McKay correspondence

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Abstract

The goal of this thesis is to establish a 1-1 correspondence between quivers created from the four following sets whenever S is the power series ring $\mathbb{C}[\![x, y]\!]$ and G is a finite subgroup of $SL(2, \mathbb{C})$ acting on S

- The Maximal Cohen-Macaulay modules of the fixed ring S^G.
- The indecomposable projective modules of the skew group algebra $S\#G$.
- The indecomposable projective modules of $\mathrm{End}_{S^G}(S)$.
- The irreducible representations of G (indecomposable $\mathbb{C}G$-modules).

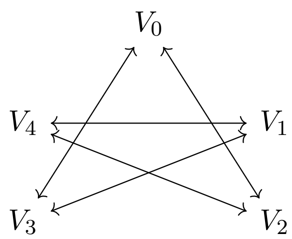
Much of the thesis will be used to define these four quivers and to develope tools to establish such a correspondence. A similar correspondence can be established for a general field k and a finite subgroup of $GL(n, k)$ with order nonzero in k, but in the general case we will only attain the MCM-modules that apear as S^G-direct summands of S. $SL(2, \mathbb{C})$ is also especially interesting because the quivers are exactly the Dynkin diagrams.

Contents

1 The McKay quiver

Definition 1.1. *Let G be a finite subgroup of $GL(n, \mathbb{C})$, and let V be the canonical representation (the one that sends g to g). Then we define the McKay quiver of G to be the quiver with verticies the irreducible representations of G, denoted V_i. For two irreducible representations V_i and V_j there is an arrow from the former to the latter if and only if V_j is a direct summand of $V \otimes V_i$.*

Example 1.1. *Let G be the group generated by $g = \begin{bmatrix} \omega^2 & 0 \\ 0 & \omega^3 \end{bmatrix}$, where ω is a primitive fifth root of unity. Then there are five different irreducible representations, the one sending g to ω, ω^2, ω^3, ω^4 respectively, and the trivial representation. Denote the representation sending g to ω^i by V_i, and let $V = V_2 \oplus V_3$ be the cannonical representation. Note that $V_i \otimes V_j = V_{i+j}$, where $i+j$ is understood to be modulo 5. Then we get the following McKay-quiver*



2 Skew group algebra S#G indecomposable projectives

This section is largely based on the book by [Leuschke and Wiegand, 2012]. This section will use definitions and theorems from representation theory as taught in the courses MA3203 - Ring Theory and MA3204 - homological algebra. Since I do not assume knowledge of this I have created appendix A. I will try to use footnotes to indicate where such theorems are used.

Definition 2.1. *If G is a subgroup of $GL_n(\mathbb{C})$, we can extend the group action of G to $\mathbb{C}[\![x_1, \cdots, x_n]\!]$. More explicitely G acts on x_i as it would the ith basis vector of $\mathbb{C}^n$, and acts on products and sums by acting on each component seperatley. We then define the skew group algebra $\mathbb{C}[\![x_1, \cdots, x_n]\!] \# G$ to be the algebra generated by elements of the form $f \cdot g$ with $f \in \mathbb{C}[\![x_1, \cdots, x_n]\!]$ and $g \in G$, and we define the multiplication by*

$$(f_1 \cdot g_1) \cdot (f_2 \cdot g_2) = (f_1 \cdot f_2^{g_1}) \cdot (g_1 \cdot g_2)$$

Where f^g denotes the image of f under the action of g.

Theorem 2.1. *We have an isomorphism of rings*

$$e\mathbb{C}[\![x,y]\!]\#Ge \simeq \mathbb{C}[\![x,y]\!]^G$$

where $e = \frac{1}{|G|}\sum_{g\in G} g$.

Proof. Let f^g denote the image of f under the action of g. Then if we let $f(x,y)g$ be an element of the skew algebra we get that $ef(x,y)ge = f(x,y)^e \cdot ege = f(x,y)^e \cdot e = e \cdot f(x,y)$. It then follows that $e\mathbb{C}[\![x,y]\!]\#Ge$ is isomorphic to the image of $\mathbb{C}[\![x,y]\!]$ under the action of e. Since $ge = g$ for all $g \in G$ it is clear that the image of e is contained in the fixed ring. For the converse you just need to notice that the fixed ring is fixed under e and thus is contained in the image. □

Lemma 2.1. *Let* $S = \mathbb{C}[\![x,y]\!]$. *An* $S\#G$*-module is projective if and only if it is projective as an* S*-module.*

Proof. Onlyifity follows from $S\#G$ being a free S-module, it is isomorphic to $\bigoplus_{g\in G} S$. Thus we need only show ifity.

First we need to see that an $S\#G$-linear map is just an S-linear map such that $f(g(m)) = g(f(m))$ for all $g \in G$. Equivalently $f(m) = g(f(g^{-1}(m)))$. This allows us to define a group action on S-linear maps by $f^g(m) = g(f(g^{-1}(m)))$. Then we can restate it as

$$\mathrm{Hom}_{S\#G}(M,N) = \mathrm{Hom}_S(M,N)^G$$

Clearly if f is $S\#G$-linear then it's in $\mathrm{Hom}_S(M,N)^G$. To see the other inclusion, let f be an S-linear map that is fixed under G. Then $f(s \cdot gm) = sf(gm) = s \cdot g(f(g^{-1}gm)) = s \cdot gf(m)$, and hence f is $S\#G$-linear. Nextly I want to show that $-^G$ is an exact functor.

If K is the kernel of a map $f : M \to N$, then the kernel of the induced map $f^G : M^G \to N^G$ is of course just $K \cap M^G$ which equals K^G. Assume f is epi and let $n \in N^G$. Consider a preimage m such that $f(m) = n$. Let $\theta = \frac{1}{|G|}\sum_{g\in G} g(m)$. Then θ is in M^G and $f(\theta) = \frac{1}{|G|}\sum_{g\in G} g(f(m)) = \frac{1}{|G|}\sum_{g\in G} n = n$.

This implies that if $\mathrm{Hom}_S(P,-)$ is exact then $\mathrm{Hom}_S(P,-)^G = \mathrm{Hom}_{S\#G}(P,-)$ is exact and our lemma follows. □

Lemma 2.2. *Let* S *be the complex power series ring in* n *variables, and* $\mathfrak{m} = \langle x_i\rangle_{i=1}^n$ *the radical of* S. *Then for any free* S*-module* N, $\mathfrak{m}N$ *is* <u>*small*</u> *in* N. *That is if* X *is a submodule of* N *such that* $X + \mathfrak{m}N = N$, *then* $X = N$.

Proof. Let N be the free module $S^{(I)} := \bigoplus_{i\in I} S_i$, where $S_i \cong S$. Assume that X is a submodule such that $X + \mathfrak{m}N = N$. We denote by 1_i the elements that is 1 at index i and 0 elsewhere. Since $\{1_i\}$ generate N, it is enough to show that X contains all of them. Since $X + \mathfrak{m}N = N$, we know that there is an $m_i \in \mathfrak{m}N$ and an $x_i \in X$ such that $x_i + m_i = 1_i$. Then we have that $x_i = 1_i - m_i$. Since the power series at index i of x_i has constant coefficient 1 it is invertible. If we multiply x_i by its inverse we get $\tilde{x}_i$ which is 1 at index i and some eleement of $\mathfrak{m}$ at index $j \neq i$, say m_{ij}. Then $\tilde{x}_i - \sum_{j \neq i} m_{ij}\tilde{x}_j$ has a unit in index i and 0 at all other indicies. Thus X contains 1_i for all i, and $X = N$. □

Theorem 2.2. *Let $S = \mathbb{C}[\![x, y]\!]$ and let $\mathfrak{m} = \langle x, y\rangle_S$ be the radical of S. Then there are bijections between the indecomposable finitely generated projective $S\#G$-modules and the indecomposable $\mathbb{C}G$-modules given by*

$$\begin{Bmatrix} \textit{indecomposable projective} \\ S\#G\textit{-modules} \end{Bmatrix} \qquad \begin{Bmatrix} \textit{indecomposable} \\ \mathbb{C}G\textit{-modules} \end{Bmatrix}$$

$$\mathcal{F} : P \longmapsto P/\mathfrak{m}P$$

$$\mathcal{G} : S \otimes_{\mathbb{C}} W \longleftarrow W$$

Where the $S\#G$-module structure on $S \otimes_{\mathbb{C}} W$ is given by $(s \cdot g) \cdot f \otimes v = sf^g \otimes v^g$.

Proof. First we should show that $S \otimes_{\mathbb{C}} W$ is an indecomposable projective $S\#G$-module and that $P/\mathfrak{m}P$ is infact an indecomposable $\mathbb{C}G$-module. Since $S \otimes_{\mathbb{C}} W$ is a free S-module it follows from lemma 2.1 that it is projective. To see that it is indecomposable we will first study it as an S-module and exploit the fact that $\mathrm{Hom}_{S\#G}(M, N) \subseteq \mathrm{Hom}_S(M, N)$.

Using lemma 2.2 we get that $\mathfrak{m}S \otimes_{\mathbb{C}} W$ is small in $S \otimes_{\mathbb{C}} W$. This means that we get that

$$\frac{S \otimes_{\mathbb{C}} W}{\mathfrak{m}S \otimes_{\mathbb{C}} W} \cong S/\mathfrak{m} \otimes_{\mathbb{C}} W \cong \mathbb{C} \otimes_{\mathbb{C}} W \cong W$$

is the top of $S \otimes_{\mathbb{C}} W$ as an S-module. Further since the projection $S \otimes_{\mathbb{C}} W \to W$ is $S\#G$-linear we have that $S \otimes_{\mathbb{C}} W$ is the projective cover of W also as $S\#G$-modules. Assume for the sake of contradiction that $S \otimes_{\mathbb{C}} W$ decomposes as $M \oplus N$ for non-zero M and N. Then W would equal $M/\mathfrak{m}M \oplus N/\mathfrak{m}N$ as an $S\#G/\langle\mathfrak{m}\rangle$-module. Since $S\#G/\langle\mathfrak{m}\rangle \cong \mathbb{C}G$ and W is indecomposable we must have that either $M/\mathfrak{m}M$ or $N/\mathfrak{m}N$ is 0. This then gives a contradiction because $\mathfrak{m}M$ and $\mathfrak{m}N$ are small in M and N. Hence we must have that $S \otimes_{\mathbb{C}} W$ is indecomposable.

It's clear that $P/\mathfrak{m}P$ is a $\mathbb{C}G$-module, because $\mathbb{C}G$ is a subring of $S\#G$. To see that it's indecomposable we will first show that it's indecomposable we will use a similar argument as above. Assume $P/\mathfrak{m}P$ decomposes as

$V \oplus W$. Then both P and $S \otimes_{\mathbb{C}} V \oplus S \otimes_{\mathbb{C}} W$ are projective covers of $\mathfrak{m}P = V \oplus W$ we get induced $S\#G$-linear epiomorphisms between them.

$$\begin{tikzcd} & S \otimes_{\mathbb{C}} P/\mathfrak{m}P \arrow[d, two heads] \arrow[dl, dashed, bend right, two heads] \\ P \arrow[r, two heads] \arrow[ur, dashed, two heads] & P/\mathfrak{m}P \end{tikzcd}$$

Now we use the fact that P is finitely generated. Since there can only be an epimorphism from a module with more or equal amount of generators P and $S \otimes_{\mathbb{C}} \mathfrak{m}P$ must have the same amount of generators and teh induced maps are in fact isomorphisms of S-modules. Since the maps are also $S\#G$-linear we have that P decomposes as $S \otimes_{\mathbb{C}} V \oplus S \otimes_{\mathbb{C}} W$. Then since P is indecomposable we must have that either $S \otimes_{\mathbb{C}} V$ or $S \otimes \mathbb{C} W$ is 0. That means that either V or W is 0, and we have shown that $P/\mathfrak{m}P$ is an indecomposable $\mathbb{C}G$-module.

To see that the given maps are bijections we will show that they are mutual inverses. First to see that $\mathcal{F}(\mathcal{G}(W)) \cong W$ we simply look at the definition

$$\frac{S \otimes_{\mathbb{C}} W}{\mathfrak{m}S \otimes_{\mathbb{C}} W} \cong S/\mathfrak{m} \otimes_{\mathbb{C}} W \cong \mathbb{C} \otimes_{\mathbb{C}} W \cong W$$

Next we consider $\mathcal{G}(\mathcal{F}(P)) = S \otimes_{\mathbb{C}} P/\mathfrak{m}P$. We have already seen that the induced map

$$\begin{tikzcd} & S \otimes_{\mathbb{C}} P/\mathfrak{m}P \arrow[d, two heads] \arrow[dl, dashed] \\ P \arrow[r, two heads] & P/\mathfrak{m}P \end{tikzcd}$$

is an isomorphism, and thus $P \cong \mathcal{G}(\mathcal{F}(P))$. $\square$

2.1 The Gabriel quiver

Definition 2.2. *For a skew group algebra $S\#G$ we define its Gabriel quiver to be the quiver with verticies as the indecomposable projective modules of $S\#G$. The arrows are given by taking the minimal projective resolution of $P/\mathfrak{m}P$, where $\mathfrak{m}$ is as defined above. If the minimal projective resolution of $P/\mathfrak{m}P$ is given by*

$$\cdots \longrightarrow Q_1 \longrightarrow Q_0 \longrightarrow 0$$

We say there is an arrow from P to P' if P' appears as a direct summand of Q_1.

Definition 2.3. *Let V be a vector space. We then define the exterior algebra $\bigwedge V$ as the associative unital graded algebra such that the multiplication is bilinear and satisfies $x \wedge y = -y \wedge x$ for any x and y in V.*

Some key properties of the exterior algebra is that $x \wedge x = 0$, and more generally that $x_1 \wedge \cdots \wedge x_p = 0$ whenever $\{x_i\}_{i=1}^p$ are linearly dependent.

The pth exterior power of V, denoted $\overset{p}{\bigwedge} V$ is the vector space of all elements that are the product of p vectors in V. If $\{x_i\}_{i=1}^n$ is a basis for V, then $x_{i_1} \wedge \cdots \wedge x_{i_p}$ where $i_1 < i_2 < \cdots < i_p$ and $1 \le i_j \le n$ forms a basis for $\overset{p}{\bigwedge} V$, thus it is $\binom{n}{p}$-dimensional.

Proposition 2.1. *If S is the ring of formal power series over $\mathbb{C}$ in n variables, and G is a finite group acting on S, let $V = \mathfrak{m}/\mathfrak{m}^2$. Then the minimal projective resolution of $\mathbb{C} \cong S/\mathfrak{m}$ is given by*

$$0 \longrightarrow S \otimes_{\mathbb{C}} \overset{n}{\bigwedge} V \xrightarrow{\partial_n} \cdots \xrightarrow{\partial_2} S \otimes_{\mathbb{C}} \overset{1}{\bigwedge} V \xrightarrow{\partial_1} S \longrightarrow 0$$

Where ∂_p is the $S\#G$-linear map defined by

$$\partial_p(s \otimes x_{i_1} \wedge x_{i_2} \wedge \cdots \wedge x_{i_p}) = \sum_{j=1}^{p} (-1)^{j+1} s x_{i_j} \otimes x_{i_1} \wedge \cdots \wedge \hat{x}_{i_j} \wedge \cdots \wedge x_{i_p}$$

Where $x_{i_1} \wedge x_{i_2} \wedge \cdots \wedge x_{i_p}$ is one of the standard basis vectors for $\overset{n}{\bigwedge} V$, namely $i_1 < i_2 < \cdots < i_p$, and $\hat{x}_j$ means that x_j is ommited.

Proof. Firts we should show that this is a projective resolution of $\mathbb{C}$. In fact the complex described above is the Koszul complex of the regular sequence[1] $(x_i)_{i=1}^n$. It's a general fact of homological algebra that the Koszul complex of a regular sequence is a projective resolution of the ring modulo the ideal generated by the regular sequence, which in this case equals $S/\langle x_i \rangle_{i=1}^n = \mathbb{C}$. refference yes/no?

Secondly we want to show that the resolution is minimal. To do this it is enough to show that for each $k \ge 1$ ∂_k is a projective cover of its image, and that $S \to \mathbb{C}$ is a projective cover. In other words we have to show is that the kernels of the maps are small. Since $\operatorname{Im} \partial_{k+1} = \operatorname{Ker} \partial_k$ and $\operatorname{Im} \partial_{k+1} \subseteq \mathfrak{m} \otimes_{\mathbb{C}} \overset{k+1}{\bigwedge} V$ it follows from lemma 2.2 that the resolution is minimal. □

Theorem 2.3. *If S is the complex power series ring in n variables and G is a fintie subgroup of $GL_n(\mathbb{C})$, then the McKay quiver of G and the Gabriel quiver of $S\#G$ are isomorphic.*

[1] Regular sequences are defined on page 15 in definition 4.2.

Proof. We have already seen that they have the same vertices, namely if V_i are the irreducible representations of G, then $S \otimes_{\mathbb{C}} V_i$ are the indecomposable projectives of $S\#G$. To see that they have the same arrows consider as above the minimal resolution of $\mathbb{C}$.

$$0 \longrightarrow S \otimes_{\mathbb{C}} \bigwedge^n V \xrightarrow{\partial_n} \cdots \xrightarrow{\partial_2} S \otimes_{\mathbb{C}} \bigwedge^1 V \xrightarrow{\partial_1} S \longrightarrow 0$$

If we tensor with V_i on the right we will get a minimal resolution of V_i (you can see that is minimal by using the exact same argumant as above).

$$\cdots \xrightarrow{\partial_2 \otimes_{\mathbb{C}} V_i} S \otimes_{\mathbb{C}} \bigwedge^1 V \otimes_{\mathbb{C}} V_i \xrightarrow{\partial_1 \otimes_{\mathbb{C}} V_i} S \otimes_{\mathbb{C}} V_i \longrightarrow 0$$

From here, since $\bigwedge^1 V = V$, we see that $P_j = S \otimes_{\mathbb{C}} V_j$ appears as a direct summand of $S \otimes_{\mathbb{C}} V \otimes_{\mathbb{C}} V_i$ exactly when V_j appears as a direct summand of $V \otimes_{\mathbb{C}} V_i$. ☐

3 The endomorphism ring of S as an S^G-module

Isomorphism to $S\#G$ implies proj SG ¡-¿ S^G direct summands of S.

In this section we will show that $S\#G$ is isomorphic to $\mathrm{End}_R(S)$ as rings, where $R = S^G$ is the fixed ring of S by G (and we have some additional assumptions on S and G). This will be the longest proof of this thesis and I have therefor decided to split it up into several steps. The proof will be done by constructing an explicit isomorphism.

$$S\#G \longrightarrow \mathrm{End}_R(S)$$

$$s \cdot g \longmapsto (t \mapsto s \cdot t^g)$$

We can easily show that this is an injective ring-homomorphism. The meat of the proof is to consider the map as a morphism of S-modules, and then using ramification theory to show that it is an epimorphism. To do this we will show that for every height one prime ideal $\mathfrak{p}$ of S if we localize at $\mathfrak{p}$ we get a socalled unramified extension of rings.

$$R_{\mathfrak{p} \cap R} \hookrightarrow S_{\mathfrak{p}}$$

We will use this to show that the short exact sequence

$$I \hookrightarrow S_{\mathfrak{p}} \otimes_{R_{\mathfrak{p} \cap R}} S_{\mathfrak{p}} \overset{\mu}{\twoheadrightarrow} S_{\mathfrak{p}}$$

where m is the multiplication map and I is the kernel, has a splitting. Whenever this happens we say the extension is seperable. We will use this splitting to construct an inverse for $S_{\mathfrak{p}}\#G \to \mathrm{End}_{R_{\mathfrak{p}\cap R}}(S_{\mathfrak{p}})$. Finally we will show that since we get an isomorphism whenever we localize at a height one prime ideal this means that the original map is an isomorphism.

Let us first begin with some definitions

Definition 3.1. *Let A and B be two local commutative rings with maximal ideal $\mathfrak{n}$ and $\mathfrak{m}$ respectively, and let $A \hookrightarrow B$ be an extension of rings. We say that the extension is unramified if the following conditions hold:*

- *B is a finitely generated A-module.*
- *$A/\mathfrak{n} \hookrightarrow B/\mathfrak{m}$ is a seperable field extension.*
- *$\mathfrak{n}B = \mathfrak{m}$*

If the two first conditions are met, and there is a positive integer e such that $\mathfrak{n}B = \mathfrak{m}^e B$, we say the extension has ramification index e when e is the smallest such number. Note that being unramified is then equivalent to having ramification index 1.

In order to show that unramified implies seperable we must first take a small detour.

Definition 3.2. *Let $A \to B$ be an extension of rings. We then define the derivation module $\Omega_{B|A}$ as the B-module with forrmal generators db for all $b \in B$ and with the following relations:*

A-linearity: $d(ab + a'b') = adb + a'db'$ for all $a, a' \in A$ and $b, b' \in B$.

Leibniz rule: $d(bc) = bdc + cdb$ for all $b, c \in B$.

Note that for any polynomial expression $f(b)$ we have that $df(b) = f'(b)db$ where f' is the formal derivative of f. Now we will show how the derivation module make a link between unramified and seperable extensions.

Proposition 3.1. *Let $A \to B$ be an unramified extension of local rings. Then $\Omega_{B|A}$ is 0.*

Proof. Keeping with the notation above we let $\mathfrak{n}$ be the maximal ideal of A and $\mathfrak{m}$ the maximal ideal of B. Furthermore let l denote $B/\mathfrak{m}$ and k denote $A/\mathfrak{n}$. Then I claim there is an exact sequence

$$\mathfrak{m}/\mathfrak{m}^2 \xrightarrow{\alpha} \Omega_{B|A} \otimes_B B/\mathfrak{m} \longrightarrow \Omega_{l|A} \longrightarrow 0$$

where $\alpha(\overline{m}) = d_{B|A}m \otimes 1$ for any m in $\mathfrak{m}$. Let's first show that α is well defined. Let $m_1 \cdot m_2$ be in $\mathfrak{m}^2$. Then we need to show that $\alpha(\overline{m_1 \cdot m_2})$ is 0.

$$\alpha(\overline{m_1 \cdot m_2}) = d_{B|A}(m_1 \cdot m_2) \otimes 1 =$$
$$m_1 d_{B|A} m_2 \otimes 1 + m_2 d_{B|A} m_1 \otimes 1 =$$
$$d_{B|A} m_2 \otimes (m_1 \cdot 1) + d_{B|A} m_1 \otimes (m_2 \cdot 1)$$

Since $l = B/\mathfrak{m}$ we have that $m_1 \cdot 1$ and $m_2 \cdot 1$ is 0 in l, thus the right hand side is 0, and α is well defined.

The map $\Omega_{B|A} \otimes_B B/\mathfrak{m} \to \Omega_{l|A}$ is just the natural projection sending $db \otimes 1$ to $d\bar{b}$, where $\bar{b}$ is teh projection of b onto l. We want to show that this is the cokernel of α. The kernel of $\Omega_{B|A} \otimes_B B/\mathfrak{m} \to \Omega_{l|A}$ is generated by $dm \otimes 1$ for $m \in \mathfrak{m}$, but this is exactly the image of α, thus the sequence is exact.

Nextly we want to show that $\Omega l|A = 0$. Since $\mathfrak{n} \subseteq \mathfrak{m}$ and l is annihalated by $\mathfrak{m}$ we have that $\Omega l|A = \Omega_{l|k}$. Let x be an element of l, and let p be its irreducible polynomial over k. Now we want to use the fact that $k \subset l$ is a seperable field extension. Remember that $k \subset l$ being seperable means that the formal derivative of p is non-zero. Now we have that

$$0 = d(p(x)) = p'(x)dx.$$

Since p' is a non-zero polynomial of lower degree than p, and p is the smallest polynomial with root x, we must have that $p'(x)$ is non-zero. This implies that $dx = 0$, and since this holds for all x it must be that $\Omega l|k = 0$.

Since $\Omega l|k = 0$ we have that α is surjective. We will now use that since $A \to B$ is unrammified $\mathfrak{n}B = \mathfrak{m}$. More specifically the map $\beta : \mathfrak{n}/\mathfrak{n}^2 \otimes_A B \to \mathfrak{m}/\mathfrak{m}^2$ is surjective. Since both α and β are surjective we have that $\alpha\beta$ is also surjective, but

$$\alpha\beta(\overline{n} \otimes b) = \alpha(\overline{nb}) = d(nb) \otimes 1 = ndb \otimes 1 = db \otimes n \cdot 1 = 0$$

for all $n \in \mathfrak{n}$ and $b \in B$. Thus the only conclusion is that $\Omega_{B|A} \otimes_B l = 0$.

Since $\Omega_{B|A} \otimes_B l = \Omega_{B|A} \otimes_B B/\mathfrak{m} = \Omega_{B|A}/\mathfrak{m}\Omega_{B|A}$ it follows from Nakayamas lemma refference or something that $\Omega_{B|A} = 0$. □

Theorem 3.1. *Let $A \to B$ be an unramified extension of local rings. Then the sequence*

$$0 \longrightarrow I \longrightarrow B \otimes_A B \xrightarrow{\mu} B \longrightarrow 0$$

splits as a short exact sequence of $B \otimes_A B$-modules. With $m(b \otimes b') = bb'$, the $B \otimes_A B$-module structure on B is given by $b \otimes b' \cdot b'' = bb'b''$, and $I = \operatorname{Ker} m$.

Proof. Firstly note that I is generated by elements on the form $b \otimes 1 - 1 \otimes b$, and that since B is fintiely generated as an A-module, I is finitely generated.

Next we want to show that $I/I^2 = \Omega_{B|A}$, which we have already seen equals 0. We will do this by constructing an epimorphism from I to $\Omega_{B|A}$ with kernel I^2 (here we again think of $\Omega_{B|A}$ as a $B \otimes_A B$-module by $b \otimes b' \cdot dc = bb'dc$). Now it is enough to show that the only relations on I/I^2 are those induced by A-linearity and the Leibniz rule. The A-linearity comes from us tensoring over A. That is $b \otimes 1 = 1 \otimes b$ in $B \otimes_A B$ exactly when b is in A. Next we also need to show that the Leibniz rule. Namely that an element in I is in I^2 if and only if it can be written on the form

$$(bc \otimes 1 - 1 \otimes bc) - (b_1 \otimes b_2)(c \otimes 1 - 1 \otimes c) - (c_1 \otimes c_2)(b \otimes 1 - 1 \otimes b)$$

where $b_1 b_2 = b$ and $c_1 c_2 = c$. It's a simple computation to check wether an element of I^2 can be written this way. Something is wrong here????

$$\begin{aligned}(b \otimes 1 - 1 \otimes b)(c \otimes 1 - 1 \otimes c) &= bc \otimes 1 - b \otimes c - c \otimes b + 1 \otimes bc \\ &= (bc \otimes 1 - 1 \otimes bc) - (c \otimes b - 1 \otimes bc) - (b \otimes c - 1 \otimes bc) \\ = (bc \otimes 1 - 1 \otimes bc) &- (1 \otimes b)(c \otimes 1 - 1 \otimes c) - (1 \otimes c)(b \otimes 1 - 1 \otimes b)\end{aligned}$$

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Now that we have shown that $I/I^2 = 0$, or rather that $I = I^2$ Nakaymas lemma refference gives that there is an $i \in I$ such that $ji = j$ for all $j \in I$. Then we can define the splitting map $B \otimes_A B \to I$ by $b \otimes b' \mapsto b \otimes b' \cdot i$. Thus the sequence

$$0 \longrightarrow I \longrightarrow B \otimes_A B \xrightarrow{\mu} B \longrightarrow 0$$

splits. □

Theorem 3.2. *Let B be a local k-algebra domain, and G a finite subgroup of $\mathrm{Aut}_k(B)$ with order relatively prime to the characteristic of k, such that the extension $B^G =: A \to B$ is unramified. Then the map*

$$B\#G \xrightarrow{\gamma} \mathrm{End}_A(B)$$

$$b \cdot g \longmapsto (a \mapsto b \cdot a^g)$$

is an isomorphism of B-modules, and isomorphism of rings.

Proof. First we to see that the map is injective, assume $b \cdot g$ and $b' \cdot g'$ map to the same endomorphism. Then $b \cdot t^g = b' \cdot t^{g'}$ for all $t \in B$. Choosing $t = 1$ we see that $b = b'$. Then since B is a domain this menas that $t^g = t^{g'}$ for all t, that is to say $g = g'$.

To see that the map is surjective we will construct a splitting. The splitting will be constructed using the following diagram:

$$\begin{tikzcd}
B\#G \arrow[r, "\gamma"] & \mathrm{End}_A(B) \arrow[d, "f\mapsto f\otimes\rho"] \\
B\otimes_A B\#G \arrow[u, "\tilde{\mu}"] & \mathrm{Hom}_B(B\otimes_A B, B\otimes_A B\#G) \arrow[l, "ev_\epsilon"]
\end{tikzcd}$$

where ρ is the modified Reinolds-opertaor

$$\rho(b) = \textstyle\sum_{g\in G} b^g \cdot g.$$

Since we assumed the extension is unrammified we have that

$$\begin{tikzcd}
0 \arrow[r] & I \arrow[r, "\iota"] & B\otimes_A B \arrow[r, "\mu"] \arrow[l, bend left, dotted, "\psi"] & B \arrow[r] & 0
\end{tikzcd}$$

splits. As indicated we denote the left splitting by ψ. Then let $\epsilon = 1\otimes 1 - \iota\psi(1\otimes 1)$. Then $\mu(\epsilon) = 1$, and $(b\otimes 1 - 1\otimes b)\epsilon = 0$. Then we define the evaluation map at ϵ by

$$ev_\epsilon : \quad \mathrm{Hom}_B(B\otimes_A B, B\otimes_A B\#G) \longrightarrow B\otimes_A B\#G$$

$$f \longmapsto f(\epsilon)$$

Lastly $\tilde{\mu} : B\otimes_A B\#G \to B\#G$ is simply the map $b\otimes c\cdot g \mapsto bc\cdot g$. We have now defined all the maps in the square

$$\begin{tikzcd}
B\#G \arrow[r, "\gamma"] & \mathrm{End}_A(B) \arrow[d, "f\mapsto f\otimes\rho"] \\
B\otimes_A B\#G \arrow[u, "\tilde{\mu}"] & \mathrm{Hom}_B(B\otimes_A B, B\otimes_A B\#G) \arrow[l, "ev_\epsilon"]
\end{tikzcd}$$

Now we want to show that the composition of the three bottom maps forms a splitting. That is for any $f \in \mathrm{End}_A(B)$ we have that $\gamma(\tilde{\mu}(ev_\epsilon(f\otimes\rho))) = f$.

Write $\epsilon = \sum\limits_i x_i \otimes y_i$. Then I claim that

$$\sum_i x_i \cdot y_i^g = \begin{cases} 1 & g = 1_G \\ 0 & \text{otherwise} \end{cases}$$

We know that

$$(b\otimes 1)\sum_i x_i \otimes y_i = (1\otimes b)\sum_i x_i \otimes y_i$$

holds for all b. Then applying the map $1\otimes g$ on both sides we get

$$\sum_i bx_i \otimes y_i^g = \sum_i x_i \otimes b^g y_i^g$$

Then by applying μ we get

$$b\sum_i x_i y_i^g = b^g \sum_i x_i y_i^g$$

Then since B is a domain we get that either $b = b^g$ or $\sum_i x_i y_i^g = 0$. If we assume that $\sum_i x_i y_i^g \neq 0$ we then get that $g = 1_G$. Then since

$$\sum_i x_i y_i = \mu(\epsilon) = 1$$

we see that my claim holds. We can now calculate $\gamma(\tilde{\mu}(ev_\epsilon(f \otimes \rho)))$:

$$\begin{aligned}
\gamma\left[\tilde{\mu}\left[(f\otimes\rho)(\epsilon)\right]\right](b) &= \\
\gamma\left[\tilde{\mu}\left[(f\otimes\rho)(\sum_i x_i \otimes y_i)\right]\right](b) &= \\
\gamma\left[\tilde{\mu}\left[\sum_i f(x_i)\otimes\rho(y_i)\right]\right](b) &= \\
\gamma\left[\sum_i f(x_i)\sum_g y_i^g \cdot g\right](b) &= \\
\gamma\left[\sum_g\sum_i f(x_i) y_i^g \cdot g\right](b) &= \\
\sum_g\left(\sum_i f(x_i) y_i^g \cdot b^g\right) &\overset{*}{=} \\
f\left(\sum_g\left(\sum_i x_i y_i^g\right)\cdot b^g\right) &\overset{**}{=} \\
f(b)
\end{aligned}$$

In (*) we use the fact that f is A-linear and that $\sum_g\sum_i y_i^g b^g$ is in A. In (**) we use the claim from above that

$$\sum_i x_i \cdot y_i^g = \begin{cases} 1 & g = 1_G \\ 0 & \text{otherwise} \end{cases}$$

This means that γ is an epimorphism and then also an isomorphism. $\square$

Definition 3.3. *Let S be a commutative ring, G a subgroup of $\mathrm{Aut}(S)$, and $\mathfrak{p}$ a prime ideal. The inertia group of $\mathfrak{p}$ is defined as*

$$T(\mathfrak{p}) = \{g \in G | s^g - s \in \mathfrak{p} \forall s\}$$

Theorem 3.3. *Let S be the complex power series ring in n variables, let G be a finite subgroup of $GL_n(\mathbb{C})$ acting on S, and let $\mathfrak{p}$ be a height one prime ideal of S. Denote by R the fixed ring S^G and let $\mathfrak{q} = R \cap \mathfrak{p}$. Then the rammification index of $R_\mathfrak{q} \subset S_\mathfrak{p}$ equals the order of the inertia group $|T(\mathfrak{p})|$.*

Proof. We write $\mathfrak{m}$ for the maximal ideal of S. Since $\mathfrak{p}$ is height one and S is a UFD we have that $\mathfrak{p} = \langle z \rangle$ for some $z \in \mathfrak{m}$. We define an inner product on $V := \mathfrak{m}/\mathfrak{m}^2$ by

$$\langle x, y \langle_G = \frac{1}{|G|} \sum_{g \in G} \langle x^g, y^g \rangle$$

□

where $\langle -, - \rangle$ is the standard inner product. Note that the action of G is orthogonal with respect to this inner product.

We write $\overline{z}$ for the representative for z in V. Since the action of G preserves degrees and that $\overline{z}^g - \overline{z} \in \langle \overline{z} \rangle$ we must have that $\overline{z}^g = a_g \cdot \overline{z}$ for some scalar $a_g \in \mathbb{C}$. Further since $x^g = x + \lambda_{g,x}\overline{z}$ for all $x \in V$ we have that g fixes the $\langle -, - \rangle_G$-orthogonal complement to $\overline{z}$ for all $g \in T(\mathfrak{p})$. This means we can choose a basis such that all elements of $T(\mathfrak{p})$ are on the form:

$$\begin{bmatrix} 1 & & & & \\ & 1 & & & \\ & & \ddots & & \\ & & & 1 & \\ & & & & a_g \end{bmatrix}$$

This means $T(\mathfrak{p})$ is isomorphic to $\{a_g\}_{g \in T(\mathfrak{p})} \leq \mathbb{C}^*$ which is a subgroup of $\mathbb{C}^*$. Since all finite subgroups of $\mathbb{C}^*$ are cyclic this implies that $T(\mathfrak{p})$ is cyclic.

wait this proof only shows that $|T(\mathfrak{p})|$ is the rammification index of $S_\mathfrak{p}^{T(\mathfrak{p})} \subset S\ldots$????? I want the rammification index for S^G, why are these the same???

Theorem 3.4. *$R_\mathfrak{q} \subset S_\mathfrak{p}$ is unrammified for all height one primes $\mathfrak{p}$ if and only if G contains no pseudoreflections, that is a non-trivial element that fixes a codimension 1 subspace.*

Proof. Firstly since we are working in characteristic 0, all field extensions are seperable, thus $R_\mathfrak{q}/\mathfrak{q} \subset S_\mathfrak{p}/\mathfrak{p}$ is seperable. Since S is a rank $|G|$ R-module, $S_\mathfrak{p}$ will be a finitely generated $R_\mathfrak{q}$-module.

We know that elements of $T(\mathfrak{p})$ can be written on the form

$$\begin{bmatrix} 1 & & & & \\ & 1 & & & \\ & & \ddots & & \\ & & & 1 & \\ & & & & a_g \end{bmatrix}.$$

Since G does not contain any pseudoreflections we must have that $a_g = 1$ and therefor $T(\mathfrak{p})$ is trivial and $|T(\mathfrak{p})| = 1$. That means that the rammification index of $R_\mathfrak{q} \subset S_\mathfrak{p}$ is 1, and the extension is unrammified. □

Note that no finite subgroup of $SL_n(\mathbb{C})$ contains pseduoreflections. In particular $R_\mathfrak{q} \subset S_\mathfrak{p}$ is unrammified when G is a finite subgroup of $SL_2(\mathbb{C})$.

Now the last piece of the puzzle is to show that this implies that

$$S\#G \xrightarrow{\gamma} \operatorname{End}_R(S)$$

is an isomorphism when $S = \mathbb{C}[\![x, y]\!]$, and G is a finite subgroup of $SL_2(\mathbb{C})$.

Lemma 3.1. *Let M and N be modules with depth$M \geq 2$ and something about depth N, and let $f : M \to N$ be a map such that $f_\mathfrak{p} : M_\mathfrak{p} \to N_\mathfrak{p}$ is an isomoprhism for all height one prime ideals $\mathfrak{p}$. Then f is an isomorphsim.*

Proof. Maybe I dont understand this proof.... □

Theorem 3.5. *Let $S = \mathbb{C}[\![x, y]\!]$ be the complex power series ring in two variables, let G be a fintie subgroup of $SL_2(\mathbb{C})$ acting on S, and let $R = S^G$ be the fixed ring. Then the map*

$$S\#G \xrightarrow{\gamma} \operatorname{End}_R(S)$$

is an isomorphism of rings.

Proof. Firstly let's show that for a prime ideal $\mathfrak{p}$ we have that $S_\mathfrak{p}^G = R_\mathfrak{q}$ where $\mathfrak{q} = R \cap \mathfrak{p}$. Assume $\frac{s}{p} \in S_\mathfrak{p}^G$ is fixed by G. Consider the fraction

$$\frac{\left(\prod_{g \neq 1} p^g\right) s}{\prod_g p^g}$$

Since we have just multiplied by $\prod_{g \neq 1} p^g$ in the nominator and the denominator it still equals $\frac{s}{p}$. The bottom is obviously fixed by g, but why is it not in $\mathfrak{q}$??? How do I know p^g is not in $\mathfrak{p}$???? You localize using the complement of the prime ideal right?? Then since the denominator is fixed and the fraction as a whole is fixed this implies that the nominator is fixed as well.

Secondly I want to show that $\mathrm{End}_R(S)_{\mathfrak{p}} = \mathrm{End}_{R_{\mathfrak{q}}}(S_{\mathfrak{p}})$. ???

From here we just need to wrap everything together. Since G does not contain any pseudoreflections we get from theorem 3.4 that the map is an isomoprhism when localizing at any height one prime ideal. Then lemma 3.1 gives us that it's an isomophism need to show depth. □

Theorem 3.6. *Let S be the complex power series ring in two variables, G be a finite subgroup of $GL_2(\mathbb{C})$, $R = S^G$ the fixed ring of S under the action of G, and $(S\#G)^G$ be the fixed ring of $S\#G$ under left multiplication by G. Then S is isomorphic to $(S\#G)^G$ as R-modules.*

Proof. To see this we will define an injective R-linear map from S to $S\#G$ and show that it's image is $(S\#G)^G$. Let $\rho : S \to S\#G$ be given by

$$\rho(s) = \sum_{g \in G} s^g \cdot g.$$

It's clear that it's injective and it is R-linear because

$$\rho(rs) = \sum_{g \in G} r^g s^g \cdot g = r \sum_{g \in G} s^g \cdot g.$$

It should also be clear that the image is contained in $(S\#G)^G$ because

$$h \cdot \rho(s) = \sum_{g \in G} h \cdot s^g \cdot g = \sum_{g \in G} s^{hg} \cdot hg = \rho(s).$$

To see that the image is all of $(S\#G)^G$ consider an arbitrary element in $(S\#G)^G$, $\psi = \sum_{g \in G} s_g \cdot g$. Since ψ is fixed under left multiplication by G we must have that

$$\sum_{g \in G} s_g^h \cdot hg = \sum_{g \in G} s_g \cdot g,$$

in particular s_h must equal s_1^h and it follows that $\psi = \rho(s_1)$. □

4 Maximal Cohen-Macaulay modules of S^G

Definition 4.1. *If R is a local ring with residual field k we define the depth of a module, M, to be the minimal n such that the extension $\mathrm{Ext}^n_R(k, M)$ is non-zero.*

Definition 4.2. *If R is a commutative ring and M is an R-module, an R-regular sequence on M is a sequence of elements of R, $r_1, r_2, \cdots r_n$ such that $M/\langle r_1, \cdots, r_i\rangle M$ is non-zero and multiplication by r_i is injective on $M/\langle r_1, \cdots, r_{i-1}\rangle M$.*

Definition 4.3. *If R is a ring, we say that $\mathfrak{p}$ is a prime ideal in R if*

1. *$\mathfrak{p}$ is a proper ideal of R.*
2. *For any two elements $a, b \in R$ such that $ab \in \mathfrak{p}$ we must have that either a is in $\mathfrak{p}$ or b is.*

Definition 4.4. *If R is a ring we define its Krull-dimesnion to be the maximum length of a chain of prime ideals in R. For example the polynomial ring $\mathbb{C}[x_1, \cdots, x_n]$ has Krull-dimension n given by the chain*

$$0 \subseteq \langle x_1 \rangle \subseteq \langle x_1, x_2 \rangle \subseteq \cdots \subseteq \langle x_1, \cdots, x_n \rangle$$

Definition 4.5. *If M is a module over a local ring R with Krull-dimension d we say that M is maximal Cohen Macaulay (MCM) if the depth of M equals d.*

Theorem 4.1. *If G is a finite subgroup of $GL_n(\mathbb{C})$, S is the complex power series ring in n variables and $R = S^G$ is the ring fixed under the action of G, then R is a direct summand of S as R-modules.*

Proof. Consider the map $\pi : S \to R$ given by

$$\pi(s) = \frac{1}{|G|} \sum_{g \in G} s^g$$

It's clear that the image of π is in R because an action from G will just permute the order of the sum. Further

$$\pi(r) = \frac{1}{|G|} \sum_{g \in G} r^g = \frac{1}{|G|} \sum_{g \in G} r = r,$$

so π splits the inclusion $R \hookrightarrow S$ which shows that R is a direct summand of S. $\square$

Definition 4.6. *Let R be a local? ring, M an R-module, and $(x_i)_{i=1}^n$ an R-regular sequence on M. Let V denote the free abelian group with formal generators $\{x_i\}_{i=1}^n$. The Koszul complex of the sequence is then defined to be*

$$0 \longrightarrow M \otimes_{\mathbb{Z}} \bigwedge^n V \xrightarrow{\partial_n} M \otimes_{\mathbb{Z}} \bigwedge^{n-1} V \xrightarrow{\partial_{n-1}} \cdots$$

$$\cdots \xrightarrow{\partial_2} M \otimes_{\mathbb{Z}} \bigwedge^1 V \xrightarrow{\partial_1} M \longrightarrow 0.$$

Theorem 4.2. *The Koszul complex of a regular sequence is exact everywhere except in degree 0.*

Proof. The proof will be inductive on the length of the regular sequence. Let's first show that it holds when the length of the sequence is 1. Let M be our module, (x_1) our regular sequence, and $V = \mathbb{Z}x_1$. Then we get the following complex

$$0 \longrightarrow M \otimes_{\mathbb{Z}} \bigwedge^1 V \xrightarrow{\partial_1} M \longrightarrow 0$$

This is isomorphic to

$$0 \longrightarrow M \xrightarrow{x_1 \cdot -} M \longrightarrow 0$$

and since multiplication by x_1 is injective on M, this is exact in degree 1. Since it is 0 in all degrees except 0 and 1, it is exact there as well.

For the inductive step we will show a relationship between regular sequences of different lengths. Let $(x_i)_{i=1}^n$ be a regular sequence on M, and let $K(s)$ be the Koszul complex of the sequnece $(x_i)_{i=1}^s$. Then there is an exact sequence of complexes

$$0 \longrightarrow K(s) \xrightarrow{\iota} K(s+1) \xrightarrow{\pi} K(s)[-1] \longrightarrow 0$$

where $K(s)[-1]$ is $K(s)$ shifted one spot to the left. The short exacts equence is given by

$$\begin{tikzcd}
\cdots \arrow[r, "\partial_3^s"] & M \otimes_{\mathbb{Z}} \bigwedge^2 V_s \arrow[r, "\partial_2^s"] \arrow[d, "\iota_2"] & M \otimes_{\mathbb{Z}} \bigwedge^1 V_s \arrow[r, "\partial_1^s"] \arrow[d, "\iota_1"] & M \arrow[r] \arrow[d, equal] & 0 \arrow[d] \\
\cdots \arrow[r, "\partial_3^{s+1}"] & M \otimes_{\mathbb{Z}} \bigwedge^2 V_{s+1} \arrow[r, "\partial_2^{s+1}"] \arrow[d, "\pi_2"] & M \otimes_{\mathbb{Z}} \bigwedge^1 V_{s+1} \arrow[r, "\partial_1^{s+1}"] \arrow[d, "\pi_1"] & M \arrow[r] \arrow[d] & 0 \arrow[d] \\
\cdots \arrow[r, "\partial_2^s"] & M \otimes_{\mathbb{Z}} \bigwedge^1 V_s \arrow[r, "\partial_1^s"] & M \arrow[r] & 0 \arrow[r] & 0
\end{tikzcd}$$

where ι is just the inclusion and $\pi_{j+1}(m \otimes x_{i_1} \wedge \cdots \wedge x_{i_j} \wedge x_{s+1}) = m \otimes x_{i_1} \wedge \cdots \wedge x_{i_j}$, and 0 if x_{s+1} does not appear in the wedge product.

This short exact sequence induces a longe exact sequence on homology

$$\begin{tikzcd}
\cdots \arrow[r] & H_2(K(s+1)) \arrow[r] & H_1(K(s)) \arrow[r] & H_1(K(s)) \arrow[dlll, rounded corners] \\
H_1(K(s+1)) \arrow[r] & H_0(K(s)) \arrow[r] & H_0(K(s)) \arrow[r] & H_0(K(s+1)) \arrow[r] & 0
\end{tikzcd}$$

Now we assume by induction that $K(s)$ is exact in all non-zero degrees. Then we get the following long exact sequence

$$
\begin{tikzcd}
\cdots \arrow[r] & H_2(K(s+1)) \arrow[r] & 0 \arrow[r] & 0 \arrow[dlll, rounded corners] & \\
H_1(K(s+1)) \arrow[r] & M/I_sM \arrow[r, "x_{s+1}\cdot -"] & M/I_s \arrow[r] & M/I_{s+1}M \arrow[r] & 0
\end{tikzcd}
$$

where $I_k = \langle x_i \rangle_{i=1}^{k}$. Then since the sequence is exact we get that $H_i(K(s+1)) = 0$ whenever $i \geq 2$, and that $H_1(K(s+1))$ is isomorphic to the kernel of multiplication by x_{s+1}. Here we use the fact that the sequence is regular, which means that multiplication by x_{s+1} is injective on M/I_sM and therefor $H_1(K(s+1)) = 0$.

Then by induction we have shown that $H_i(K(n)) = 0$ for all $i \neq 0$, or in other words that the Koszul complex is exact in all degrees except 0. $\square$

I need to show that S^G is regular CM at some point.

Proposition 4.1. *Let R be a local regular ring (a local ring where the maximal ideal is generated by a regular sequence on R) with depth of R equaling it's Krull-dimension. If M is an MCM R-module, and N is a direct summand of M then N is also MCM.*

Proof. Let $(x_i)_{i=1}^{n}$ be the regular sequence on R, let $k = R/\langle x_i \rangle_{i=1}^{n}$ be the residual field. Then we have seen that the Koszul complex is a projective resolution of k. Since $\mathrm{Ext}_R^i(k, M)$ is 0 for all $i < n$, mapping $\mathrm{Hom}_R(-, M)$ $\square$

Appendices

A Representation theory

Definition A.1. *If R is a ring and M is an abelian group, we define a representation of R to be a ring-map, φ, from R to $\mathrm{End}(M)$. We say that M is a (left) R-module, and we write rm with $r \in R$ and $m \in M$ to mean $\varphi(r)(m)$. Similarly we define a right R-module if φ goes from R to $\mathrm{End}(M)^{op}$ and we write mr for $\varphi(r)(m)$.*

Definition A.2. *If G is a group and V a complex vectorspace, we define a representation of G to be a group-map, ρ, from G to $\mathrm{Aut}_{\mathbb{C}}(V)$. When ρ is infered we say that V is a representation of G and we write gv to mean $\rho(g)(v)$. Note that representations of G exactly corresponds to representations of the ring $\mathbb{C}G$ of formal linear combinations of elements of G with multiplication given by $\lambda g \cdot \lambda' g' = (\lambda \cdot \lambda')gg'$.*

Definition A.3. *If R is a ring and M_1 and M_2 are two modules we define their direct sum, $M_1 \oplus M_2$ to be the module consisting of all pairs (m_1, m_2)*

(usually written m_1+m_2), where addition and scalar multiplication is point-wise. If a non-zero module cannot be written as the direct sum of two non-zero modules we call it indecomposable.

Definition A.4. *A submodule is a subset of a module which is also a module. A non-zero module with no non-trivial proper submodules is called simple or irreducible[2].*

Theorem A.1. *(Schur's Lemma) Let G be a group and V and W be two irreducible representations of G. If $f : V \to W$ is a G-linear map then f is a 0 if V and W are not isomorphic, and a scaling of identity (up to change of basis) if they are isomorphic.*

Proof. Start by assuming f is non-zero. Then we will show that V and W are isomorphic. Since the image of f is a non-zero subrepresentation of W and W is irreducible, we have that $\operatorname{Im} f = W$ and f is surjective. Sicne the kernel of f is a proper subrepresentation of V we must have that the kernel is 0, and that f is injective. Thus f is an isomorphism. Now assume $f : V \to V$ is a G-linear map. then we want to show that f is simply a scaling of identity. Since f is a linear map on a complex vector space it must have at least one eigen value, say $\lambda \in \mathbb{C}$. Let v be in the eigenspace λ. Sicne $f(gv) = gf(v) = \lambda gv$ for all g in G we have that gv is also in the eigenspace. This means the eigenspace is a subrepresentation, and since V is irreducible it must equal all of V. This means that f is just scaling by λ. $\square$

Definition A.5. *We call a functor left exact if for any short exact sequence*

$$0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C$$

the image of the sequence under the functor is also exact. For example for any module M the functor $\operatorname{Hom}(M, -)$ is left exact. That is the sequnce

$$0 \longrightarrow \operatorname{Hom}(M, A) \xrightarrow{f\circ -} \operatorname{Hom}(M, B) \xrightarrow{g\circ -} C$$

is exact. Dually we call a functor right exact if short exact sequnces of the form

$$A \longrightarrow B \longrightarrow C \longrightarrow 0$$

is mapped to an exact sequence. A functor that is both left exact and right exact is called exact.

[2]The word simple is used for representations of rings while irreducible is used for representations of groups. Note that for finite groups irreducible and indecomposable are equivalent.

Definition A.6. *We say that a module, P, is* projective *if for any epimorphism $f : M \twoheadrightarrow N$, and any map $g : P \to N$, there is a map $\varphi : P \to M$ such that $f\varphi = g$. Said another way, the diagram below induces the dotted arrow making the diagram commute*

$$\begin{tikzcd}
& P \arrow[d, "g"] \arrow[dl, dotted, "\varphi"'] \\
M \arrow[r, two heads, "f"'] & N
\end{tikzcd}$$

Note that P being projective is equivalent to $\mathrm{Hom}(P, -)$ *being right exact (i.e. exact).*

Projective cover + radical is small for noetherian modules
indec proj = summand of ring = idempotent of ring
projective resolution
Ext + Tor

5 Random Thoughts I need to figure out

I R=S then they have the same depth meaning M is cohen macaulay iff it's projective dimension is 0 (Auslander-Buschsbauw), but that means its a direct summand of S as S(=R)-module, which make sense. If I can show that R = S/(f) fro some polynomial f, can I show R-direct summands of S have projective dimension 1 over S? Can I show R has depth depth(S)-1? Then I also need to prove Auslander-Buschsbauw... Need to show some relation between dimension and depth.

If P is indec finitely generated projective then it is direct summand of S^n, then P must either be a direct summand of S or $S^n - 1$ then by induction P is a summand of S. Can I assume P to be finitely generated?????

$P/mP = sumV_i - > P = sumSV_i$, means all projective $S\#G$-modules can be broken down into sums.

6 questions

Why does $0 \to J \to S \otimes_R S \to S \to 0$ split?

Is it true that $dimR \leq dimS$ (this is not true for $\mathbb{Z} \subset \mathbb{Q}$), alternatively how to show that S is MCM?

Direct summand of MCM is MCM? Use Koszul complex + Ext preserves direct sums.

Are indec proj $S\#G$-modules fin.gen.? Can I state the correspondance in terms of fin.gen, indec projctives instead?

7 Disposisjon

Define McKay quiver [check]

Define $S\#G$ [check]

Correspondance with projectives [put in finitely generated to fix argument]

Gabriel Quiver [Make Koszul complex a refference]

$\mathrm{End}_R(S) \cong S\#G$ [understand the proof] $q \in S$ height one prime implies $q = (f)$ for a homogenous polynomial? Why homogenous? If $T(q)$ is non-trivial then it acts non-trivially on S/qS if f has degree bigger than 1, then all degree 1 polynomials survive in S/qS and are acted upon trivialy by $T(q)$. Therefor $T(q)$ would be trivial, so f is homogenous of degree 1. Since the group operations preseve degree $\sigma(f) = a_\sigma f$ for a nonzero constant a_σ. All finite matrix groups diagonalizeable implies $\sigma = diag(1, 1, \cdots a_\sigma)$. Therefor $T(q)$ is iso to finite subgroup of $\mathbb{C}^*$, hence cyclic. Then $p = q \cap R = (f^n)$ where n is the order of $T(q)$. Thus $q = pS_q$ if and only if $T(q)$ is trivial. $I/I^2 = \Omega_{S|R}$ is 0 iff $pS = q$, $I/I^2 = 0$ implies idempotent implies spliting. Why is $\mathrm{End}_R(S)$ reflexive? or rather why does height one iso imply iso.

MCM R-summands of S [S is MCM using dimension argument, summands are MCM using depth $\leq$ dim]

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