Examination Wed. Oct. 26, 2022, 8:30–12:30

SSY121 Introduction to Communication Engineering

- Contact person: Examiner Fredrik Brännström (070 – 872 1685) will visit the exam after approximately 1 and 3 hours.
- Instructions:
  - Write in English.
  - Use a pencil and eraser.
  - There is no page limit. Extra sheets of paper are available.
  - Solve the problems in any order (they are not ordered by difficulty).
  - Before handing in, sort the pages in problem order. Label each page with problem number and running page number. Do not hand in drafts or unused paper.
  - If any data is missing, make reasonable assumptions.
  - Chalmers' examination rules applies.
  - MP3/Music players **are not** allowed during the exam
- Allowed aids:
  - Chalmers-approved calculator
- Grading principles:
  - Explain the line of reasoning clearly. A good solution with a minor error usually gives close to full points, even if the answer is incorrect.
  - An answer without a clear motivation usually gives 0 points, even if it is correct.
  - Whenever possible, check if your answer is reasonable. An obviously unreasonable answer usually gives 0 points, even if the solution is almost correct.
- Solutions and results:
  - Solutions will be posted on the course website no later than 7 days after the exam.
  - The time and place for the exam review will be announced on the course website in Canvas.

1. **True or false statements:** [Total points: 8]
Justify *ALL* your answers using short and concise explanations (maximum 30 words per item).

(a) The only pulse that offers inter-symbol-interference (ISI) free transmission for a one-sided bandwidth (BW) $\geq \frac{1}{2T_s}$ (T_s is symbol duration) is the $\text{sinc}(t/T_s)$ pulse. [1 p]

(b) When the prior probabilities are available, for any channel, the maximum a posteriori (MAP) detector is the optimum detector in terms of minimizing the symbol error probability. [1 p]

(c) To design the optimal receiver (with minimum number of matched filters) for a digital communication system operating in an AWGN channel, the required number of matched filters is the same for both 32-QAM and 8-PSK signaling schemes. [1 p]

(d) For a communication system with limited power usage, operating in an AWGN channel, the 16-QAM constellation is a better choice than 16-PSK constellation. [1 p]

(e) In an AWGN channel with equi-probable 16-QAM signaling, the symbol error rate for an envelope detector is 0.75. (The envelope detector decides only based on the amplitude/magnitude of the received signal.) [1 p]

(f) The high SNR approximation of the union bound gives a lower bound on the symbol error probability. [1 p]

(g) Phase discontinuities in the frequency-shift keying (FSK) signals is the source of unwanted large spectral lobes, which can be avoided using continuous phase modulations (CPM). [1 p]

(h) In code-division multiple access (CDMA), different users are allotted small portions of the available bandwidth and each user transmits only in the allotted bandwidth without interfering in other user's bandwidth. [1 p]

2. **Constellations, Decision Region, SER:** [Total points: 12]
You will study the performance of different 4-ary constellations with equally likely symbols (shown in Figure 1), where the observation is given by

$$r = \sqrt{E_s} s + n,$$

where n is complex Gaussian noise with variance $N_0/2$ for both the real and the imaginary part, and $\mathbb{E}\left\{|s|^2\right\} = 1$.

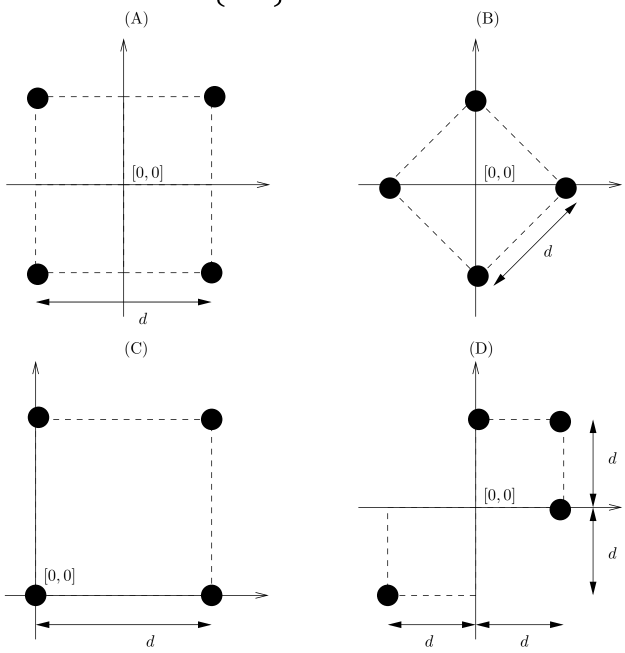

Figure 1: Four 4-ary constellations with minimum Euclidean distance d. Note that d is different from constellation to constellation.

(a) For every constellation (A), (B), (C), (D), determine d (the minimal Euclidean distance) and K (the number of neighbors at distance d). [2 p]

(b) What are the decision regions for maximum likelihood detection? You can simply draw them in a figure, making sure you mark the axes, mark the decisions made for every region, and draw the figure to scale. From the maximum likelihood criterion, derive the decision rule for constellation (C). Verify that this decision rule corresponds to the decision regions you have drawn. [2 p]

(c) For every constellation, determine an approximation of the average symbol error probability for high SNR, based on d and K, as a function of the SNR E_s/N_0. Do all symbols have the same symbol error probability? [2 p]

(d) Which of the four constellations gives the lowest symbol error probabilty for high SNR? Why? [2 p]

(e) Assume the receiver observes

$$r = e^{j\theta} \sqrt{E_s} s + n,$$

where θ is unknown. The receiver is not aware of θ, and applies maximum likelihood detection as if there were no phase θ. How does the symbol error probability change with θ? Are all constellations equally affected? Are all symbols within a constellation equally affected? [4 p]

3. **Decision Rule, Estimation** [Total points: 4]

   Suppose that you wish to detect whether the binary random variable X is 0 or 1 when $Y \sim \mathcal{N}(1,4)$ for $X = 1$ and $Y \sim \mathcal{N}(-1,1)$ for $X = 0$, where $\mathcal{N}(\mu,\sigma^2)$ denotes a Gaussian distribution with mean μ and variance σ^2.

   Let y denote the observation of Y and show that the optimal decision rule can be written as

   $$y^2 + ay \underset{\hat{x}=1}{\overset{\hat{x}=0}{\lessgtr}} b$$

   and determine the constants a and b. [4 p]

4. **Link Budget:** [Total points: 5]

   Assume you work in a telecommunication company. Your task is to deploy base stations in Gothenburg, so users that are closer than d meters from the base station can have a satisfactory downlink transmission (transmission from base station to user) with a minimum bit rate of $R_\mathrm{b} = 5$ Mbits/s and a maximum bit error rate of $P_\mathrm{e} = 10^{-4}$. The transmission uses Gray-coded 4QAM, an RRC pulse with a roll-off factor of $\alpha = 0.16$, and a carrier frequency of $f_\mathrm{c} = 2.5$ GHz. Assume an AWGN channel with noise power spectral density $N_0 = 2 \times 10^{-17}$ W/Hz. Both the transmitter and the receiver have a antenna gain $G_\mathrm{T} = G_\mathrm{R} = 3$ dB (≈ 2). Answer the following questions.

   (a) What is the minimal received power P_R at the user side to achieve the downlink quality? [3 p]

   Hint 1: The BER for the Gray-coded 4QAM is given by $Q\left(\sqrt{\frac{E_\mathrm{s}}{N_0}}\right)$.
   Hint 2: $P_R = E_s B$, *where* B *is the signal bandwidth.*

   (b) Assume the required receiver power is $P_\mathrm{R} = 10^{-9}$ W, the transmitter power is $P_\mathrm{T} = 5$ W, and that the free-space path loss model applies. What is the maximum distance, d, between the base station and the user to achieve the downlink quality? [2 p]

5. **Gram-Schmidt:** [Total points: 7]

The four waveforms shown in Figure 2 are used for communication of four equiprobable messages over an AWGN channel.

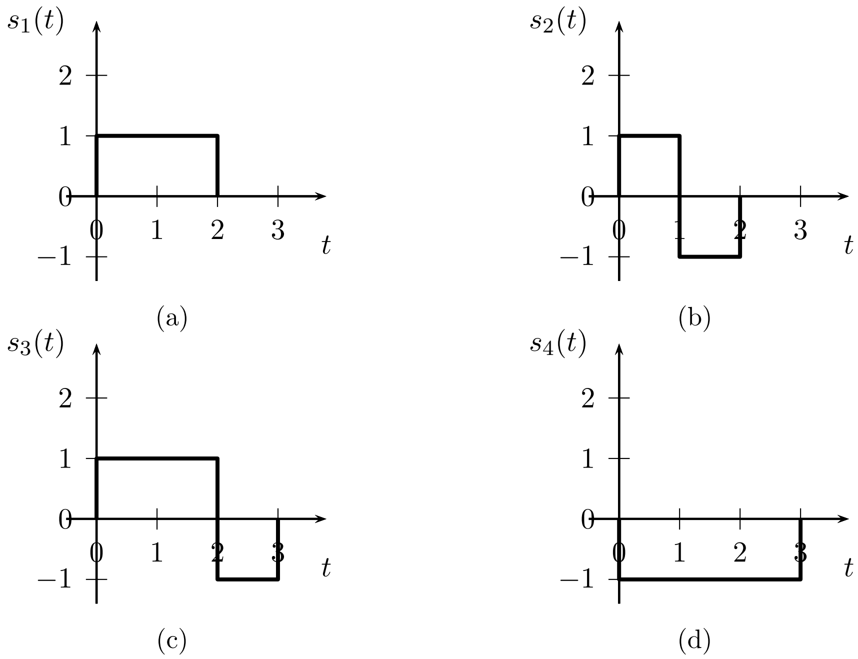

Figure 2: Signals in Problem 5

(a) Find a minimal set of orthogonal basis for the set $\{s_1(t), s_2(t), s_3(t), s_4(t)\}$. *Hint: use the Gram-Schmidt procedure.* [4 p]

(b) Express the signals $\{s_1, s_2, s_3, s_4\}$ as a linear combination of the basis functions found in (a). Draw a constellation diagram of $\{s_1, s_2, s_3, s_4\}$. *Note: The constellation diagram can be three-dimensional.* [3 p]

6. **Nyquist, ISI:** [Total points: 6]

Consider a communication system using a pulse $v(t) = \text{sinc}(at)\,\text{sinc}(bt)$, where a and b are unknowns and $a \geq b$. Answer the following questions.

(a) Sketch the frequency domain response $V(f)$ of the pulse $v(t)$ and mark important levels on both axes in terms of a and b. *Hint: A multiplication in the time domain equals a convolution in the frequency domain.* [2 p]

(b) Suppose that the pulse is to be used over an ideal baseband channel with one-side bandwidth 300 Hz. Choose a and b so that the pulse is Nyquist for 8-PSK signaling at 1500 bits/s and exactly fills the channel bandwidth. [2 p]

(c) Now, Suppose that the pulse is to be used over a passpand channel spanning the frequencies 2.40 GHz to 2.42 GHz. Assuming that we use 64 QAM signaling with a bit rate of $R_b = 60$ Mbits/s, choose a and b so that the pulse is Nyquist and exactly fills the channel bandwidth. [2 p]

7. **Synchronization:** [Total points: 6]

A manufacturer produces a communication system with an imperfect phase synchronizer, which causes a phase error of ϕ. The product is designed using BPSK modulation format. Assume an AWGN channel with noise spectral density $N_0/2$. Answer the following questions.

(a) Find the maximum phase error ϕ that the product can tolerate, to achieve a bit error probability $P_e = 10^{-5}$ at $E_b/N_0 = 10$ dB. [3 p]

(b) Assume the manufacturer produces two products (product A and B) with different phase errors $\phi_A = \pi/3$ and $\phi_B = \pi/6$, respectively. To achieve the same bit error probability, how much power in dB do we need to increase the power of product A (i.e. increase E_b of product A)? [3 p]

Formula sheet, SSY121

Version 2.1, August 10, 2016

This sheet is an allowed aid at written exams in SSY121, Introduction to Communication Engineering, at Chalmers in 2016. It will be handed out with the exam problems. Students may not bring their own copy.

Decibels

$$\left(\frac{E_1}{E_2}\right)_{\mathrm{dB}} = 10\log_{10}\frac{E_1}{E_2}$$

Energies E_{s} and E_{b}

$$E_{\mathrm{s}} = \sum_{i=1}^{M} \mathbb{P}\left[\mathsf{S} = \mathsf{s}_i\right] ||\mathsf{s}_i||^2$$

$$E_{\mathrm{s}} = E_{\mathrm{b}} \log_2 M$$

Normalized minimum distance

$$d_{\min} = \frac{D_{\min}}{\sqrt{2E_{\mathrm{b}}}}$$

Nyquist criterion

- In time domain
$$v(nT_{\mathrm{s}}) = 0, \quad n = \pm 1, \pm 2, \ldots$$
- In frequency domain
$$\sum_{n=-\infty}^{\infty} \Re\left\{V\left(f - \frac{n}{T_{\mathrm{s}}}\right)\right\} = T_{\mathrm{s}} v(0)$$
$$\sum_{n=-\infty}^{\infty} \Im\left\{V\left(f - \frac{n}{T_{\mathrm{s}}}\right)\right\} = 0,$$
where $T_{\mathrm{s}}v(0)$ is a real constant.
- If the $v(t)$ is symmetric respect to zero, the definition in frequency domain is
$$\sum_{n=-\infty}^{\infty} V\left(f - \frac{n}{T_{\mathrm{s}}}\right) = T_{\mathrm{s}} v(0)$$

T_{s}-orthogonality

- In time domain
$$\int_{-\infty}^{\infty} v(t)v(t - nT_{\mathrm{s}})dt = 0, \quad n = \pm 1, \pm 2, \ldots$$
- In frequency domain
$$\sum_{n=-\infty}^{\infty} \left|V\left(f - \frac{n}{T_{\mathrm{s}}}\right)\right|^2 = T_{\mathrm{s}} E_v$$

Sinc, Raised-cosine, and Root raised-cosine pulses

$$v_{\mathrm{sinc}}(t) = \mathrm{sinc}(t/T_{\mathrm{p}}) = \frac{\sin(\pi t/T_{\mathrm{p}})}{\pi t/T_{\mathrm{p}}}$$

$$V_{\mathrm{sinc}}(f) = \begin{cases} T_{\mathrm{p}}, & |f| < \frac{1}{2T_{\mathrm{p}}} \\ 0, & |f| \geq \frac{1}{2T_{\mathrm{p}}} \end{cases}$$

$$v_{\mathrm{RC}}(t) = \mathrm{sinc}\left(\frac{t}{T_{\mathrm{p}}}\right) \frac{\cos\left(\frac{\pi\alpha t}{T_{\mathrm{p}}}\right)}{1 - \left(\frac{2\alpha t}{T_{\mathrm{p}}}\right)^2}$$

$$V_{\mathrm{RC}}(f) = \begin{cases} T_{\mathrm{p}}, & |f| < f_1 \\ \frac{T_{\mathrm{p}}}{2}\left(1 + \cos\left[\frac{\pi T_{\mathrm{p}}}{\alpha}\left(|f| - \frac{1-\alpha}{2T_{\mathrm{p}}}\right)\right]\right), & f_1 \leq |f| < f_2 \\ 0, & |f| \geq f_2, \end{cases}$$

where $f_1 = \dfrac{1-\alpha}{2T_{\mathrm{p}}}$ and $f_2 = \dfrac{1+\alpha}{2T_{\mathrm{p}}}$

$$v_{\mathrm{RRC}}(t) = \sqrt{T_{\mathrm{p}}} \frac{\sin\left(\frac{(1-\alpha)\pi t}{T_{\mathrm{p}}}\right) + \frac{4\alpha t}{T_{\mathrm{p}}}\cos\left(\frac{(1+\alpha)\pi t}{T_{\mathrm{p}}}\right)}{\pi t\left(1 - \left(\frac{4\alpha t}{T_{\mathrm{p}}}\right)^2\right)}$$

$$V_{\mathrm{RRC}}(f) = \sqrt{V_{\mathrm{RC}}(f)}$$

Correlation receiver

$$\min_i \left\{\int_{-\infty}^{\infty} [y(t) - s_i(t)]^2 \, dt\right\}$$
$$\max_i \left\{\int_{-\infty}^{\infty} y(t)s_i(t) \, dt - \frac{E_{s_i}}{2}\right\},$$

where $E_{s_i} = \int_{-\infty}^{\infty} s_i^2(t) \, dt$.

PAM (baseband)

$$s(t) = \sum_{k=0}^{\infty} a_k v(t - kT_{\mathrm{s}}),$$

where $a_k \in \{\pm(M-1), \pm(M-3), \ldots, \pm 1\}$.

PAM (passband)

$$s(t) = \sum_{k=1}^{\infty} a_k v(t - kT_{\mathrm{s}})\sqrt{2}\cos w_c t,$$

where $a_k \in \{\pm(M-1), \pm(M-3), \ldots, \pm 1\}$

2D Modulations

$$s(t) = \sum_{k=0}^{\infty} a_k v(t - kT_{\mathrm{s}})\sqrt{2}\cos(w_c t) - \sum_{k=0}^{\infty} b_k v(t - kT_{\mathrm{s}})\sqrt{2}\sin(w_c t)$$

M-PSK

$$s(t)=\sum_{k=0}^{\infty}v(t-kT_{\mathrm{s}})\sqrt{2}\cos\left(w_c t+\frac{2i\pi}{M}\right),$$

where $i=0,1,\ldots,M-1$.

M-FSK

$$s_i(t)=\cos\left(2\pi\left[f_c+\frac{h}{2T_{\mathrm{s}}}i\right]t\right),$$

where $i=\pm(M-1),\pm(M-3),\ldots,\pm 1$.

Link budget

$$P_{\mathrm{R}}=P_{\mathrm{T}}G_{\mathrm{T}}G_{\mathrm{R}}\left(\frac{\lambda}{4\pi d}\right)^2,$$

where $c=\lambda f=3\cdot 10^8\mathrm{m/s}$.

Parabolic dish antenna

$$G_{\mathrm{Par}}=\frac{4\pi A}{\lambda^2}$$

1D Gaussian PDF (i.i.d.)

$$f_X(x)=\frac{1}{\sqrt{2\pi\sigma^2}}\exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right)$$

ND Gaussian PDF with variance σ^2 (i.i.d.)

$$f_{\mathsf{Z}}(\mathsf{z})=\frac{1}{(2\pi\sigma^2)^{N/2}}\exp\left(-\frac{||\mathsf{z}-\boldsymbol{\mu}||^2}{2\sigma^2}\right)$$

Bayes' rule

$$f_{\mathsf{X|Y}}(\mathsf{x|y})=\frac{f_{\mathsf{X}}(\mathsf{x})}{f_{\mathsf{Y}}(\mathsf{y})}f_{\mathsf{Y|X}}(\mathsf{y|x})$$

Additive white Gaussian noise (AWGN)

The following formulas are for the AWGN channel $\mathsf{Y}=\mathsf{S}+\mathsf{Z}$, where S is the transmitted symbol and Z is Gaussian noise.

Maximum likelihood (ML) detection

$$\max_i\left\{\mathbb{P}\left[\mathsf{Y}=\mathsf{y}|\mathsf{S}=\mathsf{s}_i\right]\right\}$$

Maximum a posteriori (MAP) detection

$$\max_i\left\{\mathbb{P}\left[\mathsf{S}=\mathsf{s}_i|\mathsf{Y}=\mathsf{y}\right]\right\}\equiv\max_i\left\{\mathbb{P}\left[\mathsf{S}=\mathsf{s}_i\right]\mathbb{P}\left[\mathsf{Y}=\mathsf{y}|\mathsf{S}=\mathsf{s}_i\right]\right\}$$

Pairwise error probability (PEP)

$$\mathrm{PEP}^{(i,j)}=\mathrm{Q}\left(\sqrt{\frac{D_{i,j}^2}{2N_0}}\right),$$

where $D_{i,j}^2=\|\mathsf{s}_j-\mathsf{s}_i\|^2$.

Symbol error probability (SEP) (exact)

$$P_{\mathrm{e}}=\sum_{i=1}^{M}\mathbb{P}\left[\mathsf{S}=\mathsf{s}_i\right]\sum_{j\neq i}\mathbb{P}\left[\mathsf{Y}\in\mathcal{R}_j|\mathsf{S}=\mathsf{s}_i\right]$$

SEP (union bound)

$$\begin{aligned}P_{\mathrm{e}}&\leq\sum_{i=1}^{M}\mathbb{P}\left[\mathsf{S}=\mathsf{s}_i\right]\sum_{j\neq i}\mathrm{PEP}^{(i,j)}\\&=\sum_{i=1}^{M}\mathbb{P}\left[\mathsf{S}=\mathsf{s}_i\right]\sum_{j\neq i}\mathrm{Q}\left(\sqrt{\frac{D_{i,j}^2}{2N_0}}\right)\end{aligned}$$

SEP (High-SNR approximation for equally likely symbols)

$$P_{\mathrm{e}}\approx\frac{2K}{M}\cdot\mathrm{Q}\left(\sqrt{\frac{D_{\min}^2}{2N_0}}\right),$$

where K is the number of distinct signal pairs with distance $D_{\min}=\min_{i\neq j}\{D_{i,j}\}$ and M is the constellation size.

Bit error probability (BEP) (exact)

$$B_{\mathrm{e}}=\sum_{i=1}^{M}\mathbb{P}\left[\mathsf{S}=\mathsf{s}_i\right]\sum_{j\neq i}\frac{H_{i,j}}{m}\mathbb{P}\left[\mathsf{Y}\in\mathcal{R}_j|\mathsf{S}=\mathsf{s}_i\right],$$

where $H_{i,j}$ is the Hamming distance (the number of different bits) between the labels of symbols s_i and s_j.

BEP (union bound)

$$B_{\mathrm{e}}\leq\sum_{i=1}^{M}\mathbb{P}\left[\mathsf{S}=\mathsf{s}_i\right]\sum_{j\neq i}\frac{H_{i,j}}{m}\mathrm{Q}\left(\sqrt{\frac{D_{i,j}^2}{2N_0}}\right)$$

BEP (High-SNR approximation for equally likely symbols)

$$B_{\mathrm{e}}\approx\frac{2H_{\min}}{Mm}\cdot\mathrm{Q}\left(\sqrt{\frac{D_{\min}^2}{2N_0}}\right),$$

where $H_{\min}$ is the *total* number of bits differing between signal pairs at minimum distance and $m=\log_2(M)$ is the number of bits per symbol.

Q-function

See also tables in Mathematics Handbook, where $\mathrm{Q}(x)=1-\Phi(x)$

$\mathrm{Q}(x)$	x
10^{-1}	1.2816
10^{-2}	2.3263
10^{-3}	3.0902
10^{-4}	3.7190
10^{-5}	4.2649
10^{-6}	4.7534
10^{-7}	5.1993
10^{-8}	5.6120
10^{-9}	5.9978
10^{-10}	6.3613
10^{-11}	6.7060
10^{-12}	7.0345