The Laplace Transform

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Laplace Transform

- **To evaluate the performance of an automatic control**

system commonly used mathematical tool is <u>"Laplace Transform"</u>

- **Laplace transform converts the differential equation into an algebraic equation in 's'.**

- **Laplace transform exist for almost all signals of practical interest.**

Why Laplace Transform?

Laplace Transformation

Time domain unknown f(t), d/dt, Diff Eqs

Frequency domain unknown F(s), Alg Eqs

Solve Differential Equations

Solve Algebraic Equations

Time domain known f(t)

Frequency domain known F(s)

Inverse Laplace Transform

Laplace Transform- Definition

The Laplace transform of a function, *f*(*t*), is defined as

$$F(s) = \mathrm{L}\left[f(t)\right] = \int_0^\infty f(t)e^{-st}dt \qquad (1\text{-}1)$$

where *F*(*s*) is the symbol for the Laplace transform, L is the Laplace transform operator, and *f*(*t*) is some function of time, *t*.

***Note*: The L operator transforms a time domain function *f*(*t*) into an *s* domain function, *F*(*s*). *s* is a *complex variable*:**

$s = a + bj, \quad j\ \mathrm{B}\sqrt{-1}$

Inverse Laplace Transform

By definition, the inverse Laplace transform operator, L^{-1}, converts an *s*-domain function back to the corresponding time domain function:

$$f(t) = \mathcal{L}^{-1}\left[F(s)\right]$$

Transfer Function

- The relationship between input & output of a system is

given by the transfer function.

- <u>Definition:</u> The ratio of Laplace transform of the output to the Laplace transform of the input under the assumption of zero initial conditions is defined as

<u>"Transfer Function"</u>.

The Laplace Transform

The Laplace Transform of a function, f(t), is defined as;

$$L[f(t)] = F(s) = \int_0^{\infty} f(t)e^{-st}dt$$

Eq A

The Inverse Laplace Transform is defined by

$$L^{-1}[F(s)] = f(t) = \frac{1}{2\pi j}\int_{\sigma - j\infty}^{\sigma + j\infty} F(s)e^{ts}ds$$

Eq B

*notes

The Laplace Transform

Laplace Transform of the unit step.

$$L[u(t)] = \int_0^\infty 1 e^{-st} dt = \frac{-1}{s} e^{-st} \Big|_0^\infty$$

$$L[u(t)] = \frac{1}{s}$$

The Laplace Transform of a unit step is:

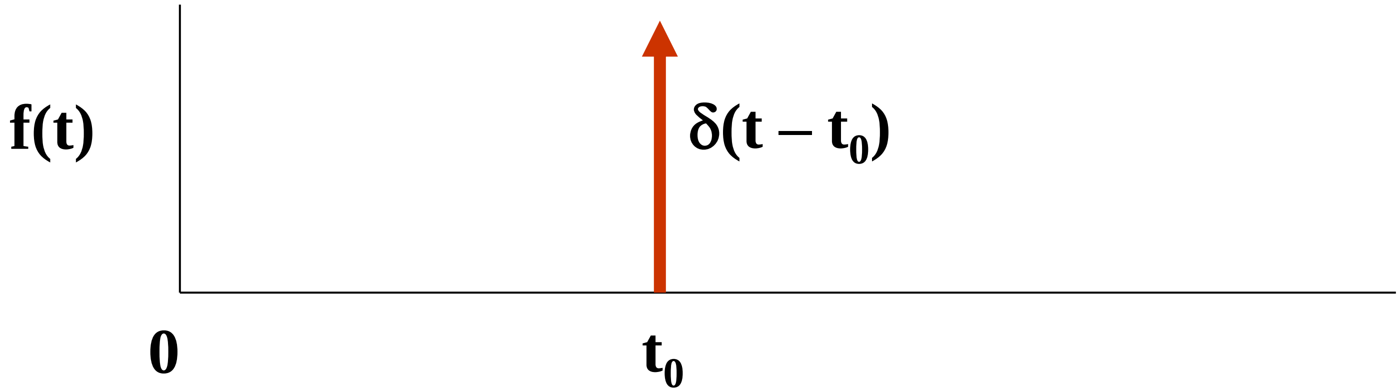
$$\frac{1}{s}$$

*notes

The Laplace Transform

The Laplace transform of a unit impulse:

Pictorially, the unit impulse appears as follows:



Mathematically:

$$\delta(t - t_0) = 0 \quad t \neq 0$$

$$\int_{t_0-\varepsilon}^{t_0+\varepsilon} \delta(t - t_0)\,dt = 1 \qquad \varepsilon > 0$$

*note

The Laplace Transform

The Laplace transform of a unit impulse:

In particular, if we let f(t) = δ(t) and take the Laplace

$$L[\delta(t)] = \int_0^\infty \delta(t) e^{-st} dt = e^{-0s} = 1$$

The Laplace Transform

Building transform pairs:

$$L[e^{-at}u(t)] = \int_0^{\infty} e^{-at}e^{-st}dt = \int_0^{\infty e} e^{-(s+a)t}dt$$

$$L[e^{-at}u(t)] = \frac{-e^{-st}}{(s+a)}\Big|_0^{\infty} = \frac{1}{s+a}$$

A transform pair

$$e^{-at}u(t) \quad \Leftrightarrow \quad \frac{1}{s+a}$$

The Laplace Transform

Building transform pairs:

$$L[\cos(wt)] = \int_0^{\infty} \frac{(e^{jwt} + e^{-jwt})}{2} e^{-st} dt$$

$$= \frac{1}{2}\left[\frac{1}{s - jw} - \frac{1}{s + jw}\right]$$

$$= \frac{s}{s^2 + w^2}$$

$$\cos(wt)u(t) \quad \Leftrightarrow \quad \frac{s}{s^2 + w^2}$$

A transform pair

The Laplace Transform

Time Shift

$$L[f(t-a)u(t-a)] = \int_a^{\infty} f(t-a)e^{-st}$$

$$Let\ x = t-a, then\ dx = dt\ and\ t = x+a$$

$$As\ t \to a,\ x \to 0\ and\ as\ t \to \infty, x \to \infty.\ So,$$

$$\int_0^{\infty} f(x)e^{-s(x+a)}dx = e^{-as}\int_0^{\infty} f(x)e^{-sx}dx$$

$$L[f(t-a)u(t-a)] = e^{-as}F(s)$$

The Laplace Transform

Frequency Shift

$$L[e^{-at} f(t)] = \int_0^\infty [e^{-at} f(t)] e^{-st} dt$$

$$= \int_0^\infty f(t) e^{-(s+a)t} dt = F(s+a)$$

$$\boxed{L[e^{-at} f(t)] = F(s+a)}$$

The Laplace Transform

Example: Using Frequency Shift

Find the $L[e^{-at}\cos(wt)]$

In this case, f(t) = cos(wt) so,

$$F(s) = \frac{s}{s^2 + w^2}$$

$$and \;\; F(s+a) = \frac{(s+a)}{(s+a)^2 + w^2}$$

$$L[e^{-at}\cos(wt)] = \frac{(s+a)}{(s+a)^2 + (w)^2}$$

The Laplace Transform

Time Integration:

Making these substitutions and carrying out The integration shows that

$$L\left[\int_0^\infty f(t)dt\right] = \frac{1}{s}\int_0^\infty f(t)e^{-st}dt$$

$$= \frac{1}{s}F(s)$$

The Laplace Transform

Time Differentiation:

If the L[f(t)] = F(s), we want to show:

$$L\left[\frac{df(t)}{dt}\right] = sF(s) - f(0)$$

Integrate by parts:

$$u = e^{-st},\ du = -se^{-st}dt\ \ and$$

$$dv = \frac{df(t)}{dt}dt = df(t),\ so\ \ v = f(t)$$

*note

The Laplace Transform

Time Differentiation:

Making the previous substitutions gives,

$$L\left[\frac{df}{dt}\right] = f(t)e^{-st}\Big|_0^\infty - \int_0^\infty f(t)\left[-se^{-st}\right]dt$$

$$= 0 - f(0) + s\int_0^\infty f(t)e^{-st}dt$$

So we have shown:

$$\boxed{L\left[\frac{df(t)}{dt}\right] = sF(s) - f(0)}$$

The Laplace Transform

Time Differentiation:

We can extend the previous to show;

$$L\left[\frac{df(t)^2}{dt^2}\right] = s^2F(s) - sf(0) - f'(0)$$

$$L\left[\frac{df(t)^3}{dt^3}\right] = s^3F(s) - s^2 f(0) - sf'(0) - f''(0)$$

$$general\ \ case$$

$$L\left[\frac{df(t)^n}{dt^n}\right] = s^nF(s) - s^{n-1} f(0) - s^{n-2} f'(0)$$

$$- \ldots - f^{(n-1)}(0)$$

The Laplace Transform

Transform Pairs:

f(t)	F(s)
$\delta(t)$	1
$u(t)$	$\frac{1}{s}$
e^{-st}	$\frac{1}{s+a}$
t	$\frac{1}{s^2}$
t^n	$\frac{n!}{s^{n+1}}$

The Laplace Transform

Transform Pairs:

f(t)	F(s)
te^{-at}	$\dfrac{1}{(s+a)^2}$
$t^n e^{-at}$	$\dfrac{n!}{(s+a)^{n+1}}$
$\sin(wt)$	$\dfrac{w}{s^2+w^2}$
$\cos(wt)$	$\dfrac{s}{s^2+w^2}$

The Laplace Transform

Transform Pairs:

f(t)	F(s)
$e^{-at}\sin(wt)$	$\dfrac{w}{(s+a)^2+w^2}$
$e^{-at}\cos(wt)$	$\dfrac{s+a}{(s+a)^2+w^2}$
$\sin(wt+\theta)$	$\dfrac{s\sin\theta+w\cos\theta}{s^2+w^2}$
$\cos(wt+\theta)$	$\dfrac{s\cos\theta-w\sin\theta}{s^2+w^2}$

The Laplace Transform

Common Transform Properties:

f(t)	F(s)
$f(t-t_0)u(t-t_0), t_0 \geq 0$	$e^{-t_o s}F(s)$
$f(t)u(t-t_0), t \geq 0$	$e^{-t_o s}L[f(t+t_0)$
$e^{-at}f(t)$	$F(s+a)$
$\dfrac{d^n f(t)}{dt^n}$	$s^n F(s) - s^{n-1}f(0) - s^{n-2}f'(0) - \ldots - s^0 f^{n-1}f(0)$
$tf(t)$	$-\dfrac{dF(s)}{ds}$
$\int_0^t f(\lambda)d\lambda$	$\dfrac{1}{s}F(s)$

The Laplace Transform

Using Matlab with Laplace transform:

Example **Use Matlab to find the transform of** te^{-4t}

The following is written in italic to indicate Matlab code

```
syms t,s
laplace(t*exp(-4*t),t,s)
ans =
      1/(s+4)^2
```

The Laplace Transform

Using Matlab with Laplace transform:

Example Use Matlab to find the inverse transform of

$$F(s) = \frac{s(s+6)}{(s+3)(s^2+6s+18)} \qquad prob.12.19$$

```
syms s t

ilaplace(s*(s+6)/((s+3)*(s^2+6*s+18)))

ans =
      -exp(-3*t)+2*exp(-3*t)*cos(3*t)
```

The Laplace Transform

Theorem: **Initial Value Theorem:**

If the function f(t) and its first derivative are Laplace transformable and f(t) Has the Laplace transform F(s), and the $\lim_{s \to \infty} sF(s)$ exists, then

$$\boxed{\lim_{s \to \infty} sF(s) = \lim_{t \to 0} f(t) = f(0)}$$

Initial Value Theorem

The utility of this theorem lies in not having to take the inverse of F(s) in order to find out the initial condition in the time domain. This is particularly useful in circuits and systems.

The Laplace Transform

Example: **Initial Value Theorem:**

Given;

$$F(s) = \frac{(s+2)}{(s+1)^2 + 5^2}$$

Find f(0)

$$f(0) = \lim_{s \to \infty} sF(s) = \lim_{s \to \infty} s\frac{(s+2)}{(s+1)^2 + 5^2} = \lim_{s \to \infty}\left[\frac{s^2 + 2s}{s^2 + 2s + 1 + 25}\right]$$

$$= \lim_{s \to \infty} \frac{s^2/s^2 + 2s/s^2}{s^2/s^2 + 2s/s^2 + (26/s^2)} = 1$$

The Laplace Transform

Theorem: **Final Value Theorem:**

If the function f(t) and its first derivative are Laplace transformable and f(t) has the Laplace transform F(s), and the $\lim_{s \to \infty} sF(s)$ **exists, then**

$$\boxed{\lim_{s \to 0} sF(s) = \lim_{t \to \infty} f(t) = f(\infty)}$$

Final Value Theorem

Again, the utility of this theorem lies in not having to take the inverse of F(s) in order to find out the final value of f(t) in the time domain. This is particularly useful in circuits and systems.

The Laplace Transform

Example: **Final Value Theorem:**

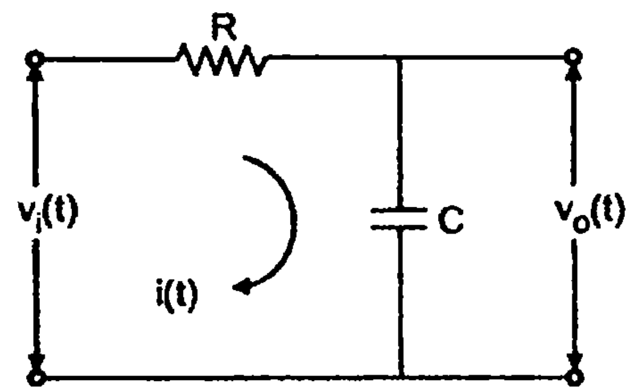
Given:

$$F(s) = \frac{(s+2)^2 - 3^2}{\left[(s+2)^2 + 3^2\right]} \qquad note\ \ F^{-1}(s) = te^{-2t}\cos 3t$$

Find $f(\infty)$.

$$f(\infty) = \lim_{s \to 0} sF(s) = \lim_{s \to 0} s \frac{(s+2)^2 - 3^2}{\left[(s+2)^2 + 3^2\right]} = 0$$

Example *For a system shown in the Fig. calculate its transfer function where $v_o(t)$ is output and $v_i(t)$ is input to the system.*



Solution : We can write for this system, equations by applying KVL as,

$$v_i(t) = R \cdot i(t) + \frac{1}{C}\int i(t)\,dt \qquad \text{... (1)}$$

and

$$v_o(t) = \frac{1}{C}\int i(t)\,dt \qquad \text{... (2)}$$

We are interested in $\frac{V_o(s)}{V_i(s)}$ where $V_o(s)$ is Laplace of $v_o(t)$ and $V_i(s)$ is Laplace of $v_i(t)$ and initial conditions are to be neglected.

So taking Laplace of above two equations and assuming initial conditions zero we can write,

$$V_i(s) = R\,I(s) + \frac{1}{sC} I(s) \qquad \ldots (3)$$

$$V_o(s) = \frac{1}{sC} I(s) \qquad \ldots (4)$$

$$\therefore \quad I(s) = sCV_o(s)$$

Substituting in equation (3),

$$V_i(s) = sCV_o(s)\left[R + \frac{1}{sC}\right]$$

$$\therefore \quad V_i(s) = sCR\,V_o(s) + V_o(s) = V_o(s)\,[1 + sCR]$$

$$\therefore \quad \boxed{\frac{V_o(s)}{V_i(s)} = \frac{1}{1 + sCR}}$$

We can represent above system as in the Fig. : which is called transfer function model of the system.