

# McKay correspondence

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## Abstract

The goal of this thesis is to establish a 1-1 correspondence between quivers created from the four following sets whenever  $G$  is a finite subgroup of  $SL(2, \mathbb{C})$  and  $S$  is the power series ring  $\mathbb{C}[[x, y]]$

- The Maximal Cohen-Macaulay modules of the fixed ring  $S^G$ .
- The indecomposable projective modules of the skew group algebra  $S \# G$ .
- The indecomposable projective modules of  $End_{S^G}(S)$ .
- The irreducible representations of  $G$  (indecomposable  $\mathbb{C}G$ -modules).

Much of the thesis will be used to define these four quivers and to develop tools to establish such a correspondence. A similar correspondence can be established for a general field  $k$  and a finite subgroup of  $GL(n, k)$  with order nonzero in  $k$ , but the case for  $SL(2, \mathbb{C})$  is the most interesting as the quivers will be extended Dynkin diagrams.

## Finite subgroups of $SL(2, \mathbb{C})$

### characters and irreducible representations

Recall that the trace of a matrix is defined to be the sum of its diagonal element and that the trace satisfies two important equations. Namely

$$tr(A + B) = tr(A) + tr(B) \text{ and } tr(AB) = tr(BA)$$

For a given representation of  $G$ ,  $\rho : G \rightarrow GL_n(\mathbb{C})$  we define its character by  $\chi_\rho : G \rightarrow \mathbb{C}$ ,  $\chi_\rho(g) = tr(\rho(g))$ .

**Proposition.** *Conjugate elements in  $G$  take the same value under a character.*

*Proof.* Let  $g$  and  $g'$  be in the same conjugacy class. Then there exists an element  $h$  such that  $h^{-1}gh = g'$ . Then we have

$$\chi(g') = \chi(h^{-1}gh) = tr(\rho(h)^{-1}\rho(g)\rho(h)) \stackrel{*}{=} tr(\rho(g)\rho(h)\rho(h)^{-1}) = tr(\rho(g)) = \chi(g)$$

In (\*) we use the fact that  $tr(AB) = tr(BA)$ . □

**Lemma.** *For a finite abelian group  $G$  any irreducible representation must be 1-dimensional.*

*Proof.* Let  $\rho : G \rightarrow GL(V)$  be an irreducible representation. Since  $G$  is abelian we have that  $\rho(g)\rho(h)v = \rho(h)\rho(g)v$ , and thus  $\rho(g)$  is a homomorphism of  $G$ -representations. Then by Schur's lemma  $\rho(g)$  must be a scalar multiplication. This implies that  $\rho$  can be written as a direct sum of 1-dimensional representations, but since  $\rho$  is irreducible  $\rho$  must be 1-dimensional.  $\square$

**Proposition.** *If  $\chi$  is the character of a representation,  $\rho$ , with dimension  $n$  of a group  $G$ , and  $g$  is an element of  $G$  with order  $m$ , then the following holds*

- (1)  $\chi(1) = n$
- (2)  $\chi(g)$  is the sum of  $m$ -th roots of unity.
- (3)  $\chi(g^{-1}) = \overline{\chi(g)}$

*Proof.*

- (1) The first result is immediate.

$$\chi(1) = \text{tr} \left( \begin{bmatrix} 1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 1 \end{bmatrix} \right) = n$$

- (2) Since  $\langle g \rangle$  is an abelian group,  $\rho$  decomposes into  $n$  1-dimensional  $\langle g \rangle$ -representations. Then there is a basis such that  $\rho(g)$  is diagonal. Since  $g$  has order  $m$  it follows that the diagonal entries of  $\rho(g)$  must be  $m$ -th roots of unity. Thus  $\chi(g) = \text{tr}(\rho(g))$  must be the sum of  $m$ -th roots of unity.
- (3) Using the same basis as above and the fact that  $\omega^{-1} = \overline{\omega}$  when  $\omega$  is a root of unity we see that  $\chi(g^{-1}) = \text{tr}(\rho(g)^{-1}) = \overline{\text{tr}(\rho(g))} = \overline{\chi(g)}$ .  $\square$

## The McKay quiver

**Definition.** *Let  $G$  be a finite subgroup of  $GL(n, \mathbb{C})$ , and let  $V$  be the canonical representation (the one that sends  $g$  to  $g$ ). Then we define the McKay quiver of  $G$  to be the quiver with vertices the irreducible representations of  $G$ , denoted  $V_i$ . For two irreducible representations  $V_i$  and  $V_j$  we say there is an arrow from the former to the latter if and only if  $V_j$  is a direct summand of  $V \otimes V_i$ .*

## Skew algebra $S\#G$ indecomposable projectives

**Definition.** If  $G$  is a subgroup of  $GL_n(\mathbb{C})$ , we can extend the group action of  $G$  to  $\mathbb{C}[[x_1, \dots, x_n]]$ . We then define the skew algebra  $\mathbb{C}[[x_1, \dots, x_n]]\#G$  to be the algebra generated by elements of the form  $f \cdot g$  with  $f \in \mathbb{C}[[x_1, \dots, x_n]]$  and  $g \in G$ , and we define the multiplication by

$$(f_1 \cdot g_1) \cdot (f_2 \cdot g_2) = (f_1 \cdot f_2^{g_1}) \cdot (g_1 \cdot g_2)$$

Where  $f^g$  denotes the image of  $f$  under the action of  $g$ .

**Theorem.** We have an isomorphism of rings

$$e\mathbb{C}[[x, y]]\#Ge \simeq \mathbb{C}[[x, y]]^G$$

where  $e = \frac{1}{|G|} \sum_{g \in G} g$ .

*Proof.* Let  $f^g$  denote the image of  $f$  under the action of  $g$ . Then if we let  $f(x, y)g$  be an element of the skew algebra we get that  $ef(x, y)ge = f(x, y)^e \cdot ege = f(x, y)^e \cdot e$ . It then follows that  $e\mathbb{C}[[x, y]]\#Ge$  is isomorphic to the image of  $e$ . Since  $ge = g$  for all  $g \in G$  it is clear that the image of  $e$  is contained in the fixed ring. For the converse you just need to notice that the fixed ring is fixed under  $e$  and thus is contained in the image.  $\square$

$End_{SG}(S)$

## 1 MCM modules

**Definition.** If  $R$  is a local ring with residual field  $k$  we define the depth of a module,  $M$ , to be the minimal  $n$  such that the extension  $Ext_R^n(k, M)$  is non-zero.

**Definition.** If  $M$  is a module over a local ring  $R$  with Krull-dimension  $d$  we say that  $M$  is maximal Cohen Macaulay (MCM) if the depth of  $M$  equals  $d$ .