Examination Wed. Oct. 26, 2022, 8:30–12:30

SSY121 Introduction to Communication Engineering

- Contact person: Examiner Fredrik Brännström (070 – 872 1685) will visit the exam after approximately 1 and 3 hours.
- Instructions:
  - Write in English.
  - Use a pencil and eraser.
  - There is no page limit. Extra sheets of paper are available.
  - Solve the problems in any order (they are not ordered by difficulty).
  - Before handing in, sort the pages in problem order. Label each page with problem number and running page number. Do not hand in drafts or unused paper.
  - If any data is missing, make reasonable assumptions.
  - Chalmers' examination rules applies.
  - MP3/Music players **are not** allowed during the exam
- Allowed aids:
  - Chalmers-approved calculator
- Grading principles:
  - Explain the line of reasoning clearly. A good solution with a minor error usually gives close to full points, even if the answer is incorrect.
  - An answer without a clear motivation usually gives 0 points, even if it is correct.
  - Whenever possible, check if your answer is reasonable. An obviously unreasonable answer usually gives 0 points, even if the solution is almost correct.
- Solutions and results:
  - Solutions will be posted on the course website no later than 7 days after the exam.
  - The time and place for the exam review will be announced on the course website in Canvas.

1. **True or false statements:** [Total points: 8]
Justify *ALL* your answers using short and concise explanations (maximum 30 words per item).

(a) The only pulse that offers inter-symbol-interference (ISI) free transmission for a one-sided bandwidth (BW) $\geq \frac{1}{2T_s}$ (T_s is symbol duration) is the $\text{sinc}(t/T_s)$ pulse. [1 p]
False. Many different pulses offer ISI-free transmission at the expense of higher BWs and the sinc(.) pulse is the only pulse that offers the ISI-free transmission for $BW = \frac{1}{2T_s}$

(b) When the prior probabilities are available, for any channel, the maximum a posteriori (MAP) detector is the optimum detector in terms of minimizing the symbol error probability. [1 p]
True. The MAP detector maximizes the posterior utilizing the prior.

(c) To design the optimal receiver (with minimum number of matched filters) for a digital communication system operating in an AWGN channel, the required number of matched filters is the same for both 32-QAM and 8-PSK signaling schemes. [1 p]
True. The minimum number of matched filters is the dimensionality of the signal space which is $N = 2$ for both QAM and PSK signaling schemes.

(d) For a communication system with limited power usage, operating in an AWGN channel, the 16-QAM constellation is a better choice than 16-PSK constellation. [1 p]
True. A 16-QAM constellation has larger minimum Euclidean distance, and hence gives lower SER at the same power.

(e) In an AWGN channel with equi-probable 16-QAM signaling, the symbol error rate for an envelope detector is 0.75. (The envelope detector decides only based on the amplitude/magnitude of the received signal.) [1 p]
False. The 16-QAM envelope has 3 amplitude levels and in each level one symbol can be detected correctly and the rest will result in error. Thus, $P_e = 1 - \frac{1}{4} \cdot \frac{1}{4} + \frac{1}{2} \cdot \frac{1}{8} + \frac{1}{4} \cdot \frac{1}{4} = \frac{13}{16}$

(f) The high SNR approximation of the union bound gives a lower bound on the symbol error probability. [1 p]
False. The union bound is an upper bound of the exact SEP, and the high SNR approximation gives an approximation of the upper bound, which could be an upper or a lower bound on the exact SER.

(g) Phase discontinuities in the frequency-shift keying (FSK) signals is the source of unwanted large spectral lobes, which can be avoided using continuous phase modulations (CPM). [1 p]
True. The large spectral side lobes come from the abrupt change in the phase of FSK signals, while there are no phase discontinuities in CPM.

(h) In code-division multiple access (CDMA), different users are allotted small portions of the available bandwidth and each user transmits only in the allotted bandwidth without interfering in other user's bandwidth. [1 p]
False. In CDMA, all the users transmit simultaneously over the entire available spectrum.

2. **Constellations, Decision Region, SER:** [Total points: 12]
You will study the performance of different 4-ary constellations with equally likely symbols (shown in Figure 1), where the observation is given by

$$r = \sqrt{E_s}s + n,$$

where n is complex Gaussian noise with variance $N_0/2$ for both the real and the imaginary part, and $\mathbb{E}\left\{|s|^2\right\} = 1$.

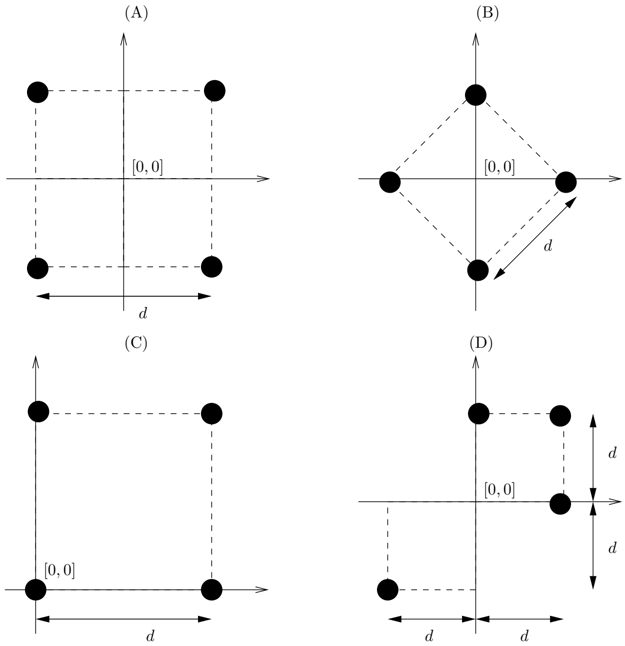

Figure 1: Four 4-ary constellations with minimum Euclidean distance d. Note that d is different from constellation to constellation.

(a) For every constellation (A), (B), (C), (D), determine d (the minimal Euclidean distance) and K (the number of neighbors at distance d). [2 p]

The distance d can be determined by the constraint $\mathbb{E}\left\{|a|^2\right\} = 1$.

A: For constellation A, the four points are $a_1 = [d/2, d/2]$, $a_2 = [-d/2, d/2]$, $a_3 = [d/2, -d/2]$, $a_4 = [-d/2, -d/2]$. Now

$$\begin{aligned} 1 &= \mathbb{E}\left\{|a|^2\right\} \\ &= \frac{1}{4}\left(|a_1|^2 + |a_2|^2 + |a_3|^2 + |a_4|^2\right) \\ &= \frac{1}{4}\left(\frac{d^2}{2} + \frac{d^2}{2} + \frac{d^2}{2} + \frac{d^2}{2}\right) \\ &= \frac{1}{2}d^2 \end{aligned}$$

so that $d = \sqrt{2}$. $K = 4$, since for every point there are two points at distance d.

B: For constellation B, since it is a rotated version of constellation A, $d = \sqrt{2}$ and $K = 4$.

C: For constellation C, the four points are $a_1 = [0, 0]$, $a_2 = [0, d]$, $a_3 = [d, 0]$,

$a_4 = [d, d]$. Now

$$\begin{aligned}1 &= \mathbb{E}\left\{|a|^2\right\} \\ &= \frac{1}{4}\left(|a_1|^2 + |a_2|^2 + |a_3|^2 + |a_4|^2\right) \\ &= \frac{1}{4}\left(0 + d^2 + d^2 + 2d^2\right) \\ &= d^2\end{aligned}$$

so that $d = 1$. $K = 4$, since for every point there are two points at distance d.

D: For constellation D, the four points are $a_1 = [-d, -d]$, $a_2 = [0, d]$, $a_3 = [d, 0]$, $a_4 = [d, d]$. Now

$$\begin{aligned}1 &= \mathbb{E}\left\{|a|^2\right\} \\ &= \frac{1}{4}\left(|a_1|^2 + |a_2|^2 + |a_3|^2 + |a_4|^2\right) \\ &= \frac{1}{4}\left(2d^2 + d^2 + d^2 + 2d^2\right) \\ &= \frac{3}{2}d^2\end{aligned}$$

so that $d = \sqrt{2/3}$ and the value of $K = 2$.

(b) What are the decision regions for maximum likelihood detection? You can simply draw them in a figure, making sure you mark the axes, mark the decisions made for every region, and draw the figure to scale. From the maximum likelihood criterion, derive the decision rule for constellation (C). Verify that this decision rule corresponds to the decision regions you have drawn. [2 p]

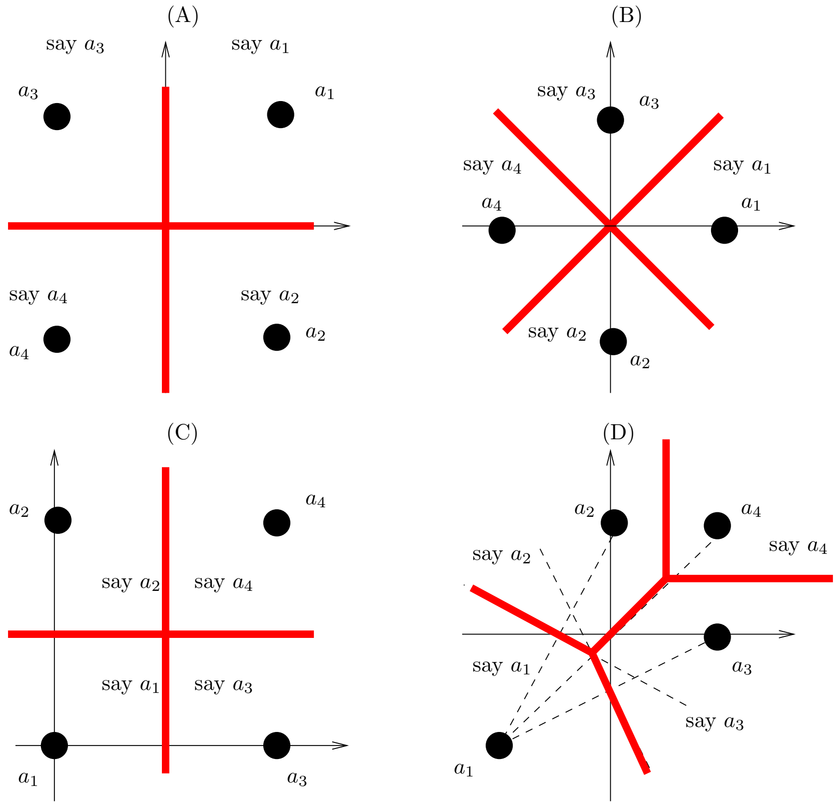

Figure 2: Decision regions shown in red solid lines, and the corresponding decisions in the form "say a_i". All a should be s in this figure.

The decision regions are drawn in Figure 2. For constellation C, the regions are found as

$$\hat{s} = \arg\min |r - s|^2 ,$$

so you will make the decision s_1, when

$$|r - s_1| < \min \{|r - s_2| , |r - s_3| , |r - s_4|\} . \tag{1}$$

By drawing the perpendicular bisector (the line between two points (say α and β) consisting of all points that are the same distance from α as from β) between s_1 and s_2, between s_1 and s_3, and between s_1 and s_4, we find the decision region for s_1 as being the left-lower quadrant in Figure 2, part C. Only the points in that quadrant satisfy (1). The remaining three decision regions can be found in a similar way.

(c) For every constellation, determine an approximation of the average symbol error probability for high SNR, based on d and K, as a function of the SNR E_s/N_0. Do all symbols have the same symbol error probability? [2 p]

The error probability can be approximated as (with $M = 4$)

$$P_s(e) \approx \frac{2K}{M} Q\left(\sqrt{\frac{d_{\min}^2 E_s}{2N_0}}\right)$$

so

$$P_s^{(A)}(e) = P_s^{(B)}(e) \quad \approx \quad 2Q\left(\sqrt{\frac{E_s}{N_0}}\right)$$

and all symbols have the same error probability. For constellation C,

$$P_s^{(C)}(e) \approx 2Q\left(\sqrt{\frac{E_s}{2N_0}}\right)$$

and again all the symbols have the same error probability. Finally, for constellation D,

$$P_s^{(D)}(e) \approx Q\left(\sqrt{\frac{E_s}{3N_0}}\right),$$

but now symbol s_1 will have smaller error probability than s_2 and s_3, which in turn have smaller error probabilities than s_4.

(d) Which of the four constellations gives the lowest symbol error probabilty for high SNR? Why? [2 p]

Constellations A and B give the same, best performance since the minimum distance among neighbors is the largest among the different constellations.

(e) Assume the receiver observes

$$r = e^{j\theta}\sqrt{E_s}s + n,$$

where θ is unknown. The receiver is not aware of θ, and applies maximum likelihood detection as if there were no phase θ. How does the symbol error probability change with θ? Are all constellations equally affected? Are all symbols within a constellation equally affected? [4 p]

In the presence of a phase error, the probability of errors increases. For constellations A and B, once $|\theta| > \pi/4$ rad (or more than 45 degrees) the error probability will be high ($P_{\text{err}} \geq 0.5$), even for high SNR. For constellations C and D, high error probabilities occur for $|\theta| > \pi/4 - \text{asin}(1/\sqrt{8}) = 0.4240$ rad (or more than 24.2952 degrees). Symbols with smaller angular separation are more affected than symbols with large angular separation.

3. **Decision Rule, Estimation** [Total points: 4]

Suppose that you wish to detect whether the binary random variable X is 0 or 1 when $Y \sim \mathcal{N}(1,4)$ for $X = 1$ and $Y \sim \mathcal{N}(-1,1)$ for $X = 0$, where $\mathcal{N}(\mu, \sigma^2)$ denotes a Gaussian distribution with mean μ and variance σ^2.

Let y denote the observation of Y and show that the optimal decision rule can be written as

$$y^2 + ay \underset{\hat{x}=1}{\overset{\hat{x}=0}{\lesseqgtr}} b$$

and determine the constants a and b. [4 p]

We use the notation

$$\mathcal{N}_x(\mu, \sigma^2) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right)$$

to denote a Gaussian distribution with argument x. Now, the random variable X is our parameter of interest, which has a certain prior $P_X(0) = p$, $P_X(1) = 1 - p$. Y is the observation, where the likelihood $p_{Y|X}(y|x)$ is directly given in the problem description, that is, $p_{Y|X}(y|0) = \mathcal{N}_y(-1,1)$ and $p_{Y|X}(y|1) = \mathcal{N}_y(1,4)$.
The optimal detector (maximum a posteriori) is given by

$$\hat{x} = \underset{x \in \{0,1\}}{\operatorname{argmax}}\, p_{Y|X}(y|x) P_X(x).$$

Because the maximization is only over two possibilities, one can rewrite the detector as

$$p_{Y|X}(y|1) P_X(1) \underset{\hat{x}=1}{\overset{\hat{x}=0}{\lesseqgtr}} p_{Y|X}(y|0) P_X(0).$$

As a first step, we plug in the likelihood and the prior to obtain

$$\frac{1}{2\sqrt{2\pi}} \exp\left(-\frac{(y-1)^2}{2 \cdot 4}\right) \cdot (1-p) \underset{\hat{x}=1}{\overset{\hat{x}=0}{\lesseqgtr}} \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{(y+1)^2}{2}\right) \cdot p.$$

Rearranging and taking the (natural) logarithm eventually leads to

$$y^2 + \frac{10}{3} y \underset{\hat{x}=1}{\overset{\hat{x}=0}{\lesseqgtr}} \frac{8}{3} \ln\left(2 \cdot \frac{p}{1-p}\right) - 1,$$

which has the desired form.

4. **Link Budget:** [Total points: 5]

Assume you work in a telecommunication company. Your task is to deploy base stations in Gothenburg, so users that are closer than d meters from the base station can have a satisfactory downlink transmission (transmission from base station to user) with a minimum bit rate of $R_\mathrm{b} = 5$ Mbits/s and a maximum bit error rate of $P_\mathrm{e} = 10^{-4}$. The transmission uses Gray-coded 4QAM, an RRC pulse with a roll-off factor of $\alpha = 0.16$, and a carrier frequency of $f_\mathrm{c} = 2.5$ GHz. Assume an AWGN channel with noise power spectral density $N_0 = 2 \times 10^{-17}$ W/Hz. Both the transmitter and the receiver have a antenna gain $G_\mathrm{T} = G_\mathrm{R} = 3$ dB (≈ 2). Answer the following questions.

(a) What is the minimal received power P_R at the user side to achieve the downlink quality? [3 p]

Hint 1: The BER for the Gray-coded 4QAM is given by $Q\left(\sqrt{\frac{E_\mathrm{s}}{N_0}}\right)$.
Hint 2: $P_R = E_s B$, *where* B *is the signal bandwidth.*

According to the Q-function table in the formula sheet

$$P_\mathrm{e} = Q\left(\sqrt{\frac{E_\mathrm{s}}{N_0}}\right) \leq 10^{-4}$$

$$\sqrt{\frac{E_\mathrm{s}}{N_0}} \geq 3.7190 \tag{2}$$

As we know

$$\frac{E_\mathrm{s}}{N_0} = \frac{P_\mathrm{R}}{BN_0} = \frac{P_\mathrm{R}}{(1+\alpha)R_\mathrm{s}N_0} = \frac{P_\mathrm{R}}{(1+\alpha)(R_\mathrm{b}/\log_2 4)N_0}. \tag{3}$$

Substituting (3) into (2), we have $P_\mathrm{R} \geq 8.02 \times 10^{-10}$ W.

(b) Assume the required receiver power is $P_\mathrm{R} = 10^{-9}$ W, the transmitter power is $P_\mathrm{T} = 5$ W, and that the free-space path loss model applies. What is the maximum distance, d, between the base station and the user to achieve the downlink quality? [2 p]

The distance d can be calculated according to the free-space path loss model in the formula sheet as

$$d \leq \frac{\lambda}{4\pi}\sqrt{\frac{P_\mathrm{T}G_\mathrm{T}G_\mathrm{R}}{P_\mathrm{R}}} \approx 1.35 \times 10^3\,\mathrm{m}.$$

where $G_\mathrm{T} = G_\mathrm{T} = 2$, $\lambda = \frac{c}{f_\mathrm{c}} = 0.12$, $P_\mathrm{T} = 5$, and $P_\mathrm{R} = 10^{-9}$. Thus, the maximum distance between each basestation is 1.35 km.

5. **Gram-Schmidt:** [Total points: 7]

The four waveforms shown in Figure 3 are used for communication of four equiprobable messages over an AWGN channel.

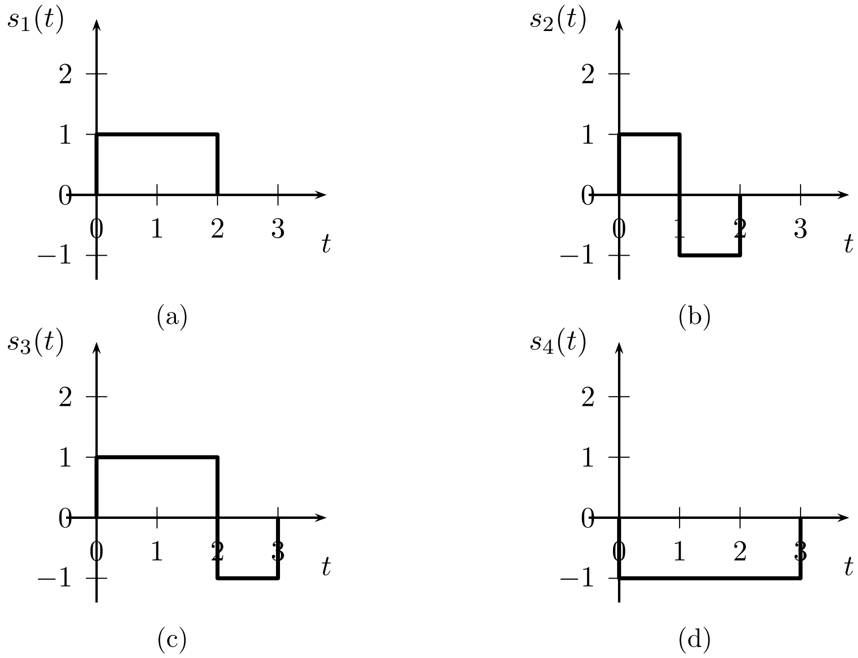

Figure 3: Signals in Problem 5

(a) Find a minimal set of orthogonal basis for the set $\{s_1(t), s_2(t), s_3(t), s_4(t)\}$. *Hint: use the Gram-Schmidt procedure.* [4 p]
Follow the Gram-Schmidt procedure.

$$
\begin{aligned}
&u_1 = s_1 = [1, 1, 0] \\
&\|u_1\| = \sqrt{2} \\
&\phi_1 = \frac{u_1}{\|u_1\|} = \frac{s_1}{\sqrt{2}} = [\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0] \\
&u_2 = s_2 - \langle s_2, \phi_1 \rangle \phi_1 = s_2 + 0 \cdot \phi_1 = [1, -1, 0] \\
&\|u_2\| = \sqrt{2} \\
&\phi_2 = \frac{u_2}{\|u_2\|} = [\frac{1}{\sqrt{2}}, \frac{-1}{\sqrt{2}}, 0] \\
&u_3 = s_3 - \langle s_3, \phi_1 \rangle \phi_1 - \langle s_3, \phi_2 \rangle \phi_2 \\
&\quad = s_3 - \sqrt{2}\phi_1 - 0\phi_2 = [0, 0, -1] \\
&\phi_3 = \frac{u_3}{\|u_3\|} = [0, 0, -1] \\
&u_4 = s_4 - \langle s_4, \phi_1 \rangle \phi_1 - \langle s_4, \phi_2 \rangle \phi_2 - \langle s_4, \phi_3 \rangle \phi_3 \\
&\quad = s_4 + \sqrt{2}\phi_1 - 0\phi_2 - 1\phi_3 = [0, 0, 0]
\end{aligned}
$$

Hence, $\{\phi_1, \phi_2, \phi_3\}$ is one of the orthonormal basis set for signals (s_1, s_2, s_3, s_4).

(b) Express the signals $\{s_1, s_2, s_3, s_4\}$ as a linear combination of the basis functions found in (a). Draw a constellation diagram of $\{s_1, s_2, s_3, s_4\}$.

Note: The constellation diagram can be three-dimensional. [3 p]

$$
\begin{aligned}
s_1 &= \sqrt{2}\phi_1, \\
s_2 &= \sqrt{2}\phi_2, \\
s_3 &= \sqrt{2}\phi_1 + \phi_3 \\
s_4 &= -\sqrt{2}\phi_1 + \phi_3.
\end{aligned}
$$

The constellation is shown in Figure 4.

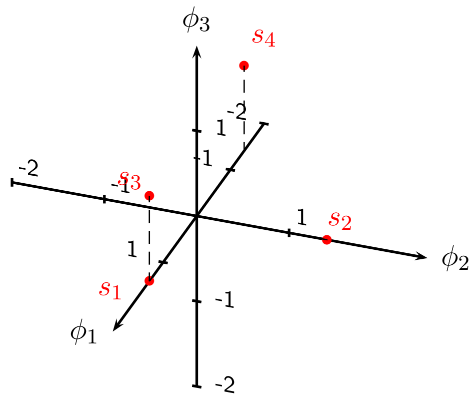

Figure 4: Constellation diagram.

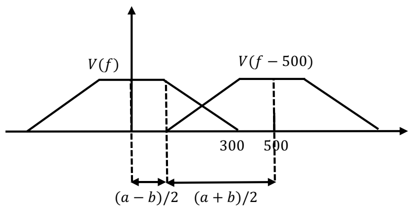

Figure 6: Nyquist problem (b).

6. **Nyquist, ISI:** [Total points: 6]

Consider a communication system using a pulse $v(t) = \text{sinc}(at)\,\text{sinc}(bt)$, where a and b are unknowns and $a \geq b$. Answer the following questions.

(a) Sketch the frequency domain response $V(f)$ of the pulse $v(t)$ and mark important levels on both axes in terms of a and b. *Hint: A multiplication in the time domain equals a convolution in the frequency domain.* [2 p]

See Figure 5.

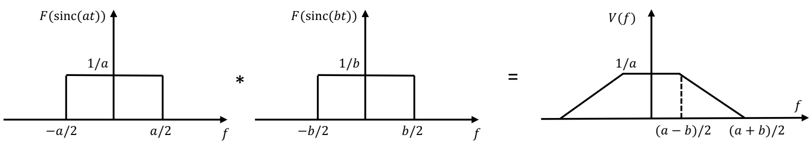

Figure 5: Frequency domain response $V(f)$ of pulse $v(t)$.

(b) Suppose that the pulse is to be used over an ideal baseband channel with one-side bandwidth 300 Hz. Choose a and b so that the pulse is Nyquist for 8-PSK signaling at 1500 bits/s and exactly fills the channel bandwidth. [2 p]

As shown in Figure 6, to fill the bandwidth, we need $(a+b)/2 = 300$. To satisfactory the Nyquist criterion for the given symbol rate $R_s = 1500/\log_2 8 = 500$, we need $(a-b)/2 + (a+b)/2 = 500$, which yields $a = 500$, $b = 100$.

(c) Now, Suppose that the pulse is to be used over a passpand channel spanning the frequencies 2.40 GHz to 2.42 GHz. Assuming that we use 64 QAM signaling with a bit rate of $R_b = 60$ Mbits/s, choose a and b so that the pulse is Nyquist and exactly fills the channel bandwidth. [2 p]

We now have a passband system of bandwidth 20 MHz, and a symbol rate

$$\frac{1}{T} = \frac{60\text{Mbps}}{\log_2 64 \text{ bits/symbol}} = 10 \text{ Msymbols/sec}$$

Note that 20 MHz is the two-sided bandwidth for the complex baseband system. Similarly, we now have $\frac{a+b}{2} = 10$ MHz and

$$\frac{a-b}{2} + \frac{a+b}{2} = 10 \text{ Msymbols/sec},$$

which yields $a = b = 10$ MHz.

7. **Synchronization:** [Total points: 6]

A manufacturer produces a communication system with an imperfect phase synchronizer, which causes a phase error of ϕ. The product is designed using BPSK modulation format. Assume an AWGN channel with noise spectral density $N_0/2$. Answer the following questions.

(a) Find the maximum phase error ϕ that the product can tolerate, to achieve a bit error probability $P_e = 10^{-5}$ at $E_b/N_0 = 10$ dB. [3 p]

The error probability of BPSK with phase error must be calculated. The received constellation is rotated by ϕ, therefore the minimum distance between the constellation points is $D_{\min} = 2\sqrt{E_s}\cos\phi$

$$P_e = Q\left(\sqrt{\frac{D_{\min}^2}{2N_0}}\right) = Q\left(\sqrt{\frac{4E_s\cos^2\phi}{2N_0}}\right) = Q\left(\sqrt{\frac{2E_s\cos^2\phi}{N_0}}\right) = Q\left(\sqrt{\frac{2E_b\cos^2\phi}{N_0}}\right)$$

Substituting P_e and E_b/N_0 in the above equation, we have

$$\begin{aligned} 10^{-5} &= Q\left(\sqrt{2\times 10^1\cos^2\phi}\right) \\ 4.2649 &= \sqrt{20\cos^2\phi} \\ \phi &= 0.3056[\text{rad}] = 17.51^\circ \end{aligned}$$

(b) Assume the manufacturer produces two products (product A and B) with different phase errors $\phi_A = \pi/3$ and $\phi_B = \pi/6$, respectively. To achieve the same bit error probability, how much power in dB do we need to increase the power of product A (i.e. increase E_b of product A)? [3 p]

Higher phase error will result in more power consumption for giving the same performance. Thus, the product **A** is more power efficient compared to product **B**.

To show how much power will be saved:

$$\begin{aligned} P_{e(\mathbf{A})} &= P_{e(\mathbf{B})} \\ Q\left(\sqrt{\frac{2E_{b(\mathbf{A})}\cos^2\phi_A}{N_0}}\right) &= Q\left(\sqrt{\frac{2E_{b(\mathbf{B})}\cos^2\phi_B}{N_0}}\right) \\ \frac{E_{b(\mathbf{A})}}{E_{b(\mathbf{B})}} &= \frac{\cos^2\phi_B}{\cos^2\phi_A} = 3 = 4.77[\text{dB}] \end{aligned}$$