Introduction to Communication Engineering SSY121, Lecture # 2

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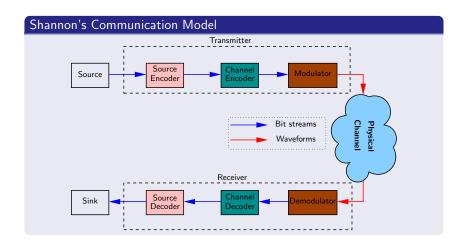
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Outline

- 1 The Big Picture
- 2 The Sampling Rx
- lacksquare A Sequence of M-ary Pulses
- The Nyquist Criterion
- **5** The Sampling Rx (cont.)
- 6 The Linear Rx for one pulse

Part I

The Sampling Rx and Nyquist Pulses



The General Diagram

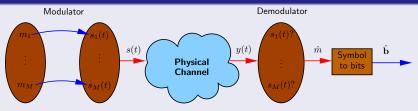


- $oldsymbol{f b}$ The length-l binary codeword ${f b}=[b_1,\ldots,b_l]$ has to be transmitted
- Each of the possible $M=2^l$ codewords is mapped to a message (symbol) $m\in\{m_1,\ldots,m_M\}$
- The modulator sends a continuous-time baseband signal s(t) through the physical channel each $T_{\rm s}$ [s], where s(t) is selected from the set of signal alternatives $\mathcal{S} = \{s_1(t), \dots, s_M(t)\}$.
- The channel introduces some distortion such that $y(t) \neq s(t)$

The General Problem

Using only the channel observation y(t) during one symbol period $T_{\rm s}$, guess what the transmitted message m was.

The General Problem



- ullet A symbol is what is transmitted in each time slot (every $T_{
 m s}$ [s])
- $T_{\rm s}$ is the symbol duration and $T_{\rm b}$ is the bit duration, where $lT_{\rm b}=T_{\rm s}$ ($T_{\rm b}=T_{\rm s}$ for binary transmission since l=1)
- \bullet The symbol rate (baud rate) is the number of symbols per second $R_{\rm s}=1/T_{\rm s}$
- The bit rate is the number of bits per second $R_{\rm b}=1/T_{\rm b}=l/T_{\rm s}=lR_{\rm s}$

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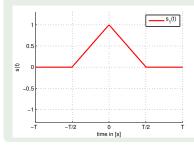
SSY121, Lecture # 2

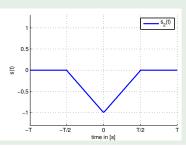
Example (Binary transmission with triangular pulse)

In binary transmission l=1, M=2, $b\in\{0,1\}$ and therefore $b=0\Rightarrow m=m_1\Rightarrow s_1(t)$, and $b=1\Rightarrow m=m_2\Rightarrow s_2(t)$. Moreover, we could use only one basic pulse such that

$$s(t) = \begin{cases} s_1(t) = +1 \cdot v(t) & \text{if } m = m_1 \\ s_2(t) = -1 \cdot v(t) & \text{if } m = m_2 \end{cases},$$

where v(t) is a triangular pulse.

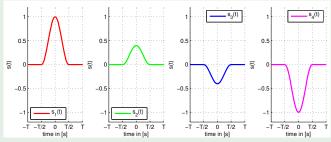




Example (4-ary transmission with arbitrary pulse)

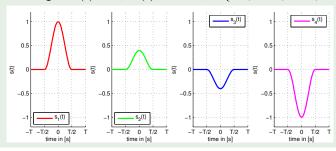
In this case l=2, M=4, $\mathbf{b} \in \{00,01,10,11\}$ and $s_i(t)=a_i \cdot v(t)$ where $a_i \in \{+1,+0.4,-0.4,-1\}$, i.e.,

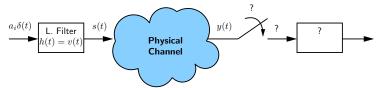
b	m	s(t)
00	m_1	$s_1(t) = +1.0 \cdot v(t)$
01	m_2	$s_2(t) = +0.4 \cdot v(t)$
10	m_3	$s_3(t) = -0.4 \cdot v(t)$
11	m_4	$s_4(t) = -1.0 \cdot v(t)$



Example (4-ary transmission with arbitrary pulse)

Transmitted signal $s(t) = a_i \cdot v(t)$ where $a_i \in \{+1, +0.4, -0.4, -1\}$





What can we do with in the receiver?

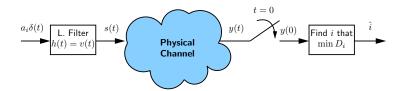
You are given a box which contains:

- a linear filter (with an impulse response you can decide)
- a "bag" with elementary signals (steps, impulses, sinusoids, ...)

The Sampling Rx

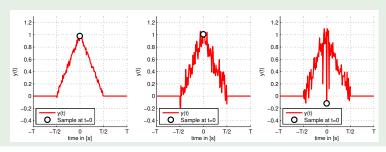
Sample the received signal y(t) at t=0 and choose the signal alternative that is the closest to the sample, i.e.,

$$\hat{i} = \arg\min_{i \in \{1,\dots,M\}} D_i, \quad \text{where} \quad D_i = |y(0) - s_i(0)|.$$



Example (Sampling Rx for binary transmission with triangular pulse)

Assume that the transmitted message is $m=m_1\Rightarrow s(t)=s_1(t)$, i.e., a positive triangular pulse. The received signal for three different channels is



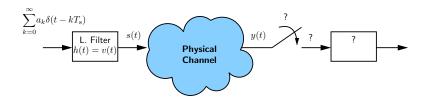
D_1	D_2	\hat{i}
+0.98-1 =0.02	+0.98+1 = 1.98	1
+1.01-1 =0.01	+1.01+1 =2.01	1
-0.12-1 =1.12	-0.12+1 =0.88	2

A Sequence of M-ary Pulses

Assume that a basic pulse v(t) is used for transmission. The transmitted signal is a linear superposition of time-shifted versions of v(t), i.e.,

$$s(t) = \sum_{k=0}^{\infty} a_k v(t - kT_s),$$

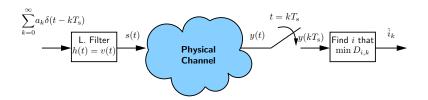
where $a_k \in \mathcal{A}$ where $|\mathcal{A}| = M$.



The Sampling Rx

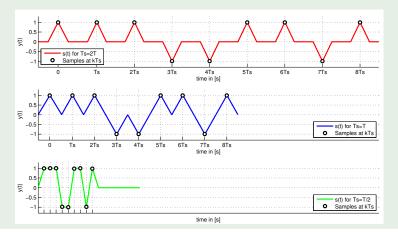
The Sampling Rx simply samples y(t) at time instants $t=kT_{\rm s}$ for $k=0,1,2,\ldots$, each time computing the M distances $D_{i,k}$ and selecting the i that gives the minimum $D_{i,k}$.

$$\hat{i}_k = \arg\min_{i \in \{1,\dots,M\}} D_{i,k}, \ \text{ where } \ D_{i,k} = |y(kT_{\mathrm{s}}) - s_i(0)|.$$



Example (Sampling Rx for binary transmission with triangular pulses)

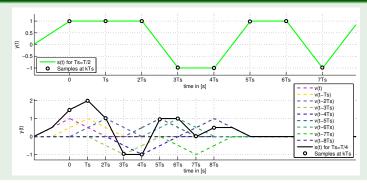
The sequence $\mathbf{b}=[1,1,1,0,0,1,1,0,1]$ is transmitted in a perfect channel using a triangular pulse of width T. The symbol rate $R_{\rm s}=1/T_{\rm s}$ is increased...



Increasing and increasing $R_{\rm s}$ forever?

In the previous example we increased $R_{\rm s}$ for a fixed BW (fixed T) and the Sampling Rx gave good results. What happens if we keep increasing it?

Example (Previous example with $T_{\rm s}=T/4$)



Intersymbol Interference

Intersymbol interference (ISI) appears if $R_{\rm s}$ is too high, i.e., when detecting the kth symbol, interference from other adjacent symbols appear.

Definition (Nyquist pulses)

If the pulse is chosen such that $v(kT_{\rm s})=0$ if $k=\pm 1,\pm 2,\pm 3,\ldots$ and $v(0)\neq 0$, the pulse is called "Nyquist pulse".

Consequences

- $y(kT_{\rm s})$ depends only on the kth transmitted symbol (since the shifted versions of v(t) coming from other symbols are zero at that time instant)
- No ISI for the Sampling Rx!
- In a perfect channel $y(kT_s) = a_k v(0) \Rightarrow D_k = 0$

Poisson's Sum Formula

$$\sum_{n=-\infty}^{\infty} V\left(f - \frac{n}{T_{\rm s}}\right) = T_{\rm s} \sum_{m=-\infty}^{\infty} v(mT_{\rm s}) e^{-\jmath 2\pi f m T_{\rm s}}$$

Definition (Nyquist pulses in the frequency domain)

A pulse v(t) is called a Nyquist pulse if and only if its Fourier transform V(f) fulfills the following two conditions

$$\sum_{n=-\infty}^{\infty} \Re \left\{ V \left(f - \frac{n}{T_{\rm s}} \right) \right\} = T_{\rm s} v(0)$$
$$\sum_{n=-\infty}^{\infty} \Im \left\{ V \left(f - \frac{n}{T_{\rm s}} \right) \right\} = 0,$$

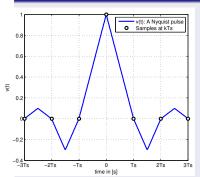
where $T_{\rm s}v(0)$ is a real constant.

A particular case

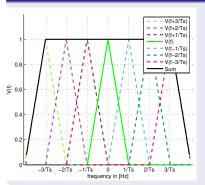
If the pulse is symmetric respect to zero, the previous definition is simplified to

$$\sum_{n=-\infty}^{\infty} V\left(f - \frac{n}{T_{\rm s}}\right) = T_{\rm s}v(0)$$

A Nyquist Pulse in t-domain



A Nyquist Pulse in f-domain



Part II

The Correlator/Matched Filter Rx

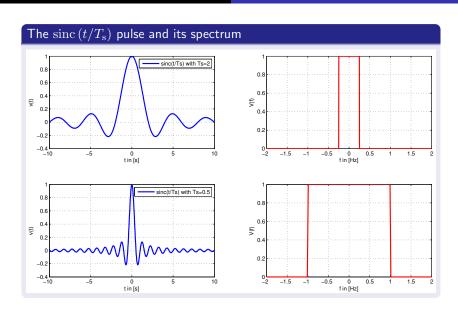
Symbol Rate BW Tradeoff

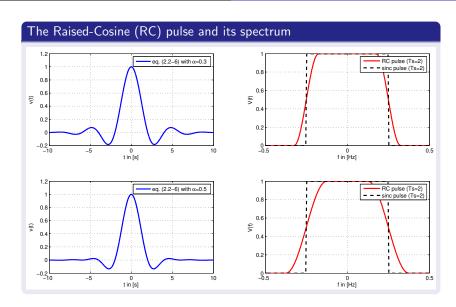
- ullet We know that for a given $R_{
 m s}$, ISI-free transmission is indeed possible
- \bullet We also know that if $R_{\rm s}$ increases ($T_{\rm s}$ decreases), the BW also increases
- ullet We are interested in maximizing $R_{
 m s}$ using the least possible BW
- The interesting question is then: Is there a fundamental limit on the BW for ISI-free transmission?

The answer is Yes, there is a fundamental lower limit

For a given $R_{\rm s}$, ISI-free transmission is possible only for BW $\geq \frac{1}{2T_{\rm s}}$, with equality if and only if $v(t)={
m sinc}\,(t/T_{\rm s})$. In other words

- Many different pulses offer ISI-free transmission at expenses of BW> $\frac{1}{2T_0}$
- Only one pulse offers ISI-free transmission for BW= $\frac{1}{2T_{\rm s}}$, the ${
 m sinc}\,(t/T_{\rm s})$ pulse
- ISI-free transmission is **not** possible for BW< $\frac{1}{2T_{\rm s}}$





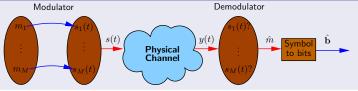
Properties of the Sampling Rx

- Very simple to implement
- Good results for good channel conditions
- Sensitive to bad channel conditions

What do we do if the channel is very noisy?

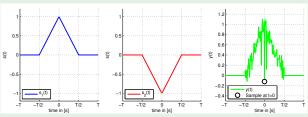
We use a better (but more complex) Rx: The linear Rx

The system we are analyzing...



Example (From the previous Example)

A binary message is transmitted using a triangular pulse. At the Rx, and based on y(t), we need to guess which message was transmitted.

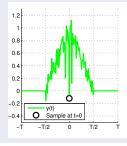


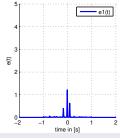
Definition (The error energy)

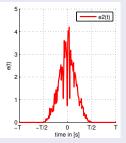
The error between the received signal y(t) and the ith signal alternative is defined as $e_i(t)=|y(t)-s_i(t)|$, and therefore, one can define the error energy as follows

$$E_{e_i} = \int_{-\infty}^{\infty} [y(t) - s_i(t)]^2 dt.$$

For the previous example







Minimizing the error energy

 The error energies can be used for decision ⇒ The problem we need to solve is

$$\hat{i} = \arg\min_{i} \left\{ \int_{-\infty}^{\infty} [y(t) - s_i(t)]^2 dt \right\}$$

It can be shown that

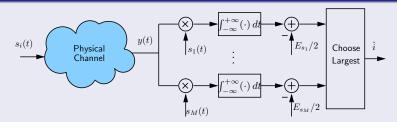
$$\min_i \biggl\{ \int_{-\infty}^\infty [y(t) - s_i(t)]^2 \, dt \biggr\} \equiv \max_i \biggl\{ \int_{-\infty}^\infty y(t) s_i(t) \, dt - \frac{E_{s_i}}{2} \biggr\},$$

where E_{s_i} is the energy of the ith signal alternative, $E_{s_i} = \int_{-\infty}^{\infty} s_i^2(t) dt$.

The problem we need to solve is then

$$\hat{i} = \arg\max_{i} \left\{ \int_{-\infty}^{\infty} y(t) s_{i}(t) dt - \frac{E_{s_{i}}}{2} \right\}$$

The Correlator Rx



Some Comments

• If all the signal alternatives have the same energy $(E_{s_1} = E_{s_2} = \ldots)$, the problem is reduced to

$$\hat{i} = \arg\max_{i} \left\{ \int_{-\infty}^{\infty} y(t) s_i(t) \, dt \right\}$$

• Since the "heart" of the processing are correlation integrals, this Rx is called the **correlator Rx** (CR)

The Correlation Integral

It can be proved that

$$\int_{-\infty}^{\infty} y(t)s_i(t) dt = y(t) * h_i(t) \Big|_{t=0},$$

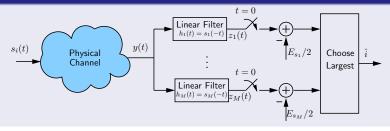
where $h_i(t) = s_i(-t)$, or in frequency domain $H(f) = S^*(f)$.

- Proof?
- The previous result implies that the correlation integrals in the previous block diagram can be replaced by linear filters and sampling at t=0.

The Causality Problem

For a linear filter to be realizable, it must be causal, i.e., if $s_i(t) \neq 0$ for $-T_1 \leq t \leq T_2$ ($T_1, T_2 > 0$), the filter with impulse response $h_i(t) = s_i(-t)$ is not causal (not realizable). To have a causal filter the impulse response must be shifted T_2 [s], i.e., $h_i(t) = s_i(-t + T_2)$. The corresponding samples will be delayed in T_2 [s].

The Matched Filter Rx



Some Comments

- Since the filters are matched to the signal alternatives, it is called the matched filter Rx (MFR)
- Both the correlator Rx (CR) and MFR can be implemented using linear filters and are therefore called linear Rx (LR)
- In terms of performance, the CR and the MFR are equivalent
- However, the performance of the CR and MFR is different than the performance of Sampling Rx

One M-ary Pulse

The transmitted signal is

$$s(t) = a_0 v(t)$$

where $a_0 \in \mathcal{A}$ is the amplitude of the transmitted symbol, and $E_{\rm v}$ is the energy of the pulse v(t).

The MF Rx for this case?

Example (MF Rx for 4-ary transmission)

Assume $\mathcal{A}=\{-3,-1,1,3\}$ and a channel which is perfect. Why do we need the gain $1/E_{\rm v}$?

A sequence of M-ary pulses

The transmitted signal is

$$s(t) = \sum_{k=0}^{\infty} a_k v(t - kT_s)$$

= $a_0 v(t) + a_1 v(t - T_s) + a_2 v(t - 2T_s) + \dots,$

where $a_k \in \mathcal{A}$ represent the amplitude of the symbol transmitted at the kth time instant.

The MF Rx for this case?

HW for L₃!!

Today's Summary

- Sampling Rx (one pulse and a train of pulses)
- Nyquist pulses. No ISI for Sampling Rx
- Rate and BW tradeoff
- Sinc pulses and RC pulses
- MFR/CR (one pulse and a train of pulses)
- Scaling by $1/E_{
 m v}$