Digital Communications
SSY125, Lectures 5 and 6

Communication over a Noisy Channel
(Chapters 4 and 5)

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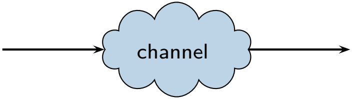
Slides prepared by Alexandre Graell i Amat

November 8, 2023

CHALMERS

digital data, voice
movie, audio
100110101



satellite link,
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In These Lectures...

- How much information can we transmit reliably over an unreliable channel?
- How do we achieve this in practice?

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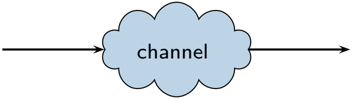

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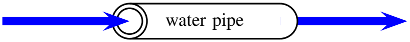
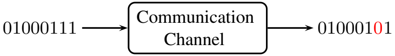
Shannon's Seminal Contribution

There is a fundamental limit, i.e., a highest rate, at which information can be transmitted reliably over the channel: channel capacity.

Model of a Noisy Communication Channel





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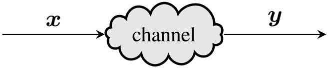

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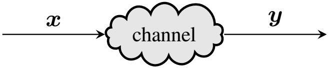

Discrete Memoryless Channel

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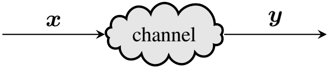

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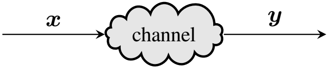

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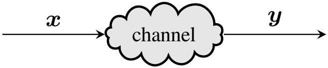

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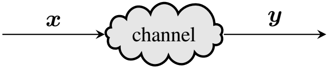

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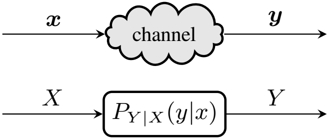

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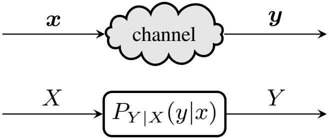

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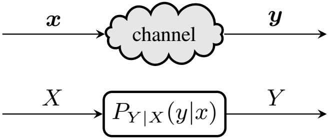

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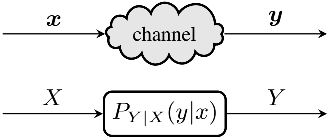

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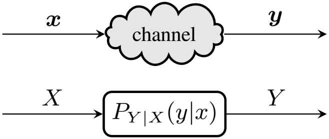

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- Channel input: $X \in \mathcal{X}$.
- Channel output: $Y \in \mathcal{Y}$.
- Entirely specified by the conditional PMF $P_{Y|X}(y|x)$.

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Average amount of uncertainty on the channel output Y:

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Definition (Conditional Entropy of X given $Y = y$)

The conditional entropy of X given the event $Y = y$ is

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&= \sum_{x \in \mathcal{X}} \sum_{y \in \mathcal{Y}} P(x, y) \log \frac{1}{P(x|y)}.
\end{aligned}$$

System Entropies

Theorem (Conditioning reduces entropy)

For any two random variables X and Y,

$$\mathsf{H}(X|Y) \leq \mathsf{H}(X).$$

with equality if and only if X and Y are statistically independent.

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Lemma

The joint entropy of two statistically independent random variables X and Y is

$$\mathsf{H}(X,Y) = \mathsf{H}(X) + \mathsf{H}(Y).$$

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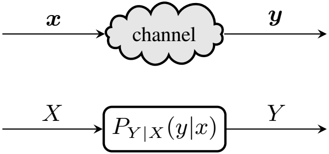
Since $P(x, y) = P(y, x)$, we can also write

$$\mathsf{H}(X, Y) = \mathsf{H}(Y) + \mathsf{H}(X|Y).$$

Information Conveyed by the Channel



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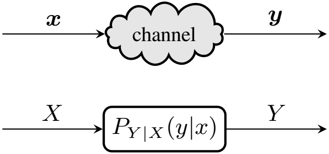

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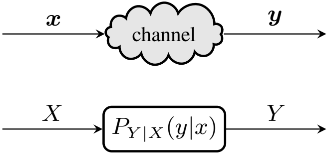

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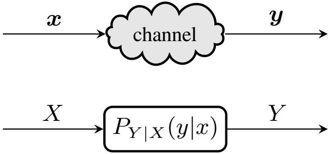

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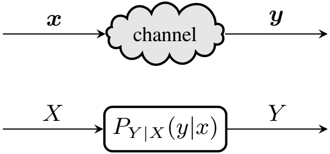

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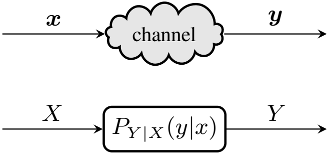

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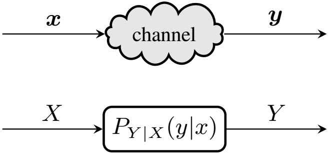

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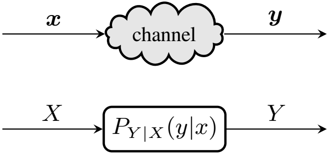

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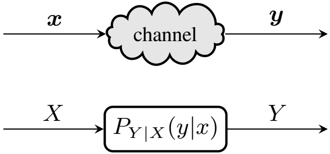

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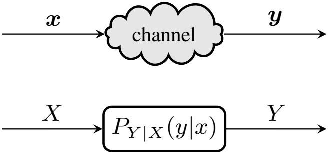

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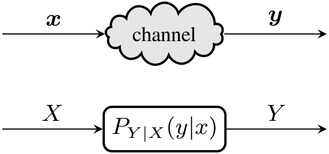

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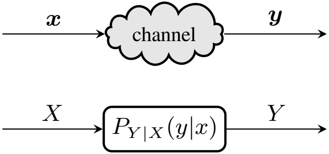

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- How much information have we conveyed over the channel?
  - *Before* receiving Y: Uncertainty on X is $\mathsf{H}(X)$
  - *After* receiving Y: Uncertainty on X is $\mathsf{H}(X|Y)$
  - Thus, the information conveyed is: $\mathsf{H}(X) - \mathsf{H}(X|Y)$!

Information Conveyed by the Channel

Definition (Mutual Information)

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By simple manipulation of the entropies,

$$\mathsf{I}(X;Y) = \sum_{x \in X} \sum_{y \in Y} P(x,y) \log \frac{P(x,y)}{P(x)P(y)}.$$

Information Conveyed by the Channel

Definition (Mutual Information)

The mutual information between X and Y is

$$\mathsf{I}(X;Y) \triangleq \mathsf{H}(X) - \mathsf{H}(X|Y).$$

- The mutual information measures the average reduction in uncertainty about X that results from learning the value of Y.

By simple manipulation of the entropies,

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The Channel Capacity

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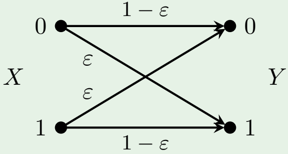
For a given channel, the channel capacity is defined to be the maximum of the mutual information, maximized over all possible input distributions $P_X(x)$,

$$\mathsf{C} \triangleq \max_{P_X} \mathsf{I}(X;Y).$$

Mutual Information and Channel Capacity

Running Example: The Binary Symmetric Channel



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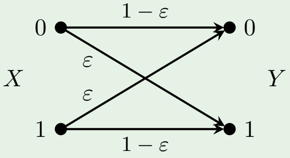

- **Input:** $X \in \mathcal{X} = \{0, 1\}$.

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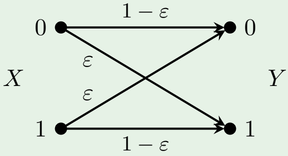
Running Example: The Binary Symmetric Channel



- Input: $X \in \mathcal{X} = \{0, 1\}$.
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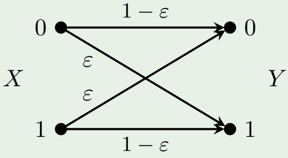

- Input: $X \in \mathcal{X} = \{0, 1\}$.
- Output: $Y \in \mathcal{Y} = \{0, 1\}$.
- Channel defined by the transition probabilities

$$P(0|0) = P(1|1) = 1 - \varepsilon$$
$$P(1|0) = P(0|1) = \varepsilon.$$

Mutual Information and Channel Capacity

Running Example: The Binary Symmetric Channel

- We compute first the mutual information,

$$\mathsf{I}(X;Y)$$

Mutual Information and Channel Capacity

Running Example: The Binary Symmetric Channel

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Mutual Information and Channel Capacity

Running Example: The Binary Symmetric Channel

- We compute first the mutual information,

$$
\begin{aligned}
\mathsf{I}(X;Y) &= \mathsf{H}(Y) - \mathsf{H}(Y|X) \\
&= \mathsf{H}(Y) - \sum_{x\in\mathcal{X}} P(x)\mathsf{H}(Y|X=x)
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$\mathsf{I}(X;Y)$ depends on P_Y, thus on P_X!

The Channel Capacity

Running Example: The Binary Symmetric Channel

$$\mathsf{I}(X;Y) = \mathsf{H}(Y) - \mathsf{H}_\mathsf{b}(\varepsilon)$$

- Let $P(X=0)=\delta$ and $P(X=1)=1-\delta$.

The Channel Capacity

Running Example: The Binary Symmetric Channel

$$\mathsf{I}(X;Y) = \mathsf{H}(Y) - \mathsf{H_b}(\varepsilon)$$

- Let $P(X=0) = \delta$ and $P(X=1) = 1-\delta$.
- The mutual information is maximized for

The Channel Capacity

Running Example: The Binary Symmetric Channel

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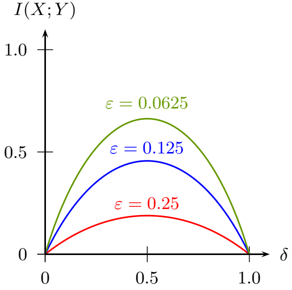

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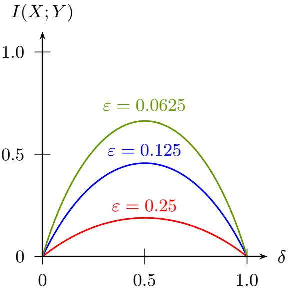

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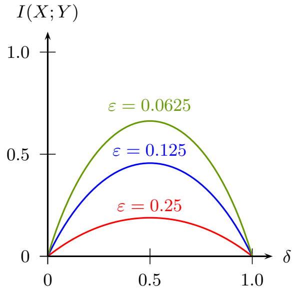

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The Channel Capacity

Running Example: The Binary Symmetric Channel

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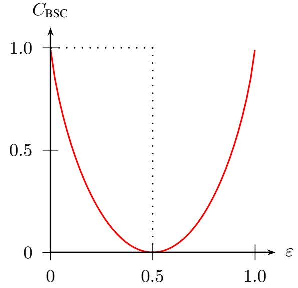

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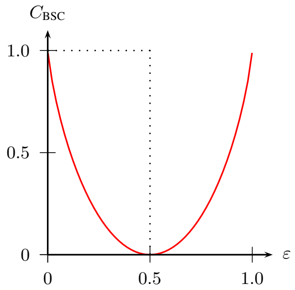

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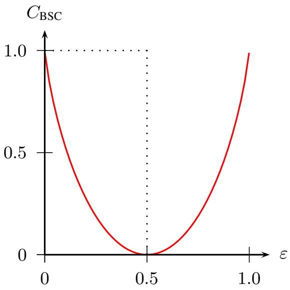

Running Example: The Binary Symmetric Channel

$$\mathsf{C}_{\mathsf{BSC}} = 1 - \mathsf{H}_{\mathsf{b}}(\varepsilon) \text{ bits.}$$

- It is maximum (1 bit) for $\varepsilon = 0$ and $\varepsilon = 1$.
- It is zero for $\varepsilon = \frac{1}{2} \longrightarrow$ The channel is useless!

The Channel Coding Theorem

Definition (Channel Capacity)

For a given channel, the channel capacity is defined to be the maximum of mutual information, maximized over all possible input distributions $P(x)$,

$$\mathsf{C} \triangleq \max_{P_X(x)} \mathsf{I}(X;Y).$$

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The channel capacity is the maximum transmission rate at which we can communicate reliably over the channel!

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Theorem (Shannon's Channel Coding Theorem)

For a discrete-time channel, it is possible to transmit information with an arbitrarily small probability of error if the communication rate R is below the channel capacity, i.e., $R < \mathsf{C}$. More precisely, for any $R \leq \mathsf{C}$, there exist a sequence of coding schemes of length N with average error probability $P_{\mathsf{e}}^{(N)}$ that tends to zero as $N \to \infty$, i.e., $P_{\mathsf{e}}^{(N)} \to 0$ as $N \to \infty$.
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Running Example: The Binary Symmetric Channel

For $\varepsilon = 0.25$, $\mathsf{C}_{\mathsf{BSC}} = 1 - \mathsf{H}_{\mathsf{b}}(\varepsilon) = 1 - \mathsf{H}_{\mathsf{b}}(0.25) = 0.1887$

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For $\varepsilon = 0.25$, $\mathsf{C}_{\mathsf{BSC}} = 1 - \mathsf{H}_{\mathsf{b}}(\varepsilon) = 1 - \mathsf{H}_{\mathsf{b}}(0.25) = 0.1887 \longrightarrow$ Reliable communication is possible as long as we transmit at a rate < 0.1887 bits per channel use.

Communication Over the AWGN Channel

The AWGN Channel

Additive White Gaussian Noise (AWGN) Channel

Continuous-time, complex AWGN channel,

$$y(t) = x(t) + n(t),$$

where $x(t)$ is bandlimited with bandwidth W and has signal power P, the symbol interval is $T = 1/W$, and $n(t)$ is complex AWGN with PSD N_0 and

$$\mathsf{SNR} \triangleq \frac{P}{\mathsf{N}_0 W}.$$

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Discrete-time AWGN channel,

$$\boldsymbol{y} = \boldsymbol{x} + \boldsymbol{n},$$

where $\boldsymbol{x} = (x_1, x_2, \ldots)$ is the transmitted sequence of constellation symbols with average energy per symbol $\mathsf{E_s}$, and $\boldsymbol{n}$ is a sequence of i.i.d. Gaussian noise random variables with zero mean and variance $\sigma^2 = \mathsf{N}_0/2$ per real dimension.

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- Signal-to-noise ratio:

$$\mathsf{SNR} \triangleq \frac{\mathsf{E_s}}{\mathsf{N_0}} = \frac{\mathsf{E_s}}{2\sigma^2}.$$

Mutual Information and Channel Capacity

Mutual Information and Channel Capacity

- For a given p_X, the amount of information that can be conveyed over the channel is given by the mutual information

$$\mathsf{I}(X;Y) = \iint p(x,y) \log \frac{p(x,y)}{p(x)p(y)} dxdy,$$

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- The ultimate limit at which we can transmit reliably is given by the channel capacity,

$$\mathsf{C} \triangleq \max_{p_X} \mathsf{I}(X;Y).$$

Capacity of the AWGN channel

Capacity of the Discrete-Time AWGN Channel

The channel capacity of the discrete-time complex AWGN channel with average energy per symbol $\mathsf{E_s}$, noise variance $\sigma^2 = \mathsf{N_0}/2$ per dimension, and $\mathsf{SNR} = \frac{\mathsf{E_s}}{2\sigma^2}$ is

$$\mathsf{C_{AWGN-D}} = \log\left(1 + \mathsf{SNR}\right) \text{ [bits/channel use] or [bits/symbol]},$$

and is achieved by a Gaussian input distribution, i.e., $X \sim \mathcal{CN}(0, \mathsf{E_s})$.

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Capacity of the Continuous-Time AWGN Channel

The channel capacity of the continuous-time AWGN channel of bandwidth W is

$$\mathsf{C_{AWGN-C}} = W \log\left(1 + \mathsf{SNR}\right) \quad [\text{bits/s}],$$

where the $\mathsf{SNR} = \frac{P}{\mathsf{N_0} W}$.

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Capacity of the Continuous-Time AWGN Channel

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where the $\mathsf{SNR} = \frac{P}{\mathsf{N_0} W}$.

The capacity $\mathsf{C_{AWGN-C}}$ depends on only two parameters, the channel bandwidth W and the SNR.

Capacity of the AWGN channel

Capacity of the Discrete-Time AWGN Channel

The channel capacity of the discrete-time real AWGN channel with average energy per symbol $\mathsf{E_s}$, noise variance $\sigma^2 = \mathsf{N_0}/2$, and $\mathsf{SNR} = \frac{\mathsf{E_s}}{\sigma^2}$ is

$$\mathsf{C_{AWGN-D}} = \frac{1}{2}\log\left(1+\mathsf{SNR}\right)\ [\text{bits/channel use}] \text{ or } [\text{bits/symbol}],$$

and is achieved by a Gaussian input distribution, i.e., $X \sim \mathcal{N}(0, \mathsf{E_s})$.

The Channel Coding Theorem for the AWGN Channel

Theorem (Channel Coding Theorem, Discrete-Time Channel)

All rates R below $\mathsf{C}_{\mathsf{AWGN-D}}$ are achievable, i.e., for every $R < \mathsf{C}_{\mathsf{AWGN-D}}$ there exists a sequence of coding schemes with vanishing error probability $P_{\mathsf{e}}^{(N)} \to 0$ as the block length $N \to \infty$. Conversely, any sequence of coding schemes of rate R and block length N with error probability $P_e^N \to 0$ must have a rate $R < \mathsf{C}_{\mathsf{AWGN-D}}$.

The Channel Coding Theorem for the AWGN Channel

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All rates R below $\mathsf{C}_{\mathsf{AWGN-D}}$ are achievable, i.e., for every $R < \mathsf{C}_{\mathsf{AWGN-D}}$ there exists a sequence of coding schemes with vanishing error probability $P_{\mathsf{e}}^{(N)} \to 0$ as the block length $N \to \infty$. Conversely, any sequence of coding schemes of rate R and block length N with error probability $P_e^N \to 0$ must have a rate $R < \mathsf{C}_{\mathsf{AWGN-D}}$.

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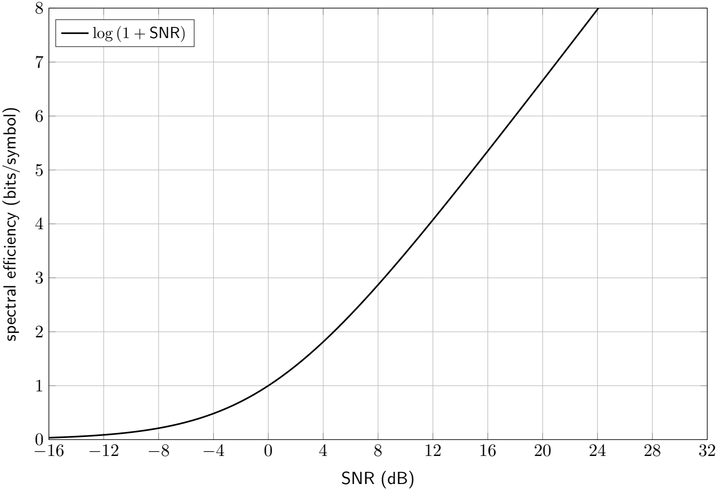
For the Continuous-Time Channel...

Reliable transmission can be achieved on the continuous-time channel at any bit rate R_{b} [bits/second] ($R_{\mathsf{b}} = R/T$) such that $R_{\mathsf{b}} < \mathsf{C}_{\mathsf{AWGN-C}}$.

AWGN Channel Capacity



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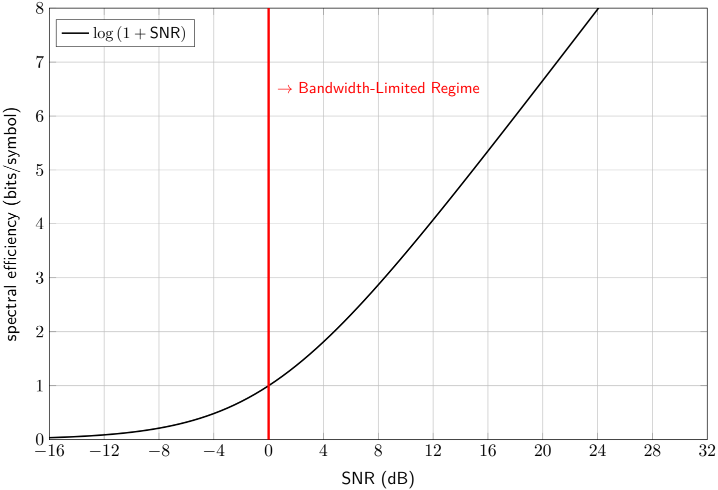

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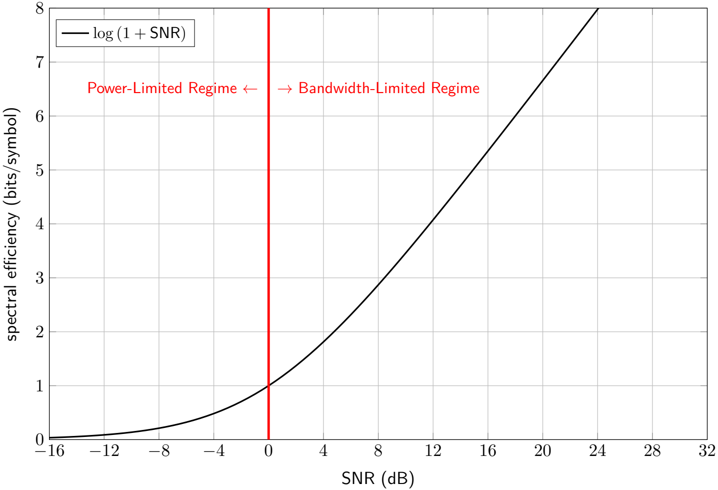

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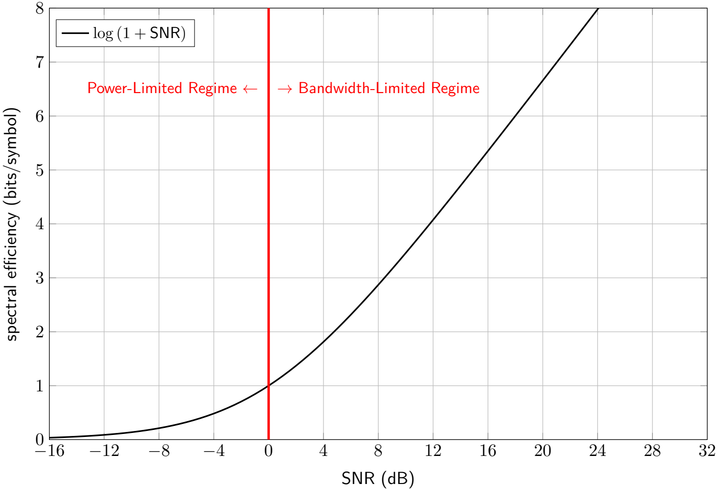

- To support spectral efficiency R, we must have SNR $> 2^R - 1$
- The capacity behaves quite differently in the two regimes

Power-Limited and Bandwidth-Limited Regimes

Power-Limited and Bandwidth-Limited Channels

Ideal band-limited AWGN channels may be classified as bandwidth-limited ($\mathsf{SNR} \gg 1$) or power-limited ($\mathsf{SNR} \ll 1$) according to whether they permit transmission at high spectral efficiencies or not.

Power-Limited Regime

Power-Limited Regime, SNR$\ll 1$ (While W Can Grow Very Large)

We can approximate channel capacity as

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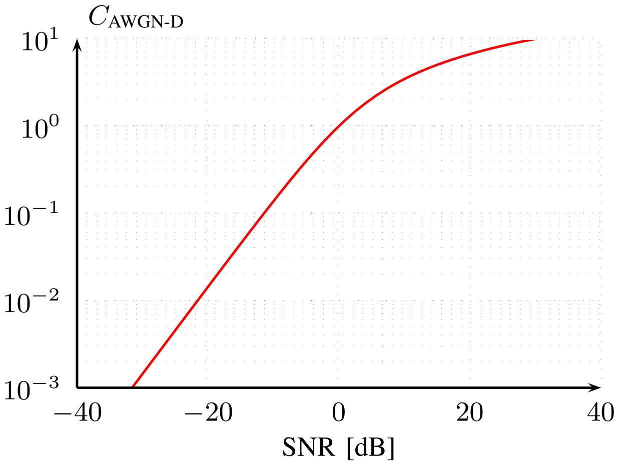
In the Power-Limited Regime

- Doubling the SNR doubles the capacity.

Capacity Curve



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In the Bandwidth-Limited Regime

- Doubling the SNR (every additional 3 dB in SNR) yields an increase in achievable spectral efficiency of only 1 (bit/s)/Hz.

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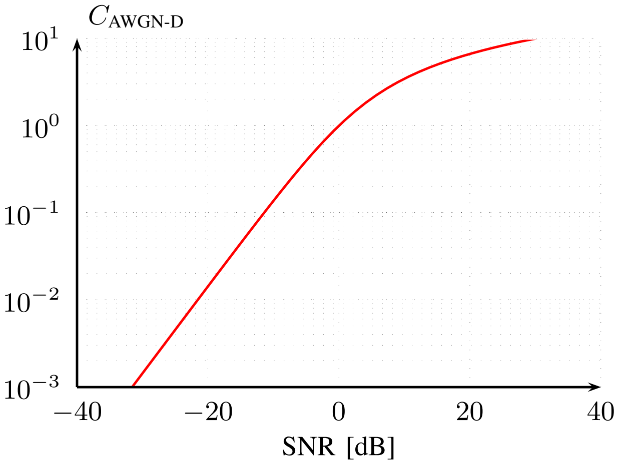

- For low SNR the gain is linear.
- For high SNR the gain is only logarithmic.

How Do We Design Good Coding Schemes in Practice?

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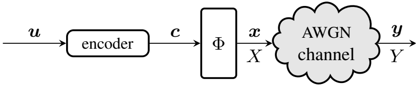

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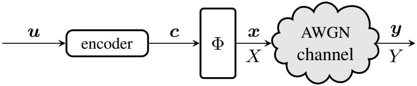

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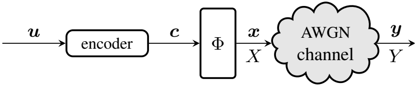

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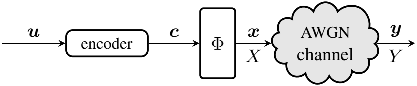

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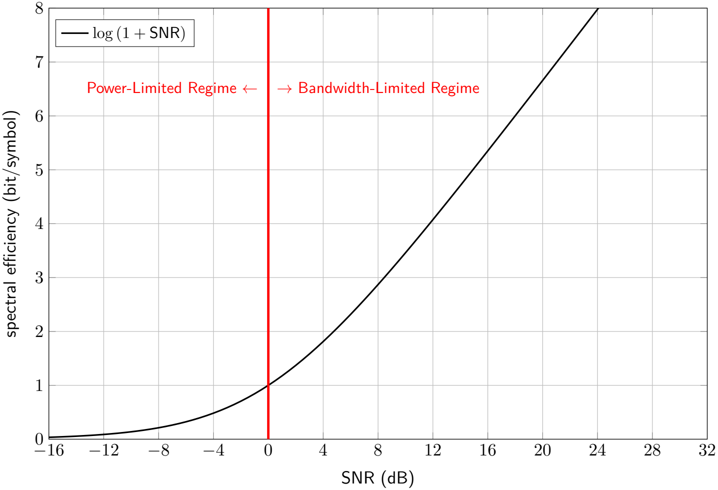
How much do we loose by restricting ourselves to M-QAM?

Mutual Information of M-QAM



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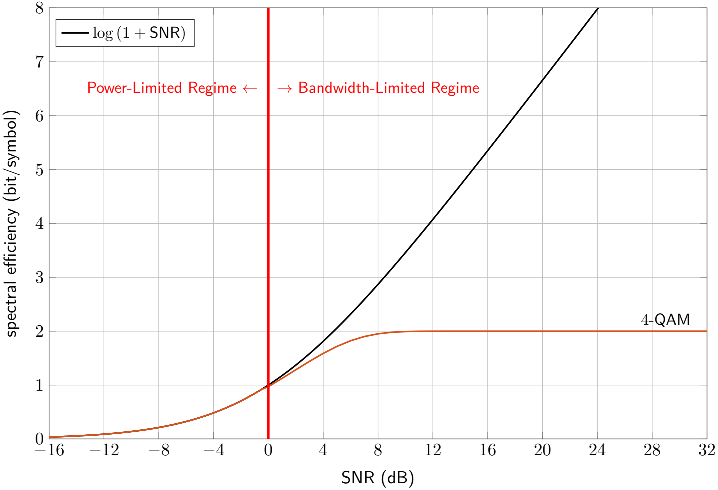

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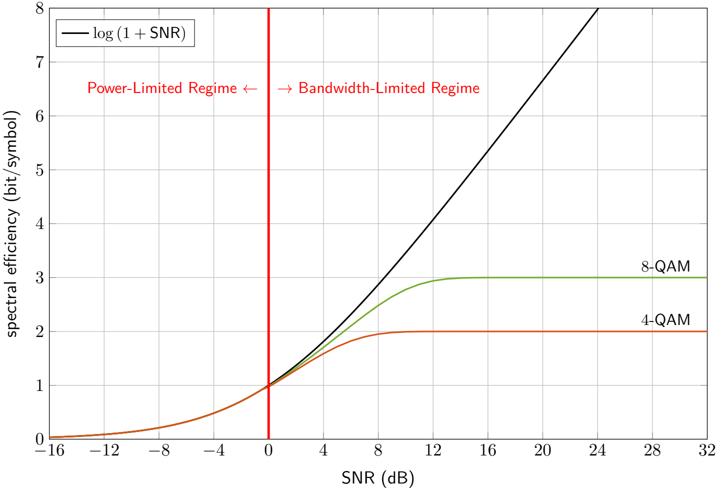

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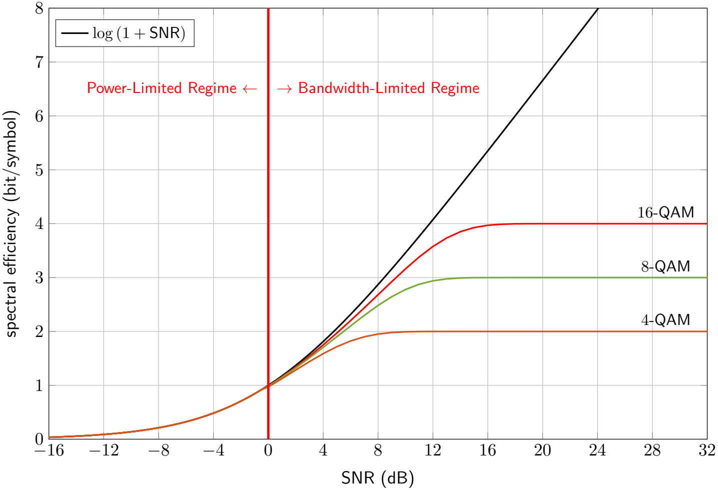

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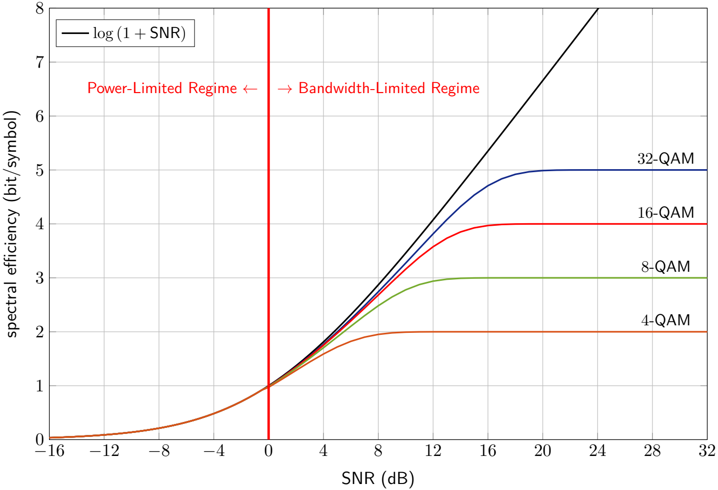

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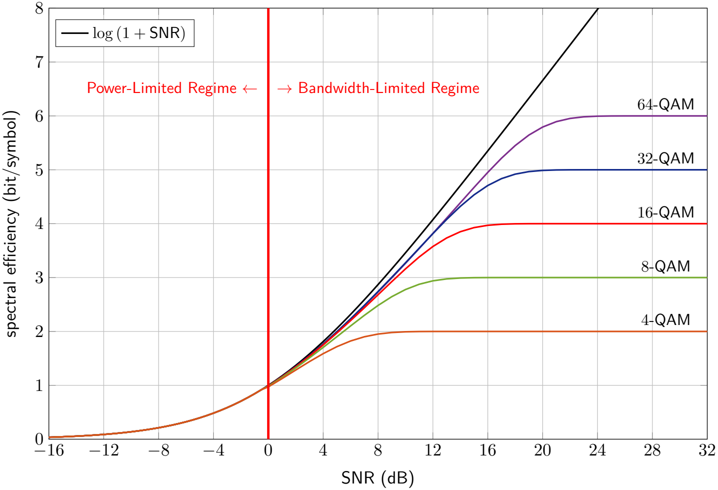

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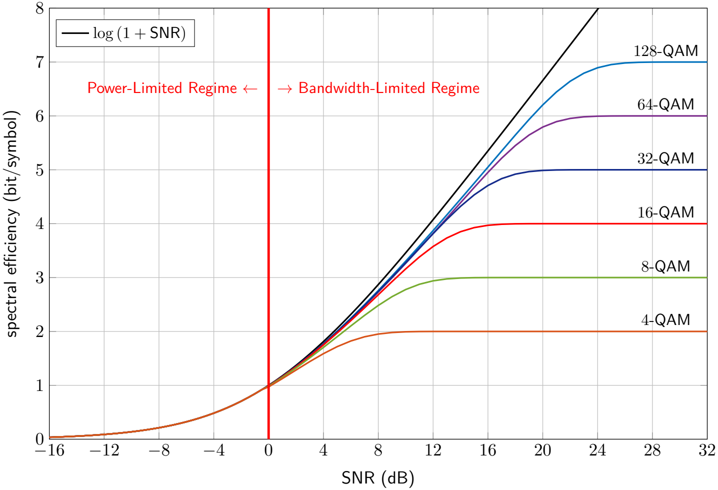

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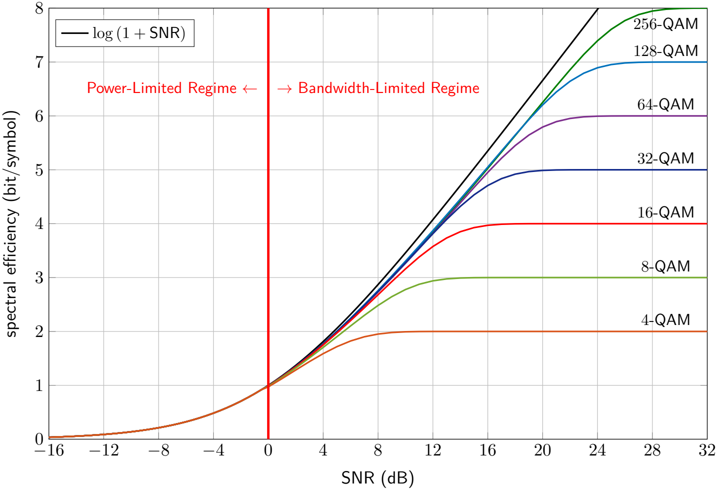

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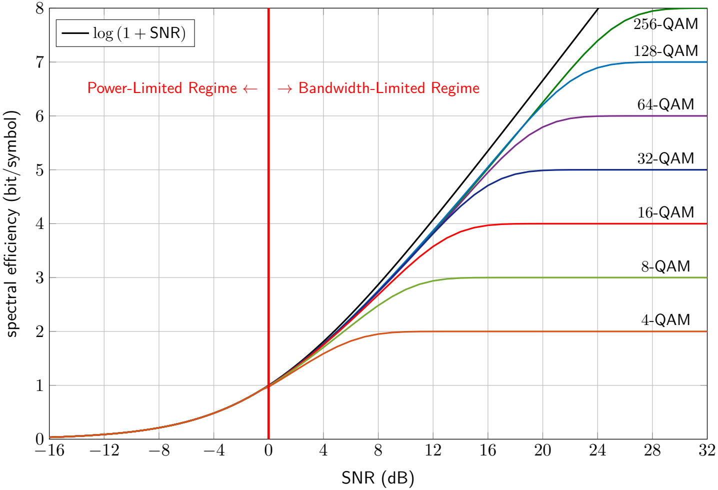

- Bandwidth-Limited: We need to use higher-order QAM constellations
- There is a gap between the mutual information curves and the capacity.

Power Efficiency and Energy per Information Bit

- To compare coded communication systems we consider the ratio

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- $\mathsf{SNR} = \frac{\mathsf{E_s}}{\mathsf{N_0}} = R\frac{\mathsf{E_b}}{\mathsf{N_0}}$.

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i.e., it is not possible to transmit reliably over the AWGN channel at $\mathsf{E_b}/\mathsf{N_0}$ smaller than -1.59 dB, even when we let $R \to 0!$.

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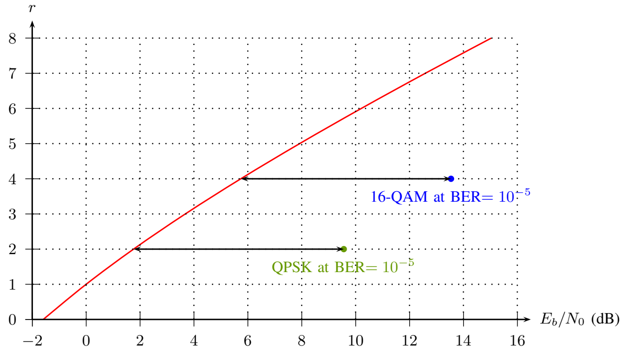

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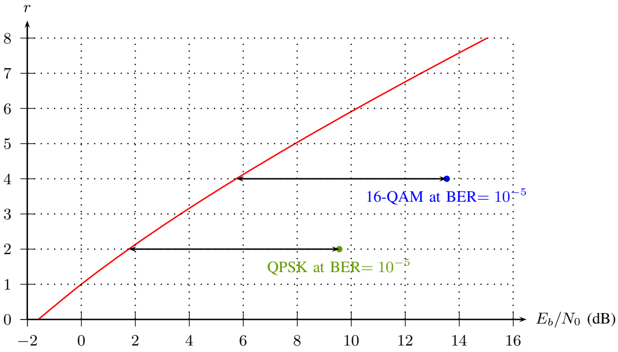

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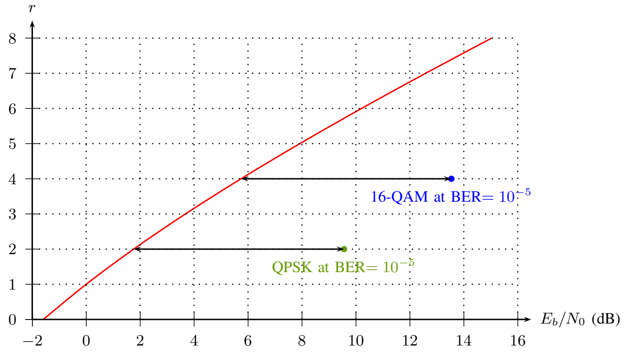

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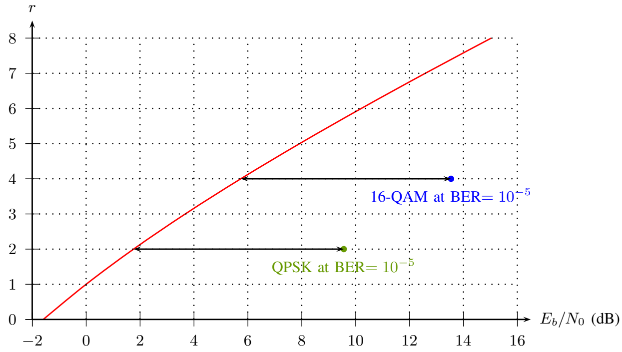

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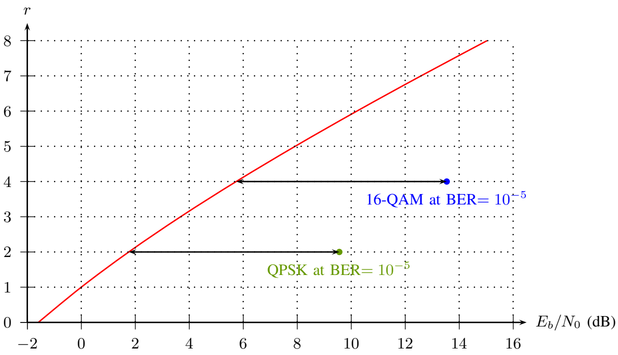

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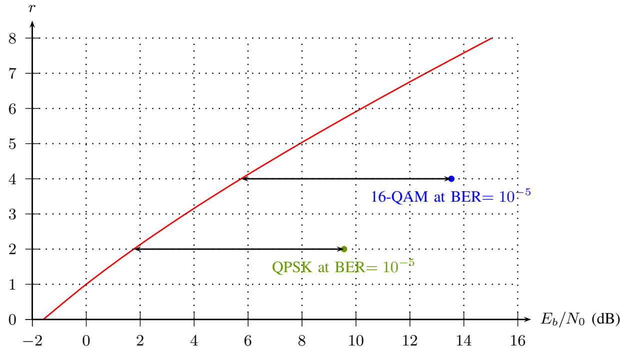

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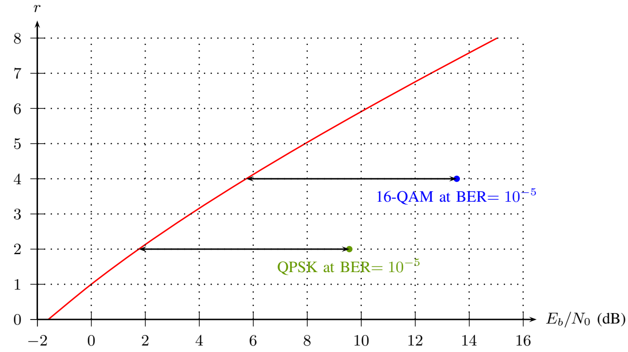

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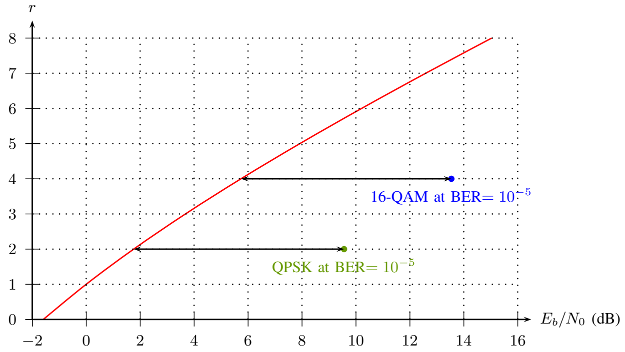

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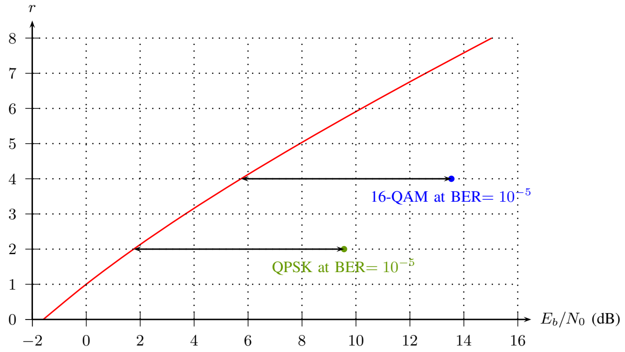

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- Decreasing R requires less $\mathsf{E_b/N_0}$, but higher bandwidth to support the same R_b.