Exercise 4 Solution

Oct. 9, 2023

Problem 1 (MSK)

Let T be a positive constant. Assume a binary frequency shift keying (FSK) modulation, i.e., the following signals are used to transmit 0 and 1:

$$s_0(t) = \begin{cases} \sqrt{\frac{2E}{T}}\cos(2\pi(f_c - \Delta)t) & \text{if } 0 \le t \le T, \\ 0 & \text{otherwise,} \end{cases}$$

$$s_1(t) = \begin{cases} \sqrt{\frac{2E}{T}}\cos(2\pi(f_c + \Delta)t) & \text{if } 0 \le t \le T, \\ 0 & \text{otherwise.} \end{cases}$$

Assume that $f_c T \gg E$.

1. Compute the energy of the signals.

$$\begin{aligned}
\int_{\infty}^{-\infty} s_0^2(t)\ \mathrm{d}t &= \int_0^T \frac{2E}{T}\cos^2(2\pi(f_c - \Delta)t)\ \mathrm{d}t \\
&= \int_0^T \frac{2E}{2T}(1 + \cos(4\pi(f_c - \Delta)t))\ \mathrm{d}t \\
&= E + \int_0^T \frac{E}{T}(\cos(4\pi(f_c - \Delta)t))\ \mathrm{d}t \\
&= E + \frac{E\sin(4\pi(f_c - \Delta)T)}{4\pi(f_c - \Delta)T} \approx E.
\end{aligned}$$

The energy for the second signal is the same and can be calculated in a similar way.

2. Find the minimal frequency shift Δ such that the signals are orthogonal.

$$\begin{aligned}
\langle s_0, s_1\rangle &= \int_{-\infty}^{\infty} s_0(t)s_1(t)\ \mathrm{d}t \\
&= \int_0^T \frac{2E}{T}\cos(2\pi(f_c - \Delta)t))\cos(2\pi(f_c + \Delta)t))\ \mathrm{d}t \\
&= \int_0^T \frac{2E}{2T}(\cos(4\pi\Delta t)) + \cos(4\pi f_c t)))\ \mathrm{d}t \\
&= \int_0^T \frac{2E}{2T}(\cos(4\pi\Delta t))\ \mathrm{d}t + \int_0^T \frac{2E}{2T}\cos(4\pi f_c t)))\ \mathrm{d}t \\
&= \frac{E}{T}\frac{\sin(4\pi\Delta t)}{4\pi\Delta}\Big|_0^T + \frac{E\sin(4\pi f_c T)}{4\pi f_c T} \\
&\approx \frac{E}{T}\frac{\sin(4\pi\Delta T)}{4\pi\Delta} \\
&= E\frac{\sin(4\pi\Delta T)}{4\pi\Delta T} \\
&= E\mathrm{sinc}(4\Delta T).
\end{aligned}$$

The correlation is zero if $4\Delta T = 1$. The minimum frequency shift when the two cosine signals are orthogonal is

$$\Delta = \frac{1}{4T}.$$

Such modulation format is called minimum-shift keying (MSK).

3. Assume that 0 and 1 are equally likely. For the found Δ, suggest the optimal detector to minimize the probability of error in the AWGN channel. Find the probability of error.
   Let us denote the receives signal by $y(t)$. An ML decoder can always be implemented as follows: choose $s_i(t)$ that minimizes

$$
\begin{aligned}
z_i = \|y(t) - s_i(t)\|^2 &= \int_{-\infty}^{\infty} (y(t) - s_i(t))^2 \, \mathrm{d}t \\
&= \int_0^T y^2(t) \, \mathrm{d}t + \int_0^T s_i^2(t) \, \mathrm{d}t - 2\int_0^T y(t)s_i(t) \, \mathrm{d}t.
\end{aligned}
$$

   The first two terms do not depend on a particular signal and hence, maximization of $\int_0^T y(t)s_i(t)\,\mathrm{d}t$ could be done instead of minimization of the distance.

   The BER can be calculated as

$$\mathrm{BER} = \mathrm{Q}\left(\sqrt{\frac{D^2}{2N_0}}\right),$$

   where the distance between the signals D is

$$
\begin{aligned}
D^2 &= \int_{-\infty}^{\infty} (s_0(t) - s_1(t))^2 \, \mathrm{d}t \\
&= \int_0^T s_0^2(t) \, \mathrm{d}t + \int_0^T s_1^2(t) \, \mathrm{d}t - 2\int_0^T s_0(t)s_1(t) \, \mathrm{d}t = 2E - 2\langle s_0, s_1\rangle.
\end{aligned}
$$

   The BER is

$$\mathrm{BER} = \mathrm{Q}\left(\sqrt{\frac{E}{N_0}}\right),$$

   which is the performance of a binary orthogonal signaling.

4. Find the optimal frequency shift to minimize the probability of error.
   As it was shown in the previous item,

$$D^2 = 2E - 2\langle s_0, s_1\rangle.$$

   To minimize the BER, the correlation between the signals need to be minimized. The function $\mathrm{sinc}(x)$ has its minimum -0.22 value at $x = 1.43$. The optimal frequency shift for coherent FSK is

$$\Delta = \frac{1.43}{4T} = 0.715\frac{1}{2T}$$

   and the BER is

$$\mathrm{BER} = \mathrm{Q}\left(\sqrt{\frac{1.22E}{N_0}}\right),$$

   $h = 2T\Delta$ is often called a modulation index. For MSK the modulation index is $h = 0.5$, for the optimal in terms of BER transmission the index is $h = 0.715$. An asymptotic gain shows how much power one can save by using a better modulation format. For the AWGN channel, the gain is usually calculated in decibels as the ratio of the arguments of the function $\mathrm{Q}(\sqrt{x})$. FSK with an optimal index gains

$$G = 10\log_{10}\left(\frac{1.22}{1}\right) = 0.85 \text{ dB}$$

   compared to MSK.

Problem 2 (Phase Error)

1. Give expressions for the BER of BPSK and QPSK with a Gray labeling in the AWGN channel in terms of E_b/N_0. Interpret the relation between these expressions.
   Note: In case of QPSK, the real part is used to decide on the first bit in the constellation, and the

imaginary part is used to decide on the second bit.

The BER expression for the two constellations are the same and given by

$$\mathrm{BER} = \mathrm{Q}\left(\sqrt{\frac{2E_b}{N_0}}\right)$$

The expression is the same because QPSK with a Gray labeling can be seen as two independent BPSK constellations in the real and the imaginary dimensions.

2. Find expressions for the BER of BPSK and QPSK with a Gray labeling in the AWGN channel in the presence of the phase error ϕ. Does the same interpretation apply here? Why?

Without loss of generality we assume $\phi \geq 0$. Let us first consider BPSK with the phase error ϕ shown in Fig. 1. The projections of the constellation points onto the real axis give the new constellation points with the coordinates

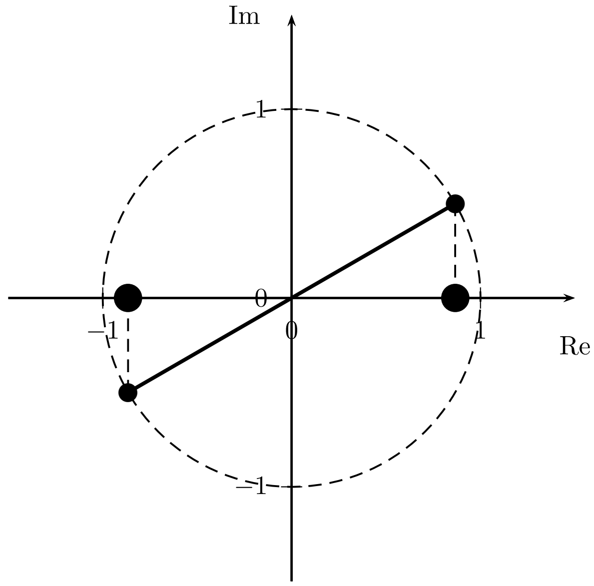

Figure 1: Rotated BPSK constellation.

$(\pm\sqrt{E_b}\cos(\phi), 0)$.

The BER is then

$$\mathrm{BER} = \mathrm{Q}\left(\sqrt{\frac{2E_b\cos^2(\phi)}{N_0}}\right).$$

The asymptotic gain is $G = 10\log_{10}(\cos^2(\phi))$.

The rotated QPSK constellation is shown in Fig. 3. The decision rule is the same as for non-rotated case, i.e., the real part is used to decide on the first bit in the constellation, and the imaginary part is used to decide on the second bit. The constellation points have coordinates $(\pm\sqrt{2E}\cos(\pi/4+\phi), 0)$ and $(\pm\sqrt{2E}\cos(\pi/4-\phi), 0)$.

The BER is then

$$\mathrm{BER} = \frac{1}{2}\left(\mathrm{Q}\left(\sqrt{\frac{4E_b\cos^2(\pi/4+\phi)}{N_0}}\right) + \mathrm{Q}\left(\sqrt{\frac{4E_b\cos^2(\pi/4-\phi)}{N_0}}\right)\right) \approx \frac{1}{2}\mathrm{Q}\left(\sqrt{\frac{4E_b\cos^2(\pi/4+\phi)}{N_0}}\right).$$

The asymptotic gain is $G = 10\log_{10}(2\cos^2(\pi/4+\phi))$.

The BER expressions are not the same because the real and the imaginary parts are not independent anymore.

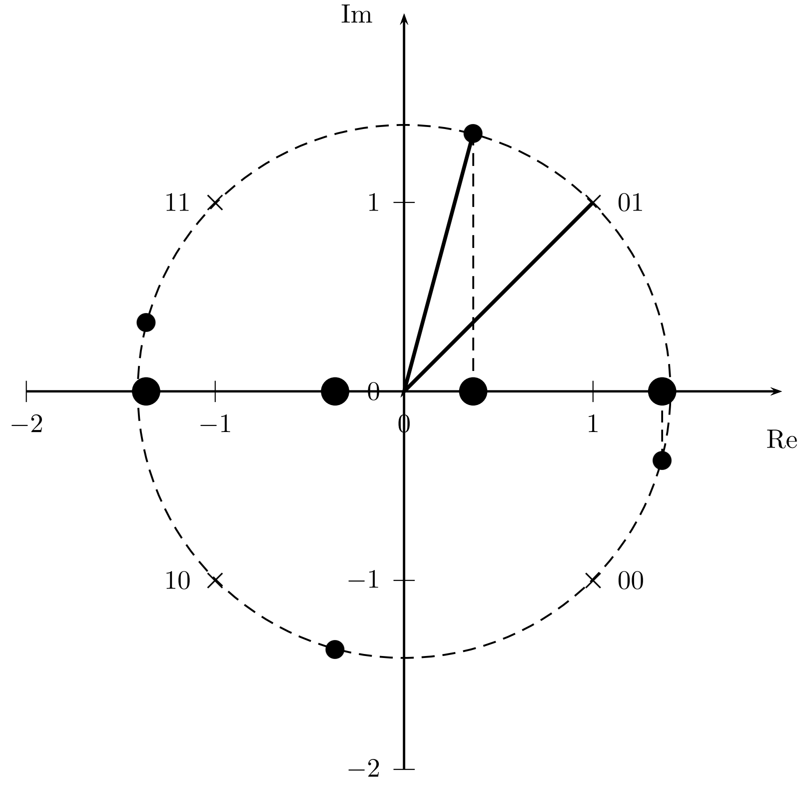

Figure 2: Rotated QPSK constellation.

3. What are the asymptotic gains for the two systems when $\phi = \pi/6$. What is the probability of error for the two systems when $\phi = \pi/4$ in the absence of noise?
   The asymptotic loss for the BPSK is
   $$G = 10 \log_{10}(\cos^2(\pi/6)) = -1.25 \text{ dB}.$$
   The asymptotic loss for QPSK is
   $$G = 10 \log_{10}(2 \cos^2(\pi/4 + \pi/6)) = -8.73 \text{ dB}.$$
   For $\phi = \pi/4$ without noise, the probability of error for BPSK is 0 and the BER for QPSK is 0.25.

Problem 3 (Gain Control Error)

Consider a communication system that uses an equally spaced M-PAM constellation $\mathcal{S} = \{\pm(M-1)d, \pm(M-3)d, \pm d\}$ with equally likely symbols, where $M = 2^m$. The parameter d is chosen to normalize the average symbol energy to one. At the receiver side, the energy of the constellation is overestimated by a factor of $\alpha^2 > 1$, i.e., the detector thinks that the constellation has the energy of α^2. The receiver calculates the new observations $Y' = Y/\sqrt{\alpha^2}$ before detection.

1. Find d.

The symbols can be written as $s_k = (2k - M - 1)d$ when $k = 1, \dots, M$.

$$\begin{aligned}
E_s &= \frac{1}{M}\sum_{k=1}^{M}(2k - M - 1)^2 d^2 = \frac{d^2}{M}\sum_{k=1}^{M}\left(4k^2 - 4(M+1)k + (M+1)^2\right) \\
&= \frac{d^2}{M}\left(4\sum_{k=1}^{M}k^2 - 4(M+1)\sum_{k=1}^{M}k + \sum_{k=1}^{M}(M+1)^2\right) \\
&= \frac{d^2}{M}\left(\frac{4M(M+1)(2M+1)}{6} - 4(M+1)\frac{(M+1)M}{2} + M(M+1)^2\right) \\
&= d^2\frac{M^2 - 1}{3}.
\end{aligned}$$

Setting the average energy to one gives

$$d = \sqrt{\frac{3}{M^2 - 1}}.$$

2. For $M = 4$, calculate the probability of symbol error for the detector with the energy estimate error if the transmitted symbol is $s_2 = -d$ over the AWGN channel with N_0. Did the probability of error increase because of the gain control error?
Since the constellation looks bigger than expected to the receiver, to properly demodulate the signal, the receiver has two options: first, stretch its reference constellation; second, shrink the observations. The second option is preferred. If the observation is denoted by Y, the receiver calculates new observation $Y' = Y/\sqrt{\alpha^2}$ before demodulation, i.e., performs gain control.

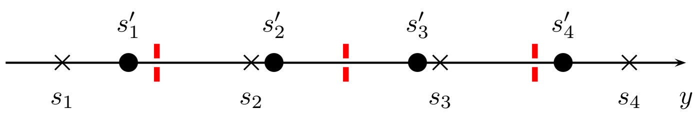

Figure 3: Decision regions for 4-PAM. The crosses show the original constellation and the dots show the constellation after scaling.

Given $Y = s_2$, the new observation $Y' \sim \mathcal{N}(-d/\alpha, \frac{N_0}{2\alpha^2})$

$$\begin{aligned}
\Pr\{\text{error}|Y = s_2\} &= \mathrm{Q}\left(\frac{d/\alpha}{\sqrt{N_0/2/\alpha^2}}\right) + \mathrm{Q}\left(\frac{2d - d/\alpha}{\sqrt{N_0/2/\alpha^2}}\right) \\
&= \mathrm{Q}\left(\sqrt{\frac{2d^2}{N_0}}\right) + \mathrm{Q}\left(\sqrt{\frac{2d^2(2\alpha - 1)^2}{N_0}}\right).
\end{aligned}$$

If we put $\alpha = 1$, the probability of error increases. Seems like the gain control error improves the performance).

3. For the same setup, find the probability of symbol error if $s_1 = -3d$ was transmitted.
Given $Y = s_1$, the new observation $Y' \sim \mathcal{N}(-3d/\alpha, \frac{N_0}{2\alpha^2})$

$$\Pr\{\text{error}|Y = s_1\} = \mathrm{Q}\left(\frac{3d/\alpha - 2d}{\sqrt{N_0/2/\alpha^2}}\right) = \mathrm{Q}\left(\sqrt{\frac{2d^2(3 - 2\alpha)^2}{N_0}}\right).$$

On average, of course, an error in gain control increases the probability of error.