Examination Wed. Oct. 27, 2021, 8:30–12:30

SSY121 Introduction to Communication Engineering

- Contact person: Examiner Fredrik Brännström (070 – 872 1685) will visit the exam after approximately 1 and 3 hours.
- Instructions:
  - Write in English.
  - Use a pencil and eraser.
  - There is no page limit. Extra sheets of paper are available.
  - Solve the problems in any order (they are not ordered by difficulty).
  - Before handing in, sort the pages in problem order. Label each page with problem number and running page number. Do not hand in drafts or unused paper.
  - If any data is missing, make reasonable assumptions.
  - Chalmers' examination rules applies.
  - MP3/Music players **are not** allowed during the exam
- Allowed aids:
  - Chalmers-approved calculator
- Grading principles:
  - Explain the line of reasoning clearly. A good solution with a minor error usually gives close to full points, even if the answer is incorrect.
  - An answer without a clear motivation usually gives 0 points, even if it is correct.
  - Whenever possible, check if your answer is reasonable. An obviously unreasonable answer usually gives 0 points, even if the solution is almost correct.
- Solutions and results:
  - Solutions will be posted on the course website no later than 7 days after the exam.
  - The time and place for the exam review will be announced on the course website in Canvas.

1. **True or false questions:** Justify *ALL* your answers using short and concise explanations (maximum 30 words per item). (Total points: 8)

   (a) For a given symbol rate $R_\mathrm{s} = 1/T_\mathrm{s}$, the convolution of a sinc pulse with its matched filter gives a T_s-orthogonal pulse. (1)

   (b) In equiprobable signaling, the ML detector is the optimum detector that minimizes the probability of error, no matter whether or not the channel is AWGN. (1)

   (c) In an 8-PSK communication system with no noise, in which the phase synchronization unit stopped working, the receiver can still tolerate a phase drift of at most $|\theta| < 15°$ (but not more) without making any errors. (1)

   (d) For a given bandwidth, 16-QAM provides lower bit rate than 16-PSK because of a larger minimum Euclidean distance. (1)

   (e) A passband signal is obtained by passing its baseband version through a passband filter. (1)

   (f) The union bound gives an upper bound on the pairwise error probability between any two constellation points. (1)

   (g) If a zero-mean one-dimensional constellation is shifted to the right so that the new constellation is no longer zero-mean, for a given energy constraint, the performance of the new constellation will be identical to the unshifted version. (1)

   (h) Consider a 16-PSK constellation with Gray labeling. Assuming perfect synchronization, the symbol error probability (SEP) increases, if the labeling is changed to a non-Gray labeling. (1)

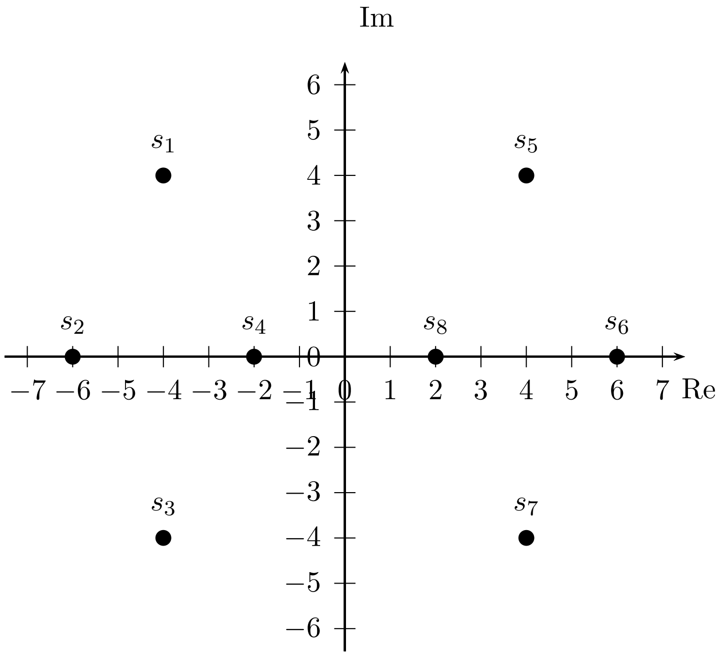

Figure 1: Problem 2.

2. Consider the constellation shown in Figure 1. The unit length on the axes is equal to d. Assume the channel to be AWGN and the constellation points to be equally likely. (Total points: 10)

   (a) Draw the decision boundaries for the maximum likelihood receiver. (2)

   (b) Find the average symbol energy in terms of d. (1)

   (c) Give an expression for the symbol-error probability at high SNR in the following form

   $$\text{SEP} = a_1 \text{Q}\left(\sqrt{\frac{k_1 E_\text{s}}{N_0}}\right),$$

   i.e., find the constants a_1 and k_1. (2)

   (d) Consider the labeling $s_1 = (000)$, $s_2 = (001)$, $s_3 = (011)$, $s_4 = (010)$, $s_5 = (100)$, $s_6 = (101)$, $s_7 = (111)$, and $s_8 = (110)$. Give an expression for the bit-error probability at high SNR in the following form

   $$\text{BEP} = a_2 \text{Q}\left(\sqrt{\frac{k_2 E_\text{b}}{N_0}}\right),$$

   i.e., find the constants a_2 and k_2. (2)

   (e) Denote the transmitted symbol by X and the detected symbol by $\hat{X}$. Find $\Pr(\text{Re}(\hat{X}) > 0 \mid \text{Re}(X) < 0)$, i.e., the probability that one of s_5, s_6, s_7, s_8 is detected, given that one of s_1, s_2, s_3, s_4 is transmitted, in terms of $\frac{E_s}{N_0}$. (3)
   Hint: You might need to use the Bayes' probability theorem:

   $$\Pr(\text{Re}(\hat{X}) > 0 \mid \text{Re}(X) < 0) = \frac{\sum_{i=1}^{4} \Pr(\text{Re}(\hat{X}) > 0 \mid X = s_i)\Pr(X = s_i)}{\Pr(\text{Re}(X) < 0)}.$$

3. In a digital communication system, the received symbol is given by $Y = X + Z$, where X takes $+d$ with probability p, and 0 with probability $2p$. Also, $Z \sim \mathcal{N}(0, \sigma^2)$. The MAP detector is used, and the detected symbol is denoted by $\hat{X}$. (Total points: 6)

   (a) Find the decision boundary y_0, so that
   $$\hat{X} = \begin{cases} d & \text{if } Y \geq y_0, \\ 0 & \text{if } Y < y_0, \end{cases}.$$ (3)

   (b) Find the SEP expression using the decision boundary y_0. (2)

   (c) Find the SEP expression if the ML detector is used. (1)

4. A line-of-sight wireless transmission with a distance of 4 km operates at a carrier frequency of $f_\mathrm{c} = 5$ GHz using BPSK and the sinc pulse. Assume the gain of the transmitter (G_T) and the receiver (G_R) are both 0 dB. Assume an AWGN channel with noise power spectral dentisty $N_0 = 10^{-20}$ W/Hz. Due to rainfall, there is a 10 dB attenuation of the received signal power. Due to hardware impairments, there is always a 30° phase error at the receiver. The communication system requires a bit rate of $R_\mathrm{b} = 10$ Mbits/s with a maximum bit error rate $P_e = 10^{-6}$. Compute the minimum transmitter power P_T to meet the requirement. (Total points: 3)

5. The four signals shown in Figure 2 are used for communication of four equiprobable messages over an AWGN channel.

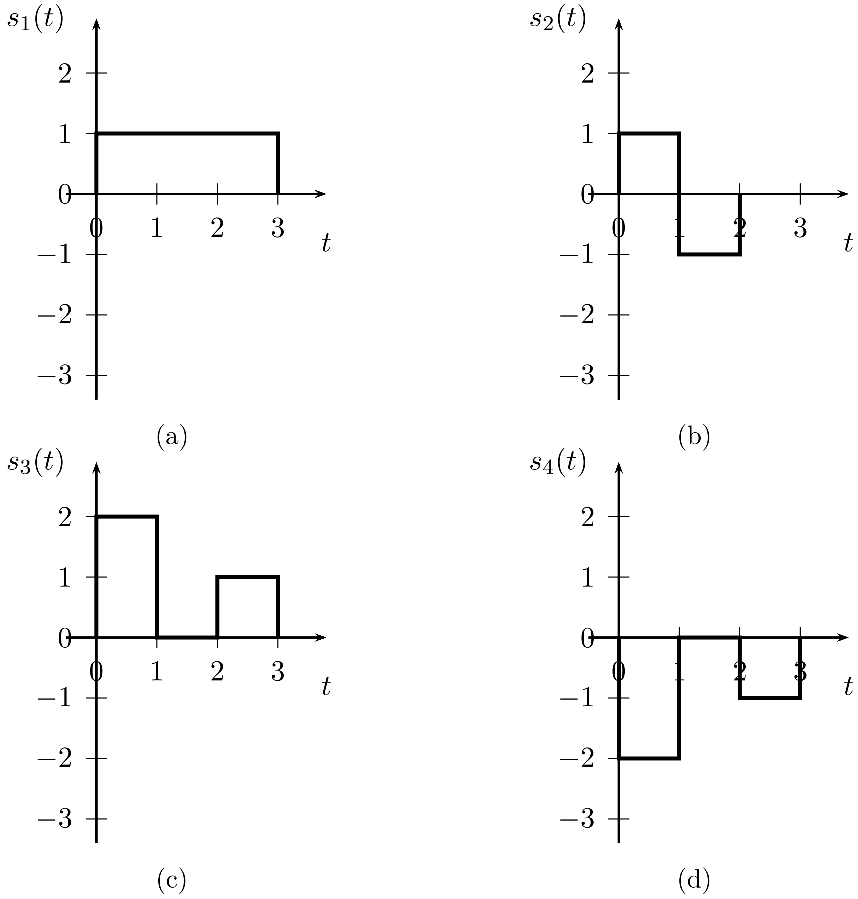

Figure 2: Signals in Problem 5

(Total points: 9)

(a) Find a **minimal set** of orthogonal basis for the space spanned by these signals. *Hint: use the Gram-Schmidt procedure.* (4)

(b) Express the signals $\{s_1, s_2, s_3, s_4\}$ as a linear combination of the basis functions found in (a). Draw a constellation diagram of $\{s_1, s_2, s_3, s_4\}$. (3)

(c) Find the energy of each signal $\{s_1, s_2, s_3, s_4\}$, the average signal energy E_s, and the average bit energy E_b. (2)

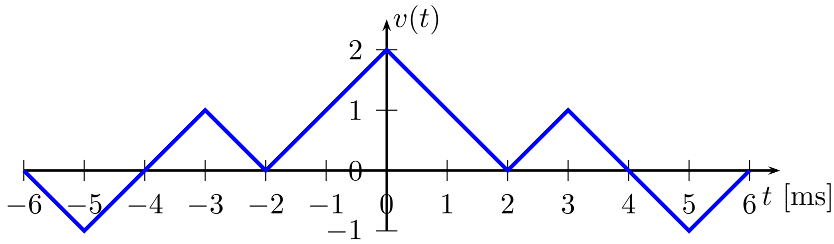

Figure 3: $v(t)$ in Problems 6(a), 6(b), and 6(c).

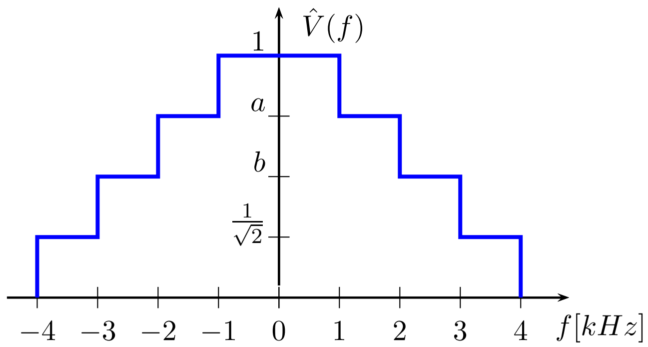

Figure 4: $\hat{V}(f)$ in Problem 6(d).

6. Consider a communication system using the pulses $v(t)$ and $\hat{V}(f)$ shown in Figure 3 and Figure 4, respectively, for binary antipodal transmission. Answer the following questions.

(Total points: 12)

(a) Find all symbol rates, R_s, for which the pulse $v(t)$ in Figure 3 satisfies the Nyquist criterion. (2)

(b) Given the information sequence $\boldsymbol{a} = [1, -1, 1]$, plot the transmitted signal $s(t) = \sum_{i=0}^{2} a_i v(t - i/R_\mathrm{s})$ for the rate $R_\mathrm{s} = 1/3$ ksymbols/s, where a_i denotes the i-th element of $\boldsymbol{a}$. (2)

(c) Considering a sampling receiver with the same sampling rate $R_\mathrm{s} = 1/3$ ksamples/s to downsample the signal $s(t)$ in (b). Find the output of the sampling receiver. In the absence of noise, is error-free transmission possible? (2)

(d) Consider the **matched filter** receiver with a **new** pulse $\hat{v}(t)$, whose spectrum $\hat{V}(f)$ is shown in Figure 4. Assuming $a > 0$ and $b > 0$, what are the values of a and b (**if any**) that give ISI-free transmissions for each of the symbol rates $R_\mathrm{s} = 3, 4, 5, 6, 7, 8$ ksymbols/s. (6)

Formula sheet, SSY121

Version 2.1, August 10, 2016

This sheet is an allowed aid at written exams in SSY121, Introduction to Communication Engineering, at Chalmers in 2016. It will be handed out with the exam problems. Students may not bring their own copy.

Decibels

$$\left(\frac{E_1}{E_2}\right)_{\mathrm{dB}} = 10\log_{10}\frac{E_1}{E_2}$$

Energies E_s and E_b

$$E_\mathrm{s} = \sum_{i=1}^{M}\mathbb{P}\left[\mathsf{S}=\mathsf{s}_i\right]||\mathsf{s}_i||^2$$

$$E_\mathrm{s} = E_\mathrm{b}\log_2 M$$

Normalized minimum distance

$$d_{\min} = \frac{D_{\min}}{\sqrt{2E_\mathrm{b}}}$$

Nyquist criterion

- In time domain
$$v(nT_\mathrm{s}) = 0, \quad n = \pm1, \pm2, \ldots$$
- In frequency domain
$$\sum_{n=-\infty}^{\infty}\Re\left\{V\left(f-\frac{n}{T_\mathrm{s}}\right)\right\} = T_\mathrm{s}v(0)$$
$$\sum_{n=-\infty}^{\infty}\Im\left\{V\left(f-\frac{n}{T_\mathrm{s}}\right)\right\} = 0,$$
where $T_\mathrm{s}v(0)$ is a real constant.
- If the $v(t)$ is symmetric respect to zero, the definition in frequency domain is
$$\sum_{n=-\infty}^{\infty}V\left(f-\frac{n}{T_\mathrm{s}}\right) = T_\mathrm{s}v(0)$$

T_s-orthogonality

- In time domain
$$\int_{-\infty}^{\infty}v(t)v(t-nT_\mathrm{s})dt = 0, \quad n = \pm1, \pm2, \ldots$$
- In frequency domain
$$\sum_{n=-\infty}^{\infty}\left|V\left(f-\frac{n}{T_\mathrm{s}}\right)\right|^2 = T_\mathrm{s}E_v$$

Sinc, Raised-cosine, and Root raised-cosine pulses

$$v_{\mathrm{sinc}}(t) = \mathrm{sinc}(t/T_\mathrm{p}) = \frac{\sin(\pi t/T_\mathrm{p})}{\pi t/T_\mathrm{p}}$$

$$V_{\mathrm{sinc}}(f) = \begin{cases} T_\mathrm{p}, & |f| < \frac{1}{2T_\mathrm{p}} \\ 0, & |f| \geq \frac{1}{2T_\mathrm{p}} \end{cases}$$

$$v_{\mathrm{RC}}(t) = \mathrm{sinc}\left(\frac{t}{T_\mathrm{p}}\right)\frac{\cos\left(\frac{\pi\alpha t}{T_\mathrm{p}}\right)}{1-\left(\frac{2\alpha t}{T_\mathrm{p}}\right)^2}$$

$$V_{\mathrm{RC}}(f) = \begin{cases} T_\mathrm{p}, & |f| < f_1 \\ \frac{T_\mathrm{p}}{2}\left(1+\cos\left[\frac{\pi T_\mathrm{p}}{\alpha}\left(|f|-\frac{1-\alpha}{2T_\mathrm{p}}\right)\right]\right), & f_1 \leq |f| < f_2 \\ 0, & |f| \geq f_2, \end{cases}$$

where $f_1 = \dfrac{1-\alpha}{2T_\mathrm{p}}$ and $f_2 = \dfrac{1+\alpha}{2T_\mathrm{p}}$

$$v_{\mathrm{RRC}}(t) = \sqrt{T_\mathrm{p}}\frac{\sin\left(\frac{(1-\alpha)\pi t}{T_\mathrm{p}}\right)+\frac{4\alpha t}{T_\mathrm{p}}\cos\left(\frac{(1+\alpha)\pi t}{T_\mathrm{p}}\right)}{\pi t\left(1-\left(\frac{4\alpha t}{T_\mathrm{p}}\right)^2\right)}$$

$$V_{\mathrm{RRC}}(f) = \sqrt{V_{\mathrm{RC}}(f)}$$

Correlation receiver

$$\min_i\left\{\int_{-\infty}^{\infty}[y(t)-s_i(t)]^2\,dt\right\}$$
$$\max_i\left\{\int_{-\infty}^{\infty}y(t)s_i(t)\,dt-\frac{E_{s_i}}{2}\right\},$$

where $E_{s_i} = \int_{-\infty}^{\infty}s_i^2(t)\,dt$.

PAM (baseband)

$$s(t) = \sum_{k=0}^{\infty}a_k v(t-kT_\mathrm{s}),$$

where $a_k \in \{\pm(M-1), \pm(M-3), \ldots, \pm1\}$.

PAM (passband)

$$s(t) = \sum_{k=1}^{\infty}a_k v(t-kT_\mathrm{s})\sqrt{2}\cos w_c t,$$

where $a_k \in \{\pm(M-1), \pm(M-3), \ldots, \pm1\}$

2D Modulations

$$s(t) = \sum_{k=0}^{\infty}a_k v(t-kT_\mathrm{s})\sqrt{2}\cos(w_c t) - \sum_{k=0}^{\infty}b_k v(t-kT_\mathrm{s})\sqrt{2}\sin(w_c t)$$

M-PSK

$$s(t) = \sum_{k=0}^{\infty} v(t - kT_\mathrm{s})\sqrt{2}\cos\left(w_c t + \frac{2i\pi}{M}\right),$$

where $i = 0, 1, \dots, M-1$.

M-FSK

$$s_i(t) = \cos\left(2\pi\left[f_c + \frac{h}{2T_\mathrm{s}} i\right] t\right),$$

where $i = \pm(M-1), \pm(M-3), \dots, \pm 1$.

Link budget

$$P_\mathrm{R} = P_\mathrm{T} G_\mathrm{T} G_\mathrm{R} \left(\frac{\lambda}{4\pi d}\right)^2,$$

where $c = \lambda f = 3 \cdot 10^8 \mathrm{m/s}$.

Parabolic dish antenna

$$G_\mathrm{Par} = \frac{4\pi A}{\lambda^2}$$

1D Gaussian PDF (i.i.d.)

$$f_X(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right)$$

ND Gaussian PDF with variance σ^2 (i.i.d.)

$$f_\mathsf{Z}(\mathsf{z}) = \frac{1}{(2\pi\sigma^2)^{N/2}} \exp\left(-\frac{||\mathsf{z} - \boldsymbol{\mu}||^2}{2\sigma^2}\right)$$

Bayes' rule

$$f_{\mathsf{X}|\mathsf{Y}}(\mathsf{x}|\mathsf{y}) = \frac{f_\mathsf{X}(\mathsf{x})}{f_\mathsf{Y}(\mathsf{y})} f_{\mathsf{Y}|\mathsf{X}}(\mathsf{y}|\mathsf{x})$$

Additive white Gaussian noise (AWGN)

The following formulas are for the AWGN channel $\mathsf{Y} = \mathsf{S} + \mathsf{Z}$, where S is the transmitted symbol and Z is Gaussian noise.

Maximum likelihood (ML) detection

$$\max_i \{\mathbb{P}[\mathsf{Y} = \mathsf{y} | \mathsf{S} = \mathsf{s}_i]\}$$

Maximum a posteriori (MAP) detection

$$\max_i \{\mathbb{P}[\mathsf{S} = \mathsf{s}_i | \mathsf{Y} = \mathsf{y}]\} \equiv \max_i \{\mathbb{P}[\mathsf{S} = \mathsf{s}_i]\, \mathbb{P}[\mathsf{Y} = \mathsf{y} | \mathsf{S} = \mathsf{s}_i]\}$$

Pairwise error probability (PEP)

$$\mathrm{PEP}^{(i,j)} = \mathrm{Q}\left(\sqrt{\frac{D_{i,j}^2}{2N_0}}\right),$$

where $D_{i,j}^2 = \|\mathsf{s}_j - \mathsf{s}_i\|^2$.

Symbol error probability (SEP) (exact)

$$P_\mathrm{e} = \sum_{i=1}^{M} \mathbb{P}[\mathsf{S} = \mathsf{s}_i] \sum_{j \neq i} \mathbb{P}[\mathsf{Y} \in \mathcal{R}_j | \mathsf{S} = \mathsf{s}_i]$$

SEP (union bound)

$$\begin{aligned} P_\mathrm{e} &\leq \sum_{i=1}^{M} \mathbb{P}[\mathsf{S} = \mathsf{s}_i] \sum_{j \neq i} \mathrm{PEP}^{(i,j)} \\ &= \sum_{i=1}^{M} \mathbb{P}[\mathsf{S} = \mathsf{s}_i] \sum_{j \neq i} \mathrm{Q}\left(\sqrt{\frac{D_{i,j}^2}{2N_0}}\right) \end{aligned}$$

SEP (High-SNR approximation for equally likely symbols)

$$P_\mathrm{e} \approx \frac{2K}{M} \cdot \mathrm{Q}\left(\sqrt{\frac{D_\mathrm{min}^2}{2N_0}}\right),$$

where K is the number of distinct signal pairs with distance $D_\mathrm{min} = \min_{i \neq j}\{D_{i,j}\}$ and M is the constellation size.

Bit error probability (BEP) (exact)

$$B_\mathrm{e} = \sum_{i=1}^{M} \mathbb{P}[\mathsf{S} = \mathsf{s}_i] \sum_{j \neq i} \frac{H_{i,j}}{m} \mathbb{P}[\mathsf{Y} \in \mathcal{R}_j | \mathsf{S} = \mathsf{s}_i],$$

where $H_{i,j}$ is the Hamming distance (the number of different bits) between the labels of symbols s_i and s_j.

BEP (union bound)

$$B_\mathrm{e} \leq \sum_{i=1}^{M} \mathbb{P}[\mathsf{S} = \mathsf{s}_i] \sum_{j \neq i} \frac{H_{i,j}}{m} \mathrm{Q}\left(\sqrt{\frac{D_{i,j}^2}{2N_0}}\right)$$

BEP (High-SNR approximation for equally likely symbols)

$$B_\mathrm{e} \approx \frac{2H_\mathrm{min}}{Mm} \cdot \mathrm{Q}\left(\sqrt{\frac{D_\mathrm{min}^2}{2N_0}}\right),$$

where H_min is the *total* number of bits differing between signal pairs at minimum distance and $m = \log_2(M)$ is the number of bits per symbol.

Q-function

See also tables in Mathematics Handbook, where $\mathrm{Q}(x) = 1 - \Phi(x)$

$\mathrm{Q}(x)$	x
10^{-1}	1.2816
10^{-2}	2.3263
10^{-3}	3.0902
10^{-4}	3.7190
10^{-5}	4.2649
10^{-6}	4.7534
10^{-7}	5.1993
10^{-8}	5.6120
10^{-9}	5.9978
10^{-10}	6.3613
10^{-11}	6.7060
10^{-12}	7.0345