1) Given $f(z) = \log_e(1+z)$ where $z = x^T x$, $x \in R^d$

Solve:

If $x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_d \end{bmatrix}$

Then,

$$x^T = \begin{bmatrix} x_1 & x_2 & \cdots\cdots & x_d \end{bmatrix}$$

$$\therefore x^T x = \begin{bmatrix} x_1^2 & x_2^2 & \cdots\cdots & x_d^2 \end{bmatrix}$$

Now, Applying the chain rule,

$$\frac{df}{dx} = \frac{df}{dz} \cdot \frac{dz}{dx}$$

$$= \frac{d}{dz}\left(\log_e(1+z)\right) \cdot \frac{d}{dx}(x^T x)$$

$$= \frac{1}{1+z} \frac{d}{dz}(1+z) \cdot \frac{d}{dx}(x_1^2 + x_2^2 + \cdots\cdots + x_d^2)$$

$$= \frac{1}{1+z} 1 \cdot (2x_1 + 2x_2 + \cdots\cdots + 2x_d)$$

$$= \frac{1}{1+z} \cdot 2(x_1 + x_2 + \cdots + x_d)$$

$$= \frac{2}{1+z} \sum_{i=1}^{i=d} x_i$$

(Ans)

② $f(z) = e^{-z/2}$, where $z = g(y)$, $g(y) = y^T S^{-1} y$, $y = h(x)$,
$h(x) = x - \mu$

solve:

using the chain rule,

$$\frac{df}{dx} = \frac{df}{dz} \cdot \frac{dz}{dy} \cdot \frac{dy}{dx} \quad \text{— ①}$$

here,
$$\frac{df}{dz} = \frac{d}{dz}\left(e^{-z/2}\right) = \frac{e^{-z/2}}{2}$$

Now,
$$\frac{dz}{dy} = \frac{d}{dy}\left(y^T S^{-1} y\right)$$

$$= \lim_{h \to 0} \frac{g(y+h) - g(y)}{h}$$

$$= \lim_{h \to 0} \frac{(y^T + h) S^{-1} (y+h) - y^T S^{-1} y}{h}$$

$$= \lim_{h \to 0} \frac{(y^T S^{-1} + h S^{-1})(y+h) - y^T S^{-1} y}{h}$$

$$= \lim_{h \to 0} \frac{y^T S^{-1} y + y^T S^{-1} h + y h S^{-1} + h^2 S^{-1} - y^T S^{-1} y}{h}$$

$$= \lim_{h \to 0} \frac{h(y^T S^{-1} + y S^{-1} + h S^{-1})}{h}$$

$$= \lim_{h \to 0} \frac{\cancel{h}(y^T S^{-1} + y S^{-1} + h S^{-1})}{\cancel{h}}$$

$$= \lim_{h \to 0} (y^T S^{-1} + y S^{-1} + h S^{-1})$$

$$= y^T S^{-1} + y S^{-1} \quad \text{(1)}$$

$$\frac{dy}{dx} = \frac{d}{dx}(x - \mu) = 1$$

from eq 1,

$$\frac{df}{dx} = \frac{df}{dz} \cdot \frac{dz}{dy} \cdot \frac{dy}{dx}$$

$$= \frac{e^{-z/2}}{2} (y^T S^{-1} + y S^{-1}) . (1)$$

$$= \frac{e^{-z/2}}{2S} (y^T + y)$$

(Ans)