Lecture 12: AVL Tree

Balanced binary tree

- The disadvantage of a binary search tree is that its height can be as large as **N-1**
- Time needed to perform insertion and deletion and many other operations can be O(N) in the worst case
- Goal is to keep the height of a binary search tree balanced
- Such trees are called balanced binary search trees. Examples are AVL tree, red-black tree

AVL Tree

- Named after **A**delson**-V**elskii and **L**andis
- The first dynamically balanced trees
- Its Binary search tree with **balance condition** in which the sub-trees of each node can differ by **at most 1** in their height i.e in the range **-1 to 1**
- **balancefactor = height(right-subtree) - height(left-subtree)**
- If balanceFactor is **negative**, the node is **"heavy on the left"** since the height of the left subtree is greater than the height of the right subtree
- With balanceFactor **positive**, the node is **"heavy on the right"**
- A balanced node has **balancefactor = 0**

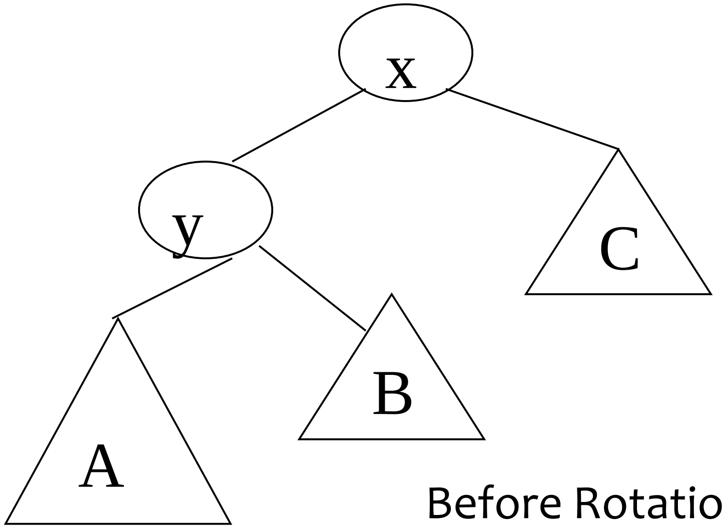
Balance Factor

- The value of the field is the **difference** between the height of the **right** and **left subtrees** of the node
- **balanceFactor = height(right-subtree) - height(left-subtree)**
- If balanceFactor is **negative**, the node is **"heavy on the left"** since the height of the left subtree is greater than the height of the right subtree
- With balanceFactor **positive**, the node is **"heavy on the right"**
- A balanced node has **balanceFactor = 0**

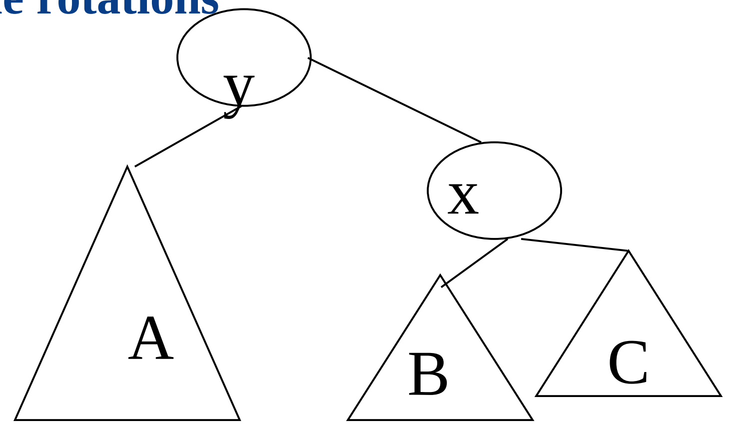
Rotations

- When the tree structure changes (e.g., **insertion or deletion**), we need to transform the tree to **restore** the **AVL tree property**

- This is done by using **single rotations** or **double rotations**



Before Rotation



After Rotation

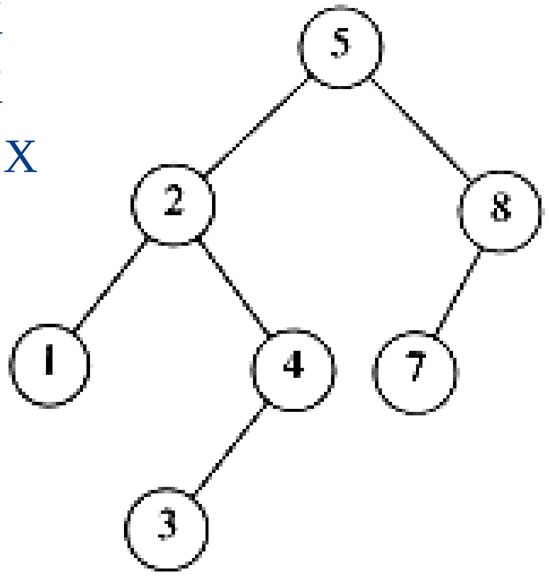
Rotations

- Since an insertion/deletion involves **adding/deleting** a single node, this can only increase/decrease the height of some subtree **by 1**
- If the AVL tree property is violated at a node x, it means that the heights of left(x) ad right(x) **differ by exactly 2**
- Rotation will be applied to x to restore the AVL tree property

Rebalancing

- Suppose the node to be rebalanced is **X**. There are **4 cases** that we might have to fix (two are the mirror images of the other two):

1. An insertion in the **left subtree** of the **left child** of X
2. An insertion in the **right subtree** of the **left child** of X
3. An insertion in the **left subtree** of the **right child** of X
4. An insertion in the **right subtree** of the **right child** of X



Balancing Operations: Rotations

- **Case 1** and **case 4** are symmetric and requires the same operation for balance
  - Cases 1,4 are handled by ***single rotation***

- **Case 2** and **case 3** are symmetric and requires the same operation for balance
  - Cases 2,3 are handled by ***double rotation***

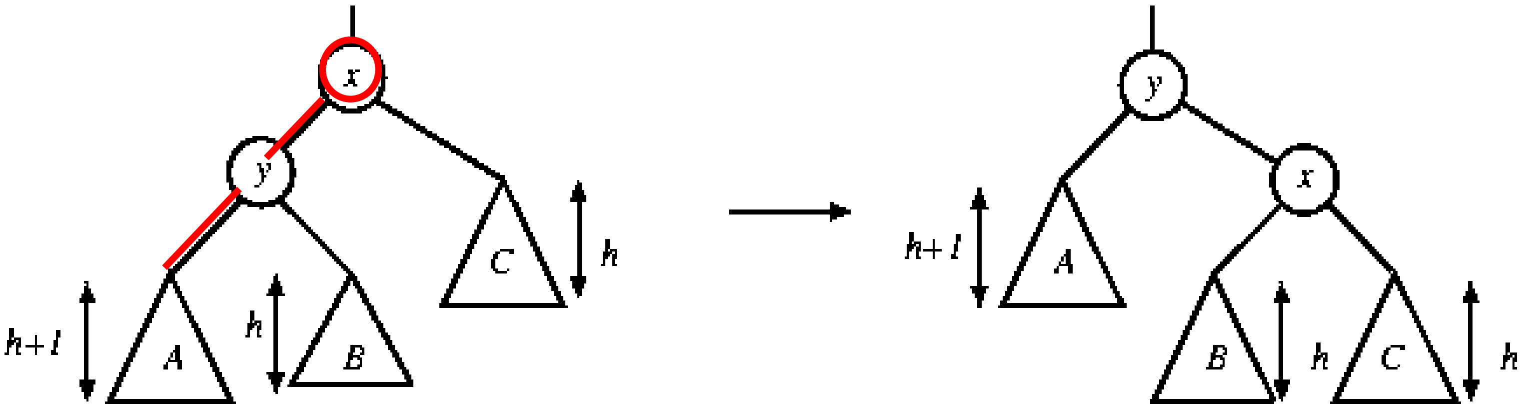
Single Rotation

- A single rotation switches the roles of the parent and child while maintaining the search order

- Rotate between a node and its child
  - Child becomes parent. Parent becomes right child in case 1, left child in case 4

Single rotation

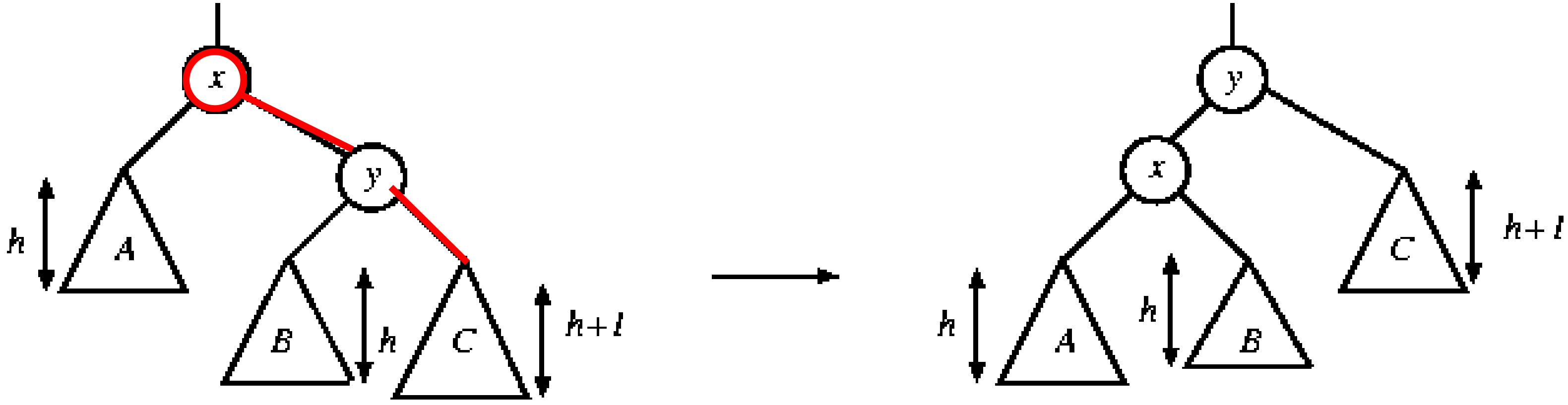
- The new key is inserted in the **subtree A**.
- The AVL-property is **violated at x**
- Height of left(x) is **h+2**
- Height of right(x) is **h**



Rotate with left child

Single rotation

- The new key is inserted in the **subtree C**
- The AVL-property is **violated at x**

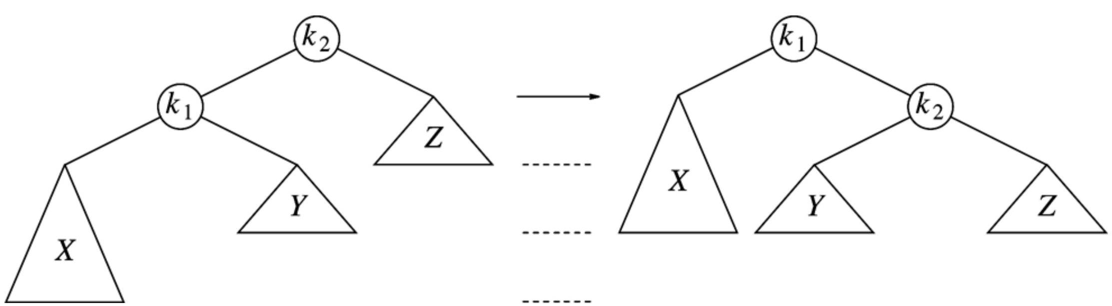


Rotate with right child

Single rotation takes O(1) time.
Insertion takes O(log N) time.

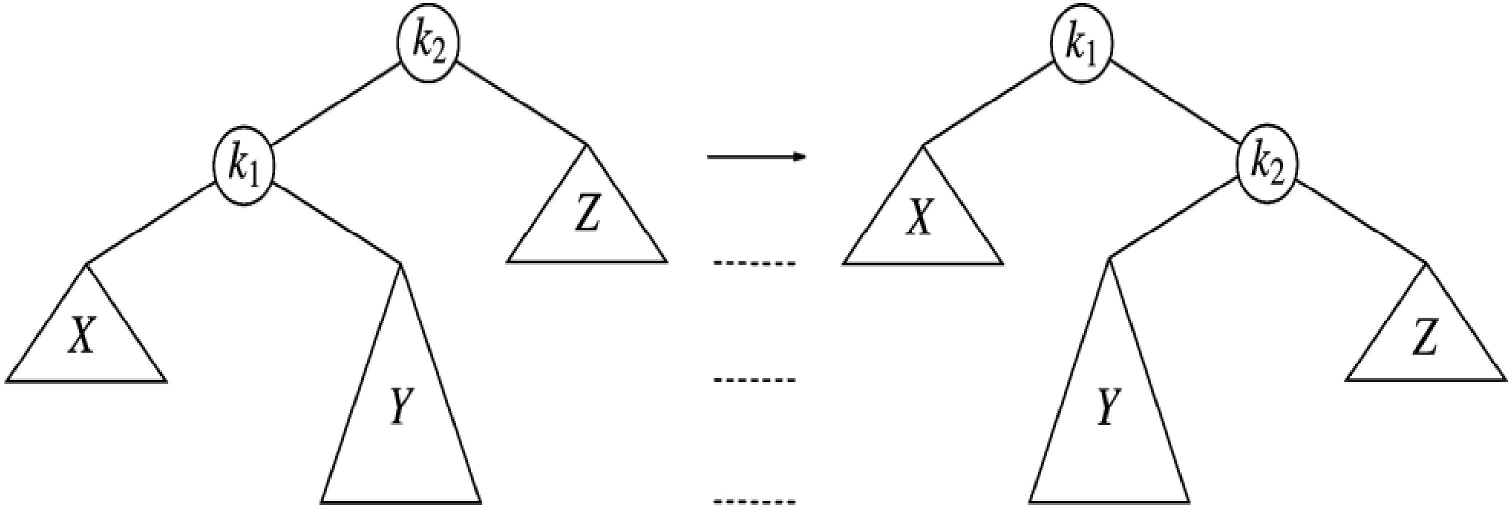
Single Rotation

```
/**
 * Rotate binary tree node with left child.
 * For AVL trees, this is a single rotation for case 1.
 * Update heights, then set new root.
 */
void rotateWithLeftChild( AvlNode * & k2 )
{
    AvlNode *k1 = k2->left;
    k2->left = k1->right;
    k1->right = k2;
    k2->height = max( height( k2->left ), height( k2->right ) ) + 1;
    k1->height = max( height( k1->left ), k2->height ) + 1;
    k2 = k1;
}
```



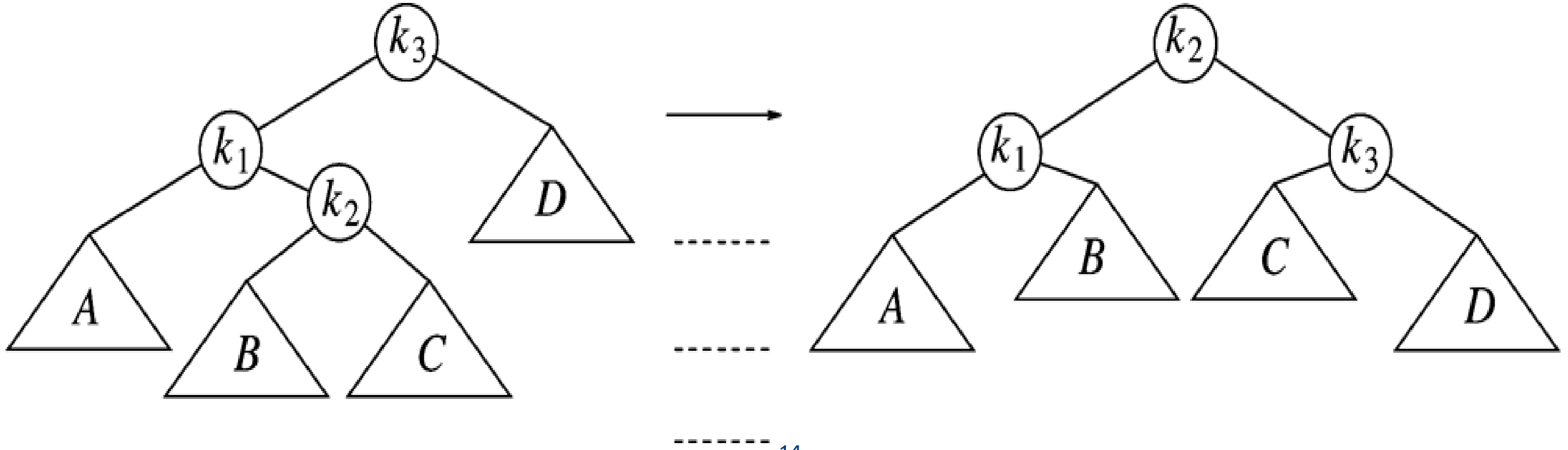
Single Rotation Will Not Work for the Other Case

- For case 2
- After single rotation, k_1 still **not balanced**
- **Double rotations** needed for **case 2** and **case 3**



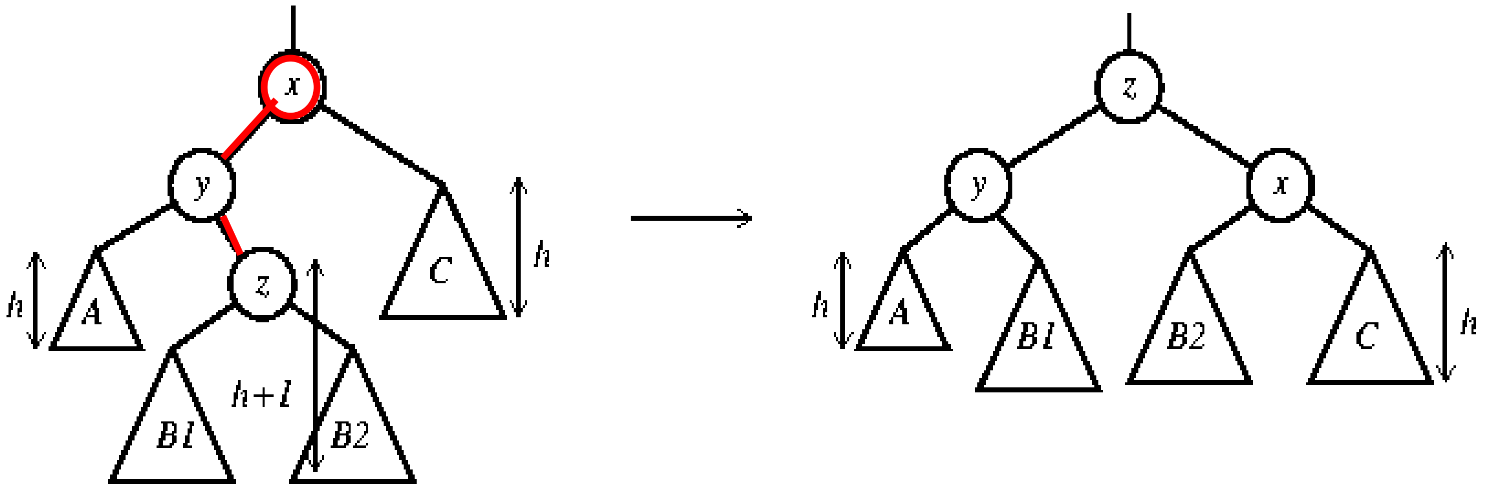
Double Rotation

- Left-right double rotation to fix case 2
- First rotate between $\mathbf{k_1}$ and $\mathbf{k_2}$
- Then rotate between $\mathbf{k_2}$ and $\mathbf{k_3}$
- Case 3 is similar



Double rotation

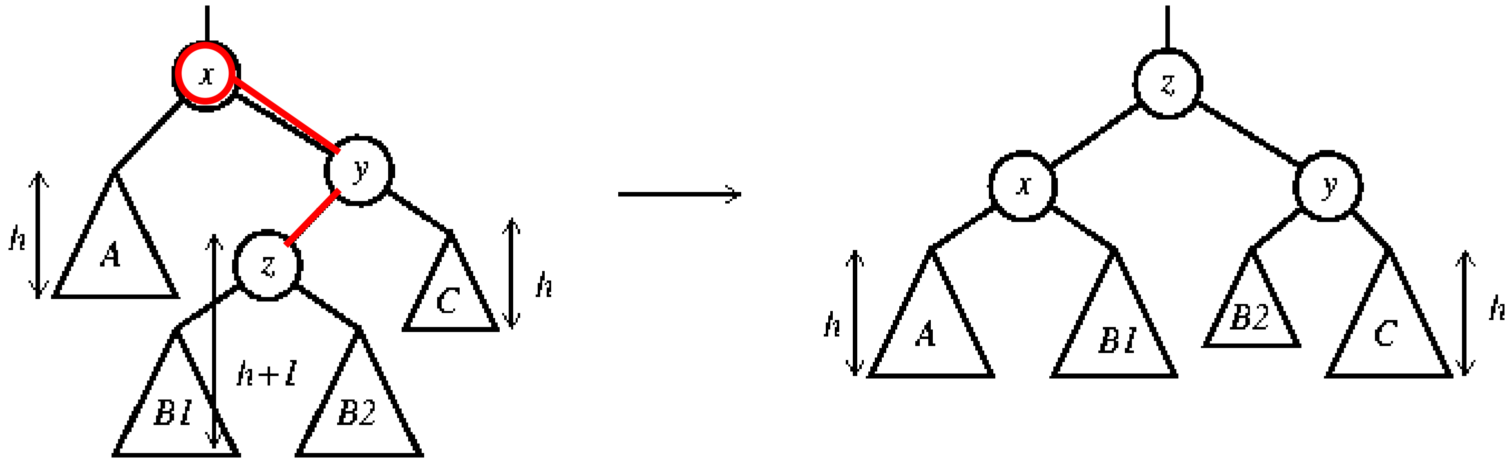
- ➢The new key is inserted in the subtree B1 or B2
- ➢The AVL-property is violated at x
- ➢x-y-z forms a zig-zag shape0



Double rotate with left child

Double rotation

- The new key is inserted in the subtree B1 or B2
- The AVL-property is violated at x



Double rotate with right child

Node declaration for AVL trees

```
template <class Comparable>
class AvlTree;

template <class Comparable>
class AvlNode
{
  Comparable element;
  AvlNode   *left;
  AvlNode   *right;
  int       height;

  AvlNode( const Comparable & theElement, AvlNode *lt,
          AvlNode *rt, int h = 0 )
    : element( theElement ), left( lt ), right( rt ),
             height( h ) { }
  friend class AvlTree<Comparable>;
};
```

Height

```
template class <Comparable>
int AvlTree<Comparable>::height( AvlNode<Comparable> *t) const
{
    return t == NULL ? -1 : t->height;
}
```

Double Rotation

```
/**
 * Double rotate binary tree node: first left child.
 * with its right child; then node k3 with new left child.
 * For AVL trees, this is a double rotation for case 2.
 * Update heights, then set new root.
 */
template <class Comparable>
void AvlTree<Comparable>::doubleWithLeftChild( AvlNode<Comparable> * & k3 ) const
{
  rotateWithRightChild( k3->left );
  rotateWithLeftChild( k3 );
}
```

```
/* Internal method to insert into a subtree.
 * x is the item to insert.
 * t is the node that roots the tree.
 */
template <class Comparable>
void AvlTree<Comparable>::insert( const Comparable & x, AvlNode<Comparable> * & t ) const
{
   if( t == NULL )
     t = new AvlNode<Comparable>( x, NULL, NULL );
   else if( x < t->element )
   {
     insert( x, t->left );
     if( height( t->left ) - height( t->right ) == 2 )
       if( x < t->left->element )
           rotateWithLeftChild( t );
       else
           doubleWithLeftChild( t );
   }
   else if( t->element < x )
   {
       insert( x, t->right );
       if( height( t->right ) - height( t->left ) == 2 )
          if( t->right->element < x )
             rotateWithRightChild( t );
          else
             doubleWithRightChild( t );
    }
    else
       ;  // Duplicate; do nothing
    t->height = max( height( t->left ), height( t->right ) ) + 1;
}
```

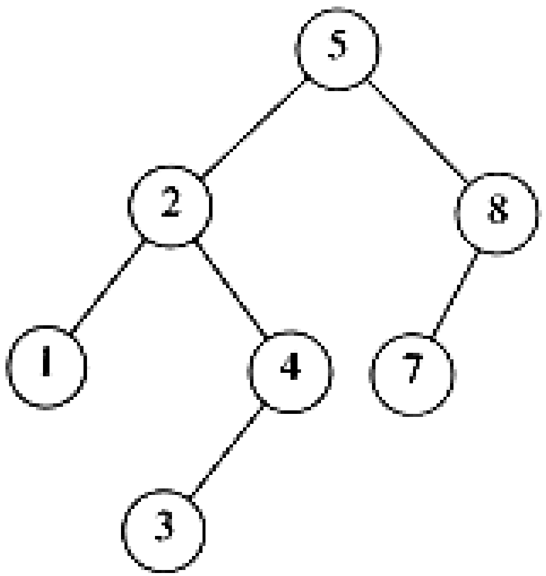
Insertion

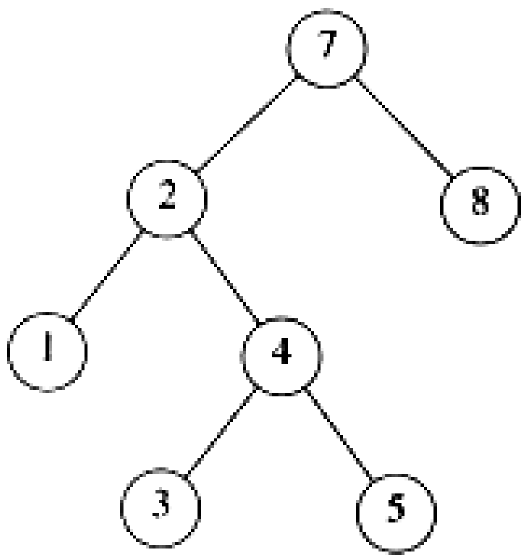
- First, insert the new key as a new leaf just as in ordinary binary search tree
- Then trace the path from the new leaf towards the root. For each node x encountered, check if heights of left(x) and right(x) differ by at most 1
- If yes, proceed to parent(x). If not, restructure by doing **either a single rotation or a double rotation**
- For insertion, once we perform a rotation at a node x, we won't need to perform any rotation at any ancestor of x

Insertion

- Let x be the node at which left(x) and right(x) differ by more than 1
- Assume that the height of x is h+3
- There are 4 cases
  - Height of left(x) is h+2 **(i.e. height of right(x) is h)**
    - Height of left(left(x)) is h+1 $\Rightarrow$ single rotate with left child
    - Height of right(left(x)) is h+1 $\Rightarrow$ double rotate with left child
  - Height of right(x) is h+2 **(i.e. height of left(x) is h)**
    - Height of right(right(x)) is h+1 $\Rightarrow$ single rotate with right child
    - Height of left(right(x)) is h+1 $\Rightarrow$ double rotate with right child

AVL tree?

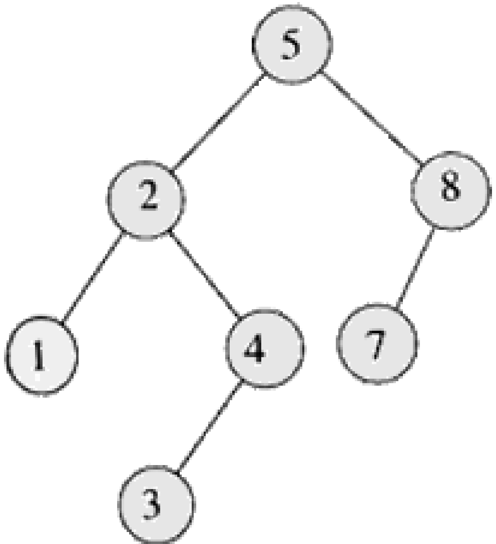




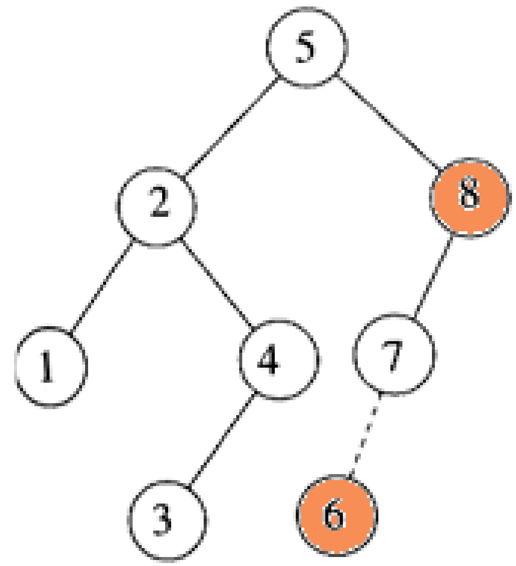
YES
Each left sub-tree has height 1 greater than each right sub-tree

NO
Left sub-tree has height 3, but right sub-tree has height 1

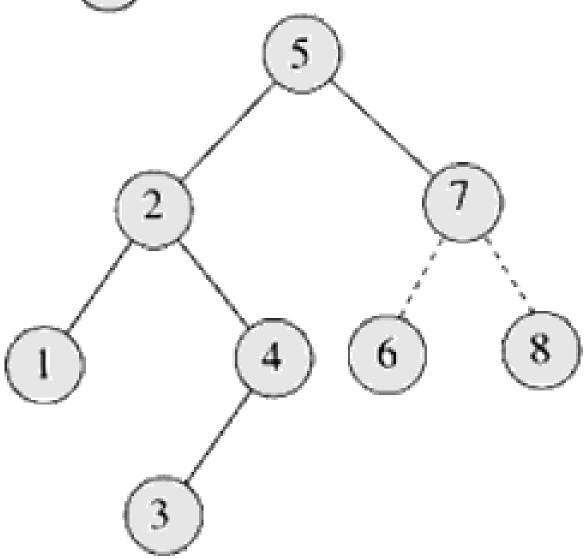
Insertion



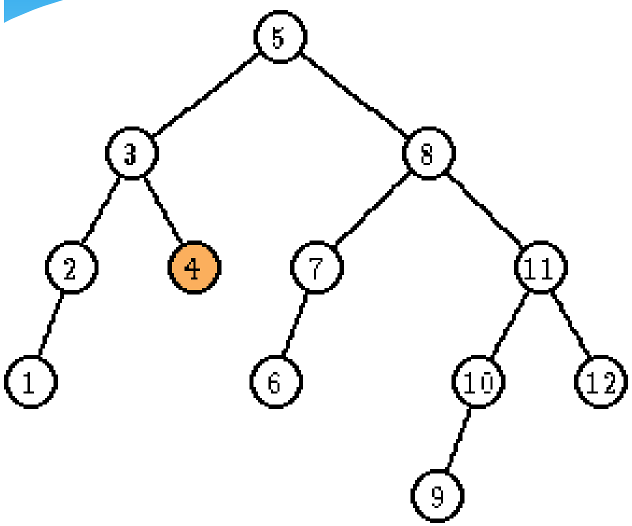
Insert 6



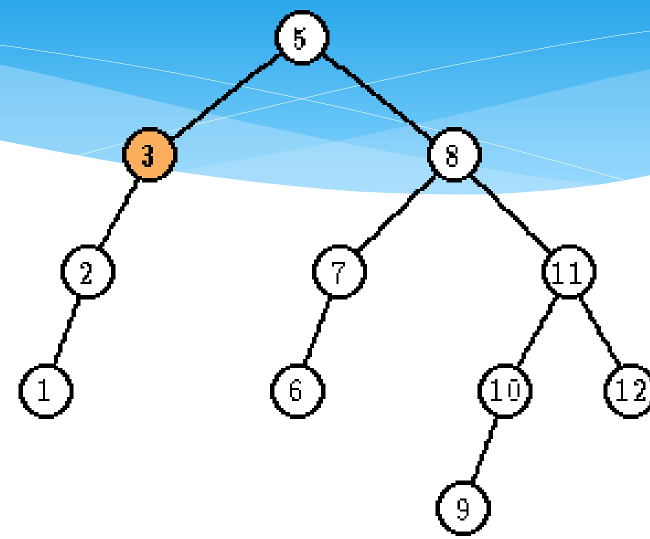
Imbalance at 8
Perform rotation with 7



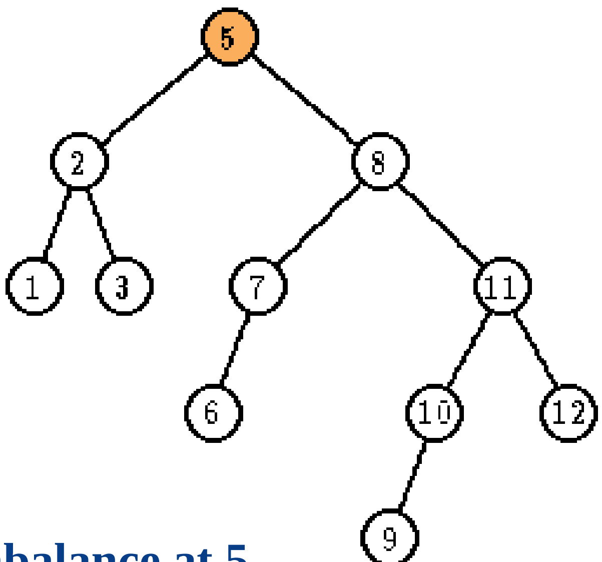
Deletion



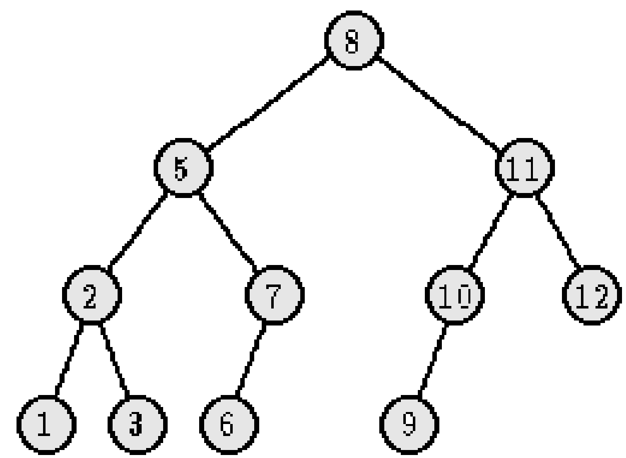
Delete 4



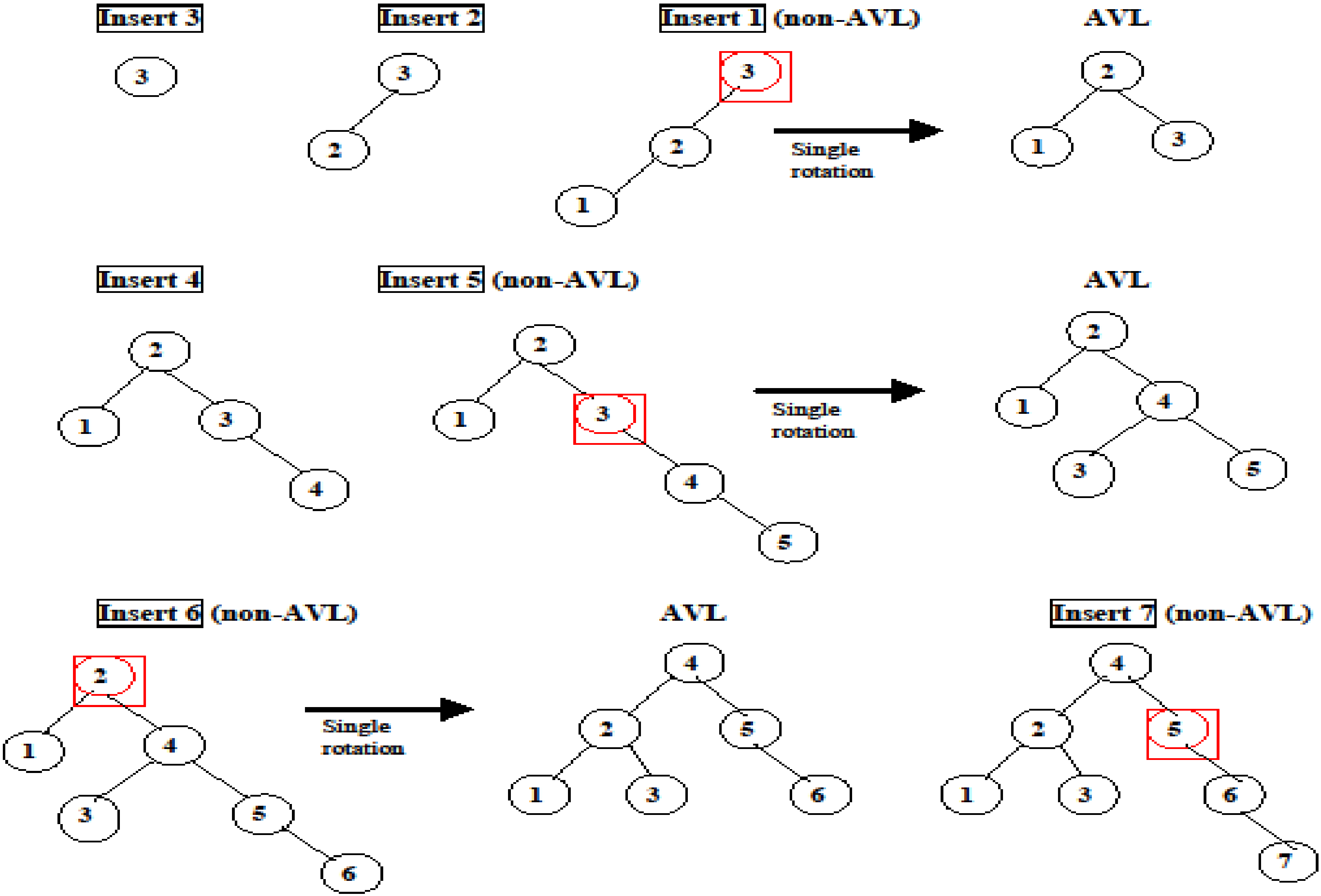
Imbalance at 3
Perform rotation with 2

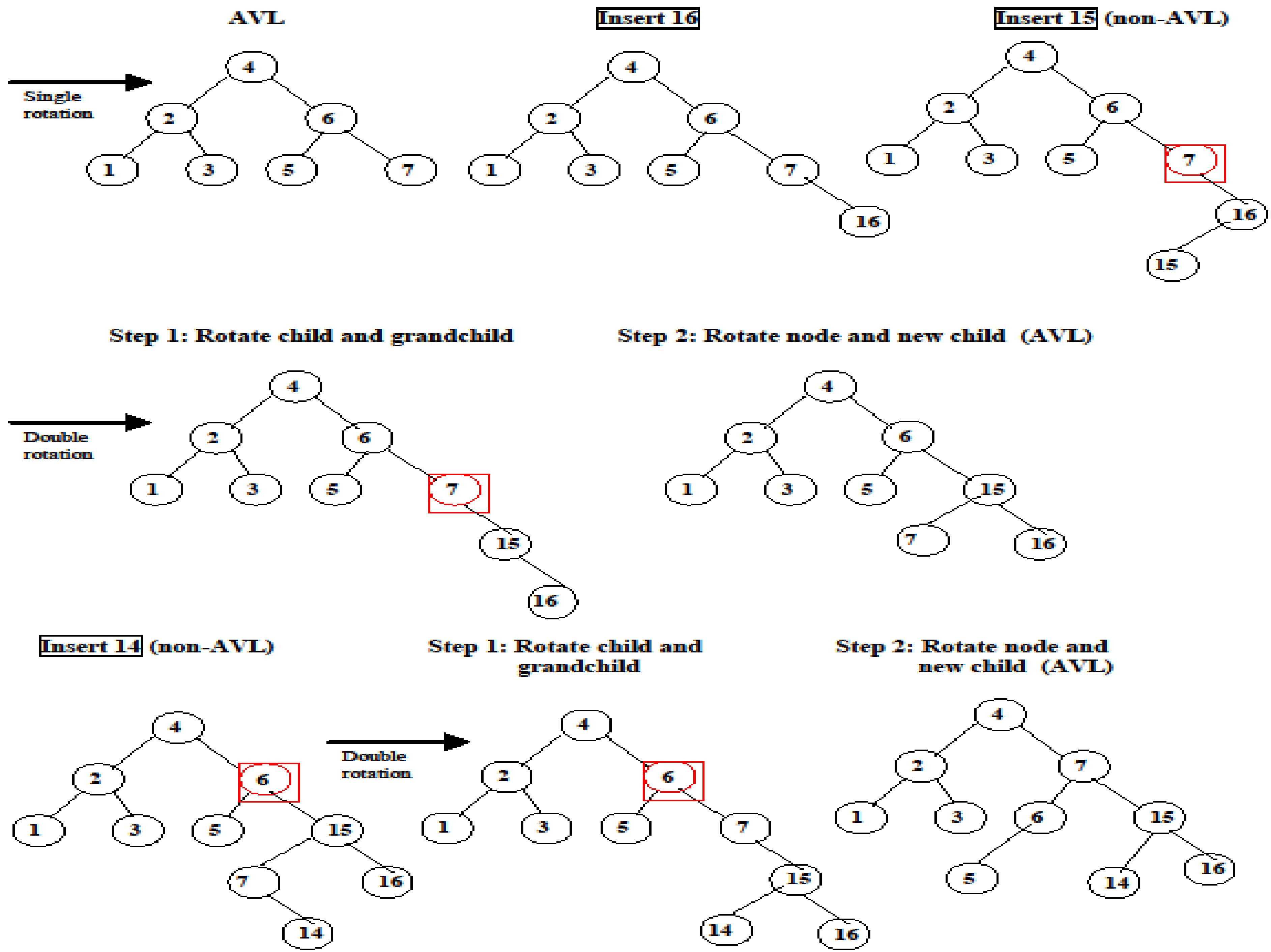


Imbalance at 5
Perform rotation with 8



Insert 3,2,1,4,5,6,7, 16,15,14




AVL
Insert 16
Insert 15 (non-AVL)
Single rotation
Step 1: Rotate child and grandchild
Step 2: Rotate node and new child (AVL)
Double rotation
Insert 14 (non-AVL)
Double rotation
Step 1: Rotate child and grandchild
Step 2: Rotate node and new child (AVL)