

# AR-species of generalized Bäckström orders

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## 0 Notation

$R$  is a complete local noetherian Dedekind ring, with maximal ideal generated by  $\pi$ .

$k := R/\pi R$  will be its residue field.

$\Lambda$  and  $\Gamma$  will be  $R$ -orders.

$A$ ,  $B$  and  $D$  will be finite dimensional  $k$ -algebras.

We might in many cases talk about Morita equivalent rings as though they are the same.

We will use that when  $\Gamma$  is hereditary every  $\Gamma$ -lattice is projective.

This is true when  $\text{Frac}(R) \otimes \Lambda$  is a semisimple  $\text{Frac}(R)$ -algebra, but it is unclear to me whether this assumption is necessary.

## 1 Definitions

**Definition 1.1** (subhereditary order).  $\Lambda$  is said to be subhereditary iff there exists a hereditary  $R$ -order  $\Gamma$ , such that

$$\text{rad } \Gamma \subset \Lambda \subset \Gamma$$

**Definition 1.2** (generalized Bäckström order). The pair  $(\Lambda, \Gamma)$  is called a generalized Bäckström order if  $\Lambda \subset \Gamma$  is subhereditary and

$$D := \begin{bmatrix} B & B \\ 0 & A \end{bmatrix}$$

is a hereditary  $k$ -algebra, where  $B = \Gamma/\text{rad } \Gamma$  and  $A = \Lambda/\text{rad } \Gamma$ . (Note that  $A$  is naturally a subalgebra of  $B$ )

**Proposition 1.3.** *Bäckström orders are generalized Bäckström.*

*Proof.* Since  $\text{rad } \Lambda = \text{rad } \Gamma$ , we have that  $A$  and  $B$  are both artinian rings with zero radical. Hence they are semisimple. Direct inspection of the ideals, using that  $A$  and  $B$  are hereditary and that  $B$  is semisimple shows that every ideal is projective.  $\square$

**Example 1.4.** Let  $\Lambda = \begin{pmatrix} R & \pi R \\ R & R \end{pmatrix}$  and  $\Gamma = \begin{pmatrix} R & \pi R \\ \pi^- R & R \end{pmatrix}$ , where  $R$ - $R$  means that the elements are equal modulo  $\pi R$ .

Then  $A = \begin{pmatrix} k & k \\ 0 & k \end{pmatrix}$ ,  $B = \begin{pmatrix} k & k \\ k & k \end{pmatrix}$  and  $D$  is morita equivalent to the path algebra of

$$\curvearrowright \bullet \longrightarrow \bullet$$

where the loop squares to 0.  $D$  is not hereditary, so  $(\Lambda, \Gamma)$  is not generalized Bäckström.

However if we keep  $\Lambda$  as before, but pick  $\Gamma = \begin{pmatrix} R & \pi R \\ R & R \end{pmatrix}$ , then  $A = k$  and  $B = k^2$ . Hence  $D$  is the path algebra of

$$\bullet \longleftarrow \bullet \longrightarrow \bullet$$

which is hereditary.

## 2 AR-species of hereditary algebras

**Definition 2.1** (Species). A species over  $k$  consist of

- an (acyclic) quiver  $Q = (Q_0, Q_1, s, t)$  with no parallel arrows,
- a labeling  $\nu: Q_1 \rightarrow \mathbb{N}^2$ ,
- a choice of finite dimensional division algebras  $f_i$  for each vertex  $i$ ,
- a choice of  $f_i$ - $f_j$  bimodules  $M_\alpha$  for each arrow  $\alpha: i \rightarrow j$  such that if  $\nu(\alpha) = (a, b)$ , then  $\dim_{f_i} M_\alpha = b$  and  $\dim_{f_j} M_\alpha = a$ .

Note that for this to be possible it must be the case that  $a \dim_k f_i = b \dim_k f_j$ .

**Definition 2.2.** Given a species let  $F = \prod_i f_i$  and  $M = \bigoplus_\alpha M_\alpha$ . Then  $M$  is an  $F$ - $F$  bimodule, so we can form the tensor algebra. We call this the tensor algebra of the species. Note that if  $Q$  is finite and acyclic, this algebra is finite dimensional.

**Proposition 2.3.** *The tensor algebra of a species is hereditary, and every finite dimensional hereditary algebra is Morita equivalent to a tensor algebra. Further, it is possible to describe the representation theory of the tensor algebra using the species, as outlined by Henrik (talk 7).*

**Definition 2.4** (AR-species). Let  $\mathcal{C}$  be an  $R$ -linear, hom-finite, krull schmidt category. Then we define a species by having the vertices be indecomposable objects of  $\mathcal{C}$ . For an indecomposable object  $X$  we define the assoicate division algebra to be  $\text{End}(X)^{\text{op}}/\text{rad End}(X)$ , and for any pair of indecomposables we define the bimodule between them to be  $\text{rad}(X, Y)/\text{rad}^2(X, Y)$ . The arrows and labeling are defined in accordance with the bimodules.

**Example 2.5.** The algebra  $\begin{pmatrix} \mathbb{R} & \mathbb{C} \\ 0 & \mathbb{C} \end{pmatrix}$  is the tensor algebra of  $\bullet \xrightarrow{(2,1)} \bullet$ . The indecomposable modules are

$$\begin{aligned} P_1 &:= \begin{pmatrix} \mathbb{R} \\ 0 \end{pmatrix} & P_2 &:= \begin{pmatrix} \mathbb{C} \\ \mathbb{C} \end{pmatrix} \\ I_1 &:= \begin{pmatrix} \mathbb{C}/\mathbb{R} \\ \mathbb{C} \end{pmatrix} & I_2 &:= \begin{pmatrix} \mathbb{C}/\mathbb{C} \\ \mathbb{C} \end{pmatrix} \end{aligned}$$

The AR-species is given by

$$\begin{array}{ccccc} & & P_2 & & I_2 \\ & \nearrow^{(2,1)} & & \searrow_{(1,2)} & \nearrow^{(2,1)} \\ P_1 & & & & I_1 \end{array}$$

The division algebras are either  $\mathbb{R}$  or  $\mathbb{C}$  and all bimodules are  $\mathbb{C}$ .

## 2.1 AR-knitting

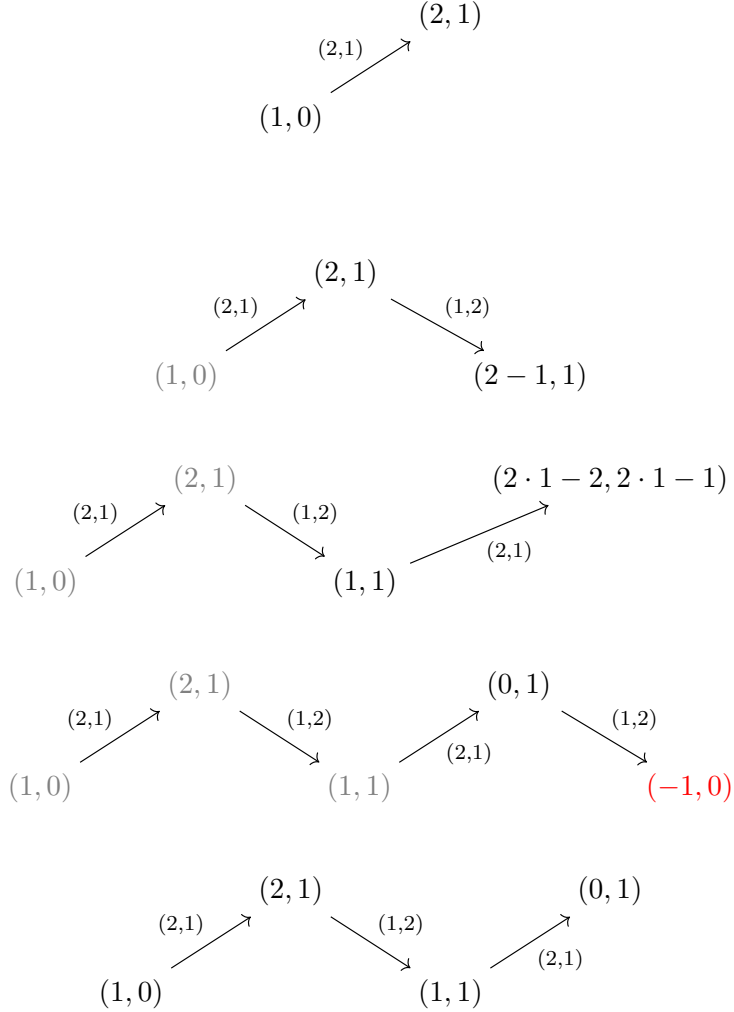
If  $D$  is the tensor algebra of a species, then there is a combinatorial methode to compute (the valued quiver of) the AR-speices of  $\text{mod } D$ :

Since  $D$  is a tensor algebra we can consider  $F = \prod f_i$  as a subalgebra. Let  $1_i$  be the vector that is 1 in coordiante  $i$  and 0 elsewhere. For a module  $M$  define the dimension vector to be  $(\dim_{f_i} 1_i M)_i$ .

Compute the dimension vectors of the projective modules  $D1_i$ . Fill in valued arrows between the projectives in the same way as the species you started with.

Starting with a source, take the sum of  $b$  copies of dimension vectors of targets for each outgoing arrow labeled  $(a, b)$  and subtract one copy of the dimension vector of the source. This gives you the dimension vector of a new vertex. For each arrow  $(a, b)$  add an arrow labeled  $(b, a)$  to the new vertex. Continue to the next object where all incoming arrows are dealt with. If a dimension vector ever becomes negative or all zeros, do not include it, but continue knitting.

**Example 2.6.** The example from before was computed as follows



### 3 AR-species of lattices

In this section  $D$  will be a finite dimensional algebra such that the socle of  $D$  is projective. We define  $\mathfrak{C}$  to be the full subcategory of  $\text{mod } D$  of modules with projective socle, and

define  $\mathfrak{C}^O$  to be the full subcategory of  $\mathfrak{C}$  consisting of objects with no simple projective summands.

**Proposition 3.1.**  *$\mathfrak{C}$  is torsionfree in a hereditary torsion pair in  $\text{mod } D$ .*

*Proof.* Let  $\mathfrak{T}$  be the subcategory of objects whose socle has no projective summands. We want to show that  $T$  is closed under quotients. Let  $T$  in  $\mathfrak{T}$  and let  $p: T \rightarrow T/K$  be some factor module. Assume that  $P$  is a projective summand of the socle of  $T/K$ . Then  $p^{-1}(P)$  surjects onto  $P$ , and since  $P$  is projective this splits. Thus  $P$  is a submodule of  $T$ , contradiction.

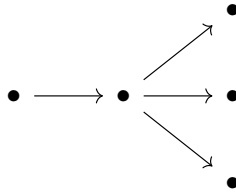
Note also that  $\mathfrak{T}$  is closed under taking subobjects and extensions, and that  $\mathfrak{C}$  is the Hom-perpendicular to  $\mathfrak{T}$ . Hence  $(\mathfrak{T}, \mathfrak{C})$  is a hereditary torsion pair.  $\square$

**Corollary 3.1.1.** *The AR-species of  $\mathfrak{C}$  can be computed by doing AR-knitting only counting dimensions of simple projectives.*

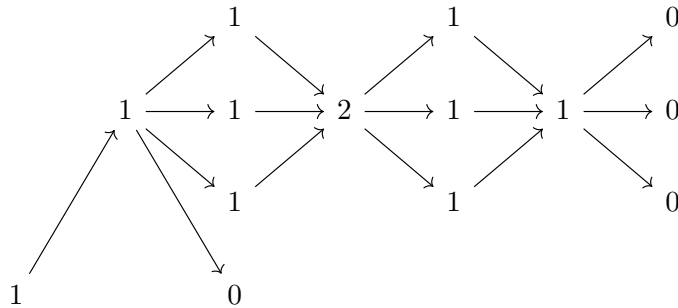
*Proof.* For any indecomposable object  $X$  in  $\mathfrak{C}$ , if we take the almost split sequence starting at  $X$  and take the torsionfree part we obtain the almost split sequence in  $\mathfrak{C}$ . This is a general result of relative Auslander–Reiten theory.

Since the socle is projective, any exact sequence in  $\mathfrak{C}$  induces a (split) exact sequence in the socle. So when knitting it is enough to consider the dimension of the socle.  $\square$

**Example 3.2.** Let  $D$  be the tensor algebra of



This algebra is representation infinite, but when doing AR-knitting counting only the simple projectives we get



Thus  $\mathfrak{C}$  is representation finite with 10 indecomposable objects.

### 3.1 Representation equivalence

**Proposition 3.3.** *Let  $(\Lambda, \Gamma)$  be a subhereditary order, and  $D$  be the corresponding  $k$ -algebra as discussed earlier. Then the functor  $F: \text{lat } \Lambda \rightarrow \text{mod } D$  given by*

$$F(X) = \begin{pmatrix} \Gamma X / \text{rad } \Gamma X \\ X / \text{rad } \Gamma X \end{pmatrix}$$

*defines a representation equivalence between  $\text{lat } \Lambda$  and  $\mathfrak{C}^O$ . I.e. it induces a bijection between their indecomposable objects.*

*Proof.* Since  $X \rightarrow \Gamma X$  is a monomorphism and  $B$  is semisimple the socle of  $F(X)$  is  $\begin{pmatrix} \Gamma X / \text{rad } \Gamma X \\ 0 \end{pmatrix}$ , which is projective. Since  $B \otimes X / \text{rad } \Gamma X = \Gamma X / \text{rad } \Gamma X$  we have that  $F(X)$  is generated by the  $X / \text{rad } \Gamma X$  component, hence has no simple summands. Thus  $F$  maps into  $\mathfrak{C}^O$ .

Let  $\begin{pmatrix} M \\ N \end{pmatrix}$  be in  $\mathfrak{C}^O$ . In particular note that  $N \rightarrow M$  is a monomorphism and  $B \otimes N = M$ . Then by lifting idempotents we can find a projective  $\Gamma$ -lattice  $Y$  such that  $Y / \text{rad } \Gamma Y = M$ . Define  $X$  to be the pullback of  $N \rightarrow M \leftarrow Y$ . Because  $N \rightarrow M$  is injective  $X / \text{rad } \Gamma X = N$ . Because  $B \otimes N = M$  we have  $\Gamma X = Y$ . Thus  $F$  is surjective on objects.

Using that all  $\Gamma$ -lattices are projective, we get that  $F$  is surjective on morphism, and using Nakayamas lemma it reflects epimorphisms.

Since  $F$  reflects epimorphisms and epimorphic endomorphisms of noetherian modules are isomorphisms,  $F$  reflects isomorphism.

Assume  $F(X)$  is indecomposable. Then since  $F$  is surjective on objects  $F(X) = F(X') \oplus F(X'') = F(X' \oplus X'')$ . Since  $F$  reflects isomorphisms  $X = X' \oplus X''$ . Hence  $F$  preserves and reflects indecomposability.

So  $F$  is a representation equivalence. □

**Proposition 3.4.** *Let  $\varphi: X \rightarrow Y$  be a homomorphism between indecomposable  $\Lambda$ -lattices.*

1. *Then if  $F(\varphi) = 0$  and  $\varphi$  is irreducible we have that  $X$  is a  $\Gamma$ -lattice and  $Y$  is projective.*

2. When  $F(\varphi)$  is nonzero it is irreducible iff  $\varphi$  is irreducible.
3. For every irreducible map  $\hat{\varphi}$  in  $\mathfrak{C}^O$ , there exists an irreducible map  $\varphi$  with  $F(\varphi) = \hat{\varphi}$ .

*Proof.* **Incomplete guess at proof**

1. If  $F(\varphi) = 0$  then  $\varphi$  maps  $X$  into  $\text{rad } \Gamma Y$ . Since  $\varphi$  is irreducible this means  $X$  is a summand of  $\text{rad } \Gamma Y$ , hence a  $\Gamma$ -lattice. **Why is  $Y$  projective...?**
2. Assume  $F(\varphi) = fg$  for  $f, g$  two nonsplit maps. Then since  $F$  is surjective on morphisms, there exists  $\psi$  and  $\xi$  such that  $F(\varphi) = F(\psi)F(\xi)$ .

We would like to say  $\varphi = \psi\xi$ . If  $X$  is not a  $\Gamma$ -lattice or  $Y$  is not projective were done, since if  $\varphi - \psi\xi$  is irreducible it must be 0. If it is not irreducible, the  $\varphi - \psi\xi \in \text{rad}^2(X, Y)$  and  $\psi\xi \in \text{rad}^2(X, Y)$  implies  $\varphi \in \text{rad}^2(X, Y)$ .

If  $X$  is a  $\Gamma$ -lattice and  $Y$  is projective, then the image of  $X$  is a  $\Gamma$ -lattice, and it seems likely that  $\text{rad } \Gamma Y$  is the maximal proper sublattice that is a  $\Gamma$ -lattice in  $Y$ , when  $Y$  projective (??) So then  $F(\varphi)$  must be 0 (or an isomorphisms).

3. Since  $F$  is surjective on morphisms, there exists such a  $\varphi$ . And by the previous point it is irreducible.

□

**Definition 3.5.** Let  $\{X_i\}$  be the set of indecomposable  $\Gamma$ -lattices. Then every  $X_i$  has a unique minimal overlattice, which is again projective indecomposable. We define the permutation  $\sigma$  of  $\Gamma$  such that  $X_{\sigma(i)}$  is the unique over lattice.

Then  $X_i$  is the radical of  $X_{\sigma(i)}$  (?)

**Proposition 3.6.** When  $\text{lat } \Lambda$  is of finite type we can reconstruct the *AR-species* of  $\text{lat } \Lambda$  from the *AR-species* of  $\mathfrak{C}$  by identifying the sink  $E_j := \begin{pmatrix} X_j / \text{rad } \Gamma X_j \\ X_j / \text{rad } \Gamma X_j \end{pmatrix}$  with the source  $S_{\sigma(j)} := \begin{pmatrix} X_{\sigma(j)} / \text{rad } \Gamma X_{\sigma(j)} \\ 0 \end{pmatrix}$ .

*Proof.* By the preceeding arguments we know the labels of arrows where the source vertex is not a  $\Gamma$ -lattice. If  $X_i$  is a  $\Gamma$ -lattice, and  $Y$  is a projective then an irreducible map  $X \rightarrow Y$  is the same as a map  $X_{\sigma(i)} \rightarrow \Gamma Y$  which maps  $X$  into  $\text{rad } \Gamma Y$  (??).

One can check that all irreducible maps starting at  $\begin{pmatrix} X_{\sigma(i)}/\text{rad } \Gamma \\ 0 \end{pmatrix}$  are given by inclusions into  $\begin{pmatrix} \Gamma P/\text{rad } \Gamma P \\ P/\text{rad } \Gamma P \end{pmatrix}$  for an indecomposable projective  $\Lambda$ -lattice  $P$ . (Note  $\begin{pmatrix} \Gamma P/\text{rad } \Gamma P \\ P/\text{rad } \Gamma P \end{pmatrix}$  are exactly the non-simple projective  $D$ -modules).  $\square$

**Example 3.7.** Let  $R = \mathbb{R}[[t]]$  and let

$$\Lambda = \begin{pmatrix} a + \pi R & b + \pi R \\ -b + \pi R & a + \pi R \end{pmatrix} \subset M_{2 \times 2}(R) = \Gamma$$

Then  $A = \mathbb{C}$  and  $D$  is Morita equivalent to  $\begin{pmatrix} \mathbb{R} & \mathbb{C} \\ 0 & \mathbb{C} \end{pmatrix}$ . Thus the AR-species of  $\text{lat } \Lambda$  is

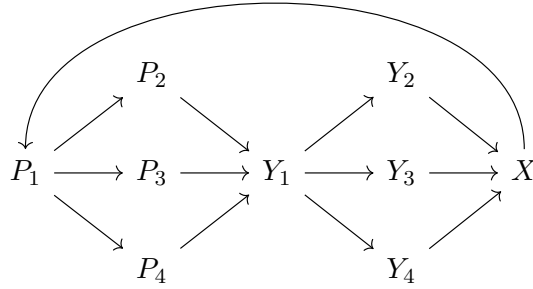
$$\bullet \begin{matrix} \xrightarrow{(1,2)} \\ \xleftarrow{(2,1)} \end{matrix} \bullet$$

The two lattices are  $X := \begin{pmatrix} R \\ R \end{pmatrix}$  and  $\Lambda$ .

**Example 3.8.** Consider

$$\Lambda = \begin{pmatrix} R & R & R & R \\ \pi R & R & \pi R & \pi R \\ \pi R & \pi R & R & \pi R \\ \pi R & \pi R & \pi R & R \end{pmatrix} \subset M_{4 \times 4}(R) = \Gamma$$

Then  $D$  is Morita equivalent to the tensor algebra in Example 3.2, hence the AR-species of  $\text{lat } \Lambda$  is given by



Where the lattices are

$$P_1 = \begin{pmatrix} R \\ \pi R \\ \pi R \\ \pi R \end{pmatrix}, P_2 = \begin{pmatrix} R \\ R \\ \pi R \\ \pi R \end{pmatrix}, P_3 = \begin{pmatrix} R \\ \pi R \\ R \\ \pi R \end{pmatrix}, P_4 = \begin{pmatrix} R \\ \pi R \\ \pi R \\ R \end{pmatrix}, X = \begin{pmatrix} R \\ R \\ R \\ R \end{pmatrix}$$



$$Y_1 = \begin{pmatrix} R & R \\ R & \pi R \\ R & R \\ \pi R & R \end{pmatrix}, Y_2 = \begin{pmatrix} R \\ \pi R \\ R \\ R \end{pmatrix}, Y_3 = \begin{pmatrix} R \\ R \\ \pi R \\ R \end{pmatrix}, Y_4 = \begin{pmatrix} R \\ R \\ R \\ \pi R \end{pmatrix}$$