

# Introduction to Communication Engineering

## SSY121, Lecture # 8

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# Outline

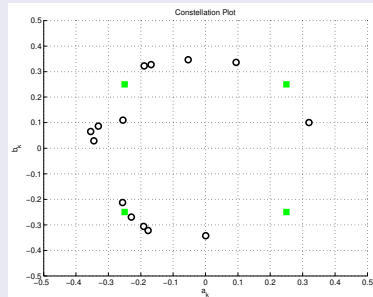
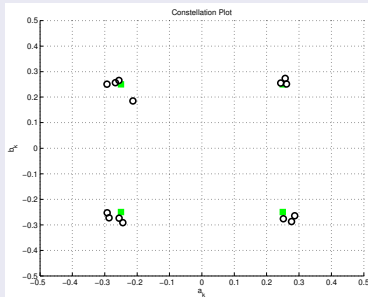
- 1 Summary of Synchronization
  - Carrier Frequency synchronization
  - Carrier Phase synchronization
  - Symbol synchronization
  - Frame synchronization
- 2 A Random Model for Communications and AWGN
  - Additive noise channels
  - Gaussian random variables
  - The AWGN channel
- 3 ML and MAP Detection in AWGN Channels
  - ML detection
  - MAP detection

# Part I

## Summary of Synchronization

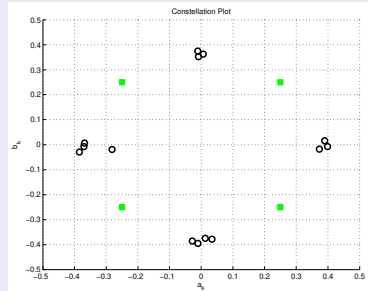
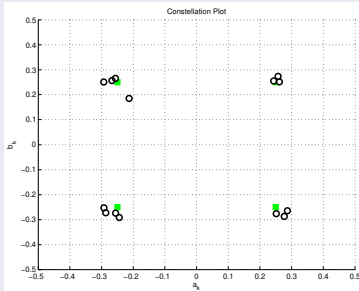
## Synchronization in Rx

- Carrier frequency synchronization:
  - The local oscillators in Rx is not exactly  $f_c$  but  $f_c + \Delta$
- Carrier phase synchronization:
  - The phase of the local oscillators in Rx is different than in Tx, i.e.,
  - $\cos(w_c t)$  and  $\sin(w_c t)$  are  $\cos(w_c t + \theta)$  and  $\sin(w_c t + \theta)$
- Symbol synchronization:
  - The output of the MFs are taken at the wrong instant, e.g.,
  - $t = 0.0T_s, 1.1T_s, 2.2T_s, \dots$  wrong sampling frequency
  - $t = 0.1T_s, 1.1T_s, 2.1T_s, \dots$  wrong timing
- Frame synchronization:
  - Locate where does the block of information bits start, i.e., where the beginning of the “sentence” starts.

Constellation Plot for QPSK, good channel, and  $\Delta = 0.1f_c$ 

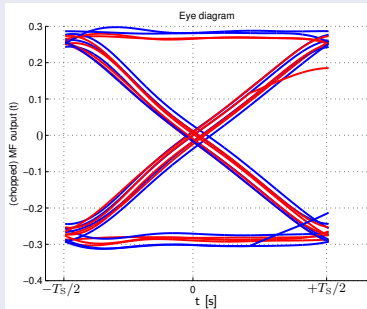
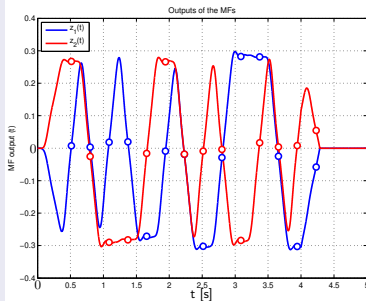
Wrong frequency  $\Rightarrow$  a “rotating” constellation

## Constellation Plot for QPSK, good channel, and $\theta = \pi/4$



Wrong phase  $\Rightarrow$  a “rotated” constellation

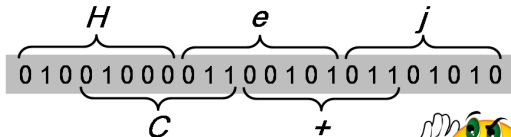
When the sampling instant is incorrect...  $t_k = kT_s + 0.5T_s$



## When the frame synchronization is incorrect...



Hej

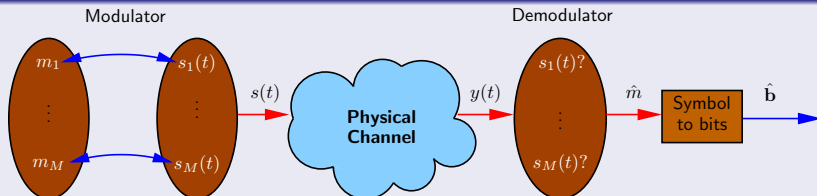
No, I  
prefer C++



## Part II

### A Random Model for Communications and AWGN

## The general problem



## The problem

- Given the observation  $y(t)$ , *guess* what the transmitted message was
- The channel will modify the transmitted signal  $s(t)$  in a random fashion
- If there is no randomness, our guesses will be always correct  $\rightarrow$  problem solved  $\rightarrow$  no need to study communication theory
- We would like to minimize the number of errors:

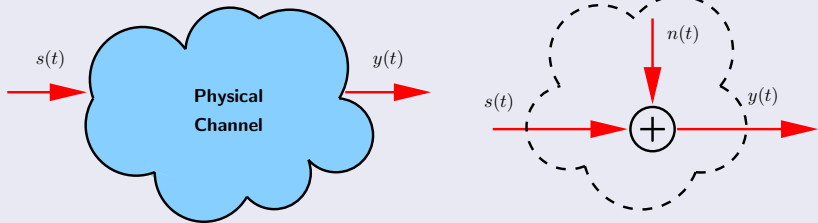
$$\min \{ \mathbb{P} [\hat{m} \neq m] \},$$

or equivalently,  $\max \{ \mathbb{P} [\hat{m} = m] \}$

## Additive white Gaussian Noise

- We need a model for the randomness of the channel
- A very common one is the additive white Gaussian noise (AWGN):
  - It represents accurately “thermal noise,” generated by movement of electrons at the receiver
  - It represents some cases of “atmospheric noise,” caused by the weather
  - The sum of many small independent contributions is Gaussian (“central limit theorem”)
  - It allows a simple analysis
- Are there other channel models?
  - Non-Gaussian noise
  - Poisson channels
  - Fading channels (multiplicative interference)
  - Noise from other users

## Additive noise



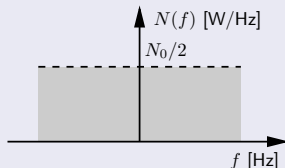
- It is additive noise because

$$y(t) = s(t) + n(t)$$

- The added noise  $n(t)$  does not depend on  $s(t)$

## White Noise

- If the noise has the following properties
  - $\mathbb{E}[n(t)] = 0$
  - $\mathbb{E}[n(t_1)n(t_2)] = \frac{N_0}{2}\delta(t_1 - t_2)$  (uncorrelated for  $t_1 \neq t_2$ )
- The result is a constant (“white”) spectrum



- Theoretically, the power of the noise is  $\int_{-\infty}^{\infty} N(f) df = \infty$  [W] ...
- When observed over any finite BW, the noise has finite power (no explosions!)
- If the noise is flat over the frequencies of interest, it will still be white for our purposes

## 1D Gaussian random variables

- Gaussian distribution is equivalent to normal distribution
- The mean value is  $\mu$  and the variance is  $\sigma^2$
- We use  $\mathcal{N}(\mu, \sigma^2)$  to denote this random variable
- The probability density function (PDF) of  $\mathcal{N}(\mu, \sigma^2)$  is

$$f_X(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right)$$

- The normalized Gaussian distribution is obtained when  $\mu = 0$  and  $\sigma^2 = 1$
- The probability density function of  $\mathcal{N}(0, 1)$  is

$$f_X(x) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right)$$

## Normalized Gaussian distribution

- The PDF of  $\mathcal{N}(0, 1)$  is

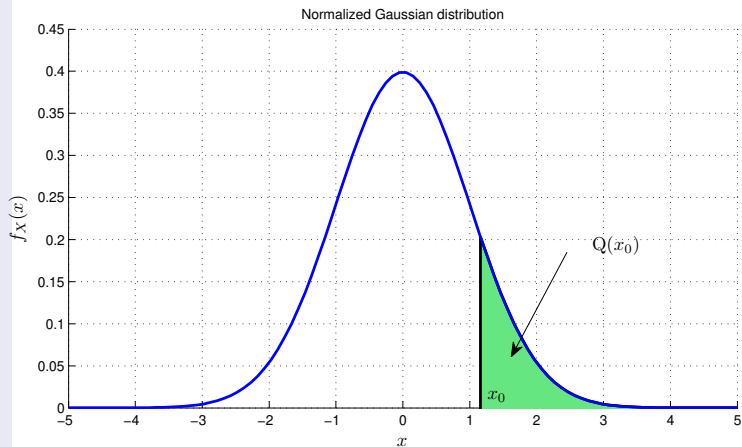
$$f_X(x) = \frac{1}{\sqrt{2\pi}} \exp(-x^2/2)$$

- The probability of  $X$  being larger than  $x_0$  is

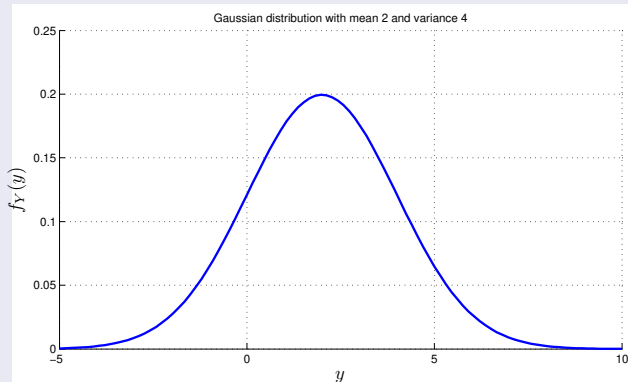
$$\mathbb{P}[X > x_0] = \int_{x_0}^{\infty} \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right) dx = Q(x_0)$$

- $Q(0) = 1/2$
- $Q(-x_0) = 1 - Q(x_0)$
- $1 - Q(x_0)$  is tabulated in Mathematics Handbook
- $Q(x_0)$  is "*qfunc*( $x_0$ )" in Matlab

## Normalized Gaussian distribution





Another Gaussian distribution ( $\mu = 2$  and  $\sigma^2 = 4$ )

The probability of  $Y \sim \mathcal{N}(\mu, \sigma^2)$  being larger than  $y_0$  is

$$\mathbb{P}[Y > y_0] = Q\left(\frac{y_0 - \mu}{\sigma}\right)$$

## 2D Gaussian random variables

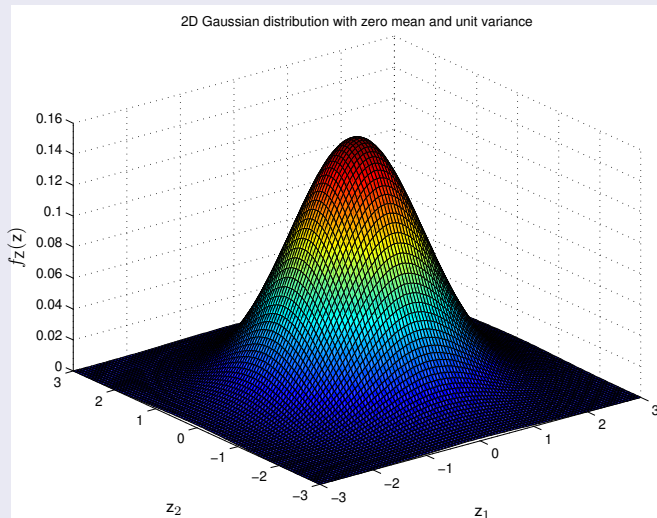
- $N$  independent Gaussian random variables with the same variance  $\sigma^2$  have a PDF given by:

$$f_Z(\mathbf{z}) = \frac{1}{(2\pi\sigma^2)^{N/2}} \exp\left(-\frac{\|\mathbf{z} - \boldsymbol{\mu}\|^2}{2\sigma^2}\right)$$

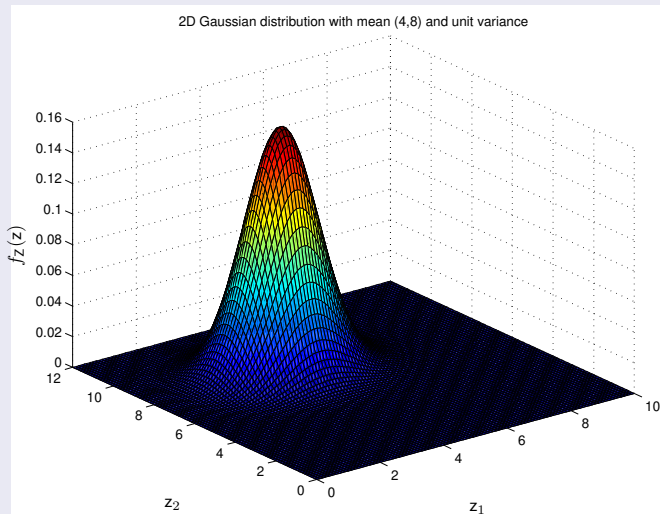
- For the 2D case, the normalized Gaussian distribution is  $\mathbf{Z} = [X_1, X_2]$ ,  $\boldsymbol{\mu} = [0, 0]$ ,  $\sigma^2 = 1$  and the PDF is

$$f_Z(\mathbf{z}) = \frac{1}{2\pi} \exp\left(-\frac{\|\mathbf{z}\|^2}{2}\right)$$

## Normalized 2D Gaussian distribution



## Not-normalized 2D Gaussian distribution



## Definition (The AWGN channel)

If the noise is *additive*, *white*, and also *Gaussian*, we talk about an AWGN channel:  $y(t) = s(t) + n(t)$

## The AWGN channel in vector space

- Or in vectorial notation ( $L_4$ )

$$\mathbf{y} = \mathbf{s} + \mathbf{n},$$

where  $\mathbf{s} = [s_1, \dots, s_N]$

- The noise can be also represented using the basis functions  $\phi_1(t), \dots, \phi_N(t)$  plus other basis functions  $\phi_{N+1}(t), \dots, \phi_{N'}(t)$ , i.e.,

$$n(t) = \sum_{i=1}^N n_i \phi_i(t) + \sum_{i=N+1}^{N'} n_i \phi_i(t)$$

- More than  $N$  basis functions are needed
- Only the  $N$  that span the signals  $s_i(t)$  are relevant for detection ( $L_4$ )

## The AWGN channel

- The continuous time channel

$$y(t) = s(t) + n(t)$$

can then be replaced by a vectorial model

$$\mathbf{y} = \mathbf{s} + \mathbf{n},$$

where  $\mathbf{s} = [s_1, \dots, s_N]$ ,  $\mathbf{n} = [n_1, \dots, n_N]$ ,  $\mathbf{y} = [y_1, \dots, y_N]$

- Each component of the received vector is  $y_i = s_i + n_i$
- If  $n(t)$  is AWGN with PSD  $N_0/2$ , what is the distribution of  $\mathbf{n}$ ?
  - $\mathbb{E}[n_i] = 0$  for all  $i$
  - $\text{var}[n_i] = N_0/2$  for all  $i$
  - $n_i$  and  $n_k$  are independent for  $i \neq k$
- Note that
  - The proof is given in Sec. 2.5.3 of [Anderson]
  - The previous result is valid for any basis functions
  - PSD and variance are different entities, but have the same value

## Summarizing...

In summary, the vectorial channel model is  $y = s + n$ , where

- $s$  is the transmitted symbol (a point in an  $N$ -dimensional Euclidean space)
- $n$  is the noise ( $N$  independent and identically distributed (i.i.d.)  $\mathcal{N}(0, N_0/2)$  components) independent of  $s$
- $y$  is the received symbol (a point in an  $N$ -dimensional Euclidean space which is a “translated” version of  $s$ )
- The PDF of the 2D noise is (zero mean and variance  $N_0/2$ )

$$f_N(n) = \frac{1}{\pi N_0} \exp\left(-\frac{\|n\|^2}{N_0}\right)$$

- For a given transmitted  $s = s_i$ , the received signal  $y$  has a similar distribution as  $n$ , but centered at  $s_i$

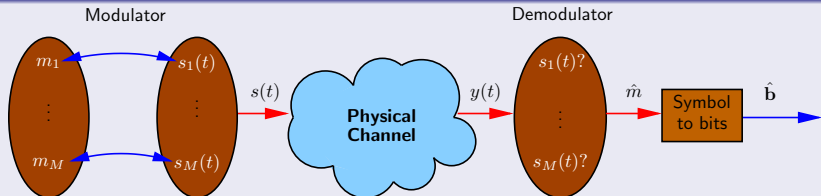
$$f_Y(y|s_i) = \frac{1}{\pi N_0} \exp\left(-\frac{\|y - s_i\|^2}{N_0}\right)$$

## Part III

# ML and MAP Detection in AWGN Channels



## The general problem



## The fundamental problem

- At the Tx, one of the possible  $M$  symbols was transmitted ( $S = s$ ).
- Given the observation of the random variable  $Y = y$ , the receiver must *guess* what the transmitted message was.
- We would like to minimize the number of errors, i.e.,

$$\min \{ \mathbb{P} [\hat{s} \neq s] \}$$

- Equivalently, choose  $i$  ( $\hat{s} = s_i$ ) which maximizes

$$\max_i \{ \mathbb{P} [S = s_i | Y = y] \}$$

## Bayes' rule

Bayes' rule tells us how to reverse conditional probabilities or PDFs:

$$f_{X|Y}(x|y) = \frac{f_X(x)}{f_Y(y)} f_{Y|X}(y|x),$$

using that the joint PDF can be factorized as

$$f_{X,Y}(x,y) = f_{X|Y}(x|y)f_Y(y) = f_{Y|X}(y|x)f_X(x)$$

## The fundamental problem

Choose  $i$  ( $\hat{s} = s_i$ ) which maximizes the conditional probabilities

$$\begin{aligned} \max_i \{ \mathbb{P}[S = s_i | Y = y] \} &\equiv \max_i \left\{ \frac{\mathbb{P}[S = s_i]}{\mathbb{P}[Y = y]} \mathbb{P}[Y = y | S = s_i] \right\} \\ &\equiv \max_i \{ \mathbb{P}[S = s_i] \mathbb{P}[Y = y | S = s_i] \} \\ &\equiv \max_i \{ \mathbb{P}[Y = y | S = s_i] \}, \end{aligned}$$

where in the last step we assume that the symbols are transmitted with equal probability, i.e.,  $\mathbb{P}[S = s_i] = \frac{1}{M}$

## ML detection

In summary, minimizing the error probability is equivalent to maximizing the conditional probabilities (or PDFs)

$$\max_i \{\mathbb{P}[Y = y | S = s_i]\},$$

which is called **maximum likelihood** (ML) detection.

## ML for the AWGN channel

For the AWGN channel, the conditional PDF is

$$f_{Y|S}(y|s_i) = \frac{1}{(\pi N_0)^{N/2}} \exp\left(-\frac{\|y - s_i\|^2}{N_0}\right),$$

and therefore,

$$\max_i \{\mathbb{P}[Y = y | S = s_i]\} \equiv \min_i \{\|y - s_i\|\}$$

## Does it look familiar?

Does the previous expression look familiar? See  $L_4$

## The Matched filter again

The matched filter (correlator) receiver is the optimal choice in two cases:

- Minimizing the error energy ( $L_2$ )
- Minimizing the error probability for AWGN channels (ML detection, previous slide)

## Maximum a posteriori detection

- For ML detection, we assumed that all the symbols are equally likely
- If they are not, the problem is

$$\max_i \{\mathbb{P}[S = s_i | Y = y]\} \equiv \max_i \{\mathbb{P}[S = s_i] \mathbb{P}[Y = y | S = s_i]\}$$

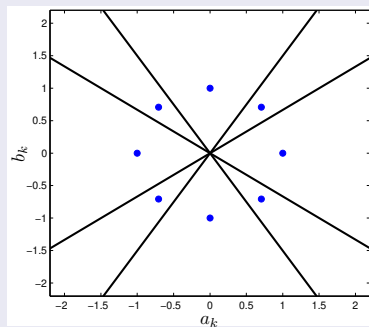
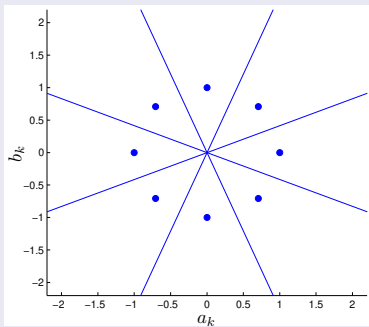
- This is called **maximum a posteriori** (MAP) detection.

## MAP detection for the AWGN channel

- If the channel is AWGN the MAP detection is

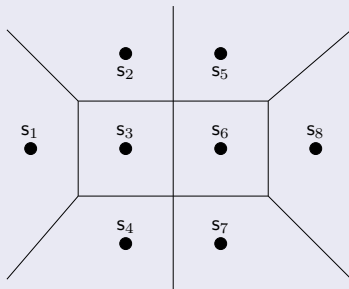
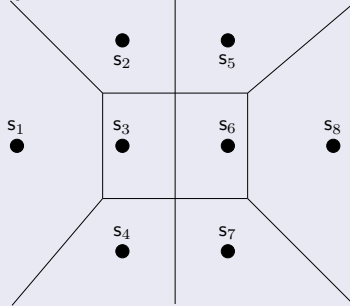
$$\begin{aligned} & \max_i \{\mathbb{P}[S = s_i | Y = y]\} \\ & \equiv \max_i \left\{ \mathbb{P}[S = s_i] \frac{1}{(\pi N_0)^{N/2}} \exp\left(-\frac{\|y - s_i\|^2}{N_0}\right) \right\} \\ & \equiv \max_i \left\{ \mathbb{P}[S = s_i] \exp\left(-\frac{\|y - s_i\|^2}{N_0}\right) \right\} \\ & \equiv \min_i \left\{ \|y - s_i\|^2 - N_0 \log \mathbb{P}[S = s_i] \right\} \end{aligned}$$

## 8-PSK with equally likely (left) and non equally likely symbols (right)



## OTTO constellation

Equally likely symbols

Symbols  $s_1$  and  $s_8$  are more likely

## Today's Summary

- Short summary of AWGN
- The MFR minimizes both the error energy and the error probability in AWGN channels
- Minimizing the error probability assuming the transmitted symbols are equally likely is called **maximum likelihood (ML)** detection
- Minimizing the error probability knowing the transmitted symbol probabilities is called **maximum a posteriori (MAP)** detection
- The decision regions for ML and MAP are different if the transmitted symbols are not equally likely