Problem Sheet 2

Last modified October 24, 2023

Note: You may work in groups of two on the homework problems. Next week, for Monday **you are required to hand in problem 1**. For Wednesday **you are required to hand in problem 3**.

Please submit your results via Canvas (one submission per group, state all group members in the comments box). State the number of the problem sheet and the name of each group member on the top of the first page. You may submit a scanned handwritten solution. If using a smartphone for scanning, ensure that it is properly readable (resolution, lighting, angle).

Problems for Monday, November 13

Problem 1

1. What is the minimum required $\mathsf{E_b}/\mathsf{N_0}$ to operate at a spectral efficiency of 0.5, 1, and 2 [bits/s/Hz]?
2. What is the minimum required $\mathsf{E_s}/\mathsf{N_0}$ to operate at a spectral efficiency of 0.5, 1, and 2 [bits/s/Hz]?
3. Assume you are bandwidth constrained to $W = 5$ kHz, the input power is limited to $P = 0.01$ W, the noise PSD is given by $\mathsf{N_0} = 10^{-6}$ W/Hz. Is reliable communication possible at a bit rate of $R_\mathsf{b} = 60$ kbits/s? What about $R_\mathsf{b} = 50$ kbits/s?
4. Assume you are power limited to $P = 1$ mW but not bandwidth constrained, and the noise PSD is given by $\mathsf{N_0} = 10^{-8}$ W/Hz. Is reliable communication possible at a bit rate of $R_\mathsf{b} = 150$ kbits/s? What is the maximum possible bit rate? What is the spectral efficiency at that bit rate?
5. The channel coding theorem states that for a fixed bit rate below capacity, we can make the error probability as low as desired by using ideal coding. Are there any disadvantages for a practical communication system when using ideal coding?
6. What happens to the capacity of the continuous-time AWGN channel if $P \to \infty$? What happens to the capacity if $W \to \infty$? Give an explanation for both results.

Problem 2

Consider the real, discrete-time AWGN channel $\boldsymbol{r} = \boldsymbol{s} + \boldsymbol{n}$. We want to gain some intuition behind the capacity formula for this channel. Recall that in the lecture, the quantities $\mathsf{E_s}$ (expected energy per symbol) and $\sigma^2 = \mathsf{N_0}/2$ (noise variance) have been defined. If we interpret $\boldsymbol{r}$, $\boldsymbol{s}$, and $\boldsymbol{n}$ as points in an N-dimensional space, we can roughly state the following for the case when N grows very large:

1. $\boldsymbol{s}$ lies almost surely within an N-dimensional sphere of squared radius $N\mathsf{E_s}$.
2. $\boldsymbol{n}$ lies almost surely within an N-dimensional sphere of squared radius $N\sigma^2 = N\mathsf{N_0}/2$.
3. $\boldsymbol{r}$ lies almost surely within an N-dimensional "output" sphere of squared radius $N\mathsf{E_s} + N\mathsf{N_0}/2$.

Questions: How many noise spheres can one pack into the output sphere, such that they are (approximately) disjoint? What does this tell you about how many different inputs $\boldsymbol{s}$ can be distinguished reliably at the receiver? What is then the number of bits per dimension N that can be transmitted reliably? Note: The volume of an N-dimensional sphere with radius R is given by

$$\frac{(\pi R^2)^{N/2}}{(N/2)!}.$$

Problems for Wednesday, November 15

Problem 3

Part I

Suppose that you wish to detect whether the binary random variable X is 0 or 1. The observation is $Y \sim \mathcal{N}(1, 4)$ for $X = 1$ and $Y \sim \mathcal{N}(-1, 1)$ for $X = 0$, where $\mathcal{N}(\mu, \sigma^2)$ denotes a Gaussian distribution with mean μ and variance σ^2.

1. Show that the optimal decision rule is equivalent to comparing a function of the form $ay^2 + by$ with a threshold.
2. Specify the rule explicitly (i.e., specify a, b and the threshold) when $P_X(0) = 1/3$.

Part II

Assume that your observation is given by $\boldsymbol{r} = \alpha\boldsymbol{s} + \boldsymbol{n}$, where $\alpha \in \mathbb{R}$ is a real constant, unknown to the receiver, $\boldsymbol{s}$ is an N-dimensional vector of real information symbols, and $\boldsymbol{n}$ contains independent and identically distributed (i.i.d.) zero-mean Gaussian random variables with variance σ^2. If you assume that the receiver knows the data symbol vector $\boldsymbol{s}$ (known as pilot-aided estimation), what is the maximum likelihood estimator for α?

Problem 4

Assume that you receive $\boldsymbol{y} = \boldsymbol{x}e^{j\phi} + \boldsymbol{n}$, where $\boldsymbol{x}$ is an N-dimensional complex vector with elements from a discrete signal constellation $\mathcal{X}$ (for example a 16-QAM constellation) and the elements of the noise vector $\boldsymbol{n}$ are i.i.d. circularly symmetric complex Gaussian random variables with variance σ^2 per dimension, $\mathbb{E}[\boldsymbol{n}\boldsymbol{n}^H] = 2\sigma^2\mathbf{I}$. If the receiver knows the data vector $\boldsymbol{x}$, what is the maximum likelihood (ML) estimator for ϕ?

Extra Problems

Problem 5

Let the joint distribution $P_{X,Y}(x, y)$ of the two random variables X and Y be given as $P_{X,Y}(0, 0) = P_{X,Y}(0, 1) = P_{X,Y}(1, 1) = 1/3$. Find

1. $\mathsf{H}(X)$, $\mathsf{H}(Y)$
2. $\mathsf{H}(X|Y)$, $\mathsf{H}(Y|X)$
3. $\mathsf{H}(X, Y)$
4. $\mathsf{H}(Y) - \mathsf{H}(Y|X)$

Problem 6

Consider a random variable X which takes on four values with probabilities $\left(\frac{1}{3}, \frac{1}{3}, \frac{1}{4}, \frac{1}{12}\right)$.

1. Construct a Huffman code for this random variable.
2. Show that there exist two different sets of optimal lengths for the codewords, namely, show that codeword length assignments $(1, 2, 3, 3)$ and $(2, 2, 2, 2)$ are both optimal.

Problem 7

Consider a discrete memoryless source whose output U takes on values in $\mathcal{U} = \{u_1, u_2, u_3\}$ with the following probability distribution:

u	u_1	u_2	u_3
$P_U(u)$	0.3	0.5	0.2

1. Use the Huffman coding algorithm to encode over two consecutive symbols of this source.
2. What is the efficiency of the code?
3. How can the code efficiency be increased? What are the disadvantages?

Problem 8

Let X and Y be random variables that take on values $x_1, x_2, \ldots, x_r$ and $y_1, y_2, \ldots, y_s$, respectively. Let $Z = X + Y$.

1. Show that $\mathsf{H}(Z|X) = \mathsf{H}(Y|X)$. Argue that if X, Y are independent, then $\mathsf{H}(Y) \leq \mathsf{H}(Z)$ and $\mathsf{H}(X) \leq \mathsf{H}(Z)$. Thus the addition of independent random variables adds uncertainty.
2. Give an example of (necessarily dependent) random variables in which $\mathsf{H}(X) > \mathsf{H}(Z)$ and $\mathsf{H}(Y) > \mathsf{H}(Z)$.