AR-species of generalized Bäckström orders

Jacob Fjeld Grevstad

November 15, 2023

0 Notation

R is a complete local noetehrian Dedekind ring, with maximal ideal genertaed by π.

$k := R/\pi R$ will be its residue field.

Λ and Γ will be R-orders.

A, B and D will be fintie dimensional k-algebras.

We might in many cases talk about Morita equivalent rings as though they are the same.

We will use that when Γ is hereditary every Γ-lattice is projective.

This is true when $\mathrm{Frac}(R) \otimes \Lambda$ is a semisimple $\mathrm{Frac}(R)$-algebra, but it is unclear to me whether this assumption is necceary.

1 Definitions

Definition 1.1 (subhereditary order)**.** Λ is said to be subhereditary iff there exists a hereditary R-order Γ, such that

$$\operatorname{rad} \Gamma \subset \Lambda \subset \Gamma$$

Definition 1.2 (generalized Bäckström order)**.** The pair (Λ, Γ) is called a generalized Bäckström order if $\Lambda \subset \Gamma$ is subhereditary and

$$D := \begin{bmatrix} B & B \\ 0 & A \end{bmatrix}$$

is a hereditary k-algebra, where $B = \Gamma / \operatorname{rad} \Gamma$ and $A = \Lambda / \operatorname{rad} \Gamma$. (Note that A is naturally a subalgebra of B)

Proposition 1.3. *Bäckström orders are generalized Bäckström.*

Proof. Since $\operatorname{rad}\Lambda = \operatorname{rad}\Gamma$, we have that A and B are both artinian rings with zero radical. Hence they are semisimple. Direct inspection of the ideals, using that A and B are hereditary and that B is semisimple shows that every ideal is projective. $\square$

Example 1.4. Let $\Lambda = \begin{pmatrix} R & \pi R \\ R & R \end{pmatrix}$ and $\Gamma = \begin{pmatrix} R & \pi R \\ \pi^{-}R & R \end{pmatrix}$, where R–R means that the elements are equal modulo πR.

Then $A = \begin{pmatrix} k & k \\ 0 & k \end{pmatrix}$, $B = \begin{pmatrix} k & k \\ k & k \end{pmatrix}$ and D is morita equivalent to the path algebra of

$$\circlearrowleft \bullet \longrightarrow \bullet$$

where the loop squares to 0. D is not hereditary, so (Λ, Γ) is not generalized Bäckström.

However if we keep Λ as before, but pick $\Gamma = \begin{pmatrix} R & \pi R \\ R & R \end{pmatrix}$, then $A = k$ and $B = k^2$. Hence D is the path algebra of

$$\bullet \longleftarrow \bullet \longrightarrow \bullet$$

which is hereditary.

2 AR-species of hereditary algebras

Definition 2.1 (Species)**.** A species over k consist of

- an (acyclic) quiver $Q = (Q_0, Q_1, s, t)$ with no parallel arrows,
- a labeling $\nu\colon Q_1 \to \mathbb{N}^2$,
- a choice of finite dimensional division algebras f_i for each vertex i,
- a choice of f_i-f_j bimodules M_α for each arrow $\alpha\colon i \to j$ such that if $\nu(\alpha) = (a, b)$, then $\dim_{f_i} M_\alpha = b$ and $\dim_{f_j} M_\alpha = a$.

Note that for this to be possible it must be the case that $a \dim_k f_i = b \dim_k f_j$.

Definition 2.2. Given a species let $F = \prod_i f_i$ and $M = \bigoplus_\alpha M_\alpha$. Then M is an F-F bimodule, so we can form the tensor algebra. We call this the tensor algebra of the species. Note that if Q is finite and acyclic, this algebra is finite dimensional.

Proposition 2.3. *The tensor algebra of a species is hereditary, and every fintie dimensional hereditary algebra is Morita equivalent to a tensor algebra. Further, it is possible to describe the representation theory of the tensor algebra using the species, as outlined by Henrik (talk 7).*

Definition 2.4 (AR-species)**.** Let $\mathcal{C}$ be an R-linear, hom-finite, krull schmidt category. Then we define a species by having the vertecies be indecomposable objects of $\mathcal{C}$. For an indecomposable object X we define the assoicate division algebra to be $\operatorname{End}(X)^{\mathrm{op}} / \operatorname{rad} \operatorname{End}(X)$, and for any pair of indecomposables we define the bimodule between them to be $\operatorname{rad}(X, Y) / \operatorname{rad}^2(X, Y)$. The arrows and labeling are defined in acordance with the bimodules.

Example 2.5. The algebra $\begin{pmatrix} \mathbb{R} & \mathbb{C} \\ 0 & \mathbb{C} \end{pmatrix}$ is the tensor algebra of $\bullet \xrightarrow{(2,1)} \bullet$. The indecomposable modules are

$$P_1 := \begin{pmatrix} \mathbb{R} \\ 0 \end{pmatrix} \quad P_2 := \begin{pmatrix} \mathbb{C} \\ \mathbb{C} \end{pmatrix}$$
$$I_1 := \begin{pmatrix} \mathbb{C}/\mathbb{R} \\ \mathbb{C} \end{pmatrix} \quad I_2 := \begin{pmatrix} \mathbb{C}/\mathbb{C} \\ \mathbb{C} \end{pmatrix}$$

The AR-species is given by

$$\begin{tikzcd}
 & P_2 \arrow[dr, "{(1,2)}"] & & I_2 \\
P_1 \arrow[ur, "{(2,1)}"] & & I_1 \arrow[ur, "{(2,1)}"'] &
\end{tikzcd}$$

The divison algebras are either $\mathbb{R}$ or $\mathbb{C}$ and all bimodules are $\mathbb{C}$.

2.1 AR-knitting

If D is the tensor algebra of a species, then there is a combinatorial methode to compute (the valued quiver of) the AR-speices of $\operatorname{mod} D$:

Since D is a tensor algebra we can consider $F = \prod f_i$ as a subalgebra. Let 1_i be the vector that is 1 in coordiante i and 0 elsewhere. For a module M define the dimension vector to be $(\dim_{f_i} 1_i M)_i$.

Compute the dimension vectors of the projective modules $D1_i$. Fill in valued arrows between the projectives in the same way as the species you started with.

Starting with a source, take the sum of b copies of dimension vectors of targets for each outgoing arrow labeled (a, b) and subtract one copy of the dimension vector of the source. This gives you the dimension vector of a new vertex. For each arrow (a, b) add an arrow labeled (b, a) to the new vertex. Continue to the next object where all incoming arrows are dealt with. If a dimension vector ever becomes negative or all zeros, do not include it, but coninue knitting.

Example 2.6. The exmaple from before was computed as follows

$$\begin{tikzcd} & (2,1) \\ (1,0) \arrow[ur, "{(2,1)}"] & \end{tikzcd}$$

$$\begin{tikzcd} & (2,1) \arrow[dr, "{(1,2)}"] & \\ (1,0) \arrow[ur, "{(2,1)}"] & & (2-1,1) \end{tikzcd}$$

$$\begin{tikzcd} & (2,1) \arrow[dr, "{(1,2)}"] & & (2\cdot 1-2, 2\cdot 1-1) \\ (1,0) \arrow[ur, "{(2,1)}"] & & (1,1) \arrow[ur, "{(2,1)}"'] & \end{tikzcd}$$

$$\begin{tikzcd} & (2,1) \arrow[dr, "{(1,2)}"] & & (0,1) \arrow[dr, "{(1,2)}"] & \\ (1,0) \arrow[ur, "{(2,1)}"] & & (1,1) \arrow[ur, "{(2,1)}"'] & & (-1,0) \end{tikzcd}$$

$$\begin{tikzcd} & (2,1) \arrow[dr, "{(1,2)}"] & & (0,1) \\ (1,0) \arrow[ur, "{(2,1)}"] & & (1,1) \arrow[ur, "{(2,1)}"'] & \end{tikzcd}$$

3 AR-species of lattices

In this section D wil be a finite dimensional algebra such that the socle of D is projective. We define $\mathfrak{C}$ to be the full subcategory of $\operatorname{mod} D$ of modules with projective socle, and define $\mathfrak{C}^O$ to be the full subcategory of $\mathfrak{C}$ consisting of objects with no simple projective summands.

Proposition 3.1. *$\mathfrak{C}$ is torsionfree in a hereditary torsion pair in* $\operatorname{mod} D$.

Proof. Let $\mathfrak{T}$ be the subcategory of objects whose socle has no projective summands. We want to show that T is closed under quotients. Let T in $\mathfrak{T}$ and let $p\colon T \to T/K$ be some factor module. Assume that P is a projective summand of the socle of T/K. Then $p^{-1}(P)$ surjects onto P, and since P is projective this splits. Thus P is a submodule of T, contradiction.

Note also that $\mathfrak{T}$ is closed under taking subobjects and extensions, and that $\mathfrak{C}$ is the Hom-perpendicular to $\mathfrak{T}$. Hence $(\mathfrak{T}, \mathfrak{C})$ is a hereditary torsion pair. $\square$

Corollary 3.1.1. *The AR-species of $\mathfrak{C}$ can be computed by doing AR-knitting only counting dimensions of simple projectives.*

Proof. For any indecomposable object X in $\mathfrak{C}$, if we take the almost split sequence starting at X and take the torsionfree part we obtain the almost split sequence in $\mathfrak{C}$. This is a general result of relative Auslander–Reiten theory.

Since the socle is projective, any exact sequence in $\mathfrak{C}$ induces a (split) exact sequence in the socle. So when knitting it is enough to consider the dimension of the socle. $\square$

Example 3.2. Let D be the tensor algebra of

$$\begin{tikzcd}
 & & \bullet \\
\bullet \arrow[r] & \bullet \arrow[ur] \arrow[r] \arrow[dr] & \bullet \\
 & & \bullet
\end{tikzcd}$$

This algebra is representation infinite, but when doing AR-knitting counting only the simple projectives we get

$$\begin{tikzcd}
 & & 1 \arrow[dr] & & 1 \arrow[dr] & & 0 \\
 & 1 \arrow[ur] \arrow[r] \arrow[dr] \arrow[ddr] & 1 \arrow[r] & 2 \arrow[ur] \arrow[r] \arrow[dr] & 1 \arrow[r] & 1 \arrow[ur] \arrow[r] \arrow[dr] & 0 \\
 & & 1 \arrow[ur] & & 1 \arrow[ur] & & 0 \\
1 \arrow[uur] & & 0 & & & &
\end{tikzcd}$$

Thus $\mathfrak{C}$ is representation finite with 10 indecomposable objects.

3.1 Representation equivalence

Proposition 3.3. *Let (Λ, Γ) be a subhereditary order, and D be the corresponding k-algebra as discussed earlier. Then the functor $F\colon \operatorname{lat}\Lambda \to \operatorname{mod} D$ given by*

$$F(X) = \begin{pmatrix} \Gamma X/\operatorname{rad}\Gamma X \\ X/\operatorname{rad}\Gamma X \end{pmatrix}$$

defines a representation equivalence between $\operatorname{lat}\Lambda$ and $\mathfrak{C}^O$. I.e. it induces a bijection between their indecomposable objects.

Proof. Since $X \to \Gamma X$ is a monomorphism and B is semisimple the socle of $F(X)$ is $\begin{pmatrix} \Gamma X/\operatorname{rad}\Gamma X \\ 0 \end{pmatrix}$, which is projective. Since $B \otimes X/\operatorname{rad}\Gamma X = \Gamma X/\operatorname{rad}\Gamma X$ we have that $F(X)$ is generated by the $X/\operatorname{rad}\Gamma X$ component, hence has no simple summands. Thus F maps into $\mathfrak{C}^O$.

Let $\begin{pmatrix} M \\ N \end{pmatrix}$ be in $\mathfrak{C}^O$. In particular note that $N \to M$ is a monomorphism and $B \otimes N = M$. Then by lifting idempotents we can find a projective Γ-lattice Y such that $Y/\operatorname{rad}\Gamma Y = M$. Define X to be the pullback of $N \to M \leftarrow Y$. Because $N \to M$ is injective $X/\operatorname{rad}\Gamma X = N$. Becuase $B \otimes N = M$ we have $\Gamma X = Y$. Thus F is surjective on objects.

Using that all Γ-lattices are projective, we get that F is surjective on morphism, and using Nakayamas lemma it reflects epimorphisms.

Since F reflects epimoprhisms and epimorphic endomorphisms of noetherian modules are isomorphisms, F reflects isomorphism.

Assume $F(X)$ is indecomposable. Then since F is surjective on objects $F(X) = F(X') \oplus F(X'') = F(X' \oplus X'')$. Since F reflects isomorphisms $X = X' \oplus X''$. Hence F preserves and reflects indecomposability.

So F is a representation equivalence. □

Proposition 3.4. *Let $\varphi\colon X \to Y$ be a homomorphism between indecomposable Λ-lattices.*

1. *Then if $F(\varphi) = 0$ and φ is irreducible we have that X is a Γ-lattice and Y is projective.*

2. *When $F(\varphi)$ is nonzero it is irreducible iff φ is irreducible.*

3. *For every irreducible map $\hat{\varphi}$ in $\mathfrak{C}^O$, there exists an irreducible map φ with $F(\varphi) = \hat{\varphi}$.*

Proof. Incomplete guess at proof

1. If $F(\varphi) = 0$ then φ maps X into $\operatorname{rad}\Gamma Y$. Since φ is irreducible this means X is a summand of $\operatorname{rad}\Gamma Y$, hence a Γ-lattice. Why is Y projective...?

2. Assume $F(\varphi) = fg$ for f, g two nonsplit maps. Then since F is surjective on morphisms, there exists ψ and ξ such that $F(\varphi) = F(\psi)F(\xi)$.

   We would like to say $\varphi = \psi\xi$. If X is not a Γ-lattice or Y is not projective were done, since if $\varphi - \psi\xi$ is irreducible it must be 0. If it is not irreducible, the $\varphi - \psi\xi \in \operatorname{rad}^2(X, Y)$ and $\psi\xi \in \operatorname{rad}^2(X, Y)$ implies $\varphi \in \operatorname{rad}^2(X, Y)$.

   If X is a Γ-lattice and Y is projective, then the image of X is a Γ-lattice, and it seems likely that $\operatorname{rad}\Gamma Y$ is the maximal proper sublattice that is a Γ-lattice in Y, when Y projective (??) So then $F(\varphi)$ must be 0 (or an isomorphisms).

3. Since F is surjective on morphisms, there exists such a φ. And by the previous point it is irreducible.

$\square$

Definition 3.5. Let $\{X_i\}$ be the set of indecomposable Γ-lattices. Then every X_i has a unique minimal overlattice, which is again projective indecomposable. We define the permutation σ of Γ such that $X_{\sigma(i)}$ is the unique over lattice.

Then X_i is the radical of $X_{\sigma(i)}$ (?)

Proposition 3.6. *When* $\operatorname{lat}\Lambda$ *is of finite type we can reconstruct the AR-species of* $\operatorname{lat}\Lambda$ *from the AR-species of $\mathfrak{C}$ by identifying the sink* $E_j := \begin{pmatrix} X_j/\operatorname{rad}\Gamma X_j \\ X_j/\operatorname{rad}\Gamma X_j \end{pmatrix}$ *with the source* $S_{\sigma(j)} := \begin{pmatrix} X_{\sigma(j)}/\operatorname{rad}\Gamma X_{\sigma(j)} \\ 0 \end{pmatrix}$.

Proof. By the preceeding arguments we know the labels of arrows where the source vertex is not a Γ-lattice. If X_i is a Γ-lattice, and Y is a projective then an irreducible map $X \to Y$ is the same as a map $X_{\sigma(i)} \to \Gamma Y$ which maps X into $\operatorname{rad}\Gamma Y$ (??).

One can check that all irreducible maps starting at $\begin{pmatrix} X_{\sigma(i)}/\operatorname{rad}\Gamma \\ 0 \end{pmatrix}$ are given by inclusions into $\begin{pmatrix} \Gamma P/\operatorname{rad}\Gamma P \\ P/\operatorname{rad}\Gamma P \end{pmatrix}$ for an indecomposable projective Λ-lattice P. (Note $\begin{pmatrix} \Gamma P/\operatorname{rad}\Gamma P \\ P/\operatorname{rad}\Gamma P \end{pmatrix}$ are exactly the non-simple projective D-modules). $\square$

Example 3.7. Let $R = \mathbb{R}[[t]]$ and let

$$\Lambda = \begin{pmatrix} a + \pi R & b + \pi R \\ -b + \pi R & a + \pi R \end{pmatrix} \subset M_{2\times 2}(R) = \Gamma$$

Then $A = \mathbb{C}$ and D is Morita equivalent to $\begin{pmatrix} \mathbb{R} & \mathbb{C} \\ 0 & \mathbb{C} \end{pmatrix}$. Thus the AR-species of $\operatorname{lat}\Lambda$ is

$$\begin{tikzcd} \bullet \arrow[r, bend left, "{(1,2)}"] & \bullet \arrow[l, bend left, "{(2,1)}"] \end{tikzcd}$$

The two lattices are $X := \begin{pmatrix} R \\ R \end{pmatrix}$ and Λ.

Example 3.8. Consider

$$\Lambda = \begin{pmatrix} R & R & R & R \\ \pi R & R & \pi R & \pi R \\ \pi R & \pi R & R & \pi R \\ \pi R & \pi R & \pi R & R \end{pmatrix} \subset M_{4\times 4}(R) = \Gamma$$

Then D is Morita equivalent to the tensor algebra in Example 3.2, hence the AR-species of $\operatorname{lat}\Lambda$ is given by

$$\begin{tikzcd} & P_2 \arrow[dr] & & Y_2 \arrow[dr] & \\ P_1 \arrow[ur] \arrow[r] \arrow[dr] & P_3 \arrow[r] & Y_1 \arrow[ur] \arrow[r] \arrow[dr] & Y_3 \arrow[r] & X \arrow[llll, bend right] \\ & P_4 \arrow[ur] & & Y_4 \arrow[ur] & \end{tikzcd}$$

Where the lattices are

$$P_1 = \begin{pmatrix} R \\ \pi R \\ \pi R \\ \pi R \end{pmatrix}, P_2 = \begin{pmatrix} R \\ R \\ \pi R \\ \pi R \end{pmatrix}, P_3 = \begin{pmatrix} R \\ \pi R \\ R \\ \pi R \end{pmatrix}, P_4 = \begin{pmatrix} R \\ \pi R \\ \pi R \\ R \end{pmatrix}, X = \begin{pmatrix} R \\ R \\ R \\ R \end{pmatrix}$$

$$Y_1 = \begin{pmatrix} R & R \\ R & \pi R \\ R & R \\ \pi R & R \end{pmatrix}, Y_2 = \begin{pmatrix} R \\ \pi R \\ R \\ R \end{pmatrix}, Y_3 = \begin{pmatrix} R \\ R \\ \pi R \\ R \end{pmatrix}, Y_4 = \begin{pmatrix} R \\ R \\ R \\ \pi R \end{pmatrix}$$