Exercise 2 Solution

Sept. 13, 2023

Problem 1 (Inner Product)

Consider the signals $s_1(t) = \frac{1}{\sqrt{T}} e^{j2\pi t/T} \mathrm{I}\{0 \le t \le T\}$ and $s_2(t) = \frac{1}{\sqrt{T}} e^{j4\pi t/T} \mathrm{I}\{0 \le t \le T\}$. Compute the following inner products

- $\langle s_1, s_1 \rangle$,

$$\begin{aligned}
\langle s_1, s_1 \rangle &= \int_{-\infty}^{\infty} s_1(t) s_1^*(t) \, \mathrm{d}t \\
&= \int_{-\infty}^{\infty} |s_1|^2(t) \, \mathrm{d}t \\
&= \|s_1\|^2 \\
&= \int_{-\infty}^{\infty} \frac{1}{\sqrt{T}} e^{j2\pi t/T} \mathrm{I}\{0 \le t \le T\} \frac{1}{\sqrt{T}} e^{-j2\pi t/T} \mathrm{I}\{0 \le t \le T\} \, \mathrm{d}t \\
&= \frac{1}{T} \int_{-\infty}^{\infty} e^{j2\pi t/T - j2\pi t/T} \mathrm{I}\{0 \le t \le T\} \, \mathrm{d}t \\
&= \frac{1}{T} \int_{-\infty}^{\infty} \mathrm{I}\{0 \le t \le T\} \, \mathrm{d}t \\
&= 1.
\end{aligned}$$

- $\langle s_1, s_2 \rangle$.

$$\begin{aligned}
\langle s_1, s_2 \rangle &= \int_{-\infty}^{\infty} \frac{1}{\sqrt{T}} e^{j2\pi t/T} \mathrm{I}\{0 \le t \le T\} \frac{1}{\sqrt{T}} e^{-j4\pi t/T} \mathrm{I}\{0 \le t \le T\} \, \mathrm{d}t \\
&= \frac{1}{T} \int_{-\infty}^{\infty} e^{-j2\pi t/T} \mathrm{I}\{0 \le t \le T\} \, \mathrm{d}t \\
&= \frac{1}{T} \frac{e^{-j2\pi t/T}}{-j2\pi/T} \Bigg|_0^T \\
&= \frac{1}{T} \frac{e^{-j2\pi} - e^0}{-j2\pi/T} \\
&= 0.
\end{aligned}$$

Problem 2 (Orthonormal Basis)

Consider the signals $s_1(t) = \mathrm{I}\{0 \le t \le 1\} - \mathrm{I}\{1 < t \le 2\}$; $s_2(t) = \mathrm{I}\{0 \le t \le 2\}$; $s_3(t) = \mathrm{I}\{0 \le t \le 0.5\} + \mathrm{I}\{1.5 \le t \le 2\}$; and $s_4(t) = \mathrm{I}\{0 \le t \le 0.5\} - \mathrm{I}\{1.5 \le t \le 2\}$.

1. Plot the signals.

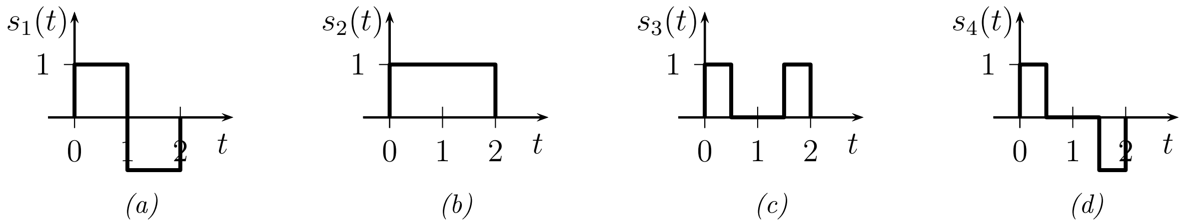


Figure 1: Problem 2.

2. Find a **minimal set** of orthonormal basis for span(s_1, s_2, s_3, s_4).
   We are going to follow Gram-Schmidt procedure (to make sure we find the minimal set):

$$
\begin{aligned}
&u_1 = s_1 \\
&\|u_1\| = \sqrt{\int_{-\infty}^{\infty} u_1^2 \ \mathrm{d}t} = \sqrt{2}, \\
&\phi_1 = \frac{u_1}{\|u_1\|} = \frac{s_1}{\sqrt{2}} = \frac{\sqrt{2}}{2} \left(\mathrm{I}\left\{0 \le t \le 1\right\} - \mathrm{I}\left\{1 < t \le 2\right\} \right), \\
&\langle s_2, \phi_1 \rangle = \int_{-\infty}^{\infty} s_2(t)\phi_1(t) \ \mathrm{d}t = 0, \\
&u_2 = s_2 - \langle s_2, \phi_1 \rangle \phi_1 = s_2, \\
&\|u_2\| = \sqrt{2} \\
&\phi_2 = \frac{u_2}{\|u_2\|} = \frac{\sqrt{2}}{2} \mathrm{I}\left\{0 \le t \le 2\right\}, \\
&\langle s_3, \phi_1 \rangle = 0, \\
&\langle s_3, \phi_2 \rangle = \frac{\sqrt{2}}{2}, \\
&u_3 = s_3 - \langle s_3, \phi_1 \rangle \phi_1 - \langle s_3, \phi_2 \rangle \phi_2 = s_3 - \frac{1}{2} s_2, \\
&u_3 = 0.5\mathrm{I}\left\{0 \le t \le 0.5\right\} - 0.5\mathrm{I}\left\{0.5 \le t \le 1.5\right\} + 0.5\mathrm{I}\left\{1.5 \le t \le 2\right\}, \\
&\|u_3\|^2 = \frac{1}{2}; \\
&\phi_3 = \frac{u_3}{\|u_3\|} = \frac{\sqrt{2}}{2} \left(\mathrm{I}\left\{0 \le t \le 0.5\right\} - \mathrm{I}\left\{0.5 \le t \le 1.5\right\} + \mathrm{I}\left\{1.5 \le t \le 2\right\} \right), \\
&\langle s_4, \phi_2 \rangle = \langle s_4, \phi_3 \rangle = 0, \\
&\langle s_4, \phi_1 \rangle = \frac{\sqrt{2}}{2}, \\
&u_4 = s_4 - \langle s_4, \phi_1 \rangle \phi_1 - \langle s_4, \phi_2 \rangle \phi_2 - \langle s_4, \phi_3 \rangle \phi_3 = s_4 - \frac{1}{2} s_1, \\
&u_4 = 0.5\mathrm{I}\left\{0 \le t \le 0.5\right\} - 0.5\mathrm{I}\left\{0.5 \le t \le 1\right\} + 0.5\mathrm{I}\left\{1 \le t \le 1.5\right\} - 0.5\mathrm{I}\left\{1.5 \le t \le 2\right\} \\
&\|u_4\|^2 = \frac{1}{2}, \\
&\phi_4 = \frac{u_4}{\|u_4\|} = \frac{\sqrt{2}}{2} \left(\mathrm{I}\left\{0 \le t \le 0.5\right\} - \mathrm{I}\left\{0.5 \le t \le 1\right\} + \mathrm{I}\left\{1 \le t \le 1.5\right\} - \mathrm{I}\left\{1.5 \le t \le 2\right\} \right).
\end{aligned}
$$

3. Express each of the signals as a linear combination of the basis functions.

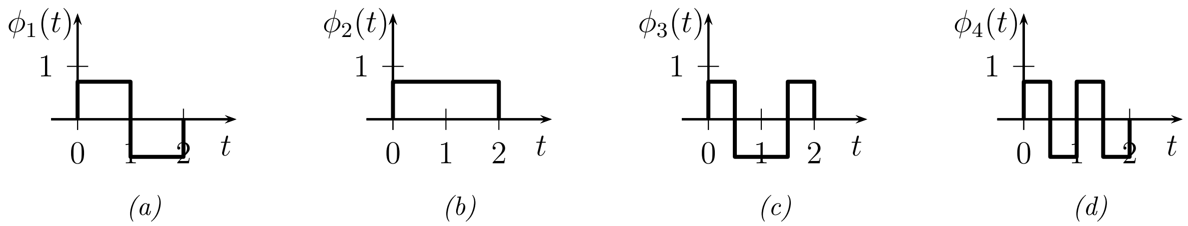


Figure 2: Problem 2.

$$
\begin{aligned}
s_1 &= \sqrt{2}\phi_1 = [\sqrt{2}, 0, 0, 0],\\
s_2 &= \sqrt{2}\phi_2 = [0, \sqrt{2}, 0, 0],\\
s_3 &= \frac{\sqrt{2}}{2}\phi_2 + \frac{\sqrt{2}}{2}\phi_3 = [0, \frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}, 0],\\
s_4 &= \frac{\sqrt{2}}{2}\phi_1 + \frac{\sqrt{2}}{2}\phi_4 = [\frac{\sqrt{2}}{2}, 0, 0, \frac{\sqrt{2}}{2}].
\end{aligned}
$$

4. Find energy of the signals using the results of the previous item.

$$
\begin{aligned}
\|s_1\|^2 &= \sqrt{2}^2 = 2,\\
\|s_2\|^2 &= \sqrt{2}^2 = 2,\\
\|s_3\|^2 &= \frac{1}{2} + \frac{1}{2} = 1,\\
\|s_4\|^2 &= \frac{1}{2} + \frac{1}{2} = 1.
\end{aligned}
$$

Problem 3 (Simplex Constellations)

A communication system uses three signals shown in Figure 3 for transmission. The signals are transmitted equiprobably. Answer the following questions.

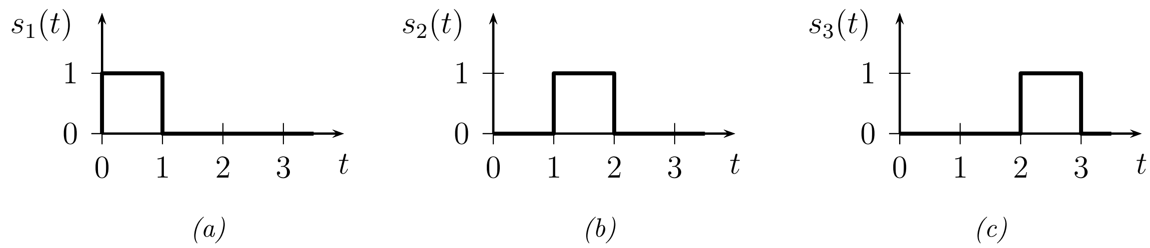


Figure 3: Signal alternatives.

1. Find an orthonormal basis $\{\phi_i(t)\}$ for these signals. Sketch a constellation diagram.
   The signals are already orthogonal and of unit energy. Thus, $\phi_j(t) = s_j(t)$.

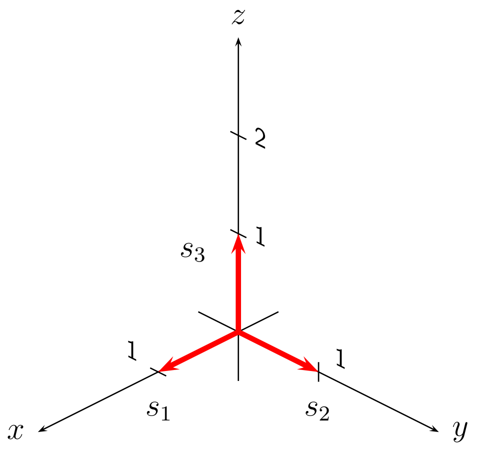


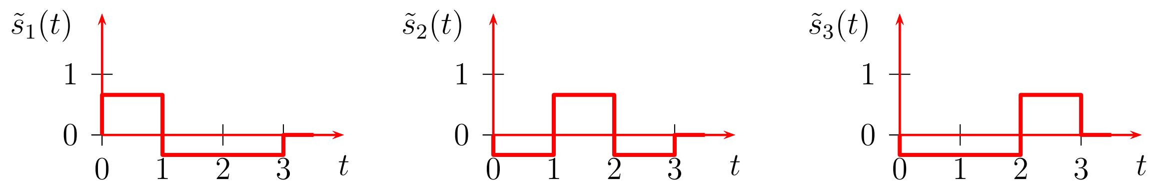
Figure 4: Constellation diagram for $\{s_j(t)\}$.

2. Find a new set of signals $\{\tilde{s}_j(t)\}$, $j = 1, 2, 3$ by subtracting the mean signal from the original signals, i.e.,

$$\tilde{s}_j(t) = s_j(t) - \frac{1}{3}\sum_{k=1}^{3} s_k(t).$$

   Plot the new signals.

   The new signals are shown in Figure 5.



(a) *(b)* *(c)*

Figure 5: The new signal alternatives.

3. Find an orthonormal basis $\{\tilde{\phi}_i(t)\}$ for the new set of signals $\{\tilde{s}_j(t)\}$ and sketch them. Express the signals as a superposition of the basis functions.

$$
\begin{aligned}
&\int_{-\infty}^{\infty} \tilde{s}_1^2(t) = \frac{2}{3},\\
&\tilde{\phi}_1(t) = \sqrt{\frac{3}{2}}\tilde{s}_1(t),\\
&\langle\tilde{\phi}_1(t), \tilde{s}_2(t)\rangle = \sqrt{\frac{3}{2}}\left(-\frac{2}{9} - \frac{2}{9} + \frac{1}{9}\right) = -\frac{1}{\sqrt{6}},\\
&u_2(t) = \tilde{s}_2(t) + \sqrt{\frac{1}{6}}\tilde{\phi}_1(t) = \tilde{s}_2(t) + \frac{1}{2}\tilde{s}_1(t),\\
&\int_{-\infty}^{\infty} u_2^2(t) = \frac{1}{2},\\
&\tilde{\phi}_2(t) = \sqrt{2}u_2(t),\\
&\langle\tilde{\phi}_2(t), \tilde{s}_2(t)\rangle = \frac{1}{\sqrt{2}}\left(0 + \frac{2}{3} + \frac{1}{3}\right) = \frac{1}{\sqrt{2}},\\
&\langle\tilde{\phi}_1(t), \tilde{s}_3(t)\rangle = \sqrt{\frac{3}{2}}\left(-\frac{2}{9} + \frac{1}{9} - \frac{2}{9}\right) = -\frac{1}{\sqrt{6}},\\
&\langle\tilde{\phi}_2(t), \tilde{s}_3(t)\rangle = \frac{1}{\sqrt{2}}\left(0 - \frac{1}{3} - \frac{2}{3}\right) = -\frac{1}{\sqrt{2}},\\
&\tilde{s}_1(t) = \sqrt{\frac{2}{3}}\tilde{\phi}_1(t),\\
&\tilde{s}_2(t) = -\frac{1}{\sqrt{6}}\tilde{\phi}_1(t) + \frac{1}{\sqrt{2}}\tilde{\phi}_2(t),\\
&\tilde{s}_3(t) = -\frac{1}{\sqrt{6}}\tilde{\phi}_1(t) - \frac{1}{\sqrt{2}}\tilde{\phi}_2(t).
\end{aligned}
$$

The basis functions are shown in Figure 6.

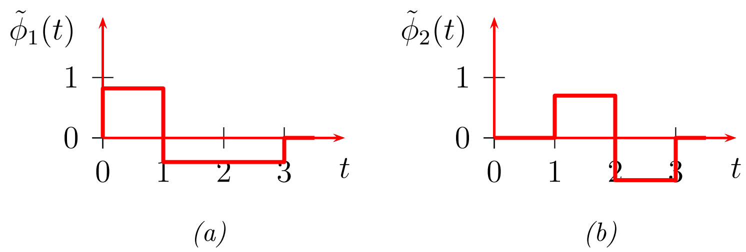


Figure 6: The basis functions $\{\tilde{\phi}_j\}(t)$.

4. Sketch a constellation diagram for the new signals and show the decision boundaries for the ML detector.

   The constellation is shown in Figure 7.

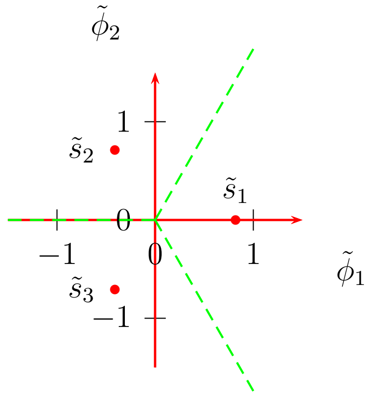


Figure 7: Constellation diagram.

Problem 4 (Energy)

Of all unit energy real signals that are bandlimited to W Hz, which one has the largest value at $t = 0$? What is its value at $t = 0$? Repeat for $t = 17$. Hint: find a good orthonormal basis for bandlimited signals and express signal energy in terms of its coordinates.

A good orthonormal basis for bandlimited signals is

$$\left\{ \frac{1}{\sqrt{T}} \text{sinc}((t - kT)/T) \right\}, k \in \mathbb{Z},$$

where $W = \frac{1}{2T}$. Any signal s bandlimited to W can be written in terms of coordinates in this basis as

$$s = [\dots, c_0, c_1, c_2, \dots],$$

where each coordinate is $c_k = \sqrt{T} s(kT)$. The energy of the signal can be written as

$$\|s\|^2 = \sum_{k=-\infty}^{\infty} c_k^2$$

and is equal to 1. Maximization of the values at $s(0)$ is equivalent to maximization of c_0^2.

$$c_0^2 = 1 - \sum_{k=-\infty, k \neq 0}^{\infty} c_k^2$$

c_0^2 is maximized if $c_k^2 = 0$ for $k \in \{\mathbb{Z} \setminus 0\}$. The maximum value is $c_0^2 = 1$. Therefore, the signal is

$$s(t) = \frac{1}{\sqrt{T}} \text{sinc}(t/T)$$

and

$$s(0) = \frac{1}{\sqrt{T}} = \sqrt{2W}.$$

If we want to maximize $s(17)$, we can use the basis

$$\left\{ \frac{1}{\sqrt{T}} \text{sinc}((t - kT - 17)/T) \right\}, k \in \mathbb{Z},$$

and the same reasoning. The answer is the same.