Problem Sheet 5

Last modified October 24, 2023

Note: You may work in groups of two on the homework problems. Next week, for Monday **you are required to hand in problem 1**. The week after, for Monday **you are required to hand in problem 3**.

Please submit your results via Canvas (one submission per group, state all group members in the comments box). State the number of the problem sheet and the name of each group member on the top of the first page. You may submit a scanned handwritten solution. If using a smartphone for scanning, ensure that it is properly readable (resolution, lighting, angle).

Problems for Monday, December 4

Problem 1

Consider an RSC encoder with generators $(1 + D + D^2, 1 + D^2)$. Find the output of the weight-1 input sequence 1000... and note that it is of infinite length. (The length is measured by the location of the last '1', or the degree-plus-1 of the polynomial representation of the sequence.) Find the output of the weight-2 input sequence 100100... and note that it is of finite length. Why does the weight-2 sequence 10000100... produce an infinite-length sequence?

Problem 2

Consider a rate 2/3 convolutional code in which two inputs $(u_{1,k}, u_{2,k})$ come in at every time k, and the three outputs $(y_{1,k}, y_{2,k}, y_{3,k})$ are emitted at every time k. The input-output relation is given by

$$\begin{aligned} y_{1,k} &= u_{1,k} + u_{1,k-1} + u_{2,k-1} \\ y_{2,k} &= u_{1,k-1} + u_{2,k} \\ y_{3,k} &= u_{1,k} + u_{2,k} \end{aligned}$$

- Draw a simple block diagram of the preceding encoding function.
- Draw a trellis section showing all possible transitions between the encoder states. Label each transition by $u_{1,k}u_{2,k}/y_{1,k}y_{2,k}y_{3,k}$.
- Find the free distance of the code.

Problems for Wednesday, December 6

Problem 3

Consider a code $\mathcal{C}$ with parity check matrix

$$\boldsymbol{H} = \begin{bmatrix} 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 & 1 \end{bmatrix}.$$

1. Draw the Tanner graph for this code and identify all cycles.
2. Specify the degree distribution for the check nodes and variable nodes. Is it a regular LDPC code?

Note: Even though $\boldsymbol{H}$ is not sparse, consider $\mathcal{C}$ to be an LDPC code.

Problem 4

Consider an LDPC code represented by the Tanner graph in Fig. 1. Find all cycles and its girth. Specify the degree distribution for the check nodes and variable nodes. Is it a regular LDPC code?

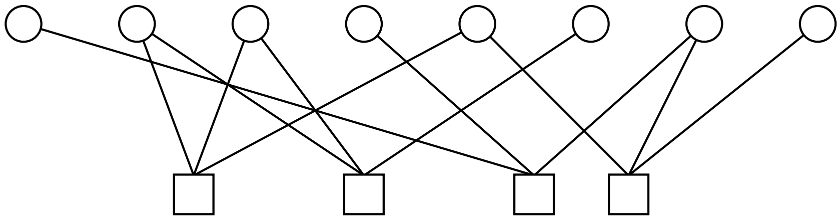

Figure 1: Tanner graph.

Extra Problems

Problem 5

Part I

Let X and Y be two independent integer-valued random variables. Let X be uniformly distributed over $\{1, 2, \ldots, 16\}$, and let $\mathrm{P}(Y = k) = 2^{-k}$, where $k = 1, 2, 3, \ldots$ The following expression may be useful:

$$\sum_{i=0}^{\infty} i a^i = \frac{a}{(1-a)^2}, \quad |a| < 1.$$

1. Find the entropies $\mathsf{H}(X)$, $\mathsf{H}(Y)$, and $\mathsf{H}(X, Y)$, in bits.

Part II

Consider a source with 5 symbols which have probabilities $P = \{0.3, 0.3, 0.2, 0.1, 0.1\}$.

1. What is the source entropy?

2. Apply the Huffman coding algorithm to this source.

3. What is the efficiency of the code?

4. Let $l_1, l_2, \ldots, l_5$ be the codeword lengths of the code and consider an alternative probability distribution $P' = \{p'_1, p'_2, \ldots, p'_5\}$ for the 5 symbols. Determine $p'_1, p'_2, \ldots, p'_5$ such that the expected codeword length $\bar{L}$ for the code,
$$\bar{L} = \sum_{i=1}^{5} p'_i l_i,$$
is equal to the entropy of the random variable defined by the distribution P'. Use the codeword lengths that you found in question 2.

Part III

Consider a channel whose input is a random variable X which takes values on $\mathcal{X} = \{0, 1\}$ with probabilities $P(X = 0) = P(X = 1) = 0.5$. The channel output is a random variable Y which takes values on $\mathcal{Y} = \{0, 2\}$. The channel is defined by the conditional distribution $P(y|x)$,

$$P(0|0) = P(2|1) = 1 - \varepsilon$$
$$P(2|0) = P(0|1) = \varepsilon.$$

1. What is the entropy of the source, $\mathsf{H}(X)$, the probability distribution of the output, $P(y)$, and the entropy of the output, $\mathsf{H}(Y)$?

2. What is the joint probability distribution for the source and the output, $P(x, y)$, and what is the joint entropy, $\mathsf{H}(X, Y)$?

3. Use $\mathsf{H}(X)$, $\mathsf{H}(Y)$, and $\mathsf{H}(X, Y)$ to compute the mutual information of this channel, $\mathsf{I}(X; Y)$, as a function of ε.

4. What is the maximum mutual information of the channel in bits?

5. For what value of ε is the mutual information minimal? What is the mutual information in this case?