**Coulomb integrals - neutral case**

For Z=0, the function *Fkl* can be expressed in terms of spherical Bessel functions and the Coulomb integrals has the form



The recurrence relations reduce to



and



*The limit* k = k' (Whelan 1986)



Direct formulas through the hypergeometric functions (Burgess et al 1970):



or



where *k*< is the smaller of *k*1 and *k*2, *k*> is the greater of *k*1 and *k*2, which is a more usual form for the integrals *I*0 (see Watson 1944, p.401).

Using the above recurrence relations or from Watson (1944) we have

,

where *p* = 0 if *k*1 < *k*2

*p* = 1 if *k*1 > *k*2 .

This formula is used in STGF program (subroutine Fdip0)

More general formula including higher multipoles is given by Chidishimo (1992) (also based on the Watson 1944). The general expressions for *λ* ≥ 1 has discontinuity at *k* = *k*':

0<*k*'<*k*



0<*k*<*k'*

.

Another expression for  is derived by Whelan (1986) and programmed by Burgess and Whelan (1987):

,

where *QL*(χ) is a Legendre function of the 2nd kind whose argument is given by χ = (*k*2 + *k'*2)/2*kk'.* These functions satisfy the same recurrence relations as the standard Legendre functions



and



Therefor it would appear that *Ql*+1 could be calculated from *Ql* ,*Ql*-1. However (as noted Burgess and Whelan 1987) for certain values of χ cancellations errors are appreciable and in this case it is more efficient to use the recurrence relations for decreasing *l*. In general the nearer χ is to 1 the less likely it is to need the decreasing *l* procedure. To evaluate Q-functions, we will use the DLEGENI program of Gil and Segura (1997).

The coefficients  are given by Seaton (1961) and expressed in terms of 3*j* and 6*j* symbols. It is more convenient to use the formula given by Somerville (1963):



where

 .

Parameters *a*,*b*,*c* are restricted by triangle relation {*a*,*b*,*c*} and *a*+*b*+*c* must be even. Value *L* in the above relations is also restricted by the relations (empirically found based on table of Seaton (1961):

(*l+l'* – *λ*)/2 ≤ *L* ≤ (*l+l'* +*λ*)/2

As programing algorithm, we will use the simplified expression



where all terms with *E*(…) = 0 are dropped.

**SUM RULES** (Burgess 1974)

 Z > 0

 Z = 0

These results are of importance in dealing with angular momentum summations of partial collision strengths.

**Another expression** for the case Z=0 (Whelan 1986):



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