# McKay correspondence

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**Abstract**

The goal of this thesis is to establish a 1-1 correspondence between quivers created from the four following sets whenever $G$ is a finite subgroup of $SL(2,\mathbb{C})$ and $S$ is the power series ring $\mathbb{C}[\![x,y]\!]$

- The Maximal Cohen-Macaulay modules of the fixed ring $S^G$.
- The indecomposable projective modules of the skew group algerba $S\#G$.
- The indecomposable projective modules of $\mathrm{End}_{S^G}(S)$.
- The irreducible representations of $G$ (indecomposable $\mathbb{C}G$-modules).

Much of the thesis will be used to define these four quivers and to develope tools to establish such a correspondence. A similar correspondence can be established for a general field $k$ and a finite subgroup of $GL(n,k)$ with order nonzero in $k$, but the case for $SL(2,\mathbb{C})$ is the most interesting as the quivers will be extended Dynkin diagrams.

## 1 Finite subgroups of SL(2,C)

## 2 Characters and irreducible representations

Sorcue: Representations and characters of groups - Gordon James, Martin Lieback

Recall that the trace of a matrix is defined to be the sum of its diagonal elements and that the trace satisfies two important equations. Namely

$$tr(A+B) = tr(A) + tr(B) \text{ and } tr(AB) = tr(BA)$$

For a given representation of $G$, $\rho : G \to GL_n(\mathbb{C})$, we define its character by $\chi_\rho : G \to \mathbb{C}$, $\chi_\rho(g) = tr(\rho(g))$.

**Proposition 2.1.** *Conjugate elements in $G$ take the same value under a character.*

*Proof.* Let $g$ and $g'$ be in the same conjugacy class. Then there exists an element $h$ such that $h^{-1}gh = g'$. Then we have

$$\chi(g') = \chi(h^{-1}gh) = tr(\rho(h)^{-1}\rho(g)\rho(h)) \overset{*}{=} tr(\rho(g)\rho(h)\rho(h)^{-1}) = tr(\rho(g)) = \chi(g)$$

In (*) we use the fact that $tr(AB) = tr(BA)$. □

**Lemma 2.1.** *For a finite abelian group $G$ any irreducible representation must be 1-dimensional.*

*Proof.* Let $\rho : G \to GL(V)$ be an irreducible representation. Since $G$ is abelian we have that $\rho(g)\rho(h)v = \rho(h)\rho(g)v$. Thus multiplication by $\rho(g)$ respects the action of $G$ and we have that $\rho(g)$ is a homomorphism of $G$-representations between $\rho$ and itself. Then by Schur's lemma $\rho(g)$ must be a scalar multiplication. In other words every matrix $\rho(g)$ for $g \in G$ is diagonal (it is a scaling of identity). This implies that $\rho$ can be written as a direct sum of 1-dimensional representations, but since $\rho$ is irreducible $\rho$ must be 1-dimensional. □

**Proposition 2.2.** *If $\chi$ is the character of a representation, $\rho$, with dimension $n$ of a group $G$, and $g$ is an element of $G$ with order $m$, then the following holds*

*(1)* $\chi(1) = n$

*(2)* $\chi(g)$ *is the sum of $m$-th roots of unity.*

*(3)* $\chi(g^{-1}) = \overline{\chi(g)}$

*Proof.*

(1) The first result is immidiate.

$$\chi(1) = tr\left(\begin{bmatrix} 1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 1 \end{bmatrix}\right) = n$$

(2) Since $\langle g \rangle$ is an abelian group, $\rho$ decomposes into $n$ 1-dimensional $\langle g \rangle$-representations. Then there is a basis such that $\rho(g)$ is diagonal. Since $g$ has order $m$ it follows that the diagonal entries of $\rho(g)$ must be $m$-th roots of unity. Thus $\chi(g) = tr(\rho(g))$ must be the sum of $m$-th roots of unity.

(3) Using the same basis as above and the fact that $\omega^{-1} = \overline{\omega}$ when $\omega$ is a root of unity we see that $\chi(g^{-1}) = tr(\rho(g)^{-1}) = \overline{tr(\rho(g))} = \overline{\chi(g)}$.

□

# 3 The McKay quiver

**Definition 3.1.** *Let $G$ be a finite subgroup of $GL(n, \mathbb{C})$, and let $V$ be the cannonical representation (the one that sends $g$ to $g$). Then we define the <u>McKay quiver</u> of $G$ to be the quiver with verticies the irreducible representations of $G$, denoted $V_i$. For two irreducible representations $V_i$ and $V_j$ we say there is an arrow from the former to the latter if and only if $V_j$ is a direct summand of $V \otimes V_i$.*

**Example 3.1.** *Let $G$ be the group generated by $g = \begin{bmatrix} \omega^2 & 0 \\ 0 & \omega^3 \end{bmatrix}$, where $\omega$ is the primitive fifth root of unity. Then there are five different irreducible representations, the one sending $g$ to $\omega$, $\omega^2$, $\omega^3$, $\omega^4$ respectively, and the trivial representation. Denote the representation sending $g$ to $\omega^i$ by $V_i$, and let $V = V_2 \oplus V_3$ be the cannonical representation. Note that $V_i \otimes V_j = V_{i+j}$, where $i+j$ is understood to be modulo 5. Then we get the following McKay-quiver*

$$\begin{tikzcd}
 & V_0 \arrow[ddl, leftrightarrow] \arrow[ddr, leftrightarrow] & \\
V_4 \arrow[rr, leftrightarrow] \arrow[ddrr, leftrightarrow, phantom] \arrow[drr, leftrightarrow] & & V_1 \arrow[dll, leftrightarrow] \\
V_3 & & V_2
\end{tikzcd}$$

# 4 Skew group algebra S#G indecomposable projectives

Source Cohen-Macaulay representations - Graham J Leuschke, Roger Wiegand

**Definition 4.1.** *If $G$ is a subgroup of $GL_n(\mathbb{C})$, we can extend the group action of $G$ to $\mathbb{C}[\![x_1, \cdots, x_n]\!]$. We then define the <u>skew group algebra $\mathbb{C}[\![x_1, \cdots, x_n]\!]\#G$</u> to be the algebra generated by elements of the form $f \cdot g$ with $f \in \mathbb{C}[\![x_1, \cdots, x_n]\!]$ and $g \in G$, and we define the multiplication by*

$$(f_1 \cdot g_1) \cdot (f_2 \cdot g_2) = (f_1 \cdot f_2^{g_1}) \cdot (g_1 \cdot g_2)$$

*Where $f^g$ denotes the image of $f$ under the action of $g$.*

**Theorem 4.1.** *We have an isomorphism of rings*

$$e\mathbb{C}[\![x, y]\!]\#Ge \simeq \mathbb{C}[\![x, y]\!]^G$$

*where $e = \frac{1}{|G|} \sum_{g \in G} g$.*

*Proof.* Let $f^g$ denote the image of $f$ under the action of $g$. Then if we let $f(x,y)g$ be an element of the skew algebra we get that $ef(x,y)ge = f(x,y)^e \cdot ege = f(x,y)^e \cdot e = e \cdot f(x,y)$. It then follows that $e\mathbb{C}[\![x,y]\!]\#Ge$ is isomorphic to the image of $\mathbb{C}[\![x,y]\!]$ under the action of $e$. Since $ge = g$ for all $g \in G$ it is clear that the image of $e$ is contained in the fixed ring. For the converse you just need to notice that the fixed ring is fixed under $e$ and thus is contained in the image. □

**Lemma 4.1.** *Let $S = \mathbb{C}[\![x,y]\!]$. An $S\#G$-module is projective if and only if it is projective as an $S$-module.*

*Proof.* Onlyifity follows from $S\#G$ being a free $S$-module, it is isomorphic to $\bigoplus_{g\in G} S$. Thus we need only show ifity.

First we need to see that an $S\#G$-linear map is just an $S$-linear map such that $f(g(m)) = g(f(m))$ for all $g \in G$. Equivalently $f(m) = g(f(g^{-1}(m)))$. This allows us to define a group action on $S$-linear maps by $f^g(m) = g(f(g^{-1}(m)))$. Then we can restate it as

$$\mathrm{Hom}_{S\#G}(M,N) = \mathrm{Hom}_S(M,N)^G$$

Clearly if $f$ is $S\#G$-linear then it's in $\mathrm{Hom}_S(M,N)^G$. To see the other inclusion, let $f$ be an $S$-linear map that is fixed under $G$. Then $f(s \cdot gm) = sf(gm) = s \cdot g(f(g^{-1}gm)) = s \cdot gf(m)$, and hence $f$ is $S\#G$-linear. Nextly I want to show that $-^G$ is an exact functor.

If $K$ is the kernel of a map $f : M \to N$, then the kernel of the inuced map $f^G : M^G \to N^G$ is of course just $K \cap M^G$ which equals $K^G$. Assume $f$ is epi and let $n \in N^G$. Consider a preimage $m$ such that $f(m) = n$. Let $\theta = \frac{1}{|G|}\sum_{g\in G} g(m)$. Then $\theta$ is in $M^G$ and $f(\theta) = \frac{1}{|G|}\sum_{g\in G} g(f(m)) = \frac{1}{|G|}\sum_{g\in G} n = n$.

This implies that if $\mathrm{Hom}_S(P,-)$ is exact then $\mathrm{Hom}_S(P,-)^G = \mathrm{Hom}_{S\#G}(P,-)$ is exact and our lemma follows. □

**Theorem 4.2.** *Let $S = \mathbb{C}[\![x,y]\!]$ and let $\mathfrak{m} = \langle x,y\rangle_S$ be the radical of $S$. Then there are bijections between the indecomposable projective $S\#G$-modules and the indecomposable $\mathbb{C}G$-modules given by*

$$\begin{Bmatrix}\textit{indecomposable projective}\\ S\#G\textit{-modules}\end{Bmatrix} \qquad \begin{Bmatrix}\textit{indecomposable}\\ \mathbb{C}G\textit{-modules}\end{Bmatrix}$$

$$\mathcal{F} : P \longmapsto P/\mathfrak{m}P$$

$$\mathcal{G} : S \otimes_{\mathbb{C}} W \longleftarrow\!\shortmid W$$

*Where the $S\#G$-module structure on $S \otimes_{\mathbb{C}} W$ is given by $(s \cdot g) \cdot f \otimes v = sf^g \otimes v^g$.*

*Proof.* First we should show that $S \otimes_{\mathbb{C}} W$ is an irreducible projective $S\#G$-module and that $P/\mathfrak{m}P$ is infact an irreducible $\mathbb{C}G$-module. Since $S \otimes_{\mathbb{C}} W$ is a free $S$-module it follows from lemma 4.1 that it is projective. To see that it is irreducible we will first study it as an $S$-module and exploit the fact that $\mathrm{Hom}_{S\#G}(M,N) \subseteq \mathrm{Hom}_S(M,N)$.

Since $\mathfrak{m}$ is the radical of $S$ we have that

$$\frac{S \otimes_{\mathbb{C}} W}{\mathfrak{m}S \otimes_{\mathbb{C}} W} \cong S/\mathfrak{m} \otimes_{\mathbb{C}} W \cong \mathbb{C} \otimes_{\mathbb{C}} W \cong W$$

$W$ is the top of $S \otimes_{\mathbb{C}} W$. Further since the projection is $S\#G$-linear we have that $S \otimes_{\mathbb{C}} W$ is the projective cover of $W$ also as $S\#G$-modules. Then since $W$ is simple it follows that $S \otimes_{\mathbb{C}} W$ is an indecomposable projective.

It's clear that $P/\mathfrak{m}P$ is a $\mathbb{C}G$-module, because $\mathbb{C}G$ is a subring of $S\#G$. To see that it's indecomposable we will first show that it's indecomposable as an $S\#G$-module. By considering $P$ and $P/\mathfrak{m}P$ as $S$-modules and using the same argument as above we see that $P$ is the projective cover of $P/\mathfrak{m}P$. Then since $P$ is indecomposable $P/\mathfrak{m}P$ must also be indecomposable as an $S\#G$-module.

To see that this implies $P/\mathfrak{m}P$ is indecomposable as a $\mathbb{C}G$-module notice that $P/\mathfrak{m}P$ is annihilated by the ideal $\langle \mathfrak{m} \rangle$. This means it's an indecomposable as $S\#G$-module if and only if it's indecomposable as an $S\#G/\langle \mathfrak{m} \rangle$-module. Then since $S\#G/\langle \mathfrak{m} \rangle \cong \mathbb{C}G$ it follows that $P/\mathfrak{m}P$ is an indecomposable $\mathbb{C}G$-module.

To see that the given maps are bijections we will show that they are mutual inverses. First to see that $\mathcal{F}(\mathcal{G}(W)) \cong W$ we simply look at the definition

$$\frac{S \otimes_{\mathbb{C}} W}{\mathfrak{m}S \otimes_{\mathbb{C}} W} \cong S/\mathfrak{m} \otimes_{\mathbb{C}} W \cong \mathbb{C} \otimes_{\mathbb{C}} W \cong W$$

Next we consider $\mathcal{G}(\mathcal{F}(P)) = S \otimes_{\mathbb{C}} P/\mathfrak{m}P$. We have already seen that it's projective. Both $P$ and $S \otimes_{\mathbb{C}} P/\mathfrak{m}P$ have a natural projection onto $P/\mathfrak{m}P$, and by projectivity we get an induced $S\#G$-linear map from $S \otimes_{\mathbb{C}} P/\mathfrak{m}P$ to $P$:

$$\begin{tikzcd}
 & S \otimes_{\mathbb{C}} P/\mathfrak{m}P \arrow[dl, dashed] \arrow[d, two heads] \\
P \arrow[r, two heads] & P/\mathfrak{m}P
\end{tikzcd}$$

Further since $\mathfrak{m}$ is the radical of $S$, both $P$ and $S \otimes_{\mathbb{C}} P/\mathfrak{m}P$ are projective covers of $P/\mathfrak{m}P$ (as $S$-modules). This means that the map is an isomorphism of $S$-modules, and therefor it is also an isomorphism of $S\#G$-modules. $\square$

## 5 $End_{S^G}(S)$

## 6 MCM modules

**Definition 6.1.** *If $R$ is a local ring with residual field $k$ we define the depth of a module, $M$, to be the minimal $n$ such that the extension $\mathrm{Ext}^n_R(k, M)$ is non-zero.*

**Definition 6.2.** *If $M$ is a module over a local ring $R$ with Krull-dimension $d$ we say that $M$ is maximal Cohen Macaulay (MCM) if the depth of $M$ equals $d$.*