McKay correspondence

Jacob Fjeld Grevstad

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Abstract

The goal of this thesis is to establish a 1-1 correspondence between quivers created from the four following sets whenever S is the power series ring $\mathbb{C}[\![x, y]\!]$ and G is a finite subgroup of $SL(2, \mathbb{C})$ acting on S

- The Maximal Cohen-Macaulay modules of the fixed ring S^G.
- The indecomposable projective modules of the skew group algebra $S\#G$.
- The indecomposable projective modules of $\mathrm{End}_{S^G}(S)$.
- The irreducible representations of G (indecomposable $\mathbb{C}G$-modules).

Much of the thesis will be used to define these four quivers and to develope tools to establish such a correspondence. A similar correspondence can be established for a general field k and a finite subgroup of $GL(n, k)$ with order nonzero in k, but in the general case we will only attain the MCM-modules that apear as S^G-direct summands of S. $SL(2, \mathbb{C})$ is also especially interesting because the quivers are exactly the Dynkin diagrams.

Contents

1 Finite subgroups of SL(2,C)

2 Characters and irreducible representations

This section is largely based on the book by [James and Liebeck, 2001].

Recall that the trace of a matrix is defined to be the sum of its diagonal elements and that the trace satisfies two important equations. Namely

$$tr(A+B) = tr(A) + tr(B) \text{ and } tr(AB) = tr(BA)$$

For a given representation of G, $\rho : G \to GL_n(\mathbb{C})$, we define its character by $\chi_\rho : G \to \mathbb{C}$, $\chi_\rho(g) = tr(\rho(g))$.

Proposition 2.1. *Conjugate elements in G take the same value under a character.*

Proof. Let g and g' be in the same conjugacy class. Then there exists an element h such that $h^{-1}gh = g'$. Then we have

$$\chi(g') = \chi(h^{-1}gh) = tr(\rho(h)^{-1}\rho(g)\rho(h)) \overset{*}{=} tr(\rho(g)\rho(h)\rho(h)^{-1}) = tr(\rho(g)) = \chi(g)$$

In (*) we use the fact that $tr(AB) = tr(BA)$. □

Lemma 2.1. *For a finite abelian group G any irreducible representation must be 1-dimensional.*

Proof. Let $\rho : G \to GL(V)$ be an irreducible representation. Since G is abelian we have that $\rho(g)\rho(h)v = \rho(h)\rho(g)v$. Thus multiplication by $\rho(g)$ respects the action of G and we have that $\rho(g)$ is a homomorphism of G-representations between ρ and itself. Then by Schur's lemma[1] $\rho(g)$ must be a scalar multiplication. In other words every matrix $\rho(g)$ for $g \in G$ is diagonal (it is a scaling of identity). This implies that ρ can be written as a direct sum of 1-dimensional representations, but since ρ is irreducible ρ must be 1-dimensional. □

Proposition 2.2. *If χ is the character of a representation, ρ, with dimension n of a group G, and g is an element of G with order m, then the following holds*

[1]Statement and proof of Schur's lemma can be found in the appendix on page 12 as theorem A.1.

(1) $\chi(1) = n$

(2) $\chi(g)$ *is the sum of* m*-th roots of unity.*

(3) $\chi(g^{-1}) = \overline{\chi(g)}$

Proof.

(1) The first result is immediate.

$$\chi(1) = tr\left(\begin{bmatrix} 1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 1 \end{bmatrix}\right) = n$$

(2) Since $\langle g \rangle$ is an abelian group, ρ decomposes into n 1-dimensional $\langle g \rangle$-representations. Then there is a basis such that $\rho(g)$ is diagonal. Since g has order m it follows that the diagonal entries of $\rho(g)$ must be m-th roots of unity. Thus $\chi(g) = tr(\rho(g))$ must be the sum of m-th roots of unity.

(3) Using the same basis as above and the fact that $\omega^{-1} = \overline{\omega}$ when ω is a root of unity we see that $\chi(g^{-1}) = tr(\rho(g)^{-1}) = \overline{tr(\rho(g))} = \overline{\chi(g)}$.

□

3 The McKay quiver

Definition 3.1. *Let G be a finite subgroup of $GL(n, \mathbb{C})$, and let V be the canonical representation (the one that sends g to g). Then we define the* <u>*McKay quiver*</u> *of G to be the quiver with verticies the irreducible representations of G, denoted V_i. For two irreducible representations V_i and V_j there is an arrow from the former to the latter if and only if V_j is a direct summand of $V \otimes V_i$.*

Example 3.1. *Let G be the group generated by $g = \begin{bmatrix} \omega^2 & 0 \\ 0 & \omega^3 \end{bmatrix}$, where ω is a primitive fifth root of unity. Then there are five different irreducible representations, the one sending g to ω, ω^2, ω^3, ω^4 respectively, and the trivial representation. Denote the representation sending g to ω^i by V_i, and let $V = V_2 \oplus V_3$ be the cannonical representation. Note that $V_i \otimes V_j = V_{i+j}$, where $i + j$ is understood to be modulo 5. Then we get the following McKay-quiver*

$$\begin{tikzcd}
 & V_0 \arrow[ddl, leftrightarrow] \arrow[ddr, leftrightarrow] & \\
V_4 \arrow[rr, leftrightarrow] \arrow[drr, leftrightarrow] & & V_1 \arrow[dll, leftrightarrow] \\
V_3 & & V_2
\end{tikzcd}$$

4 Krull-Remack-Schmidt

This section is largely based on the book by [Leuschke and Wiegand, 2012]. Here we will prove the Krull-Remack-Schmidt theorem for complete local noetherian rings.

We say a ring satisfies Krull-Remack-Schmidt if the following condition holds:

(i) Any finitely generated module can be written as the finite direct sum of indecomposable modules.

(ii) If
$$\bigoplus_{i=1}^{m} M_i \cong \bigoplus_{j=1}^{n} N_j$$
for indecomposable M_i's and N_j's, then $m = n$ and there is a permutation, $\sigma \in S_n$, such that $M_i \cong N_{\sigma(i)}$ for all $i = 1, 2, \cdots, n$.

It's clear that (i) holds for any noetherian ring, since any decomposition of a noetherian module must eventually reach an indecomposable. In this chapter we will focus on proving (ii).

5 Skew group algebra S#G indecomposable projectives

This section is largely based on the book by [Leuschke and Wiegand, 2012]. This section will use definitions and theorems from representation theory as taught in the courses MA3203 - Ring Theory and MA3204 - homological algebra. Since I do not assume knowledge of this I have created appendix A. I will try to use footnotes to indicate where such theorems are used.

Definition 5.1. *If G is a subgroup of $GL_n(\mathbb{C})$, we can extend the group action of G to $\mathbb{C}[\![x_1, \cdots, x_n]\!]$. More explicitely G acts on x_i as it would the ith basis vector of $\mathbb{C}^n$, and acts on products and sums by acting on each component seperatley. We then define the skew group algebra $\mathbb{C}[\![x_1, \cdots, x_n]\!]\#G$ to be the algebra generated by elements of the form $f \cdot g$ with $f \in \mathbb{C}[\![x_1, \cdots, x_n]\!]$ and $g \in G$, and we define the multiplication by*

$$(f_1 \cdot g_1) \cdot (f_2 \cdot g_2) = (f_1 \cdot f_2^{g_1}) \cdot (g_1 \cdot g_2)$$

Where f^g denotes the image of f under the action of g.

Theorem 5.1. *We have an isomorphism of rings*

$$e\mathbb{C}[\![x, y]\!]\#Ge \simeq \mathbb{C}[\![x, y]\!]^G$$

where $e = \frac{1}{|G|} \sum_{g \in G} g$.

Proof. Let f^g denote the image of f under the action of g. Then if we let $f(x,y)g$ be an element of the skew algebra we get that $ef(x,y)ge = f(x,y)^e \cdot ege = f(x,y)^e \cdot e = e \cdot f(x,y)$. It then follows that $e\mathbb{C}[\![x,y]\!]\#Ge$ is isomorphic to the image of $\mathbb{C}[\![x,y]\!]$ under the action of e. Since $ge = g$ for all $g \in G$ it is clear that the image of e is contained in the fixed ring. For the converse you just need to notice that the fixed ring is fixed under e and thus is contained in the image. $\square$

Lemma 5.1. *Let $S = \mathbb{C}[\![x,y]\!]$. An $S\#G$-module is projective if and only if it is projective as an S-module.*

Proof. Onlyifity follows from $S\#G$ being a free S-module, it is isomorphic to $\bigoplus_{g\in G} S$. Thus we need only show ifity.

First we need to see that an $S\#G$-linear map is just an S-linear map such that $f(g(m)) = g(f(m))$ for all $g \in G$. Equivalently $f(m) = g(f(g^{-1}(m)))$. This allows us to define a group action on S-linear maps by $f^g(m) = g(f(g^{-1}(m)))$. Then we can restate it as

$$\mathrm{Hom}_{S\#G}(M,N) = \mathrm{Hom}_S(M,N)^G$$

Clearly if f is $S\#G$-linear then it's in $\mathrm{Hom}_S(M,N)^G$. To see the other inclusion, let f be an S-linear map that is fixed under G. Then $f(s \cdot gm) = sf(gm) = s \cdot g(f(g^{-1}gm)) = s \cdot gf(m)$, and hence f is $S\#G$-linear. Nextly I want to show that $-^G$ is an exact functor.

If K is the kernel of a map $f : M \to N$, then the kernel of the induced map $f^G : M^G \to N^G$ is of course just $K \cap M^G$ which equals K^G. Assume f is epi and let $n \in N^G$. Consider a preimage m such that $f(m) = n$. Let $\theta = \frac{1}{|G|}\sum_{g\in G} g(m)$. Then θ is in M^G and $f(\theta) = \frac{1}{|G|}\sum_{g\in G} g(f(m)) = \frac{1}{|G|}\sum_{g\in G} n = n$.

This implies that if $\mathrm{Hom}_S(P,-)$ is exact then $\mathrm{Hom}_S(P,-)^G = \mathrm{Hom}_{S\#G}(P,-)$ is exact and our lemma follows. $\square$

Lemma 5.2. *Let S be the complex power series ring in n variables, and $\mathfrak{m} = \langle x_i \rangle_{i=1}^n$ the radical of S. Then for any free S-module N, $\mathfrak{m}N$ is small in N. That is if X is a submodule of N such that $X + \mathfrak{m}N = N$, then $X = N$.*

Proof. Let N be the free module $S^{(I)} := \bigoplus_{i\in I} S_i$, where $S_i \cong S$. Assume that X is a submodule such that $X + \mathfrak{m}N = N$. We denote by 1_i the elements that is 1 at index i and 0 elsewhere. Since $\{1_i\}$ generate N, it is enough to show that X contains all of them. Since $X + \mathfrak{m}N = N$, we know that there is an $m_i \in \mathfrak{m}N$ and an $x_i \in X$ such that $x_i + m_i = 1_i$. Then we have that $x_i = 1_i - m_i$. Since the power series at index i of x_i has constant coefficient 1 it is invertible. If we multiply x_i by its inverse we get $\tilde{x}_i$ which is 1 at index i and some eleement of $\mathfrak{m}$ at index $j \neq i$, say m_{ij}. Then $\tilde{x}_i - \sum_{j \neq i} m_{ij}\tilde{x}_j$ has a unit in index i and 0 at all other indicies. Thus X contains 1_i for all i, and $X = N$. $\square$

Theorem 5.2. *Let $S = \mathbb{C}[\![x, y]\!]$ and let $\mathfrak{m} = \langle x, y\rangle_S$ be the radical of S. Then there are bijections between the indecomposable projective $S\#G$-modules and the indecomposable $\mathbb{C}G$-modules given by*

$$\begin{Bmatrix} \textit{indecomposable projective} \\ S\#G\textit{-modules} \end{Bmatrix} \qquad \begin{Bmatrix} \textit{indecomposable} \\ \mathbb{C}G\textit{-modules} \end{Bmatrix}$$

$$\mathcal{F} : P \longmapsto P/\mathfrak{m}P$$

$$\mathcal{G} : S \otimes_{\mathbb{C}} W \longleftarrow\!\shortmid W$$

Where the $S\#G$-module structure on $S \otimes_{\mathbb{C}} W$ is given by $(s \cdot g) \cdot f \otimes v = sf^g \otimes v^g$.

Proof. First we should show that $S \otimes_{\mathbb{C}} W$ is an indecomposable projective $S\#G$-module and that $P/\mathfrak{m}P$ is infact an indecomposable $\mathbb{C}G$-module. Since $S \otimes_{\mathbb{C}} W$ is a free S-module it follows from lemma 5.1 that it is projective. To see that it is indecomposable we will first study it as an S-module and exploit the fact that $\mathrm{Hom}_{S\#G}(M, N) \subseteq \mathrm{Hom}_S(M, N)$.

Since $\mathfrak{m}$ is the radical of S we have that

$$\frac{S \otimes_{\mathbb{C}} W}{\mathfrak{m}S \otimes_{\mathbb{C}} W} \cong S/\mathfrak{m} \otimes_{\mathbb{C}} W \cong \mathbb{C} \otimes_{\mathbb{C}} W \cong W$$

W is the top of $S \otimes_{\mathbb{C}} W$. Further since the projection $S \otimes_{\mathbb{C}} W \to W$ is $S\#G$-linear we have that $S \otimes_{\mathbb{C}} W$ is the projective cover of W also as $S\#G$-modules. Then since W is simple it follows that $S \otimes_{\mathbb{C}} W$ is an indecomposable projective. Does it follow?

It's clear that $P/\mathfrak{m}P$ is a $\mathbb{C}G$-module, because $\mathbb{C}G$ is a subring of $S\#G$. To see that it's indecomposable we will first show that it's indecomposable as an $S\#G$-module. By considering P and $P/\mathfrak{m}P$ as S-modules and using the same argument as above we see that P is the projective cover of $P/\mathfrak{m}P$. Then since P is indecomposable $P/\mathfrak{m}P$ Is this true in general? must also be indecomposable as an $S\#G$-module.

To see that this implies $P/\mathfrak{m}P$ is indecomposable as a $\mathbb{C}G$-module notice that $P/\mathfrak{m}P$ is annihilated by the ideal $\langle\mathfrak{m}\rangle$. This means it's an indecomposable as $S\#G$-module if and only if it's indecomposable as an $S\#G/\langle\mathfrak{m}\rangle$-module. Then since $S\#G/\langle\mathfrak{m}\rangle \cong \mathbb{C}G$ it follows that $P/\mathfrak{m}P$ is an indecomposable $\mathbb{C}G$-module.

To see that the given maps are bijections we will show that they are mutual inverses. First to see that $\mathcal{F}(\mathcal{G}(W)) \cong W$ we simply look at the definition

$$\frac{S \otimes_{\mathbb{C}} W}{\mathfrak{m}S \otimes_{\mathbb{C}} W} \cong S/\mathfrak{m} \otimes_{\mathbb{C}} W \cong \mathbb{C} \otimes_{\mathbb{C}} W \cong W$$

Next we consider $\mathcal{G}(\mathcal{F}(P)) = S \otimes_{\mathbb{C}} P/\mathfrak{m}P$. We have already seen that it's projective. Both P and $S \otimes_{\mathbb{C}} P/\mathfrak{m}P$ have a natural projection onto $P/\mathfrak{m}P$, and by projectivity we get an induced $S\#G$-linear map from $S \otimes_{\mathbb{C}} P/\mathfrak{m}P$ to P:

$$\begin{tikzcd} & S \otimes_{\mathbb{C}} P/\mathfrak{m}P \arrow[dl, dashed] \arrow[d, two heads] \\ P \arrow[r, two heads] & P/\mathfrak{m}P \end{tikzcd}$$

Now by lemma 5.2 we have that $\mathfrak{m}P$ is small and that therefor the induced map is an epimorphism. Similarly we get an epimorphism in the other direction. Since $S \otimes_{\mathbb{C}} P/\mathfrak{m}P$ has finite S-rank (P finitely generated Why is P finitely generated? This would be true if P/mP is indec CG-module, so if thats clear this should be clear), it follows that the map is an isomorphism of S-modules. Since the map is $S\#G$-linear it is tehrefor also an isomorphism of $S\#G$-modules. $\square$

5.1 The Gabriel quiver

Definition 5.2. *For a skew group algebra $S\#G$ we define its Gabriel quiver to be the quiver with verticies as the indecomposable projective modules of $S\#G$. The arrows are given by taking the minimal projective resolution of $P/\mathfrak{m}P$, where $\mathfrak{m}$ is as defined above. If the minimal projective resolution of $P/\mathfrak{m}P$ is given by*

$$\cdots \longrightarrow Q_1 \longrightarrow Q_0 \longrightarrow 0$$

We say there is an arrow from P to P' if P' appears as a direct summand of Q_1.

Definition 5.3. *Let V be a vector space. We then define the exterior algebra $\bigwedge V$ as the associative unital graded algebra such that the multiplication is bilinear and satisfies $x \wedge y = -y \wedge x$ for any x and y in V.*

Some key properties of the exterior algebra is that $x \wedge x = 0$, and more generally that $x_1 \wedge \cdots \wedge x_p = 0$ whenever $\{x_i\}_{i=1}^{p}$ are linearly dependent.

The pth exterior power of V, denoted $\bigwedge^p V$ is the vector space of all elements that are the product of p vectors in V. If $\{x_i\}_{i=1}^{n}$ is a basis for V, then $x_{i_1} \wedge \cdots \wedge x_{i_p}$ where $i_1 < i_2 < \cdots < i_p$ and $1 \leq i_j \leq n$ forms a basis for $\bigwedge^p V$, thus it is $\binom{n}{p}$-dimensional.

Proposition 5.1. *If S is the ring of formal power series over $\mathbb{C}$ in n variables, and G is a finite group acting on S, let $V = \mathfrak{m}/\mathfrak{m}^2$. Then the minimal projective resolution of $\mathbb{C} \cong S/\mathfrak{m}$ is given by*

$$0 \longrightarrow S \otimes_{\mathbb{C}} \bigwedge^{n} V \xrightarrow{\partial_n} \cdots \xrightarrow{\partial_2} S \otimes_{\mathbb{C}} \bigwedge^{1} V \xrightarrow{\partial_1} S \longrightarrow 0$$

Where ∂_p is the $S\#G$-linear map defined by

$$\partial_p(s \otimes x_{i_1} \wedge x_{i_2} \wedge \cdots \wedge x_{i_p}) = \sum_{j=1}^{p} (-1)^{j+1} s x_{i_j} \otimes x_{i_1} \wedge \cdots \wedge \hat{x}_{i_j} \wedge \cdots \wedge x_{i_p}$$

Where $x_{i_1} \wedge x_{i_2} \wedge \cdots \wedge x_{i_p}$ is one of the standard basis vectors for $\bigwedge^{n} V$, namely $i_1 < i_2 < \cdots < i_p$, and $\hat{x}_j$ means that x_j is ommited.

Proof. First we should show that this is a projective resolution. Note that since the maps are $S\#G$-linear, showing that it's a minimal free resolution as an S-module implies it is a minimal projective resolution as an $S\#G$-module. Then what we need to show is

(i) $\operatorname{Cok} \partial_1 = \mathbb{C}$

(ii) $\partial_{p-1} \circ \partial_p = 0$ for all p

(iii) $\operatorname{Im} \partial_{p+1} = \operatorname{Ker} \partial_p$ for $p \geq 2$

(i) is clear since the image of ∂_1 is $\mathfrak{m}$. (ii) can be shown through a quick computation

$$\partial_{p-1} \circ \partial_p(s \otimes x_{i_1} \wedge \cdots \wedge x_{i_p}) =$$
$$\partial_{p-1} \left(\sum_{j=1}^{p} (-1)^{j+1} s x_{i_j} \otimes x_{i_1} \wedge \cdots \hat{x}_j \wedge \cdots \wedge x_{i_p} \right) =$$
$$\sum_{j=1}^{p} (-1)^{j+1} \left(\sum_{k=1}^{j-1} (-1)^{k+1} s x_{i_j} x_{i_k} \otimes x_{i_1} \wedge \cdots \hat{x}_{i_k} \wedge \cdots \wedge \cdots \hat{x}_j \wedge \cdots \wedge x_{i_p} + \right.$$
$$\left. \sum_{k=j+1}^{p} (-1)^{k} s x_{i_j} x_{i_k} \otimes x_{i_1} \wedge \cdots \hat{x}_{i_j} \wedge \cdots \wedge \cdots \hat{x}_k \wedge \cdots \wedge x_{i_p} \right)$$

From here we notice that the term with $j < k$ is canceled by the term where $k < j$, because they are the negatives of each other. Thus the composition is 0. This would then imply that $\operatorname{Im} \partial_{p+1} \subseteq \operatorname{Ker} \partial_p$, so for part (iii) we need only show that $\operatorname{Ker} \partial_p \subseteq \operatorname{Im} \partial_{p+1}$.

First some notation: let $\mathfrak{I}_p$ be the set of all tuples $(i_1, i_2, \cdots, i_p)$ with $i_1 < i_2 < \cdots < i_p$ and $1 \leq i_j \leq n$, and let x_I denote $x_{i_1} \wedge \cdots x_{i_p}$ when $I = (i_1, \cdots, i_p)$. Then assume

$$\sum_{I \in \mathfrak{I}_p} s_I \otimes x_I$$

is in the kernel of ∂_p.

Maybe just prove n=2, its simpler.... proof by induction on n on wikipedia
Maybe refferencing the proof will be better... or make an appendix on homological algebra

Secondly we want to show that the resolution is minimal. To do this it is enough to show that for each $k \geq 1$ ∂_k is a projective cover of its image, and that $S \to k$ is a projective cover. To this all we have to show is that the kernels of the maps are small. Since $\operatorname{Im} \partial_{k+1} = \operatorname{Ker} \partial_k$ and $\operatorname{Im} \partial_{k+1} \subseteq \mathfrak{m} \otimes_{\mathbb{C}} \bigwedge^{k+1} V$ it follows from lemma 5.2 that the resolution is minimal. $\square$

Theorem 5.3. *If S is the complex power series ring in n variables and G is a fintie subgroup of $GL_n(\mathbb{C})$, then the McKay quiver of G and the Gabriel quiver of $S\#G$ are isomorphic.*

Proof. We have already seen that they have the same vertices, namely if V_i are the irreducible representations of G, then $S \otimes_{\mathbb{C}} V_i$ are the indecomposable projectives of $S\#G$. To see that they have the same arrows consider as above the minimal resolution of $\mathbb{C}$.

$$0 \longrightarrow S \otimes_{\mathbb{C}} \bigwedge^n V \xrightarrow{\partial_n} \cdots \xrightarrow{\partial_2} S \otimes_{\mathbb{C}} \bigwedge^1 V \xrightarrow{\partial_1} S \longrightarrow 0$$

If we tensor with V_i on the right we will get a minimal resolution of V_i (you can see that is minimal by using the exact same argumant above and prove that $\operatorname{Tor}^n_S(\mathbb{C}, V_i)$ is non-zero).

$$\cdots \xrightarrow{\partial_2 \otimes_{\mathbb{C}} V_i} S \otimes_{\mathbb{C}} \bigwedge^1 V \otimes_{\mathbb{C}} V_i \xrightarrow{\partial_1 \otimes_{\mathbb{C}} V_i} S \otimes_{\mathbb{C}} V_i \longrightarrow 0$$

From here, since $\bigwedge^1 V = V$, we see that $P_j = S \otimes_{\mathbb{C}} V_j$ appears as a direct summand of $S \otimes_{\mathbb{C}} V \otimes_{\mathbb{C}} V_i$ exactly when V_j appears as a direct summand of $V \otimes_{\mathbb{C}} V_i$. $\square$

6 The endomorphism ring of S as an S^G-module

Isomorphism to $S\#G$ implies proj SG ¡-¿ S^G direct summands of S.

Theorem 6.1. *Let S be the complex power series ring in two variables, G be a finite subgroup of $GL_2(\mathbb{C})$, $R = S^G$ the fixed ring of S under the action of G, and $(S\#G)^G$ be the fixed ring of $S\#G$ under left multiplication by G. Then S is isomorphic to $(S\#G)^G$ as R-modules.*

Proof. To see this we will define an injective R-linear map from S to $S\#G$ and show that it's image is $(S\#G)^G$. Let $\rho : S \to S\#G$ be given by

$$\rho(s) = \sum_{g \in G} s^g \cdot g.$$

It's clear that it's injective and it is R-linear because

$$\rho(rs) = \sum_{g \in G} r^g s^g \cdot g = r \sum_{g \in G} s^g \cdot g.$$

It should also be clear that the image is contained in $(S\#G)^G$ because

$$h \cdot \rho(s) = \sum_{g \in G} h \cdot s^g \cdot g = \sum_{g \in G} s^{hg} \cdot hg = \rho(s).$$

To see that the image is all of $(S\#G)^G$ consider an arbitrary element in $(S\#G)^G$, $\psi = \sum_{g \in G} s_g \cdot g$. Since ψ is fixed under left multiplication by G we must have that

$$\sum_{g \in G} s_g^h \cdot hg = \sum_{g \in G} s_g \cdot g,$$

in particular s_h must equal s_1^h and it follows that $\psi = \rho(s_1)$. □

Theorem 6.2.

7 Maximal Cohen-Macaulay modules of S^G

Definition 7.1. *If R is a local ring with residual field k we define the depth of a module, M, to be the minimal n such that the extension $\mathrm{Ext}_R^n(k, M)$ is non-zero.*

Definition 7.2. *If R is a commutative ring and M is an R-module, an R-regular sequence on M is a sequence of elements of R, $r_1, r_2, \cdots r_n$ such that $M/\langle r_1, \cdots, r_i\rangle M$ is non-zero and multiplication by r_i is injective on $M/\langle r_1, \cdots, r_{i-1}\rangle M$.*

Definition 7.3. *If R is a ring, we say that $\mathfrak{p}$ is a prime ideal in R if*

1. *$\mathfrak{p}$ is a proper ideal of R.*

2. *For any two elements $a, b \in R$ such that $ab \in \mathfrak{p}$ we must have that either a is in $\mathfrak{p}$ or b is.*

Definition 7.4. *If R is a ring we define its Krull-dimesnion to be the maximum length of a chain of prime ideals in R. For example the polynomial ring $\mathbb{C}[x_1, \cdots, x_n]$ has Krull-dimension n given by the chain*

$$0 \subseteq \langle x_1 \rangle \subseteq \langle x_1, x_2 \rangle \subseteq \cdots \subseteq \langle x_1, \cdots, x_n \rangle$$

Definition 7.5. *If M is a module over a local ring R with Krull-dimension d we say that M is maximal Cohen Macaulay (MCM) if the depth of M equals d.*

Theorem 7.1. *If G is a finite subgroup of $GL_n(\mathbb{C})$, S is the complex power series ring in n variables and $R = S^G$ is the ring fixed under the action of G, then R is a direct summand of S as R-modules.*

Proof. Consider the map $\pi : S \to R$ given by

$$\pi(s) = \frac{1}{|G|} \sum_{g \in G} s^g$$

It's clear that the image of π is in R because an action from G will just permute the order of the sum. Further

$$\pi(r) = \frac{1}{|G|} \sum_{g \in G} r^g = \frac{1}{|G|} \sum_{g \in G} r = r,$$

so π splits the inclusion $R \hookrightarrow S$ which shows that R is a direct summand of S. $\square$

Appendices

A Representation theory

Definition A.1. *If R is a ring and M is an abelian group, we define a representation of R to be a ring-map, φ, from R to $\mathrm{End}(M)$. We say that M is a (left) R-module, and we write rm with $r \in R$ and $m \in M$ to mean $\varphi(r)(m)$. Similarly we define a right R-module if φ goes from R to $\mathrm{End}(M)^{op}$ and we write mr for $\varphi(r)(m)$.*

Definition A.2. *If G is a group and V a complex vectorspace, we define a representation of G to be a group-map, ρ, from G to $\mathrm{Aut}_{\mathbb{C}}(V)$. When ρ is infered we say that V is a representation of G and we write gv to mean $\rho(g)(v)$. Note that representations of G exactly corresponds to representations of the ring $\mathbb{C}G$ of formal linear combinations of elements of G with multiplication given by $\lambda g \cdot \lambda' g' = (\lambda \cdot \lambda') g g'$.*

Definition A.3. *If R is a ring and M_1 and M_2 are two modules we define their direct sum, $M_1 \oplus M_2$ to be the module consisting of all pairs (m_1, m_2) (usually written $m_1 + m_2$), where addition and scalar multiplication is pointwise. If a non-zero module cannot be written as the direct sum of two non-zero modules we call it indecomposable.*

Definition A.4. *A submodule is a subset of a module which is also a module. A non-zero module with no non-trivial proper submodules is called simple or irreducible[2].*

Theorem A.1. *(Schur's Lemma) Let G be a group and V and W be two irreducible representations of G. If $f : V \to W$ is a G-linear map then f is a 0 if V and W are not isomorphic, and a scaling of identity (up to change of basis) if they are isomorphic.*

Proof. Start by assuming f is non-zero. Then we will show that V and W are isomorphic. Since the image of f is a non-zero subrepresentation of W and W is irreducible, we have that $\operatorname{Im} f = W$ and f is surjective. Sicne the kernel of f is a proper subrepresentation of V we must have that the kernel is 0, and that f is injective. Thus f is an isomorphism. Now assume $f : V \to V$ is a G-linear map. then we want to show that f is simply a scaling of identity. Since f is a linear map on a complex vector space it must have at least one eigen value, say $\lambda \in \mathbb{C}$. Let v be in the eigenspace λ. Sicne $f(gv) = gf(v) = \lambda gv$ for all g in G we have that gv is also in the eigenspace. This means the eigenspace is a subrepresentation, and since V is irreducible it must equal all of V. This means that f is just scaling by λ. $\square$

Definition A.5. *We call a functor left exact if for any short exact sequence*

$$0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C$$

the image of the sequence under the functor is also exact. For example for any module M the functor $\operatorname{Hom}(M, -)$ is left exact. That is the sequnce

$$0 \longrightarrow \operatorname{Hom}(M, A) \xrightarrow{f\circ -} \operatorname{Hom}(M, B) \xrightarrow{g\circ -} C$$

is exact. Dually we call a functor right exact if short exact sequnces of the form

$$A \longrightarrow B \longrightarrow C \longrightarrow 0$$

is mapped to an exact sequence. A functor that is both left exact and right exact is called exact.

[2]The word simple is used for representations of rings while irreducible is used for representations of groups. Note that for finite groups irreducible and indecomposable are equivalent.

Definition A.6. *We say that a module, P, is* <u>projective</u> *if for any epimorphism $f : M \twoheadrightarrow N$, and any map $g : P \to N$, there is a map $\varphi : P \to M$ such that $f\varphi = g$. Said another way, the diagram below induces the dotted arrow making the diagram commute*

$$\begin{tikzcd} & P \arrow[d, "g"] \arrow[dl, dotted, "\varphi"'] \\ M \arrow[r, two heads, "f"'] & N \end{tikzcd}$$

Note that P being projective is equivalent to $\operatorname{Hom}(P, -)$ *being right exact (i.e. exact).*

Projective cover + radical is small for noetherian modules
indec proj = summand of ring = idempotent of ring
projective resolution
Ext + Tor

8 Random Thoughts I need to figure out

I R=S then they have the same depth meaning M is cohen macaulay iff it's projective dimension is 0 (Auslander-Buschsbauw), but that means its a direct summand of S as S(=R)-module, which make sense. If I can show that R = S/(f) fro some polynomial f, can I show R-direct summands of S have projective dimension 1 over S? Can I show R has depth depth(S)-1? Then I also need to prove Auslander-Buschsbauw... Need to show some relation between dimension and depth.

If P is indec finitely generated projective then it is direct summand of S^n, then P must either be a direct summand of S or $S^n - 1$ then by induction P is a summand of S. Can I assume P to be finitely generated?????

$P/mP = sumV_i- > P = sumSV_i$, means all projective $S\#G$-modules can be broken down into sums.

9 questions

Why does $0 \to J \to S \otimes_R S \to S \to 0$ split?

Is it true that $dimR \leq dimS$ (this is not true for $\mathbb{Z} \subset \mathbb{Q}$), alternatively how to show that S is MCM?

Direct summand of MCM is MCM?

Is $\mathfrak{m}P$ small for all fin.gen. S-modules P? What about projectives? If $\mathfrak{m}^{(I)}$ is small in $S^{(I)}$, then $\pi(\mathfrak{m}S^{(I)}) = \mathfrak{m}P$ is small in P.

Are indec proj $S\#G$-modules fin.gen.?

References

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