Digital Communications
Lectures 2, 3, and 4

A Measure of Information, Source Compression
(Chapters 2 and 3)

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Slides prepared by Alexandre Graell i Amat

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CHALMERS

digital data, voice
movie, audio

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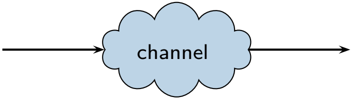

satellite link,
fiber, telephone line,
hard-disk drive, DVD

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In These Lectures...

- How do we measure information
- How can we mathematically describe an information source
- Data compression

Discrete Information Source

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The information source generates a sequence of symbols that take values on a finite alphabet $\mathcal{X}$.

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- Probability mass function (PMF) $P_X(x)$; $\Pr(X = x_i) = P_X(x_i) = p_i$,

$$p_i \geq 0 \ \forall i, \quad \text{and} \quad \sum_{x_i \in \mathcal{X}} p_i = 1.$$

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In This Course...

Discrete memoryless sources, i.e., sources where the random variables $X^{(i)}$ are independent of each other.

A Measure of Information

- How can we measure the information content of a particular outcome $X = x_i$ from the source?
- How much information does a source contain in average?

A Measure of Information

Example: Consider the following two claims (about Göteborg's weather)

- This weekend it will rain
- This weekend there will be more than 30°C

Which sentence conveys more information?

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The information associated to a particular message x is related to its probability: The less probable the message is, the more information it conveys!

A Measure of Information

A good measure of information $\mathsf{i}(x)$, associated to a symbol x should:

P: Be a decreasing function of the probability of the symbol, i.e.,

$$\mathsf{i}(x_i) > \mathsf{i}(x_j) \quad \text{if} \quad p_i < p_j$$

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Example: Consider the following three claims

- This weekend it will rain
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Does claim 3 convey more/less information than claim 2?

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- A slight change in the probability of a symbol should only cause a slight change in the information it conveys $\longrightarrow$ A measure of information should be a continuous function of the probability
- Two equiprobable symbols carry the same amount of information $\longrightarrow$ A measure of information should depend only on the probability of occurrence

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A good measure of information $\mathsf{i}(x)$ associated to a symbol x should:

P1: Not depend on the symbol itself, but only on its probability,

$$\mathsf{i}(x_i) = \mathsf{i}(x_j) \quad \text{if} \quad p_i = p_j$$

P2: Be a continuous, decreasing function of the probability of the symbol,

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Example: Rolling a dice

Information conveyed by the outcome of rolling a dice once: i_1

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For two independent random variables X and Y, the information we gain when we learn X and Y should equal the sum of the information gained if we learn X and Y separately $\longrightarrow$ The information should be additive

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P3: For two independent symbols x_i and x_j,

$$\mathsf{i}(x_i, x_j) = \mathsf{i}(x_i) + \mathsf{i}(x_j)$$

Shannon's Information Measure

Definition (Shannon's Information Content)

The Shannon information content of the outcome $X = x_i$ is

$$\mathsf{i}(x_i) = \log_a \frac{1}{p_i}.$$

The base a of the logarithm determines the unit of information. If $a = 2$, the unit of information is the bit. If the base is e, then the information unit is called nat.

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$$\begin{aligned}\mathsf{i}(x,y) &= \log \frac{1}{P(x,y)} = \log \frac{1}{P(x)P(y)} = \log \frac{1}{P(x)} + \log \frac{1}{P(y)} \\ &= \mathsf{i}(x) + \mathsf{i}(y).\end{aligned}$$

Information Content of a Source

Average Information Content of a Source

A complete characterization of the source can then be obtained by defining the average information content,

$$\mathsf{H}(X) = \sum_{i=1}^{M} p_i \mathsf{i}(x_i) = \sum_{i=1}^{M} p_i \log \frac{1}{p_i}.$$

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Definition (Entropy)

The entropy of a random variable (random symbol) X that takes values on the alphabet $\mathcal{X} = \{x_1, x_2, \ldots, x_M\}$ with probabilities $p_1, p_2, \ldots, p_M$ is defined as

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The entropy measures the average information content (or *uncertainty*) of X.

Average Information Content of a Source

Definition (The Binary Entropy Function)

Let X be a binary source with two possible values $\mathcal{X} = \{x_1, x_2\}$ such that $P(x_1) = p$ and $P(x_2) = 1 - p$. Then

$$\mathsf{H}(X) = \mathsf{H}_\mathsf{b}(p),$$

where $\mathsf{H}_\mathsf{b}(p)$ is called the binary entropy function,

$$\mathsf{H}_\mathsf{b}(p) \triangleq p \log \frac{1}{p} + (1-p) \log \frac{1}{1-p}.$$

Average Information Content of a Source

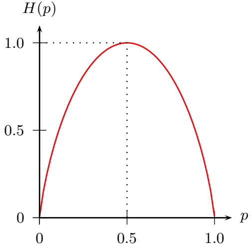

Figure: The binary entropy function as a function of the probability p.

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Lemma

The entropy of a random variable X that takes values on the alphabet $\mathcal{X} = \{x_1, \ldots, x_M\}$, $|\mathcal{X}| = M$, is bounded by

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where

$$\mathsf{H}(X) = 0 \qquad \textit{if and only if } \; p_i = 1 \; \textit{ for some } \; i,$$

$$\mathsf{H}(X) = \log M \qquad \textit{if and only if } \; p_i = \frac{1}{M} \;\; \forall i.$$

$\mathsf{H}(X) \geq 0$

$\mathsf{H}(X) \leq \log M$ (hint: use $\ln x \leq x - 1$)

Data Compression

Wh_ c_n y_u r_co_er this q_est_on?

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Most information sources contain redundancy!

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Data (Source) Compression

Find an efficient representation of the output of the source which results in zero or little redundancy

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Data (Source) Compression

Find an efficient representation of the output of the source which results in zero or little redundancy $\longrightarrow$ Encode (map) the symbols (or messages) at the output of the source to a sequence of symbols, called codeword, that is shorter in average.

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Source Encoder

- Can encode source symbols separately or groups of symbols.

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Source Encoder

- Can encode source symbols separately or groups of symbols.
- Typically requires knowledge of the statistics of the source
- We can achieve compression on average by assigning shorter codewords to more probable symbols and longer codewords to less likely ones: Variable-length encoder

Data Compression

Example: The Morse Code

The Morse code assigns the most frequent letters to shorter codewords, and less frequent letters to longer codewords:

- "e" $\longrightarrow$ "$\cdot$"
- "q" $\longrightarrow$ "$- - \cdot -$"

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- Are there efficient methods to assign codewords to source symbols?
- How can we make sure that the source code is easy to decode?

Data Compression

symbol	probability	$\mathcal{C}_1$	$\mathcal{C}_2$	$\mathcal{C}_3$	$\mathcal{C}_4$	$\mathcal{C}_5$	$\mathcal{C}_6$
a	$1/2$	000	00	00	0	0	0
b	$1/4$	001	00	01	01	01	10
c	$1/8$	101	10	10	001	011	110
d	$1/8$	111	11	11	111	111	111

Example: Alphabet $\{a, b, c, d\}$ with probabilities $\{1/2, 1/4, 1/8, 1/8\}$

We would like to design a good (binary) source code to compress the language produced by this alphabet.

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- $\mathcal{C}_1$ Not efficient
- $\mathcal{C}_2$ Not uniquely decodable
- $\mathcal{C}_3$ Uniquely decodable, not efficient

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b	$1/4$	001	00	01	01	01	10
c	$1/8$	101	10	10	001	011	110
d	$1/8$	111	11	11	111	111	111

Example: Alphabet $\{a, b, c, d\}$ with probabilities $\{1/2, 1/4, 1/8, 1/8\}$

We would like to design a good (binary) source code to compress the language produced by this alphabet.

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- $\mathcal{C}_4$ Not uniquely decodable

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- $\mathcal{C}_6$ Uniquely decodable, instantaneous (prefix-free)

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Example

If $\Sigma = \{a, b, c\}$ then a, ab, aac and $bbac$ are strings over Σ of lengths one, two, three and four respectively.

Data Compression

Definition (Binary Source Code)

Consider a random variable X which takes values on $\mathcal{X} = \{x_1, \ldots, x_M\}$.

- A binary encoder $\mathcal{E}$ is a function $\mathcal{E} : \mathcal{X} \to \{0,1\}^+$ that maps each source symbol $x_i \in \mathcal{X}$ to a binary sequence $\boldsymbol{c}_i \in \{0,1\}^+$ (codeword). The number of bits in $\boldsymbol{c}_i$ is called its length and is denoted by ℓ_i.

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Example: Code $\mathcal{C}_6$

x_i		$\boldsymbol{c}_i$	ℓ_i
a	$\to$	0	1
b	$\to$	10	2
c	$\to$	110	3
d	$\to$	111	3

Definition (Extended Code)

The extended code $\mathcal{C}^+$ is the set of binary sequences resulting from the mapping $\mathcal{E}^+ : \mathcal{X}^+ \to \{0,1\}^+$ from a sequence of symbols in the alphabet $\mathcal{X}$ to a binary sequence obtained by concatenating the corresponding codewords, such that, for a sequence of symbols of length n, at times $t = 1, \ldots, n$,

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$$\begin{array}{ccc} aab & \to & 0010 \\ bca & \to & 101100 \end{array}$$

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symbol	$\mathcal{C}_4$	$\mathcal{C}_5$	$\mathcal{C}_6$
a	0	0	0
b	01	01	10
c	001	011	110
d	111	111	111

$\mathcal{C}_4$ is not uniquely decodable: $ab \to 001$ and $c \to 001$.

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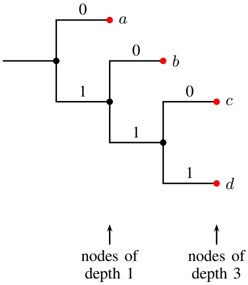
- Prefix-free code $\Longrightarrow$ Uniquely decodable

Binary Code Tree



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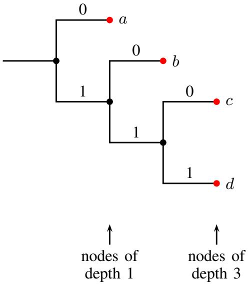

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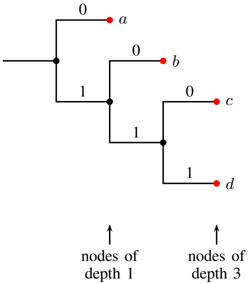

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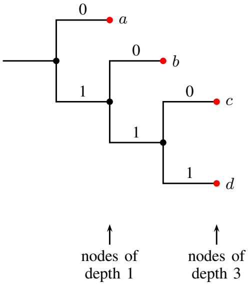

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- Every codeword can be represented by a particular path through the tree

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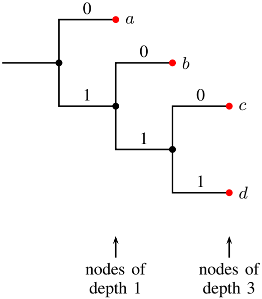

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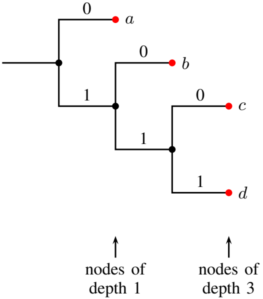

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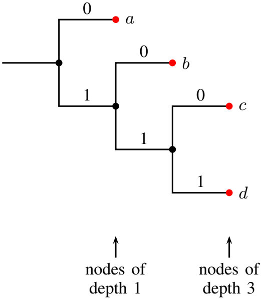

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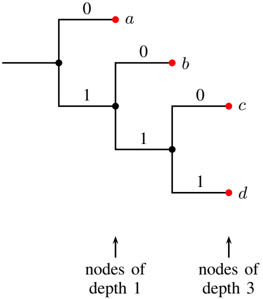

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Kraft's Inequality

Theorem (Kraft's Inequality)

There exists a binary prefix-free code of cardinality M and codeword lengths $\ell_1, \ell_2, \ldots, \ell_M$ if and only if

$$\sum_{i=1}^{M} 2^{-\ell_i} \leq 1.$$

Kraft's Inequality: proof

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$\Longrightarrow$ We can restrict ourselves to prefix-free codes with no loss in performance!

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$$\bar{L} = \frac{1}{2} \cdot 1 + \frac{1}{4} \cdot 2 + 2 \cdot \frac{1}{8} \cdot 3 = 1.75 \text{ bits.}$$

The Limits of Source Coding

- What is the best achievable compression?

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$$\bar{L} \geq \mathsf{H}(X).$$

Proof: (hint: use $\ln x \leq x - 1$)

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Theorem (Source Coding Theorem for a Single Random Symbol)

Let X be a random variable generated by a discrete memoryless source with entropy $\mathsf{H}(X)$. There exists a prefix-free code $\mathcal{C}$ with average codeword length satisfying

$$\mathsf{H}(X) \leq \bar{L}(\mathcal{C}) < \mathsf{H}(X) + 1.$$

Proof:

Optimal Source Coding: Huffman Coding

- Given a probability distribution P_X, how can we design an optimal prefix-free code? (i.e., with minimum $\bar{L}$)

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5. Starting from the root of the obtained tree, assign the binary symbols 0 and 1 to each pair of branches that arise from each node. The codeword for each symbol is read as the binary sequence from the root to the leaf associated to the symbol

Optimal Source Coding: Huffman Coding

p	x
0.5	x_1
0.15	x_2
0.15	x_3
0.10	x_4
0.05	x_5
0.05	x_6

Example (Huffman Coding)

Coding of $\mathcal{X} = \{x_1, x_2, x_3, x_4, x_5, x_6\}$, $P_X = \{0.5, 0.15, 0.15, 0.10, 0.05, 0.05\}$.
$\mathsf{H}(X) = 2.08548$ bits.

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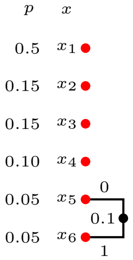

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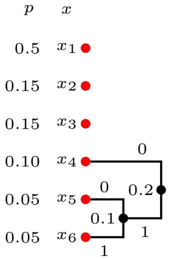

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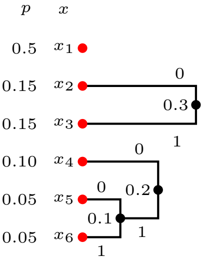

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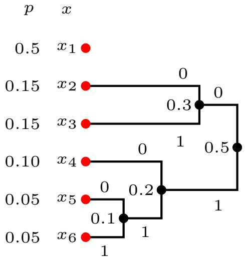

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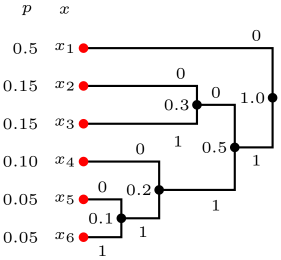

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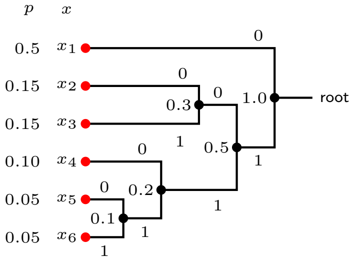

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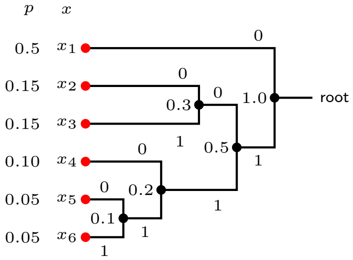

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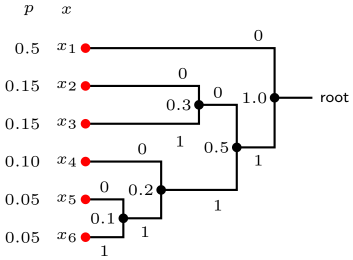

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Source compression by encoding each output of the source independently using an optimal prefix-free (Huffman) code.

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We can do better than encoding each symbol separately!

Encoding Blocks of Symbols

What We Have Achieved up to Now...

Source compression by encoding each output of the source independently using an optimal prefix-free (Huffman) code.

- Is there a more efficient approach?

Encoding Blocks of Symbols

We can do better than encoding each symbol separately! $\longrightarrow$ Encoding blocks of symbols.

Encoding Blocks of Symbols

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$\boxed{\text{Source}} \longrightarrow Y \in \mathcal{Y} = \{y_1, y_2, \ldots, y_{M^\nu}\} \sim \mathcal{X}^\nu$

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where $x^{(j)} \in \mathcal{X}$.

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We design a code for an equivalent source Y with alphabet $\mathcal{Y} = \{y_1, y_2, \ldots, y_{M^\nu}\}$, $|\mathcal{Y}| = M^\nu$, consisting of all possible tuples $(x^{(1)}, x^{(2)}, \ldots, x^{(\nu)})$ of length ν.

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For a memoryless source, the probability of a message y_i is

$$p_i = \prod_{j=1}^{\nu} p_{i,j},$$

where $p_{i,j}$ is the probability of the jth symbol of y_i.

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We can build a code for Y with average length $\bar{L}_\nu$,

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The average codeword length necessary to describe one source symbol, $\bar{L}_\nu/\nu$!

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The Source Coding Theorem

Theorem (Source Coding Theorem)

Let X be a random variable generated by a discrete memoryless source with entropy $\mathsf{H}(X)$. There exists a prefix-free code $\mathcal{C}$ that encodes messages of length ν symbols with average codeword length per source symbol $\frac{\bar{L}_\nu}{\nu}$ satisfying

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However...encoding large blocks of symbols requires long codes $\longrightarrow$ difficult to construct, delay.

The Source Coding Theorem

Efficiency of a Source Code

$$\eta = \frac{\nu \mathsf{H}(X)}{\bar{L}_\nu}, \quad 0 \leq \eta \leq 1.$$

The Source Coding Theorem

Source $\longrightarrow X \in \{x_1, x_2, x_3\}$

Source $\longrightarrow Y \in \{y_1, y_2, \ldots, y_9\} = \{x_1x_1, x_1x_2, \ldots, x_3x_3\}$

Example: Source X, $\mathcal{X} = \{x_1, x_2, x_3\}$, with probabilities $\{0.45, 0.35, 0.20\}$. $\mathsf{H}(X) = 1.513$ bits.

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- $\mathcal{C}_1$: Encode symbols separately using a Huffman code. $\bar{L} = 1.55$ bits, $\eta = 0.976$.
- $\mathcal{C}_2$: Encode pairs of symbols, i.e., we have equivalent source $Y = (X^1, X^2)$, with $\mathcal{Y} = \{y_1, y_2, \ldots, y_9\} = \{x_1x_1, x_1x_2, x_1x_3, x_2x_1, x_2x_2, x_2x_3, x_3x_1, x_3x_2, x_3x_3\}$, $|\mathcal{Y}| = |\mathcal{X}|^2 = 9$, and encode messages Y using a Huffman code. $\frac{\bar{L}_\nu}{\nu} = 1.53375$ bits, $\eta = 0.989$.

The Source Coding Theorem

Table: Huffman code, encoding symbols separately, $\bar{L} = 1.55$, $\eta = 0.976$

symbol	probability	codeword
x_1	0.45	1
x_2	0.35	00
x_3	0.20	01

Table: Huffman code, encoding blocks of two symbols, $\bar{L}_\nu = 3.0675$, $\eta = 0.989$

symbol	probability	codeword
x_1x_1	0.2025	10
x_1x_2	0.1575	001
x_2x_1	0.1575	010
x_2x_2	0.1225	011
x_1x_3	0.09	111
x_3x_1	0.09	0000
x_2x_3	0.07	0001
x_3x_2	0.07	1100
x_3x_3	0.04	1101

Further Discussion

Source Compression using Huffman Codes

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- Used to compress sound, video or image (jpeg compression).