# Poles and Zeros of a Transfer Function

## 1. Transfer Function

A transfer function is a mathematical model whose purpose is to describe a linear time invariant dynamic system. For systems whose dimension is finite, the transfer function is simply a rational function of a complex variable (in this lab, this complex variable is $s$, in the Laplace domain). In other words, a transfer function $H(s)$ is the ratio, in Laplace domain, between the output $Y(s)$ and input $X(s)$ of a system. To illustrate this concept, consider the following example:

Suppose a system can be modeled by means of the following differential equation (second order system):

$$a_0 \frac{d^2y}{dt^2} + a_1 \frac{dy}{dt} + a_2 y(t) = b_0 \frac{dx}{dt} + b_1 x(t) \quad (1)$$

Transforming the above equation to the Laplace domain, we get:

| Time domain | Laplace domain |
|---|---|
| $y(t)$ | $Y(s)$ |
| $dy/dt$ | $sY(s)$ |
| $d^2y/dt^2$ | $s^2Y(s)$ |
| $x(t)$ | $X(s)$ |
| $dx/dt$ | $sX(s)$ |

$$a_0 s^2 Y(s) + a_1 s Y(s) + a_2 Y(s) = b_0 s X(s) + b_1 X(s)$$

$$Y(s).(a_0 s^2 + a_1 s + a_2) = X(s).(b_0 s + b_1)$$

Therefore, the transfer function of the system is:

$$\boxed{H(s) = \frac{Y(s)}{X(s)} = \frac{b_0 s + b_1}{a_0 s^2 + a_1 s + a_2}} \quad (2)$$

Figure 1 below is the block diagram that represents the system, where $H(s)$ is the transfer function, and $X(s)$ and $Y(s)$ are the input and output of the system, respectively.

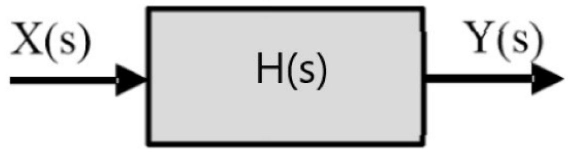

Figure 1 – System block diagram.

## 2. Poles and zeros

Poles and zeros of a transfer function are important concepts that help us to analyze and manipulate systems, particularly when the stability of the system is to be verified.

In summary, poles are the values of $s$ that cause the transfer function to go to infinity, in other words, to find the poles, the denominator of the transfer function has to be set to zero, and the values of $s$ that satisfy this equation are the poles of the system.

Likewise, zeros are the values of $s$ that cause the transfer function to go to zero, in other words, to find the zeros, the numerator of the transfer function has to be set to zero, and the values of $s$ that satisfy this equation are the zeros of the system.

For example, let us consider the transfer function defined in (2) with $b_0 = 1, b_1 = 3, a_0 = 1, a_1 = -5$ and $a_2 = 6$.

$$H(s) = \frac{s+3}{s^2 - 5s + 6}$$

- Finding the poles of the system:

$$s^2 - 5s + 6 = 0$$

  Therefore, the poles are at $s = 3$ and $s = 2$

- Finding the zeros of the system:

$$s + 3 = 0$$

  Therefore, there is one zero at $s = -3$

## 3. Stability

The stability of a system depends on its poles.

- A system is said to be stable if the system response is bound to a finite value (a constant value) as time goes to infinity. In terms of poles, a system is stable if its poles are equal or less than zero.
- Particularly, a system is said to be asymptotically stable if the real part of all poles is strictly less than zero.
- Finally, a system is unstable if at least one pole is greater than zero.

## 4. MATLAB commands

The following examples involve the necessary MATLAB commands to solve this lab.

***1. Transfer function***

$$H(s) = \frac{s+3}{s^2 - 5s + 6}$$

This is how we write the numerator and denominator in MATLAB:

Numerator = $s + 3 \rightarrow [1 \quad 3]$
Denominator = $s^2 - 5s + 6 \rightarrow [1 - 5 \quad 6]$

We use the ***tf*** command to call the transfer function in MATLAB.

Enter in MATLAB:

```
H = tf([1 3], [1 -5 6])
```

Running the line of code above, we get:

```
H =

      s + 3
  -------------
  s^2 - 5 s + 6
```

A transfer function can be written in terms of its poles and zeros, as follows, where $z$ are the zeros and $p$ are the poles.

$$H(s) = \frac{(s - z_1).(s - z_2) \dots (s\_z_m)}{(s - p_1).(s - p_2) \dots (s\_p_n)} \quad (3)$$

Suppose we want to write the following transfer function in the above form.

$$H(s) = \frac{s^2 - 23s + 132}{s^3 - 6s^2 - 9s + 14}$$

Numerator = $s^2 - 23s + 132$

```
>> z = roots([1  -23  132])
z =
   12
   11
```

Denominator = $s^3 - 6s^2 - 9s + 14$

```
>> p = roots([1   -6   -9   14])
p =
         7
        -2
         1
```

Once we obtain the poles and zeros we can write the transfer function in the form described in (3), as follows:

$$H(s) = \frac{(s-12).(s-11)}{(s-7).(s-(-2)).(s-1)}$$

$$\boxed{H(s) = \frac{(s-12).(s-11)}{(s-7).(s+2).(s-1)}}$$

## 2. ***Poles, zeros and stability***

The above section shows us one way to find the poles and zeros of a transfer function. Let's look at another method:

Suppose we want to find the poles and zeros directly from the following transfer function:

$$H(s) = \frac{s^2 - 23s + 132}{s^3 - 6s^2 - 9s + 14}$$

To do so, the following commands can be used:

```
>> H = tf([1 -23 132], [1 -6 -9 14]);
>> [p,z] = pzmap(H)
```

Note: pzmap command is the pole-zero plot of dynamic system

Which yields the following poles and zeros:

```
p =
        7
       -2
        1
z =
   12
   11
```

Using this same transfer function, we can also plot the pole-zero map and the impulse response, as follows:

***Pole-zero map***

```
pzmap(H)
```

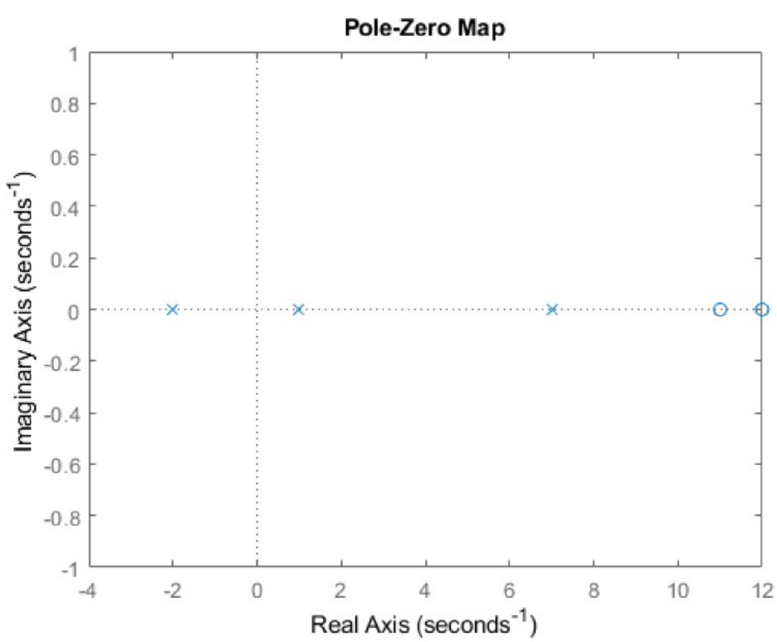


Figure 2 – Pole-zero location, where the axis are the poles and the circles are the zeros.

***Impulse response***

```
impulse(H)
```

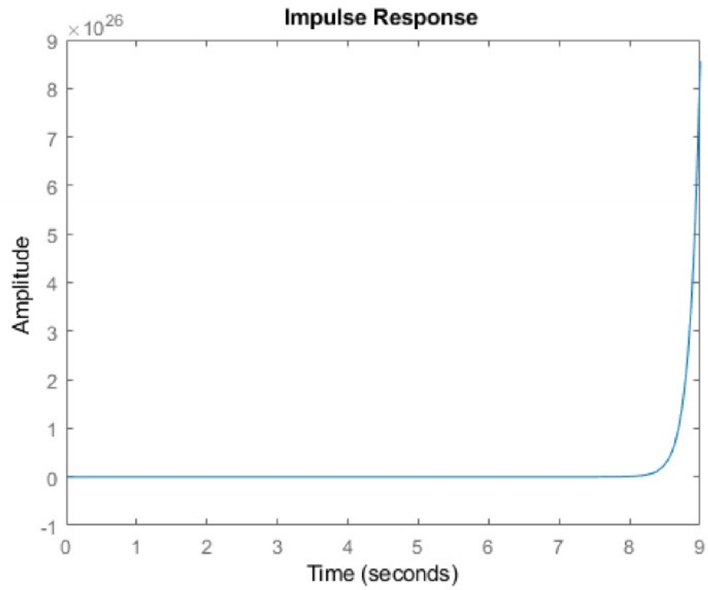


Figure 3 – Response of the system to an impulse.

As it can be seen in Figure 3, the system response to an impulse grows unboundedly as time progresses, therefore the system is unstable (notice that this system has two poles that cause it to be unstable, namely, $s = 7$ and $s = 1$)

Figure 3 shows the behavior of an unstable system. Figure 4 shows the behavior of a system that is stable (but not asymptotically stable). Notice that the response of the system is bounded to a constant value, in this case to 1, but does not decay to zero as time progresses.

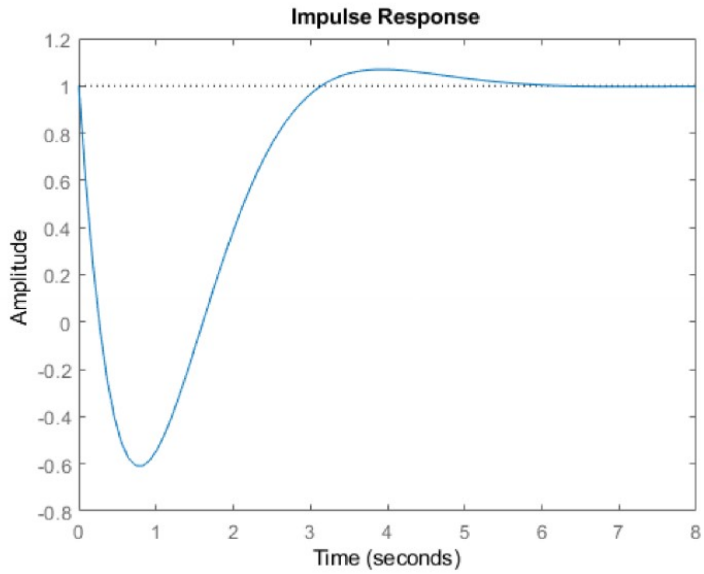


Figure 4 – Impulse response of a stable system.

Figure 5 shows the behavior of a system that is asymptotically stable. Notice that the response of the system to an impulse decays to zero as time progresses.

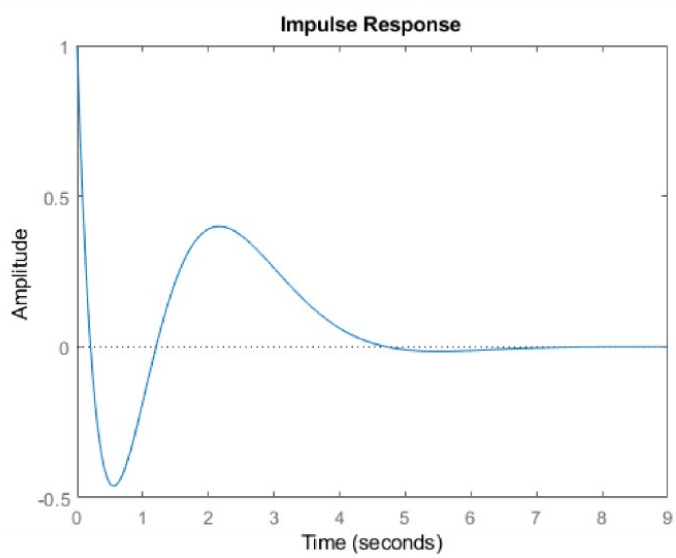


Figure 5 – Impulse response of an asymptotically stable system.

## 5. Exercises

Given the following differential equations,

- Find the corresponding transfer functions
- Find the poles and zeros and plot the pole-zero map
- Determine if the system is stable, asymptotically stable or unstable by both analyzing the poles and zeros and by plotting the impulse response of the system.

1. $$\frac{d^2y}{dt^2} + 4\frac{dy}{dt} + 3y(t) = \frac{dx}{dt} + 3x(t)$$

$y(s)(s^2+4s+3) = X(s)(s+3) \to$ stable

2. $$\frac{d^3y}{dt^3} + 3\frac{d^2y}{dt^2} + \frac{dy}{dt} - 5y(t) = \frac{d^2x}{dt^2} - 7\frac{dx}{dt} + 12x(t)$$

$y(s)(s^3+3s^2-s-5) = X(s)(s^2-7s+12) \to$ unstable

3. $$\frac{d^4y}{dt^4} + 3\frac{d^3y}{dt^3} + \frac{d^2y}{dt^2} - 5\frac{dy}{dt} = \frac{d^2x}{dt^2} - 10\frac{dx}{dt} + 21x(t)$$

$y(s)(s^4+3s^3+s^2-5s) = X(s)(s^2-10s+21) \to$ unstable

4. Design a third order transfer function that produces a stable (but not asymptotically stable) impulse response. The transfer function should have two zeros. Plot the impulse response, pole-zero map and explain why the response is stable. Hint: Choose one of the poles to be at $s = 0$.

as t increases the TF approaches constant nonzero value

5. Design a third order transfer function that produces an asymptotically stable impulse response. The transfer function should have two zeros. Plot the impulse response, pole-zero map and explain why the response is asymptotically stable.

as t increases the TF aproaches a constant zero

6. Design a third order transfer function that produces an unstable impulse response. The transfer function should have two zeros. Plot the impulse response, pole-zero map and explain why the response is unstable.

as t increases, the TF increases to infinity

**Consider the RLC circuit in Figure 6.**

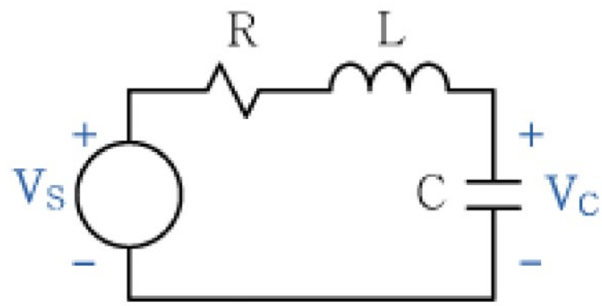


Figure 6 – Series RLC circuit.

Transforming the circuit above to Laplace domain, it becomes:

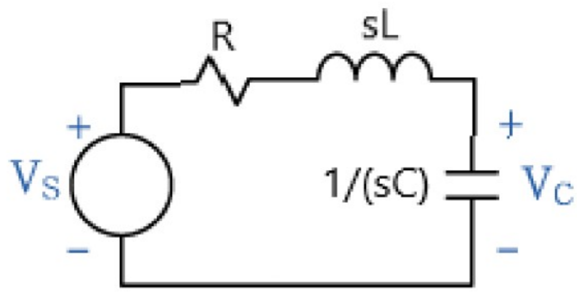


Figure 7 – Series RLC circuit in Laplace Domain.

$$\frac{Vc(s)}{Vs(s)} = \frac{1/LC}{s^2 + \frac{R}{L}s + \frac{1}{sC}}$$

7. Find the transfer function $H(s) = \frac{V_C(s)}{V_S(s)}$ of the circuit. Show all the steps you took to find your answer. The denominator of the transfer function should be in the form $s^2 + (something).s + something$.

8. For $R = 40\Omega$, $L = 3\ mH$ and $C = 5\ \mu F$, find the impulse response. Using step(H), find the step response of the system.

9. Find the poles, zeros and plot the pole-zero map. Is the system, stable, unstable or asymptotically stable? Explain your answer.

asymptotically stable because impulse aproaches zero

10. Maintain the values of $L$ and $C$ as $L = 3\ mH$ and $C = 5\ \mu F$. Set the resistance value to $R = 1\Omega$, $R = 2\Omega$, $R = 3\Omega$, and $R = 10\Omega$. What happens to the impulse response in terms of oscillations as the resistance varies?

as the resistance increases the number of oscillations decrease

***Pole-zero cancelation***

As stated before, a pole whose value is greater than zero (to the right-half of the $s$ - plane), produces an unstable response. To cause an unstable system to become stable, we need to get rid of the unstable pole. One easy way to achieve this goal is to introduce a zero that will *cancel* the unstable pole. For instance, consider the following transfer function with two unstable poles, one at $s = 4$ and another at $s = 6$:

$$H(s) = \frac{1}{(s-4)(s-6)(s+3)(s+7)(s+10)}$$

Introducing two zeros in the system, one at $s = 4$ and another at $s = 6$, in other words, multiplying the transfer function by $(s-4)(s-6)$, we get a new system that is stable:

$$H_{new}(s) = \frac{(s-4)(s-6)}{(s-4)(s-6)(s+3)(s+7)(s+10)}$$

$$\boxed{H_{new}(s) = \frac{1}{(s+3)(s+7)(s+10)}}$$

11. The transfer function shown below produces an unstable impulse response. Demonstrate this fact by plotting the impulse response and finding the poles of $H(s)$.

$$H(s) = \frac{s^2 - 7s + 12}{s^3 + s^2 - 2}$$

12. Write $H(s)$ in the form described in Equation (3).

$$H(s) = \frac{(s-3)(s-4)}{(s-(-1+1j))(s-(-1))(s-(1+1j))}$$

13. Perform pole-zero cancelation to get rid of the unstable pole and plot the impulse response of the new system. Is the new system stable?

$$H_{new}(s) = \frac{(s-3)(s-4)}{(s-(1+1j))(s-(1+1j))}$$

yes, asymptotically stable