence $H_o = \frac{1}{2} K_y.$
- Substituting *Ho* $\mathbf{H} = \begin{cases} \frac{1}{2} K_y \mathbf{a}_x, & z > 0 \\ -\frac{1}{2} K_y \mathbf{a}_x, & z < 0 \end{cases}$

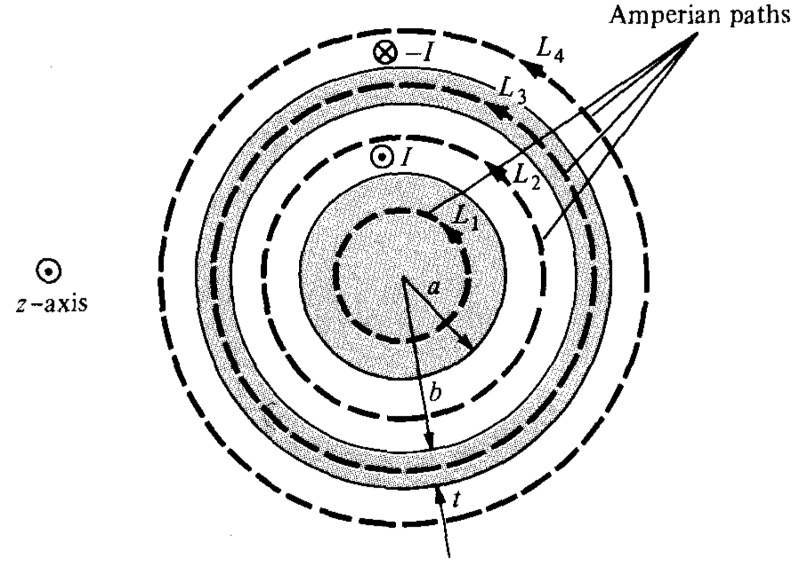
- In general, for an infinite sheet of current density K A/m,.

$$\boxed{\mathbf{H} = \frac{1}{2}\mathbf{K} \times \mathbf{a}_n}$$

- Where $\mathbf{a}_n$ is a unit normal vector directed from the current sheet to the point of interest.

Infinitely Long Coaxial Transmission Line

- Consider an infinitely long transmission line consisting of two concentric cylinders having their axes along the z-axis. The cross section of the line is shown in Figure. where the z-axis is out of the page.



- The inner conductor has radius *a* and carries current I while the outer conductor has inner radius *b* and thickness *t* and carries return current - I .
- We want to determine H everywhere assuming that current is uniformly distributed in both conductors.
- we apply Ampere's law along the Amperian path for each of possible regions

$$0 \le \rho \le a, a \le \rho \le b, b \le \rho \le b + t, \text{ and } \rho \ge b + t.$$

For region $0 \le \rho \le a$, we apply Ampere's law to path L_1, giving

$$\oint_{L_1} \mathbf{H} \cdot d\mathbf{l} = I_{enc} = \int \mathbf{J} \cdot d\mathbf{S}$$

Since the current is uniformly distributed over the cross section,

$$\mathbf{J} = \frac{I}{\pi a^2} \mathbf{a}_z, \qquad d\mathbf{S} = \rho \, d\phi \, d\rho \, \mathbf{a}_z$$

$$I_{enc} = \int \mathbf{J} \cdot d\mathbf{S} = \frac{I}{\pi a^2} \iint \rho \, d\phi \, d\rho = \frac{I}{\pi a^2} \pi \rho^2 = \frac{I\rho^2}{a^2}$$

- Hence

$$H_\phi \int dl = H_\phi 2\pi\rho = \frac{I\rho^2}{a^2}$$

$$H_\phi = \frac{I\rho}{2\pi a^2}$$

For region $a \leq \rho \leq b$, we use path L_2 as the Amperian path,

$$\oint_{L_2} \mathbf{H} \cdot d\mathbf{l} = I_{\text{enc}} = I$$

$$H_\phi 2\pi\rho = I$$

or

$$H_\phi = \frac{I}{2\pi\rho}$$

since the whole current I is enclosed by L_2.

For region $b \leq \rho \leq b + t$, we use path L_3, getting

$$\oint \mathbf{H} \cdot d\mathbf{l} = H_\phi \cdot 2\pi\phi = I_{enc}$$

$$I_{enc} = I + \int \mathbf{J} \cdot d\mathbf{S}$$

and $\mathbf{J}$ in this case is the current density (current per unit area) of the outer conductor and is along $-\mathbf{a}_z$, that is,

$$\mathbf{J} = -\frac{I}{\pi[(b + t)^2 - b^2]} \mathbf{a}_z$$

- Thus

$$I_{enc} = I - \frac{I}{\pi[(b+t)^2 - t^2]} \int_{\phi=0}^{2\pi} \int_{\rho=b}^{\rho} \rho \, d\rho \, d\phi$$

$$= I\left[1 - \frac{\rho^2 - b^2}{t^2 + 2bt}\right]$$

- Substituting this in eq.

$$H_\phi = \frac{I}{2\pi\rho}\left[1 - \frac{\rho^2 - b^2}{t^2 + 2bt}\right]$$

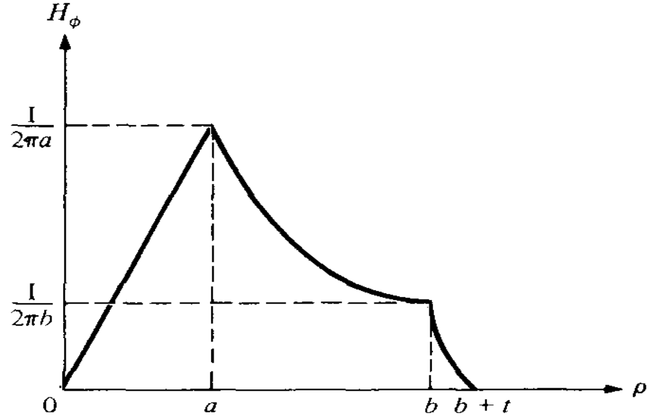
For region $\rho \geq b + t$, we use path L_4, getting

$$\oint_{L_4} \mathbf{H} \cdot d\mathbf{l} = I - I = 0$$

or

$$H_\phi = 0$$

$$
\mathbf{H} = \begin{cases} \dfrac{I\rho}{2\pi a^2}\,\mathbf{a}_\phi, & 0 \le \rho \le a \\[2ex] \dfrac{I}{2\pi\rho}\,\mathbf{a}_\phi, & a \le \rho \le b \\[2ex] \dfrac{I}{2\pi\rho}\left[1 - \dfrac{\rho^2 - b^2}{t^2 + 2bt}\right]\mathbf{a}_\phi, & b \le \rho \le b + t \\[2ex] 0, & \rho \ge b + t \end{cases}
$$



MAGNETIC FLUX DENSITY—MAXWELL'S EQUATION

The magnetic flux density **B** is similar to the electric flux density **D**. As $\mathbf{D} = \varepsilon_o \mathbf{E}$ in free space, the magnetic flux density **B** is related to the magnetic field intensity **H** according to

$$\boxed{\mathbf{B} = \mu_o \mathbf{H}}$$

where μ_o is a constant known as the *permeability of free space*. The constant is in henrys/meter (H/m) and has the value of

$$\boxed{\mu_o = 4\pi \times 10^{-7} \text{ H/m}}$$

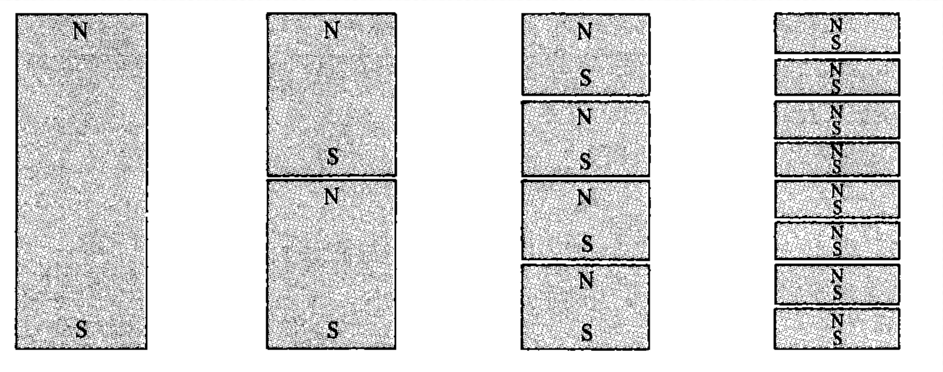
The magnetic flux through a surface S is given by

$$\boxed{\Psi = \int_S \mathbf{B} \cdot d\mathbf{S}}$$

where the magnetic flux Ψ is in webers (Wb) and the magnetic flux density is in webers/square meter (Wb/m^2) or teslas.

- In an electrostatic field, the flux passing through a closed surface is the same as the charge enclosed; that is, $\Psi = \oint \mathbf{D} \cdot d\mathbf{S} = Q.$

- This is due to the fact that *it is not possible to have isolated magnetic poles (or magnetic charges).*



- An **isolated magnetic** charge does not exist.

Thus the total flux through a closed surface in a magnetic field must be zero; that is,

$$\boxed{\oint \mathbf{B} \cdot d\mathbf{S} = 0}$$

This equation is referred to as the *law of conservation of magnetic flux* or *Gauss's law for magnetostatic fields* just as $\oint \mathbf{D} \cdot d\mathbf{S} = Q$ is Gauss's law for electrostatic fields. Although the magnetostatic field is not conservative, magnetic flux is conserved.

By applying the divergence theorem to eq. (7.33), we obtain

$$\oint_S \mathbf{B} \cdot d\mathbf{S} = \int_v \nabla \cdot \mathbf{B} \, dv = 0 \qquad \boxed{\nabla \cdot \mathbf{B} = 0}$$

- This equation is the fourth Maxwell's equation to be derived.

TABLE 7.2 Maxwell's Equations for Static EM Fields

Differential (or Point) Form	Integral Form	Remarks
$\nabla \cdot \mathbf{D} = \rho_v$	$\oint_S \mathbf{D} \cdot d\mathbf{S} = \int_v \rho_v \, dv$	Gauss's law
$\nabla \cdot \mathbf{B} = 0$	$\oint_S \mathbf{B} \cdot d\mathbf{S} = 0$	Nonexistence of magnetic monopole
$\nabla \times \mathbf{E} = 0$	$\oint_L \mathbf{E} \cdot d\mathbf{l} = 0$	Conservativeness of electrostatic field
$\nabla \times \mathbf{H} = \mathbf{J}$	$\oint_L \mathbf{H} \cdot d\mathbf{l} = \int_S \mathbf{J} \cdot d\mathbf{S}$	Ampere's law

Example

Planes $z = 0$ and $z = 4$ carry current $\mathbf{K} = -10\mathbf{a}_x$ A/m and $\mathbf{K} = 10\mathbf{a}_x$ A/m, respectively. Determine $\mathbf{H}$ at

(a) (1, 1, 1)

(b) (0, −3, 10)

Solution

Let the parallel current sheets be as in Figure 7.14. Also let

$$\mathbf{H} = \mathbf{H}_o + \mathbf{H}_4$$

where $\mathbf{H}_o$ and $\mathbf{H}_4$ are the contributions due to the current sheets $z = 0$ and $z = 4$, respectively. We make use of eq. (7.23).

(a) At (1, 1, 1), which is between the plates ($0 < z = 1 < 4$),

$$\mathbf{H}_o = 1/2\ \mathbf{K} \times \mathbf{a}_n = 1/2\ (-10\mathbf{a}_x) \times \mathbf{a}_z = 5\mathbf{a}_y \text{ A/m}$$

$$\mathbf{H}_4 = 1/2\ \mathbf{K} \times \mathbf{a}_n = 1/2\ (10\mathbf{a}_x) \times (-\mathbf{a}_z) = 5\mathbf{a}_y \text{ A/m}$$

Hence,

$$\mathbf{H} = 10\mathbf{a}_y \text{ A/m}$$

Hence,

$$\mathbf{H} = 10\mathbf{a}_y \text{ A/m}$$

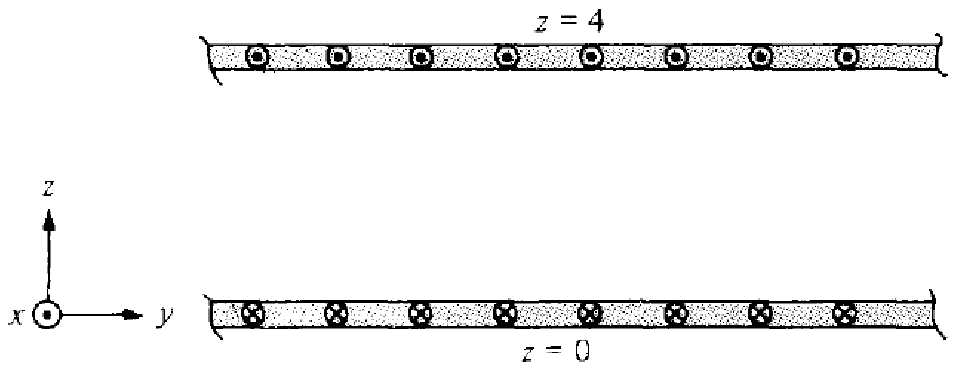


Figure 7.14 For Example 7.5; parallel infinite current sheets.

(b) At $(0, -3, 10)$, which is above the two sheets ($z = 10 > 4 > 0$),

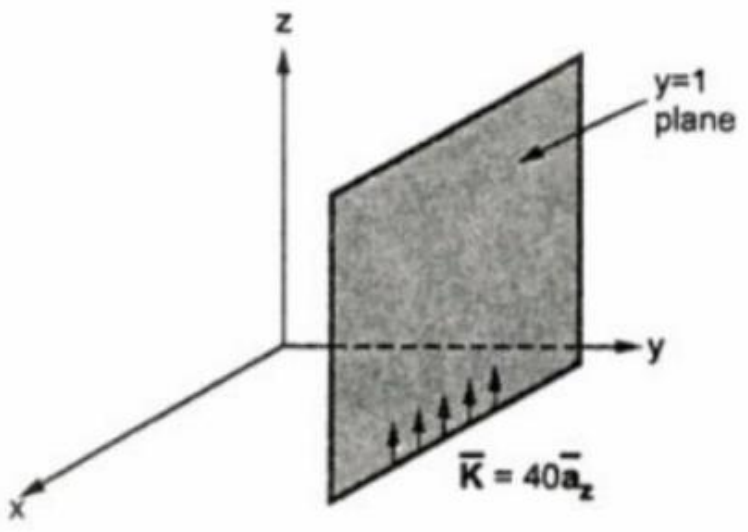
$$\mathbf{H}_o = 1/2\,(-10\mathbf{a}_x) \times \mathbf{a}_z = 5\mathbf{a}_y \text{ A/m}$$

$$\mathbf{H}_4 = 1/2\,(10\mathbf{a}_x) \times \mathbf{a}_z = -5\mathbf{a}_y \text{ A/m}$$

Hence,

$$\mathbf{H} = 0 \text{ A/m}$$

The plane $y = 1$ carries current density $\overline{K} = 40\,\overline{a}_z$ A/m. Find $\overline{H}$ at A (0,0,0) and B(1,5,−2).

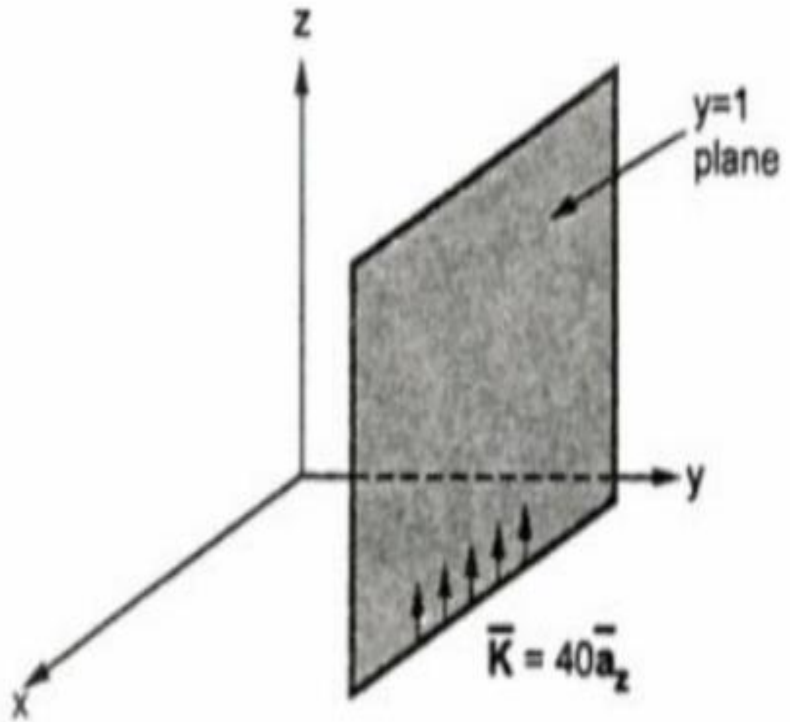


Solution : The sheet is located at $y = 1$ on which $\overline{K}$ is in $\bar{a}_z$ direction. The sheet is infinite

The $\overline{H}$ will be in x direction.

a) Point A (0, 0, 0)

$\bar{a}_N = -\bar{a}_y$ normal to current sheet at Point A



$$\therefore \quad \overline{H} = \frac{1}{2}\overline{K} \times \bar{a}_N$$

$$= \frac{1}{2}\left[40\bar{a}_z \times -\bar{a}_y\right]$$

Now $\quad \bar{a}_z \times \bar{a}_y = -\bar{a}_x$

$$\therefore \quad \overline{H} = \frac{1}{2}[+40]\bar{a}_x = 20\,\bar{a}_x \ \text{A/m}$$

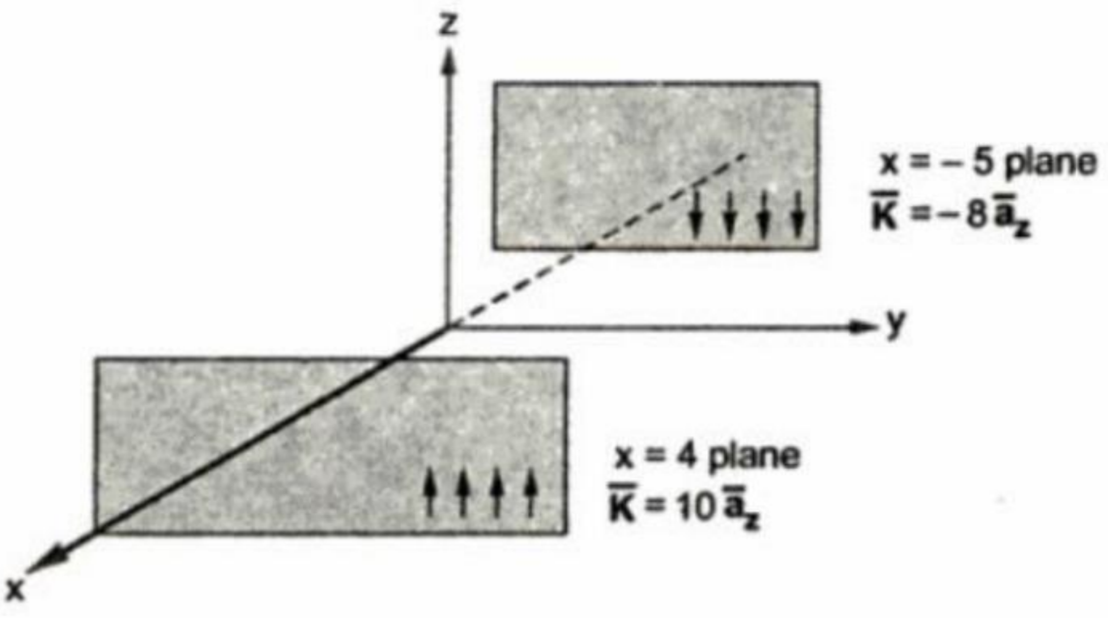
b) Point B $(1,5,-2)$

This is to the right of the plane as $y = 5$ for B.

$\therefore \quad \bar{a}_N = \bar{a}_y$ normal to sheet at point B

$$\therefore \quad \overline{H} = \frac{1}{2}\,\overline{K}\times\bar{a}_N = \frac{1}{2}\left[40\,\bar{a}_z \times \bar{a}_y\right] = -20\,\bar{a}_x \quad \text{A/m}$$

A current sheet $\overline{K} = 10\,\overline{a}_z$ A/m lies in the $x = 4$ m plane and a second sheet $\overline{K} = -8\overline{a}_z$ A/m is at $x = -5$ m plane. Find $\overline{H}$ in all the regions.



Region 1 $\quad x > 4$

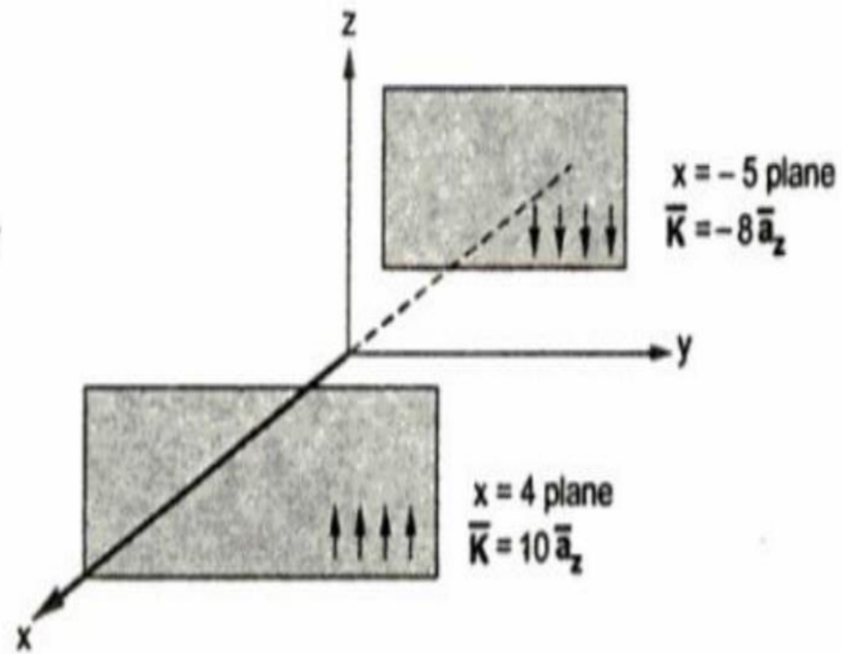
For $x > 4$, $\quad \bar{a}_N = +\bar{a}_x$ for both the sheets

$$\therefore \quad \bar{H}_1 = \frac{1}{2}\bar{K}\times\bar{a}_N = \frac{1}{2}[-8\bar{a}_z\times\bar{a}_x] = -4\,\bar{a}_y$$

and

$$\bar{H}_2 = \frac{1}{2}\bar{K}\times\bar{a}_N = \frac{1}{2}[10\bar{a}_z\times\bar{a}_x] = +5\,\bar{a}_y$$

$$\therefore \quad \bar{H} = -4\,\bar{a}_y + 5\,\bar{a}_y = \bar{a}_y \text{ A/m}$$



Region 2 $x < 4$ but $x > -5$

For this, $\bar{a}_N = +\bar{a}_x$ for sheet at $x = -5$

$$\therefore \quad \bar{H}_1 = \frac{1}{2}\bar{K}\times\bar{a}_N = \frac{1}{2}[-8\bar{a}_z\times\bar{a}_x] = -4\,\bar{a}_y$$

For this, $\bar{a}_N = -\bar{a}_x$ for sheet at $x = 4$

$$\therefore \quad \bar{H}_2 = \frac{1}{2}\bar{K}\times\bar{a}_N = \frac{1}{2}[10\bar{a}_z\times\bar{a}_x] = -5\,\bar{a}_y$$

$$\therefore \quad \bar{H} = -4\,\bar{a}_y - 5\,\bar{a}_y = -9\,\bar{a}_y \text{ A/m}$$

Region 3 For $x < -5$

For this region, for both the sheets $\bar{a}_N = -\bar{a}_x$

$$\therefore \quad \bar{H}_1 = \frac{1}{2}\bar{K}\times\bar{a}_N = \frac{1}{2}[-8\bar{a}_z\times-\bar{a}_x] = +4\,\bar{a}_y$$

$$\therefore \quad \bar{H}_2 = \frac{1}{2}\bar{K}\times\bar{a}_N = \frac{1}{2}[10\bar{a}_z\times-\bar{a}_x] = -5\,\bar{a}_y$$

$$\therefore \quad \bar{H} = +4\,\bar{a}_y - 5\,\bar{a}_y = -\bar{a}_y \text{ A/m}$$

Plane $y = 1$ carries current $\mathbf{K} = 50\mathbf{a}_z$ mA/m. Find $\mathbf{H}$ at

(a) $(0, 0, 0)$

(b) $(1, 5, -3)$

$$\mathbf{H} = \frac{1}{2}\mathbf{K} \times \mathbf{a}_n$$

(a) $$H(0,0,0) = \frac{1}{2}50a_z \times (-a_y) = \underline{\underline{25a_x}} \quad \text{mA/m}$$

(b) $$H(1,5,-3) = \frac{1}{2}50a_z \times a_y = \underline{\underline{-25a_x}} \quad \text{mA/m}$$

One of the following is not a source of magnetostatic fields:

(a) A dc current in a wire

(b) A permanent magnet

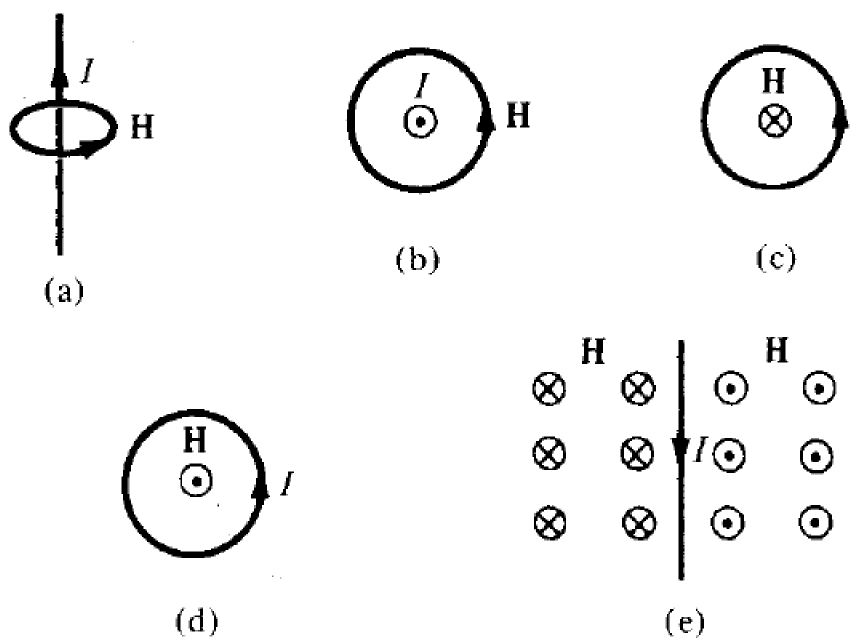
(c) An accelerated charge

(d) An electric field linearly changing with time

(e) A charged disk rotating at uniform speed

- C

Identify the configuration in Figure 7.22 that is not a correct representation of I and $\mathbf{H}$.



Consider points A, B, C, D, and E on a circle of radius 2 as shown in Figure 7.23. The items in the right list are the values of $\mathbf{a}_\phi$ at different points on the circle. Match these items with the points in the list on the left.

(a)	A	(i)	$\mathbf{a}_x$
(b)	B	(ii)	$-\mathbf{a}_x$
(c)	C	(iii)	$\mathbf{a}_y$
(d)	D	(iv)	$-\mathbf{a}_y$
(e)	E	(v)	$\dfrac{\mathbf{a}_x + \mathbf{a}_y}{\sqrt{2}}$
		(vi)	$\dfrac{-\mathbf{a}_x - \mathbf{a}_y}{\sqrt{2}}$
		(vii)	$\dfrac{-\mathbf{a}_x + \mathbf{a}_y}{\sqrt{2}}$
		(viii)	$\dfrac{\mathbf{a}_x - \mathbf{a}_y}{\sqrt{2}}$

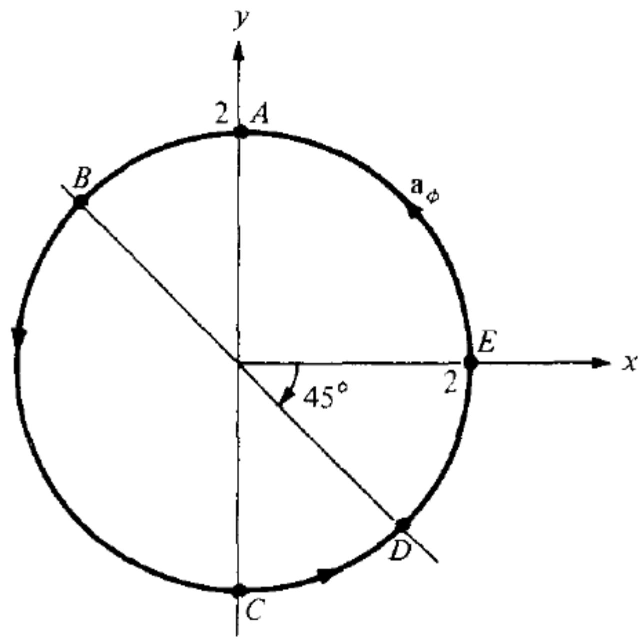


Figure 7.23

(a)-(ii), (b)-(vi), (c)-(i), (d)-(v), (e)-(iii), The z-axis carries filamentary current of 10π A along $\mathbf{a}_z$. Which of these is incorrect?

(a) $\mathbf{H} = -\mathbf{a}_x$ A/m at (0, 5, 0)

(b) $\mathbf{H} = \mathbf{a}_\phi$ A/m at (5, $\pi/4$, 0)

(c) $\mathbf{H} = -0.8\mathbf{a}_x - 0.6\mathbf{a}_y$ at (−3, 4, 0)

(d) $\mathbf{H} = -\mathbf{a}_\phi$ at (5, $3\pi/2$, 0)

- d

Plane $y = 0$ carries a uniform current of $30\mathbf{a}_z$ mA/m. At $(1, 10, -2)$, the magnetic field intensity is

(a) $-15\mathbf{a}_x$ mA/m

(b) $15\mathbf{a}_x$ mA/m

(c) $477.5\mathbf{a}_y$ μA/m

(d) $18.85\mathbf{a}_y$ nA/m

(e) None of the above

- a

Which of these statements is not characteristic of a static magnetic field?

(a) It is solenoidal.

(b) It is conservative.

(c) It has no sinks or sources.

(d) Magnetic flux lines are always closed.

(e) The total number of flux lines entering a given region is equal to the total number of flux lines leaving the region.

- b

Two identical coaxial circular coils carry the same current I but in opposite directions. The magnitude of the magnetic field $\mathbf{B}$ at a point on the axis midway between the coils is

(a) Zero

(b) The same as that produced by one coil

(c) Twice that produced by one coil

(d) Half that produced by one coil.

- a

Thanks a lot !