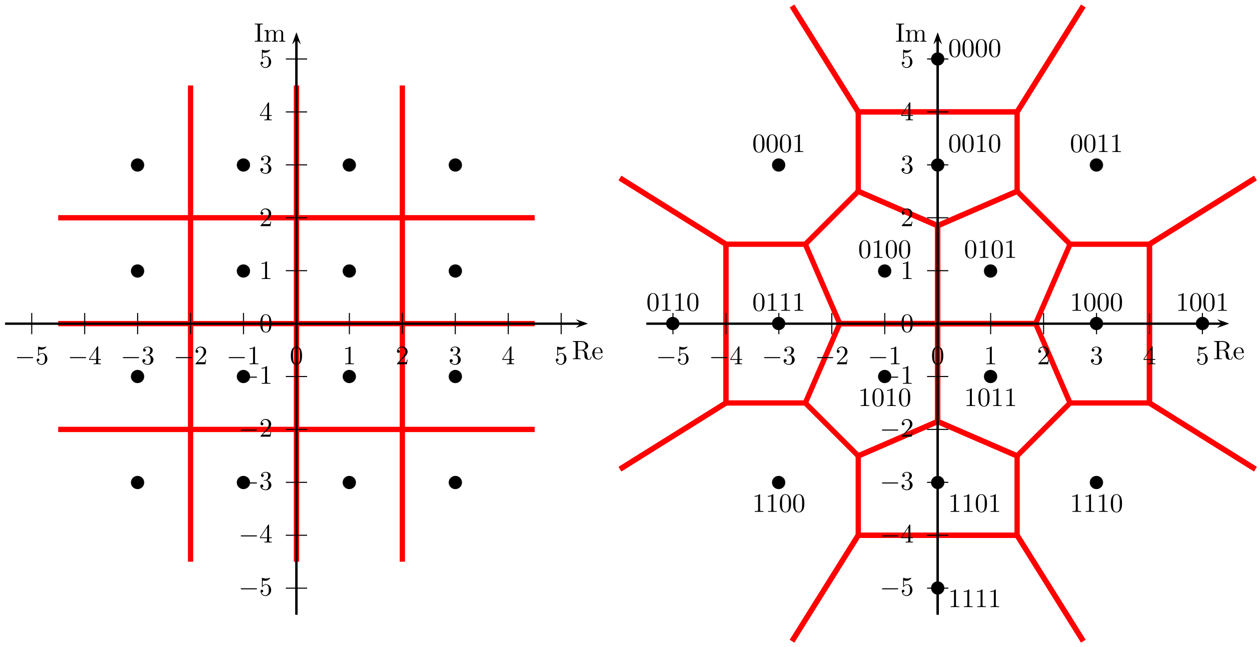
Exercise 3 Solution

Sept. 27, 2023

Problem 1 (Constellations)

Two 16-point QAM signal sets are shown in Figure 1. The first one is a standard square 4×4 constellation; the second one is the V.29 constellation. These constellations have a minimum squared distance of $D^2 = 4$.



(a) *(b)*

Figure 1: Problem 1.

1. Compute the average energy of each constellation if all points are equiprobable.

$$E_1 = \frac{4 \cdot 2 + 8 \cdot 10 + 4 \cdot 18}{16} = 10.$$

$$E_2 = \frac{4 \cdot 2 + 4 \cdot 9 + 4 \cdot 18 + 4 \cdot 25}{16} = \frac{216}{16} = 13.5.$$

2. Find a high-snr approximation for the SER. Compare the power efficiencies of the two constellations in dB.
The high-snr approximation for SER is given by

$$\text{SER} \approx \frac{2K}{M} \text{Q}\left(\sqrt{\frac{D_{\min}^2}{2N_0}}\right).$$

We have

$$\text{SER}_1 \approx 3\text{Q}\left(\sqrt{\frac{E_s}{5N_0}}\right)$$

For the second constellation, the number of pairs of points at minimum distance is 8.

$$\text{SER}_2 \approx \frac{2 \cdot 8}{16} \text{Q}\left(\sqrt{\frac{E_s}{6.75N_0}}\right) = \text{Q}\left(\sqrt{\frac{E_s}{6.75N_0}}\right).$$

The asymptotic gain shows how much more energy one system should spend compared to another system in order to achieve the same bit error rate. It is usually expressed in dB.

$$G = 10\log_{10}\left(\frac{E_2}{E_1}\right) = 1.3 \text{ dB}$$

3. Find a high-snr approximation for the BER for the constellation in Fig.1(b).
A bit error occurs only if a symbol error happens. Therefore, the argument of the q-function is the same. The question is how many bits are in error when a symbol error happens. The high SNR approximation is given by

$$\text{BER} \approx \frac{2H_{\min}}{Mm} \cdot \text{Q}\left(\sqrt{\frac{D^2_{\min}}{2N_0}}\right),$$

where $H_{\min}$ is the total number of bits differing between signal pairs at minimum distance.

$$\frac{2H_{\min}}{Mm} = \frac{2(6\cdot 1 + 2\cdot 3)}{16\cdot 4} = \frac{24}{64} = \frac{3}{8}.$$

The BER is

$$\text{BER} \approx \frac{3}{8}\text{Q}\left(\sqrt{\frac{E_s}{6.75N_0}}\right).$$

4. Sketch the decision regions of a minimum-distance detector for the two constellations.
See Fig. 6. Note that the boundaries always intersect in one point. Perpendicular bisectors of the sides of a triangle intersect at a point (circumcenter).

Problem 2 (Sandwiching the BER)

Consider a message $\mathbf{D}$ of k bits, i.e., $\mathbf{D} = [D_1, \dots, D_k]$. The bits are not necessarily independent and equally likely. Show that the probability of bit error rate is bounded as

$$\frac{1}{k}\Pr[\hat{\mathbf{D}} \neq \mathbf{D}] \leq \frac{1}{k}\sum_{j=1}^{k}\Pr[\hat{D}_j \neq D_j] \leq \Pr[\hat{\mathbf{D}} \neq \mathbf{D}],$$

where $\Pr[\hat{\mathbf{D}} \neq \mathbf{D}]$ is the probability that a message error occurs.

When the error $\hat{\mathbf{D}} \neq \mathbf{D}$ happens, this may or may not result in an error like $\hat{D}_j \neq D_j$ for all $j = 1, \dots, k$. This means that

$$\Pr[\hat{D}_j \neq D_j] \leq \Pr[\hat{\mathbf{D}} \neq \mathbf{D}]$$

for all $j = 1, \dots, k$. Summing up the left-hand side (LHS) and the right-hand side (RHS) over all j we obtain

$$\sum_{j=1}^{k}\Pr[\hat{D}_j \neq D_j] \leq \sum_{j=1}^{k}\Pr[\hat{\mathbf{D}} \neq \mathbf{D}].$$

Since the probability of the message error does not depend on k, the RHS is

$$\sum_{j=1}^{k}\Pr[\hat{\mathbf{D}} \neq \mathbf{D}] = k\Pr[\hat{\mathbf{D}} \neq \mathbf{D}],$$

which gives the upper bound. The lower bound is obtained as follows. There is at least one bit error when a message error occurs, i.e.,

$$\Pr[\hat{\mathbf{D}} \neq \mathbf{D}] \leq \sum_{j=1}^{k}\Pr[\hat{D}_j \neq D_j].$$

This gives a lower bound. *Interpretation:* The BER for any constellation and any labeling can be bounded using the SER as

$$\frac{1}{k}\text{SER} \leq \text{BER} \leq \text{SER}.$$

Problem 3 (MAP for BPSK)

Assume a discrete-time Gaussian channel. The received symbol is given by $Y = X + Z$, where X takes on a value $-d$ with probability 0.75 and d with probability 0.25 and $Z \sim \mathcal{N}(0, \sigma^2)$.

- Find the BER expression if the MAP demodulation is used.
  Let's label $-d$ with 0 and d with 1. We define a threshold y_0 as the point such that

$$\hat{B} = \begin{cases} 1 & \text{if } Y \geq y_0, \\ 0 & \text{if } Y < y_0, \end{cases}$$

The threshold can be found as

$$\Pr\{B = 0 | Y = y_0\} = \Pr\{B = 1 | Y = y_0\}.$$

This can be written as

$$\begin{aligned} \Pr\{Y = y_0 | B = 0\} \Pr\{B = 0\} &= \Pr\{Y = y_0 | B = 1\} \Pr\{B = 1\}. \\ \frac{1}{\sqrt{2\pi\sigma^2}} \mathrm{e}^{-\frac{(y_0+d)^2}{2\sigma^2}} \Pr\{B = 0\} &= \frac{1}{\sqrt{2\pi\sigma^2}} \mathrm{e}^{-\frac{(y_0-d)^2}{2\sigma^2}} \Pr\{B = 1\}. \end{aligned}$$

By taking the natural logarithm of both sides

$$\begin{aligned} -\frac{(y_0+d)^2}{2\sigma^2} + \ln(\Pr\{B = 0\}) &= -\frac{(y_0-d)^2}{2\sigma^2} + \ln(\Pr\{B = 1\}), \\ \frac{2dy_0}{\sigma^2} &= \ln\left(\frac{\Pr\{B = 0\}}{\Pr\{B = 1\}}\right) \\ y_0 &= \frac{\sigma^2}{2d} \ln\left(\frac{\Pr\{B = 0\}}{\Pr\{B = 1\}}\right) \end{aligned}$$

Comment: $y_0 > 0$ if $\Pr\{B = 0\} > \Pr\{B = 1\}$, i.e., the decision region for the likeliest point is "larger".

$$\begin{aligned} \Pr\{\text{error} | B = 0\} &= \mathrm{Q}\left(\frac{d + y_0}{\sigma}\right) \\ &= \mathrm{Q}\left(\frac{d + \frac{\sigma^2}{2d} \ln\left(\frac{\Pr\{B=0\}}{\Pr\{B=1\}}\right)}{\sigma}\right) \\ &= \mathrm{Q}\left(\frac{d}{\sigma}\left(1 + \frac{\sigma^2}{2d^2} \ln\left(\frac{\Pr\{B = 0\}}{\Pr\{B = 1\}}\right)\right)\right) \end{aligned}$$

Analogously,

$$\Pr\{\text{error} | B = 1\} = \mathrm{Q}\left(\frac{d}{\sigma}\left(1 - \frac{\sigma^2}{2d^2} \ln\left(\frac{\Pr\{B = 0\}}{\Pr\{B = 1\}}\right)\right)\right)$$

We note that

$$\begin{aligned} E_s &= d^2, \\ \sigma^2 &= \frac{N_0}{2}. \end{aligned}$$

Thus,

$$\frac{d}{\sigma} = \sqrt{\frac{2E_s}{N_0}}.$$

The final probability of error is

$$\begin{aligned} \Pr\{\text{error}\} &= \Pr\{B = 0\} \Pr\{\text{error} | B = 0\} + \Pr\{B = 1\} \Pr\{\text{error} | B = 1\} \\ &= \Pr\{B = 0\} \mathrm{Q}\left(\sqrt{\frac{2E_s}{N_0}}\left(1 + \frac{N_0}{4E_s} \ln\left(\frac{\Pr\{B = 0\}}{\Pr\{B = 1\}}\right)\right)\right) \\ &+ \Pr\{B = 1\} \mathrm{Q}\left(\sqrt{\frac{2E_s}{N_0}}\left(1 - \frac{N_0}{4E_s} \ln\left(\frac{\Pr\{B = 0\}}{\Pr\{B = 1\}}\right)\right)\right). \end{aligned}$$

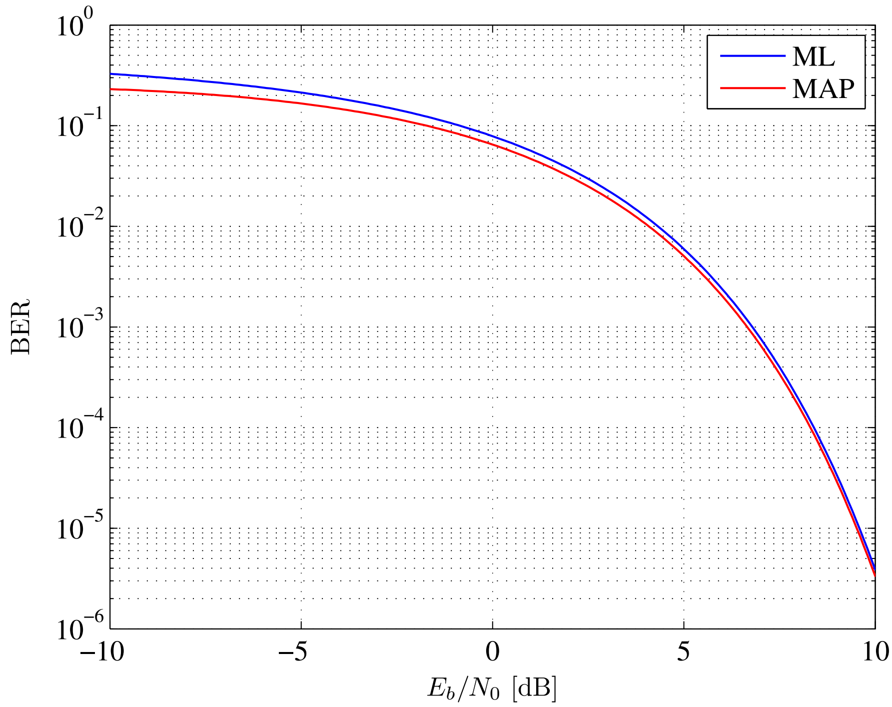


Figure 2: Problem 3. BER for BPSK.

- Find the BER expression if the ML demodulation is used.
  For the ML demodulator, the threshold is right between the points, i.e., $y_0 = 0$. This gives us

$$\Pr\{\text{error}\} = \Pr\{B=0\}\mathrm{Q}\left(\sqrt{\frac{2E_s}{N_0}}\right) + \Pr\{B=1\}\mathrm{Q}\left(\sqrt{\frac{2E_s}{N_0}}\right) = \mathrm{Q}\left(\sqrt{\frac{2E_s}{N_0}}\right).$$

Problem 4 (MAP for QPSK)

A QPSK constellation is used for transmission. The constellation points s_i, $i = 1, \ldots 4$ are defined as $s_i = \mathrm{e}^{j(i-1)\pi/2 + j\pi/4}$. Let $\Pr[s_1] = 0.4$ and $\Pr[s_2] = \Pr[s_3] = \Pr[s_4] = 0.2$. The SNR $E_s/N_0 = 0$ dB.

1. Sketch carefully the Maximum-Likelihood decision regions for the constellation points.
   The ML decision boundaries between two points are straight lines equidistant from the points. See Fig. 4(a).

2. Sketch carefully the Maximum a posteriori decision regions for the constellation points.
   To find the boundaries, wee need to find the thresholds between each pair of the constellation points. Let us start with the pair (s_1, s_2). Let's introduce a new coordinate system (y, z) with the origin between the points and axis y passing through (s_1, s_2). The boundary can be found from

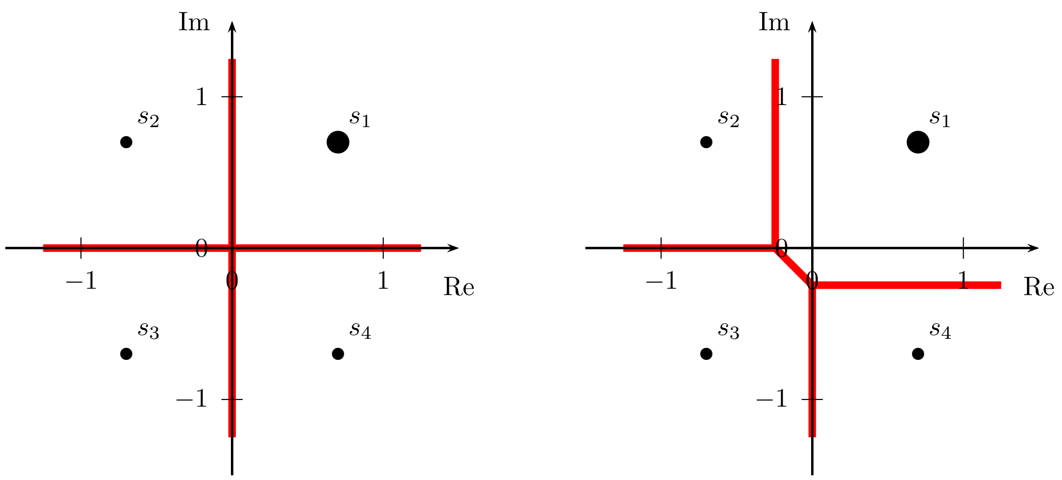
$$\Pr\{X = s_1 | Y = y_0, Z = z_0\} = \Pr\{X = s_2 | Y = y_0, Z = z_0\}.$$

   This cab be written as

$$\Pr\{Y = y_0 | X = s_1\} \Pr\{X = s_1\} = \Pr\{Y = y_0 | X = s_2\} \Pr\{X = s_2\}.$$
$$\frac{1}{2\pi\sigma^2}\mathrm{e}^{-\frac{(y_0+s_1)^2}{2\sigma^2} - \frac{(z_0)^2}{2\sigma^2}} \Pr\{X = s_1\} = \frac{1}{2\pi\sigma^2}\mathrm{e}^{-\frac{(y_0+s_2)^2}{2\sigma^2} - \frac{(z_0)^2}{2\sigma^2}} \Pr\{X = s_2\}.$$

   As we can see, z_0 is irrelevant for guessing on the symbol, and therefore, the problem is reduced to the binary case. The solution for y_0 from Problem 5 gives as

$$y_0 = \frac{\sigma^2}{2d} \ln\left(\frac{\Pr\{X = s_2\}}{\Pr\{X = s_1\}}\right), \tag{0.1}$$



(a) ML decision regions.

(b) MAP decision regions.

Figure 3: Problem 4.

Using the fact that d in this case is $d = \frac{\sqrt{2}}{2}$ and $\sigma^2 = \frac{N_0}{2} = \frac{N_0 E_s}{2E_s} = \frac{N_0}{2E_s}$ gives

$$y_0 = \frac{N_0}{2\sqrt{2}E_s} \ln\left(\frac{\Pr\{X = s_2\}}{\Pr\{X = s_1\}}\right)$$

$$y_0 = \frac{1}{2\sqrt{2} \cdot 10^{0/10}} \ln(0.5) \approx -0.245.$$

Obviously, repeating the same exercise for the pair (s_1, s_4) will give the same result. For the pairs (s_2, s_3) and (s_3, s_4) the decision boundaries will go between the points since they have the same probability.

The threshold for the pair (s_1, s_3) can be calculated using (0.1) and

Using $d = 1$ gives

$$y_0 = \frac{N_0}{4E_s} \ln\left(\frac{\Pr\{X = s_3\}}{\Pr\{X = s_1\}}\right)$$

$$y_0 = \frac{1}{4 \cdot 10^{0/10}} \ln(0.5) \approx -0.173.$$

The resulting decision regions are shown in Fig. 3(b).