# Examination Wed. Oct. 31, 2018, 08:30–12:30

SSY121 Introduction to Communication Engineering

- Contact persons: Mohammad Nazari (031 - 772 1771) will visit the exam after approximately 1 and 3 hours.
- Instructions:
  - Write in English.
  - Use a pencil and eraser.
  - There is no page limit. Extra sheets of paper are available.
  - Solve the problems in any order (they are not ordered by difficulty).
  - Before handing in, sort the pages in problem order. Label each page with problem number and running page number. Do not hand in drafts or unused paper.
  - If any data is missing, make reasonable assumptions.
  - Chalmers' examination rules apply.
  - MP3/Music players **are not** allowed during the exam
- Allowed aids:
  - Mathematics Handbook by Råde and Westergren (any edition, including Beta) or equivalent
  - Chalmers-approved calculator can be borrowed by the exam administrators
- Grading principles:
  - Explain the line of reasoning clearly. A good solution with a minor error usually gives close to full points, even if the answer is incorrect.
  - An answer without a clear motivation usually gives 0 points, even if it is correct.
  - Whenever possible, check if your answer is reasonable. An obviously unreasonable answer usually gives 0 points, even if the solution is almost correct.
- Solutions and results:
  - Solutions will be posted on the course website no later than 7 days after the exam.
  - The grading can be reviewed on Thursday November 15, 2018, at 13:00–14:00 in E2 Blue Room (3340) on floor 3 in the EDIT building.

1. **True or false questions:** Justify *ALL* your answers using short and concise explanations (maximum 30 words per item). (Total points: 10)

   (a) In a digital communication system, a *maximum likelihood* (ML) detector uses the prior knowledge about the data source to make the decision. (1)

   (b) The matched filter receiver maximizes the SNR at the output of the matched filter. (1)

   (c) In a noiseless transmission of BPSK symbols, a phase error does not affect the performance. (1)

   (d) A communication system uses BPSK signaling over an AWGN channel and operates at a very high SNR, i.e., no bits are in error. Because of a technical problem at the RF front side of the transmitter, nothing is transmitted during some periods. This randomly happens in half of the time, and neither the transmitter nor the receiver are aware of it. The BER of this faulty system system is 0.5. (1)

   (e) Although the pulse $\text{sinc}(t/T)$ fulfills both the Nyquist criterion and $T$-orthogonality for a symbol period $T$, and has the smallest bandwidth for a fixed $T$, an RRC pulse is preferred in practical communication systems. (1)

   (f) The part of noise which cannot be projected in the dimensions of the signals does not affect the detection. (1)

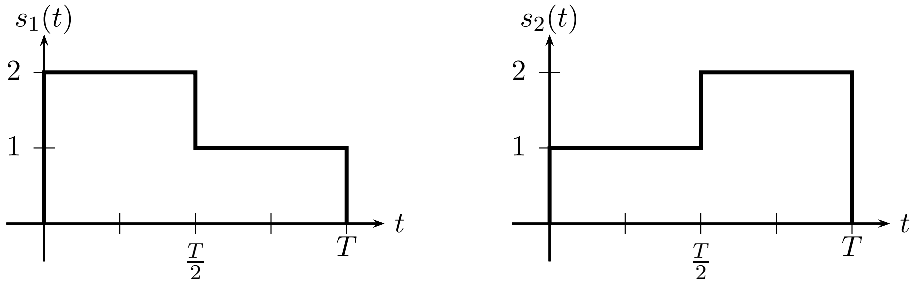
   (g) Suppose that rectangular pulses of duration $T$ are used for transmission of binary information over an AWGN channel using BPSK modulation, and a matched filter receiver is used at the receiver side. If a timing error $\Delta t = \frac{T}{2}$ occurs at the receiver, the BER tends to 0.5, i.e., the receiver decides completely randomly, when the SNR goes to infinity. (1)

   (h) One can approximately write that $\text{BER} = \log_2 M \times \text{SER}$ (where $M > 2$ is the number of symbols), when anti-Gray labeling is used. (1)

   (i) QPSK uses two orthogonal basis functions, but 8-PSK uses three orthogonal basis functions since 3 bits are mapped to every symbol. (1)

   (j) In OFDM systems, inter-symbol interference (ISI) can be completely eliminated using a cyclic prefix of length greater than the maximum delay spread. (1)

2. A digital communication system, operating at a bitrate of $R_\mathrm{b}$ bits/s, uses binary signaling with the equiprobable signals shown below, over an AWGN channel with noise power spectral density of $\frac{N_0}{2}$. (Total points: 9)




(a) (b)

   (a) What is $\frac{E_\mathrm{b}}{N_0}$ for this system (in terms of $N_0$ and $R_\mathrm{b}$), where $E_\mathrm{b}$ is the energy per bit? (2)
   (b) What is the error probability for this system (in terms of $N_0$ and $R_\mathrm{b}$)? *Hint: You need to find the distance between two signals.* (2)
   (c) By how many decibels does this system underperform a binary antipodal signaling system (normal BPSK) with the same $\frac{E_b}{N_0}$? (1)
   (d) Now assume that this system is augmented with two more signals $s_3(t) = -s_1(t)$ and $s_4(t) = -s_2(t)$ to result in a 4-ary system with equiprobable symbols. Using the union bound, find a bound on the symbol error probability of the resulted 4-ary system. (4)

3. A satellite in synchronous orbit is used to communicate with an earth station at a distance of $40,000$ km. The satellite has an antenna with a gain of 10 dB and the receiver antenna has a gain of 1 dB. The transmitter power is 1.6 W, and the carrier frequency is chosen so that $\frac{\lambda}{4\pi} = 1$. Assume that the power spectral density of noise is $N_0 = 4.14 \cdot 10^{-21}\mathrm{W/Hz}$. (Total points: 6)

   (a) What is the carrier frequency $f_\mathrm{c}$? (1)
   (b) Determine the bitrate $R_\mathrm{b}$ in order to have a BER of at most $10^{-5}$, if BPSK with sinc pulses are used as the signaling scheme. (4)
   (c) Find the one sided bandwidth required to have ISI-free transmission using a raised cosine pulse with roll-off factor $\alpha = 0.5$, with the bitrate obtained in the previous part. (1)

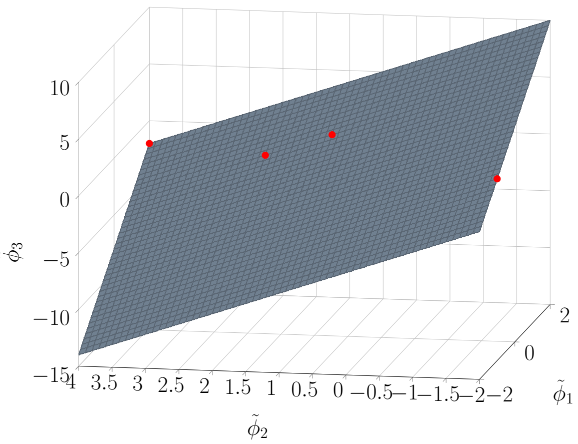



Figure 2: Symbol constellation.

4. Consider the constellation points given by the coordinates $s_1 = (1, 2, -1)$, $s_2 = (2, 4, -2)$, $s_3 = (-1, -2, 1)$, and $s_4 = (1, 1, 1)$. The constellation points are given in terms of the basis functions:

$$\begin{aligned}\tilde{\phi}_1(t) &= c_1 I\{0 \le t < 2\},\\ \tilde{\phi}_2(t) &= c_2(I\{0 \le t < 1\} - I\{1 \le t < 2\})\\ \tilde{\phi}_3(t) &= ?,\end{aligned}$$

where $c_1$ and $c_2$ are constants. The indicator function $I(a \le t < b)$ is equal to one for $t$ between $a$ and $b$ $(a < b)$, and equal to zero otherwise. The constellation points are located on a plane as shown in Fig. 2

(Total points: 10)

(a) Find the constants $c_1$ and $c_2$ in order to make $\tilde{\phi}_1(t)$ and $\tilde{\phi}_2(t)$ have unit energy. Also, propose a third basis function such that $\tilde{\phi}_1(t)$, $\tilde{\phi}_2(t)$, and $\tilde{\phi}_3(t)$ form an orthonormal basis. (4)

(b) Argue why the above basis is not a minimal basis for the constellation points. (1)

(c) Provide a minimal basis for the constellation points and express the points in the new basis. Sketch the new constellation. (5)

5. Consider a noiseless channel and a communication system using QPSK and matched filtering using a pulse that fulfills the Nyquist criterion after the matched filter. The symbol time is $T_\mathrm{s} = 1$ s and root-raised cosine pulses are used.

   The transmitter sends the symbols $[s_0, s_1, s_2, s_3] = [1+j, 1+j, 1+j, 1+j]/\sqrt{2}$ where $j$ is the imaginary unit. The transmitted passband signal is given as

$$x(t) = \sqrt{2}\sum_{k=0}^{3} \left(v(t - kT_\mathrm{s})a_k \cos(2\pi f_\mathrm{c} t) + v(t - kT_\mathrm{s})b_k \sin(2\pi f_\mathrm{c} t)\right)$$

   where $a_k$ and $b_k$ are the real and imaginary part of $s_k$, respectively, $f_\mathrm{c}$ is the carrier frequency, and $v(t)$ is the pulse used in the transmission.

   (Total points: 8)

   (a) Assume that a phase error of $\Delta$ radians and that a frequency error of $\phi$ radians per second is present at the receiver. The symbols are sampled at the correct sampling instants. Derive the expression for the received signal after sampling the matched filter output.
   *Hint: the identities below may be useful*

$$\begin{aligned}
2\sin(a)\cos(b) &= \sin(a+b) + \sin(a-b),\\
2\cos(a)\sin(b) &= \sin(a+b) - \sin(a-b),\\
2\sin(a)\sin(b) &= \cos(a-b) - \cos(a+b),\\
2\cos(a)\cos(b) &= \cos(a+b) + \cos(a-b).
\end{aligned}$$

   (5)

   (b) Plot the four received symbols in an IQ plot if $\Delta = \pi/3$ and $\phi = \pi/4$. (3)

6. Some questions relating to WiFi: (Total points: 5)

   (a) Briefly describe why a channel code is used. (2)

   (b) Briefly describe the spacial mapper (SM) block in the transmitter. (2)

   (c) 802.11ax increased the largest constellation from 256QAM to 1024QAM compared to 802.11ac. How much does the bitrate increase in percentage due to this increase? (1)

# Formula sheet, SSY121

Version 2.1, August 10, 2016

This sheet is an allowed aid at written exams in SSY121, Introduction to Communication Engineering, at Chalmers in 2016. It will be handed out with the exam problems. Students may not bring their own copy.

**Decibels**

$$\left(\frac{E_1}{E_2}\right)_{\text{dB}} = 10\log_{10}\frac{E_1}{E_2}$$

**Energies $E_\mathrm{s}$ and $E_\mathrm{b}$**

$$E_\mathrm{s} = \sum_{i=1}^{M}\mathbb{P}\left[\mathsf{S}=\mathsf{s}_i\right]||\mathsf{s}_i||^2$$

$$E_\mathrm{s} = E_\mathrm{b}\log_2 M$$

**Normalized minimum distance**

$$d_{\min} = \frac{D_{\min}}{\sqrt{2E_\mathrm{b}}}$$

**Nyquist criterion**

- In time domain

$$v(nT_\mathrm{s}) = 0, \quad n = \pm1, \pm2, \ldots$$

- In frequency domain

$$\sum_{n=-\infty}^{\infty}\Re\left\{V\left(f-\frac{n}{T_\mathrm{s}}\right)\right\} = T_\mathrm{s}v(0)$$
$$\sum_{n=-\infty}^{\infty}\Im\left\{V\left(f-\frac{n}{T_\mathrm{s}}\right)\right\} = 0,$$

  where $T_\mathrm{s}v(0)$ is a real constant.

- If the $v(t)$ is symmetric respect to zero, the definition in frequency domain is

$$\sum_{n=-\infty}^{\infty}V\left(f-\frac{n}{T_\mathrm{s}}\right) = T_\mathrm{s}v(0)$$

**$T_\mathrm{s}$-orthogonality**

- In time domain

$$\int_{-\infty}^{\infty}v(t)v(t-nT_\mathrm{s})dt = 0, \quad n = \pm1, \pm2, \ldots$$

- In frequency domain

$$\sum_{n=-\infty}^{\infty}\left|V\left(f-\frac{n}{T_\mathrm{s}}\right)\right|^2 = T_\mathrm{s}E_v$$

**Sinc, Raised-cosine, and Root raised-cosine pulses**

$$v_{\text{sinc}}(t) = \text{sinc}(t/T_\mathrm{p}) = \frac{\sin(\pi t/T_\mathrm{p})}{\pi t/T_\mathrm{p}}$$

$$V_{\text{sinc}}(f) = \begin{cases} T_\mathrm{p}, & |f| < \frac{1}{2T_\mathrm{p}} \\ 0, & |f| \geq \frac{1}{2T_\mathrm{p}} \end{cases}$$

$$v_{\text{RC}}(t) = \text{sinc}\left(\frac{t}{T_\mathrm{p}}\right)\frac{\cos\left(\frac{\pi\alpha t}{T_\mathrm{p}}\right)}{1-\left(\frac{2\alpha t}{T_\mathrm{p}}\right)^2}$$

$$V_{\text{RC}}(f) = \begin{cases} T_\mathrm{p}, & |f| < f_1 \\ \frac{T_\mathrm{p}}{2}\left(1+\cos\left[\frac{\pi T_\mathrm{p}}{\alpha}\left(|f|-\frac{1-\alpha}{2T_\mathrm{p}}\right)\right]\right), & f_1 \leq |f| < f_2 \\ 0, & |f| \geq f_2, \end{cases}$$

where $f_1 = \dfrac{1-\alpha}{2T_\mathrm{p}}$ and $f_2 = \dfrac{1+\alpha}{2T_\mathrm{p}}$

$$v_{\text{RRC}}(t) = \sqrt{T_\mathrm{p}}\frac{\sin\left(\frac{(1-\alpha)\pi t}{T_\mathrm{p}}\right)+\frac{4\alpha t}{T_\mathrm{p}}\cos\left(\frac{(1+\alpha)\pi t}{T_\mathrm{p}}\right)}{\pi t\left(1-\left(\frac{4\alpha t}{T_\mathrm{p}}\right)^2\right)}$$

$$V_{\text{RRC}}(f) = \sqrt{V_{\text{RC}}(f)}$$

**Correlation receiver**

$$\min_i\left\{\int_{-\infty}^{\infty}[y(t)-s_i(t)]^2\,dt\right\}$$
$$\max_i\left\{\int_{-\infty}^{\infty}y(t)s_i(t)\,dt-\frac{E_{s_i}}{2}\right\},$$

where $E_{s_i} = \int_{-\infty}^{\infty}s_i^2(t)\,dt$.

**PAM (baseband)**

$$s(t) = \sum_{k=0}^{\infty}a_k v(t-kT_\mathrm{s}),$$

where $a_k \in \{\pm(M-1), \pm(M-3), \ldots, \pm1\}$.

**PAM (passband)**

$$s(t) = \sum_{k=1}^{\infty}a_k v(t-kT_\mathrm{s})\sqrt{2}\cos w_c t,$$

where $a_k \in \{\pm(M-1), \pm(M-3), \ldots, \pm1\}$

**2D Modulations**

$$s(t) = \sum_{k=0}^{\infty}a_k v(t-kT_\mathrm{s})\sqrt{2}\cos(w_c t) - \sum_{k=0}^{\infty}b_k v(t-kT_\mathrm{s})\sqrt{2}\sin(w_c t)$$

**$M$-PSK**

$$s(t) = \sum_{k=0}^{\infty} v(t-kT_\mathrm{s})\sqrt{2}\cos\left(w_c t + \frac{2i\pi}{M}\right),$$

where $i = 0, 1, \ldots, M-1$.

**$M$-FSK**

$$s_i(t) = \cos\left(2\pi\left[f_c + \frac{h}{2T_\mathrm{s}}i\right]t\right),$$

where $i = \pm(M-1), \pm(M-3), \ldots, \pm 1$.

**Link budget**

$$P_\mathrm{R} = P_\mathrm{T} G_\mathrm{T} G_\mathrm{R}\left(\frac{\lambda}{4\pi d}\right)^2,$$

where $c = \lambda f = 3\cdot 10^8 \mathrm{m/s}$.

**Parabolic dish antenna**

$$G_\mathrm{Par} = \frac{4\pi A}{\lambda^2}$$

**1D Gaussian PDF (i.i.d.)**

$$f_X(x) = \frac{1}{\sqrt{2\pi\sigma^2}}\exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right)$$

**$N$D Gaussian PDF with variance $\sigma^2$ (i.i.d.)**

$$f_\mathsf{Z}(\mathsf{z}) = \frac{1}{(2\pi\sigma^2)^{N/2}}\exp\left(-\frac{||\mathsf{z}-\boldsymbol{\mu}||^2}{2\sigma^2}\right)$$

**Bayes' rule**

$$f_{\mathsf{X}|\mathsf{Y}}(\mathsf{x}|\mathsf{y}) = \frac{f_\mathsf{X}(\mathsf{x})}{f_\mathsf{Y}(\mathsf{y})} f_{\mathsf{Y}|\mathsf{X}}(\mathsf{y}|\mathsf{x})$$

## Additive white Gaussian noise (AWGN)

The following formulas are for the AWGN channel $\mathsf{Y} = \mathsf{S}+\mathsf{Z}$, where $\mathsf{S}$ is the transmitted symbol and $\mathsf{Z}$ is Gaussian noise.

**Maximum likelihood (ML) detection**

$$\max_i \{\mathbb{P}[\mathsf{Y}=\mathsf{y}|\mathsf{S}=\mathsf{s}_i]\}$$

**Maximum a posteriori (MAP) detection**

$$\max_i \{\mathbb{P}[\mathsf{S}=\mathsf{s}_i|\mathsf{Y}=\mathsf{y}]\} \equiv \max_i \{\mathbb{P}[\mathsf{S}=\mathsf{s}_i]\,\mathbb{P}[\mathsf{Y}=\mathsf{y}|\mathsf{S}=\mathsf{s}_i]\}$$

**Pairwise error probability (PEP)**

$$\mathrm{PEP}^{(i,j)} = \mathrm{Q}\left(\sqrt{\frac{D_{i,j}^2}{2N_0}}\right),$$

where $D_{i,j}^2 = \|\mathsf{s}_j - \mathsf{s}_i\|^2$.

**Symbol error probability (SEP) (exact)**

$$P_\mathrm{e} = \sum_{i=1}^{M}\mathbb{P}[\mathsf{S}=\mathsf{s}_i]\sum_{j\neq i}\mathbb{P}[\mathsf{Y}\in\mathcal{R}_j|\mathsf{S}=\mathsf{s}_i]$$

**SEP (union bound)**

$$\begin{aligned} P_\mathrm{e} &\leq \sum_{i=1}^{M}\mathbb{P}[\mathsf{S}=\mathsf{s}_i]\sum_{j\neq i}\mathrm{PEP}^{(i,j)} \\ &= \sum_{i=1}^{M}\mathbb{P}[\mathsf{S}=\mathsf{s}_i]\sum_{j\neq i}\mathrm{Q}\left(\sqrt{\frac{D_{i,j}^2}{2N_0}}\right) \end{aligned}$$

**SEP (High-SNR approximation for equally likely symbols)**

$$P_\mathrm{e} \approx \frac{2K}{M}\cdot\mathrm{Q}\left(\sqrt{\frac{D_\mathrm{min}^2}{2N_0}}\right),$$

where $K$ is the number of distinct signal pairs with distance $D_\mathrm{min} = \min_{i\neq j}\{D_{i,j}\}$ and $M$ is the constellation size.

**Bit error probability (BEP) (exact)**

$$B_\mathrm{e} = \sum_{i=1}^{M}\mathbb{P}[\mathsf{S}=\mathsf{s}_i]\sum_{j\neq i}\frac{H_{i,j}}{m}\mathbb{P}[\mathsf{Y}\in\mathcal{R}_j|\mathsf{S}=\mathsf{s}_i],$$

where $H_{i,j}$ is the Hamming distance (the number of different bits) between the labels of symbols $\mathsf{s}_i$ and $\mathsf{s}_j$.

**BEP (union bound)**

$$B_\mathrm{e} \leq \sum_{i=1}^{M}\mathbb{P}[\mathsf{S}=\mathsf{s}_i]\sum_{j\neq i}\frac{H_{i,j}}{m}\mathrm{Q}\left(\sqrt{\frac{D_{i,j}^2}{2N_0}}\right)$$

**BEP (High-SNR approximation for equally likely symbols)**

$$B_\mathrm{e} \approx \frac{2H_\mathrm{min}}{Mm}\cdot\mathrm{Q}\left(\sqrt{\frac{D_\mathrm{min}^2}{2N_0}}\right),$$

where $H_\mathrm{min}$ is the *total* number of bits differing between signal pairs at minimum distance and $m = \log_2(M)$ is the number of bits per symbol.

**Q-function**

See also tables in Mathematics Handbook, where $\mathrm{Q}(x) = 1-\Phi(x)$

| $\mathrm{Q}(x)$ | $x$ |
|---|---|
| $10^{-1}$ | 1.2816 |
| $10^{-2}$ | 2.3263 |
| $10^{-3}$ | 3.0902 |
| $10^{-4}$ | 3.7190 |
| $10^{-5}$ | 4.2649 |
| $10^{-6}$ | 4.7534 |
| $10^{-7}$ | 5.1993 |
| $10^{-8}$ | 5.6120 |
| $10^{-9}$ | 5.9978 |
| $10^{-10}$ | 6.3613 |
| $10^{-11}$ | 6.7060 |
| $10^{-12}$ | 7.0345 |