Examination Wed. Oct. 31, 2018, 08:30–12:30

SSY121 Introduction to Communication Engineering

- Contact persons: Mohammad Nazari (031 - 772 1771) will visit the exam after approximately 1 and 3 hours.
- Instructions:
  - Write in English.
  - Use a pencil and eraser.
  - There is no page limit. Extra sheets of paper are available.
  - Solve the problems in any order (they are not ordered by difficulty).
  - Before handing in, sort the pages in problem order. Label each page with problem number and running page number. Do not hand in drafts or unused paper.
  - If any data is missing, make reasonable assumptions.
  - Chalmers' examination rules apply.
  - MP3/Music players **are not** allowed during the exam
- Allowed aids:
  - Mathematics Handbook by Råde and Westergren (any edition, including Beta) or equivalent
  - Chalmers-approved calculator can be borrowed by the exam administrators
- Grading principles:
  - Explain the line of reasoning clearly. A good solution with a minor error usually gives close to full points, even if the answer is incorrect.
  - An answer without a clear motivation usually gives 0 points, even if it is correct.
  - Whenever possible, check if your answer is reasonable. An obviously unreasonable answer usually gives 0 points, even if the solution is almost correct.
- Solutions and results:
  - Solutions will be posted on the course website no later than 7 days after the exam.
  - The grading can be reviewed on Thursday November 15, 2018, at 13:00–14:00 in E2 Blue Room (3340) on floor 3 in the EDIT building.

1. **True or false questions:** Justify *ALL* your answers using short and concise explanations (maximum 30 words per item). (Total points: 10)

   (a) In a digital communication system, a *maximum likelihood* (ML) detector uses the prior knowledge about the data source to make the decision. (1)

   False. The *maximum a posteriori* (MAP) detector uses the prior information to make the decision.

   (b) The matched filter receiver maximizes the SNR at the output of the matched filter. (1)

   True. The matched filter receiver minimizes the symbol error rate, i.e., maximize the SNR.

   (c) In a noiseless transmission of BPSK symbols, a phase error does not affect the performance. (1)

   False. If the phase error is in the interval $\frac{\pi}{2} < \phi < \frac{3\pi}{2}$ all symbols are in error.

   (d) A communication system uses BPSK signaling over an AWGN channel and operates at a very high SNR, i.e., no bits are in error. Because of a technical problem at the RF front side of the transmitter, nothing is transmitted during some periods. This randomly happens in half of the time, and neither the transmitter nor the receiver are aware of it. The BER of this faulty system system is 0.5. (1)

   False. In half of cases, the BER is equal to 0.5 because the receiver only receives the noise, and in the other half, the BER is assumed zero due to the high SNR. The BER is therefore equal to 0.25.

   (e) Although the pulse $\mathrm{sinc}(t/T)$ fulfills both the Nyquist criterion and T-orthogonality for a symbol period T, and has the smallest bandwidth for a fixed T, an RRC pulse is preferred in practical communication systems. (1)

   True. Because the sinc pulse has significant power in its secondary lobes.

   (f) The part of noise which cannot be projected in the dimensions of the signals does not affect the detection. (1)

   True. According to the theorem of irrelevance, only noise in the dimensions of the signals affects the detection.

   (g) Suppose that rectangular pulses of duration T are used for transmission of binary information over an AWGN channel using BPSK modulation, and a matched filter receiver is used at the receiver side. If a timing error $\Delta t = \frac{T}{2}$ occurs at the receiver, the BER tends to 0.5, i.e., the receiver decides completely randomly, when the SNR goes to infinity. (1)

   False. The BER severely increases due to the timing error $\Delta t = \frac{T}{2}$, and it tends to 0.25 when SNR goes to infinity.

   (h) One can approximately write that $\mathrm{BER} = \log_2 M \times \mathrm{SER}$ (where $M > 2$ is the number of symbols), when anti-Gray labeling is used. (1)

   False. The BER is upper bounded by the SER for any labeling.

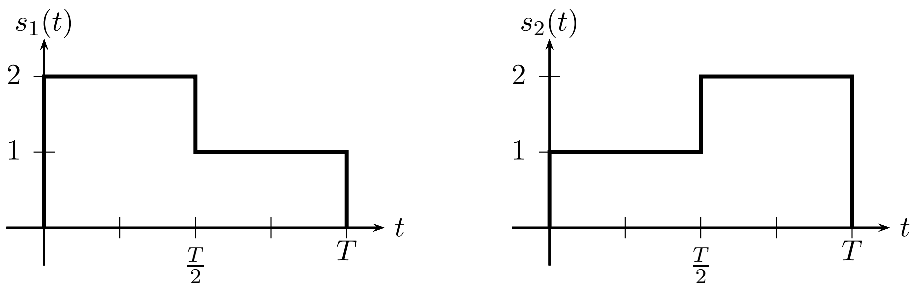
   (i) QPSK uses two orthogonal basis functions, but 8-PSK uses three orthogonal basis functions since 3 bits are mapped to every symbol. (1)

   False. 8-PSK also uses two orthogonal basis functions.

(j) In OFDM systems, inter-symbol interference (ISI) can be completely eliminated using a cyclic prefix of length greater than the maximum delay spread. (1)

True. That is why OFDM is less sensitive to ISI that single carrier communication.

2. A digital communication system, operating at a bitrate of R_b bits/s, uses binary signaling with the equiprobable signals shown below, over an AWGN channel with noise power spectral density of $\frac{N_0}{2}$. (Total points: 9)



(a) (b)

(a) What is $\frac{E_\mathrm{b}}{N_0}$ for this system (in terms of N_0 and R_b), where E_b is the energy per bit? (2)
The two signals have the same energy $E_\mathrm{s} = E_\mathrm{b} = \int_0^{T/2} 4\mathrm{d}t + \int_{T/2}^{T} 1\mathrm{d}t = 5T/2$. Also, $T = 1/R_\mathrm{s} = 1/R_\mathrm{b}$, so we obtain $\frac{E_b}{N_0} = \frac{5}{2R_\mathrm{b}N_0}$.

(b) What is the error probability for this system (in terms of N_0 and R_b)? *Hint: You need to find the distance between two signals.* (2)
The probability of error is $P_e = Q(\sqrt{\frac{D_{1,2}^2}{2N_0}})$, where $D_{1,2}$ is the Euclidean distance between the two signals as follows

$$D_{1,2}^2 = \int_0^T (s_1(t) - s_2(t))^2 \mathrm{d}t = \int_0^{T/2} 1^2 \mathrm{d}t + \int_{T/2}^{T} (-1)^2 \mathrm{d}t = T,$$

and hence, $P_e = Q(\sqrt{\frac{T}{2N_0}}) = Q(\sqrt{\frac{1}{2R_\mathrm{b}N_0}})$.

(c) By how many decibels does this system underperform a binary antipodal signaling system (normal BPSK) with the same $\frac{E_b}{N_0}$? (1)
From the previous parts, we have $P_e = Q(\sqrt{\frac{1}{2R_\mathrm{b}N_0}}) = Q(\sqrt{\frac{1}{10}\frac{2E_b}{N_0}})$, while for binary antipodal signaling, $P_e = Q(\sqrt{\frac{2E_b}{N_0}})$. So, the current system underperforms the binary antipodal signaling by a factor of 10, or equivalently $10\log_{10} 10 = 10$ dB.

(d) Now assume that this system is augmented with two more signals $s_3(t) = -s_1(t)$ and $s_4(t) = -s_2(t)$ to result in a 4-ary system with equiprobable symbols. Using the union bound, find a bound on the symbol error probability of the resulted 4-ary system. (4)

We have

$$
\begin{aligned}
D_{1,2}^2 = D_{3,4}^2 &= \int_0^T (s_1(t) - s_2(t))^2 \mathrm{d}t = T, \\
D_{1,3}^2 = D_{2,4}^2 &= \int_0^T (s_1(t) - s_3(t))^2 \mathrm{d}t = 10T, \\
D_{1,4}^2 = D_{2,3}^2 &= \int_0^T (s_1(t) - s_4(t))^2 \mathrm{d}t = 9T, \\
\mathrm{UB} &= \frac{1}{M} \sum_{k=1}^{M} \sum_{l=1, l\neq k}^{M} \mathrm{PEP}(s_k, s_l) \\
&= Q\left(\sqrt{\frac{T}{2N_0}}\right) + Q\left(\sqrt{\frac{9T}{2N_0}}\right) + Q\left(\sqrt{\frac{10T}{2N_0}}\right) \\
&= Q\left(\sqrt{\frac{1}{2R_\mathrm{s}N_0}}\right) + Q\left(\sqrt{\frac{9}{2R_\mathrm{s}N_0}}\right) + Q\left(\sqrt{\frac{10}{2R_\mathrm{s}N_0}}\right) \\
&= Q\left(\sqrt{\frac{1}{R_\mathrm{b}N_0}}\right) + Q\left(\sqrt{\frac{9}{R_\mathrm{b}N_0}}\right) + Q\left(\sqrt{\frac{10}{R_\mathrm{b}N_0}}\right),
\end{aligned}
$$

since $T = 1/R_\mathrm{s}$ and $2R_\mathrm{s} = R_\mathrm{b}$.

3. A satellite in synchronous orbit is used to communicate with an earth station at a distance of $40,000$ km. The satellite has an antenna with a gain of 10 dB and the receiver antenna has a gain of 1 dB. The transmitter power is 1.6 W, and the carrier frequency is chosen so that $\frac{\lambda}{4\pi} = 1$. Assume that the power spectral density of noise is $N_0 = 4.14 \cdot 10^{-21}$W/Hz. (Total points: 6)

   (a) What is the carrier frequency f_c? (1)
   The carrier frequency is $f_c = c/\lambda = 3 \cdot 10^8/(4\pi) = 2.3873$ GHz.

   (b) Determine the bitrate R_b in order to have a BER of at most 10^{-5}, if BPSK with sinc pulses are used as the signaling scheme. (4)
   The BER for BPSK signaling is given by $Q(\sqrt{\frac{2E_b}{N_0}})$ which needs to be less than or equal to 10^{-5}. Therefore, according to the table in the formula sheet, $\sqrt{\frac{2E_b}{N_0}} \geq 4.2649$ or $\frac{E_b}{N_0} \geq 9.0947$. Furthermore, the received power is

$$
\begin{aligned}
P_R &= P_T G_T G_R \left(\frac{\lambda}{4\pi d}\right)^2 \\
&= 1.6 \cdot 10^{1/10} \cdot 10^{10/10} \left(\frac{1}{40 \cdot 10^6}\right)^2 = 1.2589 \cdot 10^{-14} \text{ W},
\end{aligned}
$$

   Since the bandwidth is $B = R_b$, one can write

$$
\frac{E_b}{N_0} = \frac{P_R}{R_b N_0} = \frac{1.2589 \cdot 10^{-14}}{R_b \cdot 4.14 \cdot 10^{-21}} \geq 9.0947.
$$

   Then, $R_b \leq 0.3344$ Mbits/s.

   (c) Find the one sided bandwidth required to have ISI-free transmission using a raised cosine pulse with roll-off factor $\alpha = 0.5$, with the bitrate obtained in the previous part. (1)
   For ISI-free transmission, since $R_s = R_b$ we need to have

$$
W_{\min} = \frac{\alpha + 1}{2} R_b = 0.75 R_b = 0.75 \cdot 0.3344 \cdot 10^6 = 250.8 \text{ kHz}
$$

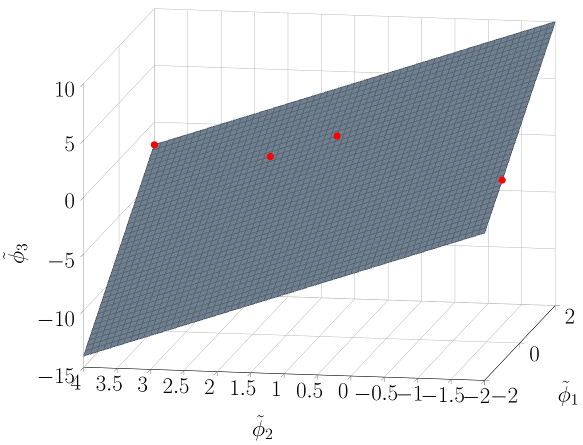


Figure 2: Symbol constellation.

4. Consider the constellation points given by the coordinates $s_1 = (1, 2, -1)$, $s_2 = (2, 4, -2)$, $s_3 = (-1, -2, 1)$, and $s_4 = (1, 1, 1)$. The constellation points are given in terms of the basis functions:

$$\tilde\phi_1(t) = c_1 I\{0 \le t < 2\},$$
$$\tilde\phi_2(t) = c_2(I\{0 \le t < 1\} - I\{1 \le t < 2\})$$
$$\tilde\phi_3(t) = ?,$$

where c_1 and c_2 are constants. The indicator function $I(a \le t < b)$ is equal to one for t between a and b ($a < b$), and equal to zero otherwise. The constellation points are located on a plane as shown in Fig. 2

(Total points: 10)

(a) Find the constants c_1 and c_2 in order to make $\tilde\phi_1(t)$ and $\tilde\phi_2(t)$ have unit energy. Also, propose a third basis function such that $\tilde\phi_1(t)$, $\tilde\phi_2(t)$, and $\tilde\phi_3(t)$ form an orthonormal basis. (4)

Let $\phi_1(t) = I\{0 \le t < 2\}$ and $\phi_2(t) = I\{0 \le t < 1\} - I\{1 \le t < 2\}$. The constants c_1 and c_2 are obtained from the norms of ϕ_1 and ϕ_2 as

$$\|\phi\|_1^2 = \int_0^2 dt = [t]_0^2 = 2 \Rightarrow c_1 = \sqrt{\frac{1}{2}}$$
$$\|\phi\|_2^2 = \int_{-\infty}^{\infty} |\phi_2(t)|^2 dt = [t]_0^2 = 2 \Rightarrow c_2 = \sqrt{\frac{1}{2}}.$$

A possible solution for $\tilde\phi_3(t)$ is

$$\phi_3(t) = I\{0 \le t < 0.5\} - I\{0.5 \le t < 1\} + I\{1 \le t < 1.5\} - I\{1.5 \le t < 2\}.$$

The normalization is given by

$$\|\phi_3\|^2 = 2 \Rightarrow c_3 = \sqrt{\frac{1}{2}}.$$

We then have $\tilde\phi_3(t) = \phi_3(t)/\sqrt{2}$.

(b) Argue why the above basis is not a minimal basis for the constellation points. (1)

It can be seen from Fig. 2 and from the coordinates of the constellation points that all the points are located in a plane, hence two basis functions should suffice.

(c) Provide a minimal basis for the constellation points and express the points in the new basis. Sketch the new constellation. (5)

We will carry out the Gram-Schmidt orthogonalization procedure to find a minimal basis $\{\tilde{\theta}_1, \tilde{\theta}_2\}$.

Let $\theta_1 = s_1 = \tilde{\phi}_1 + 2\tilde{\phi}_2 - \tilde{\phi}_3$. The energy of θ_1 is easily found, since $\{\tilde{\phi}_k\}_{k=1}^3$ are orthonormal, as

$$\|\theta_1\|^2 = \left\|\tilde{\phi}_1\right\|^2 + 4\left\|\tilde{\phi}_2\right\|^2 + \left\|\tilde{\phi}_3\right\|^2 = 6.$$

Hence, from the Gram-Schmidth procedure, we have our first basis function as

$$\tilde{\theta}_1 = \frac{\theta_1}{\|\theta_1\|} = \sqrt{\frac{1}{6}}(\tilde{\phi}_1 + 2\tilde{\phi}_2 - \tilde{\phi}_3).$$

For the next step, we notice that $s_2 = 2s_1$ and $s_3 = -s_1$. Therefore, s_2 and s_3 can also be expressed solely with $\tilde{\theta}_1$. The last constellation point, however, needs another basis function. Let $\theta_2 = s_4 - \langle s_4, \tilde{\theta}_1\rangle\tilde{\theta}_1$. We can evaluate the inner product (using again the orthonormal property) as

$$\begin{aligned}
\langle s_4, \tilde{\theta}_1\rangle &= \left\langle \tilde{\phi}_1 + \tilde{\phi}_2 + \tilde{\phi}_3, \sqrt{\frac{1}{6}}(\tilde{\phi}_1 + 2\tilde{\phi}_2 - \tilde{\phi}_3) \right\rangle \\
&= \sqrt{\frac{1}{6}}\left(\left\|\tilde{\phi}_1\right\|^2 + 2\left\|\tilde{\phi}_2\right\|^2 - \left\|\tilde{\phi}_3\right\|^2\right) \\
&= \frac{2}{\sqrt{6}}.
\end{aligned}$$

This gives us

$$\begin{aligned}
\theta_2 &= s_4 - \frac{2}{\sqrt{6}}\tilde{\theta}_1 \\
&= \tilde{\phi}_1 + \tilde{\phi}_2 + \tilde{\phi}_3 - \frac{2}{6}\left(\tilde{\phi}_1 + 2\tilde{\phi}_2 - \tilde{\phi}_3\right) \\
&= \frac{2}{3}\tilde{\phi}_1 + \frac{1}{3}\tilde{\phi}_2 + \frac{4}{3}\tilde{\phi}_3.
\end{aligned}$$

We also need the energy of θ_2 as

$$\|\theta_2\|^2 = \frac{4}{9}\left\|\tilde{\phi}_1\right\|^2 + \frac{1}{9}\left\|\tilde{\phi}_2\right\|^2 + \frac{16}{9}\left\|\tilde{\phi}_3\right\|^2 = \frac{7}{3}.$$

Finally, we obtain the second basis function as

$$\tilde{\theta}_2 = \frac{\theta_2}{\|\theta_2\|} = \sqrt{\frac{3}{7}}\theta_2.$$

We note that

$$\begin{aligned}
s_1 &= \|\theta_1\|\,\tilde{\theta}_1 = \sqrt{6}\tilde{\theta}_1 \\
s_4 &= \theta_2 + \langle s_4, \tilde{\theta}_1\rangle\tilde{\theta}_1 = \theta_2 + \frac{2}{\sqrt{6}}\tilde{\theta}_1 = \sqrt{\frac{7}{3}}\tilde{\theta}_2 + \frac{2}{\sqrt{6}}\tilde{\theta}_1.
\end{aligned}$$

Therefore, we have that

$$s_1 = \begin{bmatrix}\sqrt{6} \\ 0\end{bmatrix}, s_2 = \begin{bmatrix}2\sqrt{6} \\ 0\end{bmatrix}, s_3 = \begin{bmatrix}-\sqrt{6} \\ 0\end{bmatrix}, s_4 = \begin{bmatrix}\frac{2}{\sqrt{6}} \\ \sqrt{\frac{7}{3}}\end{bmatrix}.$$

The symbol constellation is shown in Fig. 3.

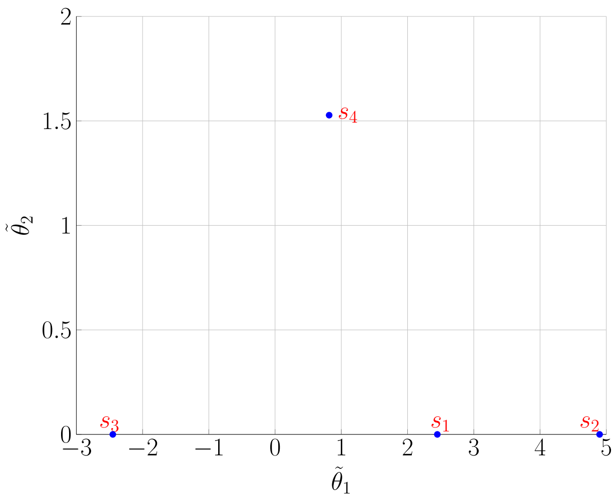


Figure 3: New constellation symbols.

5. Consider a noiseless channel and a communication system using QPSK and matched filtering using a pulse that fulfills the Nyquist criterion after the matched filter. The symbol time is $T_s = 1$ s and root-raised cosine pulses are used.

   The transmitter sends the symbols $[s_0, s_1, s_2, s_3] = [1+j, 1+j, 1+j, 1+j]/\sqrt{2}$ where j is the imaginary unit. The transmitted passband signal is given as

   $$x(t) = \sqrt{2} \sum_{k=0}^{3} \left(v(t - kT_s)a_k \cos(2\pi f_c t) + v(t - kT_s)b_k \sin(2\pi f_c t)\right)$$

   where a_k and b_k are the real and imaginary part of s_k, respectively, f_c is the carrier frequency, and $v(t)$ is the pulse used in the transmission.

   (Total points: 8)

   (a) Assume that a phase error of Δ radians and that a frequency error of ϕ radians per second is present at the receiver. The symbols are sampled at the correct sampling instants. Derive the expression for the received signal after sampling the matched filter output.
   Hint: the identities below may be useful

   $$\begin{aligned} 2\sin(a)\cos(b) &= \sin(a+b) + \sin(a-b), \\ 2\cos(a)\sin(b) &= \sin(a+b) - \sin(a-b), \\ 2\sin(a)\sin(b) &= \cos(a-b) - \cos(a+b), \\ 2\cos(a)\cos(b) &= \cos(a+b) + \cos(a-b). \end{aligned}$$

   (5)

   We shall write the in-phase and the quadrature component of the received signals separately. The in-phase component is obtained by multiplying the received signal with a cosine and then by matched filtering (alternatively lowpass filtering before matched filtering). Since we have both a phase error and a frequency error, we can write the in-phase component before matched filtering as

$$
\begin{aligned}
y_{\mathrm{I}}(t) &= \sqrt{2}x(t)\cos(2\pi f_{\mathrm{c}}t + \phi t + \Delta) \\
&= 2\sum_{k=0}^{3} v(t-kT_{\mathrm{s}})\left[a_k\cos(2\pi f_{\mathrm{c}}t) + b_k\sin(2\pi f_{\mathrm{c}}t)\right]\cos(2\pi f_{\mathrm{c}}t + \phi t + \Delta) \\
&= \sum_{k=0}^{3} v(t-kT_{\mathrm{s}})a_k\left[\cos(4\pi f_{\mathrm{c}}t + \phi t + \Delta) + \cos(\phi t + \Delta)\right] \\
&\quad + v(t-kT_{\mathrm{s}})b_k\left[\sin(4\pi f_{\mathrm{c}}t + \phi t + \Delta) + \sin(\phi t + \Delta)\right].
\end{aligned}
$$

After matched filtering (also a low-pass filter) and sampling at $0, T_{\mathrm{s}}, 2T_{\mathrm{s}}, 3T_{\mathrm{s}}$, we obtain

$$
y_{\mathrm{I,MF}}(t) = \sum_{k=0}^{3}(a_k\cos(\phi t + \Delta) + b_k\sin(\phi t + \Delta))\delta(t-kT_{\mathrm{s}}).
$$

Similarly, the quadrature component can be written as

$$
y_{\mathrm{Q,MF}}(t) = \sum_{k=0}^{3}(-a_k\sin(\phi t + \Delta) + b_k\cos(\phi t + \Delta))\delta(t-kT_{\mathrm{s}}).
$$

By adding the in-phase and quadrature phase, we obtain

$$
\begin{aligned}
&y_{\mathrm{MF}}(t) = y_{\mathrm{I,MF}}(t) + jy_{\mathrm{Q,MF}}(t) \\
&= \sum_{k=0}^{3}\left(a_k\left[\cos(\phi t + \Delta) - j\sin(\phi t + \Delta)\right] + jb_k\left[\cos(\phi t + \Delta) - j\sin(\phi t + \Delta)\right]\right)\delta(t-kT_{\mathrm{s}}) \\
&= \sum_{k=0}^{3} s_k e^{-j(\phi t + \Delta)}\delta(t-kT_{\mathrm{s}}).
\end{aligned}
$$

Hence, the kth output symbol is given as

$$
\hat{s}_k = y_{\mathrm{MF}}[kT_{\mathrm{s}}] = s_k e^{-j(\phi kT_{\mathrm{s}} + \Delta)} \tag{1}
$$

for $k = 0, 1, 2, 3$.

(b) Plot the four received symbols in an IQ plot if $\Delta = \pi/3$ and $\phi = \pi/4$. (3)

The received symbols are given as

$$
\begin{aligned}
\hat{s}_0 &= s_0 e^{-j\pi/3} \\
\hat{s}_1 &= s_1 e^{-j(\pi/4 + \pi/3)} \\
\hat{s}_2 &= s_2 e^{-j(2\pi/4 + \pi/3)} \\
\hat{s}_3 &= s_3 e^{-j(3\pi/4 + \pi/3)}.
\end{aligned}
$$

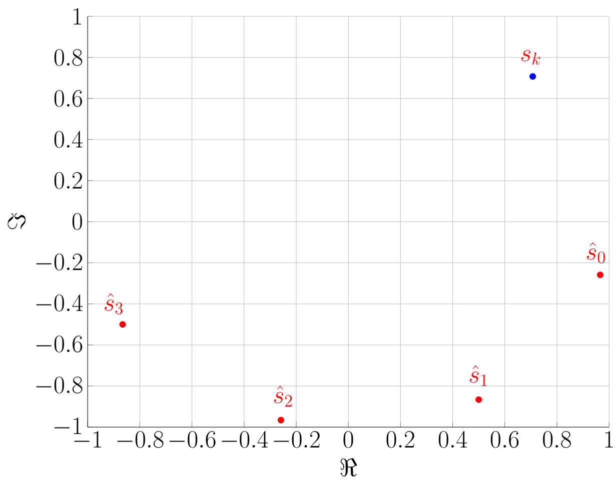


Figure 4: Symbols with frequency- and phase error.

6. Some questions relating to WiFi: (Total points: 5)

(a) Briefly describe why a channel code is used. (2)

The channel code in the transmitter can either be a binary convolutional code or a low density parity check (LDPC) code. It adds redundant bits to protect the bits from errors introduced in the channel, so that the channel decoder in the receiver can still potentially receive the information bits correctly.

(b) Briefly describe the spacial mapper (SM) block in the transmitter. (2)

The SM maps the N_{SS} spatial streams to N_{TX} antennas for sub-carrier k using a complex matrix Q_k of size $N_{\mathrm{TX}} \times N_{\mathrm{SS}}$.

(c) 802.11ax increased the largest constellation from 256QAM to 1024QAM compared to 802.11ac. How much does the bitrate increase in percentage due to this increase? (1)

256QAM carries 8 bits/symbol and 1024QAM carries 10 bits/symbol, so the increase in bitrate is $10/8 = 1.25$, i.e., an increase of 25%.