McKay correspondence

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Abstract

The goal of this thesis is to establish a 1-1 correspondence between quivers created from the four following sets whenever G is a finite subgroup of $SL(2, \mathbb{C})$ and S is the power series ring $\mathbb{C}[\![x, y]\!]$

- The Maximal Cohen-Macaulay modules of the fixed ring S^G.
- The indecomposable projective modules of the skew group algerba $S\#G$.
- The indecomposable projective modules of $\mathrm{End}_{S^G}(S)$.
- The irreducible representations of G (indecomposable $\mathbb{C}G$-modules).

Much of the thesis will be used to define these four quivers and to develope tools to establish such a correspondence. A similar correspondence can be established for a general field k and a finite subgroup of $GL(n, k)$ with order nonzero in k, but the case for $SL(2, \mathbb{C})$ is the most interesting as the quivers will be extended Dynkin diagrams.

Contents

1 Finite subgroups of SL(2,C)

2 Characters and irreducible representations

This section is largely based on the book by [James and Liebeck, 2001].

Recall that the trace of a matrix is defined to be the sum of its diagonal elements and that the trace satisfies two important equations. Namely

$$tr(A+B) = tr(A) + tr(B) \text{ and } tr(AB) = tr(BA)$$

For a given representation of G, $\rho : G \to GL_n(\mathbb{C})$, we define its character by $\chi_\rho : G \to \mathbb{C}$, $\chi_\rho(g) = tr(\rho(g))$.

Proposition 2.1. *Conjugate elements in G take the same value under a character.*

Proof. Let g and g' be in the same conjugacy class. Then there exists an element h such that $h^{-1}gh = g'$. Then we have

$$\chi(g') = \chi(h^{-1}gh) = tr(\rho(h)^{-1}\rho(g)\rho(h)) \overset{*}{=} tr(\rho(g)\rho(h)\rho(h)^{-1}) = tr(\rho(g)) = \chi(g)$$

In (*) we use the fact that $tr(AB) = tr(BA)$. □

Lemma 2.1. *For a finite abelian group G any irreducible representation must be 1-dimensional.*

Proof. Let $\rho : G \to GL(V)$ be an irreducible representation. Since G is abelian we have that $\rho(g)\rho(h)v = \rho(h)\rho(g)v$. Thus multiplication by $\rho(g)$ respects the action of G and we have that $\rho(g)$ is a homomorphism of G-representations between ρ and itself. Then by Schur's lemma[1] $\rho(g)$ must be a scalar multiplication. In other words every matrix $\rho(g)$ for $g \in G$ is diagonal (it is a scaling of identity). This implies that ρ can be written as a direct sum of 1-dimensional representations, but since ρ is irreducible ρ must be 1-dimensional. □

Proposition 2.2. *If χ is the character of a representation, ρ, with dimension n of a group G, and g is an element of G with order m, then the following holds*

(1) $\chi(1) = n$

(2) $\chi(g)$ *is the sum of m-th roots of unity.*

[1] Statement and proof of Schur's lemma can be found in the appendix A.1

(3) $\chi(g^{-1}) = \overline{\chi(g)}$

Proof.

(1) The first result is immidiate.

$$\chi(1) = tr\left(\begin{bmatrix} 1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 1 \end{bmatrix}\right) = n$$

(2) Since $\langle g \rangle$ is an abelian group, ρ decomposes into n 1-dimensional $\langle g \rangle$-representations. Then there is a basis such that $\rho(g)$ is diagonal. Since g has order m it follows that the diagonal entries of $\rho(g)$ must be m-th roots of unity. Thus $\chi(g) = tr(\rho(g))$ must be the sum of m-th roots of unity.

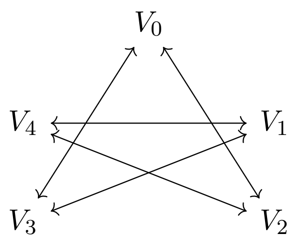
(3) Using the same basis as above and the fact that $\omega^{-1} = \overline{\omega}$ when ω is a root of unity we see that $\chi(g^{-1}) = tr(\rho(g)^{-1}) = \overline{tr(\rho(g))} = \overline{\chi(g)}$.

□

3 The McKay quiver

Definition 3.1. *Let G be a finite subgroup of $GL(n, \mathbb{C})$, and let V be the cannonical representation (the one that sends g to g). Then we define the* <u>*McKay quiver*</u> *of G to be the quiver with verticies the irreducible representations of G, denoted V_i. For two irreducible representations V_i and V_j we say there is an arrow from the former to the latter if and only if V_j is a direct summand of $V \otimes V_i$.*

Example 3.1. *Let G be the group generated by $g = \begin{bmatrix} \omega^2 & 0 \\ 0 & \omega^3 \end{bmatrix}$, where ω is the primitive fifth root of unity. Then there are five different irreducible representations, the one sending g to ω, ω^2, ω^3, ω^4 respectively, and the trivial representation. Denote the representation sending g to ω^i by V_i, and let $V = V_2 \oplus V_3$ be the cannonical representation. Note that $V_i \otimes V_j = V_{i+j}$, where $i + j$ is understood to be modulo 5. Then we get the following McKay-quiver*



4 Krull-Remack-Schmidt

This section is largely based on the book by [Leuschke and Wiegand, 2012]. Here we will prove the Krull-Remack-Schmidt theorem for complete local noetherian rings.

We say a ring satisfies Krull-Remack-Schmidt if the following condition holds:

(i) Any finitely generated module can be written as the finite direct sum of indecomposable modules.

(ii) If

$$\bigoplus_{i=1}^{m} M_i \cong \bigoplus_{j=1}^{n} N_j$$

for indecomposable M_i's and N_j's, then $m = n$ and there is a permutation, $\sigma \in S_n$, such that $M_i \cong N_{\sigma(i)}$ for all $i = 1, 2, \cdots, n$.

It's clear that (i) golds for any noetherian ring, since any decomposition of a noetherian module must eventually reach an indecomposable. In this chapter we will focus on proving (ii).

5 Skew group algebra S#G indecomposable projectives

This section is largely based on the book by [Leuschke and Wiegand, 2012]. This section will use definitions and theorems from representation theory as taught in the courses MA3203 - Ring Theory and MA3204 - homological algebra. Since I do not assume knowledge of this I have created appendix A. I will try to use footnotes to indicate where such theorems are used.

Definition 5.1. *If G is a subgroup of $GL_n(\mathbb{C})$, we can extend the group action of G to $\mathbb{C}[\![x_1, \cdots, x_n]\!]$. We then define the* <u>*skew group algebra* $\mathbb{C}[\![x_1, \cdots, x_n]\!]\#G$</u> *to be the algebra generated by elements of the form $f \cdot g$ with $f \in \mathbb{C}[\![x_1, \cdots, x_n]\!]$ and $g \in G$, and we define the multiplication by*

$$(f_1 \cdot g_1) \cdot (f_2 \cdot g_2) = (f_1 \cdot f_2^{g_1}) \cdot (g_1 \cdot g_2)$$

Where f^g denotes the image of f under the action of g.

Theorem 5.1. *We have an isomorphism of rings*

$$e\mathbb{C}[\![x, y]\!]\#Ge \simeq \mathbb{C}[\![x, y]\!]^G$$

where $e = \frac{1}{|G|} \sum_{g \in G} g$.

Proof. Let f^g denote the image of f under the action of g. Then if we let $f(x, y)g$ be an element of the skew algebra we get that $ef(x, y)ge = f(x, y)^e \cdot ege = f(x, y)^e \cdot e = e \cdot f(x, y)$. It then follows that $e\mathbb{C}[\![x, y]\!]\#Ge$ is isomorphic to the image of $\mathbb{C}[\![x, y]\!]$ under the action of e. Since $ge = g$ for all $g \in G$ it is clear that the image of e is contained in the fixed ring. For the converse you just need to notice that the fixed ring is fixed under e and thus is contained in the image. $\square$

Lemma 5.1. *Let $S = \mathbb{C}[\![x, y]\!]$. An $S\#G$-module is projective if and only if it is projective as an S-module.*

Proof. Onlyifity follows from $S\#G$ being a free S-module, it is isomorphic to $\bigoplus_{g\in G} S$. Thus we need only show ifity.

First we need to see that an $S\#G$-linear map is just an S-linear map such that $f(g(m)) = g(f(m))$ for all $g \in G$. Equivalently $f(m) = g(f(g^{-1}(m)))$. This allows us to define a group action on S-linear maps by $f^g(m) = g(f(g^{-1}(m)))$. Then we can restate it as

$$\mathrm{Hom}_{S\#G}(M, N) = \mathrm{Hom}_S(M, N)^G$$

Clearly if f is $S\#G$-linear then it's in $\mathrm{Hom}_S(M, N)^G$. To see the other inclusion, let f be an S-linear map that is fixed under G. Then $f(s \cdot gm) = sf(gm) = s \cdot g(f(g^{-1}gm)) = s \cdot gf(m)$, and hence f is $S\#G$-linear. Nextly I want to show that $-^G$ is an exact functor.

If K is the kernel of a map $f : M \to N$, then the kernel of the inuced map $f^G : M^G \to N^G$ is of course just $K \cap M^G$ which equals K^G. Assume f is epi and let $n \in N^G$. Consider a preimage m such that $f(m) = n$. Let $\theta = \frac{1}{|G|}\sum_{g\in G} g(m)$. Then θ is in M^G and $f(\theta) = \frac{1}{|G|}\sum_{g\in G} g(f(m)) = \frac{1}{|G|}\sum_{g\in G} n = n$.

This implies that if $\mathrm{Hom}_S(P, -)$ is exact then $\mathrm{Hom}_S(P, -)^G = \mathrm{Hom}_{S\#G}(P, -)$ is exact and our lemma follows. $\square$

Theorem 5.2. *Let $S = \mathbb{C}[\![x, y]\!]$ and let $\mathfrak{m} = \langle x, y\rangle_S$ be the radical of S. Then there are bijections between the indecomposable projective $S\#G$-modules and the indecomposable $\mathbb{C}G$-modules given by*

$$\left\{\begin{matrix}\textit{indecomposable projective}\\ S\#G\textit{-modules}\end{matrix}\right\} \qquad \left\{\begin{matrix}\textit{indecomposable}\\ \mathbb{C}G\textit{-modules}\end{matrix}\right\}$$

$$\mathcal{F} : P \longmapsto P/\mathfrak{m}P$$

$$\mathcal{G} : S \otimes_{\mathbb{C}} W \longleftarrow W$$

Where the $S\#G$-module structure on $S \otimes_{\mathbb{C}} W$ is given by $(s \cdot g) \cdot f \otimes v = sf^g \otimes v^g$.

Proof. First we should show that $S \otimes_{\mathbb{C}} W$ is an irreducible projective $S\#G$-module and that $P/\mathfrak{m}P$ is infact an irreducible $\mathbb{C}G$-module. Since $S \otimes_{\mathbb{C}} W$ is a free S-module it follows from lemma 5.1 that it is projective. To see that it is irreducible we will first study it as an S-module and exploit the fact that $\mathrm{Hom}_{S\#G}(M, N) \subseteq \mathrm{Hom}_S(M, N)$.

Since $\mathfrak{m}$ is the radical of S we have that

$$\frac{S \otimes_{\mathbb{C}} W}{\mathfrak{m}S \otimes_{\mathbb{C}} W} \cong S/\mathfrak{m} \otimes_{\mathbb{C}} W \cong \mathbb{C} \otimes_{\mathbb{C}} W \cong W$$

W is the top of $S \otimes_{\mathbb{C}} W$. Further since the projection is $S\#G$-linear we have that $S \otimes_{\mathbb{C}} W$ is the projective cover of W also as $S\#G$-modules. Then since W is simple it follows that $S \otimes_{\mathbb{C}} W$ is an indecomposable projective.

It's clear that $P/\mathfrak{m}P$ is a $\mathbb{C}G$-module, because $\mathbb{C}G$ is a subring of $S\#G$. To see that it's indecomposable we will first show that it's indecomposable as an $S\#G$-module. By considering P and $P/\mathfrak{m}P$ as S-modules and using the same argument as above we see that P is the projective cover of $P/\mathfrak{m}P$. Then since P is indecomposable $P/\mathfrak{m}P$ must also be indecomposable as an $S\#G$-module.

To see that this implies $P/\mathfrak{m}P$ is indecomposable as a $\mathbb{C}G$-module notice that $P/\mathfrak{m}P$ is annihilated by the ideal $\langle \mathfrak{m} \rangle$. This means it's an indecomposable as $S\#G$-module if and only if it's indecomposable as an $S\#G/\langle \mathfrak{m} \rangle$-module. Then since $S\#G/\langle \mathfrak{m} \rangle \cong \mathbb{C}G$ it follows that $P/\mathfrak{m}P$ is an indecomposable $\mathbb{C}G$-module.

To see that the given maps are bijections we will show that they are mutual inverses. First to see that $\mathcal{F}(\mathcal{G}(W)) \cong W$ we simply look at the definition

$$\frac{S \otimes_{\mathbb{C}} W}{\mathfrak{m}S \otimes_{\mathbb{C}} W} \cong S/\mathfrak{m} \otimes_{\mathbb{C}} W \cong \mathbb{C} \otimes_{\mathbb{C}} W \cong W$$

Next we consider $\mathcal{G}(\mathcal{F}(P)) = S \otimes_{\mathbb{C}} P/\mathfrak{m}P$. We have already seen that it's projective. Both P and $S \otimes_{\mathbb{C}} P/\mathfrak{m}P$ have a natural projection onto $P/\mathfrak{m}P$, and by projectivity we get an induced $S\#G$-linear map from $S \otimes_{\mathbb{C}} P/\mathfrak{m}P$ to P:

$$\begin{tikzcd} & S \otimes_{\mathbb{C}} P/\mathfrak{m}P \arrow[dl, dashed] \arrow[d, two heads] \\ P \arrow[r, two heads] & P/\mathfrak{m}P \end{tikzcd}$$

Further since $\mathfrak{m}$ is the radical of S, both P and $S \otimes_{\mathbb{C}} P/\mathfrak{m}P$ are projective covers of $P/\mathfrak{m}P$ (as S-modules). This means that the map is an isomorphism of S-modules, and therefor it is also an isomorphism of $S\#G$-modules. $\square$

5.1 The Gabriel quiver

Definition 5.2. *For a skew group algebra $S\#G$ we define its* <u>*Gabriel quiver*</u> *to be the quiver with verticies as the indecomposable projective modules of $S\#G$. The arrows are given by taking the minimal projective resolution of $P/\mathfrak{m}P$, where $\mathfrak{m}$ is as defined above. If the minimal projective resolution of $P/\mathfrak{m}P$ is given by*

$$\cdots \longrightarrow Q_1 \longrightarrow Q_0 \longrightarrow 0$$

We say there is an arrow from P to P' if P' appears as a direct summand of Q_1.

Definition 5.3. *Exterior algebra ....*

Proposition 5.1. *If S is the ring of formal power series over $\mathbb{C}$ in n variables, and G is a finite group acting on S, let $V = \mathfrak{m}/\mathfrak{m}^2$. Then the minimal projective resolution of $\mathbb{C} \cong S/\mathfrak{m}$ is given by*

$$0 \longrightarrow S \otimes_{\mathbb{C}} \bigwedge^n V \xrightarrow{\partial_n} \cdots \xrightarrow{\partial_2} S \otimes_{\mathbb{C}} \bigwedge^1 V \xrightarrow{\partial_1} S \longrightarrow 0$$

Where ∂_p is the $S\#G$-linear map defined by

$$\partial_p(s \otimes x_{i_1} \wedge x_{i_2} \wedge \cdots \wedge x_{i_p}) = \sum_{j=1}^{p} (-1)^{j+1} s x_{i_j} \otimes x_{i_1} \wedge \cdots \wedge \hat{x}_{i_j} \wedge \cdots \wedge x_{i_p}$$

Where $x_{i_1} \wedge x_{i_2} \wedge \cdots \wedge x_{i_p}$ is one of the standard basis vectors for $\bigwedge^n V$, namely $i_1 < i_2 < \cdots < i_p$, and $\hat{x}_j$ means that x_j is ommited.

Proof. First we should show that this is a projective resolution. Note that since the maps are $S\#G$-linear, showing that it's a minimal free resolution as an S-module implies it is a minimal projective resolution as an $S\#G$-module. Then what we need to show is

(i) $\operatorname{Cok} \partial_1 = \mathbb{C}$

(ii) $\partial_{p-1} \circ \partial_p = 0$ for all p

(iii) $\operatorname{Im} \partial_{p+1} = \operatorname{Ker} \partial_p$ for $p \geq 2$

(i) is clear since the image of ∂_1 is $\mathfrak{m}$. (ii) can be shown through a quick computation

$$\partial_{p-1} \circ \partial_p(s \otimes x_{i_1} \wedge \cdots \wedge x_{i_p}) =$$

$$\partial_{p-1} \left(\sum_{j=1}^{p} (-1)^{j+1} s x_{i_j} \otimes x_{i_1} \wedge \cdots \hat{x}_j \wedge \cdots \wedge x_{i_p} \right) =$$

$$\sum_{j=1}^{p} (-1)^{j+1} \left(\sum_{k=1}^{j-1} (-1)^{k+1} s x_{i_j} x_{i_k} \otimes x_{i_1} \wedge \cdots \hat{x}_{i_k} \wedge \cdots \wedge \cdots \hat{x}_j \wedge \cdots \wedge x_{i_p} + \sum_{k=j+1}^{p} (-1)^{k} s x_{i_j} x_{i_k} \otimes x_{i_1} \wedge \cdots \hat{x}_{i_j} \wedge \cdots \wedge \cdots \hat{x}_k \wedge \cdots \wedge x_{i_p} \right)$$

From here we notice that the term with $j < k$ is canceled by the term where $k < j$, because they are the negatives of each other. Thus the composition is 0. This would then imply that $\operatorname{Im} \partial_{p+1} \subseteq \operatorname{Ker} \partial_p$, so for part (iii) we need only show that $\operatorname{Ker} \partial_p \subseteq \operatorname{Im} \partial_{p+1}$.

First some notation: let $\mathfrak{I}_p$ be the set of all tuples $(i_1, i_2, \cdots, i_p)$ with $i_1 < i_2 < \cdots < i_p$ and $1 \leq i_j \leq n$, and let x_I denote $x_{i_1} \wedge \cdots x_{i_p}$ when $I = (i_1, \cdots, i_p)$. Then assume

$$\sum_{I \in \mathfrak{I}_p} s_I \otimes x_I$$

is in the kernel of ∂_p.

□

6 The endomorphism ring of S as an S^G-module

Isomorphism to $S\#G$ implies proj SG ¡-¿ S^G direct summands of S.

Theorem 6.1. *Let S be the complex power series ring in two variables, G be a finite subgroup of $GL_2(\mathbb{C})$, $R = S^G$ the fixed ring of S under the action of G, and $(S\#G)^G$ be the fixed ring of $S\#G$ under left multiplication by G. Then S is isomorphic to $(S\#G)^G$ as R-modules.*

Proof. To see this we will define an injective R-linear map from S to $S\#G$ and show that it's image is $(S\#G)^G$. Let $\rho : S \to S\#G$ be given by

$$\rho(s) = \sum_{g \in G} s^g \cdot g.$$

It's clear that it's injective and it is R-linear because

$$\rho(rs) = \sum_{g \in G} r^g s^g \cdot g = r \sum_{g \in G} s^g \cdot g.$$

It should also be clear that the image is contained in $(S\#G)^G$ becuase

$$h \cdot \rho(s) = \sum_{g \in G} h \cdot s^g \cdot g = \sum_{g \in G} s^{hg} \cdot hg = \rho(s).$$

To see that the image is all of $(S\#G)^G$ consider an arbitrary element in $(S\#G)^G$, $\psi = \sum_{g \in G} s_g \cdot g$. Since ψ is fixed under left multiplication by G we must have that

$$\sum_{g \in G} s_g^h \cdot hg = \sum_{g \in G} s_g \cdot g,$$

in particular s_h must equal s_1^h and it follows that $\psi = \rho(s_1)$. □

Theorem 6.2.

7 Maximal Cohen-Macaulay modules of S^G

Definition 7.1. *If R is a local ring with residual field k we define the depth of a module, M, to be the minimal n such that the extension $\mathrm{Ext}^n_R(k, M)$ is non-zero.*

Definition 7.2. *If R is a commutative ring and M is an R-module, a regular sequence is a sequence of elements of R, $r_1, r_2, \cdots r_n$ such that $M/\langle r_1, \cdots, r_i\rangle M$ is non-zero and multiplication by r_i is injective on $M/\langle r_1, \cdots, r_{i-1}\rangle M$.*

Definition 7.3. *If M is a module over a local ring R with Krull-dimension d we say that M is maximal Cohen Macaulay (MCM) if the depth of M equals d.*

Theorem 7.1. *If G is a finite subgroup of $GL_n(\mathbb{C})$, S is the formal power series ring in n variables and $R = S^G$ is the ring fixed under the action of G, then R is a sirect summand of S as R-modules.*

Proof. Consider the map $\pi : S \to R$ given by

$$\pi(s) = \frac{1}{|G|} \sum_{g \in G} s^g$$

It's clear that the image of π is in R becuase an action from G wil just permute the order of teh sum. Further

$$\pi(r) = \frac{1}{|G|} \sum_{g \in G} r^g = \frac{1}{|G|} \sum_{g \in G} r = r,$$

so π splits the inclusion $R \hookrightarrow S$ which shows that R is a direct summand of S. □

Appendices

A Representation theory

Define rep of ring

Define group representation this is the same as $\mathbb{C}G$-rep

Theorem A.1.

Schur's lemma
Projective module + exact hom
indec module + direct sum
indec proj = summand of ring = idempotent of ring
projective resolution
Ext

8 Random Thoughts I need to figure out

I R=S then they have the same depth meaning M is cohen macaulay iff it's projective dimension is 0 (Auslander-Buschsbauw), but that means its a direct summand of S as S(=R)-module, which make sense. If I can show that R = S/(f) fro some polynomial f, can I show R-direct summands of S have projective dimension 1 over S? Can I show R has depth depth(S)-1? Then I also need to prove Auslander-Buschsbauw...

If P is indec projective then it is teh direct summand of a free module. Then there is a non-zero map to the ring. If P is not a summand of the ring then P decomposes, contradiction. Why do we need KRS???????

References

[James and Liebeck, 2001] James, G. and Liebeck, M. (2001). *Representations and characters of groups.* Cambridge University Press, New York, second edition.

[Leuschke and Wiegand, 2012] Leuschke, G. J. and Wiegand, R. (2012). *Cohen-Macaulay representations*, volume 181 of *Mathematical Surveys and Monographs.* American Mathematical Society, Providence, RI.