Examination Wed. Oct. 27, 2021, 8:30–12:30

SSY121 Introduction to Communication Engineering

- Contact person: Examiner Fredrik Brännström (070 – 872 1685) will visit the exam after approximately 1 and 3 hours.
- Instructions:
  - Write in English.
  - Use a pencil and eraser.
  - There is no page limit. Extra sheets of paper are available.
  - Solve the problems in any order (they are not ordered by difficulty).
  - Before handing in, sort the pages in problem order. Label each page with problem number and running page number. Do not hand in drafts or unused paper.
  - If any data is missing, make reasonable assumptions.
  - Chalmers' examination rules applies.
  - MP3/Music players **are not** allowed during the exam
- Allowed aids:
  - Chalmers-approved calculator
- Grading principles:
  - Explain the line of reasoning clearly. A good solution with a minor error usually gives close to full points, even if the answer is incorrect.
  - An answer without a clear motivation usually gives 0 points, even if it is correct.
  - Whenever possible, check if your answer is reasonable. An obviously unreasonable answer usually gives 0 points, even if the solution is almost correct.
- Solutions and results:
  - Solutions will be posted on the course website no later than 7 days after the exam.
  - The time and place for the exam review will be announced on the course website in Canvas.

1. **True or false questions:** Justify *ALL* your answers using short and concise explanations (maximum 30 words per item). (Total points: 8)

   (a) For a given symbol rate $R_s = 1/T_s$, the convolution of a sinc pulse with its matched filter gives a T_s-orthogonal pulse. (1)
   True. It gives the same sinc pulse which is a T_s- orthogonal pulse.

   (b) In equiprobable signaling, the ML detector is the optimum detector that minimizes the probability of error, no matter whether or not the channel is AWGN. (1)
   True. The ML detector minimizes the probability of error for any channel.

   (c) In an 8-PSK communication system with no noise, in which the phase synchronization unit stopped working, the receiver can still tolerate a phase drift of at most $|\theta| < 15^\circ$ (but not more) without making any errors. (1)
   False. The receiver can tolerate a phase drift of at most $|\theta| < 22.5^\circ$, without making any errors, in a noiseless case.

   (d) For a given bandwidth, 16-QAM provides lower bit rate than 16-PSK because of a larger minimum Euclidean distance. (1)
   False. They provide the same bit rate.

   (e) A passband signal is obtained by passing its baseband version through a passband filter. (1)
   False. A passband signal is obtained by up-conversion of its baseband version.

   (f) The union bound gives an upper bound on the pairwise error probability between any two constellation points. (1)
   False. It gives an upper bound on the exact SEP.

   (g) If a zero-mean one-dimensional constellation is shifted to the right so that the new constellation is no longer zero-mean, for a given energy constraint, the performance of the new constellation will be identical to the unshifted version. (1)
   False. For a given energy constraint, the minimum distance of a non-zero-mean constellation is smaller than its zero-mean counterpart. Hence, the performance of the non-zero-mean constellation will be worse.

   (h) Consider a 16-PSK constellation with Gray labeling. Assuming perfect synchronization, the symbol error probability (SEP) increases, if the labeling is changed to a non-Gray labeling. (1)
   False. The symbol error probability does not change if the binary labeling is changed, while the bit error probability does.

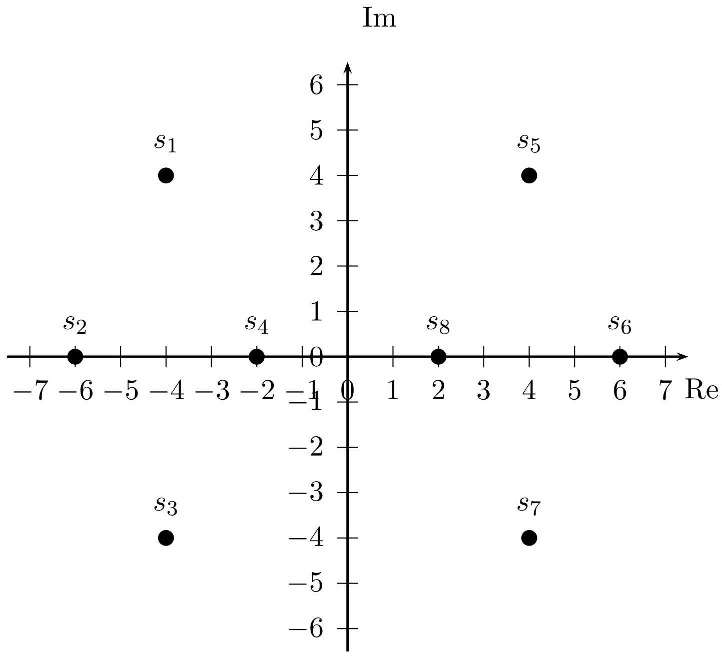

Figure 1: Problem 2.

2. Consider the constellation shown in Figure 1. The unit length on the axes is equal to d. Assume the channel to be AWGN and the constellation points to be equally likely. (Total points: 10)

   (a) Draw the decision boundaries for the maximum likelihood receiver. (2)

   See Fig. 2.

   (b) Find the average symbol energy in terms of d. (1)

$$E_\mathrm{s} = \frac{4 \cdot 32d^2 + 2 \cdot 36d^2 + 2 \cdot 4d^2}{8} = 26d^2.$$

   (c) Give an expression for the symbol-error probability at high SNR in the following form

$$\mathrm{SEP} = a_1 \mathrm{Q}\left(\sqrt{\frac{k_1 E_\mathrm{s}}{N_0}}\right),$$

   i.e., find the constants a_1 and k_1. (2)
   The number of pairs at minimum distance is $K = 3$ and $D_\mathrm{min} = 4d$.

$$\mathrm{SEP} = \frac{2K}{M}\mathrm{Q}\left(\sqrt{\frac{D_\mathrm{min}^2}{2N_0}}\right) = \frac{2 \times 3}{8}\mathrm{Q}\left(\sqrt{\frac{8d^2}{N_0}}\right) = \frac{3}{4}\mathrm{Q}\left(\sqrt{\frac{4E_\mathrm{s}}{13N_0}}\right).$$

   Therefore,

$$a_1 = \frac{3}{4},\ k_1 = \frac{4}{13}.$$

   (d) Consider the labeling $s_1 = (000)$, $s_2 = (001)$, $s_3 = (011)$, $s_4 = (010)$, $s_5 = (100)$, $s_6 = (101)$, $s_7 = (111)$, and $s_8 = (110)$. Give an expression

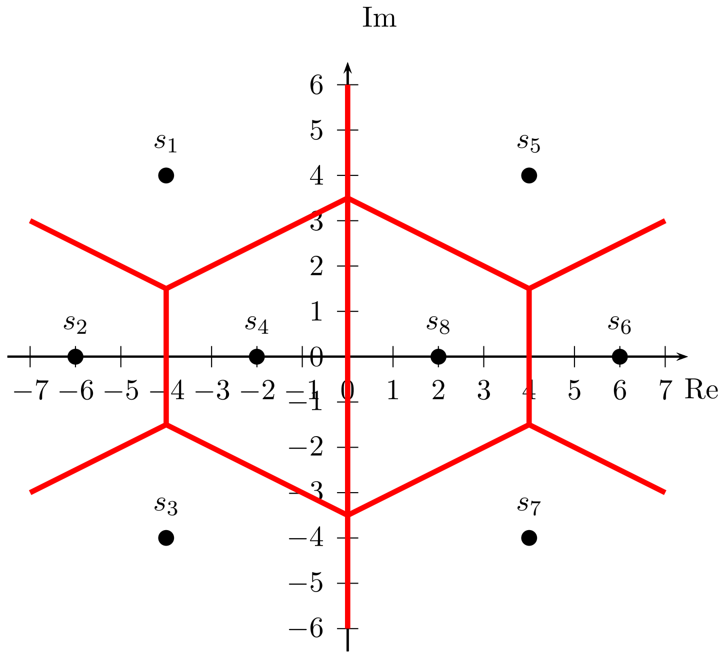

Figure 2: Problem 2 solution.

for the bit-error probability at high SNR in the following form

$$\text{BEP} = a_2 \text{Q}\left(\sqrt{\frac{k_2 E_\text{b}}{N_0}}\right),$$

i.e., find the constants a_2 and k_2. (2)

$$E_\text{s} = 3E_\text{b}.$$

The number of bits differing in the pairs at minimum distance is $H_\text{min} = 5$.

$$\text{BEP} = \frac{2H_\text{min}}{M \cdot m}\text{Q}\left(\sqrt{\frac{D_\text{min}^2}{2N_0}}\right) = \frac{2 \cdot 5}{8 \cdot 3}\text{Q}\left(\sqrt{\frac{4 \cdot 3E_\text{b}}{13N_0}}\right) = \frac{5}{12}\text{Q}\left(\sqrt{\frac{12E_\text{b}}{13N_0}}\right).$$

Therefore,

$$a_2 = \frac{5}{12},\ k_2 = \frac{12}{13}.$$

(e) Denote the transmitted symbol by X and the detected symbol by $\hat{X}$. Find $\Pr(\text{Re}(\hat{X}) > 0 \mid \text{Re}(X) < 0)$, i.e., the probability that one of s_5, s_6, s_7, s_8 is detected, given that one of s_1, s_2, s_3, s_4 is transmitted, in terms of $\frac{E_s}{N_0}$. (3)
Hint: You might need to use the Bayes' probability theorem:

$$\Pr(\text{Re}(\hat{X}) > 0 \mid \text{Re}(X) < 0) = \frac{\sum_{i=1}^{4}\Pr(\text{Re}(\hat{X}) > 0 \mid X = s_i)\Pr(X = s_i)}{\Pr(\text{Re}(X) < 0)}.$$

We have

$$\Pr(\mathrm{Re}(\hat{X}) > 0|X = s_1) = \Pr(n > 4d) = \mathrm{Q}\left(\sqrt{\frac{16d^2}{N_0/2}}\right) = \mathrm{Q}\left(\sqrt{\frac{32E_s}{26N_0}}\right),$$

$$\Pr(\mathrm{Re}(\hat{X}) > 0|X = s_2) = \Pr(n > 6d) = \mathrm{Q}\left(\sqrt{\frac{36d^2}{N_0/2}}\right) = \mathrm{Q}\left(\sqrt{\frac{72E_s}{26N_0}}\right),$$

$$\Pr(\mathrm{Re}(\hat{X}) > 0|X = s_3) = \Pr(n > 4d) = \mathrm{Q}\left(\sqrt{\frac{16d^2}{N_0/2}}\right) = \mathrm{Q}\left(\sqrt{\frac{32E_s}{26N_0}}\right),$$

$$\Pr(\mathrm{Re}(\hat{X}) > 0|X = s_4) = \Pr(n > 2d) = \mathrm{Q}\left(\sqrt{\frac{4d^2}{N_0/2}}\right) = \mathrm{Q}\left(\sqrt{\frac{8E_s}{26N_0}}\right),$$

$$\Pr(X = s_1) = \Pr(X = s_2) = \Pr(X = s_3) = \Pr(X = s_4) = \frac{1}{8},$$

$$\Pr(\mathrm{Re}(X) < 0) = \sum_{i=1}^{4} \Pr(X = s_i) = 0.5.$$

3. In a digital communication system, the received symbol is given by $Y = X + Z$, where X takes $+d$ with probability p, and 0 with probability $2p$. Also, $Z \sim \mathcal{N}(0, \sigma^2)$. The MAP detector is used, and the detected symbol is denoted by $\hat{X}$. (Total points: 6)

   (a) Find the decision boundary y_0, so that

$$\hat{X} = \begin{cases} d & \text{if } Y \geq y_0, \\ 0 & \text{if } Y < y_0, \end{cases}. \tag{3}$$

The threshold y_0 satisfies the following equation:

$$\Pr\{X = 0|Y = y_0\} = \Pr\{X = d|Y = y_0\}.$$

This can be written as

$$\Pr\{Y = y_0|X = 0\} \Pr\{X = 0\} = \Pr\{Y = y_0|X = d\} \Pr\{X = d\}.$$
$$\frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{y_0^2}{2\sigma^2}} \times 2p = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(y_0-d)^2}{2\sigma^2}} \times p.$$

By taking the natural logarithm of both sides

$$-\frac{y_0^2}{2\sigma^2} + \ln(2) = -\frac{(y_0 - d)^2}{2\sigma^2},$$
$$y_0 = \frac{d}{2} + \frac{\sigma^2}{d} \ln 2.$$

   (b) Find the SEP expression using the decision boundary y_0. (2)

$$\Pr\{\text{error}|X = 0\} = \Pr\{Y \geq y_0|X = 0\} = \mathrm{Q}\left(\frac{y_0}{\sigma}\right)$$
$$\Pr\{\text{error}|X = d\} = \Pr\{Y < y_0|X = d\} = \mathrm{Q}\left(\frac{d - y_0}{\sigma}\right)$$

We note that

$$\Pr\{X = 0\} + \Pr\{X = d\} = 2p + p = 1 \rightarrow p = \frac{1}{3}.$$

Thus,

$$\begin{aligned}\Pr\{\text{error}\} &= \Pr\{X = 0\} \Pr\{\text{error}|X = 0\} + \Pr\{X = d\} \Pr\{\text{error}|X = d\} \\ &= \frac{2}{3}\mathrm{Q}\left(\frac{y_0}{\sigma}\right) + \frac{1}{3}\mathrm{Q}\left(\frac{d - y_0}{\sigma}\right).\end{aligned}$$

   (c) Find the SEP expression if the ML detector is used. (1)

If ML detector is used, $y_0 = d/2$, and the error probability is given by

$$\Pr\{\text{error}\} = \mathrm{Q}\left(\frac{d}{2\sigma}\right).$$

4. A line-of-sight wireless transmission with a distance of 4 km operates at a carrier frequency of $f_c = 5$ GHz using BPSK and the sinc pulse. Assume the gain of the transmitter (G_T) and the receiver (G_R) are both 0 dB. Assume an AWGN channel with noise power spectral dentisty $N_0 = 10^{-20}$ W/Hz. Due to rainfall, there is a 10 dB attenuation of the received signal power. Due to hardware impairments, there is always a 30° phase error at the receiver. The communication system requires a bit rate of $R_b = 10$ Mbits/s with a maximum bit error rate $P_e = 10^{-6}$. Compute the minimum transmitter power P_T to meet the requirement. (Total points: 3)

The BEP for BPSK with a phase error ϕ is given by $Q(\sqrt{\frac{2E_b \cos^2\phi}{N_0}})$, which needs to be $< 10^{-6}$. According to the table in the formula sheet, $\sqrt{\frac{2E_b\cos^2\phi}{N_0}} > 4.7534$ or $\frac{E_b}{N_0} = 15.0632$.

Since we use the sinc pulse and BPSK, the bandwidth B is equal to the bit rate. We have

$$\frac{E_b}{N_0} = \frac{P_R}{BN_0} = \frac{P_R}{R_b N_0} = \frac{P_R}{10^7 \cdot 10^{-20}} \geq 15.0632.$$

The received signal power P_R can be calculated according to the free-space path loss model as

$$P_R = P_T G_T G_R \left(\frac{\lambda}{4\pi d}\right)^2 \cdot \frac{1}{10} \geq 15.0632 \cdot 10^{-13}.$$

where $G_T = G_R = 1$, $d = 4 \times 10^3$, $\lambda = \frac{c}{f_c} = 0.06$, and $\frac{1}{10}$ is the attenuation because of rainfall. Thus, $P_T \geq 10.5719$ W.

5. The four signals shown in Figure 3 are used for communication of four equiprobable messages over an AWGN channel.

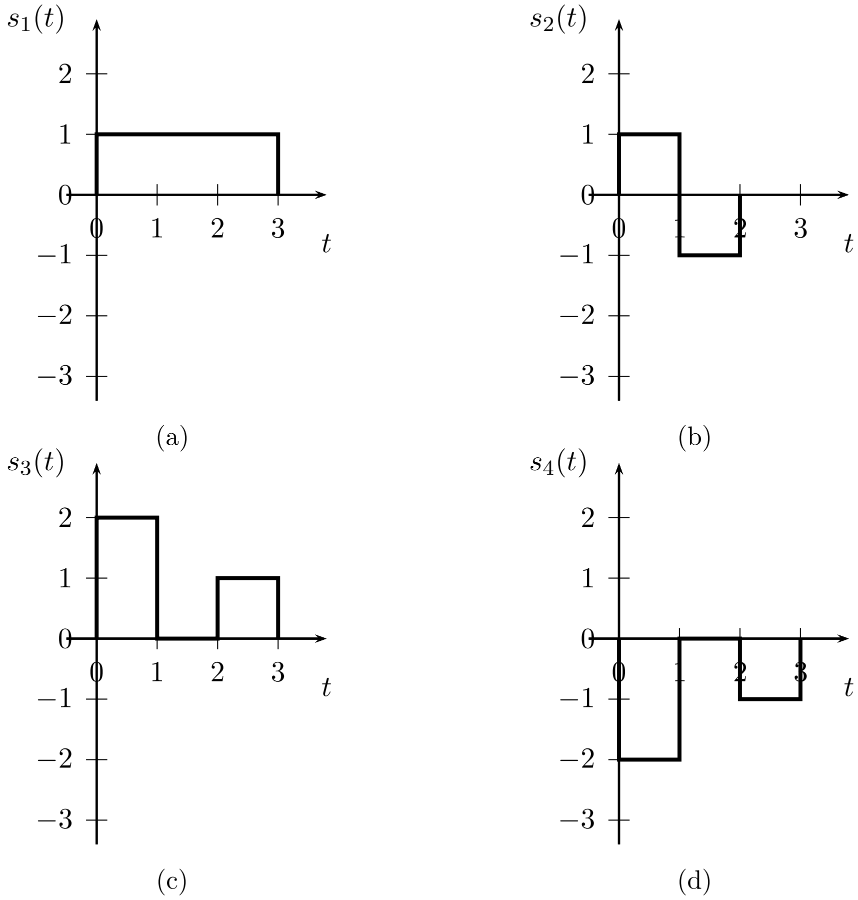

Figure 3: Signals in Problem 5

(Total points: 9)

(a) Find a **minimal set** of orthogonal basis for the space spanned by these signals. *Hint: use the Gram-Schmidt procedure.* (4)

Follow the Gram-Schmidt procedure.

$$
\begin{aligned}
&u_1 = s_1 = [1, 1, 1] \\
&\|u_1\| = \sqrt{3} \\
&\phi_1 = \frac{u_1}{\|u_1\|} = \frac{s_1}{\sqrt{3}} = [\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}] \\
&u_2 = s_2 - \langle s_2, \phi_1 \rangle \phi_1 = s_2 + 0 \cdot \phi_1 = [1, -1, 0] \\
&\|u_2\| = \sqrt{2} \\
&\phi_2 = \frac{u_2}{\|u_2\|} = [\frac{1}{\sqrt{2}}, \frac{-1}{\sqrt{2}}, 0] \\
&u_3 = s_3 - \langle s_3, \phi_1 \rangle \phi_1 - \langle s_3, \phi_2 \rangle \phi_2 \\
&\quad = s_3 - \sqrt{3}\phi_1 - \sqrt{2}\phi_2 = [0, 0, 0] \\
&u_4 = s_4 - \langle s_4, \phi_1 \rangle \phi_1 - \langle s_4, \phi_2 \rangle \phi_2 \\
&\quad = s_4 + \sqrt{3}\phi_1 + \sqrt{2}\phi_2 = [0, 0, 0]
\end{aligned}
$$

Hence, (ϕ_1, ϕ_2) is one of the orthonormal basis set for span (s_1, s_2, s_3, s_4).

(b) Express the signals $\{s_1, s_2, s_3, s_4\}$ as a linear combination of the basis functions found in (a). Draw a constellation diagram of $\{s_1, s_2, s_3, s_4\}$. (3)

$$\begin{aligned} s_1 &= \sqrt{3}\phi_1, \\ s_2 &= \sqrt{2}\phi_2, \\ s_3 &= \sqrt{3}\phi_1 + \sqrt{2}\phi_2 \\ s_4 &= -\sqrt{3}\phi_1 - \sqrt{2}\phi_2. \end{aligned}$$

The constellation is shown in Figure 4.

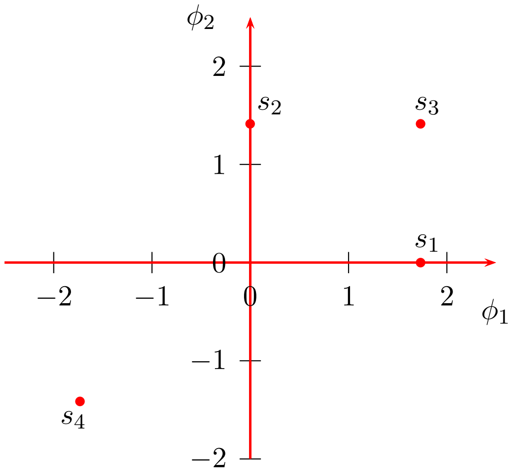

Figure 4: Constellation diagram.

(c) Find the energy of each signal $\{s_1, s_2, s_3, s_4\}$, the average signal energy E_s, and the average bit energy E_b. (2)

$$\begin{aligned} E_{\mathrm{s}_1} &= \sqrt{3}^2 = 3, \\ E_{\mathrm{s}_2} &= \sqrt{2}^2 = 2, \\ E_{\mathrm{s}_3} &= \sqrt{3}^2 + \sqrt{2}^2 = 5, \\ E_{\mathrm{s}_4} &= \sqrt{3}^2 + \sqrt{2}^2 = 5, \\ E_\mathrm{s} &= \frac{E_{\mathrm{s}_1} + E_{\mathrm{s}_2} + E_{\mathrm{s}_3} + E_{\mathrm{s}_4}}{4} = \frac{15}{4}, \\ E_\mathrm{b} &= \frac{E_\mathrm{s}}{\log_2 4} = \frac{15}{8}. \end{aligned}$$

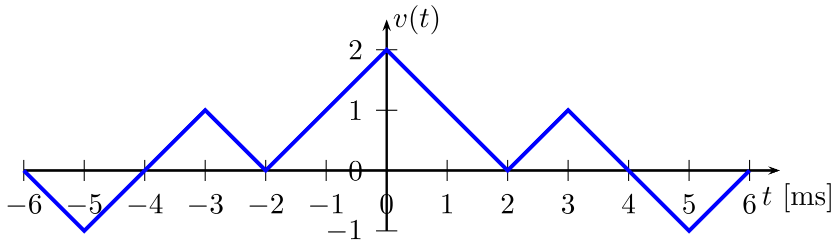

Figure 5: $v(t)$ in Problems 6(a), 6(b), and 6(c).

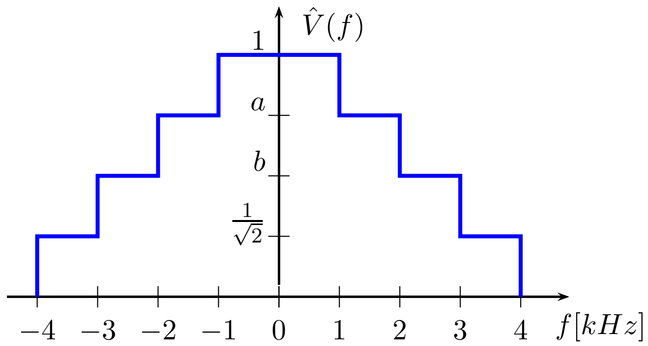

Figure 6: $\hat{V}(f)$ in Problem 6(d).

6. Consider a communication system using the pulses $v(t)$ and $\hat{V}(f)$ shown in Figure 5 and Figure 6, respectively, for binary antipodal transmission. Answer the following questions.

(Total points: 12)

(a) Find all symbol rates, R_s, for which the pulse $v(t)$ in Figure 5 satisfies the Nyquist criterion. (2)

The rates are:

$$R_s = 1/2 \times 10^{-3} = 0.5 \text{ ksymbols/s}$$
$$R_s = 1/4 \times 10^{-3} = 0.25 \text{ ksymbols/s}$$
$$\text{and any } R_s \leq 1/6 \times 10^{-3} \approx 0.167 \text{ ksymbols/s}$$

(b) Given the information sequence $\boldsymbol{a} = [1, -1, 1]$, plot the transmitted signal $s(t) = \sum_{i=0}^{2} a_i v(t - i/R_s)$ for the rate $R_s = 1/3$ ksymbols/s, where a_i denotes the i-th element of $\boldsymbol{a}$. (2)

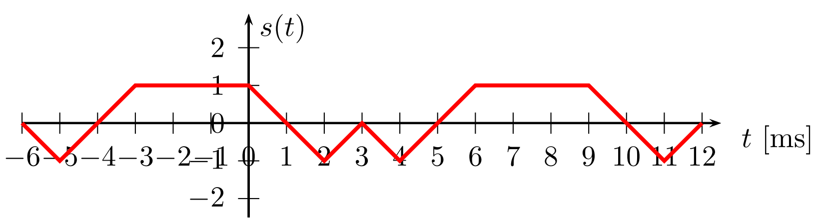

Figure 7: The transmitted signal in Problem 6(b).

(c) Considering a sampling receiver with the same sampling rate $R_\mathrm{s} = 1/3$ ksamples/s to downsample the signal $s(t)$ in (b). Find the output of the sampling receiver. In the absence of noise, is error-free transmission possible? (2)

The output is $\boldsymbol{y} = [1, 0, 1]$. Thus, error-free transmission is not possible even in the absence of noise.

(d) Consider the **matched filter** receiver with a **new** pulse $\hat{v}(t)$, whose spectrum $\hat{V}(f)$ is shown in Figure 6. Assuming $a > 0$ and $b > 0$, what are the values of a and b (**if any**) that give ISI-free transmissions for each of the symbol rates $R_\mathrm{s} = 3, 4, 5, 6, 7, 8$ ksymbols/s. (6)

For a matched filter receiver, ISI-free transmission is possible When $\hat{v}(t)$ is a T_s-Orthogonal pulse, i.e., $\hat{V}(f)$ should fulfil

$$\sum_{n=-\infty}^{\infty} |\hat{V}(f - nR_\mathrm{s})|^2 = A, \tag{1}$$

where R_s is the symbol rate and A is a constant.
If $R_\mathrm{s} = 3$ [ksymbs/s], (1) is met for any a and b that fulfil $1 + 1/2 + b^2 = 2a^2$.
If $R_\mathrm{s} = 4$ [ksymbs/s], (1) is met for any a and b that fulfil $a^2 + b^2 = 1 + 1/2$.
If $R_\mathrm{s} = 5$ [ksymbs/s], (1) is met when $a^2 + \frac{1}{2} = 2b^2 = 1 \to a = b = \frac{1}{\sqrt{2}}$.
If $R_\mathrm{s} = 6$ [ksymbs/s], (1) is met when $b^2 + \frac{1}{2} = a^2 = 1 \to a = 1$ and $b = \frac{1}{\sqrt{2}}$.
If $R_\mathrm{s} = 7$ [ksymbs/s], (1) is met when $1 = a^2 = b^2 \to a = b = 1$.
If $R_\mathrm{s} = 8$ [ksymbs/s], no values of a and b satisfy (1).