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McKay correspondence

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Overview

— Statement of the McKay correspondance
— History and motivation
— examples and proofs

McKay correspondance

The theorem involves $G \leq SL_2(\mathbb{C})$ finite group, $S = \mathbb{C}[\![x, y]\!]$, and $R = S^G$. It establishes a correspondence between:

- The irreducible representations of G;
- The indecomposable projective modules of the skew group algebra $S\#G$;
- The indecomposable projective modules of the endomorphism ring $\mathrm{End}_R(S)$;
- The indecomposable MCM modules of R.

History and motivation

- Kleinian singularities, $\mathbb{C}^2/G$
- Resolution graphs ADE Dynkin diagrams
- McKay connected the geometry to representation theory
- Herzog showed that the MCM R-modules are direct summands of S
- Auslander showed the relationship between the projective $S\#G$-modules and the MCM R-modules

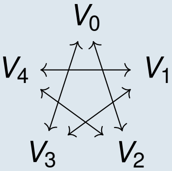
The McKay quiver

Definition

The McKay quiver of G has verticies the irreducible representations of G, and an arrow from W to W' iff W' appears as a direct summand of $V \otimes_{\mathbb{C}} W$, where V is the cannonical representation.

Example

$G = \langle g \rangle \cong \mathbb{Z}/5\mathbb{Z}$, $g = \begin{pmatrix} \omega^2 & 0 \\ 0 & \omega^3 \end{pmatrix}$, $\omega = \exp(2\pi i/5)$,



Non-Dynkin example

Example

$G = \langle \mu, \rho \rangle \cong S_3$, $\mu = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $\rho = \begin{pmatrix} \exp(2\pi i/3) & 0 \\ 0 & \exp(-2\pi i/3) \end{pmatrix}$

$$\begin{tikzcd} V_0 \arrow[r, leftrightarrow] & V \arrow[loop above] \arrow[r, leftrightarrow] & V_\mu \end{tikzcd}$$

The skew group algebra

Definition

If S is an algebra and G is a subgroup of $\mathrm{Aut}(S)$, then the *skew group algebra* $S\#G$ is the algebra generated by $s \cdot g$ with $s \in S$ and $g \in G$. The multiplication is given by

$$(s \cdot g)(t \cdot h) = st^g \cdot gh$$

Example

Let $S = \mathbb{C}[\![x, y]\!]$, and $G = \left\{ \mathrm{I} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, g := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right\}$.

Then $x \cdot \mathrm{I} + y \cdot g$ and $(x - y) \cdot g$ are in $S\#G$, and their product is

$$(x \cdot \mathrm{I} + y \cdot g)((x - y) \cdot g) = x(x - y) \cdot g + y(y - x) \cdot \mathrm{I}$$

Gabriel quiver

Definition

The *Gabriel quiver* of an algebra, A (where projective covers exist) has verticies the simple A-modules. If M and N are simple A-modules with minimal projective resolutions:

$$\cdots \to P_1^M \to P_0^M \to 0 \qquad \text{and} \qquad \cdots \to P_1^N \to P_0^N \to 0.$$

Then there is an arrow from M to N iff P_0^N is a direct summand of P_1^M.

Example of Gabriel quiver

Example

$$S = \mathbb{C}[\![x, y]\!],\ G = \langle g \rangle,\ g = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},\ \mathfrak{m} = \langle x, y \rangle_{S\#G}$$

$$S\#G \cong S \otimes_{\mathbb{C}} \langle \mathrm{I} + g \rangle \oplus S \otimes_{\mathbb{C}} \langle \mathrm{I} - g \rangle$$

$$\begin{tikzcd} \langle \mathrm{I} + g \rangle \arrow[loop left] \arrow[r, leftrightarrow] & \langle \mathrm{I} - g \rangle \arrow[loop right] \end{tikzcd}$$

The indecomposable projective $S\#G$-modules

Theorem

There's a correspondance between the irreducible G representations and the indecomposable projective $S\#G$-modules given by

$$\left\{\begin{array}{c}\textit{indecomposable projective}\\ S\#G\textit{-modules}\end{array}\right\} \longleftrightarrow \left\{\begin{array}{c}\textit{irreducible}\\ G\textit{ representations}\end{array}\right\}$$

$$\mathcal{F} : P \longmapsto P/\mathfrak{m}P$$

$$\mathcal{G} : S \otimes_{\mathbb{C}} W \longleftarrow W$$

McKay and Gabriel quiver

Theorem

The McKay quiver of G and the Gabriel quiver of $S\#G$ are isomorphic when G is finite subgroup of $GL_n(\mathbb{C})$.

Proof.

Let $V = \mathfrak{m}/\mathfrak{m}^2$. Then the minimal projective resolution of $\mathbb{C} = S/\mathfrak{m}$ is

$$0 \longrightarrow S \otimes_{\mathbb{C}} \bigwedge^n V \xrightarrow{\partial_n} \cdots \xrightarrow{\partial_2} S \otimes_{\mathbb{C}} \bigwedge^1 V \xrightarrow{\partial_1} S \longrightarrow 0.$$

Applying $- \otimes_{\mathbb{C}} W$ we get

$$\cdots \longrightarrow S \otimes_{\mathbb{C}} V \otimes W \longrightarrow S \otimes_{\mathbb{C}} W \longrightarrow 0.$$

$\square$

Questions?