Examination Wed. Oct. 28, 2020, 14:00–18:00

SSY121 Introduction to Communication Engineering

- Contact person: Examiner Fredrik Brännström (070 – 872 1685)
- Instructions:
  - Write in English.
  - Use a pencil and eraser.
  - There is no page limit.
  - Solve the problems in any order (they are not ordered by difficulty).
  - If any data is missing, make reasonable assumptions.
  - Make sure that each paper is clearly marked with your name, exam problem number and running page number.
  - Scan or photograph your solutions. Make sure to have a good lightning and preferably use a document scanning app, e.g., CamScanner or Genius Scan.
  - Name your image files Problem_YY_Page_XX.
    Example: Problem_01_Page_02.jpg
  - If you want, you can combine images for the same problem into a single document (e.g. PDF). *Example*: Problem_YY.pdf
  - You can also combine all problems into one single file.
    Example: Exam_SSY121_Fredrik.pdf
- Allowed aids:
  - All aids are allowed. However, **it is not permitted to cooperate with or take help from another person**.
- Grading principles:
  - Explain the line of reasoning clearly. A good solution with a minor error usually gives close to full points, even if the answer is incorrect.
  - An answer without a clear motivation usually gives 0 points, even if it is correct.
  - Whenever possible, check if your answer is reasonable. An obviously unreasonable answer usually gives 0 points, even if the solution is almost correct.
- Solutions and results:
  - After the exams are graded, a date and time for an online grading review will be posted on the course website in Canvas.

1. Consider a communication system using the following four signals for transmission,

$$\begin{aligned} s_1(t) &= +2\mathrm{I}\{0 \le t \le 1\} - 2\mathrm{I}\{1 \le t \le 2\} + 2\mathrm{I}\{2 \le t \le 3\}, \\ s_2(t) &= -2\mathrm{I}\{0 \le t \le 1\} + 2\mathrm{I}\{4 \le t \le 5\}, \\ s_3(t) &= +2\mathrm{I}\{2 \le t \le 3\} + 2\mathrm{I}\{4 \le t \le 5\}, \\ s_4(t) &= +2\mathrm{I}\{0 \le t \le 1\} + 2\mathrm{I}\{3 \le t \le 5\}, \end{aligned}$$

where the indicator function $\mathrm{I}\{\cdot\}$ is equal to 1 when the condition is fulfilled.

(Total points: 7)

(a) Draw the signals $s_1(t), s_2(t), s_3(t)$, and $s_4(t)$. (1)

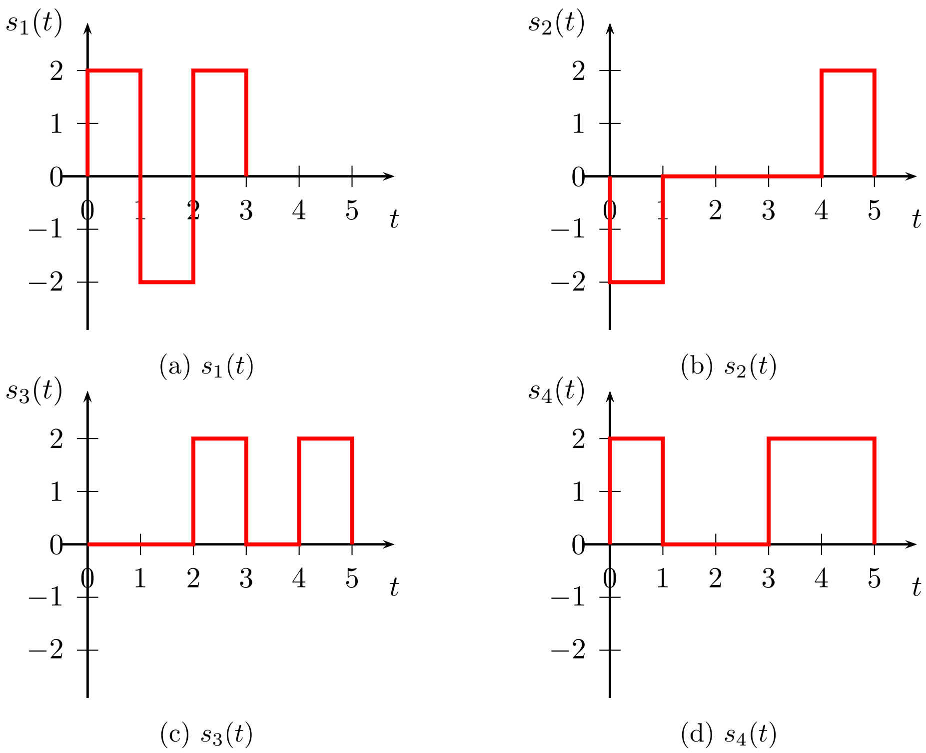

Figure 1: Signals in problem 1

(b) Find a **minimal set** of orthonormal basis for $\{s_1(t), s_2(t), s_3(t), s_4(t)\}$. (4)

Use Gram-Schmidt method to make sure find a minimal set:

$u_1 = s_1$

$\|u_1\| = \sqrt{\int_{-\infty}^{\infty} u_1^2 \, \mathrm{d}t} = \sqrt{12},$

$\phi_1 = \frac{u_1}{\|u_1\|} = \frac{s_1}{\sqrt{12}} = \frac{\sqrt{3}}{3}(\mathrm{I}\{0 \le t \le 1\} - \mathrm{I}\{1 \le t \le 2\} + \mathrm{I}\{2 \le t \le 3\}),$

$\langle s_2, \phi_1 \rangle = \int_{-\infty}^{\infty} s_2(t)\phi_1(t) \, \mathrm{d}t = -\frac{2\sqrt{3}}{3},$

$u_2 = s_2 - \langle s_2, \phi_1 \rangle \phi_1 = -\frac{4}{3}\mathrm{I}\{0 \le t \le 1\} - \frac{2}{3}\mathrm{I}\{1 \le t \le 2\} + \frac{2}{3}\mathrm{I}\{2 \le t \le 3\}) + 2\mathrm{I}\{4 \le t \le 5\}),$

$\|u_2\| = \frac{2\sqrt{15}}{3}$

$\phi_2 = \frac{u_2}{\|u_2\|} = -\frac{2\sqrt{15}}{15}\mathrm{I}\{0 \le t \le 1\} - \frac{\sqrt{15}}{15}\mathrm{I}\{1 \le t \le 2\} + \frac{\sqrt{15}}{15}\mathrm{I}\{2 \le t \le 3\}) + \frac{3\sqrt{15}}{15}\mathrm{I}\{4 \le t \le 5\}),$

$\langle s_3, \phi_1 \rangle = \frac{2\sqrt{3}}{3},$

$\langle s_3, \phi_2 \rangle = \frac{8\sqrt{15}}{15},$

$u_3 = s_3 - \langle s_3, \phi_1 \rangle \phi_1 - \langle s_3, \phi_2 \rangle \phi_2$

$u_3 = \frac{2}{5}\mathrm{I}\{0 \le t \le 1\} + \frac{6}{5}\mathrm{I}\{1 \le t \le 2\} + \frac{4}{5}\mathrm{I}\{2 \le t \le 3\}) + \frac{2}{5}\mathrm{I}\{4 \le t \le 5\}),$

$\|u_3\| = \frac{2\sqrt{15}}{5},$

$\phi_3 = \frac{\sqrt{15}}{15}\mathrm{I}\{0 \le t \le 1\} + \frac{\sqrt{15}}{5}\mathrm{I}\{1 \le t \le 2\} + \frac{2\sqrt{15}}{15}\mathrm{I}\{2 \le t \le 3\}) + \frac{\sqrt{15}}{15}\mathrm{I}\{4 \le t \le 5\}),$

$\langle s_4, \phi_1 \rangle = \frac{2\sqrt{3}}{3},$

$\langle s_4, \phi_2 \rangle = \frac{2\sqrt{15}}{15},$

$\langle s_4, \phi_3 \rangle = \frac{4\sqrt{15}}{15},$

$u_4 = s_4 - \langle s_4, \phi_1 \rangle \phi_1 - \langle s_4, \phi_2 \rangle \phi_2 - \langle s_4, \phi_3 \rangle \phi_3,$

$u_4 = \frac{4}{3}\mathrm{I}\{0 \le t \le 1\} - \frac{4}{3}\mathrm{I}\{2 \le t \le 3\} + 2\mathrm{I}\{3 \le t \le 4\}) + \frac{4}{3}\mathrm{I}\{4 \le t \le 5\}),$

$\|u_4\| = \frac{2\sqrt{21}}{3},$

$\phi_4 = \frac{2\sqrt{21}}{21}\mathrm{I}\{0 \le t \le 1\} - \frac{2\sqrt{21}}{21}\mathrm{I}\{2 \le t \le 3\} + \frac{\sqrt{21}}{7}\mathrm{I}\{3 \le t \le 4\}) + \frac{2\sqrt{21}}{21}\mathrm{I}\{4 \le t \le 5\}).$

(c) Represent the set of signals $\{s_1(t), s_2(t), s_3(t), s_4(t)\}$ in terms of the basis functions. (2)

$$\begin{aligned}
s_1 &= 2\sqrt{3}\phi_1, \\
s_2 &= -\frac{2}{3}\sqrt{3}\phi_1 + \frac{2}{3}\sqrt{15}\phi_2, \\
s_3 &= \frac{2}{3}\sqrt{3}\phi_1 + \frac{8}{15}\sqrt{15}\phi_2 + \frac{2}{5}\sqrt{15}\phi_3, \\
s_4 &= \frac{2}{3}\sqrt{3}\phi_1 + \frac{2}{15}\sqrt{15}\phi_2 + \frac{4}{15}\sqrt{15}\phi_3 + \frac{2}{3}\sqrt{21}\phi_4.
\end{aligned}$$

2. Assume a pulse

$$v(t) = \begin{cases} 1 & 0 \le t < 0.1 \text{ and } 0.3 \le t < 0.4 \\ 0, & \text{otherwise} \end{cases}$$

with a duration of 1 second is used to modulate the BPSK symbol $a \in \{-1, +1\}$ to create the transmitted signal $s(t) = av(t)$. A matched filter is used at the receiver. (Total points: 10)

(a) Sketch the output of the matched filter when $s(t)$ is received without noise for $a = 1$ and write down its expression. (2)
The sketch of the MF output $z(t)$ when $v(t)$ (see Figure 2) is transmitted is shown in Figure 3

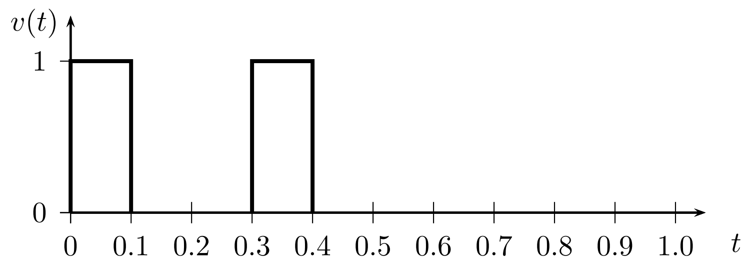

Figure 2: $v(t)$

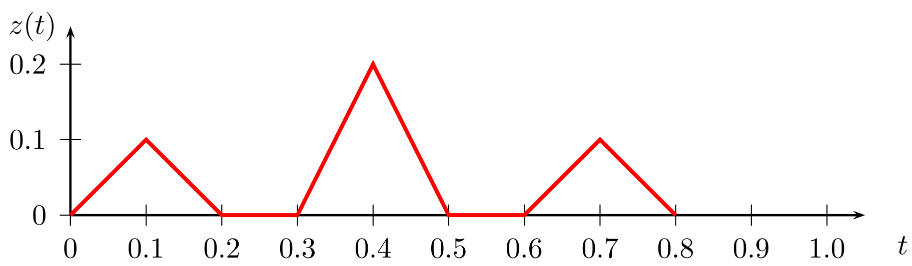

Figure 3: The output of MF when $v(t)$ is received

and its expression is given by

$$z(t) = \begin{cases} t & 0 \le t < 0.1 \\ -t + 0.2 & 0.1 \le t < 0.2 \\ 2t - 0.6 & 0.3 \le t < 0.4 \\ -2t + 1 & 0.4 \le t < 0.5 \\ t - 0.6 & 0.6 \le t < 0.7 \\ -t + 0.8 & 0.7 \le t < 0.8 \\ 0, & \text{otherwise} \end{cases}$$

(b) Sketch the eye-diagram for the case $a \in \{-1, +1\}$ assuming a noise-free channel. Find in what range the timing error could be in order to have a zero symbol error rate. (2)
The eye-diagram is shown in Figure 4. We can conclude that when the timing error is in the ranges $(-0.4, -0.2)$, $(-0.1, 0.1)$, or $(0.2, 0.4)$, there are no symbol errors.

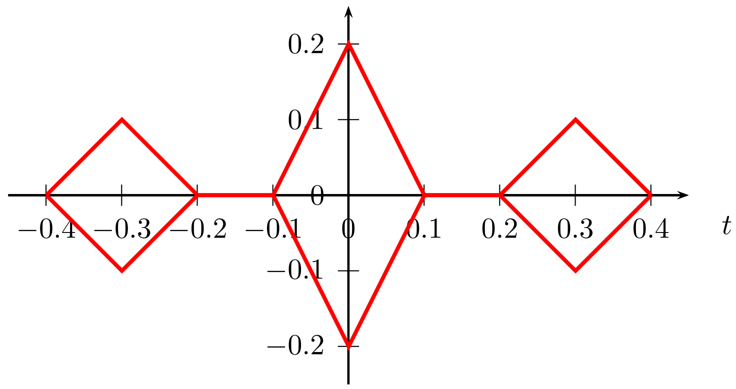

Figure 4: Eye-diagram

(c) Suppose the transmitted signal $s(t)$ when $a = 1$ is passed through a channel with a frequency response $H(f) = 1 + e^{-j2\pi f t_0} + e^{-j4\pi f t_0}$. Sketch the channel output $y(t)$ when $t_0 = 0.1$ and write down its expression. (3)

Solution 1: Take the inverse Fourier transform of $H(f)$, we obtain

$$h(t) = \mathcal{F}^{-1}[H(f)] = \delta(t) + \delta(t - t_0) + \delta(t - 2t_0)$$

So,

$$\begin{aligned} y(t) =& s(t) * h(t) = s(t) + s(t - 0.1) + s(t - 0.2) \\ =& \begin{cases} 1 & 0 \le t < 0.6 \\ 0, & \text{otherwise} \end{cases} \end{aligned}$$

Solution 2: we have $Y(f) = S(f)H(f)$, where $Y(f)$ and $S(f)$ denote the Fourier transform of $y(t)$ and $s(t)$, respectively. Then

$$Y(f) = S(f)H(f) = S(f) + S(f)e^{-j2\pi f t_0} + S(f)e^{-j4\pi f t_0},$$

Then $y(t) = \mathcal{F}^{-1}[Y(f))] = s(t) + s(t - t_0) + s(t - 2t_0) = \begin{cases} 1 & 0 \le t < 0.6 \\ 0, & \text{otherwise} \end{cases}$

The sketch of $y(t)$ is shown in Figure. 5

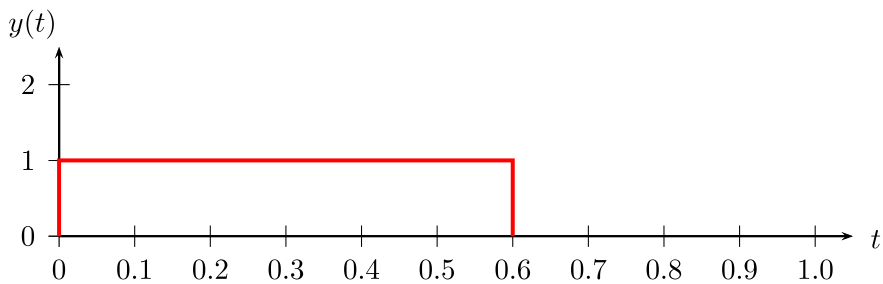

Figure 5: Channel output $y(t)$

(d) The channel output $y(t)$ from (c) is now passed through the filter matched to $v(t)$. Sketch the matched filter output when $t_0 = 0.1$ assuming no noise in the receiver and write down its expression. (3)

Since $y(t) = s(t) + s(t - 0.1) + s(t - 0.2)$, the MF output $y(t) = z(t) + z(t - 0.1) + z(t - 0.2)$, where $z(t)$ is the MF output of $s(t)$ in problem (a). The sketch is shown in Figure 6, where the solid, dashed, and dotted gray curves represent $z(t)$, $z(t-0.1)$, and $z(t-0.2)$, respectively.

Alternatively, one can directly get the MF output of $y(t)$ by convolving $y(t)$ from problem (b) with $v(-t)$, which would give the same sketch as Figure. 6.

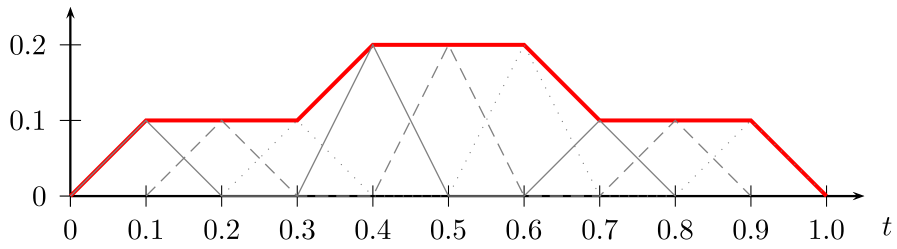

Figure 6: The MF output of $y(t)$

The expression of the matched filter output of $y(t)$ is

$$\begin{cases} t+0.1 & 0 \le t < 0.1 \\ 0.1 & 0.1 \le t < 0.3 \\ t-0.2 & 0.3 \le t < 0.4 \\ 0.2 & 0.4 \le t < 0.6 \\ -t+0.8 & 0.6 \le t < 0.7 \\ 0.1 & 0.7 \le t < 0.9 \\ -t+1 & 0.9 \le t < 1 \\ 0, & \text{otherwise} \end{cases}$$

3. A spacecraft located $50,000$ km from earth is sending data at a rate of R bits/sec using a BPSK modulation scheme. The antenna gain at the transmitter side and the transmitted power are 0 dB and 10 W, respectively. The earth station receiver uses a parabolic antenna, 200 m in diameter, and the noise temperature of the receiver front end is $T = 300$ K. (Total points: 4)

   (a) Determine the received power level. (2)

   The received power is written as:

$$P_{\mathrm{R}} = P_{\mathrm{T}} G_{\mathrm{T}} G_{\mathrm{R}} \left(\frac{\lambda}{4\pi d} \right)^2 ,$$

   where $P_{\mathrm{T}} = 10$, $G_{\mathrm{R}} = 0\,\mathrm{dB} = 1$, and

$$G_{\mathrm{R}} = \left(\frac{\pi D}{\lambda} \right)^2 .$$

   Substituting the latter equation in the former one, we have:

$$P_{\mathrm{R}} = P_{\mathrm{T}} G_{\mathrm{T}} \left(\frac{D}{4d} \right)^2 .$$

   Hence, the received power level is

$$P_{\mathrm{R}} = 10 \times 1 \times \left(\frac{200}{4 \times 50 \times 10^6} \right)^2 = 10^{-11}\,\mathrm{W}.$$

   (b) If an average SNR of $\frac{E_b}{N_0} = 10$ dB is required, determine the maximum bit rate that the spacecraft can transmit with. (2)

   If sinc pulses are used, $R = B$, where B is the bandwidth.

$$\frac{E_b}{N_0} = \frac{P_{\mathrm{R}}/B}{N_0} = \frac{1}{B} \frac{P_{\mathrm{R}}}{N_0} \rightarrow B = \frac{P_{\mathrm{R}}}{N_0} \left(\frac{E_b}{N_0} \right)^{-1}$$

   We need to find the noise power density N_0:

$$N_0 = kT = 1.38 \times 10^{-23} \times 300 = 4.14 \times 10^{-21}\,\mathrm{W/Hz}.$$

   Noting $\frac{E_b}{N_0} = 10\,\mathrm{dB} = 10$, we have:

$$R = B = \frac{P_{\mathrm{R}}}{N_0} \left(\frac{E_b}{N_0} \right)^{-1} = \frac{10^{-12}}{4.14 \times 10^{-21}} = 241.5\,\mathrm{Mbits/sec}.$$

4. Consider a communication system using a pulse $v(t)$ having a spectrum $V(f)$ shown in Figure 7, where $a > 0$ and $0 < b < 1$. (Total points: 3)

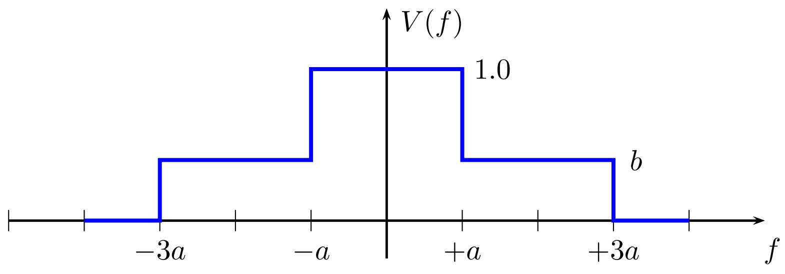

Figure 7: Spectrum of the pulse.

(a) What is the maximum symbol rate expressed in a to achieve ISI-free communication using a matched filter receiver when $b = 0.4$? (1)

Define the auxiliary function

$$Z(f) = \sum_{n=-\infty}^{+\infty} V(f - nR)^2 ,$$

where $R = 1/T_s$ is the symbol rate. ISI-free communication using a matched filter is achieved if $Z(f) = K$, i.e., independent of f. Clearly, if $R > 6a$, $Z(f) = 0$ for some f. If $R = 4a$, $Z(f) = 1.0$ or $Z(f) = 2 \cdot 0.4^2 = 0.32$ depending on f. However, if $R = 2a$, $Z(f) = 1 + 2 \cdot 0.4^2 = 1.32$ for all f.

(b) What value of b, $0 < b < 1$, maximizes the symbol rate to achieve ISI-free communication using a matched filter receiver, and what is the maximum rate expressed in a for this case? (2)

If $R = 4a$, $Z(f) = 1.0$ or $Z(f) = 2 \cdot b^2$ depending on f. This means that if $1 = 2b^2$, $Z(f) = 1.0$ for all values of f. This means the the maximum rate for ISI-free communication is $R = 4a$, which happens only if $b = 1/\sqrt{2} = 0.7071$.

5. In a binary communication system, the symbols $s_1 = +A$ and $s_2 = -A$ with the *a priori* probabilities $p(s_1) = p_1$ and $p(s_2) = p_2$, where $p_1 + p_2 = 1$, are sent over a fading communication channel with additive noise. The received symbol r can be written as

$$r = \rho s + n,$$

where s is the transmitted symbol and n is the noise variable characterized by the probability density function

$$p(n) = \frac{1}{\sqrt{2\sigma^2}} \exp\left(\frac{-|n|}{\sigma/\sqrt{2}}\right).$$

The communication channel undergoes a random fading process described by the random variable ρ taking the values 0 and 1, with equal probability.

(Total points: 12)

(a) Determine the optimal *decision rule* in this communication scheme in the general case. (3)

The receiver makes a decision in favor of s_1 if $p(s_1|r) > p(s_2|r)$, and in favor of s_2, otherwise. We first find the decision region for s_1; then everywhere else in the space belongs to the decision region of s_2.

$$p(s_1|r) > p(s_2|r)$$
$$p(s_1)p(r|s_1) > p(s_2)p(r|s_2)$$
$$p_1 p(r| + A) > p_1 p(r| - A)$$

Now we need to condition on the value of ρ:

$$p_1 \left(\frac{1}{2} p(r| + A, \rho = 0) + \frac{1}{2} p(r| + A, \rho = 1) \right) > p_2 \left(\frac{1}{2} p(r| - A, \rho = 0) + \frac{1}{2} p(r| - A, \rho = 1) \right).$$

Then we can write the decision rule as:

$$p_1 \left(\exp\left(\frac{-|r|}{\sigma/\sqrt{2}}\right) + \exp\left(\frac{-|r - A|}{\sigma/\sqrt{2}}\right) \right) > p_2 \left(\exp\left(\frac{-|r|}{\sigma/\sqrt{2}}\right) + \exp\left(\frac{-|r + A|}{\sigma/\sqrt{2}}\right) \right).$$

(b) Determine and plot the optimal *decision regions* in case $p_1 = p_2$. (3)

You can either start finding the optimal decision region from scratch, or use the below expression. Just to emphasize that this does not necessarily depend on the previous part. With the equiprobable signaling we have the following decision rule:

$$\exp\left(\frac{-|r|}{\sigma/\sqrt{2}}\right) + \exp\left(\frac{-|r - A|}{\sigma/\sqrt{2}}\right) > \exp\left(\frac{-|r|}{\sigma/\sqrt{2}}\right) + \exp\left(\frac{-|r + A|}{\sigma/\sqrt{2}}\right),$$

which is simplified further as:

$$\exp\left(\frac{-|r - A|}{\sigma/\sqrt{2}}\right) > \exp\left(\frac{-|r + A|}{\sigma/\sqrt{2}}\right).$$

Now we can take the natural logarithm from both sides of the above inequality and obtain:

$$|r - A| < |r + A|,$$

meaning that the receiver makes a decision in favor of $+A$ if the received symbol r is closer to $+A$ than to $-A$, in the Euclidean signal space. This is exactly the same as the minimum distance detector. In this case, the decision threshold is zero, and the decision regions are as in Figure 8.

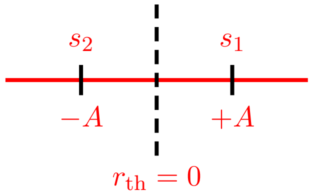

Figure 8: Decision Regions

(c) Find the error probability of the optimal receiver as a function of A and σ when $p_1 = p_2$. *If you have not found the decision boundary in (b), you can assume it is equal to* r_{th}, *and still solve this part.* (4)

$$p_{\text{e}} = \frac{1}{2} p_{\text{e}|s_1} + \frac{1}{2} p_{\text{e}|s_2},$$

where

$$p_{\text{e}|\text{s}_2} = \frac{1}{2} p_{\text{e}|\text{s}_2,\rho=0} + \frac{1}{2} p_{\text{e}|\text{s}_2,\rho=1},$$

and equivalently,

$$p_{\text{e}|\text{s}_1} = \frac{1}{2} p_{\text{e}|\text{s}_1,\rho=0} + \frac{1}{2} p_{\text{e}|\text{s}_1,\rho=1}.$$

We have

$$\begin{aligned} p_{\text{e}|\text{s}_2,\rho=1} = \text{Prob}(n > A + r_{\text{th}}) &= \int_{A+r_{\text{th}}}^{+\infty} \frac{1}{\sqrt{2\sigma^2}} \text{e}^{\frac{-|n|}{\sigma/\sqrt{2}}} \text{d}n \\ &= \int_{A+r_{\text{th}}}^{+\infty} \frac{1}{\sqrt{2\sigma^2}} \text{e}^{\frac{-n}{\sigma/\sqrt{2}}} \text{d}n \\ &= \frac{1}{2} \exp\left(-\frac{(A + r_{\text{th}})\sqrt{2}}{\sigma}\right), \end{aligned}$$

where clearly $A + r_{\text{th}} > 0$, because $r_{\text{th}} > -A$, and

$$\begin{aligned} p_{\text{e}|s_1,\rho=1} &= \text{Prob}(n < -A + r_{\text{th}}) \\ &= \text{Prob}(n > A - r_{\text{th}}) \\ &= \frac{1}{2} \exp\left(-\frac{(A - r_{\text{th}})\sqrt{2}}{\sigma}\right), \end{aligned}$$

where we have used the symmetry of the noise pdf, and that $A > r_{\text{th}}$. Now we can write:

$$\begin{aligned} p_{\text{e}} &= \frac{1}{4} p_{\text{e}|\text{s}_2,\rho=1} + \frac{1}{4} p_{\text{e}|\text{s}_1,\rho=1} + \frac{1}{4} p_{\text{e}|\text{s}_2,\rho=0} + \frac{1}{4} p_{\text{e}|\text{s}_1,\rho=0} \\ &= \frac{1}{4} p_{\text{e}|\text{s}_2,\rho=1} + \frac{1}{4} p_{\text{e}|\text{s}_1,\rho=1} + \frac{1}{4} (\text{Prob}(n > r_{\text{th}}) + \text{Prob}(n < r_{\text{th}})) \\ &= \frac{1}{8} \exp\left(-\frac{(A + r_{\text{th}})\sqrt{2}}{\sigma}\right) + \frac{1}{8} \exp\left(-\frac{(A - r_{\text{th}})\sqrt{2}}{\sigma}\right) + \frac{1}{4} \end{aligned}$$

Since $r_{\text{th}} = 0$, the error probability is

$$p_{\text{e}} = \frac{1}{4} + \frac{1}{4} \exp\left(-\frac{A\sqrt{2}}{\sigma}\right).$$

Note that the noise is not Gaussian, so we do not have Q functions in this part.

(d) Assume now that the additive noise is normally distributed with the probability density function (2)

$$p(n) = \frac{1}{\sqrt{2\pi\sigma^2}} \mathrm{e}^{\frac{-|n|^2}{2\sigma^2}},$$

and the decision rule for the optimal receiver is given by

$$\hat{s} = \begin{cases} s_1, & r > r_{\text{th}} \\ s_2, & r < r_{\text{th}} \end{cases}.$$

Find the error probability as a function of A and σ when $p_1 = p_2$.

$$p_{\mathrm{e}} = \frac{1}{2} p_{\mathrm{e}|s_1} + \frac{1}{2} p_{\mathrm{e}|s_2}$$

where

$$p_{\mathrm{e}|\mathrm{s}_2,\rho=1} = \mathrm{Prob}(n > A + r_{\text{th}}) = \mathrm{Q}\left(\frac{A + r_{\text{th}}}{\sigma}\right),$$

$$p_{\mathrm{e}|\mathrm{s}_1,\rho=1} = \mathrm{Prob}(n < -A + r_{\text{th}}) = \mathrm{Q}\left(\frac{A - r_{\text{th}}}{\sigma}\right).$$

Thus,

$$p_{\mathrm{e}} = \frac{1}{4}\mathrm{Q}\left(\frac{A + r_{\text{th}}}{\sigma}\right) + \frac{1}{4}\mathrm{Q}\left(\frac{A - r_{\text{th}}}{\sigma}\right) + \frac{1}{4}.$$

Since $r_{\text{th}} = 0$, the error probability is

$$p_{\mathrm{e}} = \frac{1}{4} + \frac{1}{2}\mathrm{Q}\left(\frac{A}{\sigma}\right),$$

6. Consider the 16-point QAM constellation shown in Figure 9, with the minimum distance of $D_{\min} = 2d$. This constellation is used for communication over an AWGN channel with noise variance σ^2. Consider the following relations between the *a priori* probabilities:

$$p(s_i) = p(s_j) \ \forall i, j \in \{1, 4, 13, 16\},$$
$$p(s_k) = p(s_l) \ \forall k, l \in \{2, 3, 5, 6, 7, 8, 9, 10, 11, 12, 14, 15\},$$
$$p(s_1) = 2p(s_2).$$

(Total points: 12)

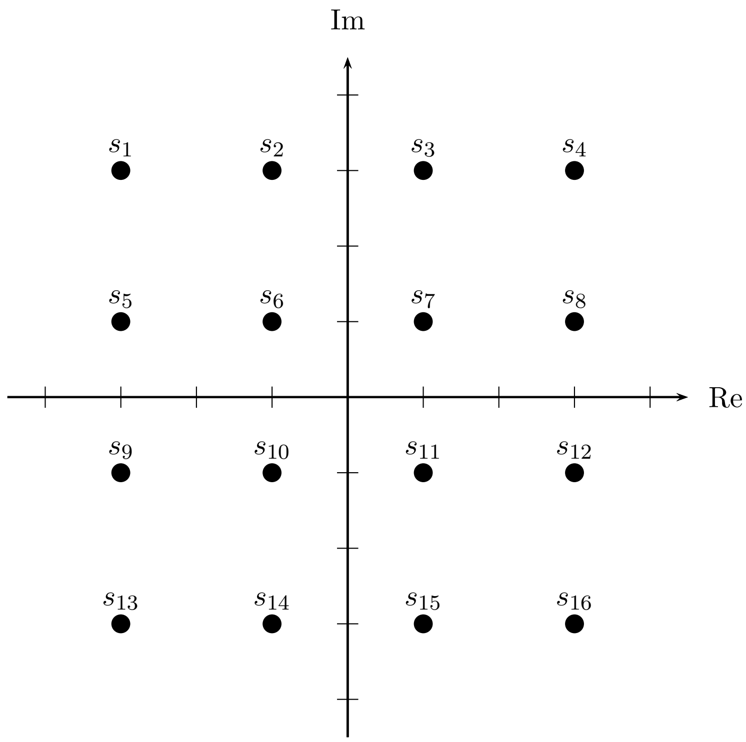

Figure 9: 16-point QAM constellation

(a) Find the *decision boundaries*, i.e., thresholds, as functions of d and σ^2 and specify the *decision regions* by drawing the boundaries for the MAP receiver. (4)

For any pair of neighbouring symbols, if they are equiprobable, then the boundary is the perpendicular bi-sector of the line connecting the symbols. However, if they are not equiprobable, the decision threshold is obtained by writing the MAP decision rule. We obtain the decision boundary between s_3 and s_4, i.e., r_{th_1}, which gives all the thresholds, due to symmetry.

The decision is in favor of s_4 if $p(s_4|r) > p(s_3|r)$, i.e.,

$$p(s_4)p(r|s_4) = p(s_3)p(r|s_3),$$
$$\frac{p(r|s_4)}{p(r|s_3)} > \frac{p(s_3)}{p(s_4)},$$
$$\exp\left(-\frac{(r-3d)^2}{2\sigma^2} + \frac{(r-d)^2}{2\sigma^2}\right) > \frac{p(s_3)}{p(s_4)},$$
$$-\frac{(r-3d)^2}{2\sigma^2} + \frac{(r-d)^2}{2\sigma^2} > \ln\frac{p(s_3)}{p(s_4)} = \ln\frac{1}{2} = -\ln 2,$$

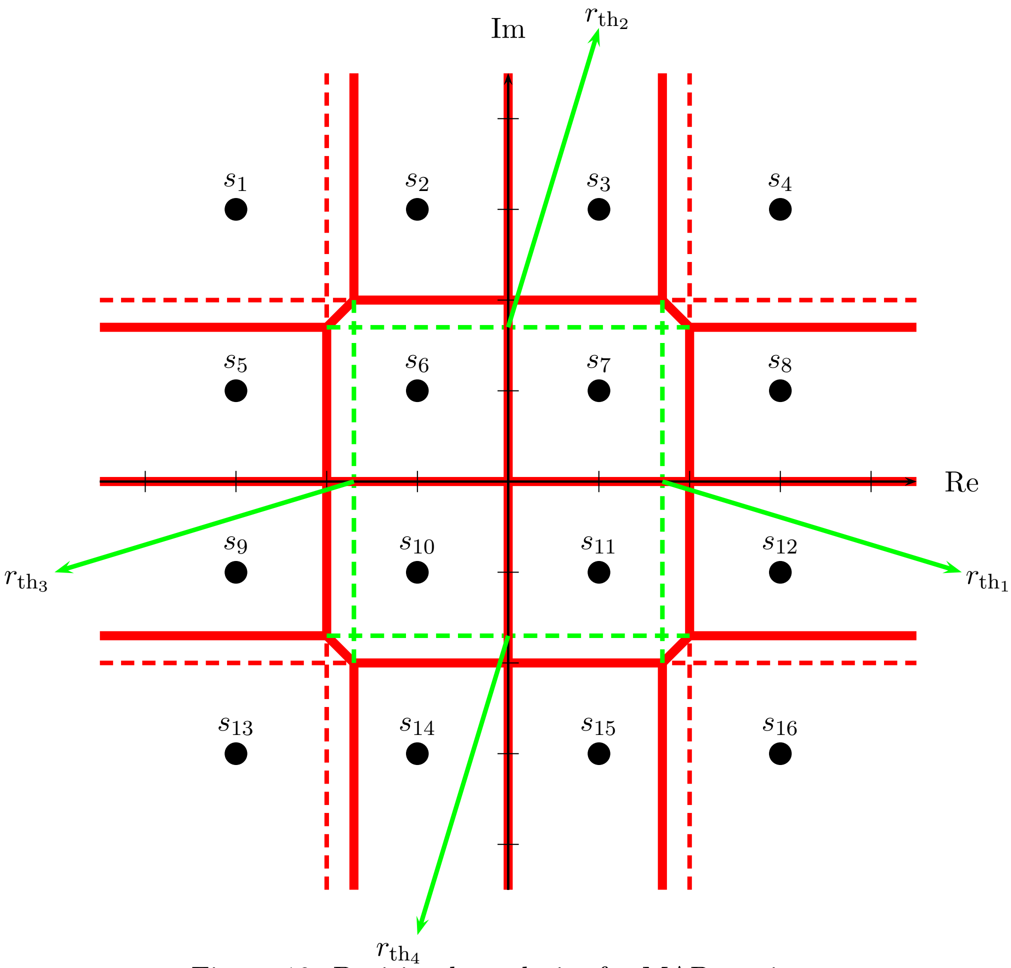

Figure 10: Decision boundaries for MAP receiver.

which can be simplified to

$$r > 2d - \frac{\sigma^2}{2d}\ln 2.$$

Therefore:

$$r_{\mathrm{th}_1} = 2d - \frac{\sigma^2}{2d}\ln 2.$$

Due to the symmetry of the constellation:

$$r_{\mathrm{th}_2} = r_{\mathrm{th}_1}$$
$$r_{\mathrm{th}_3} = -r_{\mathrm{th}_1}$$
$$r_{\mathrm{th}_4} = -r_{\mathrm{th}_1}$$

See Figure 10 for a sketch of the decision regions.

(b) Compute the average energy of the constellation in terms of d, both for the case above with unequal probabilities and in the case when all signal alternatives are equally likely. (3)

We need to obtain the *a priori* probabilities. As $\sum_{i=1}^{16} p(s_i) = 1$, it is easy to see that

$$p(s_i) = 0.1 \ \forall i \in \{1, 4, 13, 16\}$$
$$p(s_j) = 0.05 \ \forall j \in \{2, 3, 5, 6, 7, 8, 9, 10, 11, 12, 14, 15\}$$

There are three energy levels for the signals in this constellation:

$$E_{s_i} = 18d^2 \ \forall i \in \{1,4,13,16\}$$
$$E_{s_j} = 2d^2 \ \forall j \in \{6,7,10,11\}$$
$$E_{s_k} = 10d^2 \ \forall k \in \{2,3,5,8,9,12,14,15\}$$

Then we have:

$$E_{s,\text{equiprobable}} = \sum_{i=1}^{16} \frac{1}{16} E_{s_i} = 10d^2$$
$$E_{s,\text{non-equiprobable}} = \sum_{i=1}^{16} p(s_i) E_{s_i} = 11.6d^2$$

(c) If the transmitted symbol is denoted by s and the detected symbol by $\hat{s}$, find the exact conditional probability $\text{Prob}(\hat{s} = s_3 | s = s_1)$ as a function of d and σ. Assume that the noise components are independent and identically distributed. (3)

$$\begin{aligned}\text{Prob}(\hat{s} = s_3 | s = s_1) &= \text{Prob}(0 < r_1 < r_{\text{th}_1}\ ,\ r_2 > 2d | s = s_1) \\ &= \text{Prob}(0 < -3d + n_1 < r_{\text{th}_1}\ ,\ 3d + n_2 > 2d | s = s_1) \\ &= \text{Prob}(3d < n_1 < 3d + r_{\text{th}_1}\ ,\ n_2 > -d | s = s_1),\end{aligned}$$

where r_1 and n_1 are components of the received symbol and noise along the Re axis, and r_2 and n_2 are components of the received symbol and noise along the Im axis. The noise components are independent, and hence:

$$\begin{aligned}\text{Prob}(\hat{s} = s_3 | s = s_1) &= \text{Prob}(3d < n_1 < 3d + r_{\text{th}_1}) \times \text{Prob}(n_2 > -d) \\ &= (\text{Prob}(n_1 > 3d) - \text{Prob}(n_1 > 3d + r_{\text{th}_1})) \times (1 - \text{Prob}(n_2 > d)) \\ &= \left(\mathrm{Q}\left(\frac{3d}{\sigma}\right) - \mathrm{Q}\left(\frac{3d + r_{\text{th}_1}}{\sigma}\right)\right) \times \left(1 - \mathrm{Q}\left(\frac{d}{\sigma}\right)\right)\end{aligned}$$

(d) Assume that the exact SER for the non-equiprobable case is given by $P_\text{e}(d, \sigma^2) = P$. Obtain the exact SER $\tilde{P}_\text{e}(d, \sigma^2)$ as a function of P for the new constellation with the signals $\tilde{s}_i = s_i - \mathbf{b}$, $i = 1, \cdots, 16$, where $\mathbf{b} = [-1, 1]$. Assume that $p(\tilde{s}_i) = p(s_i)$, $i = 1, \cdots, 16$. (2)

For the new constellation we have:

$$\tilde{D}_{i,j} = |\tilde{s}_i - \tilde{s}_j| = |s_i - \mathbf{b} - (s_j - \mathbf{b})| = |s_i - s_j| = D_{i,j}.$$

As the distance between the symbols do not change, and they only shift in the space, then the probability of error as a function of d and σ^2 does not change, i.e.,

$$\tilde{P}_\text{e}(d, \sigma^2) = P_\text{e}(d, \sigma^2) = P.$$