Digital Communications
SSY125, Lecture 7

Analysis of Linear Modulations (Chapter 6)

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Slides prepared by Alexandre Graell i Amat

November 13, 2022

CHALMERS

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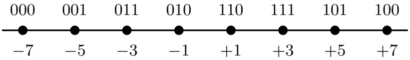

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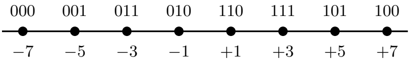

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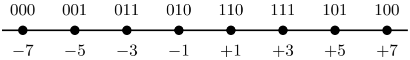

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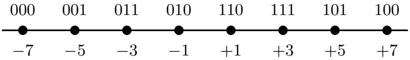

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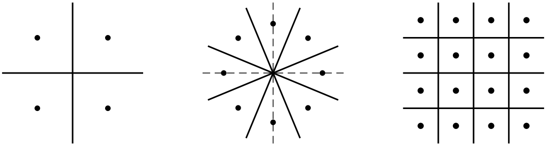

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The Union Bound

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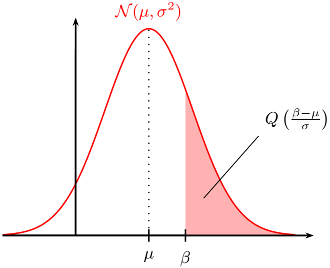

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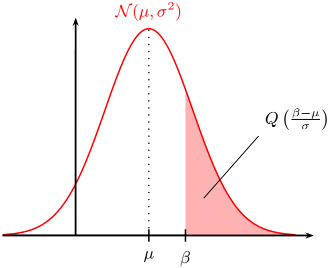

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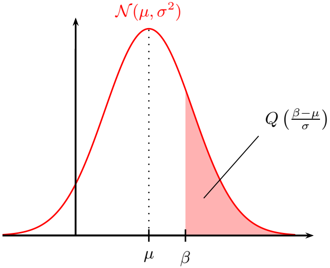

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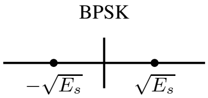
- For $0 < a \ll b < c$,

$$\mathrm{Q}(a) + \mathrm{Q}(b) + \mathrm{Q}(c) \approx \mathrm{Q}(a).$$

Symbol Error Probability of BPSK



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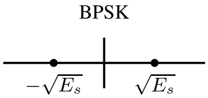

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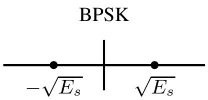

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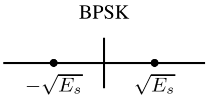

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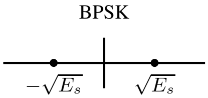

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$$\begin{aligned}
P_{\mathsf{s}}^{\mathsf{BPSK}} &= \sum_{x \in \mathcal{X}} \Pr(\hat{x} \neq x | x) P(x) \\
&= \Pr(\mathsf{X}_2 | \mathsf{X}_1) P(\mathsf{X}_1) + \Pr(\mathsf{X}_1 | \mathsf{X}_2) P(\mathsf{X}_2) \\
&= \frac{1}{2}\Pr(\mathsf{X}_2|\mathsf{X}_1) + \frac{1}{2}\Pr(\mathsf{X}_1|\mathsf{X}_2),
\end{aligned}$$

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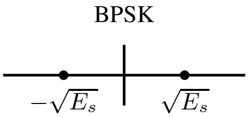

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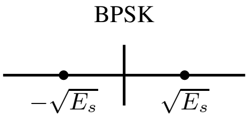

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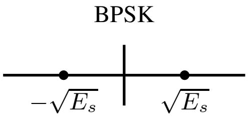

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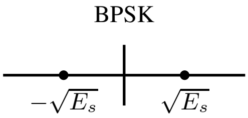

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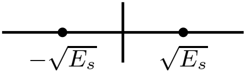

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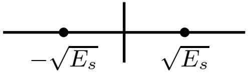

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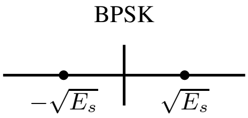

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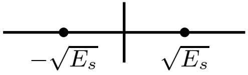

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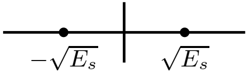

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Due to symmetry, $\Pr(\hat{x} = \mathsf{X}_1 | x = \mathsf{X}_2) = \Pr(\hat{x} = \mathsf{X}_2 | x = \mathsf{X}_1)$, and

$$P_{\mathsf{s}}^{\mathsf{BPSK}} = \mathsf{Q}\left(\sqrt{\frac{2\mathsf{E_s}}{\mathsf{N_0}}}\right) = \mathsf{Q}\left(\sqrt{\frac{2\mathsf{E_b}}{\mathsf{N_0}}}\right).$$

Symbol Error Probability and Euclidean Distance

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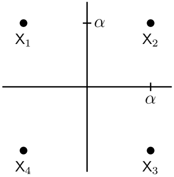
- $\mathsf{Pr}(\hat{x} = \mathsf{X}_2 | x = \mathsf{X}_1)$ depends on $d_\mathsf{E}(\mathsf{X}_1, \mathsf{X}_2)$ $\Rightarrow$ Construct constellations with high distance between constellation points!

Symbol Error Probability of 4-QAM



- $\mathcal{X} = \{\mathsf{X}_1, \mathsf{X}_2, \mathsf{X}_3, \mathsf{X}_4\}$ with $\alpha = \sqrt{\mathsf{E_s}/2}$ such that the average energy per symbol is $\mathsf{E_s}$.

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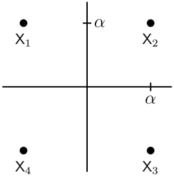

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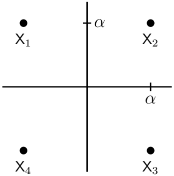

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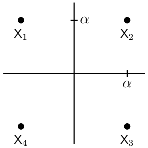
$$\hat{x} = \begin{cases} \mathsf{X}_1 & \text{if } y_\mathsf{I} < 0 \text{ and } y_\mathsf{Q} > 0 \\ \mathsf{X}_2 & \text{if } y_\mathsf{I} > 0 \text{ and } y_\mathsf{Q} > 0 \\ \mathsf{X}_3 & \text{if } y_\mathsf{I} > 0 \text{ and } y_\mathsf{Q} < 0 \\ \mathsf{X}_4 & \text{if } y_\mathsf{I} < 0 \text{ and } y_\mathsf{Q} < 0 \end{cases},$$

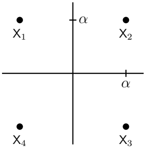
where y_I and y_Q are the in-phase and quadrature components of y.

Analysis of Linear Modulations



$$P_s^{4QAM} = \Pr(\hat{x} \neq X_1 | x = X_1) =$$

Analysis of Linear Modulations



$$P_{\mathsf{s}}^{\mathsf{4QAM}} = \mathsf{Pr}(\hat{x} \neq \mathsf{X}_1 | x = \mathsf{X}_1) = \mathsf{Pr}(Y_{\mathsf{I}} > 0 \cup Y_{\mathsf{Q}} < 0 | \mathsf{X}_1)$$

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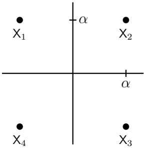

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\end{aligned}
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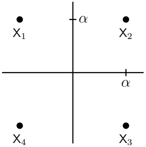

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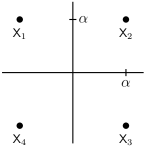

$$
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P_s^{4\mathsf{QAM}} &= \mathsf{Pr}(\hat{x} \neq \mathsf{X}_1 | x = \mathsf{X}_1) = \mathsf{Pr}(Y_\mathsf{I} > 0 \cup Y_\mathsf{Q} < 0 | \mathsf{X}_1) \\
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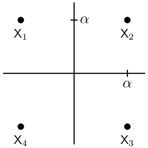

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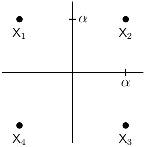

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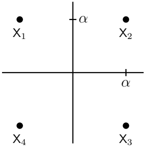

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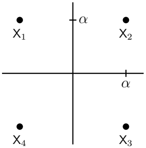
Analysis of Linear Modulations



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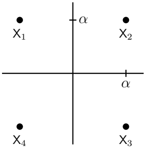
Analysis of Linear Modulations



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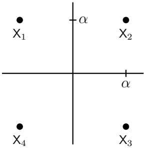

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$$

By symmetry, $\Pr(Y_{\mathsf{Q}} < 0 | \mathsf{X}_1) = \Pr(Y_{\mathsf{I}} > 0 | \mathsf{X}_1)$, and

$$
P_{\mathsf{s}}^{\mathsf{4QAM}} = 2\mathsf{Q}\left(\sqrt{\frac{2\mathsf{E}_{\mathsf{b}}}{\mathsf{N}_0}}\right) - \left(\mathsf{Q}\left(\sqrt{\frac{2\mathsf{E}_{\mathsf{b}}}{\mathsf{N}_0}}\right)\right)^2.
$$

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For general constellations, the exact symbol error probability P_s is hard to compute

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&\leq \frac{1}{M} \sum_{i=1}^{M} \sum_{j \neq i} \mathsf{Pr}(\hat{x} = \mathsf{X}_j | x = \mathsf{X}_i).
\end{aligned}$$

Upper Bound on the Symbol Error Probability

Using

$$\Pr(\hat{x} = \mathsf{X}_j | x = \mathsf{X}_i) = \mathsf{Q}\left(\sqrt{\frac{d_{\mathsf{E}}^2(\mathsf{X}_i, \mathsf{X}_j)}{2\mathsf{N}_0}}\right)$$

the symbol error probability of an M-ary constellation can be upperbounded as

$$P_{\mathsf{s}}^{(M)} \leq \frac{1}{M} \sum_{i=1}^{M} \sum_{j \neq i} \mathsf{Q}\left(\sqrt{\frac{d_{\mathsf{E}}^2(\mathsf{X}_i, \mathsf{X}_j)}{2\mathsf{N}_0}}\right).$$

Upper bound on P_s for 4-QAM

Example: 4-QAM

$$P_\mathsf{s}^{4\mathsf{QAM}} \leq \frac{1}{4}\sum_{i=1}^{4}\sum_{j\neq i} \mathsf{Q}\left(\sqrt{\frac{d_\mathsf{E}^2(\mathsf{X}_i,\mathsf{X}_j)}{2\mathsf{N}_0}}\right)$$

Upper bound on P_s for 4-QAM

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$$P_\mathsf{s}^{\mathsf{4QAM}} \leq \frac{1}{4}\sum_{i=1}^{4}\sum_{j\neq i}\mathsf{Q}\left(\sqrt{\frac{d_\mathsf{E}^2(\mathsf{X}_i,\mathsf{X}_j)}{2\mathsf{N}_0}}\right) = \sum_{j=2}^{4}\mathsf{Q}\left(\sqrt{\frac{d_\mathsf{E}^2(\mathsf{X}_1,\mathsf{X}_j)}{2\mathsf{N}_0}}\right)$$

$$= \mathsf{Q}\left(\sqrt{\frac{d_\mathsf{E}^2(\mathsf{X}_1,\mathsf{X}_2)}{2\mathsf{N}_0}}\right) + \mathsf{Q}\left(\sqrt{\frac{d_\mathsf{E}^2(\mathsf{X}_1,\mathsf{X}_3)}{2\mathsf{N}_0}}\right) + \mathsf{Q}\left(\sqrt{\frac{d_\mathsf{E}^2(\mathsf{X}_1,\mathsf{X}_4)}{2\mathsf{N}_0}}\right).$$

Upper bound on P_{s} for 4-QAM

Example: 4-QAM

$$P_{\mathsf{s}}^{4\mathsf{QAM}} \leq \frac{1}{4}\sum_{i=1}^{4}\sum_{j\neq i}\mathsf{Q}\left(\sqrt{\frac{d_{\mathsf{E}}^2(\mathsf{X}_i,\mathsf{X}_j)}{2\mathsf{N}_0}}\right) = \sum_{j=2}^{4}\mathsf{Q}\left(\sqrt{\frac{d_{\mathsf{E}}^2(\mathsf{X}_1,\mathsf{X}_j)}{2\mathsf{N}_0}}\right)$$
$$= \mathsf{Q}\left(\sqrt{\frac{d_{\mathsf{E}}^2(\mathsf{X}_1,\mathsf{X}_2)}{2\mathsf{N}_0}}\right) + \mathsf{Q}\left(\sqrt{\frac{d_{\mathsf{E}}^2(\mathsf{X}_1,\mathsf{X}_3)}{2\mathsf{N}_0}}\right) + \mathsf{Q}\left(\sqrt{\frac{d_{\mathsf{E}}^2(\mathsf{X}_1,\mathsf{X}_4)}{2\mathsf{N}_0}}\right).$$

Using $d_{\mathsf{E}}^2(\mathsf{X}_1,\mathsf{X}_2) = d_{\mathsf{E}}^2(\mathsf{X}_1,\mathsf{X}_4) = ||\alpha(-1+j) - \alpha(1+j)||^2 = 4\alpha^2 = 2\mathsf{E}_{\mathsf{s}}$ and $d_{\mathsf{E}}^2(\mathsf{X}_1,\mathsf{X}_3) = ||\alpha(-1+j) - \alpha(1-j)||^2 = 8\alpha^2 = 4\mathsf{E}_{\mathsf{s}}$,

Upper bound on P_s for 4-QAM

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Using $d_\mathsf{E}^2(\mathsf{X}_1,\mathsf{X}_2) = d_\mathsf{E}^2(\mathsf{X}_1,\mathsf{X}_4) = ||\alpha(-1+j) - \alpha(1+j)||^2 = 4\alpha^2 = 2\mathsf{E}_\mathsf{s}$ and $d_\mathsf{E}^2(\mathsf{X}_1,\mathsf{X}_3) = ||\alpha(-1+j) - \alpha(1-j)||^2 = 8\alpha^2 = 4\mathsf{E}_\mathsf{s}$,

$$P_\mathsf{s}^{\mathsf{4QAM}} \leq 2\mathsf{Q}\left(\sqrt{\frac{\mathsf{E}_\mathsf{s}}{\mathsf{N}_0}}\right) + \mathsf{Q}\left(\sqrt{\frac{2\mathsf{E}_\mathsf{s}}{\mathsf{N}_0}}\right)$$
$$= 2\mathsf{Q}\left(\sqrt{\frac{2\mathsf{E}_\mathsf{b}}{\mathsf{N}_0}}\right) + \mathsf{Q}\left(\sqrt{\frac{4\mathsf{E}_\mathsf{b}}{\mathsf{N}_0}}\right).$$

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$$P_{\mathsf{s}}^{(M)} \leq \frac{1}{M} \sum_{i=1}^{M} \sum_{j \neq i} \mathsf{Q}\left(\sqrt{\frac{d_{\mathsf{E}}^2(\mathsf{X}_i, \mathsf{X}_j)}{2\mathsf{N}_0}} \right).$$

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- The dominant terms are the ones with smaller $d_{\mathsf{E}}(\mathsf{X}_i, \mathsf{X}_j)$
- Let

$$d_{\mathsf{E,min}}(\mathsf{X}_i) = \min_{\mathsf{X}_j \neq \mathsf{X}_i} d_{\mathsf{E}}(\mathsf{X}_i, \mathsf{X}_j)$$

and $A_{\mathsf{min}}(\mathsf{X}_i)$ the number of constellation points at distance $d_{\mathsf{E,min}}(\mathsf{X}_i)$ from X_i (nearest neighbors of X_i).

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  and $A_{\mathsf{min}}(\mathsf{X}_i)$ the number of constellation points at distance $d_{\mathsf{E,min}}(\mathsf{X}_i)$ from X_i (nearest neighbors of X_i). Then,

$$P_{\mathsf{s}}^{(M)} \approx \frac{1}{M}\sum_{i=1}^{M} A_{\min}(\mathsf{X}_i)\mathsf{Q}\left(\sqrt{\frac{d_{\mathsf{E,min}}^2(\mathsf{X}_i)}{2\mathsf{N}_0}}\right).$$

Nearest Neighbor Approximation

- Further approximate P_s considering only the minimum Euclidean distance,

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$$\begin{aligned} P_{\mathsf{s}}^{(M)} &\approx \frac{1}{M} \sum_{i=1}^{M} A_{\min}(\mathsf{X}_i) \mathsf{Q}\left(\sqrt{\frac{d^2_{\mathsf{E,min}}(\mathsf{X}_i)}{2\mathsf{N}_0}}\right) \\ &\leq \frac{1}{M} \sum_{i=1}^{M} A_{\mathsf{min}}(\mathsf{X}_i) \mathsf{Q}\left(\sqrt{\frac{d^2_{\mathsf{E,min}}}{2\mathsf{N}_0}}\right) \end{aligned}$$

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where $\bar{A}_{\mathsf{min}} = \frac{1}{M}\sum_{i=1}^{M} A_{\mathsf{min}}(\mathsf{X}_i)$ is the average number of nearest neighbors.

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&\leq \frac{1}{M}\sum_{i=1}^{M} A_{\mathsf{min}}(\mathsf{X}_i)\mathsf{Q}\left(\sqrt{\frac{d^2_{\mathsf{E,min}}}{2\mathsf{N}_0}}\right)\\
&= \left(\frac{1}{M}\sum_{i=1}^{M} A_{\mathsf{min}}(\mathsf{X}_i)\right)\mathsf{Q}\left(\sqrt{\frac{d^2_{\mathsf{E,min}}}{2\mathsf{N}_0}}\right) = \bar{A}_{\mathsf{min}}\mathsf{Q}\left(\sqrt{\frac{d^2_{\mathsf{E,min}}}{2\mathsf{N}_0}}\right),
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where $\bar{A}_{\mathsf{min}} = \frac{1}{M}\sum_{i=1}^{M} A_{\mathsf{min}}(\mathsf{X}_i)$ is the average number of nearest neighbors.

- Requires only knowledge of $\bar{A}_{\mathsf{min}}$ and $d_{\mathsf{E,min}}$!

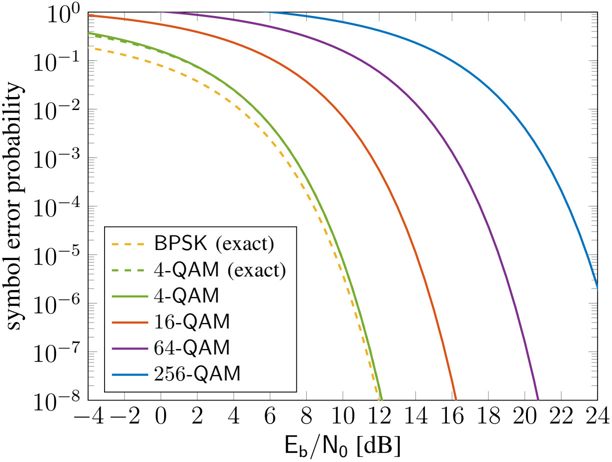
Nearest Neighbor Approximation

Nearest Neighbor Approximation for squared M-QAM

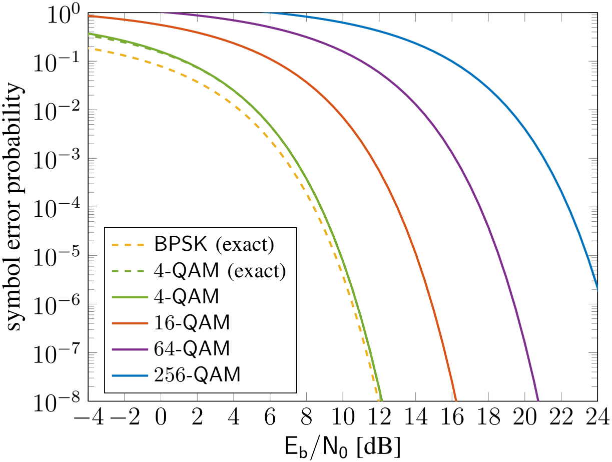
For squared M-QAM constellations, there are 4 constellation points with 2 neighbors, $4(\sqrt{M}-2)$ points with 3 neighbors and the remaining points have 4 neighbors, hence $\bar{A}_{\mathsf{min}} = 4 - 4/\sqrt{M}$. Furthermore, $d_{\mathsf{E,min}} = \sqrt{\frac{6\mathsf{E_s}}{M-1}}$. Hence,

$$
\begin{aligned}
P_{\mathsf{s}}^{M\mathsf{QAM}} &\approx \left(4 - \frac{4}{\sqrt{M}}\right)\mathsf{Q}\left(\sqrt{\frac{3\mathsf{E_s}}{(M-1)\mathsf{N_0}}}\right) \\
&= \left(4 - \frac{4}{\sqrt{M}}\right)\mathsf{Q}\left(\sqrt{\frac{3\mathsf{E_b}\log M}{(M-1)\mathsf{N_0}}}\right).
\end{aligned}
$$

Nearest Neighbor Approximation of P_s of QAM

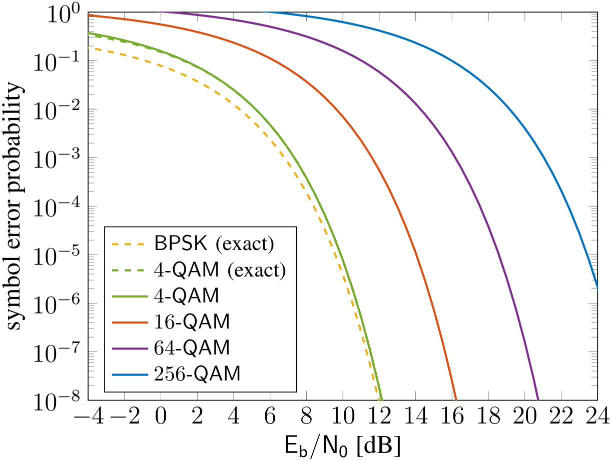


Nearest Neighbor Approximation of P_s of QAM



- The power efficiency decreases with M.

Nearest Neighbor Approximation of P_s of QAM



- The power efficiency decreases with M.
- The spectral efficiency increases with M.

Bit Error Probability of Linear Modulations

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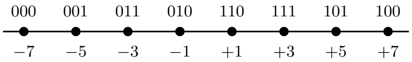
$$\mathsf{L}(x) = (b_1(x), \ldots, b_m(x))$$

be the m-bit labeling associated to constellation symbol $x \in \mathcal{X}$.

Bit Error Probability of Linear Modulations



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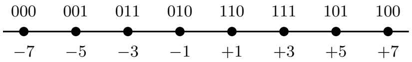

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- Example: $\mathsf{L}(+1) = (1, 1, 0)$, $\mathsf{L}(+5) = (1, 0, 1)$.

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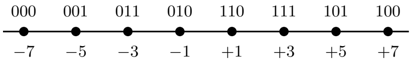

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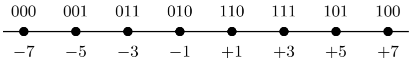

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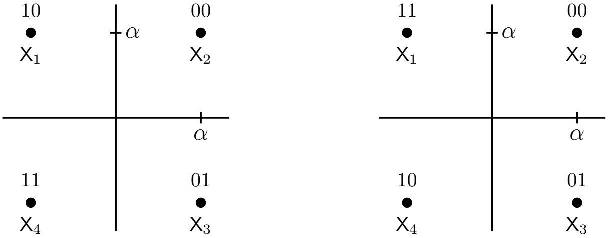
Bit Error Probability of 4-QAM with Gray and Lexicographic Labeling



Gray Labeling

- e_{b_i}: event that bit i is decoded in error.

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Gray Labeling

- e_{b_i}: event that bit i is decoded in error.

$$p_1 = \mathsf{Pr}(\mathsf{e}_{b_1}) = \sum_{x \in \mathcal{X}} \mathsf{Pr}(\mathsf{e}_{b_1} | x) P(x)$$

Gray Labeling

- e_{b_i}: event that bit i is decoded in error.

$$p_1 = \mathsf{Pr}(\mathsf{e}_{b_1}) = \sum_{x\in\mathcal{X}} \mathsf{Pr}(\mathsf{e}_{b_1}|x)P(x) = \frac{1}{4}\sum_{x\in\mathcal{X}} \mathsf{Pr}(\mathsf{e}_{b_1}|x)$$

Gray Labeling

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$$p_1 = \mathsf{Pr}(\mathsf{e}_{b_1}) = \sum_{x \in \mathcal{X}} \mathsf{Pr}(\mathsf{e}_{b_1}|x) P(x) = \frac{1}{4} \sum_{x \in \mathcal{X}} \mathsf{Pr}(\mathsf{e}_{b_1}|x)$$
$$= \frac{1}{4}(\mathsf{Pr}(\mathsf{e}_{b_1}|\mathsf{X}_1) + \mathsf{Pr}(\mathsf{e}_{b_1}|\mathsf{X}_2) + \mathsf{Pr}(\mathsf{e}_{b_1}|\mathsf{X}_3) + \mathsf{Pr}(\mathsf{e}_{b_1}|\mathsf{X}_4))$$

Gray Labeling

- e_{b_i}: event that bit i is decoded in error.

$$
\begin{aligned}
p_1 = \mathsf{Pr}(\mathsf{e}_{b_1}) &= \sum_{x \in \mathcal{X}} \mathsf{Pr}(\mathsf{e}_{b_1}|x)P(x) = \frac{1}{4}\sum_{x \in \mathcal{X}} \mathsf{Pr}(\mathsf{e}_{b_1}|x) \\
&= \frac{1}{4}(\mathsf{Pr}(\mathsf{e}_{b_1}|\mathsf{X}_1) + \mathsf{Pr}(\mathsf{e}_{b_1}|\mathsf{X}_2) + \mathsf{Pr}(\mathsf{e}_{b_1}|\mathsf{X}_3) + \mathsf{Pr}(\mathsf{e}_{b_1}|\mathsf{X}_4)) \\
&= \frac{1}{4}(\mathsf{Pr}(b_1(\hat{x}) = 0|\mathsf{X}_1) + \mathsf{Pr}(b_1(\hat{x}) = 1|\mathsf{X}_2) \\
&\qquad + \mathsf{Pr}(b_1(\hat{x}) = 1|\mathsf{X}_3) + \mathsf{Pr}(b_1(\hat{x}) = 0|\mathsf{X}_4)),
\end{aligned}
$$

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$$
\begin{aligned}
p_1 = \mathsf{Pr}(\mathsf{e}_{b_1}) &= \sum_{x \in \mathcal{X}} \mathsf{Pr}(\mathsf{e}_{b_1}|x)P(x) = \frac{1}{4}\sum_{x \in \mathcal{X}} \mathsf{Pr}(\mathsf{e}_{b_1}|x) \\
&= \frac{1}{4}(\mathsf{Pr}(\mathsf{e}_{b_1}|\mathsf{X}_1) + \mathsf{Pr}(\mathsf{e}_{b_1}|\mathsf{X}_2) + \mathsf{Pr}(\mathsf{e}_{b_1}|\mathsf{X}_3) + \mathsf{Pr}(\mathsf{e}_{b_1}|\mathsf{X}_4)) \\
&= \frac{1}{4}(\mathsf{Pr}(b_1(\hat{x}) = 0|\mathsf{X}_1) + \mathsf{Pr}(b_1(\hat{x}) = 1|\mathsf{X}_2) \\
&\qquad + \mathsf{Pr}(b_1(\hat{x}) = 1|\mathsf{X}_3) + \mathsf{Pr}(b_1(\hat{x}) = 0|\mathsf{X}_4)),
\end{aligned}
$$

- All terms are $\mathsf{Q}\left(\sqrt{\frac{2\mathsf{E_b}}{\mathsf{N_0}}}\right)$, thus $p_1 = \mathsf{Q}\left(\sqrt{\frac{2\mathsf{E_b}}{\mathsf{N_0}}}\right)$.

Gray Labeling

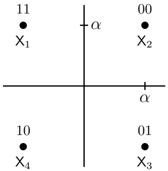
- e_{b_i}: event that bit i is decoded in error.

$$
\begin{aligned}
p_1 &= \Pr(\mathsf{e}_{b_1}) = \sum_{x\in\mathcal{X}} \Pr(\mathsf{e}_{b_1}|x)P(x) = \frac{1}{4}\sum_{x\in\mathcal{X}} \Pr(\mathsf{e}_{b_1}|x) \\
&= \frac{1}{4}(\Pr(\mathsf{e}_{b_1}|\mathsf{X}_1) + \Pr(\mathsf{e}_{b_1}|\mathsf{X}_2) + \Pr(\mathsf{e}_{b_1}|\mathsf{X}_3) + \Pr(\mathsf{e}_{b_1}|\mathsf{X}_4)) \\
&= \frac{1}{4}(\Pr(b_1(\hat{x}) = 0|\mathsf{X}_1) + \Pr(b_1(\hat{x}) = 1|\mathsf{X}_2) \\
&\qquad + \Pr(b_1(\hat{x}) = 1|\mathsf{X}_3) + \Pr(b_1(\hat{x}) = 0|\mathsf{X}_4)),
\end{aligned}
$$

- All terms are $\mathsf{Q}\left(\sqrt{\frac{2\mathsf{E_b}}{\mathsf{N_0}}}\right)$, thus $p_1 = \mathsf{Q}\left(\sqrt{\frac{2\mathsf{E_b}}{\mathsf{N_0}}}\right)$.
- By symmetry, $p_2 = p_1$, hence,

$$
P_{\mathsf{b}}^{\mathsf{4QAM-Gray}} = \frac{1}{2}(p_1 + p_2) = \mathsf{Q}\left(\sqrt{\frac{2\mathsf{E_b}}{\mathsf{N_0}}}\right).
$$

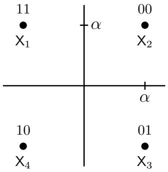
Lexicographic Labeling



$$p_1 = \mathsf{Q}\left(\sqrt{\frac{2\mathsf{E_b}}{\mathsf{N_0}}}\right)$$

$$p_2 =$$

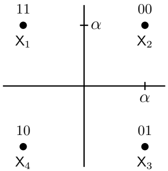
Lexicographic Labeling



$$p_1 = \mathsf{Q}\left(\sqrt{\frac{2\mathsf{E_b}}{\mathsf{N_0}}}\right)$$

$$p_2 = 2\mathsf{Q}\left(\sqrt{\frac{2\mathsf{E_b}}{\mathsf{N_0}}}\right)\left(1 - \mathsf{Q}\left(\sqrt{\frac{2\mathsf{E_b}}{\mathsf{N_0}}}\right)\right).$$

Lexicographic Labeling



$$p_1 = \mathsf{Q}\left(\sqrt{\frac{2\mathsf{E_b}}{\mathsf{N_0}}}\right)$$

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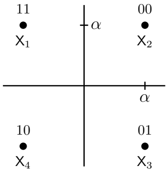
Thus,

$$P_{\mathsf{b}}^{\mathsf{4QAM-Lex}} = \frac{1}{2}(p_1 + p_2) = \frac{3}{2}\mathsf{Q}\left(\sqrt{\frac{2\mathsf{E_b}}{\mathsf{N_0}}}\right) - \left(\mathsf{Q}\left(\sqrt{\frac{2\mathsf{E_b}}{\mathsf{N_0}}}\right)\right)^2.$$

Lexicographic Labeling



$$p_1 = \mathsf{Q}\left(\sqrt{\frac{2\mathsf{E_b}}{\mathsf{N_0}}}\right)$$

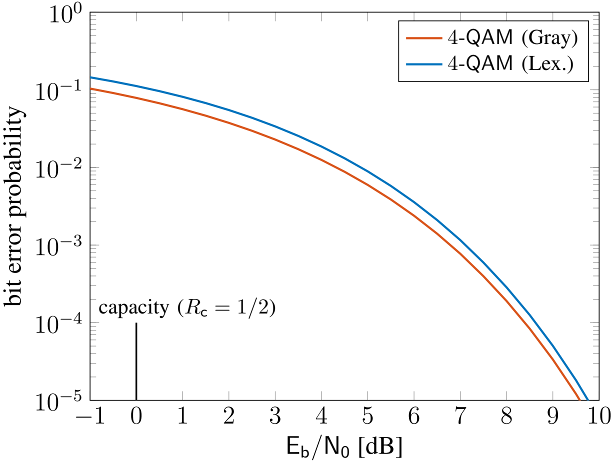
$$p_2 = 2\mathsf{Q}\left(\sqrt{\frac{2\mathsf{E_b}}{\mathsf{N_0}}}\right)\left(1 - \mathsf{Q}\left(\sqrt{\frac{2\mathsf{E_b}}{\mathsf{N_0}}}\right)\right).$$

Thus,

$$P_{\mathsf{b}}^{4\mathsf{QAM-Lex}} = \frac{1}{2}(p_1 + p_2) = \frac{3}{2}\mathsf{Q}\left(\sqrt{\frac{2\mathsf{E_b}}{\mathsf{N_0}}}\right) - \left(\mathsf{Q}\left(\sqrt{\frac{2\mathsf{E_b}}{\mathsf{N_0}}}\right)\right)^2.$$

- Due to the lack of symmetry, $p_1 \neq p_2$.

Bit Error Probability for 4-QAM



Bit Error Probability of M-ary Constellations

- For general M-ary constellations and arbitrary labelings, the computation of P_b is cumbersome $\longrightarrow$ Nearest neighbor approximation.