Introduction to Communication Engineering
SSY121, Lecture # 10

Fredrik Brännström
Fredrik.Brannstrom@chalmers.se

Communication Systems Group
Department of Electrical Engineering
Chalmers University of Technology
Göteborg, Sweden

Mon Sept 25, 2023

Information

Project Quiz, Wed Oct 11, at 10:00-10:20 (first 20 min)

10 questions (10 points) is answered individually. The quiz is 20 minutes long, and it is composed of 10 multiple choice questions which evaluate the understanding of the scientific base of the project.

Written exam Mon Oct 23 at 14:00-18:00

Sign up for the exam before Sun Oct 8! So far only 24 signed up!

Part I

Review: Performance Analysis

ML and MAP detectors for the AWGN channel

For the AWGN channel, the conditional PDF is

$$f_{\mathsf{Y}|\mathsf{S}}(\mathsf{y}|\mathsf{s}_i) = \frac{1}{(\pi N_0)^{N/2}} \exp\left(-\frac{||\mathsf{y}-\mathsf{s}_i||^2}{N_0}\right),$$

and therefore,

$$\hat{i}_{\mathsf{ML}} = \arg\max_i \left\{\mathbb{P}\left[\mathsf{Y}=\mathsf{y}|\mathsf{S}=\mathsf{s}_i\right]\right\} \equiv \min_i \left\{||\mathsf{y}-\mathsf{s}_i||\right\}$$

$$\hat{i}_{\mathsf{MAP}} = \arg\max_i \left\{\mathbb{P}\left[\mathsf{S}=\mathsf{s}_i|\mathsf{Y}=\mathsf{y}\right]\right\} \equiv \min_i \left\{||\mathsf{y}-\mathsf{s}_i||^2 - N_0 \log \mathbb{P}\left[\mathsf{S}=\mathsf{s}_i\right]\right\}$$

PEP for an AWGN channel in N dimensions

$$\mathrm{PEP}^{(i,j)} = \mathbb{P}\left[||\mathsf{Y}-\mathsf{s}_j|| < ||\mathsf{Y}-\mathsf{s}_i|| \middle| \mathsf{S}=\mathsf{s}_i \right] = \mathrm{Q}\left(\sqrt{\frac{D_{i,j}^2}{2N_0}} \right),$$

where $N_0/2$ the variance of the noise, and $D_{i,j}^2$ is the squared Euclidean distance between s_i and s_j, i.e., $D_{i,j}^2 = \|\mathsf{s}_j - \mathsf{s}_i\|^2$.

Symbol Error Probability (equally likely symbols)

$$\begin{aligned} P_\mathrm{e} &= \frac{1}{M}\sum_{i=1}^{M}\sum_{j\neq i}\mathbb{P}\left[\mathsf{Y}\in\mathcal{R}_j|\mathsf{S}=\mathsf{s}_i\right] \\ &\leq \frac{1}{M}\sum_{i=1}^{M}\sum_{j\neq i}\mathrm{Q}\left(\sqrt{\frac{D_{i,j}^2}{2N_0}}\right) \approx \frac{2K}{M}\cdot\mathrm{Q}\left(\sqrt{\frac{D_{\min}^2}{2N_0}}\right), \end{aligned}$$

where $D_{\min} = \min_{i\neq j}\{D_{i,j}\}$ is the *minimum distance* of the constellation, and K is the number of signal pairs at minimum distance.

Symbol Error Probability (equally likely symbols)

$$P_{\mathrm{e}} = \frac{1}{M}\sum_{i=1}^{M}\sum_{j\neq i}\mathbb{P}\left[\mathsf{Y}\in\mathcal{R}_j|\mathsf{S}=\mathsf{s}_i\right]$$

$$\leq \frac{1}{M}\sum_{i=1}^{M}\sum_{j\neq i}\mathrm{Q}\left(\sqrt{\frac{D_{i,j}^2}{2N_0}}\right) \approx \frac{2K}{M}\cdot\mathrm{Q}\left(\sqrt{\frac{D_{\min}^2}{2N_0}}\right)$$

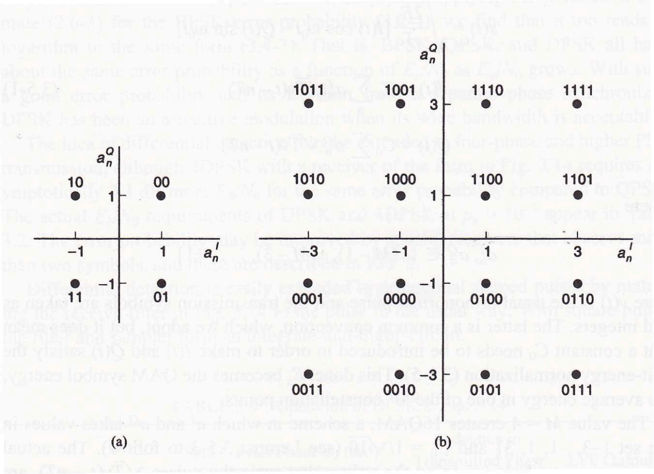
Bit Error Probability (equally likely symbols)

$$B_{\mathrm{e}} = \frac{1}{M}\sum_{i=1}^{M}\sum_{j\neq i}\frac{H_{i,j}}{m}\mathbb{P}\left[\mathsf{Y}\in\mathcal{R}_j|\mathsf{S}=\mathsf{s}_i\right]$$

$$\leq \frac{1}{M}\sum_{i=1}^{M}\sum_{j\neq i}\frac{H_{i,j}}{m}\mathrm{Q}\left(\sqrt{\frac{D_{i,j}^2}{2N_0}}\right) \approx \frac{2H_{\min}}{Mm}\cdot\mathrm{Q}\left(\sqrt{\frac{D_{\min}^2}{2N_0}}\right),$$

where $H_{i,j}$ is the Hamming distance (the number of different bits) between the labels of symbols s_i and s_j, and $H_{\min}$ is the total number of bits differing between signal pairs at minimum distance.

Example (16-QAM from (Anderson))

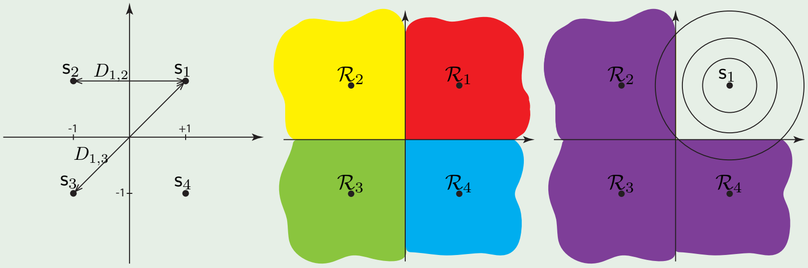


In (b), what is $K =$ $H_{1,11} =$ $H_{\min} =$ $H_{\min} =$ (Gray)?

Part II

Review: Performance Analysis

Example (Exact SEP for QPSK)

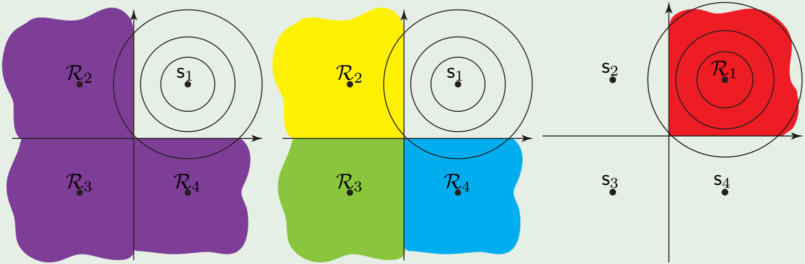


$$E_{\mathrm{s}} = 2, \quad D_{1,2}^2 = D_{1,4}^2 = 4 = 2E_{\mathrm{s}}, \quad \text{and } D_{1,3}^2 = 8 = 4E_{\mathrm{s}}$$

$$P_{\mathrm{e}} = \frac{1}{M}\sum_{i=1}^{M}\sum_{j\neq i}\mathbb{P}\left[\mathsf{Y}\in\mathcal{R}_j|\mathsf{S}=\mathsf{s}_i\right] = \frac{1}{M}\sum_{i=1}^{M}\mathbb{P}\left[\mathsf{Y}\notin\mathcal{R}_i|\mathsf{S}=\mathsf{s}_i\right]$$

$$= \mathbb{P}\left[\mathsf{Y}\notin\mathcal{R}_1|\mathsf{S}=\mathsf{s}_1\right] \qquad \text{(since QPSK is symmetric)}$$

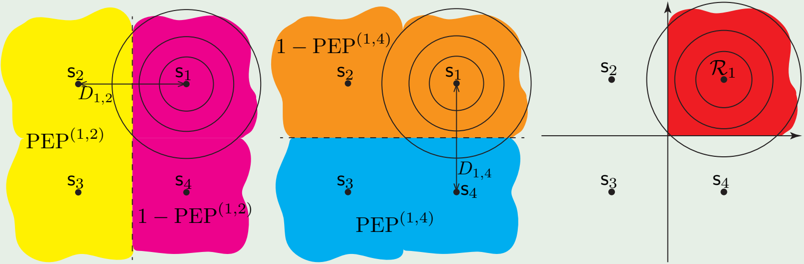
Example (Exact SEP for QPSK)



- What is $P_\mathrm{e} = \mathbb{P}\left[\mathsf{Y} \notin \mathcal{R}_1 | \mathsf{S} = \mathsf{s}_1\right]$?

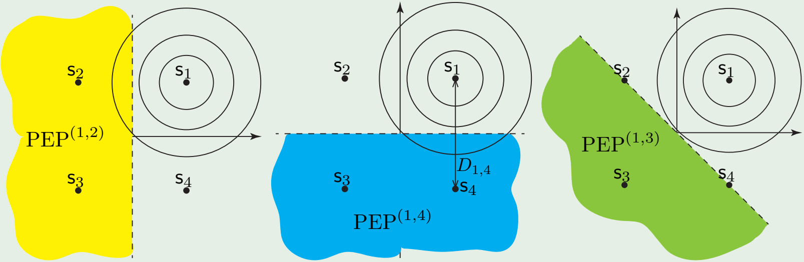
$$
\begin{aligned}
P_\mathrm{e} = \mathbb{P}\left[\mathsf{Y} \notin \mathcal{R}_1 | \mathsf{S} = \mathsf{s}_1\right] &= \sum_{j=2}^{4} \mathbb{P}\left[\mathsf{Y} \in \mathcal{R}_j | \mathsf{S} = \mathsf{s}_1\right] \\
&= 1 - \mathbb{P}\left[\mathsf{Y} \in \mathcal{R}_1 | \mathsf{S} = \mathsf{s}_1\right]
\end{aligned}
$$

Example (Exact SEP for QPSK)



- $D_{1,2}^2 = D_{1,4}^2 = 2E_\mathrm{s}$
- $\mathrm{PEP}^{(1,2)} = \mathrm{Q}\left(\sqrt{\frac{D_{1,2}^2}{2N_0}}\right) = \mathrm{Q}\left(\sqrt{\frac{E_\mathrm{s}}{N_0}}\right) = \mathrm{PEP}^{(1,4)}$
- $\mathbb{P}\left[\mathsf{Y} \in \mathcal{R}_1 | \mathsf{S} = \mathsf{s}_1\right] = (1 - \mathrm{PEP}^{(1,2)})(1 - \mathrm{PEP}^{(1,4)})$
  $= \left(1 - \mathrm{Q}\left(\sqrt{\frac{E_\mathrm{s}}{N_0}}\right)\right)^2 = 1 + \mathrm{Q}\left(\sqrt{\frac{E_\mathrm{s}}{N_0}}\right)^2 - 2\mathrm{Q}\left(\sqrt{\frac{E_\mathrm{s}}{N_0}}\right)$
- $P_\mathrm{e} = 1 - \mathbb{P}\left[\mathsf{Y} \in \mathcal{R}_1 | \mathsf{S} = \mathsf{s}_1\right] = 2\mathrm{Q}\left(\sqrt{\frac{E_\mathrm{s}}{N_0}}\right) - \mathrm{Q}\left(\sqrt{\frac{E_\mathrm{s}}{N_0}}\right)^2$

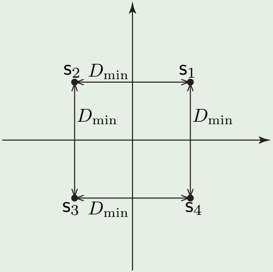
Example (Upper Bound of SEP for QPSK)



- $D_{1,2}^2 = D_{1,4}^2 = 2E_\mathrm{s}$, and $D_{1,3}^2 = 4E_\mathrm{s}$
- $\mathrm{PEP}^{(1,2)} = \mathrm{PEP}^{(1,4)} = \mathrm{Q}\left(\sqrt{\frac{D_{1,2}^2}{2N_0}}\right) = \mathrm{Q}\left(\sqrt{\frac{E_\mathrm{s}}{N_0}}\right)$
- $\mathrm{PEP}^{(1,3)} = \mathrm{Q}\left(\sqrt{\frac{D_{1,3}^2}{2N_0}}\right) = \mathrm{Q}\left(\sqrt{\frac{2E_\mathrm{s}}{N_0}}\right)$

$$P_\mathrm{e} \leq \frac{1}{M}\sum_{i=1}^{M}\sum_{j\neq i}\mathrm{PEP}^{(i,j)} = \sum_{j=2}^{4}\mathrm{PEP}^{(1,j)} = 2\mathrm{Q}\left(\sqrt{\frac{E_\mathrm{s}}{N_0}}\right) + \mathrm{Q}\left(\sqrt{\frac{2E_\mathrm{s}}{N_0}}\right)$$

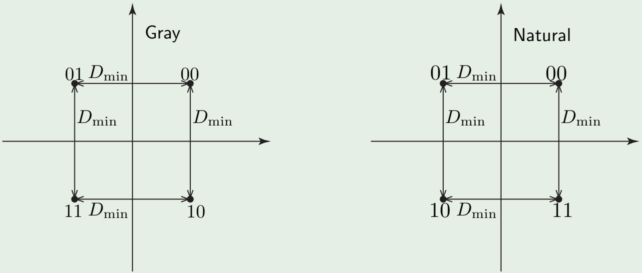
Example (Approximation of SEP for QPSK)



- $D^2_{\min} = D^2_{1,2} = D^2_{2,3} = D^2_{3,4} = D^2_{1,4} = 2E_{\mathrm{s}}$
- Number of signal pairs at minimum distance: $K = 4$

$$P_{\mathrm{e}} \approx \frac{2K}{M} \cdot \mathrm{Q}\left(\sqrt{\frac{D^2_{\min}}{2N_0}}\right) = 2\mathrm{Q}\left(\sqrt{\frac{E_{\mathrm{s}}}{N_0}}\right)$$

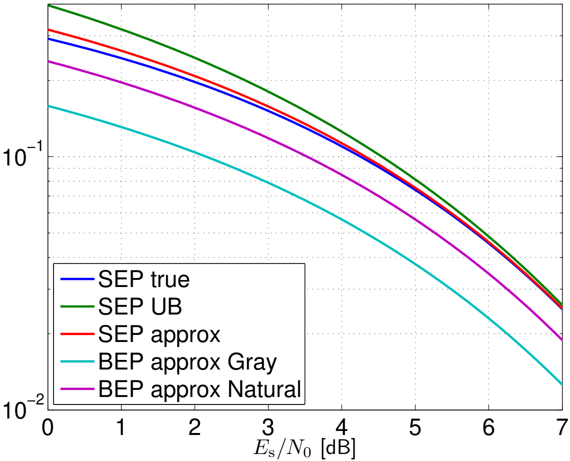
Example (Approximation of BEP for QPSK)



- $E_\mathrm{s} = 2E_\mathrm{b}, \quad D_{\min}^2 = 2E_\mathrm{s} = 4E_\mathrm{b}$
- Total number of bits differing between signal pairs at minimum distance: $H_{\min} = 4$ (Gray) $\quad H_{\min} = 6$ (Natural)

$$B_\mathrm{e} \approx \frac{2H_{\min}}{Mm} \cdot \mathrm{Q}\left(\sqrt{\frac{D_{\min}^2}{2N_0}}\right) = \mathrm{Q}\left(\sqrt{\frac{E_\mathrm{s}}{N_0}}\right) = \mathrm{Q}\left(\sqrt{\frac{2E_\mathrm{b}}{N_0}}\right) \quad \text{(Gray)}$$

$$B_\mathrm{e} \approx \frac{3}{2}\mathrm{Q}\left(\sqrt{\frac{E_\mathrm{s}}{N_0}}\right) = \frac{3}{2}\mathrm{Q}\left(\sqrt{\frac{2E_\mathrm{b}}{N_0}}\right) \quad \text{(Natural)}$$



$$P_\mathrm{e} = 2\mathrm{Q}\left(\sqrt{\frac{E_\mathrm{s}}{N_0}}\right) - \mathrm{Q}\left(\sqrt{\frac{E_\mathrm{s}}{N_0}}\right)^2 \leq 2\mathrm{Q}\left(\sqrt{\frac{E_\mathrm{s}}{N_0}}\right) + \mathrm{Q}\left(\sqrt{\frac{2E_\mathrm{s}}{N_0}}\right) \approx 2\mathrm{Q}\left(\sqrt{\frac{E_\mathrm{s}}{N_0}}\right)$$

$$B_\mathrm{e} \approx \mathrm{Q}\left(\sqrt{\frac{E_\mathrm{s}}{N_0}}\right) \quad \text{(Gray)}, \qquad B_\mathrm{e} \approx \frac{3}{2}\mathrm{Q}\left(\sqrt{\frac{E_\mathrm{s}}{N_0}}\right) \quad \text{(Natural)}$$