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# McKay correspondence

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# Overview

— Statement of the McKay correspondance
— History and motivation
— examples and proofs

# McKay correspondance

The theorem involves $G \leq SL_2(\mathbb{C})$, $S = \mathbb{C}[\![x, y]\!]$, and $R = S^G$. It establishes a correspondence between:

- The irreducible representations of $G$;
- The indecomposable projective modules of the skew group algebra $S \# G$;
- The indecomposable projective modules of the endomorphism ring $\mathrm{End}_R(S)$;
- The indecomposable MCM modules of $R$.

## History and motivation

- Kleinian singularities, $\mathbb{C}^2/G$
- Resolution graphs ADE Dynkin diagrams
- McKay connected the geometry to representation theory
- Herzog showed that the MCM $R$-modules are direct summands of $S$
- Auslander showed the relationship between the projective $S\#G$-modules and the MCM $R$-modules

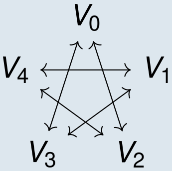
# The McKay quiver

## Definition

The McKay quiver of $G$ has verticies the irreducible representations of $G$, and an arrow from $W$ to $W'$ iff $W'$ appears as a direct summand of $V \otimes_{\mathbb{C}} W$, where $V$ is the cannonical representation.

## Example

$G = \langle g \rangle \cong \mathbb{Z}/5\mathbb{Z}$, $g = \begin{pmatrix} \omega^2 & 0 \\ 0 & \omega^3 \end{pmatrix}$, $\omega = \exp(2\pi i/5)$,



# Non-Dynkin example

## Example

$G = \langle \mu, \rho \rangle \cong S_3$, $\mu = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $\rho = \begin{pmatrix} \exp(2\pi i/3) & 0 \\ 0 & \exp(-2\pi i/3) \end{pmatrix}$

$$
\begin{tikzcd}
V_0 \arrow[r, leftrightarrow] & V \arrow[loop above] \arrow[r, leftrightarrow] & V_\mu
\end{tikzcd}
$$

# The skew group algebra

## Definition

If $S$ is an algebra and $G$ is a subgroup of $\mathrm{Aut}(S)$, then the *skew group algebra* $S\#G$ is the algebra generated by $s \cdot g$ with $s \in S$ and $g \in G$. The multiplication is given by

$$(s \cdot g)(t \cdot h) = st^g \cdot gh$$

## Example

Let $S = \mathbb{C}[\![x, y]\!]$, and $G = \left\{ e := \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, g := \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \right\}$.
Then $x \cdot e + y \cdot g$ and $(x - y) \cdot g$ are in $S\#G$, and their product is

$$(x \cdot e + y \cdot g)((x - y) \cdot g) = x(x - y) \cdot g - y(x + y) \cdot e$$

## Gabriel quiver

### Definition

The *Gabriel quiver* of an algebra, $A$ (where projective covers exist) has verticies the simple $A$-modules. If $M$ and $N$ are simple $A$-modules with minimal projective resolutions:

$$\cdots \to P_1^M \to P_0^M \to 0 \qquad \text{and} \qquad \cdots \to P_1^N \to P_0^N \to 0.$$

Then there is an arrow from $M$ to $N$ iff $P_0^N$ is a direct summand of $P_1^M$.

# Example of Gabriel quiver

## Example

$$S = \mathbb{C}[\![x, y]\!],\ G = \langle g \rangle,\ g = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},\ \mathfrak{m} = \langle x, y \rangle_{S\#G}$$

$$S\#G \cong S \otimes_{\mathbb{C}} \langle I + g \rangle \oplus S \otimes_{\mathbb{C}} \langle I - g \rangle$$

$$\begin{tikzcd} \langle I + g \rangle \arrow[loop left] \arrow[r, leftrightarrow] & \langle I - g \rangle \arrow[loop right] \end{tikzcd}$$

## The indecomposable projective $S\#G$-modules

### Theorem

*There's a correspondance between the irreducible G representations and the indecomposable projective $S\#G$-modules given by*

$$\left\{\begin{matrix}\textit{indecomposable projective} \\ S\#G\textit{-modules}\end{matrix}\right\} \longleftrightarrow \left\{\begin{matrix}\textit{irreducible} \\ \mathbb{C}G\textit{ representations}\end{matrix}\right\}$$

$$\mathcal{F} : P \longmapsto P/\mathfrak{m}P$$

$$\mathcal{G} : S \otimes_{\mathbb{C}} W \longleftarrow\!\shortmid W$$

# McKay and Gabriel quiver

**Theorem**

*The McKay quiver of $G$ and the Gabriel quiver of $S\#G$ are isomorphic.*

**Proof.**

*Let* $V = \mathfrak{m}/\mathfrak{m}^2$. *Then the minimal projective resolution of* $\mathbb{C} = S/\mathfrak{m}$ *is*

$$0 \longrightarrow S \otimes_{\mathbb{C}} \bigwedge^n V \xrightarrow{\partial_n} \cdots \xrightarrow{\partial_2} S \otimes_{\mathbb{C}} \bigwedge^1 V \xrightarrow{\partial_1} S \longrightarrow 0.$$

*Applying* $- \otimes_{\mathbb{C}} W$ *we get*

$$\cdots \longrightarrow S \otimes_{\mathbb{C}} V \otimes W \longrightarrow S \otimes_{\mathbb{C}} W \longrightarrow 0.$$

$\square$

# Questions?