Solution Sheet 3

Last modified October 26, 2022

Problem 1

1. The average energy per symbol is:

$$\begin{aligned}\mathbb{E}[|s|^2] &= \frac{1}{6}\sum_{i=1}^{6}|s_i|^2\\ &= \frac{1}{6}(4r_1^2 + 2r_2^2).\end{aligned}$$

Therefore r_1 and r_2 must satisfy:

$$\begin{aligned}\frac{1}{6}(4r_1^2 + 2r_2^2) &= 1\\ 2r_1^2 + r_2^2 &= 3\\ r_2 &= \sqrt{3 - 2r_1^2}.\end{aligned}$$

2. The average energy per symbol is:

$$\begin{aligned}\mathbb{E}[|s|^2] &= \frac{1}{6}\sum_{i=1}^{6}|s_i|^2\\ &= \frac{1}{6}(2A^2 + 4(A^2 + B^2)).\end{aligned}$$

Therefore A and B must satisfy:

$$\begin{aligned}\frac{1}{6}(2A^2 + 4(A^2 + B^2)) &= 1\\ 3A^2 + 2B^2 &= 3\\ A &= \sqrt{1 - \frac{2}{3}B^2}.\end{aligned}$$

3. The ML decision regions for constellation a) are drawn in Figure 1.

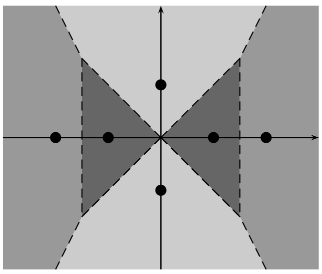


Figure 1

4. The ML decision regions for constellation b) are drawn in Figure 2.

5. The nearest neighbor approximation to the symbol error probability is given by:

$$P_{\mathsf{s}}^{(6)} \approx \bar{A}_{\mathsf{min}}\mathsf{Q}\left(\sqrt{\frac{d_{\mathsf{E,min}}^2}{2\mathsf{N}_0}}\right).$$

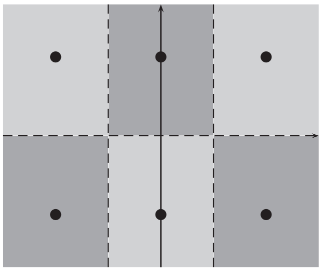


Figure 2

It can be seen that, with $r_1 = \sqrt{1/2}$ and $r_2 = \sqrt{2}$, the minimum distance between constellation points is $d_{\mathsf{E,min}} = \sqrt{2} - \sqrt{1/2} = \sqrt{1/2}$. Also, there are 4 symbols which have 1 neighbor at distance $d_{\mathsf{E,min}}$ (the four symbols in the x axis) and 2 symbols that have no neighbors at distance $d_{\mathsf{E,min}}$ (the two symbols in the y axis). Therefore,

$$\bar{A}_{\mathsf{min}} = \frac{1}{6}(4 \cdot 1 + 2 \cdot 0) = \frac{2}{3}.$$

Finally, we obtain

$$\begin{aligned} P_{\mathsf{s}}^{(6)} &\approx \bar{A}_{\mathsf{min}} \mathsf{Q}\left(\sqrt{\frac{d^2_{\mathsf{E,min}}}{2\mathsf{N}_0}}\right) \\ &= \frac{2}{3}\mathsf{Q}\left(\sqrt{\frac{\mathsf{E_s}}{4\mathsf{N}_0}}\right). \end{aligned} \tag{1}$$

6. It can be seen that, with $A = \sqrt{1/3}$ and $B = 1$, the minimum distance between constellation points is $d_{\mathsf{E,min}} = 1$. Also, there are 4 symbols which have 1 neighbor at distance $d_{\mathsf{E,min}}$ (the four symbols in the corners) and 2 symbols that have 2 neighbors at distance $d_{\mathsf{E,min}}$ (the two symbols in the y axis). Therefore,

$$\bar{A}_{\mathsf{min}} = \frac{1}{6}(4 \cdot 1 + 2 \cdot 2) = \frac{4}{3}.$$

Finally, we obtain

$$\begin{aligned} P_{\mathsf{s}}^{(6)} &\approx \bar{A}_{\mathsf{min}} \mathsf{Q}\left(\sqrt{\frac{d^2_{\mathsf{E,min}}}{2\mathsf{N}_0}}\right) \\ &= \frac{4}{3}\mathsf{Q}\left(\sqrt{\frac{\mathsf{E_s}}{2\mathsf{N}_0}}\right). \end{aligned} \tag{2}$$

7. We want to compare the two expressions in (1) and (2) given the same E_s/N_0. Let $x = \sqrt{\frac{E_s}{4N_0}}$ and plot $P_1 = \frac{2}{3}\mathsf{Q}(x)$ and $P_2 = \frac{4}{3}\mathsf{Q}(\sqrt{2}x)$, as shown in 3. Suppose the intersection of P_1 and P_2 is when $x = c$. So when $0 < x < c$, P_1 is smaller, which means constellation (a) is more power efficient, when $x > c$, P_2 is smaller, implying constellation (b) is more power efficient.

Problem 2

1. For $\mathbb{E}[|S|^2] = \mathsf{E_s}$, we require

$$\begin{aligned} \frac{1}{4}\left\{A^2 + A^2 + A^2 + 0\right\} &= \mathsf{E_s} \\ \Rightarrow A &= 2\sqrt{\mathsf{E_s}/3}. \end{aligned}$$

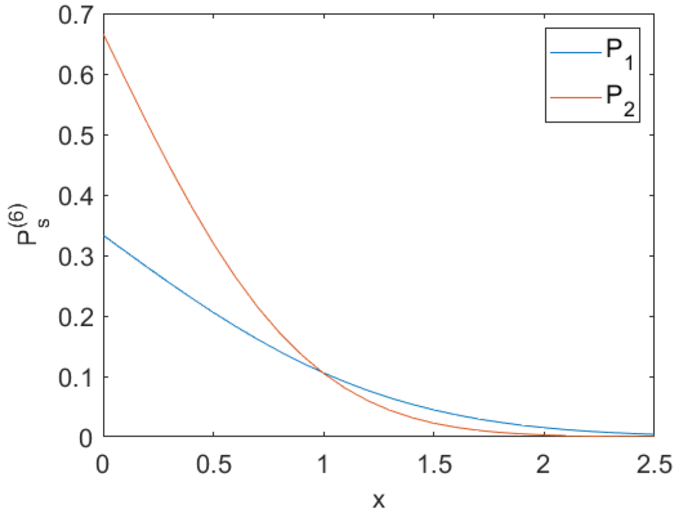


Figure 3

2. The decision regions are shown in Fig. 4 for equal priors (left) and $\phi_1 = \pi/2$, $\phi_2 = \pi$.

3. The decision regions are shown in Fig. 4 for unequal priors (right) and $\phi_1 = \pi/2$, $\phi_2 = \pi$.

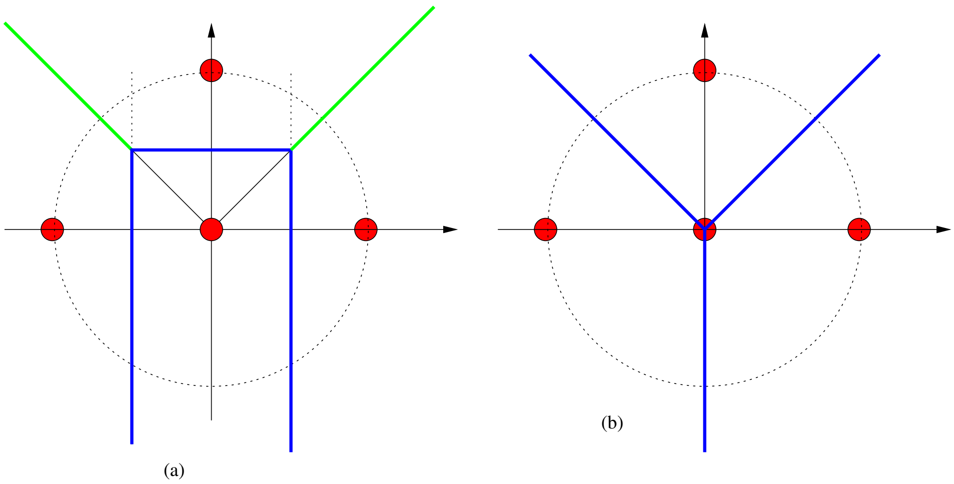


Figure 4: Decision regions

4. The distance between 0 and any other point is A. The distance between two points on a circle with relative angle ϕ is (from some simple trigonometry): $d = 2A\sin(\phi/2)$.

5. The average distance is maximized when the angle is 120 degrees $= 2\pi/3$, in which case the distance is $d = A\sqrt{3} = 2\sqrt{\mathsf{E_s}}$.

6. Assuming $A = 2\sqrt{\mathsf{E_s}/3}$, $\phi_1 = \pi/2$, $\phi_2 = \pi$, and equiprobable symbols, the error probability can be upper

bounded by

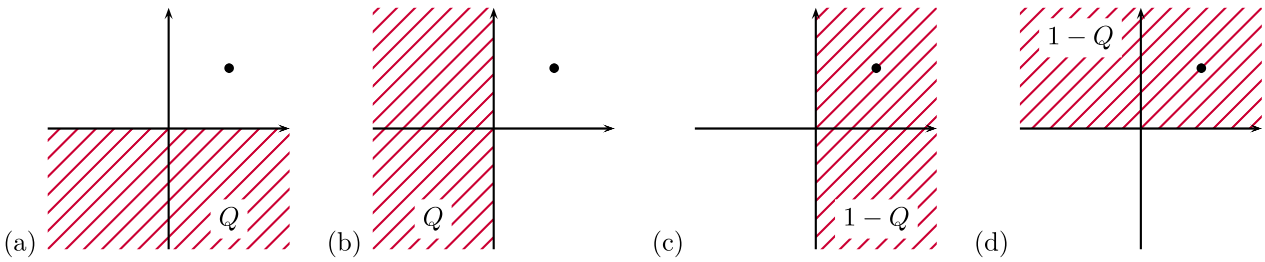
$$P_{\mathsf{s}}^{(4)} \leq \frac{3}{2}\mathsf{Q}\left(\sqrt{\frac{A^2}{2\mathsf{N}_0}}\right) + \mathsf{Q}\left(\sqrt{\frac{A^2}{\mathsf{N}_0}}\right) + \frac{1}{2}\mathsf{Q}\left(\sqrt{\frac{2A^2}{\mathsf{N}_0}}\right)$$
$$= \frac{3}{2}\mathsf{Q}\left(\sqrt{\frac{2\mathsf{E}_\mathsf{s}}{3\mathsf{N}_0}}\right) + \mathsf{Q}\left(\sqrt{\frac{4\mathsf{E}_\mathsf{s}}{3\mathsf{N}_0}}\right) + \frac{1}{2}\mathsf{Q}\left(\sqrt{\frac{8\mathsf{E}_\mathsf{s}}{3\mathsf{N}_0}}\right).$$

The error probability is dominated by the terms with minimal Euclidean distance $\Rightarrow$ The term with $\mathsf{Q}\left(\sqrt{\frac{A^2}{2\mathsf{N}_0}}\right)$ dominates.

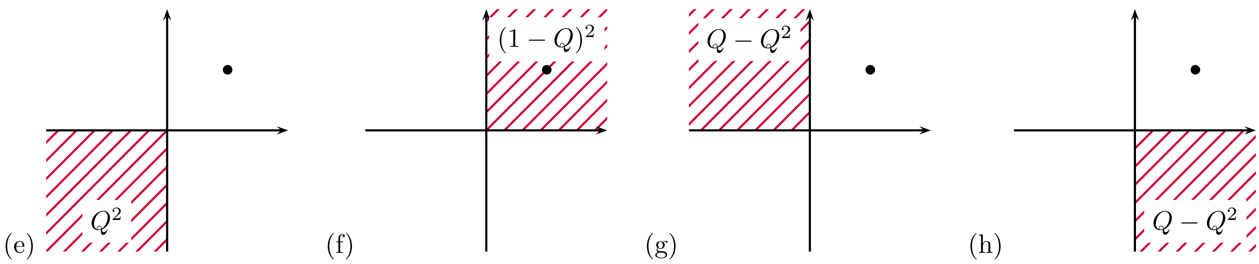
Problem 3

Part I (uncoded)

We start by defining the abbreviation $\mathsf{Q} \triangleq \mathsf{Q}\left(\sqrt{\frac{\mathsf{E}_s}{\mathsf{N}_0}}\right)$ in order to make our solution more readable. Let us assume that the constellation point x in the upper right quadrant of the complex plane is transmitted. In general, the probability that the observation y falls inside a certain region is then given by integrating the conditional probability density function $p(y|x)$ over this region. The simplest regions and the corresponding probablities for y to fall inside those regions are shown below (the black dot corresponds to the transmitted point x):



From these basic regions, we can immediately obtain new regions by intersecting, which corresponds to a multiplication of the corresponding probabilities (Q-functions):



The probabilities for the above regions (e)–(h) are sometimes referred to as *symbol transition probabilities*, because they tell us how likely it is to move from one symbol (the transmitted one) to another particular symbol (the detected one) during transmission. Similarly, we can obtain the following regions and probabilities:

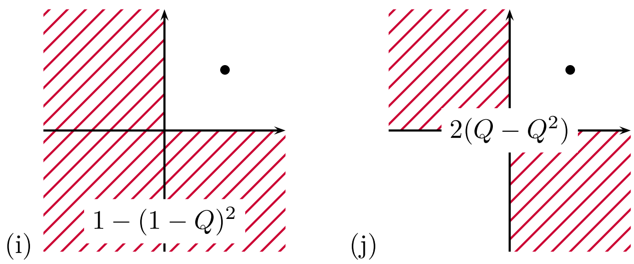


Fig. (i) is obtained by taking the inverse of (f). Fig. (j) is obtained by adding (g) and (h). Keep in mind that for QPSK, it is sufficient to consider only the regions assuming that one (arbitrary) point was transmitted. This is because if we assume that some other point is transmitted, the above regions are simply rotated (by either 90, 180, or 270 degrees) and the corresponding probabilities are identical.
Now one can start identifying the different error probabilities by relating them to the above figures. For example, the average symbol error probability for QPSK is reflected in Fig. (i), independent of the binary mapping. Finally, we obtain the following table:

	P_s	$P_{\mathsf{b},1}$	$P_{\mathsf{b},2}$	$P_\mathsf{b} = (P_{\mathsf{b},1} + P_{\mathsf{b},2})/2$
Gray	$1-(1-\mathsf{Q})^2$	Q	Q	Q
Lexicographical	$1-(1-\mathsf{Q})^2$	Q	$2(\mathsf{Q}-\mathsf{Q}^2)$	$3\mathsf{Q}/2-\mathsf{Q}^2$

Notice that in order to obtain the bit error probabilities in above table, one may explicitly write down all symbol transition probabilities and connect them to the corresponding binary mapping. For the Gray mapping, this leads to the following three tables, where the second and third table correspond to the first and second bit of the mapping, respectively (the convention here is to interpret the row as the transmitted bits and column as the received/detected ones):

	00	01	10	11
00	$(1-\mathsf{Q})^2$	$\mathsf{Q}-\mathsf{Q}^2$	$\mathsf{Q}-\mathsf{Q}^2$	Q^2
01	$\mathsf{Q}-\mathsf{Q}^2$	$(1-\mathsf{Q})^2$	Q^2	$\mathsf{Q}-\mathsf{Q}^2$
10	$\mathsf{Q}-\mathsf{Q}^2$	Q^2	$(1-\mathsf{Q})^2$	$\mathsf{Q}-\mathsf{Q}^2$
11	Q^2	$\mathsf{Q}-\mathsf{Q}^2$	$\mathsf{Q}-\mathsf{Q}^2$	$(1-\mathsf{Q})^2$

	0	1
0	$1-\mathsf{Q}$	Q
1	Q	$1-\mathsf{Q}$

	0	1
0	$1-\mathsf{Q}$	Q
1	Q	$1-\mathsf{Q}$

Similarly, for the lexicographical mapping we obtain the following tables:

	00	01	10	11
00	$(1-\mathsf{Q})^2$	$\mathsf{Q}-\mathsf{Q}^2$	Q^2	$\mathsf{Q}-\mathsf{Q}^2$
01	$\mathsf{Q}-\mathsf{Q}^2$	$(1-\mathsf{Q})^2$	$\mathsf{Q}-\mathsf{Q}^2$	Q^2
10	Q^2	$\mathsf{Q}-\mathsf{Q}^2$	$(1-\mathsf{Q})^2$	$\mathsf{Q}-\mathsf{Q}^2$
11	$\mathsf{Q}-\mathsf{Q}^2$	Q^2	$\mathsf{Q}-\mathsf{Q}^2$	$(1-\mathsf{Q})^2$

	0	1
0	$1-\mathsf{Q}$	Q
1	Q	$1-\mathsf{Q}$

	0	1
0	$1-2(\mathsf{Q}-\mathsf{Q}^2)$	$2(\mathsf{Q}-\mathsf{Q}^2)$
1	$2(\mathsf{Q}-\mathsf{Q}^2)$	$1-2(\mathsf{Q}-\mathsf{Q}^2)$

Part II (coded)

- The length is $n = 6$ and the dimension is $k = \log_2(4) = 2$. Thus the code rate is $R_\mathsf{c} = 1/3$. By adding all codeword pairs, one may verify that the code is linear. Thus the minimum distance corresponds to the minimum Hamming weight, which is $d_\mathsf{min} = 3$. To transmit one codeword, we need to use the channel three times, that is, we need to transmit three QPSK symbols.

- Here we can use the result that for the Gray mapping the channel decomposes into two independent binary symmetric channels. For a binary symmetric channel, it was shown in the lecture that the ML decoder finds the codeword with minimum Hamming distance to the received bit word. Thus, the most likely codeword in this case is (000000).

- (Bonus question) It is important to realize that minimizing the Hamming distance does *not* correspond to ML decoding here. Notationwise, we introduce a lower index i to refer to a particular use of the channel. The ML codeword is given by

$$\begin{aligned} c_{\mathrm{ML}} &= \arg\max_{c\in\mathcal{C}} p(\hat{\boldsymbol{c}}|c) \\ &= \arg\max_{c\in\mathcal{C}} \prod_{i=1}^{3} p(\hat{c}_i^{(1)}, \hat{c}_i^{(2)}|c_i^{(1)}, c_i^{(2)}) \\ &= \arg\max_{c\in\mathcal{C}} \sum_{i=1}^{3} \log p(\hat{c}_i^{(1)}, \hat{c}_i^{(2)}|c_i^{(1)}, c_i^{(2)}) \\ &= \arg\max_{c\in\mathcal{C}} \sum_{i=1}^{3} \lambda_i, \end{aligned}$$

where the second equality follows from the fact that the channel is memoryless (from QPSK symbol to QPSK symbol), and for the second equality we just applied the log function, which does not change the argument of the maximization. For the last equality, we have introduced λ_i as our decoding metric:

$$\lambda_i \triangleq \log p(\hat{c}_i^{(1)}, \hat{c}_i^{(2)}|c_i^{(1)}, c_i^{(2)}).$$

One can now directly use the above transition table for the lexicographical mapping to find the ML codeword.

A major simplification is obtained by scaling the entire table (that is, each entry) by $(1-\mathsf{Q})^2$ before taking the log and then removing the common factor. For example, if we normalize $P(00|00) = (1-\mathsf{Q})^2$ by $(1-\mathsf{Q})^2$, we get 1 of which the logarithm is 0. The same we do for the rest of all entries in the table. After rearranging, we then find that it is equivalent to work with the following metric (which has to be minimized), similar to, but not quite equal to the Hamming distance :

	00	01	10	11
00	0	1	2	1
01	1	0	1	2
10	2	1	0	1
11	1	2	1	0

Thus, decoding the observation (100001) amounts to calculating the four cumulative metrics

$$\begin{aligned} (000000) &\to 2+0+1 = 3, \\ (010101) &\to 1+1+0 = 2, \\ (101010) &\to 0+2+1 = 3, \\ (111111) &\to 1+1+2 = 4. \end{aligned}$$

Thus, the second codeword (010101) has the smallest metric, and hence is the ML codeword.

Problem 4

Note that we can write

$$p(y|b_i = 0) = \sum_{s\in\mathcal{S}} p(y, s|b_i = 0) = \sum_{s\in\mathcal{S}} p(y|s, b_i = 0) \cdot p(s|b_i = 0) = \sum_{s\in\mathcal{S}_{i,0}} p(y|s)$$

For the first bit position we have

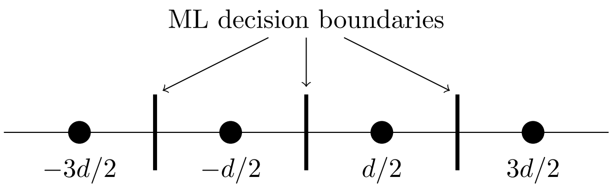
$$\begin{aligned}
L_1 &= \log \frac{p(y|x_1) + p(y|x_2)}{p(y|x_3) + p(y|x_4)} \\
&= \log \frac{\exp\left(\frac{ay_I + ay_Q}{\sigma^2}\right) + \exp\left(\frac{ay_I - ay_Q}{\sigma^2}\right)}{\exp\left(\frac{-ay_I - ay_Q}{\sigma^2}\right) + \exp\left(\frac{-ay_I + ay_Q}{\sigma^2}\right)} \\
&= \log \frac{\exp(ay_I/\sigma^2)\big(\exp(ay_Q/\sigma^2) + \exp(-ay_Q/\sigma^2)\big)}{\exp(-ay_I/\sigma^2)(\exp(-ay_Q/\sigma^2) + \exp(ay_Q/\sigma^2))} \\
&= \frac{2ay_I}{\sigma^2}.
\end{aligned}$$

Similarly, for the second bit position we have

$$\begin{aligned}
L_2 &= \log \frac{p(y|x_1) + p(y|x_3)}{p(y|x_2) + p(y|x_4)} \\
&= \log \frac{\exp\left(\frac{ay_I + ay_Q}{\sigma^2}\right) + \exp\left(\frac{-ay_I - ay_Q}{\sigma^2}\right)}{\exp\left(\frac{ay_I - ay_Q}{\sigma^2}\right) + \exp\left(\frac{-ay_I + ay_Q}{\sigma^2}\right)} \\
&= \log \frac{\cosh\left(a\frac{y_I + y_Q}{\sigma^2}\right)}{\cosh\left(a\frac{y_I - y_Q}{\sigma^2}\right)}.
\end{aligned}$$

Problem 5

The 4-PAM constellation and ML decision boundaries are shown in the figure below.



Let $n \sim \mathcal{N}(0, \sigma^2)$ denote the noise sample corrupting the received signal. Conditioned on an outer signal point being sent (consider the leftmost point without loss of generality), the error probability is

$$\Pr(\text{error} \,|\, \text{outer}) = \Pr(n > d/2) = \mathsf{Q}\left(\frac{d}{2\sigma}\right).$$

Conditioned on one of the two inner points being sent, the error probability is

$$\Pr(\text{error} \,|\, \text{inner}) = \Pr(|n| > d/2) = 2\mathsf{Q}\left(\frac{d}{2\sigma}\right).$$

Note that $\mathsf{E_s} = ((d/2)^2 + (3d/2)^2)/2$ and $\mathsf{E_b} = \mathsf{E_s}/\log_2 4$, so that $d^2/\mathsf{E_b} = 8/5$. We therefore can write

$$P_{\mathsf{s}}^{\mathsf{4PAM}} = \frac{1}{2}(\Pr(\text{error} \,|\, \text{outer}) + \Pr(\text{error} \,|\, \text{inner})) = \frac{3}{2}\mathsf{Q}\left(\frac{d}{2\sigma}\right) = \frac{3}{2}\mathsf{Q}\left(\sqrt{\frac{4\mathsf{E_b}}{5\mathsf{N_0}}}\right).$$

16-QAM can be viewed as a product of two 4-PAM constellations sent in parallel over independent AWGN channels, so that a symbol error occurs if either of the two 4-PAM symbols are received incorrectly. Thus,

$$P_{\mathsf{s}}^{\mathsf{16QAM}} = 1 - \left(1 - P_{\mathsf{s}}^{\mathsf{4PAM}}\right)^2 = 2P_{\mathsf{s}}^{\mathsf{4PAM}} - \left(P_{\mathsf{s}}^{\mathsf{4PAM}}\right)^2.$$

Using the 4-PAM symbol error probability bound yields the following bound on the symbol error probability of 16-QAM:

$$P_{\mathsf{s}}^{\mathsf{16QAM}} = 3\mathsf{Q}\left(\sqrt{\frac{4\mathsf{E_b}}{5\mathsf{N_0}}}\right) - \frac{9}{4}\mathsf{Q}^2\left(\sqrt{\frac{4\mathsf{E_b}}{5\mathsf{N_0}}}\right).$$

Problem 6

Assuming equiprobable points, the second moment of these constellations is

$$\frac{1}{4}(4 \cdot (a^2 + b^2)) = a^2 + b^2 = 1,$$

and from this constraint, b can be expressed as

$$b = \sqrt{1 - a^2}.$$

Moreover, the minimum Euclidean distance of these constellations can be found by computing

$$d_{\mathsf{E,min}} = \min\Big(2a, 2b, \sqrt{(2a)^2 + (2b)^2}\Big),$$

but $\sqrt{(2a)^2 + (2b)^2} \geq 2a$ and $\sqrt{(2a)^2 + (2b)^2} \geq 2b$, and therefore

$$d_{\mathsf{E,min}} = \min(2a, 2b). \tag{3}$$

The largest minimum distance can thus be found as

$$\max_{a \in [0,1]} d_{\mathsf{E,min}} = \max_{a \in [0,1]} \min(2a, 2b).$$

A plot of $2a$, $2b$, and (3) as functions of a can be seen in Figure 5. Clearly, the maximum of $d_{\mathsf{E,min}}$ occurs at the intersection of $2a$ and $2b$, i.e., when

$$2a = 2b = 2\sqrt{1 - a^2} \qquad \Rightarrow \qquad a = b = \sqrt{1/2}.$$

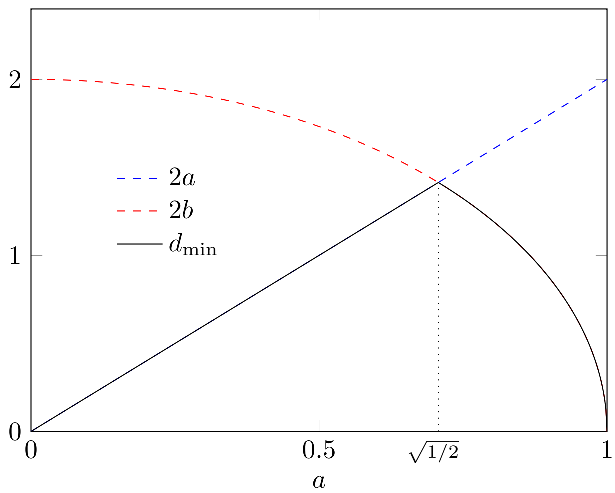


Figure 5