McKay correspondence

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Finite subgroups of SL(2,C)

characters and irreducible representations

Recall that the trace of a matrix is defined to be the sum of its diagonal element and that the trace satisfies two important equations. Namely

$$tr(A+B) = tr(A) + tr(B) \text{ and } tr(AB) = tr(BA)$$

For a given representation of G, $\rho : G \to GL_n(\mathbb{C})$ we define its characther by $\chi_\rho : G \to \mathbb{C}$, $\chi_\rho(g) = tr(\rho(g))$.

Proposition. *Conjugate elements in G take the same value under a character.*

Bevis. Let g and g' be in the same conjugacy class. Then there exists an element h such that $h^{-1}gh = g'$. Then we have

$$\chi(g') = \chi(h^{-1}gh) = tr(\rho(h)^{-1}\rho(g)\rho(h)) \stackrel{*}{=} tr(\rho(g)\rho(h)\rho(h)^{-1}) = tr(\rho(g)) = \chi(g)$$

In (*) we use the fact that $tr(AB) = tr(BA)$. □

Proposition. *The dimension of the representation with character χ is $\chi(1)$.*

Bevis.

$$\chi(1) = tr\left(\begin{bmatrix} 1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 1 \end{bmatrix}\right) = n$$

□

Lemma. *For a finite abelian group G any irreducible representation must be 1-dimensional.*

Bevis. Let $\rho : G \to GL(V)$ be an irreducible representation. Since G is abelian we have that $\rho(g)\rho(h)v = \rho(h)\rho(g)v$, and thus $\rho(g)$ is a homomorphism of G-representations. Then by Schur's lemma $\rho(g)$ must be a scalar multiplication. This implies that ρ can be written as a direct sum of 1-dimensional representations, but since ρ is irreducible ρ must be 1-dimensional. □

The McKay quiver

Skew algebra S#G indecomposable projectives

Definition. *If G is a subgroup of $GL_n(\mathbb{C})$, we can extend the group action of G to $\mathbb{C}[\![x_1, \cdots, x_n]\!]$. We then define the* skew algebra $\mathbb{C}[\![x_1, \cdots, x_n]\!]\#G$ *to be the algebra generated by elements of the form $f \cdot g$ with $f \in \mathbb{C}[\![x_1, \cdots, x_n]\!]$ and $g \in G$, and we define the multiplication by*

$$(f_1 \cdot g_1) \cdot (f_2 \cdot g_2) = (f_1 \cdot f_2^{g_1}) \cdot (g_1 \cdot g_2)$$

Where f^g denotes the image of f under the action of g.

Theorem. *We have an isomorphism of rings*

$$e\mathbb{C}[\![x, y]\!]\#Ge \simeq \mathbb{C}[\![x, y]\!]^G$$

where $e = \frac{1}{|G|}\sum_{g \in G} g$.

Bevis. Let f^g denote the image of f under the action of g. Then if we let $f(x, y)g$ be an element of the skew algebra we get that $ef(x, y)ge = f(x, y)^e \cdot ege = f(x, y)^e \cdot e$. It then follows that $e\mathbb{C}[\![x, y]\!]\#Ge$ is isomorphic to the image of e. Since $ge = g$ for all $g \in G$ it is clear that the image of e is contained in the fixed ring. For the converse you just need to notice that the fixed ring is fixed under e and thus is contained in the image. $\square$

S^G-direct summands of $End_{S^G}(S)$