

# McKay correspondence

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## Abstract

The goal of this thesis is to establish a 1-1 correspondence between quivers created from the four following sets whenever  $S$  is the power series ring  $\mathbb{C}[[x, y]]$  and  $G$  is a finite subgroup of  $SL(2, \mathbb{C})$  acting on  $S$

- The Maximal Cohen-Macaulay modules of the fixed ring  $S^G$ .
- The indecomposable projective modules of the skew group algebra  $S\#G$ .
- The indecomposable projective modules of  $\text{End}_{S^G}(S)$ .
- The irreducible representations of  $G$  (indecomposable  $\mathbb{C}G$ -modules).

Much of the thesis will be used to define these four quivers and to develop tools to establish such a correspondence. A similar correspondence can be established for a general field  $k$  and a finite subgroup of  $GL(n, k)$  with order nonzero in  $k$ , but in the general case we will only attain the MCM-modules that appear as  $S^G$ -direct summands of  $S$ .  $SL(2, \mathbb{C})$  is also especially interesting because the quivers are exactly the Dynkin diagrams.

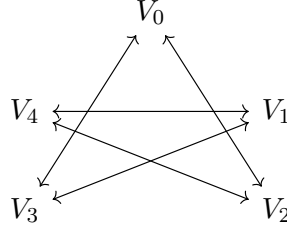
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## 1 The McKay quiver

**Definition 1.1.** Let  $G$  be a finite subgroup of  $GL(n, \mathbb{C})$ , and let  $V$  be the canonical representation (the one that sends  $g$  to  $g$ ). Then we define the McKay quiver of  $G$  to be the quiver with vertices the irreducible representations of  $G$ , denoted  $V_i$ . For two irreducible representations  $V_i$  and  $V_j$  there is an arrow from the former to the latter if and only if  $V_j$  is a direct summand of  $V \otimes V_i$ .

**Example 1.1.** Let  $G$  be the group generated by  $g = \begin{bmatrix} \omega^2 & 0 \\ 0 & \omega^3 \end{bmatrix}$ , where  $\omega$  is a primitive fifth root of unity. Then there are five different irreducible representations, the one sending  $g$  to  $\omega, \omega^2, \omega^3, \omega^4$  respectively, and the trivial representation. Denote the representation sending  $g$  to  $\omega^i$  by  $V_i$ , and let  $V = V_2 \oplus V_3$  be the canonical representation. Note that  $V_i \otimes V_j = V_{i+j}$ , where  $i+j$  is understood to be modulo 5. Then we get the following McKay-quiver



## 2 Skew group algebra $S \# G$ indecomposable projectives

This section is largely based on the book by [Leuschke and Wiegand, 2012]. This section will use definitions and theorems from representation theory as taught in the courses MA3203 - Ring Theory and MA3204 - homological algebra. Since I do not assume knowledge of this I have created appendix A. I will try to use footnotes to indicate where such theorems are used.

**Definition 2.1.** If  $G$  is a subgroup of  $GL_n(\mathbb{C})$ , we can extend the group action of  $G$  to  $\mathbb{C}[[x_1, \dots, x_n]]$ . More explicitly  $G$  acts on  $x_i$  as it would the  $i$ th basis vector of  $\mathbb{C}^n$ , and acts on products and sums by acting on each component separately. We then define the skew group algebra  $\mathbb{C}[[x_1, \dots, x_n]] \# G$  to be the algebra generated by elements of the form  $f \cdot g$  with  $f \in \mathbb{C}[[x_1, \dots, x_n]]$  and  $g \in G$ , and we define the multiplication by

$$(f_1 \cdot g_1) \cdot (f_2 \cdot g_2) = (f_1 \cdot f_2^{g_1}) \cdot (g_1 \cdot g_2)$$

Where  $f^g$  denotes the image of  $f$  under the action of  $g$ .

**Theorem 2.1.** *We have an isomorphism of rings*

$$e\mathbb{C}[[x, y]]\#Ge \simeq \mathbb{C}[[x, y]]^G$$

where  $e = \frac{1}{|G|} \sum_{g \in G} g$ .

*Proof.* Let  $f^g$  denote the image of  $f$  under the action of  $g$ . Then if we let  $f(x, y)g$  be an element of the skew algebra we get that  $ef(x, y)ge = f(x, y)^e \cdot ege = f(x, y)^e \cdot e = e \cdot f(x, y)$ . It then follows that  $e\mathbb{C}[[x, y]]\#Ge$  is isomorphic to the image of  $\mathbb{C}[[x, y]]$  under the action of  $e$ . Since  $ge = g$  for all  $g \in G$  it is clear that the image of  $e$  is contained in the fixed ring. For the converse you just need to notice that the fixed ring is fixed under  $e$  and thus is contained in the image.  $\square$

**Lemma 2.1.** *Let  $S = \mathbb{C}[[x, y]]$ . An  $S\#G$ -module is projective if and only if it is projective as an  $S$ -module.*

*Proof.* Onlyifity follows from  $S\#G$  being a free  $S$ -module, it is isomorphic to  $\bigoplus_{g \in G} S$ . Thus we need only show ifity.

First we need to see that an  $S\#G$ -linear map is just an  $S$ -linear map such that  $f(g(m)) = g(f(m))$  for all  $g \in G$ . Equivalently  $f(m) = g(f(g^{-1}(m)))$ . This allows us to define a group action on  $S$ -linear maps by  $f^g(m) = g(f(g^{-1}(m)))$ . Then we can restate it as

$$\text{Hom}_{S\#G}(M, N) = \text{Hom}_S(M, N)^G$$

Clearly if  $f$  is  $S\#G$ -linear then it's in  $\text{Hom}_S(M, N)^G$ . To see the other inclusion, let  $f$  be an  $S$ -linear map that is fixed under  $G$ . Then  $f(s \cdot gm) = sf(gm) = s \cdot g(f(g^{-1}gm)) = s \cdot gf(m)$ , and hence  $f$  is  $S\#G$ -linear. Nextly I want to show that  $-^G$  is an exact functor.

If  $K$  is the kernel of a map  $f : M \rightarrow N$ , then the kernel of the induced map  $f^G : M^G \rightarrow N^G$  is of course just  $K \cap M^G$  which equals  $K^G$ . Assume  $f$  is epi and let  $n \in N^G$ . Consider a preimage  $m$  such that  $f(m) = n$ . Let  $\theta = \frac{1}{|G|} \sum_{g \in G} g(m)$ . Then  $\theta$  is in  $M^G$  and  $f(\theta) = \frac{1}{|G|} \sum_{g \in G} g(f(m)) = \frac{1}{|G|} \sum_{g \in G} n = n$ .

This implies that if  $\text{Hom}_S(P, -)$  is exact then  $\text{Hom}_S(P, -)^G = \text{Hom}_{S\#G}(P, -)$  is exact and our lemma follows.  $\square$

**Lemma 2.2.** *Let  $S$  be the complex power series ring in  $n$  variables, and  $\mathfrak{m} = \langle x_i \rangle_{i=1}^n$  the radical of  $S$ . Then for any free  $S$ -module  $N$ ,  $\mathfrak{m}N$  is small in  $N$ . That is if  $X$  is a submodule of  $N$  such that  $X + \mathfrak{m}N = N$ , then  $X = N$ .*

*Proof.* Let  $N$  be the free module  $S^{(I)} := \bigoplus_{i \in I} S_i$ , where  $S_i \cong S$ . Assume that  $X$  is a submodule such that  $X + \mathfrak{m}N = N$ . We denote by  $1_i$  the elements

that is 1 at index  $i$  and 0 elsewhere. Since  $\{1_i\}$  generate  $N$ , it is enough to show that  $X$  contains all of them. Since  $X + \mathfrak{m}N = N$ , we know that there is an  $m_i \in \mathfrak{m}N$  and an  $x_i \in X$  such that  $x_i + m_i = 1_i$ . Then we have that  $x_i = 1_i - m_i$ . Since the power series at index  $i$  of  $x_i$  has constant coefficient 1 it is invertible. If we multiply  $x_i$  by its inverse we get  $\tilde{x}_i$  which is 1 at index  $i$  and some element of  $\mathfrak{m}$  at index  $j \neq i$ , say  $m_{ij}$ . Then  $\tilde{x}_i - \sum_{j \neq i} m_{ij} \tilde{x}_j$  has a unit in index  $i$  and 0 at all other indices. Thus  $X$  contains  $1_i$  for all  $i$ , and  $X = N$ .  $\square$

**Theorem 2.2.** *Let  $S = \mathbb{C}[[x, y]]$  and let  $\mathfrak{m} = \langle x, y \rangle_S$  be the radical of  $S$ . Then there are bijections between the indecomposable finitely generated projective  $S\#G$ -modules and the indecomposable  $\mathbb{C}G$ -modules given by*

$$\left\{ \begin{array}{c} \text{indecomposable projective} \\ S\#G\text{-modules} \end{array} \right\} \quad \left\{ \begin{array}{c} \text{indecomposable} \\ \mathbb{C}G\text{-modules} \end{array} \right\}$$

$$\mathcal{F} : P \longmapsto P/\mathfrak{m}P$$

$$\mathcal{G} : S \otimes_{\mathbb{C}} W \longleftarrow W$$

Where the  $S\#G$ -module structure on  $S \otimes_{\mathbb{C}} W$  is given by  $(s \cdot g) \cdot f \otimes v = sf^g \otimes v^g$ .

*Proof.* First we should show that  $S \otimes_{\mathbb{C}} W$  is an indecomposable projective  $S\#G$ -module and that  $P/\mathfrak{m}P$  is infact an indecomposable  $\mathbb{C}G$ -module. Since  $S \otimes_{\mathbb{C}} W$  is a free  $S$ -module it follows from lemma 2.1 that it is projective. To see that it is indecomposable we will first study it as an  $S$ -module and exploit the fact that  $\text{Hom}_{S\#G}(M, N) \subseteq \text{Hom}_S(M, N)$ .

Using lemma 2.2 we get that  $\mathfrak{m}S \otimes_{\mathbb{C}} W$  is small in  $S \otimes_{\mathbb{C}} W$ . This means that we get that

$$\frac{S \otimes_{\mathbb{C}} W}{\mathfrak{m}S \otimes_{\mathbb{C}} W} \cong S/\mathfrak{m} \otimes_{\mathbb{C}} W \cong \mathbb{C} \otimes_{\mathbb{C}} W \cong W$$

is the top of  $S \otimes_{\mathbb{C}} W$  as an  $S$ -module. Further since the projection  $S \otimes_{\mathbb{C}} W \rightarrow W$  is  $S\#G$ -linear we have that  $S \otimes_{\mathbb{C}} W$  is the projective cover of  $W$  also as  $S\#G$ -modules. Assume for the sake of contradiction that  $S \otimes_{\mathbb{C}} W$  decomposes as  $M \oplus N$  for non-zero  $M$  and  $N$ . Then  $W$  would equal  $M/\mathfrak{m}M \oplus N/\mathfrak{m}N$  as an  $S\#G/\langle \mathfrak{m} \rangle$ -module. Since  $S\#G/\langle \mathfrak{m} \rangle \cong \mathbb{C}G$  and  $W$  is indecomposable we must have that either  $M/\mathfrak{m}M$  or  $N/\mathfrak{m}N$  is 0. This then gives a contradiction because  $\mathfrak{m}M$  and  $\mathfrak{m}N$  are small in  $M$  and  $N$ . Hence we must have that  $S \otimes_{\mathbb{C}} W$  is indecomposable.

It's clear that  $P/\mathfrak{m}P$  is a  $\mathbb{C}G$ -module, because  $\mathbb{C}G$  is a subring of  $S\#G$ . To see that it's indecomposable we will first show that it's indecomposable we will use a similar argument as above. Assume  $P/\mathfrak{m}P$  decomposes as

$V \oplus W$ . Then both  $P$  and  $S \otimes_{\mathbb{C}} V \oplus S \otimes_{\mathbb{C}} W$  are projective covers of  $\mathfrak{m}P = V \oplus W$  we get induced  $S\#G$ -linear epimorphisms between them.

$$\begin{array}{ccc} & S \otimes_{\mathbb{C}} P/\mathfrak{m}P & \\ \swarrow \text{dashed} & \downarrow & \\ P & \twoheadrightarrow & P/\mathfrak{m}P \end{array}$$

Now we use the fact that  $P$  is finitely generated. Since there can only be an epimorphism from a module with more or equal amount of generators  $P$  and  $S \otimes_{\mathbb{C}} \mathfrak{m}P$  must have the same amount of generators and the induced maps are in fact isomorphisms of  $S$ -modules. Since the maps are also  $S\#G$ -linear we have that  $P$  decomposes as  $S \otimes_{\mathbb{C}} V \oplus S \otimes_{\mathbb{C}} W$ . Then since  $P$  is indecomposable we must have that either  $S \otimes_{\mathbb{C}} V$  or  $S \otimes_{\mathbb{C}} W$  is 0. That means that either  $V$  or  $W$  is 0, and we have shown that  $P/\mathfrak{m}P$  is an indecomposable  $\mathbb{C}G$ -module.

To see that the given maps are bijections we will show that they are mutual inverses. First to see that  $\mathcal{F}(\mathcal{G}(W)) \cong W$  we simply look at the definition

$$\frac{S \otimes_{\mathbb{C}} W}{\mathfrak{m}S \otimes_{\mathbb{C}} W} \cong S/\mathfrak{m} \otimes_{\mathbb{C}} W \cong \mathbb{C} \otimes_{\mathbb{C}} W \cong W$$

Next we consider  $\mathcal{G}(\mathcal{F}(P)) = S \otimes_{\mathbb{C}} P/\mathfrak{m}P$ . We have already seen that the induced map

$$\begin{array}{ccc} & S \otimes_{\mathbb{C}} P/\mathfrak{m}P & \\ \swarrow \text{dashed} & \downarrow & \\ P & \twoheadrightarrow & P/\mathfrak{m}P \end{array}$$

is an isomorphism, and thus  $P \cong \mathcal{G}(\mathcal{F}(P))$ . □

## 2.1 The Gabriel quiver

**Definition 2.2.** For a skew group algebra  $S\#G$  we define its *Gabriel quiver* to be the quiver with vertices as the indecomposable projective modules of  $S\#G$ . The arrows are given by taking the minimal projective resolution of  $P/\mathfrak{m}P$ , where  $\mathfrak{m}$  is as defined above. If the minimal projective resolution of  $P/\mathfrak{m}P$  is given by

$$\cdots \longrightarrow Q_1 \longrightarrow Q_0 \longrightarrow 0$$

We say there is an arrow from  $P$  to  $P'$  if  $P'$  appears as a direct summand of  $Q_1$ .

**Definition 2.3.** Let  $V$  be a vector space. We then define the exterior algebra  $\bigwedge V$  as the associative unital graded algebra such that the multiplication is bilinear and satisfies  $x \wedge y = -y \wedge x$  for any  $x$  and  $y$  in  $V$ .

Some key properties of the exterior algebra is that  $x \wedge x = 0$ , and more generally that  $x_1 \wedge \cdots \wedge x_p = 0$  whenever  $\{x_i\}_{i=1}^p$  are linearly dependent.

The  $p$ th exterior power of  $V$ , denoted  $\bigwedge^p V$  is the vector space of all elements that are the product of  $p$  vectors in  $V$ . If  $\{x_i\}_{i=1}^n$  is a basis for  $V$ , then  $x_{i_1} \wedge \cdots \wedge x_{i_p}$  where  $i_1 < i_2 < \cdots < i_p$  and  $1 \leq i_j \leq n$  forms a basis for  $\bigwedge^p V$ , thus it is  $\binom{n}{p}$ -dimensional.

**Proposition 2.1.** If  $S$  is the ring of formal power series over  $\mathbb{C}$  in  $n$  variables, and  $G$  is a finite group acting on  $S$ , let  $V = \mathfrak{m}/\mathfrak{m}^2$ . Then the minimal projective resolution of  $\mathbb{C} \cong S/\mathfrak{m}$  is given by

$$0 \longrightarrow S \otimes_{\mathbb{C}} \bigwedge^n V \xrightarrow{\partial_n} \cdots \xrightarrow{\partial_2} S \otimes_{\mathbb{C}} \bigwedge^1 V \xrightarrow{\partial_1} S \longrightarrow 0$$

Where  $\partial_p$  is the  $S \# G$ -linear map defined by

$$\partial_p(s \otimes x_{i_1} \wedge x_{i_2} \wedge \cdots \wedge x_{i_p}) = \sum_{j=1}^p (-1)^{j+1} s x_{i_j} \otimes x_{i_1} \wedge \cdots \wedge \hat{x}_{i_j} \wedge \cdots \wedge x_{i_p}$$

Where  $x_{i_1} \wedge x_{i_2} \wedge \cdots \wedge x_{i_p}$  is one of the standard basis vectors for  $\bigwedge^n V$ , namely  $i_1 < i_2 < \cdots < i_p$ , and  $\hat{x}_j$  means that  $x_j$  is omitted.

*Proof.* Firsts we should show that this is a projective resolution of  $\mathbb{C}$ . In fact the complex described above is the Koszul complex of the regular sequence<sup>1</sup>  $(x_i)_{i=1}^n$ . It's a general fact of homological algebra that the Koszul complex of a regular sequence is a projective resolution of the ring modulo the ideal generated by the regular sequence, which in this case equals  $S/\langle x_i \rangle_{i=1}^n = \mathbb{C}$ .  
reference yes/no?

Secondly we want to show that the resolution is minimal. To do this it is enough to show that for each  $k \geq 1$   $\partial_k$  is a projective cover of its image, and that  $S \rightarrow \mathbb{C}$  is a projective cover. In other words we have to show is that the kernels of the maps are small. Since  $\text{Im } \partial_{k+1} = \text{Ker } \partial_k$  and  $\text{Im } \partial_{k+1} \subseteq \mathfrak{m} \otimes_{\mathbb{C}} \bigwedge^{k+1} V$  it follows from lemma 2.2 that the resolution is minimal.  $\square$

**Theorem 2.3.** If  $S$  is the complex power series ring in  $n$  variables and  $G$  is a finite subgroup of  $GL_n(\mathbb{C})$ , then the McKay quiver of  $G$  and the Gabriel quiver of  $S \# G$  are isomorphic.

<sup>1</sup>Regular sequences are defined on page 9 in definition 4.2.

*Proof.* We have already seen that they have the same vertices, namely if  $V_i$  are the irreducible representations of  $G$ , then  $S \otimes_{\mathbb{C}} V_i$  are the indecomposable projectives of  $S \# G$ . To see that they have the same arrows consider as above the minimal resolution of  $\mathbb{C}$ .

$$0 \longrightarrow S \otimes_{\mathbb{C}} \bigwedge^n V \xrightarrow{\partial_n} \cdots \xrightarrow{\partial_2} S \otimes_{\mathbb{C}} \bigwedge^1 V \xrightarrow{\partial_1} S \longrightarrow 0$$

If we tensor with  $V_i$  on the right we will get a minimal resolution of  $V_i$  (you can see that is minimal by using the exact same argument as above).

$$\cdots \xrightarrow{\partial_2 \otimes_{\mathbb{C}} V_i} S \otimes_{\mathbb{C}} \bigwedge^1 V \otimes_{\mathbb{C}} V_i \xrightarrow{\partial_1 \otimes_{\mathbb{C}} V_i} S \otimes_{\mathbb{C}} V_i \longrightarrow 0$$

From here, since  $\bigwedge^1 V = V$ , we see that  $P_j = S \otimes_{\mathbb{C}} V_j$  appears as a direct summand of  $S \otimes_{\mathbb{C}} V \otimes_{\mathbb{C}} V_i$  exactly when  $V_j$  appears as a direct summand of  $V \otimes_{\mathbb{C}} V_i$ .  $\square$

### 3 The endomorphism ring of $S$ as an $S^G$ -module

Isomorphism to  $S \# G$  implies  $\text{proj } S \# G$  is  $S^G$  direct summands of  $S$ .

In this section we will show that  $S \# G$  is isomorphic to  $\text{End}_R(S)$  as rings, where  $R = S^G$  is the fixed ring of  $S$  by  $G$  (and we have some additional assumptions on  $S$  and  $G$ ). This will be the longest proof of this thesis and I have therefor decided to split it up into several steps. The proof will be done by constructing an explicit isomorphism.

$$S \# G \quad \text{End}_R(S)$$

$$s \cdot g \quad (t \mapsto s \cdot t^g)$$

We can easily show that this is an injective ring-homomorphism. The meat of the proof is to consider the map as a morphism of  $S$ -modules, and then using ramification theory to show that it is an epimorphism. To do this we will show that for every height one prime ideal  $\mathfrak{p}$  of  $S$  if we localize at  $\mathfrak{p}$  we get a so-called unramified extension of rings.

$$R_{\mathfrak{p} \cap R} \hookrightarrow S_{\mathfrak{p}}$$

We will use this to show that the short exact sequence

$$I \hookrightarrow S_{\mathfrak{p}} \otimes_{R_{\mathfrak{p} \cap R}} S_{\mathfrak{p}} \xrightarrow{m} S_{\mathfrak{p}}$$

where  $m$  is the multiplication map and  $I$  is the kernel, has a splitting. Whenever this happens we say the extension is separable. We will use this splitting to construct an inverse for  $S_{\mathfrak{p}} \# G \rightarrow \text{End}_{R_{\mathfrak{p} \cap R}}(S_{\mathfrak{p}})$ . Finally we will

show that since we get an isomorphism whenever we localize at a height one prime ideal this means that the original map is an isomorphism.

Let us first begin with some definitions

**Definition 3.1.** Let  $A$  and  $B$  be two local commutative rings with maximal ideal  $\mathfrak{n}$  and  $\mathfrak{m}$  respectively, and let  $A \hookrightarrow B$  be an extension of rings. We say that the extension is unramified if the following conditions hold:

- $B$  is a finitely generated  $A$ -module.
- $A/\mathfrak{n} \hookrightarrow B/\mathfrak{m}$  is a separable field extension.
- $\mathfrak{n}B = \mathfrak{m}$

If the two first conditions are met, and there is a positive integer  $e$  such that  $\mathfrak{n}^e B = \mathfrak{m}$ , we say the extension has ramification index  $e$ . Note that being unramified is then equivalent to having ramification index 1.

In order to show that unramified implies separable we must first take a small detour.

**Definition 3.2.** Let  $A \rightarrow B$  be an extension of rings. We then define the derivation module  $\Omega_{B|A}$  as the  $B$ -module with formal generators  $db$  for all  $b \in B$  and with the following relations:

*A-linearity:*  $d(ab + a'b') = adb + a'db'$  for all  $a, a' \in A$  and  $b, b' \in B$ .

*Leibniz rule:*  $d(bc) = bdc + cdb$  for all  $b, c \in B$ .

Note that for any polynomial expression  $f(b)$  we have that  $df(b) = f'(b)db$  where  $f'$  is the formal derivative of  $f$ .

**Theorem 3.1.** Let  $S$  be the complex power series ring in two variables,  $G$  be a finite subgroup of  $GL_2(\mathbb{C})$ ,  $R = S^G$  the fixed ring of  $S$  under the action of  $G$ , and  $(S \# G)^G$  be the fixed ring of  $S \# G$  under left multiplication by  $G$ . Then  $S$  is isomorphic to  $(S \# G)^G$  as  $R$ -modules.

*Proof.* To see this we will define an injective  $R$ -linear map from  $S$  to  $S \# G$  and show that its image is  $(S \# G)^G$ . Let  $\rho : S \rightarrow S \# G$  be given by

$$\rho(s) = \sum_{g \in G} s^g \cdot g.$$

It's clear that it's injective and it is  $R$ -linear because

$$\rho(rs) = \sum_{g \in G} r^g s^g \cdot g = r \sum_{g \in G} s^g \cdot g.$$



It should also be clear that the image is contained in  $(S\#G)^G$  because

$$h \cdot \rho(s) = \sum_{g \in G} h \cdot s^g \cdot g = \sum_{g \in G} s^{hg} \cdot hg = \rho(s).$$

To see that the image is all of  $(S\#G)^G$  consider an arbitrary element in  $(S\#G)^G$ ,  $\psi = \sum_{g \in G} s_g \cdot g$ . Since  $\psi$  is fixed under left multiplication by  $G$  we must have that

$$\sum_{g \in G} s_g^h \cdot hg = \sum_{g \in G} s_g \cdot g,$$

in particular  $s_h$  must equal  $s_1^h$  and it follows that  $\psi = \rho(s_1)$ .  $\square$

**Theorem 3.2.**

## 4 Maximal Cohen-Macaulay modules of $S^G$

**Definition 4.1.** If  $R$  is a local ring with residual field  $k$  we define the depth of a module,  $M$ , to be the minimal  $n$  such that the extension  $\text{Ext}_R^n(k, M)$  is non-zero.

**Definition 4.2.** If  $R$  is a commutative ring and  $M$  is an  $R$ -module, an  $R$ -regular sequence on  $M$  is a sequence of elements of  $R$ ,  $r_1, r_2, \dots, r_n$  such that  $M/\langle r_1, \dots, r_i \rangle M$  is non-zero and multiplication by  $r_i$  is injective on  $M/\langle r_1, \dots, r_{i-1} \rangle M$ .

**Definition 4.3.** If  $R$  is a ring, we say that  $\mathfrak{p}$  is a prime ideal in  $R$  if

1.  $\mathfrak{p}$  is a proper ideal of  $R$ .
2. For any two elements  $a, b \in R$  such that  $ab \in \mathfrak{p}$  we must have that either  $a$  is in  $\mathfrak{p}$  or  $b$  is.

**Definition 4.4.** If  $R$  is a ring we define its Krull-dimension to be the maximum length of a chain of prime ideals in  $R$ . For example the polynomial ring  $\mathbb{C}[x_1, \dots, x_n]$  has Krull-dimension  $n$  given by the chain

$$0 \subseteq \langle x_1 \rangle \subseteq \langle x_1, x_2 \rangle \subseteq \dots \subseteq \langle x_1, \dots, x_n \rangle$$

**Definition 4.5.** If  $M$  is a module over a local ring  $R$  with Krull-dimension  $d$  we say that  $M$  is maximal Cohen Macaulay (MCM) if the depth of  $M$  equals  $d$ .

**Theorem 4.1.** If  $G$  is a finite subgroup of  $GL_n(\mathbb{C})$ ,  $S$  is the complex power series ring in  $n$  variables and  $R = S^G$  is the ring fixed under the action of  $G$ , then  $R$  is a direct summand of  $S$  as  $R$ -modules.

*Proof.* Consider the map  $\pi : S \rightarrow R$  given by

$$\pi(s) = \frac{1}{|G|} \sum_{g \in G} s^g$$

It's clear that the image of  $\pi$  is in  $R$  because an action from  $G$  will just permute the order of the sum. Further

$$\pi(r) = \frac{1}{|G|} \sum_{g \in G} r^g = \frac{1}{|G|} \sum_{g \in G} r = r,$$

so  $\pi$  splits the inclusion  $R \hookrightarrow S$  which shows that  $R$  is a direct summand of  $S$ .  $\square$

**Definition 4.6.** Let  $R$  be a *local* ring,  $M$  an  $R$ -module, and  $(x_i)_{i=1}^n$  an  $R$ -regular sequence on  $M$ . Let  $V$  denote the free abelian group with formal generators  $\{x_i\}_{i=1}^n$ . The Koszul complex of the sequence is then defined to be

$$\begin{aligned} 0 &\longrightarrow M \otimes_{\mathbb{Z}} \bigwedge^n V \xrightarrow{\partial_n} M \otimes_{\mathbb{Z}} \bigwedge^{n-1} V \xrightarrow{\partial_{n-1}} \dots \\ &\dots \xrightarrow{\partial_2} M \otimes_{\mathbb{Z}} \bigwedge^1 V \xrightarrow{\partial_1} M \longrightarrow 0. \end{aligned}$$

**Theorem 4.2.** The Koszul complex of a regular sequence is exact everywhere except in degree 0.

*Proof.* The proof will be inductive on the length of the regular sequence. Let's first show that it holds when the length of the sequence is 1. Let  $M$  be our module,  $(x_1)$  our regular sequence, and  $V = \mathbb{Z}x_1$ . Then we get the following complex

$$0 \longrightarrow M \otimes_{\mathbb{Z}} \bigwedge^1 V \xrightarrow{\partial_1} M \longrightarrow 0$$

This is isomorphic to

$$0 \longrightarrow M \xrightarrow{x_1 \cdot -} M \longrightarrow 0$$

and since multiplication by  $x_1$  is injective on  $M$ , this is exact in degree 1. Since it is 0 in all degrees except 0 and 1, it is exact there as well.

For the inductive step we will show a relationship between regular sequences of different lengths. Let  $(x_i)_{i=1}^n$  be a regular sequence on  $M$ , and let  $K(s)$  be the Koszul complex of the sequence  $(x_i)_{i=1}^s$ . Then there is an exact sequence of complexes

$$0 \longrightarrow K(s) \xrightarrow{\iota} K(s+1) \xrightarrow{\pi} K(s)[-1] \longrightarrow 0$$

where  $K(s)[-1]$  is  $K(s)$  shifted one spot to the left. The short exacts sequence is given by

$$\begin{array}{ccccccccc}
\cdots & \xrightarrow{\partial_3^s} & M \otimes_{\mathbb{Z}} \bigwedge^2 V_s & \xrightarrow{\partial_2^s} & M \otimes_{\mathbb{Z}} \bigwedge^1 V_s & \xrightarrow{\partial_1^s} & M & \longrightarrow & 0 \\
& & \downarrow \iota_2 & & \downarrow \iota_1 & & \parallel & & \downarrow \\
\cdots & \xrightarrow{\partial_3^{s+1}} & M \otimes_{\mathbb{Z}} \bigwedge^2 V_{s+1} & \xrightarrow{\partial_2^{s+1}} & M \otimes_{\mathbb{Z}} \bigwedge^1 V_{s+1} & \xrightarrow{\partial_1^{s+1}} & M & \longrightarrow & 0 \\
& & \downarrow \pi_2 & & \downarrow \pi_1 & & \downarrow & & \downarrow \\
\cdots & \xrightarrow{\partial_2^s} & M \otimes_{\mathbb{Z}} \bigwedge^1 V_s & \xrightarrow{\partial_1^s} & M & \longrightarrow & 0 & \longrightarrow & 0
\end{array}$$

where  $\iota$  is just the inclusion and  $\pi_{j+1}(m \otimes x_{i_1} \wedge \cdots \wedge x_{i_j} \wedge x_{s+1}) = m \otimes x_{i_1} \wedge \cdots \wedge x_{i_j}$ , and 0 if  $x_{s+1}$  does not appear in the wedge product.

This short exact sequence induces a long exact sequence on homology

[illegible]

Now we assume by induction that  $K(s)$  is exact in all non-zero degrees. Then we get the following long exact sequence

[illegible]

where  $I_k = \langle x_i \rangle_{i=1}^k$ . Then since the sequence is exact we get that  $H_i(K(s+1)) = 0$  whenever  $i \geq 2$ , and that  $H_1(K(s+1))$  is isomorphic to the kernel of multiplication by  $x_{s+1}$ . Here we use the fact that the sequence is regular, which means that multiplication by  $x_{s+1}$  is injective on  $M/I_s M$  and therefore  $H_1(K(s+1)) = 0$ .

Then by induction we have shown that  $H_i(K(n)) = 0$  for all  $i \neq 0$ , or in other words that the Koszul complex is exact in all degrees except 0.  $\square$

I need to show that  $S^G$  is regular CM at some point.

**Proposition 4.1.** *Let  $R$  be a local regular ring (a local ring where the maximal ideal is generated by a regular sequence on  $R$ ) with depth of  $R$  equaling it's Krull-dimension. If  $M$  is an MCM  $R$ -module, and  $N$  is a direct summand of  $M$  then  $N$  is also MCM.*

*Proof.* Let  $(x_i)_{i=1}^n$  be the regular sequence on  $R$ , let  $k = R/\langle x_i \rangle_{i=1}^n$  be the residual field. Then we have seen that the Koszul complex is a projective resolution of  $k$ . Since  $\text{Ext}_R^i(k, M)$  is 0 for all  $i < n$ , mapping  $\text{Hom}_R(-, M)$

□

## Appendices

### A Representation theory

**Definition A.1.** If  $R$  is a ring and  $M$  is an abelian group, we define a representation of  $R$  to be a ring-map,  $\varphi$ , from  $R$  to  $\text{End}(M)$ . We say that  $M$  is a (left)  $R$ -module, and we write  $rm$  with  $r \in R$  and  $m \in M$  to mean  $\varphi(r)(m)$ . Similarly we define a right  $R$ -module if  $\varphi$  goes from  $R$  to  $\text{End}(M)^{op}$  and we write  $mr$  for  $\varphi(r)(m)$ .

**Definition A.2.** If  $G$  is a group and  $V$  a complex vectorspace, we define a representation of  $G$  to be a group-map,  $\rho$ , from  $G$  to  $\text{Aut}_{\mathbb{C}}(V)$ . When  $\rho$  is infered we say that  $V$  is a representation of  $G$  and we write  $gv$  to mean  $\rho(g)(v)$ . Note that representations of  $G$  exactly corresponds to representations of the ring  $\mathbb{C}G$  of formal linear combinations of elements of  $G$  with multiplication given by  $\lambda g \cdot \lambda' g' = (\lambda \cdot \lambda') gg'$ .

**Definition A.3.** If  $R$  is a ring and  $M_1$  and  $M_2$  are two modules we define their direct sum,  $M_1 \oplus M_2$  to be the module consisting of all pairs  $(m_1, m_2)$  (usually written  $m_1 + m_2$ ), where addition and scalar multiplication is point-wise. If a non-zero module cannot be written as the direct sum of two non-zero modules we call it indecomposable.

**Definition A.4.** A submodule is a subset of a module which is also a module. A non-zero module with no non-trivial proper submodules is called simple or irreducible<sup>2</sup>.

**Theorem A.1.** (Schur's Lemma) Let  $G$  be a group and  $V$  and  $W$  be two irreducible representations of  $G$ . If  $f : V \rightarrow W$  is a  $G$ -linear map then  $f$  is a 0 if  $V$  and  $W$  are not isomorphic, and a scaling of identity (up to change of basis) if they are isomorphic.

*Proof.* Start by assuming  $f$  is non-zero. Then we will show that  $V$  and  $W$  are isomorphic. Since the image of  $f$  is a non-zero subrepresentation of  $W$  and  $W$  is irreducible, we have that  $\text{Im } f = W$  and  $f$  is surjective. Since the kernel of  $f$  is a proper subrepresentation of  $V$  we must have that the

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<sup>2</sup>The word simple is used for representations of rings while irreducible is used for representations of groups. Note that for finite groups irreducible and indecomposable are equivalent.

kernel is 0, and that  $f$  is injective. Thus  $f$  is an isomorphism. Now assume  $f : V \rightarrow V$  is a  $G$ -linear map. then we want to show that  $f$  is simply a scaling of identity. Since  $f$  is a linear map on a complex vector space it must have at least one eigen value, say  $\lambda \in \mathbb{C}$ . Let  $v$  be in the eigenspace  $\lambda$ . Since  $f(gv) = gf(v) = \lambda gv$  for all  $g$  in  $G$  we have that  $gv$  is also in the eigenspace. This means the eigenspace is a subrepresentation, and since  $V$  is irreducible it must equal all of  $V$ . This means that  $f$  is just scaling by  $\lambda$ .  $\square$

**Definition A.5.** We call a functor left exact if for any short exact sequence

$$0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C$$

the image of the sequence under the functor is also exact. For example for any module  $M$  the functor  $\text{Hom}(M, -)$  is left exact. That is the sequence

$$0 \longrightarrow \text{Hom}(M, A) \xrightarrow{f \circ -} \text{Hom}(M, B) \xrightarrow{g \circ -} C$$

is exact. Dually we call a functor right exact if short exact sequences of the form

$$A \longrightarrow B \longrightarrow C \longrightarrow 0$$

is mapped to an exact sequence. A functor that is both left exact and right exact is called exact.

**Definition A.6.** We say that a module,  $P$ , is projective if for any epimorphism  $f : M \twoheadrightarrow N$ , and any map  $g : P \rightarrow N$ , there is a map  $\varphi : P \rightarrow M$  such that  $f\varphi = g$ . Said another way, the diagram below induces the dotted arrow making the diagram commute

$$\begin{array}{ccc} & & P \\ & \nearrow \varphi & \downarrow g \\ M & \xrightarrow{f} & N \end{array}$$

Note that  $P$  being projective is equivalent to  $\text{Hom}(P, -)$  being right exact (i.e. exact).

Projective cover + radical is small for noetherian modules

indecomposable projective = summand of ring = idempotent of ring

projective resolution

$\text{Ext} + \text{Tor}$

## 5 Random Thoughts I need to figure out

If  $R=S$  then they have the same depth meaning  $M$  is Cohen-Macaulay iff its projective dimension is 0 (Auslander-Buchsbaum), but that means its a direct summand of  $S$  as  $S(=R)$ -module, which make sense. If I can show that  $R = S/(f)$  for some polynomial  $f$ , can I show  $R$ -direct summands of  $S$  have projective dimension 1 over  $S$ ? Can I show  $R$  has depth  $\text{depth}(S)-1$ ? Then I also need to prove Auslander-Buchsbaum... Need to show some relation between dimension and depth.

If  $P$  is indecomposable finitely generated projective then it is direct summand of  $S^n$ , then  $P$  must either be a direct summand of  $S$  or  $S^n - 1$  then by induction  $P$  is a summand of  $S$ . Can I assume  $P$  to be finitely generated????

$P/mP = \sum V_i \rightarrow P = \sum SV_i$ , means all projective  $S\#G$ -modules can be broken down into sums.

## 6 questions

Why does  $0 \rightarrow J \rightarrow S \otimes_R S \rightarrow S \rightarrow 0$  split?

Is it true that  $\dim R \leq \dim S$  (this is not true for  $\mathbb{Z} \subset \mathbb{Q}$ ), alternatively how to show that  $S$  is MCM?

Direct summand of MCM is MCM? Use Koszul complex + Ext preserves direct sums.

Are indecomposable proj  $S\#G$ -modules fin.gen.? Can I state the correspondence in terms of fin.gen, indecomposable projectives instead?

## 7 Disposition

Define McKay quiver [check]

Define  $S\#G$  [check]

Correspondence with projectives [put in finitely generated to fix argument]

Gabriel Quiver [Make Koszul complex a reference]

$\text{End}_R(S) \cong S\#G$  [understand the proof]  $q \in S$  height one prime implies  $q = (f)$  for a homogeneous polynomial? Why homogeneous? If  $T(q)$  is non-trivial then it acts non-trivially on  $S/qS$  if  $f$  has degree bigger than 1, then all degree 1 polynomials survive in  $S/qS$  and are acted upon trivially by  $T(q)$ . Therefore  $T(q)$  would be trivial, so  $f$  is homogeneous of degree 1. Since the group operations preserve degree  $\sigma(f) = a_\sigma f$  for a nonzero constant  $a_\sigma$ . All finite matrix groups diagonalizable implies  $\sigma = \text{diag}(1, 1, \dots, a_\sigma)$ . Therefore  $T(q)$  is iso to finite subgroup of  $\mathbb{C}^*$ , hence cyclic. Then  $p = q \cap R = (f^n)$  where  $n$  is the order of  $T(q)$ . Thus  $q = pS_q$  if and only if  $T(q)$  is trivial.  $I/I^2 = \Omega_{S|R}$  is 0 iff  $pS = q$ ,  $I/I^2 = 0$  implies idempotent implies splitting. Why is  $\text{End}_R(S)$  reflexive? or rather why does height one iso imply iso.

MCM  $R$ -summands of  $S$  [ $S$  is MCM using dimension argument, summands are MCM using  $\text{depth} \leq \dim$ ]

## References

- [James and Liebeck, 2001] James, G. and Liebeck, M. (2001). *Representations and characters of groups*. Cambridge University Press, New York, second edition.
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