Solution of Assignment 2

Due Date: December 3, 2017

December 25, 2017

1 Compute the low bound of congestion when embedding a hypercube in a ring with same number of nodes.

Proof: For a k-cube, There are 2^k nodes.
Hence, $\frac{k2^k}{2} = k2^{\text{k-1}}$ edges,
For this the Congestion must satisfy
$C \geq \frac{k2^{\text{k-1}}}{2^k}$.

Similarly ,
Low Bound Congestion (n) =
6 $\quad n = 4$
$(5 * 2^{(n-2)} - 2 = 3)$ n is even
$(5 * 2^{(n-2)} - 1 = 3)$ n is odd

2 Prove that i-cube contains 2j * 2k mesh where i =j+k

Proof:

Graph Theory According to one of the principles of graph theory
If a set of components graphs are subgraphs of another set, the product graphs will have the same relationship.

1. A line is a sub-graph of a cube and 2^j line is the subgraph of j-cube —— (A)

2. An i-cube is a product graph of an j-cube and a k-cube where $i = j + k$ — (B)

3. Also a $2^j * 2^k$ mesh is a product graph of 2^j and 2^k line. — (C)

As i-cube = j-cube * k-cube from (B), Then
Mesh = 2j * 2k from (C), Hence
j-cube contains 2^j lines and k-cube contains 2^k line.
So i-cube also contains 2j * 2k mesh according to the principle from (A).

Proof by Induction

1. When i =1, it's obvious that it contains 1 * 2 or 2 * 1 mesh.

2. Suppose when i=n, the n-qube contains $2^j * 2^k$ mesh where n= j+k.

3. When i=n+1, we need to prove a (n+1)-cube contains $2^{j+1} * 2^k$ mesh and $2^j * 2^{k+1}$ mesh.

4. We know a (k+1)-cube is composed of two k-qubes $Q_1(n)$ and $Q_2(n)$, with each vertex in $Q_1(n)$ connecting to a vertex in $Q_1(n)$ for once.

5. $Q_1(n)$ and $Q_2(n)$ contains $2^j * 2^k$ mesh with n=j+k.

6. If we connect these two meshes by rows, we can get a $2^{j+1} * 2^k$ mesh and if we connect these two meshes by columns, we will get a $2^j * 2^{k+1}$ mesh.

7. Hence, the case where i=n+1 is proved.
   The i-hypercube contains $2^i * 2^k$ mesh and $i = j + k$.

3 Prove that the diameter of faulty -cube is i+1, Note that there are at most i - 1 faulty nodes in faulty i-cube

Prove by Induction Note that we can consider only the case where there are i-1 nodes because $\leq$ i-1 faulty nodes are easier cases.

1. i=3 , when i=3, the diameter of a 3-cube is 4.

2. i=k, Now we suppose when i=k, the faulty diameter is k+1.
   i=k+1, when i=k+1 ,a(k+1)-cube H_{k+1} consists of two k-cube $H_{k,1}$ and $H_{k,2}$. Note that a (k+1) faulty cube has at most k faulty nodes.
   We can assume k= $k_1 + k_2$, where k_1, k_2 is the number of faulty nodes in $H_{k,1}$ and $H_{k,2}$, respectively.

   if $k_i = 0$, then all the faulty nodes come from $H_{k,2}$ and each node in k_2 can connect to a node $H_{k,1}$. Given two nodes n,m:
   n,m $\in H_{k,1}$. Then $dis(m,n) \leq k$.
   n,m $\in H_{k,2}$. We begin from n, and then go past the path between n and it's corresponding node in $H_{k,1}$ after reaching the corresponding node of m and then comes to m, Hence $dis(m,n) \leq 1 + k + 1$= $k+2$.
   n $\in H_{k,1}$ and m $\in H_{k,2}$. Similarly we have dis(m,n) $\leq$ k+1.
   if $k_1 > 0$, without loss of generosity we rule that $k_1 = k-1, k_2 = 1$. First , we know every two nodes $H_(k,1)$ and $H_(k,2)$ has a distance <= k+1 according to the hypothesis. Hence, we only need to prove nodes between two cubes has a maximum distance of k+1.
   Suppose $n \in H_{k,1}$ and $n \in H_{k,2}$. Suppose the corresponding node of n in $H_{k,2}$ in P(n) and the corresponding node of m in $H_{k,1}$ is Q(m).
   (i) if both P(n) and Q(m) are faulty nodes. Note that P(n) is the only faulty node in $H_{k,2}$. Note that when $k > 3$, One faulty node won't affect the diameter (can be proved by induction). Hence, we can begin from n, then goes to n's neighbor and then goes to $H_{k,2}$ and finally goes to m. The distance satisfies
   dis(m,n) $\leq$ 1+1+k = k+2
   (ii) If one of P(n) and Q(m) is a faulty node. The longest path should be $n \rightarrow m$ or $m \rightarrow Q(m) \rightarrow n$. Then dis(m,n) $\leq k+2$.
   (iii) if neither Q(m) nor P(n) is a faulty node. Then dis(m,n) $\leq$ k+2.

Another Method to Prove faulty cubes i+1

Proof The node connectivity of an i-cube is i. The diameter of faulty i-cube is $i + 1$, it can be shown in following way.
Lemma —: Between any two nodes u and v in the i-cube, there exist exactly i node-disjoint paths, $d_H(u, v)$ (Hamming distance) of these paths are of length $d_H(u, v)$, and the remaining $i - d_H(u, v)$ paths are of length $d_H(u, v) + 2$.

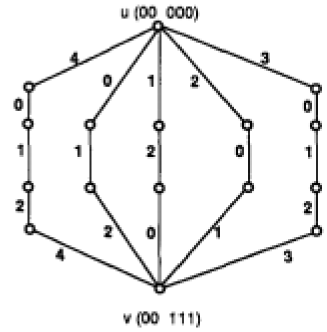


Fig. 1. The node-disjoint paths between two nodes in a 5-cube.

Course files/Semester 1/Computer Networks/graph.pdf
This figure illustrate all the node-disjoints paths between nodes u and v in a five dimensional hypercube. Circles are labeled edges represent nodes and links, respectively.

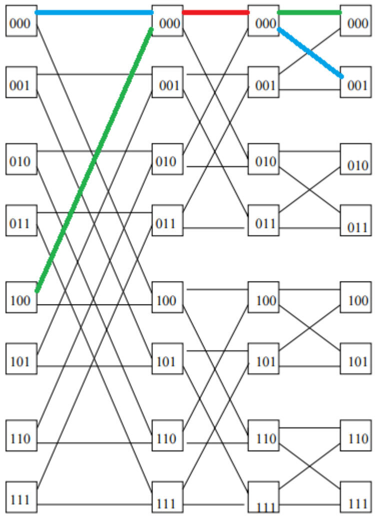
Lemma 2: In an i-cube, d$_{i-1} = i + 1$
Proof: The proof is established by showing that in the presence of (i-1) node failures, the diameter can at most be (i+1). Suppose nodes u and v are located at two ends of the diameter of a damaged i-cube with (i-1) faults. Depending on $d_H(u, v)$, three cases are distinguised.
In the first case if $d_H(u, v) = i$, according to Lemma 1, there will be disjoint paths between u and v, each of length i.
In the worst case, anyone of (i-1) faults belong to a distinct path, yielding one path operational, and thus the distance of u and v in the presence of faults remains the same.
In the second case if $d_H(u, v) = i \leq (i - 2)$,there will be (i-j)$(j \geq 2)$ paths of length (i-j) and j paths of length (i-j+2) between u and v. Assuming the worst case, the only path which does not include any fault is of length (i-j+2) yielding a physical distance of (i-j+2) between u and v in the presense of faults. The length of this path is (i+1). by examining the three cases, we conclude that the diameter of a damaged cube can at most be equal to the longest diameter occurring in all cases (i.e , n+1).

Reference: Combinatorial Analysis of the fault-diameter of the n-cube, IEEE Transaction on Computers, Vol 42, No.1, Jan 1993.

4 Figure out a permutation that cannot be supported by 8 butterfly network.

Solution

The permutation $(0, 1, 2, 3, 4, 5, 6, 7) \to (1, 3, 6, 7, 0, 2, 5, 4)$, the link between Node 0 and Node 1 in path P1 and between Node 4 and Node 0 in path P2 becomes contented.



Course files/Semester 1/Computer Networks/Butterfly.pdf

Figure 1: Butterfly Network, $N = 8$

Figure shows the contented path between P1 and node 4 and node 0 in path P2. The contention is shown by red color. Path P1 is shown by blue and path P2 is shown by green color respectively.

Another Way to proof it is that : A permutation with input in node 4 and output in node 2. At the same time, we need input in node 6 and output in node 6.
Then a collsion will occurs. Hence , the butterfly network doesn't support such permutations.

5 Calculate how many permutations n * n Omega network could support

Solution: Omega Network refers to the multi-stage interconnection network using 2*2 switch boxes and a perfect shuffle interconnect pattern between the stages As we know, an $n * n$ Omega network contains $log_2 n$ with $\frac{n}{2}$ switches per stage.
So, in total $\frac{n}{2}$ $log_2 n$ switches. Since every switch has two settings, the number of permutations n*n Omega network could support is $2^{\frac{n}{2} log2n} = n^{\frac{n}{2}}$

Another Method of Proving the permutations for Omega network is: In n *n Omega network, each stage has $\frac{n}{2}$ switches and there are log(n) stages , we have $\frac{n}{2} * log(n)$ switches, and each switches has 2 methods which can avoid blocking.
(Method 1: entry 0 $\rightarrow$ exit 0, entry 1 $\rightarrow$ exit 1;)
(Method 2: entry 0 $\rightarrow$ exit 1, entry 1 $\rightarrow$ exit 0).
So n*n Omega network can support $2^{(\frac{n}{2} * log(n))} = n^{\frac{n}{2}}$ permutations.