McKay correspondence

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Abstract

The goal of this thesis is to establish a 1-1 correspondence between quivers created from the four following sets whenever S is the power series ring $\mathbb{C}[\![x, y]\!]$ and G is a finite subgroup of $SL(2, \mathbb{C})$ acting on S.

- The Maximal Cohen-Macaulay modules of the fixed ring S^G;
- The indecomposable projective modules of the skew group algebra $S\#G$;
- The indecomposable projective modules of $\operatorname{End}_{S^G}(S)$;
- The irreducible representations of G (indecomposable $\mathbb{C}G$-modules).

Much of the thesis will be used to define these four quivers and to develop tools to establish such a correspondence.

Contents

Introduction

The McKay correspondence can be described as an isomorphism between four quivers related to a finite subgroup $G \leq SL_2(\mathbb{C})$ of the special linear group of complex matricies. If we let $S = \mathbb{C}[\![x, y]\!]$ be the complex power series ring in two variables, and $R = S^G$ the subring of S fixed by G, then there are two rings relevant to the theorem. Those are $S\#G$, the skew group algebra related to the group action of G on S, and $\mathrm{End}_R(S)$ the endomorphism ring of S as an R-module. Then the McKay correspondence is the isomorphism between:

- The McKay quiver formed by the irreducible representations of G;
- The Gabriel quiver formed by the indecomposable projective modules of the skew group algebra $S\#G$;
- The Gabriel quiver formed by the indecomposable projective modules of the endomorphism ring $\mathrm{End}_R(S)$;
- The Auslander-Reiten quiver formed by the indecomposable maximal Cohen-Macaulay R-modules.

In fact the skew group algebra and the endomorphism ring are isomorphic, thus the isomorphism between the two Gabriel quivers is evident.

The story of the McKay correspondence starts in 1884 with Klein. He studied the finite subgroups $G \leq SL_2(\mathbb{C})$, and gave a characterization of the isolated quotient singularities $\mathbb{C}^2/G$, known today as Kleinian singularities [Kle84]. Later it was shown that the resolution graphs for these singularities gave a correspondence with the ADE Dynkin diagrams [DV34, Art66].

McKay observed that one could obtain the resolution graphs for $\mathbb{C}^2/G$ purely by studying the representation theory of G [McK83]. Specifically the resolution graph is isomorphic to the reduced McKay quiver. This established an important bridge between the geometric and algebraic views of these quotient singularities. Many other important results have aided in the understanding of this correspondence, both from a geometric point of view [GSV81, AV85, EK85], and from an algebraic point of view [Aus86, AR89].

In this thesis we focus on the algebraic side. The story starts with Herzog in 1978 [Her78] when he showed that the ring $R = \mathbb{C}[\![x, y]\!]^G$ has finitely many indecomposable maximal Cohen-Macaulay modules and that they all arise as direct summands of $\mathbb{C}[\![x, y]\!]$. The work of Auslander [Aus86] extended the McKay correspondence to be between the indecomposable projective modules of the skew algebra $S\#G$, and the indecomposable R-direct summands of $\mathbb{C}[\![x,y]\!]$. Since the indecomposable projective modules of $S\#G$ are in correspondence with the irreducible representations of G, this ties in nicely with McKay's observation. In our specific case when G is in $SL_2(\mathbb{C})$ we have that $S\#G$ is isomorphic to $\mathrm{End}_R(S)$ [AG60, Aus62].

The goal of this thesis is to outline part of the McKay correspondence. The strategy for the proof in this thesis can be divided into 4 parts:

- Define the McKay quiver with verticies the irreducible representations of G;
- Show that the irreducible representations of G exactly corresponds to the indecomposable projective modules of the skew group algebra $S\#G$, and that this correspondance means the Gabriel quiver equals the McKay quiver;
- Show that $S\#G$ is isomorphic to the endomorphism ring $\mathrm{End}_R(S)$;
- Show that R-direct summands of S are MCM modules, and use the isomorphism $S\#G \cong \mathrm{End}_R(S)$ to show that the R-direct summands of S corresponds to indecomposable projective modules that appear as summands of $S\#G$.

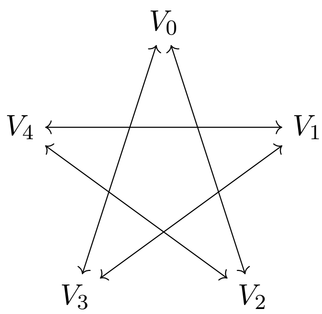
Something that is omitted from this thesis is the definition of the Auslander-Reiten quiver for the R-direct summands of S and why it is isomorphic to the Gabriel quiver of $S\#G$. The interested reader may look in [LW12] for further reading, specifically the isomorphism between the McKay quiver and the Auslander-Reiten quiver of the MCM R-modules is given in [LW12, Proposition 13.22].

1 The McKay quiver

For a given group, G, and a representation of that group the McKay quiver uses that representation to establish relations between the irreducible representations of G. In the special case that G is a linear group we have a natural choice of representation to use. This leads us to define the McKay quiver as below.

Definition 1.1. Let G be a finite subgroup of $GL_n(\mathbb{C})$, and let V be the canonical representation (the one that sends every element to itself). Then we define the *McKay quiver* of G to be the quiver with vertices the irreducible representations of G, denoted V_i. For two irreducible representations V_i and V_j there is an arrow from the former to the latter if and only if V_j is a direct summand of $V \otimes V_i$.

Example 1.2. Let G be the group generated by $g = \begin{pmatrix} \omega^2 & 0 \\ 0 & \omega^3 \end{pmatrix}$, where ω is a primitive fifth root of unity. Then there are five different irreducible representations, the one sending g to ω, ω^2, ω^3, ω^4 respectively, and the trivial representation. Denote the representation sending g to ω^i by V_i, and let $V = V_2 \oplus V_3$ be the canonical representation. Note that $V_i \otimes V_j = V_{i+j}$, where $i+j$ is understood to be modulo 5. Then we get the following McKay quiver



One of the remarks that sparked the interest for the McKay correspondence is that the McKay quivers for the finite subgroups of $SL_2(\mathbb{C})$ are exactly the ADE Dynkin diagrams, if we exclude the trivial representation. Including the trivial representation we get an extended ADE Dynkin diagram. The proof of this can be done on a case by case since there are up to change of basis only five families of subgroups of $SL_2(\mathbb{C})$. A derivation of these groups can be found in [CS14]. Here we simply list them in table 1 on the following page. We write ζ for the primitive fifth root of unity $\exp(2\pi i/5)$, and n is an integer strictly larger than 1.

Table 1: The finite subgroups of $SL_2(\mathbb{C})$.

G	generators in $SL_2(\mathbb{C})$
$\mathbb{Z}/n\mathbb{Z}$	$\begin{pmatrix} \exp(2\pi i/n) & 0 \\ 0 & \exp(-2\pi i/n) \end{pmatrix}$
BD_{4n}	$\begin{pmatrix} \exp(\pi i/n) & 0 \\ 0 & \exp(-\pi i/n) \end{pmatrix}, \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$
BT_{24}	$\begin{pmatrix} \frac{i+1}{2} & -\frac{i+1}{2} \\ \frac{-i+1}{2} & \frac{-i+1}{2} \end{pmatrix}, \begin{pmatrix} \frac{i+1}{2} & \frac{i+1}{2} \\ -\frac{-i+1}{2} & \frac{-i+1}{2} \end{pmatrix}$
BO_{48}	$BT_{24}, \begin{pmatrix} \frac{1+i}{\sqrt{2}} & 0 \\ 0 & \frac{1-i}{\sqrt{2}} \end{pmatrix}$
BI_{120}	$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \begin{pmatrix} \zeta^3 & 0 \\ 0 & \zeta^2 \end{pmatrix}, \frac{1}{\sqrt{5}} \begin{pmatrix} -\zeta+\zeta^4 & \zeta^2-\zeta^3 \\ \zeta^2-\zeta^3 & \zeta-\zeta^4 \end{pmatrix}$

The groups have a natural action on the power series ring $S = \mathbb{C}[\![x,y]\!]$ by changing variables. If we let R be the ring fixed by this action then R is isomorphic to $\mathbb{C}[\![u,v,w]\!]/\langle f\rangle$ for some irreducible polynomial f. This is also shown on a case by case basis and can be found in [Boc]. In table 2 below, we list the groups, their fixed rings, and the extended Dynkin diagrams that is their McKay quiver.

Table 2: The fixed rings of the finite subgroups of $SL_2(\mathbb{C})$.

McKay quiver	G	$R = S^G$
$\tilde{A}_{n-1}$	$\mathbb{Z}/n\mathbb{Z}$	$\mathbb{C}[\![u,v,w]\!]/\langle uv - w^n\rangle$
$\tilde{D}_{n+2}$	BD_{4n}	$\mathbb{C}[\![u,v,w]\!]/\langle u^{n+1} + v^2 - uw^2\rangle$
$\tilde{E}_6$	BT_{24}	$\mathbb{C}[\![u,v,w]\!]/\langle u^4 + v^3 + w^2\rangle$
$\tilde{E}_7$	BO_{48}	$\mathbb{C}[\![u,v,w]\!]/\langle u^3v + v^3 + w^2\rangle$
$\tilde{E}_8$	BI_{120}	$\mathbb{C}[\![u,v,w]\!]/\langle u^5 + v^3 + w^2\rangle$

Since the finite subgroups of $SL_2(\mathbb{C})$ are conjugate to finite subgroups in $SU(2)$, and there is a 2-1 map from $SU(2)$ to $SO(3)$ we can associate the finite subgroups of $SL_2(\mathbb{C})$ to finite subgroups of $SO(3)$. Specifically each finite subgroup of $SL_2(\mathbb{C})$ has a surjective group homomorphism to a finite subgroup of $SO(3)$ with kernel either of order 2 or 1. Since $SO(3)$ is the group of 3-dimensional roatations, the subgroups of $SO(3)$ is exactly the symmetry groups of solid figures. In figure 1 below, there are some illustrations of solid figures labeled with the extended Dynkin diagrams coming from the subgroup of $SL_2(\mathbb{C})$ which their symmetry group can be associated with.

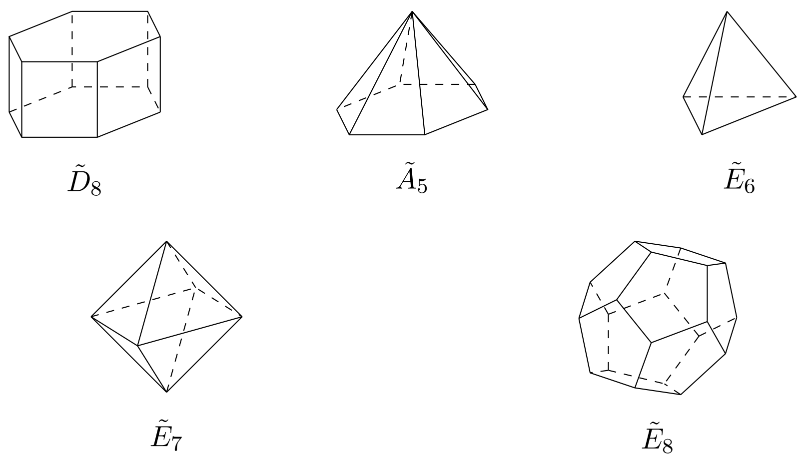


Figure 1: Three dimensional solids labeled by corresponding extended Dynkin diagrams.

2 Indecomposable projectives of the skew group algebra S#G

This section is largely based on the book by Leuschke and Wiegand [LW12]. Here we will define the skew group algebra (definition 2.1) and the Gabriel quiver (definition 2.6). We will show in theorem 2.5 that the irreducible representations of G are in correspondence with the indecomposable projective modules of the skew group algebra. Finally in theorem 2.9 we will show that the Gabriel quiver is isomorphic to the McKay quiver.

This section will use definitions and theorems from representation theory as taught in the courses MA3203 - Ring Theory and MA3204 - Homological Algebra. Since we do not assume knowledge of this we have created appendix A. We will try to use footnotes to indicate where such theorems are used.

Definition 2.1. Let S be an algebra and G a subgroup of $\mathrm{Aut}(S)$ acting on S. Then we define the *skew group algebra* $S\#G$ to be the algebra generated by elements of the form $f \cdot g$ with $f \in S$ and $g \in G$. the multiplication in $S\#G$ is given by

$$(f_1 \cdot g_1) \cdot (f_2 \cdot g_2) = (f_1 \cdot f_2^{g_1}) \cdot (g_1 \cdot g_2),$$

where f^g denotes the image of f under the action of g.

The skew group algebra is also sometimes called the twisted group algebra, because the multiplication is "twisted" by the action of G. Now we will define an action by $GL_n(\mathbb{C})$ on $\mathbb{C}[\![x_1, \cdots, x_n]\!]$ so we can construct the skew group algebra we are interested in.

Definition 2.2. If $S = \mathbb{C}[\![x_1, \cdots, x_n]\!]$ is the complex power series ring in n variables and G is a subgroup of $GL_n(\mathbb{C})$ we want to extend the action of G on $\mathbb{C}^n$ to S. We say that G acts by *linear change of variables* if G acts on x_i as it would on the ith basis vector of $\mathbb{C}^n$, and acts on products and sums by acting on each component separately.

Now that we have a group action of $GL_n(\mathbb{C})$ on the power series ring we can construct the $S\#G$ we are interested in, but first we will show a general statement about skew group algebras for finite groups.

Lemma 2.3. *Let S be a complex algebra, and G a finite group acting (faithfully) on S. An $S\#G$-module is projective*[1] *if and only if it is projective as an S-module.*

[1] The definition of projective can be found in definition A.6 on page 31.

Proof. Only-ifity follows from $S\#G$ being a free S-module, it is isomorphic to $\bigoplus_{g\in G} S$. Thus we need only show ifity.

First we need to see that an $S\#G$-linear map is just an S-linear map, $f : M \to N$ between $S\#G$-modules, such that $f(g(m)) = g(f(m))$ for all $g \in G$ and all $m \in M$. Equivalently $f(m) = g(f(g^{-1}(m)))$. This allows us to define a group action on S-linear maps by $f^g(m) = g(f(g^{-1}(m)))$. Then we just need to show

$$\mathrm{Hom}_{S\#G}(M,N) = \mathrm{Hom}_S(M,N)^G.$$

Clearly if f is $S\#G$-linear then it's in $\mathrm{Hom}_S(M,N)^G$. To see the other inclusion, let f be an S-linear map that is fixed under G. Then $f(s \cdot gm) = sf(gm) = s \cdot g(f(g^{-1}gm)) = s \cdot gf(m)$, and hence f is $S\#G$-linear.

Nextly we want to show that $-^G$ is an exact functor[2]. If K is the kernel of a map $f : M \to N$, then the kernel of the induced map $f^G : M^G \to N^G$ is just $K \cap M^G$ which equals K^G, and hence $-^G$ preserves kernels. Assume f is epi and let $n \in N^G$. Consider a preimage m such that $f(m) = n$. Let $\theta = \frac{1}{|G|}\sum_{g\in G} g(m)$. Then θ is in M^G and $f(\theta) = \frac{1}{|G|}\sum_{g\in G} g(f(m)) = \frac{1}{|G|}\sum_{g\in G} n = n$. Since $-^G$ preserves kernels and epimorphisms it is exact.

Recall that a module being projective is equivalent to its covariant Hom-functor being exact. So if P is projective as an S-module then $\mathrm{Hom}_S(P,-)$ is exact. Using our above result we get $\mathrm{Hom}_S(P,-)^G = \mathrm{Hom}_{S\#G}(P,-)$ is exact and the lemma follows. $\square$

This shows that if we want to study projective $S\#G$-modules it is, for many purposes, sufficient to study them as S-modules. The following lemma holds for any local ring, but here we prove it only for the case when S is a complex power series ring.

Lemma 2.4. *Let S be the complex power series ring in n variables, and $\mathfrak{m} = \langle x_i \rangle_{i=1}^n$ the radical of S. Then for any free S-module N, $\mathfrak{m}N$ is small in N. That is, if X is a submodule of N such that $X + \mathfrak{m}N = N$, then $X = N$.*

Proof. Let N be the free module $S^{(I)} := \bigoplus\limits_{i\in I} S_i$, where $S_i \cong S$. Assume that X is a submodule such that $X + \mathfrak{m}N = N$. We denote by 1_i the elements that is 1 at index i and 0 elsewhere. Since $\{1_i\}$ generate N, it is enough to show that X contains all of them. Since $X + \mathfrak{m}N = N$, we know that there is an $m_i \in \mathfrak{m}N$ and an $x_i \in X$ such that $x_i + m_i = 1_i$. Then we have that

[2] The definition of an exact functor can be found in definition A.5 on page 30.

$x_i = 1_i - m_i$. Since the power series at index i of x_i has constant coefficient 1 it is invertible. If we multiply x_i by its inverse we get $\tilde{x}_i$ which is 1 at index i and some element of $\mathfrak{m}$ at index $j \neq i$, say m_{ij}. Then $\tilde{x}_i - \sum\limits_{j \neq i} m_{ij}\tilde{x}_j$ has a unit in index i and 0 at all other indices. Thus X contains 1_i for all i, and $X = N$. □

Theorem 2.5. *Let $S = \mathbb{C}[\![x,y]\!]$, G be a finite subgroup of $GL_2(\mathbb{C})$, and $\mathfrak{m} = \langle x, y\rangle_S$ be the radical of S. Then there are bijections between the indecomposable*[3] *projective $S\#G$-modules and the indecomposable $\mathbb{C}G$-modules given by*

$$\left\{\begin{matrix}\textit{indecomposable projective}\\ S\#G\textit{-modules}\end{matrix}\right\} \longrightarrow \left\{\begin{matrix}\textit{indecomposable}\\ \mathbb{C}G\textit{-modules}\end{matrix}\right\}$$

$$\mathcal{F} : P \longmapsto P/\mathfrak{m}P$$

$$\mathcal{G} : S \otimes_{\mathbb{C}} W \longleftarrow W$$

where the $S\#G$-module structure on $S \otimes_{\mathbb{C}} W$ is given by $(s \cdot g) \cdot f \otimes v = sf^g \otimes v^g$.

Proof. First we should show that $S \otimes_{\mathbb{C}} W$ is an indecomposable projective $S\#G$-module and that $P/\mathfrak{m}P$ is in fact an indecomposable $\mathbb{C}G$-module. Since $S \otimes_{\mathbb{C}} W$ is a free S-module it follows from lemma 2.3 that it is projective. To see that it is indecomposable we will first study it as an S-module and exploit the fact that $\operatorname{Hom}_{S\#G}(M,N) \subseteq \operatorname{Hom}_S(M,N)$.

Using lemma 2.4 we get that $\mathfrak{m}S \otimes_{\mathbb{C}} W$ is small in $S \otimes_{\mathbb{C}} W$. This means that we get that

$$\frac{S \otimes_{\mathbb{C}} W}{\mathfrak{m}S \otimes_{\mathbb{C}} W} \cong S/\mathfrak{m} \otimes_{\mathbb{C}} W \cong \mathbb{C} \otimes_{\mathbb{C}} W \cong W$$

and therefore $S \otimes_{\mathbb{C}} W \to W$ is a projective cover[4] of W as S-modules. Further since the projection $S \otimes_{\mathbb{C}} W \to W$ is $S\#G$-linear we have that $S \otimes_{\mathbb{C}} W$ is the projective cover of W also as $S\#G$-modules. Assume, for the sake of contradiction, that $S \otimes_{\mathbb{C}} W$ decomposes as $M \oplus N$ for non-zero M and N. Then W would equal $M/\mathfrak{m}M \oplus N/\mathfrak{m}N$ as an $S\#G/\langle\mathfrak{m}\rangle$-module.

[3] The definition of indecomposable can be found in definition A.3 on page 30.
[4] The definition of projective cover can be found in definition A.7 on page 31.

Since $S\#G/\langle\mathfrak{m}\rangle \cong \mathbb{C}G$ and W is indecomposable we must have that either $M/\mathfrak{m}M$ or $N/\mathfrak{m}N$ is 0. This then gives a contradiction because $\mathfrak{m}M$ and $\mathfrak{m}N$ are small in M and N. Hence we must have that $S \otimes_{\mathbb{C}} W$ is indecomposable.

It's clear that $P/\mathfrak{m}P$ is a $\mathbb{C}G$-module, because $\mathbb{C}G$ is a subring of $S\#G$. To see that it's indecomposable we will use a similar argument as above. Assume $P/\mathfrak{m}P$ decomposes as $V \oplus W$. Then both P and $S \otimes_{\mathbb{C}} V \oplus S \otimes_{\mathbb{C}} W$ are projective covers of $P/\mathfrak{m}P = V \oplus W$, and we get an induced $S\#G$-linear epimorphism between them.

$$\begin{tikzcd} & S \otimes_{\mathbb{C}} P/\mathfrak{m}P \arrow[d, two heads, "\pi_S"] \\ P \arrow[ur, dashed, two heads, "\varphi"] \arrow[r, two heads, "\pi_P"] & P/\mathfrak{m}P \end{tikzcd}$$

Since $S \otimes_{\mathbb{C}} P/\mathfrak{m}P$ is projective there exists an $S\#G$-linear map $\psi : P \to S \otimes_{\mathbb{C}} P/\mathfrak{m}P$ such that $\varphi \circ \psi = id_{S\otimes_{\mathbb{C}} P/\mathfrak{m}P}$. Then we have that

$$\pi_S = \pi_S \circ id_{S\otimes_{\mathbb{C}} P/\mathfrak{m}P} = \pi_S \circ \varphi \circ \psi = \pi_P \circ \psi.$$

Since π_P is a projective cover and π_S is an epimorphism this means that ψ is an epimorphism. Since $\varphi \circ \psi = id_{S\otimes_{\mathbb{C}} P/\mathfrak{m}P}$ it follows that φ and ψ are mutually inverse isomorphisms and that $P \cong S \otimes_{\mathbb{C}} P/\mathfrak{m}P$. This means that P decomposes as $S \otimes_{\mathbb{C}} V \oplus S \otimes_{\mathbb{C}} W$. Then since P is indecomposable we must have that either $S \otimes_{\mathbb{C}} V$ or $S \otimes_{\mathbb{C}} W$ is 0. That means that either V or W is 0, and we have shown that $P/\mathfrak{m}P$ is an indecomposable $\mathbb{C}G$-module.

To see that the given maps are bijections we will show that they are mutual inverses. First to see that $\mathcal{F}(\mathcal{G}(W)) \cong W$ we simply look at the definition

$$\frac{S \otimes_{\mathbb{C}} W}{\mathfrak{m}S \otimes_{\mathbb{C}} W} \cong S/\mathfrak{m} \otimes_{\mathbb{C}} W \cong \mathbb{C} \otimes_{\mathbb{C}} W \cong W$$

Next we consider $\mathcal{G}(\mathcal{F}(P)) = S \otimes_{\mathbb{C}} P/\mathfrak{m}P$. We have already seen that the induced map

$$\begin{tikzcd} & S \otimes_{\mathbb{C}} P/\mathfrak{m}P \arrow[d, two heads] \\ P \arrow[ur, dashed] \arrow[r, two heads] & P/\mathfrak{m}P \end{tikzcd}$$

is an isomorphism, and thus $P \cong \mathcal{G}(\mathcal{F}(P))$. $\square$

2.1 The Gabriel quiver

Now that we have seen that the indecomposable $\mathbb{C}G$-modules and the indecomposable projective $S\#G$-modules are in correspondence we will construct the Gabriel quiver and show that it corresponds to the McKay quiver.

Definition 2.6. For a skew group algebra $S\#G$ we define its *Gabriel quiver* to be the quiver with vertices as the indecomposable projective modules of $S\#G$. The arrows are given by taking the minimal projective resolution[5] of $P/\mathfrak{m}P$, where $\mathfrak{m}$ is as defined above. If the minimal projective resolution of $P/\mathfrak{m}P$ is given by

$$\cdots \longrightarrow Q_1 \longrightarrow Q_0 \longrightarrow 0$$

We say there is an arrow from P to P', if P' appears as a direct summand of Q_1.

Definition 2.7. Let V be a vector space. We then define the exterior algebra $\bigwedge V$ as the associative unital graded algebra such that the multiplication is bilinear and satisfies $x \wedge y = -y \wedge x$ for any x and y in V.

Some key properties of the exterior algebra is that $x \wedge x = 0$, and more generally that $x_1 \wedge \cdots \wedge x_p = 0$ whenever $\{x_i\}_{i=1}^{p}$ are linearly dependent.

The pth exterior power of V, denoted $\bigwedge^p V$ is the vector space of all elements that are the product of p vectors in V. If $\{x_i\}_{i=1}^{n}$ is a basis for V, then $x_{i_1} \wedge \cdots \wedge x_{i_p}$, where $i_1 < i_2 < \cdots < i_p$ and $1 \leq i_j \leq n$ form a basis for $\bigwedge^p V$, thus it is $\binom{n}{p}$-dimensional.

Proposition 2.8. *If S is the ring of formal power series over $\mathbb{C}$ in n variables, and G is a finite group acting on S, let $V = \mathfrak{m}/\mathfrak{m}^2$. Then the minimal projective resolution of $\mathbb{C} \cong S/\mathfrak{m}$ as an S-module is given by*

$$0 \longrightarrow S \otimes_{\mathbb{C}} \bigwedge^n V \xrightarrow{\partial_n} \cdots \xrightarrow{\partial_2} S \otimes_{\mathbb{C}} \bigwedge^1 V \xrightarrow{\partial_1} S \longrightarrow 0.$$

Here ∂_p is the $S\#G$-linear map defined by

$$\partial_p(s \otimes x_{i_1} \wedge x_{i_2} \wedge \cdots \wedge x_{i_p}) = \sum_{j=1}^{p} (-1)^{j+1} s x_{i_j} \otimes x_{i_1} \wedge \cdots \wedge \hat{x}_{i_j} \wedge \cdots \wedge x_{i_p},$$

[5] The definition of a minimal projective resolution can be found in definition A.8 on page 31.

where $x_{i_1} \wedge x_{i_2} \wedge \cdots \wedge x_{i_p}$ is one of the standard basis vectors for $\bigwedge^n V$, namely $i_1 < i_2 < \cdots < i_p$, and $\hat{x}_j$ means that x_j is omitted.

Proof. First we should show that this is a projective resolution of $\mathbb{C}$. In fact the complex described above is the Koszul complex of the regular sequence[6] $(x_i)_{i=1}^n$. The Koszul complex of a regular sequence is a projective resolution of the ring modulo the ideal generated by the regular sequence [Sta19, Tag 062F], which in this case equals $S/\langle x_i \rangle_{i=1}^n = \mathbb{C}$.

Secondly we want to show that the resolution is minimal. To do this it is enough to show that for each $k \geq 1$, ∂_k is a projective cover of its image, and that $S \to \mathbb{C}$ is a projective cover of $\mathbb{C}$. In other words we have to show that the kernels of the maps are small. Since $\operatorname{Im} \partial_{k+1} = \operatorname{Ker} \partial_k$ and $\operatorname{Im} \partial_{k+1} \subseteq \mathfrak{m} \otimes_{\mathbb{C}} \bigwedge^{k+1} V$ it follows from lemma 2.4 that the resolution is minimal. $\square$

Since $V = \mathfrak{m}/\mathfrak{m}^2 = \langle x_1, x_2, \cdots, x_n \rangle_{\mathbb{C}}$ is exactly the canonical representation of G the relationship between the McKay quiver and the Gabriel quiver should be apparent. Now we move to the next theorem for a formal argument.

Theorem 2.9. *If S is the complex power series ring in n variables and G is a finite subgroup of $GL_n(\mathbb{C})$, then the McKay quiver of G and the Gabriel quiver of $S\#G$ are isomorphic.*

Proof. We have already seen in theorem 2.5 that they have the same vertices, namely if V_i are the irreducible representations of G, then $S \otimes_{\mathbb{C}} V_i$ are the indecomposable projectives of $S\#G$. To see that they have the same arrows consider as above the minimal resolution of $\mathbb{C}$:

$$0 \longrightarrow S \otimes_{\mathbb{C}} \bigwedge^n V \xrightarrow{\partial_n} \cdots \xrightarrow{\partial_2} S \otimes_{\mathbb{C}} \bigwedge^1 V \xrightarrow{\partial_1} S \longrightarrow 0.$$

If we tensor with V_i on the right we will get a minimal resolution of V_i:

$$\cdots \xrightarrow{\partial_2 \otimes_{\mathbb{C}} V_i} S \otimes_{\mathbb{C}} \bigwedge^1 V \otimes_{\mathbb{C}} V_i \xrightarrow{\partial_1 \otimes_{\mathbb{C}} V_i} S \otimes_{\mathbb{C}} V_i \longrightarrow 0.$$

[6] Regular sequences are defined on page 13 in definition 3.2.

From here, since $\bigwedge^1 V = V$, we see that $P_j = S \otimes_{\mathbb{C}} V_j$ appears as a direct summand of $S \otimes_{\mathbb{C}} V \otimes_{\mathbb{C}} V_i$ exactly when V_j appears as a direct summand of $V \otimes_{\mathbb{C}} V_i$. $\square$

3 The endomorphism ring of S as an S^G-module

This section is largely based on the article by Iyama and Takahashi [IT13] and the book by Leuschke and Wiegand [LW12].

In this section we will show that $S\#G$ is isomorphic to $\mathrm{End}_R(S)$ as rings, where S is the complex power series ring in 2 variables, G is a finite subgroup of $SL_2(\mathbb{C})$, and $R = S^G$ is the fixed ring of S by G. This will be the longest proof in this thesis and we have therefore decided to split it up into several steps. The proof will be done by constructing an explicit isomorphism:

$$S\#G \longrightarrow \mathrm{End}_R(S)$$

$$s \cdot g \longmapsto (t \mapsto s \cdot t^g).$$

We can easily show that this is an injective ring-homomorphism. The meat of the proof is to consider the map as a morphism of R-modules, and then use ramification theory to show that it is an epimorphism. We will show in theorem 3.18 that for every height one prime ideal $\mathfrak{p}$ of S if we localize[7] at $\mathfrak{p}$ we get a so-called unramified extension of rings

$$R_{\mathfrak{p}\cap R} \hookrightarrow S_{\mathfrak{p}}.$$

Then theorem 3.10 will give us that the extension is separable. That is, we have a split exact sequence

$$I \hookrightarrow S_{\mathfrak{p}} \otimes_{R_{\mathfrak{p}\cap R}} S_{\mathfrak{p}} \xrightarrow{\mu} S_{\mathfrak{p}},$$

where μ is the multiplication map and I is the kernel of μ. Now writing $\mathfrak{q} = \mathfrak{p} \cap R$, and $S_{\mathfrak{q}}$ for $R_{\mathfrak{q}} \otimes_R S$, what we really want is a split exact sequence

$$I \hookrightarrow S_{\mathfrak{q}} \otimes_{R_{\mathfrak{q}}} S_{\mathfrak{q}} \xrightarrow{\mu} S_{\mathfrak{q}}.$$

[7] Prime ideals and localization will be defined later in definition 3.4 and definition 3.5.

We will use this splitting in theorem 3.11 to construct an inverse for $S_{\mathfrak{q}}\#G \to \mathrm{End}_{R_{\mathfrak{q}}}(S_{\mathfrak{q}})$. Finally we will show that since we get an isomorphism whenever we localize at a height one prime ideal, lemma 3.19 and theorem 3.20 will give us that the map is an isomorphism.

3.1 Commutative algebra

Definition 3.1. If A is a local ring with residual field k we define the *depth* of a module, M, to be the minimal n such that $\mathrm{Ext}^n_A(k, M)$ is non-zero[8]. We write $\mathrm{depth}_A(M)$ for this or simply $\mathrm{depth}(M)$ when which ring we are using is clear.

Definition 3.2. If R is a commutative ring and M is an R-module, an *R-regular sequence* on M is a sequence of elements of R, $r_1, r_2, \cdots r_n$ such that $M/\langle r_1, \cdots, r_n\rangle M$ is non-zero and multiplication by r_i is injective on $M/\langle r_1, \cdots, r_{i-1}\rangle M$.

Proposition 3.3. *For a module over a local ring all maximal regular sequences have the same length. That length is the depth of the module.*

Proof. The proof is omitted here, but can be found in [Sta19, Tag 00LW] and [Sta19, Tag 090R]. □

Definition 3.4. If R is a ring, we say that $\mathfrak{p}$ is a *prime ideal* in R if

1. $\mathfrak{p}$ is a proper ideal of R.
2. For any two elements $a, b \in R$ such that $ab \in \mathfrak{p}$ we must have that either a is in $\mathfrak{p}$ or b is.

The *height* of $\mathfrak{p}$ is the length of the longest chain of prime ideals contained in $\mathfrak{p}$:

$$\mathfrak{p}_0 \subset \mathfrak{p}_1 \subset \cdots \subset \mathfrak{p}_n = \mathfrak{p}.$$

Definition 3.5. Let R be a commutative ring, M an R-module, and $\mathfrak{p} \subset R$ a prime ideal of R. Then we define the *loclization of M at $\mathfrak{p}$*, written as $M_{\mathfrak{p}}$, to be the set of formal fractions $\frac{m}{r}$ with m in M and r in $R \setminus \mathfrak{p}$. Two such fractions $\frac{m}{r}$ and $\frac{n}{s}$ are considered the same if there is an element $t \in R \setminus \mathfrak{p}$ such that $t(sm - rn) = 0$.

Note that if we localize R at $\mathfrak{p}$ we get an R-algebra with the multiplication $\frac{r}{s} \cdot \frac{r'}{s'} = \frac{rr'}{ss'}$, as you would expect. In fact it's enough to understand $R_{\mathfrak{p}}$ because $M_{\mathfrak{p}} = R_{\mathfrak{p}} \otimes_R M$.

[8] The extension group is defined in definition A.9 on page 32.

3.2 Unramified and separable extensions

Definition 3.6. Let A and B be two local commutative rings with maximal ideal $\mathfrak{m}$ and $\mathfrak{n}$ respectively, and let $A \hookrightarrow B$ be an extension of rings. We say that the extension is *unramified* if the following conditions hold:

- B is a finitely generated A-module;
- $A/\mathfrak{n} \hookrightarrow B/\mathfrak{m}$ is a separable field extension;
- $\mathfrak{m}B = \mathfrak{n}$.

If the two first conditions are met, and there is a positive integer e such that $\mathfrak{m}B = \mathfrak{n}^e B$, we say the extension has *ramification index* e when e is the smallest such number. Note that being unramified is then equivalent to having ramification index 1.

Definition 3.7. Let $A \hookrightarrow B$ be an extension of commutative rings. We say the extension is *separable* if the sequence

$$0 \longrightarrow \operatorname{Ker}\mu \longrightarrow B \otimes_A B \xrightarrow{\mu} B \longrightarrow 0$$

splits as a short exact sequence of $B \otimes_A B$-modules. Here μ is the multiplication map given by $\mu(b \otimes b') = bb'$, and the $B \otimes_A B$-module structure on B is given by $b \otimes b' \cdot b'' = bb'b''$.

In order to show that unramified extensions are separable we must first take a small detour.

Definition 3.8. Let $A \to B$ be an extension of rings. We then define the *derivation module* $\Omega_{B|A}$ as the B-module with formal generators db for all $b \in B$ and with the following relations:

$$A\text{-linearity: } d(ab + a'b') = adb + a'db' \text{ for all } a, a' \in A \text{ and } b, b' \in B.$$

$$\text{Leibniz rule: } d(bc) = bdc + cdb \text{ for all } b, c \in B.$$

Note that for any polynomial f we have that $df(b) = f'(b)db$ where f' is the formal derivative of f. Now we will show how the derivation module makes a link between unramified extensions and the splitting of our sequence.

Proposition 3.9. *Let $A \to B$ be an unramified extension of local rings. Then $\Omega_{B|A}$ is 0.*

Proof. Keeping with the notation above we let $\mathfrak{m}$ be the maximal ideal of A and $\mathfrak{n}$ the maximal ideal of B. Furthermore let l denote $B/\mathfrak{n}$ and k denote $A/\mathfrak{m}$. Then we claim there is an exact sequence

$$\mathfrak{n}/\mathfrak{n}^2 \xrightarrow{\alpha} \Omega_{B|A} \otimes_B B/\mathfrak{n} \longrightarrow \Omega_{l|A} \longrightarrow 0$$

where $\alpha(\overline{n}) = dn \otimes 1$ for any n in $\mathfrak{n}$. Let's first show that α is well defined. Let $n_1 \cdot n_2$ be in $\mathfrak{n}^2$. Then we need to show that $\alpha(\overline{n_1 \cdot n_2})$ is 0.

$$\begin{gathered}\alpha(\overline{n_1 \cdot n_2}) = d(n_1 \cdot n_2) \otimes 1 = \\ n_1 dn_2 \otimes 1 + n_2 dn_1 \otimes 1 = \\ dn_2 \otimes (n_1 \cdot 1) + dn_1 \otimes (n_2 \cdot 1)\end{gathered}$$

Since $l = B/\mathfrak{n}$ we have that $n_1 \cdot 1$ and $n_2 \cdot 1$ is 0 in l, thus the right hand side is 0, and α is well defined.

The map $\Omega_{B|A} \otimes_B B/\mathfrak{n} \to \Omega_{l|A}$ is just the natural projection sending $db \otimes 1$ to $d\bar{b}$, where $\bar{b}$ is the projection of b onto l. We want to show that this is the cokernel of α. The kernel of $\Omega_{B|A} \otimes_B B/\mathfrak{n} \to \Omega_{l|A}$ is generated by $dn \otimes 1$ for $n \in \mathfrak{n}$, but this is exactly the image of α, thus the sequence is exact.

Nextly we want to show that $\Omega_{l|A} = 0$. Since $\mathfrak{m} \subseteq \mathfrak{n}$ and l is annihilated by $\mathfrak{n}$ we have that $\Omega_{l|A} = \Omega_{l|k}$. Let x be an element of l, and let p be its irreducible polynomial over k. Now we want to use the fact that $k \subset l$ is a separable field extension. Remember that $k \subset l$ being separable means that the formal derivative of p is non-zero. Now we have that

$$0 = d(p(x)) = p'(x)dx.$$

Since p' is a non-zero polynomial of lower degree than p, and p is the smallest polynomial with root x, we must have that $p'(x)$ is non-zero. This implies that $dx = 0$, and since this holds for all x it must be that $\Omega_{l|k} = 0$.

Since $\Omega_{l|k} = 0$ we have that α is surjective. We will now use that since $A \to B$ is unramified $\mathfrak{m}B = \mathfrak{n}$. More specifically the map $\beta : \mathfrak{m}/\mathfrak{m}^2 \otimes_A B \to \mathfrak{n}/\mathfrak{n}^2$ is surjective. Since both α and β are surjective we have that $\alpha\beta$ is also surjective, but

$$\alpha\beta(\overline{m} \otimes b) = \alpha(\overline{mb}) = d(mb) \otimes 1 = mdb \otimes 1 = db \otimes m \cdot 1 = 0$$

for all $m \in \mathfrak{m}$ and $b \in B$. Thus the only conclusion is that $\Omega_{B|A} \otimes_B l = 0$.

Since $\Omega_{B|A} \otimes_B l = \Omega_{B|A} \otimes_B B/\mathfrak{m} = \Omega_{B|A}/\mathfrak{m}\Omega_{B|A}$ it follows from Nakayama's lemma [Sta19, Tag 07RC] that $\Omega_{B|A} = 0$. $\square$

Theorem 3.10. *Let $A \to B$ be an unramified extension of local rings. Then the extension is separable. That is, the sequence*

$$0 \longrightarrow I \longrightarrow B \otimes_A B \xrightarrow{\mu} B \longrightarrow 0$$

splits as a short exact sequence of $B \otimes_A B$-modules.

Proof. Firstly note that I is generated by elements on the form $b \otimes 1 - 1 \otimes b$, and that since B is finitely generated as an A-module, I is finitely generated.

Next we want to show that $I/I^2 = \Omega_{B|A}$, which we have already seen equals 0. Since $\Omega_{B|A}$ is a B-module we need a B-module structure on I/I^2. Since $(b \otimes 1)i - (1 \otimes b)i$ is in I^2 for $i \in I$, we have that $(b \otimes 1)i = (1 \otimes b)i \mod I^2$. Then I/I^2 is generated by $(c \otimes 1 - 1 \otimes c)$ as a B-module with the B-module action given by $b \cdot i := (1 \otimes b)i$.

Now to see that $I/I^2 = \Omega_{B|A}$ we will show that the relations on $(b \otimes 1 - 1 \otimes b)$ in I/I^2 are exactly the same as those for db in $\Omega_{B|A}$, thus that $(db \mapsto (b \otimes 1 - 1 \otimes b))$ is an isomorphism.

A-linearity follows from the fact that we are tensoring over A, that is

$$(ab \otimes 1 - 1 \otimes ab) = (b \otimes a - 1 \otimes ab) =$$
$$(1 \otimes a)(b \otimes 1 - 1 \otimes b) = a \cdot (b \otimes 1 - 1 \otimes b).$$

The Leibniz rule $dbc - bdc - cdb = 0$ follows from a similar computation:

$$\begin{aligned}
&(b \otimes 1 - 1 \otimes b)(c \otimes 1 - 1 \otimes c) \\
&= bc \otimes 1 - b \otimes c - c \otimes b + 1 \otimes bc \\
&= (bc \otimes 1 - 1 \otimes bc) - (c \otimes b - 1 \otimes bc) - (b \otimes c - 1 \otimes bc) \\
&= (bc \otimes 1 - 1 \otimes bc) - (1 \otimes b)(c \otimes 1 - 1 \otimes c) - (1 \otimes c)(b \otimes 1 - 1 \otimes b).
\end{aligned}$$

Thus we see that $(bc \otimes 1 - 1 \otimes bc) - b \cdot (c \otimes 1 - 1 \otimes c) - c \cdot (b \otimes 1 - 1 \otimes b)$ generates I^2.

Now that we have shown that $I/I^2 = \Omega_{B|A} = 0$, or rather that $I = I^2$. Nakayama's lemma gives that there is an $i \in I$ such that $ji = j$ for all $j \in I$. Then we can define the splitting map $B \otimes_A B \to I$ by $b \otimes b' \mapsto b \otimes b' \cdot i$. Thus the sequence

$$0 \longrightarrow I \longrightarrow B \otimes_A B \xrightarrow{\mu} B \longrightarrow 0$$

splits. □

Theorem 3.11. *Let B be a local k-algebra domain, and G a finite subgroup of $\mathrm{Aut}_k(B)$ with order relatively prime to the characteristic of k, and denote by A the fixed ring B^G. If the short exact sequence*

$$0 \longrightarrow I \longrightarrow B \otimes_A B \xrightarrow{\mu} B \longrightarrow 0$$

splits, then the map

$$B\#G \xrightarrow{\gamma} \mathrm{End}_A(B)$$

$$b \cdot g \longmapsto (a \mapsto b \cdot a^g)$$

is an isomorphism of A-modules, and isomorphism of rings.

Proof. First in order to see that the map is injective, assume $b \cdot g$ and $b' \cdot g'$ map to the same endomorphism. Then $b \cdot t^g = b' \cdot t^{g'}$ for all $t \in B$. Choosing $t = 1$ we see that $b = b'$. Then since B is a domain this means that $t^g = t^{g'}$ for all t, that is to say $g = g'$.

To see that the map is surjective we will construct a splitting. The splitting will be constructed using the following diagram:

$$\begin{tikzcd}
B\#G \arrow[r, "\gamma"] & \mathrm{End}_A(B) \arrow[d, "f \mapsto f \otimes \rho"] \\
B \otimes_A B\#G \arrow[u, "\tilde{\mu}"] & \mathrm{Hom}_B(B \otimes_A B, B \otimes_A B\#G) \arrow[l, "ev_\epsilon"']
\end{tikzcd}$$

where ρ is the modified Reinolds-operator

$$\rho(b) = \sum_{g \in G} b^g \cdot g.$$

Since we assumed the extension is unramified we have that

$$\begin{tikzcd}
0 \arrow[r] & I \arrow[r, "\iota"] & B \otimes_A B \arrow[r, "\mu"] \arrow[l, bend left, dotted, "\psi"] & B \arrow[r] & 0
\end{tikzcd}$$

splits. As indicated we denote the left splitting by ψ. Then let $\epsilon = 1 \otimes 1 - \iota\psi(1 \otimes 1)$ in $B \otimes_A B$. Then $\mu(\epsilon) = 1$, and $(b \otimes 1 - 1 \otimes b)\epsilon = 0$. Then we define the evaluation map at ϵ by

$$\begin{aligned} ev_\epsilon : \qquad \mathrm{Hom}_B(B \otimes_A B, B \otimes_A B\#G) &\longrightarrow B \otimes_A B\#G \\ f &\longmapsto f(\epsilon). \end{aligned}$$

Lastly $\tilde{\mu} : B \otimes_A B\#G \to B\#G$ is simply the map $b \otimes c \cdot g \mapsto bc \cdot g$. We have now defined all the maps in the square.

Now we want to show that the composition of the three bottom maps forms a splitting. That is for any $f \in \mathrm{End}_A(B)$ we have that $\gamma(\tilde{\mu}(ev_\epsilon(f \otimes \rho))) = f$.

Write $\epsilon = \sum\limits_i x_i \otimes y_i$. Then we claim that

$$\sum_i x_i y_i^g = \begin{cases} 1 & g = 1_G \\ 0 & \text{otherwise} \end{cases}.$$

We know that

$$(b \otimes 1) \sum_i x_i \otimes y_i = (1 \otimes b) \sum_i x_i \otimes y_i$$

holds for all b. Then applying the map $1 \otimes g$ on both sides we get

$$\sum_i b x_i \otimes y_i^g = \sum_i x_i \otimes b^g y_i^g.$$

Then by applying μ we get

$$b \sum_i x_i y_i^g = b^g \sum_i x_i y_i^g.$$

Then since B is a domain we get that either $b = b^g$ or $\sum_i x_i y_i^g = 0$. If we assume that $\sum_i x_i y_i^g \neq 0$ then we must have that $b = b^g$ for all $b \in B$ and we then get that $g = 1_G$. Then since

$$\sum_i x_i y_i = \mu(\epsilon) = 1$$

we see that the claim holds. We can now calculate $\gamma(\tilde{\mu}(ev_\epsilon(f \otimes \rho)))$:

$$
\begin{aligned}
&\gamma\left[\tilde{\mu}\left[(f \otimes \rho)(\epsilon)\right]\right](b) = \\
&\gamma\left[\tilde{\mu}\left[(f \otimes \rho)\left(\sum_i x_i \otimes y_i\right)\right]\right](b) = \\
&\gamma\left[\tilde{\mu}\left[\sum_i f(x_i) \otimes \rho(y_i)\right]\right](b) = \\
&\gamma\left[\sum_i f(x_i) \sum_g y_i^g \cdot g\right](b) = \\
&\gamma\left[\sum_g \sum_i f(x_i) y_i^g \cdot g\right](b) = \\
&\sum_g \left(\sum_i f(x_i) y_i^g \cdot b^g\right) \overset{*}{=} \\
&f\left(\sum_g \left(\sum_i x_i y_i^g\right) \cdot b^g\right) \overset{**}{=} \\
&f(b).
\end{aligned}
$$

In (*) we use the fact that f is A-linear and that $\sum_g y_i^g b^g$ is in A. In (**) we use the claim from above that

$$
\sum_i x_i y_i^g = \begin{cases} 1 & g = 1_G \\ 0 & \text{otherwise} \end{cases}.
$$

This means that γ is an epimorphism and then also an isomorphism. □

3.3 Unramified extensions in codimension 1

Proposition 3.12. *Let S be the complex power series ring in n variables, G a finite subgroup of $GL_n(\mathbb{C})$ acting on S by linear change of variables, and $R = S^G$ the fixed ring of S. Then S is a finitely generated R-module of rank $|G|$.*

Proof. The proof is omitted here, but can be found in [LW12, Proposition 5.4]. □

Definition 3.13. Let S be a commutative ring, G a subgroup of $\mathrm{Aut}(S)$, and $\mathfrak{p}$ a prime ideal. The *inertia group of* $\mathfrak{p}$ is defined as

$$
T(\mathfrak{p}) = \{g \in G \mid s^g - s \in \mathfrak{p} \ \ \forall s \in S\}.
$$

Definition 3.14. Let S be a commutative ring, G a subgroup of $\mathrm{Aut}(S)$, and $\mathfrak{p}$ a prime ideal. The *decomposition group of* $\mathfrak{p}$ is defined as

$$D(\mathfrak{p}) = \{g \in G \mid g(\mathfrak{p}) = \mathfrak{p}\}.$$

Lemma 3.15. *Let S be the complex power series ring in n variables, let G be a finite subgroup of $GL_n(\mathbb{C})$ acting on S, and let $\mathfrak{p}$ be a height one prime ideal of S. Denote by R the fixed ring S^G and let $\mathfrak{q} = R \cap \mathfrak{p}$. Let e be the ramification index of $R_\mathfrak{q} \subset S_\mathfrak{p}$, and let f be the degree of the field extension $R_\mathfrak{q}/\mathfrak{q} \subset S_\mathfrak{p}/\mathfrak{p}$. Then the order of the decomposition group $|D(\mathfrak{p})|$ is ef.*

Proof. Let $\{\mathfrak{p}_i\}$ be the set of prime ideals in S lying over $\mathfrak{q}$. The group G acts on the set by permuting the ideals. We will show that this group action is transitive. Assume for the sake of contradiction that it is not and that there is a prime ideal $\mathfrak{p}_t$ such that $g(\mathfrak{p}) \neq \mathfrak{p}_t$ for all $g \in G$. Then by the prime avoidance lemma [Eis95, Lemma 3.2] we have that there is an $a \in \mathfrak{p}_t$ such that $a^g \notin \mathfrak{p}$ for any $g \in G$. Now consider $x = \prod_{g \in G} a^g$. Clearly x is in R and thus in $\mathfrak{q}$, but since none of the factors of x are in $\mathfrak{p}$ we must have that $x \notin \mathfrak{p}$. This is a contradiction, thus the action of G is transitive on $\{\mathfrak{p}_i\}$. Since the G is a finite group we have that $\{\mathfrak{p}_i\}$ is a finite set, say $r := |\{\mathfrak{p}_i\}|$.

Now let $\mathfrak{p}_t$ be a prime ideal lying over $\mathfrak{q}$ and let g be an element of G such that $g(\mathfrak{p}) = \mathfrak{p}_t$. Consider $R_\mathfrak{q}$ and $S_\mathfrak{p}$ as subsets of their respective fields of fractions. Then we have $g(S_\mathfrak{p}) = S_{\mathfrak{p}_t}$ and $g(\mathfrak{p}^e S_\mathfrak{p}) = \mathfrak{p}_t^e S_{\mathfrak{p}_t}$. This means that the ramification index, e, and degree of field extension, f, only rely on $\mathfrak{q}$ and not which prime ideal lying over $\mathfrak{q}$ we choose.

Now we want show that there is an isomorphism of rings $S/\mathfrak{q}S \cong \prod_{i=1}^r S/\mathfrak{p}_i^e S$, given by $(x + \mathfrak{q}S \mapsto (x + \mathfrak{p}_i^e S)_{i=1}^r)$. The map is clearly injective since $\mathfrak{q}S \subseteq \mathfrak{p}_i^e S$. The fact that it's surjective is a consequence of the approximation lemma [Ser79, p.12], which states that for any finite set of prime ideals $\{\mathfrak{p}_i\}$, integers $\{e_i\}$, and elements of your ring $\{x_i\}$ there is an element x such that $x - x_i \in \mathfrak{p}_i^{e_i}$.

Taking the field of fractions of $S/\mathfrak{q}S$ and $\prod_{i=1}^r S/\mathfrak{p}_i^e S$, and comparing $A_\mathfrak{q}/\mathfrak{q}A_\mathfrak{q}$-dimension, we see that $|G| = efr$.

The orbit-stabilizer theorem states that the size of an orbit is the same as the index of the stabilizer group. Note that $D(\mathfrak{p})$ is exactly the stabilizer of $\mathfrak{p}$. Then since G acts transitively we have that $|\{\mathfrak{p}_i\}| = |G|/|D(\mathfrak{p})|$. In particular this set is finite, say $r := |\{\mathfrak{p}_i\}|$, and $|G| = |D(\mathfrak{p})| \cdot r$. Since the order of G is $|G| = efr$?? [Ser79, Proposition 19] it follows that $|D(\mathfrak{p})| = ef$.

I still don't fully understand this, maybe need to just cite or see if we can figure it out. $\square$

Lemma 3.16. *Let S be the complex power series ring in n variables, let G be a finite subgroup of $GL_n(\mathbb{C})$ acting on S, and let $\mathfrak{p}$ be a height one prime ideal of S. Denote by R the fixed ring S^G and let $\mathfrak{q} = R \cap \mathfrak{p}$. Then the ramification index of $R_\mathfrak{q} \subset S_\mathfrak{p}$, denoted by e, divides the order of the inertia group $|T(\mathfrak{p})|$.*

Proof. Since $\mathfrak{p}$ is invariant under the action of $D(\mathfrak{p})$, also its complement will be. This means the group action on $S_\mathfrak{p}$ given by $g(\frac{s}{t}) = \frac{s^g}{t^g}$ is well defined whenever g is in $D(\mathfrak{p})$. This also gives a well defined group action on $S_\mathfrak{p}/\mathfrak{p}S_\mathfrak{p}$. Since this action fixes $R_\mathfrak{q}/\mathfrak{q}R_\mathfrak{q}$ we get a map from $D(\mathfrak{p})$ to the Galois group of the field extension $R_\mathfrak{q}/\mathfrak{q}R_\mathfrak{q} \subset S_\mathfrak{p}/\mathfrak{p}S_\mathfrak{p}$ [Cox12, Theorem 7.1.1]. If we can show that the kernel of this map is $T(\mathfrak{p})$, then we get that $|D(\mathfrak{p})/T(\mathfrak{p})|$ divides the order of the Galois group which equals the order of the field extension [Cox12, Theorem 7.1.5]. Then since $|D(\mathfrak{p})| = ef$ we get that $ef\Big||T(\mathfrak{p})|f$, and that e divides $T(\mathfrak{p})$.

First we see that $T(\mathfrak{p})$ is contained in the kernel. Since $\frac{s}{t} = \frac{\prod_{g\neq 1} t^g s}{\prod_g t^g}$ where g ranges over the elements of $T(\mathfrak{p})$, we have that all fractions in $S_\mathfrak{p}$ can be written with a denominator invariant under $T(\mathfrak{p})$. Then since $s^g - s \in \mathfrak{p}$ whenever s is in S and g is in $T(\mathfrak{p})$ we get that $\left(\frac{s}{t}\right)^g - \frac{s}{t} \in \mathfrak{p}S_\mathfrak{p}$ for all $\frac{s}{t} \in S_\mathfrak{p}$ and $g \in T(\mathfrak{p})$. Thus the action of $T(\mathfrak{p})$ on $S_\mathfrak{p}/\mathfrak{p}S_\mathfrak{p}$ is trivial.

To see the converse assume $g \in D(\mathfrak{p})$ acts trivially on $S_\mathfrak{p}/\mathfrak{p}S_\mathfrak{p}$. Then in particular we have that $\frac{s^g}{1} - \frac{s}{1} \in \mathfrak{p}S_\mathfrak{p}$ for all $s \in S$. This means that $s^g - s \in \mathfrak{p}$, which is exactly the condition for g to be in $T(\mathfrak{p})$.

This shows that $T(\mathfrak{p})$ is the kernel of the map, and thus that the ramification index divides the order of $T(\mathfrak{p})$. $\square$

Theorem 3.17. *Let S be the complex power series ring in n variables, let G be a finite subgroup of $GL_n(\mathbb{C})$ acting on S, and let $\mathfrak{p}$ be a height one prime ideal of S. Denote by R the fixed ring S^G and let $\mathfrak{q} = R\cap\mathfrak{p}$. Then the ramification index of $R_\mathfrak{q} \subset S_\mathfrak{p}$ equals the order of the inertia group $|T(\mathfrak{p})|$.*

Proof. We write $\mathfrak{m}$ for the maximal ideal of S. Since $\mathfrak{p}$ is height one and S is a UFD we have that $\mathfrak{p} = \langle z \rangle$ for some $z \in \mathfrak{m}$. We define an inner product on $V := \mathfrak{m}/\mathfrak{m}^2$ by

$$\langle x, y \rangle_G = \frac{1}{|G|} \sum_{g\in G} \langle x^g, y^g \rangle$$

where $\langle -, - \rangle$ is the standard inner product. Note that the action of G is orthogonal with respect to this inner product.

We write $\overline{z}$ for the representative for z in V. Since the action of G preserves degrees and that $\overline{z}^g - \overline{z} \in \langle \overline{z} \rangle$ we must have that $\overline{z}^g = a_g \cdot \overline{z}$ for some scalar $a_g \in \mathbb{C}$. Further since $x^g = x + \lambda_{g,x}\overline{z}$ for all $x \in V$ and $g \in T(\mathfrak{p})$, and g is an orthogonal operator we have that g fixes the $\langle -, - \rangle_G$-orthogonal complement to $\overline{z}$. This means we can choose a basis such that all elements of $T(\mathfrak{p})$ are on the form:

$$\begin{pmatrix} 1 & & & & \\ & 1 & & & \\ & & \ddots & & \\ & & & 1 & \\ & & & & a_g \end{pmatrix}$$

This means $T(\mathfrak{p})$ is isomorphic to $\{a_g\}_{g \in T(\mathfrak{p})} \leq \mathbb{C}^*$ which is a subgroup of $\mathbb{C}^*$. Since all finite subgroups of $\mathbb{C}^*$ are cyclic this implies that $T(\mathfrak{p})$ is cyclic. Let s be the order of $T(\mathfrak{p})$. Then

$$\sigma := \begin{pmatrix} 1 & & & & \\ & 1 & & & \\ & & \ddots & & \\ & & & 1 & \\ & & & & \exp(2\pi i/s) \end{pmatrix}$$

generates $T(\mathfrak{p})$. Consider the ring $S^{T(\mathfrak{p})}$. We have that $R \subset S^{T(\mathfrak{p})}$, and $\mathfrak{q} \subset S^{T(\mathfrak{p})} \cap \mathfrak{p}$. Then we have that $R_\mathfrak{q} \subset S^{T(\mathfrak{p})}_{S^{T(\mathfrak{p})} \cap \mathfrak{p}}$, and the ramification index of $R_\mathfrak{q} \subset S_\mathfrak{p}$ is the product of the ramification index of $R_\mathfrak{q} \subset S^{T(\mathfrak{p})}_{S^{T(\mathfrak{p})} \cap \mathfrak{p}}$ and of $S^{T(\mathfrak{p})}_{S^{T(\mathfrak{p})} \cap \mathfrak{p}} \subset S_\mathfrak{p}$. Then since $(S^{T(\mathfrak{p})} \cap \mathfrak{p})S = z^s S = \langle z \rangle^s S$, we have that the ramification index of $R_\mathfrak{q} \subset S_\mathfrak{p}$ is divisible by the order of $T(\mathfrak{p})$. Since we have already seen that the ramification index divides $|T(\mathfrak{p})|$ this implies that $e = |T(\mathfrak{p})|$. □

Theorem 3.18. *$R_\mathfrak{q} \subset S_\mathfrak{p}$ is unramified for all height one primes $\mathfrak{p}$ if and only if G contains no pseudoreflections, that is a non-trivial element that fixes a codimension 1 subspace.*

Proof. Firstly, since we are working in characteristic 0, all field extensions are separable. Thus $R_\mathfrak{q}/\mathfrak{q} \subset S_\mathfrak{p}/\mathfrak{p}$ is separable. Since S is a rank $|G|$ R-module, $S_\mathfrak{p}$ will be a finitely generated $R_\mathfrak{q}$-module.

We know that elements of $T(\mathfrak{p})$ can be written on the form

$$\begin{pmatrix} 1 & & & & \\ & 1 & & & \\ & & \ddots & & \\ & & & 1 & \\ & & & & a_g \end{pmatrix}.$$

Since G does not contain any pseudoreflections we must have that $a_g = 1$ and therefore $T(\mathfrak{p})$ is trivial and $|T(\mathfrak{p})| = 1$. That means that the ramification index of $R_\mathfrak{q} \subset S_\mathfrak{p}$ is 1, and the extension is unramified. □

Note that no finite subgroup of $SL_n(\mathbb{C})$ contains pseudoreflections. In particular $R_\mathfrak{q} \subset S_\mathfrak{p}$ is unramified when G is a finite subgroup of $SL_2(\mathbb{C})$.

Now the last piece of the puzzle is to show that this implies that

$$S\#G \xrightarrow{\gamma} \mathrm{End}_R(S)$$

is an isomorphism when $S = \mathbb{C}[\![x, y]\!]$, and G is a finite subgroup of $SL_2(\mathbb{C})$.

Lemma 3.19. *Let S be a local ring and let M and N be S-modules that satisfies the Serre's criterion (S_2) and (S_1) respectively. In other words M and N satisfies* $\operatorname{depth} M_\mathfrak{p} \geq \min\{2, \operatorname{height}(\mathfrak{p})\}$ *and* $\operatorname{depth} N_\mathfrak{p} \geq \min\{1, \operatorname{height}(\mathfrak{p})\}$ *for all prime ideals $\mathfrak{p} \subset S$. Let $f : M \to N$ be a monomorphism such that $f_\mathfrak{p} : M_\mathfrak{p} \to N_\mathfrak{p}$ is an epimorphism for all height one prime ideals. Then f is an isomorphism.*

Proof. Assume f is not an epimorphism. Then f has a cokernel $C \neq 0$, and we have a short exact sequence

$$0 \longrightarrow M \xrightarrow{f} N \longrightarrow C \longrightarrow 0$$

Now we choose $\mathfrak{p}$ to be the annihilator of a submodule $\langle \tilde{c} \rangle$ for some non-zero $\tilde{c} \in C$. We want to show that $\mathfrak{p}$ has height at least 2. If $\mathfrak{p}$ had height one then since $f_\mathfrak{p}$ is epi we would have that $C_\mathfrak{p} = 0$. This is equivalent to saying that for every $c \in C$ there is some element $s \notin \mathfrak{p}$ such that $sc = 0$. This is impossible since $\mathfrak{p}$ is the annihilator of $\langle \tilde{c} \rangle$, thus if $s\tilde{c} = 0$ then s is in $\mathfrak{p}$. The same argument works for a height 0 prime ideal since they are contained in height one prime ideals.

Thus $\mathfrak{p}$ has height at least 2 and depth $M_{\mathfrak{p}} \geq 2$, depth $N_{\mathfrak{p}} \geq 1$. Now we want to show that $C_{\mathfrak{p}}$ has depth 0, using regular sequences. Recall that the depth of a module is the length of the longest regular sequence. Since $\mathfrak{p}$ annihilates some $c \in C$ multiplication by $p \in \mathfrak{p}$ cannot be injective on $C_{\mathfrak{p}}$, because $\frac{c}{1}$ will be in the kernel. Multiplication by any element not in $\mathfrak{p}$ will be epimorphic since $s \cdot \frac{c}{s \cdot t} = \frac{c}{t}$, thus no regular sequence exist on $C_{\mathfrak{p}}$.

Now we consider the short exact sequence

$$0 \longrightarrow M_{\mathfrak{p}} \xrightarrow{f_{\mathfrak{p}}} N_{\mathfrak{p}} \longrightarrow C_{\mathfrak{p}} \longrightarrow 0$$

and take its long exact sequence of $\operatorname{Ext}_S(k, -)$ where k is the residual field of S.

$$\cdots \longrightarrow \operatorname{Hom}_S(k, N_{\mathfrak{p}}) \longrightarrow \operatorname{Hom}_S(k, C_{\mathfrak{p}}) \longrightarrow \operatorname{Ext}^1_S(k, M_{\mathfrak{p}}) \longrightarrow \cdots$$

Since depth $N_{\mathfrak{p}} \geq 1$ and depth $M_{\mathfrak{p}} \geq 2$ we have that $\operatorname{Hom}_S(k, N_{\mathfrak{p}})$ and $\operatorname{Ext}^1_S(k, M_{\mathfrak{p}})$ is 0. Then by exactness we get that $\operatorname{Hom}_S(k, C_{\mathfrak{p}}) = 0$. This contradicts the fact that depth $C_{\mathfrak{p}} = 0$, and thus our assumption that $C \neq 0$ is wrong. Therefore f is an epimorphism and thus also an isomorphism. $\square$

Theorem 3.20. *Let $S = \mathbb{C}[\![x, y]\!]$ be the complex power series ring in two variables, let G be a finite subgroup of $SL_2(\mathbb{C})$ acting on S, and let $R = S^G$ be the fixed ring. Then the map*

$$S\#G \xrightarrow{\gamma} \operatorname{End}_R(S)$$

is an isomorphism of rings.

Proof. Let $\mathfrak{q}$ be a height one prime ideal of R and let $\mathfrak{p}$ be a prime ideal in S lying over $\mathfrak{q}$. Then since G is in $SL_2(\mathbb{C})$ it can't contain any pseudoreflections, thus by theorem 3.18 the extension $R_{\mathfrak{q}} \subset S_{\mathfrak{p}}$ is unramified. Then by theorem 3.10 we have that

$$0 \longrightarrow I \longrightarrow S_{\mathfrak{p}} \otimes_{R_{\mathfrak{q}}} S_{\mathfrak{p}} \xrightarrow{\mu} S_{\mathfrak{p}} \longrightarrow 0$$

is a split exact sequence. Since $S_{\mathfrak{q}} := R_{\mathfrak{q}} \otimes_R S$ lies between $R_{\mathfrak{q}}$ and $S_{\mathfrak{p}}$ the proposition [AB59, Proposition A.2] gives that

$$0 \longrightarrow I \longrightarrow S_{\mathfrak{q}} \otimes_{R_{\mathfrak{q}}} S_{\mathfrak{q}} \xrightarrow{\mu} S_{\mathfrak{q}} \longrightarrow 0$$

also splits. Then by theorem 3.11 we have that the map

$$S_{\mathfrak{q}}\#G \xrightarrow{\gamma} \mathrm{End}_{R_{\mathfrak{q}}}(S_{\mathfrak{q}}) \tag{1}$$

is an isomorphism of rings. Now we want to show that (1) is the localization of

$$S\#G \xrightarrow{\gamma} \mathrm{End}_R(S).$$

That $S_{\mathfrak{q}}\#G$ is the localization of $S\#G$ is clear to see. What we need to show is that $\mathrm{End}_{R_{\mathfrak{q}}}(S_{\mathfrak{q}})$ is the localization of $\mathrm{End}_R(S)$, that is $R_{\mathfrak{q}} \otimes_R \mathrm{End}_R(S) \cong \mathrm{End}_{R_{\mathfrak{q}}}(S_{\mathfrak{q}})$. We can construct an explicit isomorphism by

$$R_{\mathfrak{q}} \otimes_R \mathrm{End}_R(S) \longrightarrow \mathrm{End}_{R_{\mathfrak{q}}}(S_{\mathfrak{q}})$$

$$\tfrac{1}{r} \otimes \varphi \longmapsto \left(\tfrac{s}{t} \mapsto \tfrac{\varphi(s)}{rt}\right)$$

It should be clear that the map is injective. To see surjectivity let $s_1, s_2, \cdots, s_n$ be generators of S as an R-module, and let ψ be in $\mathrm{End}_{R_{\mathfrak{q}}}(S_{\mathfrak{q}})$. Write $\frac{t_i}{r_i}$ for $\psi(s_i)$, and let φ be the map in $\mathrm{End}_R(S)$ sending s_i to $t_i \prod_{j \neq i} r_j$. Then $\frac{1}{\prod r_i} \otimes \varphi$ is a preimage of ψ, and thus $R_{\mathfrak{q}} \otimes_R \mathrm{End}_R(S) \cong \mathrm{End}_{R_{\mathfrak{q}}}(S_{\mathfrak{q}})$.

This means that for each height one prime of R, the localization of γ is an isomorphism. Then lemma 3.19 reduces the problem of showing that γ is an isomorphism to showing that $S\#G$ satisfies (S_2) and $\mathrm{End}_R(S)$ satisfies (S_1). Since S is a reflexive R-module it follows from [Sta19, Tag 0AVB] that it satisfies (S_2). Since $S\#G$ is just the direct sum of copies of S as an R-module it will also satisfy (S_2). Since $R_{\mathfrak{q}} \otimes_R \mathrm{End}_R(S) \cong \mathrm{End}_{R_{\mathfrak{q}}}(S_{\mathfrak{q}})$, and S satisfies (S_2) it follows from [Sta19, Tag 0AV5] that $\mathrm{End}_R(S)$ satisfies (S_2). Then it also satisfies (S_1). Thus γ is an isomorphism which is what we wanted to show. □

4 Maximal Cohen-Macaulay modules of S^G

Definition 4.1. If R is a ring we define its *Krull dimension* to be the maximum length of a chain of prime ideals in R.

Example 4.2. For example the power series ring $\mathbb{C}[\![x_1, \cdots, x_n]\!]$ has Krull dimension n given by the chain

$$0 \subseteq \langle x_1 \rangle \subseteq \langle x_1, x_2 \rangle \subseteq \cdots \subseteq \langle x_1, \cdots, x_n \rangle.$$

Definition 4.3. If M is a module over a local ring R with Krull dimension d we say that M is *maximal Cohen-Macaulay (MCM)* if the depth of M equals d. If a ring is MCM as a module over itself we say that it's a *Cohen-Macaulay ring (CM)*. Note that the ring must be CM for it to have any MCM modules.

Theorem 4.4. *If G is a finite subgroup of $GL_n(\mathbb{C})$, S is the complex power series ring in n variables and $R = S^G$ is the ring fixed under the action of G, then R is a direct summand of S as R-modules.*

Proof. Consider the map $\pi : S \to R$ given by

$$\pi(s) = \frac{1}{|G|} \sum_{g \in G} s^g.$$

It's clear that the image of π is in R because an action from G will just permute the order of the sum. Further

$$\pi(r) = \frac{1}{|G|} \sum_{g \in G} r^g = \frac{1}{|G|} \sum_{g \in G} r = r,$$

so π splits the inclusion $R \hookrightarrow S$ which shows that R is a direct summand of S. $\square$

Since we are interested in the MCM modules that appear as R-direct summands of S, it is nice to see that R itself is such a module.

Proposition 4.5. *Let R be a local CM ring. If M is an MCM R-module, and N is a direct summand of M then N is also MCM.*

Proof. We write M as $N \oplus X$. Since M is MCM we have that $0 = \operatorname{Ext}^i_R(M) = \operatorname{Ext}^i_R(N) \oplus \operatorname{Ext}^i_R(X)$ for all i less than the Krull dimension of R. This means the depth of N is greater than or equal to the Krull dimension of R. Since the depth of a module cannot exceed the Krull dimension of the ring [Sta19, Tag 0BK4] we have that N is MCM. $\square$

Lemma 4.6. *If S is a commutative local ring with finite depth, G a finite subgroup of $\operatorname{Aut}(S)$, and $R = S^G$ is the fixed ring, then*

$$\operatorname{depth}_S S \leq \operatorname{depth}_R R, \text{ and } \operatorname{depth}_S S = \operatorname{depth}_R S.$$

Proof. The proof will be by induction on $\operatorname{depth}_S S$. For the base case let $\operatorname{depth}_S S = 0$. Since depth is always bigger than or equal to 0 we have that $\operatorname{depth}_S S \leq \operatorname{depth}_R R$. Also since R is a subring of S we have that $\operatorname{depth}_R S \leq \operatorname{depth}_S S$. Thus if $\operatorname{depth}_S S = 0$ we have $\operatorname{depth}_S S = \operatorname{depth}_R S = 0$.

Before we can prove the induction step we first need to show some properties of the map $N : S \to S$ given by $N(s) = \prod_{g \in G} s^g$. Note that for any $s \in S$ we have that $N(s) \in R$. Further if s is a non-unit and not a zero divisor, then $N(s)$ will also be a non-unit and not a zero divisor. Another property of $N(s)$ we will use is that $(S/N(s)S)^G = R/N(s)R$ whenever $N(s)$ is a not a zero divisor. To see this note that since $(N(s)S)^G \subset N(s)S$, we have that

$$(s + N(s)S)^g = s^g + N(s)S,$$

and thus

$$(S/N(s)S)^G = S/\left(N(s)S \cap R\right).$$

Then what's left to show is that $N(s)S \cap R = N(s)R$. Assume $N(s)t$ is in $N(s)S \cap R$. Then $(N(s)t)^g = N(s)t$ for all $g \in G$. Since $N(s)$ is in R we have that $(N(s)t)^g = N(s)t^g$. Now using the fact that $N(s)$ is not a zero divisor we get that $t^g = t$ for all $g \in G$, which means that t is in R, and thus $N(s)t$ is in $N(s)R$.

We are now ready to show the induction step. Assume the statement holds whenever $\operatorname{depth}_S S < n$. Then we want to show that it holds when $\operatorname{depth}_S S = n$. Since the $\operatorname{depth}_S S > 0$ there exists a non-unit which is not a zero divisor $s \in S$. Then $N(s)$ is also a non-unit and not a zero divisor. This means $N(s)$ is the first element of some maximal regular sequence on S. Since all maximal regular sequences on S has the same length this means that

$$\operatorname{depth}_S S/N(S)S = \operatorname{depth}_S S - 1.$$

Further since $N(s)$ is in R the same argument gives that

$$\operatorname{depth}_R S/N(s)S = \operatorname{depth}_R S - 1.$$

Since every element in $N(s)S$ annihilates $S/N(s)S$ we can consider S-regular sequences on $S/N(s)S$ as $S/N(s)$-regular sequences and vice versa. Thus

$$\operatorname{depth}_S S/N(s)S = \operatorname{depth}_{S/N(s)S} S/N(s)S.$$

Using that $(S/N(s)S)^G = R/N(s)R$, the same argument gives that

$$\operatorname{depth}_R S/N(s)S = \operatorname{depth}_{R/N(s)R} S/N(s)S.$$

Now we apply the induction hypothesis. Since $S/N(s)S$ is a local ring with depth less than n we have that

$$\operatorname{depth}_{S/N(s)S} S/N(s)S \leq \operatorname{depth}_{R/N(s)R} R/N(s)R$$

and that
$$\operatorname{depth}_{S/N(s)S} S/N(s)S = \operatorname{depth}_{R/N(s)R} S/N(s)S.$$
This means that
$$\operatorname{depth}_S S - 1 \leq \operatorname{depth}_R R - 1$$
and that
$$\operatorname{depth}_S S - 1 = \operatorname{depth}_R S - 1.$$
Thus we can conclude that $\operatorname{depth}_S S \leq \operatorname{depth}_R R$ and $\operatorname{depth}_S S = \operatorname{depth}_R S$. By induction the statement holds for all n, and thus we have that
$$\operatorname{depth}_S S \leq \operatorname{depth}_R R, \text{ and } \operatorname{depth}_S S = \operatorname{depth}_R S$$
whener $\operatorname{depth}_S S$ is finite. □

Now that we have shown a general relationship between the depths of local rings fixed under group actions we can apply it to our special case of $S = \mathbb{C}[\![x, y]\!]$ and G being in $SL_2(\mathbb{C})$.

Theorem 4.7. *Let S be the complex power series ring in two variables, G a finite subgroup of $SL_2(\mathbb{C})$ acting on S by linear change of variables, and $R = S^G$ the fixed ring. Then R is CM and S is an MCM R-module.*

Proof. Since S is the complex power series ring in two variables we have that $\dim S = \operatorname{depth}_S S = 2$. By lemma 4.6 we have that $\operatorname{depth}_S S \leq \operatorname{depth}_R R$. Since the depth of a module never exceeds the Krull dimension of the ring we have that $\operatorname{depth}_S R \leq \dim S$. Lastly we saw in the first section that R always equals $\mathbb{C}[\![u, v, w]\!]/\langle f \rangle$ for an irreducible polynomial f. Thus we have that $\dim R = 2 = \dim S$. Putting this together we get
$$\dim R = \dim S = \operatorname{depth}_S S \leq \operatorname{depth}_R R \leq \dim_R$$
and thus $\dim R = \operatorname{depth}_R R$, which means R is CM. Lemma 4.6 also gives us that $\operatorname{depth}_S S = \operatorname{depth}_R S$, which means that $\operatorname{depth}_R S = \dim S = \dim R$, so S is MCM. □

Note that proposition 4.5 gives that all the R-direct summands of S are MCM R-modules. This holds true if we let G be any finite subgroup of $GL_n(\mathbb{C})$. The proof of the case where $G \subset GL_n(\mathbb{C})$ goes in much the same way, but we need some algebraic geometry to show that $\dim R = \dim S$. An interesting fact that we will not prove here is that when G is a finite subgroup of $SL_2(\mathbb{C})$ the converse holds.

Theorem 4.8. *Let S be the complex power series ring in two variables, G a finite subgroup of $SL_2(\mathbb{C})$ acting on S by linear change of variables, and $R = S^G$ the fixed ring. Then there is a one-to-one correspondence between the indecomposable projective $S\#G$-modules and the indecomposable MCM R-modules appearing as R-direct summands of S.*

Proof. Let $\mathfrak{m}$ denote the maximal ideal of S. We saw in theorem 2.5 that there's a one-to-one correspondence between the finitely generated projective $S\#G$-modules and the finitely generated $\mathbb{C}G$-modules. In particular we get that $S\#G \cong S \otimes_{\mathbb{C}} S\#G/\mathfrak{m}S\#G$. Since $S\#G/\mathfrak{m}S\#G \cong \mathbb{C}G$ and all indecomposable $\mathbb{C}G$ modules appear as a direct summand of $\mathbb{C}G$ we have that all indecomposable projective $S\#G$-modules appear as a direct summand of $S\#G$. The direct summands of $S\#G$ are exactly the primitive idempotents of $\mathrm{End}_{S\#G}(S\#G) \cong S\#G^{op}$. Since $S\#G$ is isomorphic to $S\#G^{op}$ by the isomorphism $(s \cdot g \mapsto s^{g^{-1}} \cdot g^{-1})$, we have that the indecomposable projective $S\#G$-modules correspond to primitive idempotents of $S\#G$.

In theorem 3.20 we showed that $S\#G \cong \mathrm{End}_R(S)$. Thus the primitive idempotents of $S\#G$ corresponds to the primitive idempotents of $\mathrm{End}_R(S)$ which are exactly the projections onto the indecomposable R-direct summands of S. Thus the indecomposable projective $S\#G$-modules are in one-to-one correspondence with the indecomposable R-direct summands of S. Since we showed in theorem 4.7 that S is MCM it follows from proposition 4.5 that the direct summands of S also are MCM. □

Theorem 4.9. *Let S, G, and R be as defined above. Then all indecomposable MCM R-modules are direct summands of S.*

Proof. The proof is ommited here, but can be found in [LW12, Theorem 6.3] or in Herzog's original paper [Her78]. □

This last theorem concludes the part of the McKay correspondence that we will prove in this thesis. Another interesting part of the correspondence we have not shown is that the Auslander-Reiten quiver of the indecomposable MCM modules appearing as an R-direct summand of S is isomorphic to the Gabriel quiver. If the reader is interested they can read further in [LW12, Chapter 5, Chapter 6, Chapter 13].

Appendices

A Representation theory

Definition A.1. If R is a ring and M is an abelian group, we define a *representation of* R to be a ring-map, φ, from R to $\mathrm{End}(M)$. Then we say that M is a (left) R*-module*, and we write rm with $r \in R$ and $m \in M$ to mean $\varphi(r)(m)$. Similarly we define a right R-module if φ goes from R to $\mathrm{End}(M)^{op}$ and we write mr for $\varphi(r)(m)$.

Definition A.2. If G is a group and V a complex vector space, we define a *representation of* G to be a group-map, ρ, from G to $\mathrm{Aut}_{\mathbb{C}}(V)$. When ρ is inferred we say that V is a representation of G and we write gv to mean $\rho(g)(v)$. Note that representations of G exactly corresponds to representations of the ring $\mathbb{C}G$ of formal linear combinations of elements of G with multiplication given by $\lambda g \cdot \lambda' g' = (\lambda \cdot \lambda')gg'$.

Definition A.3. If R is a ring and M_1 and M_2 are two modules we define their *direct sum*, $M_1 \oplus M_2$ to be the module consisting of all pairs (m_1, m_2) (usually written $m_1 + m_2$), where addition and scalar multiplication is pointwise. If a non-zero module cannot be written as the direct sum of two non-zero modules we call it *indecomposable.*

Definition A.4. A *submodule* is a subset of a module which is also a module. A non-zero module with no non-trivial proper submodules is called *simple.*

Definition A.5. We call a functor *left exact* if for any short exact sequence

$$0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C$$

the image of the sequence under the functor is also exact. For example for any module M the functor $\mathrm{Hom}(M, -)$ is left exact. That is the sequence

$$0 \longrightarrow \mathrm{Hom}(M, A) \xrightarrow{f\circ -} \mathrm{Hom}(M, B) \xrightarrow{g\circ -} C$$

is exact. Dually we call a functor *right exact* if short exact sequences of the form

$$A \longrightarrow B \longrightarrow C \longrightarrow 0$$

is mapped to an exact sequence. A functor that is both left exact and right exact is called *exact*.

Definition A.6. We say that a module, P, is *projective* if for any epimorphism $f : M \twoheadrightarrow N$, and any map $g : P \to N$, there is a map $\varphi : P \to M$ such that $f\varphi = g$. Said another way, the diagram below induces the dotted arrow, making the diagram commute.

$$\begin{tikzcd} & P \arrow[d, "g"] \arrow[dl, dotted, "\varphi"'] \\ M \arrow[r, two heads, "f"'] & N \end{tikzcd}$$

Note that P being projective is equivalent to $\mathrm{Hom}(P, -)$ being right exact (i.e. exact).

Definition A.7. If M is a module and $f : P \to M$ is a homomorphism we say that f is a *projective cover* of M if

- P is projective;
- f is an epimorphism;
- For any homomorphism $g : X \to P$, if $f \circ g$ is an epimorphism then g is an epimorphism.

The last condition for a projective cover is equivalent to the kernel of f being *small*. That is for any submodule X of P, if $X + \mathrm{Ker}\, f = P$ then $X = P$. For a module M the choice of P in the projective cover is unique (but the choice of f might not be). Therefore it is normal to refer to P as *the* projective cover of M.

Definition A.8. If M is a module we say that a *projective resolution* of M is a sequence

$$\cdots \xrightarrow{\partial_2} P_2 \xrightarrow{\partial_1} P_1 \xrightarrow{\partial_0} P_0 \longrightarrow 0$$

such that P_i is projective for all i, the sequence is exact around every P_i for $i \geq 1$, and that $\mathrm{Cok}\, \partial_0 = M$.

We call a projective resolution *minimal* if each map ∂_i as well as the cokernel map $P_0 \to M$ is a projective cover of its image. In a minimal resolution the objects P_i are uniquely determined.

Definition A.9. If M and N are modules then the ith extension group, $\mathrm{Ext}^i(M,N)$, is constructed in the following way.

- Take a projective resolution of M

$$\cdots \xrightarrow{\partial_2} P_2 \xrightarrow{\partial_1} P_1 \xrightarrow{\partial_0} P_0 \longrightarrow 0;$$

- Apply $\mathrm{Hom}(-,N)$

$$0 \longrightarrow \mathrm{Hom}(P_0,N) \xrightarrow{-\circ\partial_0} \mathrm{Hom}(P_1,N) \xrightarrow{-\circ\partial_1} \mathrm{Hom}(P_2,N) \xrightarrow{-\circ\partial_2} \cdots;$$

- Now $\mathrm{Ext}^i(M,N)$ is the homology at position $-i$, that is $\mathrm{Ext}^i(M,N) = \mathrm{Ker}(-\circ\partial_i)/\,\mathrm{Im}(-\circ\partial_{i-1})$.

Note that the definition of $\mathrm{Ext}^i(M,N)$ is independent of choice of projective resolution. An important property of Ext that is used in this thesis is the long exact sequence in Ext-groups.

Theorem A.10. *For any module M and short exact sequence of modules:*

$$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$$

there are long exact sequences:

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{Hom}(M,A) \arrow[r] & \mathrm{Hom}(M,B) \arrow[r] & \mathrm{Hom}(M,C) \arrow[dlll, out=-10, in=170] & \\
\mathrm{Ext}^1(M,A) \arrow[r] & \mathrm{Ext}^1(M,B) \arrow[r] & \mathrm{Ext}^1(M,C) \arrow[r] & \mathrm{Ext}^2(M,A) \arrow[r] & \cdots
\end{tikzcd}$$

and

$$\begin{tikzcd}
0 & \mathrm{Hom}(A,M) \arrow[l] & \mathrm{Hom}(B,M) \arrow[l] & \mathrm{Hom}(C,M) \arrow[l] & \\
\mathrm{Ext}^1(A,M) \arrow[urrr, out=170, in=-10] & \mathrm{Ext}^1(B,M) \arrow[l] & \mathrm{Ext}^1(C,M) \arrow[l] & \mathrm{Ext}^2(A,M) \arrow[l] & \cdots \arrow[l]
\end{tikzcd}$$

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