HOMEWORK - 2

1) Inner product is given by:

$$\langle x(t), y(t) \rangle = \int_{-\infty}^{\infty} x(t)\, y^*(t)\, dt$$

convolution is given by:

$$x(t) * y(t) = \int_{-\infty}^{\infty} x(\tau) \cdot y(t-\tau)\, d\tau$$

We can say, $y^*(t) = y(-(t-\tau))$

$\Rightarrow \quad y(t) = \overleftarrow{y^*}(t-\tau)$

$\Rightarrow$ Therefore:

$$x(t) * y(t) = \langle x(t)\, \overleftarrow{y}(t-\tau) \rangle$$

2] $x \in [0, 2]$

For uniformly distributed R.V. $\Rightarrow$ Area under curve = 1

a) $\int_{-\infty}^{\infty} A e^{-2y}\, dy = 1$

$$\left. \frac{A e^{-2y}}{-2} \right]_0^{\infty} = 1 \Rightarrow \boxed{A = 2}$$

b) Given : $Z = X + Y$

$f_Y(y) = 2e^{-2y}$

$$f_X(x) = \begin{cases} \frac{1}{2}, & 0 < x, 2 \\ 0 & \text{otherwise} \end{cases}$$

Consider unit step function : $u(t)$

$$f_X(x) = \frac{1}{2}\left[u(t) - u(t-2)\right]$$

Using laplace transformation :

Unit step function $u(t) \longrightarrow \frac{1}{s}$

$\Rightarrow u(t-t_0) \longrightarrow \frac{1}{s}e^{-t_0 s}$

Convolution is given by : $y(t) = x(t) * H(t)$

Laplace is given by : $Y(s) = X(s) \cdot H(s)$

We know : $f(z) = f(x) * f(y)$

$$L\{f(z)\} = L\{f(x)\} \cdot L\{f(y)\}$$

$$= L\{\tfrac{1}{2}[u(t) - u(t-2)]\} \cdot L\{e^{-2y}u(y)\}$$

On splitting ; we get $\Rightarrow$

$$f(z) = \frac{1}{2}\left[u(z)\left(1 - e^{-2z}\right) - \left(1 - e^{-2(z-2)}\right)u(z-2)\right]$$

So we can write it:

$$f(z) = \begin{cases} \frac{1}{2}\left(1 - e^{-2z}\right) & \text{for } 0 \le z \le 2 \\ \frac{1}{2}\left(e^{-2z} + e^{-2(z-2)}\right) & \text{for otherwise} \end{cases}$$

3)

a) Minimum distance $\Rightarrow \| s_i - s_{ii} \|$

We also have: $\tilde{s}_j = s_j - b$

distance $= \| \tilde{s}_i - \tilde{s}_{ii} \|$

$= \cancel{\ldots}$

$= \| s_i - b - (s_{ii} - b) \|$

$= \| s_i - s_{ii} \|$

b) Energy (E) for $\tilde{s} = \frac{1}{M} \sum_{j=1}^{M} \tilde{s}_j$

$= \frac{1}{M} \sum_{j=1}^{M} | s_j - b |^2$

Differentiate and equate to 0

$$\frac{dE}{db} = \frac{2}{M} \sum_{j=1}^{M} (s_j - b) = 0$$

$$\Rightarrow \boxed{\frac{1}{M} \sum_{j=1}^{M} s_j = b}$$

c) Let d be the distance b/w the constellation points. Energy = distance b/w points

$$E_{BPSK} = \frac{1}{2}\left(\|d\|^2 + \|-d\|^2\right)$$

$$\boxed{E_{BPSK} = d^2}$$

For $\tilde{S} \Rightarrow E_{\tilde{S}} = \frac{1}{2}\left[\|S_i - b\|^2 + \|S_{ii} - b\|^2\right]$

If $M = 2$ for BPSK — $b = \frac{1}{2}\sum_{j=1}^{M} S_j \Rightarrow$ ~~b~~

$b = \frac{1}{2}(S_i + S_{ii})$

$$E_{\tilde{S}} = \frac{1}{2}\left[\|S_i - \frac{1}{2}(S_i + S_{ii})\|^2 + \|S_{ii} - \frac{1}{2}(S_i + S_{ii})\|^2\right]$$

$$\Rightarrow \boxed{E_{\tilde{S}} = \frac{d^2}{4}}$$