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McKay correspondence

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31 May 2019

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Overview

— Statement of the McKay correspondance
— History and motivation
— Examples and proofs

McKay correspondance

The theorem involves $G \leq SL_2(\mathbb{C})$ finite group, $S = \mathbb{C}[\![x, y]\!]$, and $R = S^G$. It establishes a correspondence between:

— The irreducible representations of G;

— The indecomposable projective modules of the skew group algebra $S\#G$;

— The indecomposable projective modules of the endomorphism ring $\mathrm{End}_R(S)$;

— The indecomposable MCM modules of R.

History and motivation

- $G \leq SL_2(\mathbb{C})$, $S = \mathbb{C}[\![x, y]\!]$, $R = S^G$
- Kleinian singularities, $\mathbb{C}^2/G$
- Resolution graphs ADE Dynkin diagrams
- McKay connected the geometry to representation theory
- Herzog showed that the MCM R-modules are direct summands of S
- Auslander showed the relationship between the projective $S\#G$-modules and the MCM R-modules

The McKay quiver

Definition

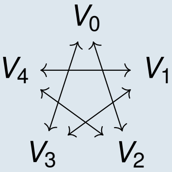
A representation of a group G is a group-homomorphsim $G \to GL_n(\mathbb{C})$.

Definition

The McKay quiver of G has verticies the irreducible representations of G, and an arrow from W to W' iff W' appears as a direct summand of $V \otimes_{\mathbb{C}} W$, where V is the cannonical representation.

Example

$G = \langle g \rangle \cong \mathbb{Z}/5\mathbb{Z}$, $g = \begin{pmatrix} \omega^2 & 0 \\ 0 & \omega^3 \end{pmatrix}$, $\omega = \exp(2\pi i/5)$,



Non-Dynkin example

Example

$$G = \langle g \rangle \cong \mathbb{Z}/2\mathbb{Z},\ g = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$\begin{tikzcd} V_0 \arrow[loop left] \arrow[r, leftrightarrow] & V_{-1} \arrow[loop right] \end{tikzcd}$$

The skew group algebra

Definition

If S is an algebra and G is a subgroup of $\mathrm{Aut}(S)$, then the *skew group algebra* $S\#G$ is the algebra generated by $s \cdot g$ with $s \in S$ and $g \in G$. The multiplication is given by

$$(s \cdot g)(t \cdot h) = st^g \cdot gh$$

Example

Let $S = \mathbb{C}[\![x, y]\!]$, and $G = \left\{ \mathrm{I} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, g := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right\}$.
Then $x \cdot \mathrm{I} + y \cdot g$ and $(x - y) \cdot g$ are in $S\#G$, and their product is

$$(x \cdot \mathrm{I} + y \cdot g)((x - y) \cdot g) = x(x - y) \cdot g + y(y - x) \cdot \mathrm{I}$$

Gabriel quiver

Definition

The *Gabriel quiver* of an algebra, A (where projective covers exist) has verticies the indecomposable projective A-modules. If P and Q are indecomposable projectives there's an arrow from P to Q if Q is a direct summand of the projetive cover of the radical of P.

$$\begin{tikzcd} P_1 \arrow[d, bend right, "\pi"'] \arrow[dr, two heads] & & P \\ Q \arrow[u, bend right, "\iota"'] & RadP \arrow[ur, hook] & \end{tikzcd}$$

Example of Gabriel quiver

Example

$$S = \mathbb{C}[\![x, y]\!],\ G = \langle g \rangle,\ g = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},\ \mathfrak{m} = \langle x, y \rangle_{S\#G}$$

$$S\#G \cong S \otimes_{\mathbb{C}} \langle \mathrm{I} + g \rangle \oplus S \otimes_{\mathbb{C}} \langle \mathrm{I} - g \rangle$$

$$\begin{tikzcd} S \otimes \langle \mathrm{I} + g \rangle \arrow[loop left] \arrow[r, leftrightarrow] & S \otimes \langle \mathrm{I} - g \rangle \arrow[loop right] \end{tikzcd}$$

The indecomposable projective $S\#G$-modules

Theorem

There's a correspondance between the irreducible G representations and the indecomposable projective $S\#G$-modules given by

$$\left\{\begin{matrix}\text{indecomposable projective} \\ S\#G\text{-modules}\end{matrix}\right\} \longleftrightarrow \left\{\begin{matrix}\text{irreducible} \\ G \text{ representations}\end{matrix}\right\}$$

$$P \longmapsto P/\mathfrak{m}P$$

$$S \otimes_{\mathbb{C}} W \longleftarrow W$$

Where $S = \mathbb{C}[\![x_1, \cdots, x_n]\!]$, $G \leq GL_n(\mathbb{C})$, *and* $\mathfrak{m} = \langle x_1, \cdots, x_n \rangle_{S\#G}$.

McKay and Gabriel quiver

Theorem

The McKay quiver of G and the Gabriel quiver of $S\#G$ are isomorphic when G is finite subgroup of $GL_n(\mathbb{C})$.

Proof.

Let $V = \mathfrak{m}/\mathfrak{m}^2$. *Then the minimal projective resolution of* $\mathbb{C} = S/\mathfrak{m}$ *is*

$$0 \longrightarrow S \otimes_{\mathbb{C}} \bigwedge^{n} V \xrightarrow{\partial_n} \cdots \xrightarrow{\partial_2} S \otimes_{\mathbb{C}} \bigwedge^{1} V \xrightarrow{\partial_1} S \longrightarrow 0.$$

Applying $- \otimes_{\mathbb{C}} W$ *we get*

$$\cdots \longrightarrow S \otimes_{\mathbb{C}} V \otimes W \longrightarrow S \otimes_{\mathbb{C}} W \longrightarrow 0.$$

$\square$

Summary

When $G \leq SL_2(\mathbb{C})$ is a finite group, $S = \mathbb{C}[\![x, y]\!]$, and $R = S^G$ there's a correspondance between:

- **The irreducible representations of** G**;**
- **The indecomposable projective modules of the skew group algebra** $S \# G$**;**
- The indecomposable projective modules of the endomorphism ring $\mathrm{End}_R(S)$;
- The indecomposable MCM modules of R.

There's isomorphisms of quivers:

- **The McKay quiver of** G**;**
- **The Gabriel quiver of** $S \# G$**;**
- The Gabriel quiver of $\mathrm{End}_R(S)$;
- The Auslander-Reiten quiver of the MCM R-modules.

Questions?