Digital Communications
SSY125, Lecture 13

Low-Density Parity-Check Codes
(Chapter 11)

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LDPC Codes

- Introduced by Robert Gallager in 1961.
- Rediscovered in 1996 by Dave MacKay.
- One of the most celebrated code constructions, adopted in most of the communication standards.

LDPC Codes: Main Definitions

- An LDPC code is a binary linear block code.
- Can be described by a (typically large) $m \times n$ parity-check matrix $\boldsymbol{H}$ (recall: n is the code length).
- Property: $\boldsymbol{H}$ has a low density of ones.

LDPC Codes: Main Definitions

- Notation: $w_{\mathsf{r},i}$ and $w_{\mathsf{c},i}$ are the weight of the i-th row and the i-th column of $\boldsymbol{H}$.
- Low density: $w_{\mathsf{r},i} \ll n$ and $w_{\mathsf{c},i} \ll m$.

Two main classes of LDPC codes:

- Regular LDPC codes: Same number of ones per row and per column. Then,

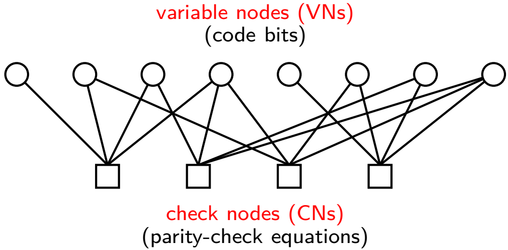
$$R_{\mathsf{c}}^{\mathsf{reg}} \geq 1 - \frac{w_{\mathsf{c}}}{w_{\mathsf{r}}}.$$

- Irregular LDPC codes: Number of ones not the same for all rows and/or for all columns. Then,

$$R_{\mathsf{c}}^{\mathsf{irreg}} \geq 1 - \frac{\bar{w}_{\mathsf{c}}}{\bar{w}_{\mathsf{r}}},$$

where $\bar{w}_{\mathsf{r}} = \frac{1}{m}\sum_{i=1}^{m} w_{\mathsf{r},i}$ and $\bar{w}_{\mathsf{c}} = \frac{1}{n}\sum_{i=1}^{n} w_{\mathsf{c},i}$.

Graphical Representation of LDPC Codes: The Bipartite Graph



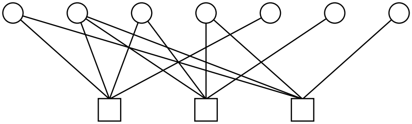
- LDPC codes (and any linear code) can be represented in a graphical, compact form: Bipartite graph (or Tanner graph) (Tanner, 1981).
- Two types of nodes:
  - Variable nodes (VNs): Represent the code bits.
  - Check nodes (or constraint nodes) (CNs): Represent the parity-check equations that the code bits satisfy.
- There exist an edge between a VN and a CN if the corresponding code bit participates in the corresponding parity-check equation.

The Bipartite Graph (the $(7,4)$ Hamming Code)

Parity-check matrix representation:

$$\boldsymbol{H} = \begin{pmatrix} 1 & 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 & 0 & 0 & 1 \end{pmatrix}$$

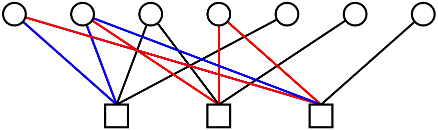
Bipartite-graph representation:



- neighborhood of a VN v, $\mathcal{N}(\mathsf{v})$: The set of all CNs v is connected to.
- neighborhood of a CN c, $\mathcal{N}(\mathsf{c})$: The set of all VNs c is connected to.

$\mathcal{N}(\mathsf{v}_1) = \{\mathsf{c}_1, \mathsf{c}_3\},\ \mathcal{N}(\mathsf{c}_3) = \{\mathsf{v}_1, \mathsf{v}_2, \mathsf{v}_4, \mathsf{v}_7\}$

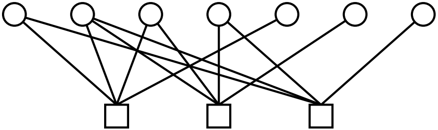
The Bipartite Graph



- A code is fully specified by both $\boldsymbol{H}$ and the bipartite graph.
- The values of the bits connected to the same CN must sum up to zero.
- A bipartite graph may contain cycles.
- Cycle: a sequence of edges starting and ending in the same node that form a closed path.
- The number of edges of the cycle is called its length.
- Girth of the graph: The length of the shortest cycle.

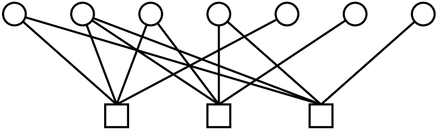
For this example: Girth 4.

Degree Distributions



- Node degree: Number of edges adjacent to the node.
- If the i-th column of $\boldsymbol{H}$ has Hamming weight $w_{\mathsf{c},i}$, the corresponding VN has degree $d_{\mathsf{v},i} \triangleq \deg(\mathsf{v}_i) = w_{\mathsf{c},i}$.
- If the i-th row of $\boldsymbol{H}$ has Hamming weight $w_{\mathsf{r},i}$, the corresponding CN has degree $d_{\mathsf{c},i} \triangleq \deg(\mathsf{c}_i) = w_{\mathsf{r},i}$.
- Regular LDPC codes: all VNs and all CNs have the same degree, i.e., $d_{\mathsf{v},i} = d_{\mathsf{v}}\ \forall i$ and $d_{\mathsf{c},i} = d_{\mathsf{c}}\ \forall i$.

Degree Distributions



- The VN and CN degree distributions can be expressed in polynomial form.
- Node-perspective VN degree distribution:

$$\Lambda(x) = \sum_i \Lambda_i x^i,$$

Λ_i: the probability that a VN has degree i (fraction of VNs of degree i).

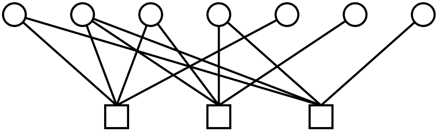
- Node-perspective CN degree distribution:

$$\mathsf{P}(x) = \sum_i \mathsf{P}_i x^i,$$

P_i: the probability that a CN has degree i (fraction of CNs of degree i).

- For regular LDPC codes: $\Lambda(x) = x^{d_\mathsf{v}}$ and $\mathsf{P}(x) = x^{d_\mathsf{c}}$.

Degree Distributions



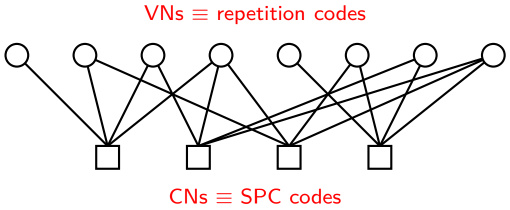
For the $(7,4)$ Hamming code:

$$\Lambda(x) = \frac{3}{7}x + \frac{3}{7}x^2 + \frac{1}{7}x^3$$

and

$$\mathrm{P}(x) = x^4.$$

LDPC Codes as a Network of Repetition and SPC Codes



- For a VN the bits associated to its adjacent edges must all be equal $\rightarrow$ A VN of degree d_{v} can be interpreted as a $(d_{\mathsf{v}}, 1)$ repetition code.
- All bits associated to the adjacent edges of a CN must sum up to zero $\rightarrow$ A CN of degree d_{c} is a $(d_{\mathsf{c}}, d_{\mathsf{c}} - 1)$ single parity-check (SPC) code.

LDPC codes can be interpreted as a network of connected repetition and SPC codes!

Decoding LDPC Codes: Belief Propagation Decoding

- Goal: Compute the log-APPs $L_{\mathsf{APP}}(u_i|\boldsymbol{y})$ based on the received (noisy) sequence $\boldsymbol{y}$,

$$L_{\mathsf{APP}}(u_i|\boldsymbol{y}) \triangleq \ln \frac{P_{\mathsf{APP}}(u_i = 0|\boldsymbol{y})}{P_{\mathsf{APP}}(u_i = 1|\boldsymbol{y})}.$$

MAP decoding of LDPC codes is unfeasible!

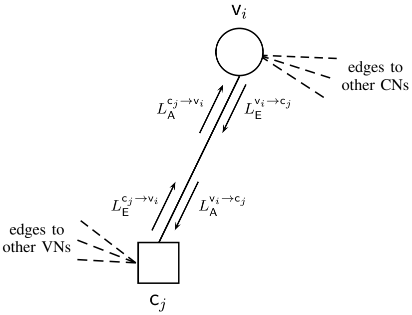
Decoding LDPC Codes

Suboptimal iterative decoding algorithm, based on the iterative exchange of messages along the edges of the bipartite graph: message passing decoding or belief propagation (BP) decoding.

Idea

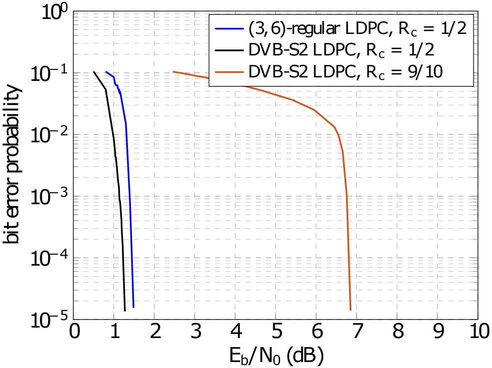
Interpret the LDPC code as a network of connected repetition and SPC codes $\rightarrow$ decode each repetition and SPC code optimally (using MAP decoding) and exchange extrinsic information between component decoders iteratively.

Decoding LDPC Codes: Belief Propagation Decoding



- The VNs and the CNs exchange extrinsic information iteratively along the edges of the bipartite graph.
- The extrinsic information is used as a priori information by the neighboring node.

Performance of Low-Density Parity-Check Codes



- The performance of LDPC codes depends greatly on the degree distributions $\Lambda(x)$ and $\mathrm{P}(x)$.
- Irregular LDPC codes perform closer to capacity than regular ones.
- To yield low error floors, short cycles in the graph must be avoided $\rightarrow$ construct codes with large girth.