Exercise 1

Aug. 30, 2023

Problem 1 (LTI Systems)

Consider an LTI system whose response to the signal $x(t)$ in Figure 1(a) is the signal $y(t)$ in Figure 1(b). Sketch the response of the system to the input signal $z(t)$ shown in Figure 1(c).

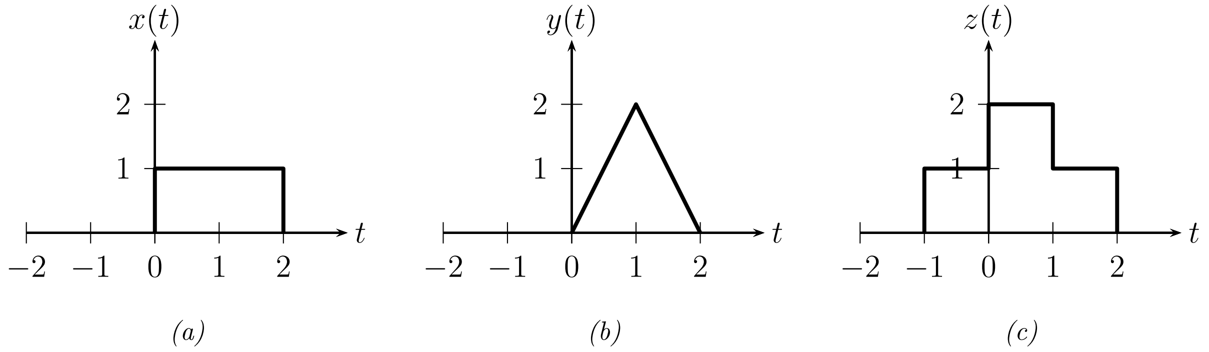

Figure 1: Problem 1.

The signal $z(t)$ can be represented as a linear combination of $x(t)$ and its shifted version $x(t+1)$, i.e.,

$$z(t) = x(t+1) + x(t).$$

Using properties of LTI systems, we conclude that the response of the LTI system to $z(t)$ is also a linear combination of $y(t)$ and its shifted version $y(t+1)$, i.e.,

$$r(t) = y(t+1) + y(t).$$

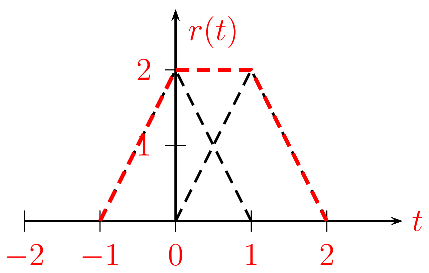

Figure 2: The output of the LTI system.

Problem 2 (Fourier Transform Properties)

Find the Fourier transform of the signal $x(t)$ shown in Figure 2, in two ways as mentioned below.

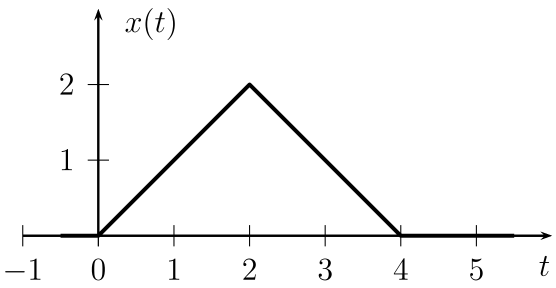

Figure 3: Problem 2.

1. Calculate the Fourier transform using the definition.

$$
\begin{aligned}
X(f) &= \int_{-\infty}^{\infty} x(t)\mathrm{e}^{-j2\pi ft}\,\mathrm{d}t \\
&= \int_{0}^{2} t\mathrm{e}^{-j2\pi ft}\,\mathrm{d}t + \int_{2}^{4}(4-t)\mathrm{e}^{-j2\pi ft}\,\mathrm{d}t \\
&= \int_{-2}^{0}(\tau+2)\mathrm{e}^{-j2\pi f(\tau+2)}\,\mathrm{d}\tau + \int_{0}^{2}(2-\tau)\mathrm{e}^{-j2\pi f(\tau+2)}\,\mathrm{d}\tau \\
&= \mathrm{e}^{-j4\pi f}\left[\int_{-2}^{0}\tau\mathrm{e}^{-j2\pi f\tau}\,\mathrm{d}\tau - \int_{0}^{2}\tau\mathrm{e}^{-j2\pi f\tau}\,\mathrm{d}\tau + 2\int_{-2}^{2}\mathrm{e}^{-j2\pi f\tau}\,\mathrm{d}\tau\right] \\
&= \mathrm{e}^{-j4\pi f}\left[\left.\frac{\tau\mathrm{e}^{-j2\pi f\tau}}{-j2\pi f}\right|_{-2}^{0} - \left.\frac{\mathrm{e}^{-j2\pi f\tau}}{(j2\pi f)^2}\right|_{-2}^{0} - \left.\frac{\tau\mathrm{e}^{-j2\pi f\tau}}{-j2\pi f}\right|_{0}^{2} + \left.\frac{\mathrm{e}^{-j2\pi f\tau}}{(j2\pi f)^2}\right|_{0}^{2} + 2\left.\frac{\mathrm{e}^{-j2\pi f\tau}}{-j2\pi f}\right|_{-2}^{2}\right] \\
&= \mathrm{e}^{-j4\pi f}\left[\frac{\mathrm{e}^{-j4\pi f}+\mathrm{e}^{j4\pi f}-2}{(j2\pi f)^2}\right] \\
&= \mathrm{e}^{-j4\pi f}\left[\frac{(\mathrm{e}^{-j2\pi f}-\mathrm{e}^{j2\pi f})^2}{(j2\pi f)^2}\right] \\
&= \mathrm{e}^{-j4\pi f}\left[\frac{(\mathrm{e}^{j2\pi f}-\mathrm{e}^{-j2\pi f})^2}{(j2\pi f)^2}\right] \\
&= \mathrm{e}^{-j4\pi f}\left[\frac{\mathrm{e}^{j2\pi f}-\mathrm{e}^{-j2\pi f}}{j2\pi f}\right]^2 \\
&= \mathrm{e}^{-j4\pi f}\left[\frac{\sin(2\pi f)}{\pi f}\right]^2 \\
&= 4\mathrm{e}^{-j4\pi f}\left[\frac{\sin(2\pi f)}{2\pi f}\right]^2 \\
&= 4\mathrm{e}^{-j4\pi f}\mathrm{sinc}^2(2f).
\end{aligned}
$$

2. Use the Fourier transform properties.
   $x(t)$ can be represented as a convolution of the rectangular pulse $y(t) = \mathrm{rect}(\frac{t-1}{2}) = \mathrm{I}\{0 < t < 2\}$ with itself, i.e.,

$$x(t) = y(t) * y(t).$$

The Fourier transform of such a pulse is

$$Y(f) = 2\mathrm{e}^{-j2\pi f}\mathrm{sinc}(2f).$$

The Fourier transform of a convolution is the product of Fourier transforms, i.e.,

$$X(f) = \left(2\mathrm{e}^{-j2\pi f}\mathrm{sinc}(2f)\right)^2 = 4\mathrm{e}^{-4j\pi f}\mathrm{sinc}^2(2f).$$

Problem 3 (Fourier)

Let $x(t)$ be a signal that is band-limited to W.

1. Show that if $f > W$, then

$$\int_{-\infty}^{\infty} x(t)\cos(2\pi ft)\ \mathrm{d}t = \int_{-\infty}^{\infty} x(t)\sin(2\pi ft)\ \mathrm{d}t = 0.$$

The fact that the signal is band-limited means that

$$X(f) = 0 \text{ for } |f| > W.$$

This can be rewritten as

$$\int_{-\infty}^{\infty} x(t)\mathrm{e}^{-j2\pi ft}\ \mathrm{d}t = 0 \text{ for } |f| > W.$$

Using Euler's equation for the complex exponential we get

$$\int_{-\infty}^{\infty} x(t)(\cos(2\pi ft) - j\sin(2\pi ft))\ \mathrm{d}t = 0 \text{ for } |f| > W$$

or

$$\int_{-\infty}^{\infty} x(t)\cos(2\pi ft)\ \mathrm{d}t - j\int_{-\infty}^{\infty} x(t)\sin(2\pi ft)\ \mathrm{d}t = 0 \text{ for } |f| > W.$$

A complex number is zero iff the real part and the imaginary part are both zero, which implies

$$\int_{-\infty}^{\infty} x(t)\cos(2\pi ft)\ \mathrm{d}t = \int_{-\infty}^{\infty} x(t)\sin(2\pi ft)\ \mathrm{d}t = 0 \quad \text{for } |f| > W.$$

Interpretation: the inner product of a slowly varying signal and a harmonic signal of high frequency is zero.

2. Show that if $f > W/2$, then

$$\int_{-\infty}^{\infty} x(t)\cos^2(2\pi ft)\ \mathrm{d}t = \frac{1}{2}\int_{-\infty}^{\infty} x(t)\ \mathrm{d}t.$$

Using trigonometric identities, we can express cosine square as

$$\cos^2(2\pi ft) = \frac{1}{2} + \frac{1}{2}\cos(4\pi ft).$$

The rest of the proof is similar to the previous item.

Problem 4 (Fourier)

Prove that

$$\text{sinc}(2Wt)\cos(2\pi Wt) = \text{sinc}(4Wt).$$

$$\begin{aligned}\text{sinc}(2Wt)\cos(2\pi Wt) &= \frac{\sin(2\pi Wt)\cos(2\pi Wt)}{2\pi Wt} \\ &= \frac{\sin(2\pi Wt + 2\pi Wt) + \sin(2\pi Wt - 2\pi Wt)}{2 \cdot 2\pi Wt} \\ &= \frac{\sin(4\pi Wt)}{4\pi Wt} = \text{sinc}(4Wt).\end{aligned}$$

Illustrate this identity in the frequency domain.

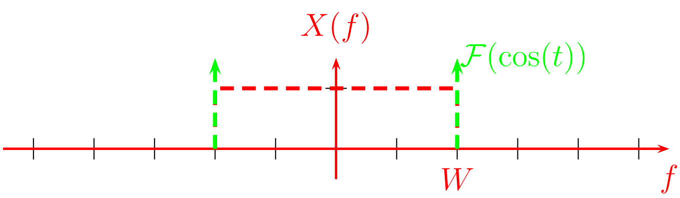

Figure 4: The signals in the frequency domain.

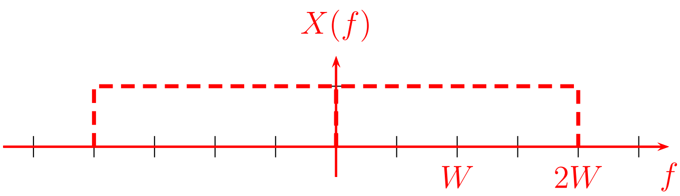

Figure 5: The result of the convolution.

Problem 5 (Nyquist pulse)

Let $v(t)$ be a continuous signal with limited energy, i.e., $\int_{-\infty}^{\infty} v^2(t)\,\mathrm{d}t < \infty$ and $v(0) = 1$. Define $g(t) = v(t)\text{sinc}(t/T)$.

1. Show that $g(t)$ is a Nyquist pulse for the time interval T.

$$g(0) = v(0)\text{sinc}(0) = 1 \text{ and}$$

$$g(kT) = v(kT)\text{sinc}(k) = v(kT)0 = 0, \text{for all } k \neq 0.$$

This means that the pulse satisfies the Nyquist criterion in the time domain.

2. Argue that the raised-cosine pulse is a Nyquist pulse.
   The raised-cosine pulse is defined in the time domain as
   $$r(t) = \text{sinc}\left(\frac{t}{T}\right)\frac{\cos\left(\frac{\pi\beta t}{T}\right)}{1-\frac{4\beta^2t^2}{T^2}}.$$
   Since $r(t)$ is of form $v(t)\text{sinc}(t/T)$, it satisfies the Nyquist criterion.

3. Find the Fourier transform $G(f)$ as a function of $V(f)$ and show that it satisfies the Nyquist criterion in the frequency domain.
   The Fourier transform of $g(t)$ is given by the convolution of $V(f)$ with a rectangular pulse $T\mathrm{I}\left\{-\frac{1}{2T} < f < \frac{1}{2T}\right\}$.
   $$\begin{aligned} G(f) &= \int_{-\infty}^{\infty} V(\tau)T\,\mathrm{I}\left\{-\frac{1}{2T} < f-\tau < \frac{1}{2T}\right\}\,\mathrm{d}\tau \\ &= T\int_{-\infty}^{\infty} V(\tau)\,\mathrm{I}\left\{f-\frac{1}{2T} < \tau < f+\frac{1}{2T}\right\}\,\mathrm{d}\tau \\ &= T\int_{f-\frac{1}{2T}}^{f+\frac{1}{2T}} V(\tau)\,\mathrm{d}\tau. \end{aligned}$$
   Let us check the Nyquist criterion in the frequency domain.
   $$\begin{aligned} \sum_{k=-\infty}^{\infty} G\left(f-\frac{k}{T}\right) &= \sum_{k=-\infty}^{\infty} T\int_{f-\frac{k}{T}-\frac{1}{2T}}^{f-\frac{k}{T}+\frac{1}{2T}} V(\tau)\,\mathrm{d}\tau \\ &= T\int_{-\infty}^{\infty} V(\tau)\,\mathrm{d}\tau \\ &= T\int_{-\infty}^{\infty} V(\tau)\mathrm{e}^{j2\pi t\tau}\,\mathrm{d}\tau\Big|_{t=0} \\ &= Tv(0) = T, \end{aligned}$$
   i.e., the signal satisfies the Nyquist criterion in the frequency domain.

Problem 6 (Nyquist Pulse)

The pulses are defined in frequency domain and their spectra are shown in the below figure (frequency is in MHz).

$$X_1(f) = \begin{cases} 2-0.5|f| & \text{if } |f| \le 4 \\ 0 & \text{o.w.} \end{cases} \qquad X_2(f) = \begin{cases} 2 & \text{if } 0 \le f \le 3 \\ -2 & \text{if } -3 \le f < 0 \end{cases}$$

1. Which pulse(s) satisfy the Nyquist criterion and for which symbol rate?
   $x_1(t)$ satisfies the Nyquist criterion for symbol rate $R_1 = 4$ MHz. However, $x_2(t)$ does not satisfy the Nyquist criterion, because although for $R_2 = 3$ MHz, the sum of shifted versions of $X_2(f)$ is a constant, it is equal to zero, meaning that $x_2(kT) = 0$ for all k, while we need to have $x_2(kT) = 0$ for $k \neq 0$ in the Nyquist criterion.

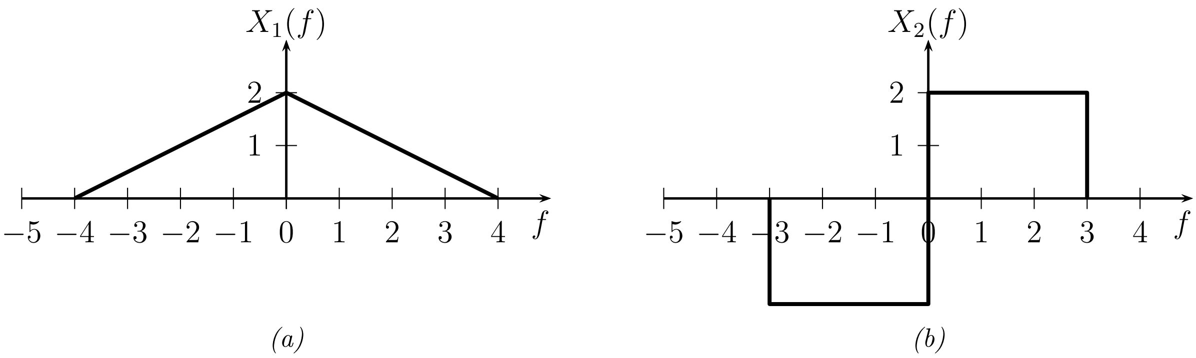

Figure 6: Problem 6.

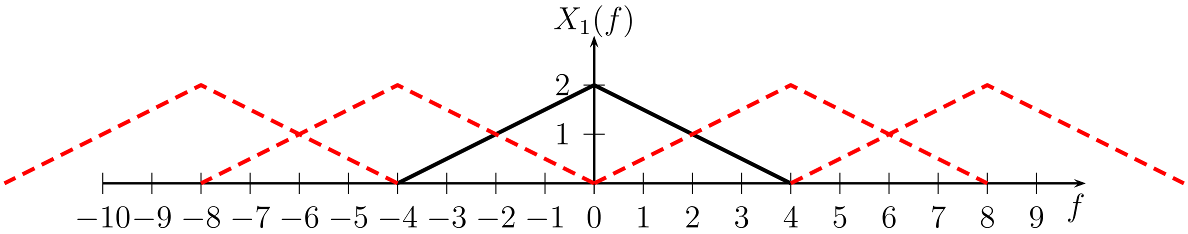

Figure 7

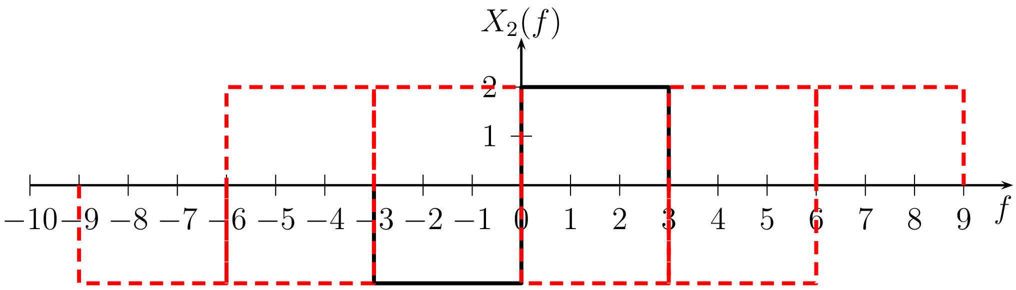

Figure 8

2. Find the value at $t = 0$ and the energy for these signals.

- For the Nyquist pulse $x_1(t)$, one can write

$$\sum_{k=-\infty}^{\infty} X_1(f - \frac{k}{T_1}) = T_1 x_1(0).$$

  Therefore,

$$x_1(0) = \frac{\sum_{k=-\infty}^{\infty} X_1(f - \frac{k}{T_1})}{T_1} = R_1 \sum_{k=-\infty}^{\infty} X_1(f - \frac{k}{T_1}).$$

  To obtain the energy, we use the parseval's theorem:

$$E_1 = \int_{-\infty}^{\infty} x_1^2(t)\mathrm{d}t = \int_{-\infty}^{\infty} X_1^2(f)\mathrm{d}f$$

- Since $x_2(t)$ is not a Nyquist pulse, we need to find the value of the signal at $t = 0$

in another way. We know that

$$x_2(t) = \int_{-\infty}^{\infty} X_2(f)\mathrm{e}^{j2\pi ft}\mathrm{d}f.$$

Therefore

$$x_2(0) = \int_{-\infty}^{\infty} X_2(f)\mathrm{d}f = 0.$$

For the energy, we can still use parseval's theorem write

$$E_2 = \int_{-\infty}^{\infty} X_2^2(f)\mathrm{d}f.$$