CHAPTER 17

OSCILLATIONS

Each day we encounter many kinds of oscillatory motion. Common examples include the swinging pendulum of a clock, a person bouncing on a trampoline, and a vibrating guitar string. Examples on the microscopic scale are vibrating atoms in the quartz crystal of a wristwatch and vibrating molecules of air that transmit sound waves. In addition to these mechanical oscillations, we can also have electromagnetic oscillations, such as electrons surging back and forth in circuits that are responsible for transmitting and receiving radio or TV signals.

These oscillating systems—whether mechanical, electromagnetic, or other types—have a common mathematical formulation and are most easily expressed in terms of sine and cosine functions. In this chapter we concentrate on mechanical oscillations and their description. Later in this book we deal with various kinds of waves and with electromagnetic oscillations, which use the same mathematical description.

17-1 OSCILLATING SYSTEMS

Imagine an oscillating system, such as the pendulum of a clock or a mass on a spring. What must be the properties of the force that produces such oscillations?

If you displace a pendulum in one direction from its equilibrium position, the force (which is due to gravity) pushes it back toward equilibrium. If you displace it in the other direction, the force still acts toward the equilibrium position. *No matter what the direction of the displacement, the force always acts in a direction to restore the system to its equilibrium position.* Such a force is called a *restoring force*. (The equilibrium position is the kind we called *stable* in Chapter 12; the system tends to return to equilibrium when slightly displaced.)

Let us consider a simple example. Suppose we have a particle that is free to move only in the x direction, and let the particle experience a force of constant magnitude F_m that acts in the $+x$ direction when $x < 0$ and in the $-x$ direction when $x > 0$, as shown in Fig. 17-1*a*. The force, which is shown in Fig. 17-1*b*, is similar to the forces that give the piecewise constant accelerations we considered in Chapter 2.

A particle of mass m initially at rest at coordinate $x = +x_m$ experiences a force whose x component is $-F_m$, and the corresponding x component of the acceleration of the particle is $-a_m = -F_m/m$. The particle moves toward its equilibrium position at $x = 0$ and reaches that position with velocity $v_x = -v_m$. When it passes through the origin to negative x, the force becomes $+F_m$, and the acceleration is $+a_m$. The particle slows and comes to rest for an instant at $x = -x_m$ before reversing its motion through the origin and returning eventually to $x = +x_m$. In the absence of friction and other dissipative forces, the cycle repeats endlessly.

Figure 17-2 shows the resulting motion, plotted in the style of the examples we considered in Chapter 2. The position $x(t)$ consists of a sequence of smoothly joined segments of parabolas, as is always the case for motion at constant acceleration. The particle oscillates back and forth between $x = +x_m$ and $x = -x_m$. The magnitude of the maximum displacement from equilibrium (x_m in this case) is called the *amplitude* of the motion. The time necessary for one complete cycle (a complete repetition of the motion) is called the *period T,* as indicated in Fig. 17-2. The

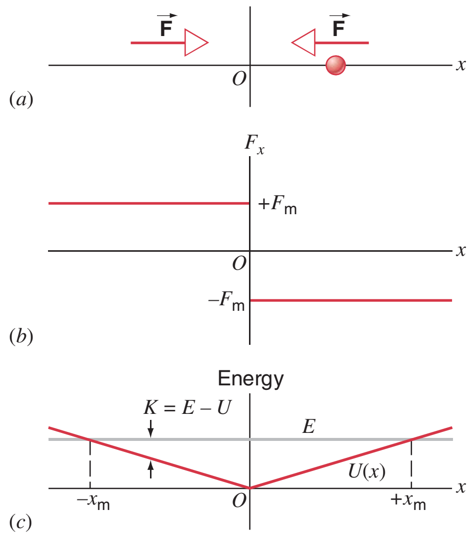

FIGURE 17-1. (a) A particle is acted on by a constant force $\vec{\mathbf{F}}$ that is always directed toward the origin. (b) A plot of this piecewise constant force, equal to $+F_m$ when $x < 0$ and to $-F_m$ when $x > 0$. Any real force of this type must be represented by a continuous function, even though it may be very steep as it goes through $x = 0$. (c) The potential energy corresponding to this force. If the system has total mechanical energy E, then at any location the difference $E - U$ gives the kinetic energy.

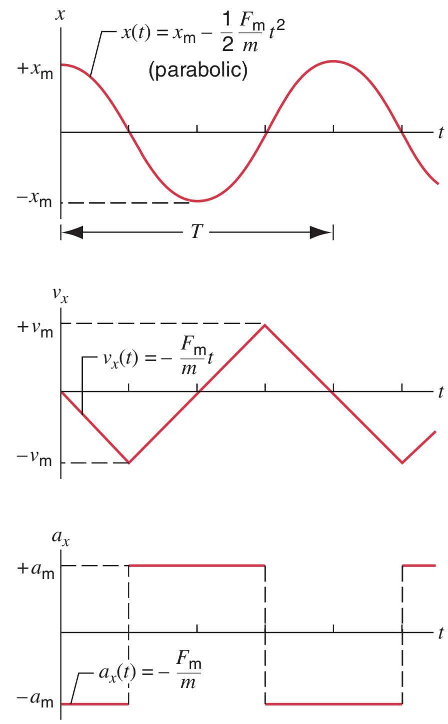

FIGURE 17-2. The position, velocity, and acceleration of the particle of Fig. 17-1 are plotted as functions of the time. The acceleration consists of alternating horizontal segments with values $+F_m/m$ and $-F_m/m$, the velocity consists of alternating linear segments with slopes $+F_m/m$ and $-F_m/m$, and the position consists of smoothly joined sections of parabolas. Because the force $F_x(x)$ is in reality a continuous function, $a_x(t)$ is also continuous, the horizontal segments having steep connections. Moreover, the sharp corners of $v_x(t)$ are rounded. The curves shown, however, are excellent approximations if the force changes from $+F_m$ to $-F_m$ over a very short interval.

number of cycles per unit time is called the *frequency* f. The frequency and the period are reciprocals of one another:

$$f = 1/T. \tag{17-1}$$

Period is measured in time units (seconds, for instance), whereas frequency is measured in the SI unit of hertz (Hz),* where 1 Hz = 1 cycle/s. Thus, for example, an oscillation with a period of $T = 5$ s has a frequency $f = 0.2$ Hz.

So far we have used a dynamical description of the oscillation. Often a description in terms of energy is useful. Figure 17-1c shows the potential energy corresponding to the force of Fig. 17-1b. Note that, as indicated by the expression $F = -dU/dx$, the negative of the slope $U(x)$ gives the force. The mechanical energy $E = K + U$ remains constant for an isolated system. At every point, the difference $E - U$ gives the kinetic energy K at that point. If we extended the graph to sufficiently large displacements, we would eventually reach locations where $E = U$ and thus $K = 0$. At these points, as Fig. 17-2 shows, the velocity is zero and the position is $x = \pm x_m$. These points are called the *turning points* of the motion.

Figures 17-1b and 17-1c illustrate two equivalent ways of describing the conditions for oscillation: the force must always act to restore the particle to equilibrium, and the potential energy must have a minimum at the equilibrium position.

The case of constant acceleration is always pleasant to work with, because the mathematics is simple, but it is seldom an accurate description of nature. Figure 17-3a shows an example of a more realistic force that can produce oscillatory motion. Such a force is responsible for the binding of molecules containing two atoms. The force increases rapidly as we try to push one atom close to the other; this repulsive component keeps the molecule from collapsing. As we try to pull the atoms to larger spacings, the force tends to oppose our efforts; this force may be an electrostatic force between two opposite electric charges, but often it is more complex and involves the spatial distribution of electronic orbits in atoms.

Figure 17-3b shows the corresponding potential energy function $U(x)$. Note that, as was the case in Fig. 17-1, the force changes sign at the equilibrium position, and the potential energy has a minimum at that position. Note that in this case the turning points (labeled x_1 and x_2 in Fig. 17-3) are *not* symmetrically located about the equilibrium position. If we were to stretch the molecule a bit beyond its equilibrium configuration and release it (which often occurs when a molecule absorbs infrared radiation), it would execute periodic motion about equilibrium, although the mathematical description would be more complex than that of Fig. 17-2. The study of these oscillations is an important

* The frequency unit is named after Heinrich Hertz (1857–1894), whose research provided the experimental confirmation of electromagnetic waves.

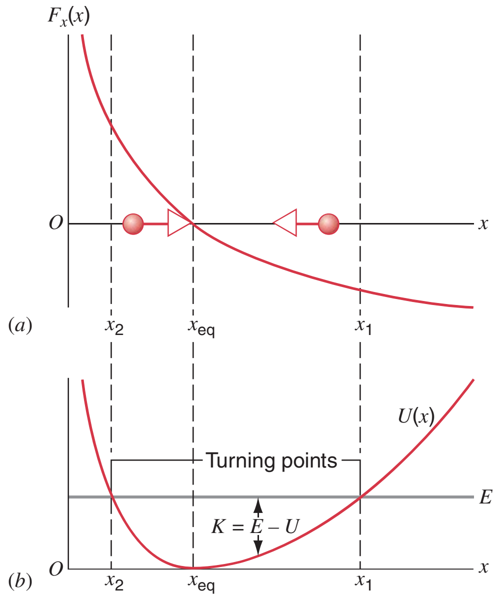

FIGURE 17-3. (*a*) The force that acts on a particle oscillating between the limits x_1 and x_2. Note that the force always tends to push the particle toward its equilibrium position, as in Fig. 17-1. Such a force might act on an atom in a molecule. (*b*) The potential energy corresponding to this force.

technique for learning about molecular structure, as we discuss in Section 17-9.

17-2 THE SIMPLE HARMONIC OSCILLATOR

The motion of a particle in a complex system, such as an atom in the vibrating molecule discussed in the previous section, is easier to analyze if we consider the motion to be a superposition of *harmonic* oscillations, which can be described in terms of sine and cosine functions.

Consider an oscillating system in one dimension, consisting of a particle subject to a force

$$F_x(x) = -kx, \tag{17-2}$$

in which k is a constant and x is the displacement of the particle from its equilibrium position. Such an oscillating system is called a *simple harmonic oscillator,* and its motion is called *simple harmonic motion.* The potential energy corresponding to this force is

$$U(x) = \tfrac{1}{2}kx^2. \tag{17-3}$$

The force and potential energy are of course related by $F_x(x) = -dU/dx$. As indicated by Eq. 17-2 and plotted in Fig. 17-4*a*, the force acting on the particle is directly proportional to the displacement but is opposite to it in direction. Equation 17-3 shows that the potential energy varies as the square of the displacement, as illustrated by the parabolic curve in Fig. 17-4*b*.

You will recognize Eqs. 17-2 and 17-3 as the expressions for the force and potential energy of an "ideal" spring

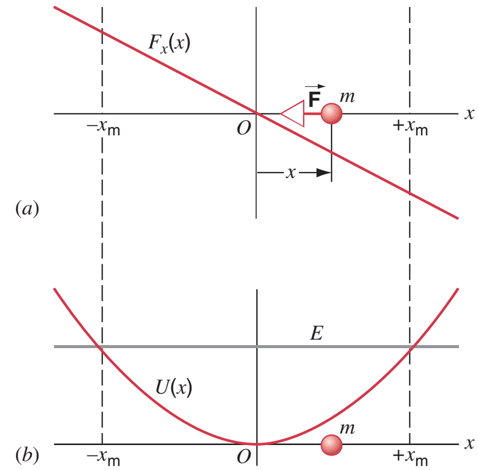

FIGURE 17-4. (*a*) The force and (*b*) the corresponding potential energy of a simple harmonic oscillator. Note the similarities and differences with Fig. 17-3.

of force constant k, compressed or extended by a distance x; see Section 11-4. Hence, *a body of mass m attached to an ideal spring of force constant k and free to move over a frictionless horizontal surface is an example of a simple harmonic oscillator* (see Fig. 17-5). Note that there is a position (the equilibrium position; see Fig. 17-5*b*) in which the spring exerts no force on the body. If the body is displaced to the right (as in Fig. 17-5*a*), the force exerted by

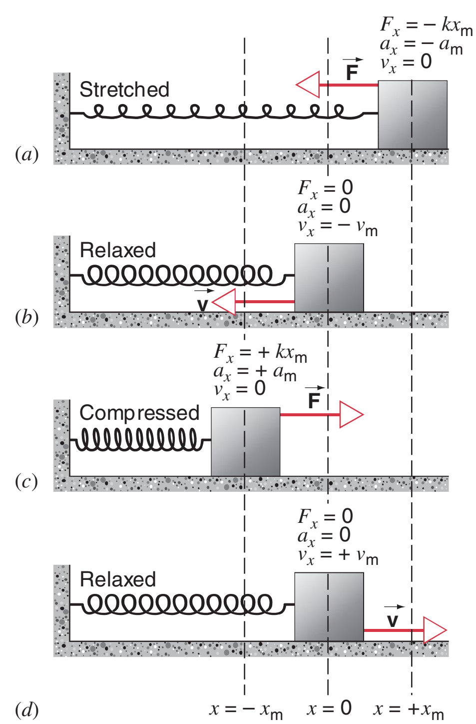

FIGURE 17-5. A simple harmonic oscillator, consisting of a spring acting on a body that slides on a frictionless horizontal surface. In (*a*), the spring is stretched so that the body has its maximum displacement from equilibrium. In (*c*) the spring is fully compressed. In (*b*) and (*d*), the body is passing through equilibrium with its maximum speed, and the spring is relaxed.

the spring on the body points to the left. If the body is displaced to the left (as in Fig. 17-5*c*), the force points to the right. In each case the force is a *restoring* force. (It is in this case a *linear* restoring force—that is, proportional to the first power of x.)

Let us apply Newton's second law, $\Sigma F_x = ma_x$, to the motion of Fig. 17-5. For ΣF_x we substitute $-kx$ and for the acceleration a_x we put in d^2x/dt^2 $(= dv_x/dt)$. This gives us

$$-kx = m\frac{d^2x}{dt^2}$$

or

$$\frac{d^2x}{dt^2} + \frac{k}{m}x = 0. \tag{17-4}$$

Equation 17-4 is called the *equation of motion* of the simple harmonic oscillator. Its solution, which we describe in the next section, is a function $x(t)$ that describes the position of the oscillator as a function of the time, in analogy with Fig. 17-2*a*, which represents the variation of position with time of a different oscillator.

The simple harmonic oscillator problem is important for two reasons. First, many problems involving mechanical vibrations at small amplitudes reduce to that of the simple harmonic oscillator, or to a combination of such oscillators. This is equivalent to saying that if we consider a small enough portion of a restoring force curve near the equilibrium position, Fig. 17-3*a*, for instance, it becomes arbitrarily close to a straight line, which, as Fig. 17-4*a* shows, is characteristic of simple harmonic motion. Or, in other words, the potential energy curve of Fig. 17-3*b* is very nearly parabolic near the equilibrium position.

Second, as we have indicated, equations like Eq. 17-4 occur in many physical problems in acoustics, optics, mechanics, electrical circuits, and even atomic physics. The simple harmonic oscillator exhibits features common to many physical systems.

17-3 SIMPLE HARMONIC MOTION

Let us now solve the equation of motion of the simple harmonic oscillator,

$$\frac{d^2x}{dt^2} + \frac{k}{m}x = 0. \tag{17-4}$$

We derived Eq. 17-4 for a spring force $F_x = -kx$ (where the force constant k is a measure of the stiffness of the spring) acting on a particle of mass m. We shall see later that other oscillating systems are governed by similar equations of motion, in which the constant k is related to other physical features of the system. We can use the oscillating mass–spring system as our prototype.

Equation 17-4 gives a relation between a function of the time $x(t)$ and its second time derivative d^2x/dt^2. Our goal is to find a function $x(t)$ that satisfies this relation. We begin by rewriting Eq. 17-4 as

$$\frac{d^2x}{dt^2} = -\left(\frac{k}{m}\right)x. \tag{17-5}$$

Equation 17-5 requires that $x(t)$ be a function whose second derivative is the negative of the function itself, except for a constant factor k/m. We know from calculus that the sine and cosine functions have this property. For example,

$$\frac{d}{dt}\cos\omega t = -\omega\sin\omega t$$

and

$$\frac{d^2}{dt^2}\cos\omega t = \frac{d}{dt}(-\omega\sin\omega t) = -\omega^2\cos\omega t.$$

The second derivative of a cosine (or of a sine) gives us back the original function multiplied by a negative factor $-\omega^2$. This property is not affected if we multiply the cosine function by any constant. We choose the constant to be x_m, so that the maximum value of x (the amplitude of the motion) will be x_m.

We write a tentative solution to Eq. 17-5 as

$$x = x_m\cos(\omega t + \phi). \tag{17-6}$$

Here, since

$$\begin{aligned} x_m\cos(\omega t + \phi) &= x_m\cos\phi\cos\omega t - x_m\sin\phi\sin\omega t \\ &= A\cos\omega t + B\sin\omega t, \end{aligned}$$

where $A = x_m\cos\phi$ and $B = -x_m\sin\phi$, the constant ϕ allows for any combination of sine and cosine solutions.

With the (as yet) unknown constants x_m, ω, and ϕ, we have written as general a solution to Eq. 17-5 as we can. To determine these constants such that Eq. 17-6 is actually the solution of Eq. 17-5, we differentiate Eq. 17-6 twice with respect to the time. We have

$$\frac{dx}{dt} = -\omega x_m\sin(\omega t + \phi)$$

and

$$\frac{d^2x}{dt^2} = -\omega^2x_m\cos(\omega t + \phi).$$

Putting this into Eq. 17-5, we obtain

$$-\omega^2x_m\cos(\omega t + \phi) = -\frac{k}{m}x_m\cos(\omega t + \phi).$$

Therefore, if we choose the constant ω such that

$$\omega^2 = \frac{k}{m}, \tag{17-7}$$

then Eq. 17-6 is in fact a solution of the equation of motion of a simple harmonic oscillator.

The constants x_m and ϕ are still undetermined and therefore still completely arbitrary. This means that *any* choice of x_m and ϕ whatsoever will satisfy Eq. 17-5, so that a large variety of motions (all of which have the same ω) is possible for the oscillator. We shall see later that x_m and ϕ are determined for a particular harmonic motion by how the motion starts.

Let us find the physical significance of the constant ω. If we increase the time t in Eq. 17-6 by $2\pi/\omega$, the function becomes

$$\begin{aligned} x &= x_m \cos[\omega(t + 2\pi/\omega) + \phi] \\ &= x_m \cos(\omega t + 2\pi + \phi) \\ &= x_m \cos(\omega t + \phi). \end{aligned}$$

That is, the function merely repeats itself after a time $2\pi/\omega$. Therefore $2\pi/\omega$ is the period of the motion T. Since $\omega^2 = k/m$, we have

$$T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{m}{k}}. \tag{17-8}$$

Hence all motions given by Eq. 17-5 have the same period of oscillation, which is determined only by the mass m of the oscillating particle and the force constant k of the spring. The frequency f of the oscillator is the number of complete vibrations per unit time and is given by

$$f = \frac{1}{T} = \frac{1}{2\pi}\sqrt{\frac{k}{m}}. \tag{17-9}$$

Hence

$$\omega = 2\pi f = \frac{2\pi}{T}. \tag{17-10}$$

The quantity ω is called the *angular frequency;* it differs from the frequency f by a factor 2π. It has the dimension of reciprocal time (the same as angular speed), and its unit is the radian/second. In Section 17-6 we give a geometric meaning to this angular frequency.

The constant x_m has a simple physical meaning. The cosine function takes on values from -1 to $+1$. The displacement x from the central equilibrium position $x = 0$ therefore has a maximum value of x_m; see Eq. 17-6. We call x_m the *amplitude* of the motion. Because x_m is not fixed by Eq. 17-4, motions of various amplitudes are possible, but all have the same frequency and period. *The frequency of a simple harmonic motion is independent of the amplitude of the motion.*

The quantity $(\omega t + \phi)$ is called the *phase* of the motion. The constant ϕ is called the *phase constant.* Two motions may have the same amplitude and frequency but differ in phase. If $\phi = -\pi/2 = -90°$, for example,

$$\begin{aligned} x = x_m \cos(\omega t + \phi) &= x_m \cos(\omega t - 90°) \\ &= x_m \sin \omega t \end{aligned}$$

so that the displacement is zero at the time $t = 0$. If $\phi = 0$, on the other hand, the displacement $x = x_m \cos \omega t$ has its maximum value $x = x_m$ at the time $t = 0$. Other initial displacements correspond to other phase constants. See Sample Problem 17-3 for an example of the method of finding x_m and ϕ from the initial displacement and velocity.

The amplitude x_m and the phase constant ϕ of the oscillation are determined by the initial position and velocity of the particle. These two initial conditions will specify x_m and ϕ exactly (except that ϕ may be increased or decreased by any multiple of 2π without changing the motion). Once the motion has started, however, the particle will continue to oscillate with a constant amplitude and phase constant at a fixed frequency, unless other forces disturb the system.

In Fig. 17-6 we plot the displacement x versus the time t for several simple harmonic motions described by Eq. 17-6. Three comparisons are made. In Fig. 17-6*a*, the two curves have the same amplitude and frequency but differ in phase by $\phi = \pi/4$ or 45°. In Fig. 17-6*b*, the two curves have the same frequency and phase constant but differ in amplitude by a factor of 2. In Fig. 17-6*c*, the curves have the same amplitude and phase constant but differ in frequency by a factor of $\frac{1}{2}$ or in period by a factor of 2. Study these curves carefully to become familiar with the terminology used in simple harmonic motion.

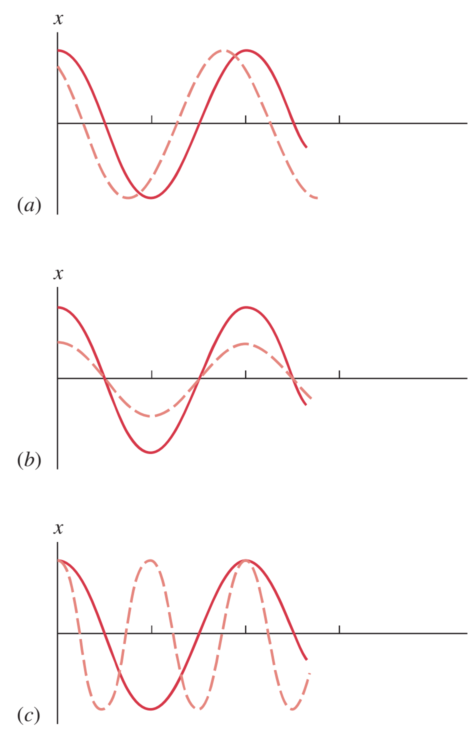

FIGURE 17-6. (*a*) Comparison of the motions of two simple harmonic oscillators of the same amplitude and frequency but differing in phase constant by 45°. If the motion is represented by Eq. 17-6, then the solid curve has $\phi = 0°$ and the dashed curve has $\phi = 45°$. (*b*) Two simple harmonic motions of the same phase constant and frequency but differing in amplitude by a factor of 2. (*c*) Two simple harmonic motions of the same amplitude and phase constant (0°) but differing in frequency by a factor of 2. The solid curve has twice the *period,* and therefore half the *frequency,* of the dashed curve.

Another distinctive feature of simple harmonic motion is the relation between the displacement, the velocity, and the acceleration of the oscillating particle. Let us compare these quantities. In Fig. 17-7 we plot separately the displacement x versus the time t, the velocity $v_x = dx/dt$ versus the time t, and the acceleration $a_x = dv_x/dt = d^2x/dt^2$ versus the time t. The equations of these curves are

$$x = x_m \cos(\omega t + \phi),$$

$$v_x = \frac{dx}{dt} = -\omega x_m \sin(\omega t + \phi), \qquad (17\text{-}11)$$

$$a_x = \frac{dv_x}{dt} = -\omega^2 x_m \cos(\omega t + \phi).$$

For the case plotted we have taken $\phi = 0$. The units and scale of displacement, velocity, and acceleration are omitted for simplicity of comparison. The displacement, velocity, and acceleration all oscillate harmonically. Note that the maximum displacement (amplitude) is x_m, the maximum speed (velocity amplitude) is ωx_m, and the maximum acceleration (acceleration amplitude) is $\omega^2 x_m$.

When the displacement is a maximum in either direction, the speed is zero because the velocity must now change its direction. The acceleration at this instant, like the restoring force, has a maximum magnitude but is directed opposite to the displacement. When the displacement is zero, the speed of the particle is a maximum and the acceleration is zero, corresponding to a zero restoring force. The speed increases as the particle moves toward the equilibrium position and then decreases as it moves out to the maximum displacement. Compare Fig. 17-7 with Fig. 17-2, and note their similarities and differences.

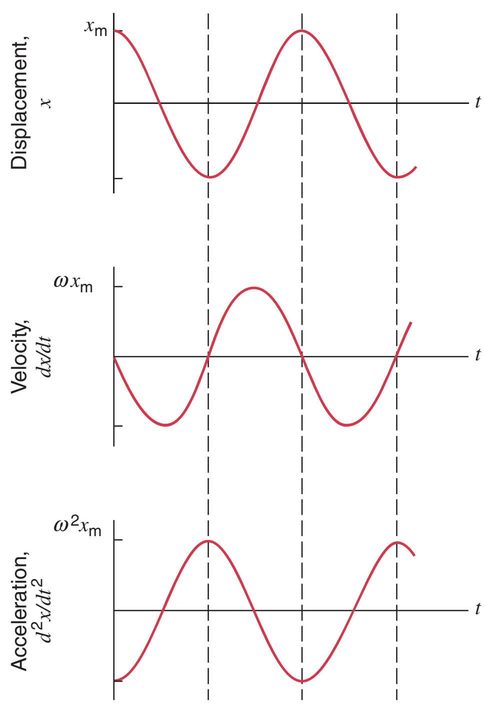

FIGURE 17-7. The displacement, velocity, and acceleration of a simple harmonic oscillator, according to Eqs. 17-11.

SAMPLE PROBLEM 17-1. A certain spring hangs vertically. When a body of mass $M = 1.65$ kg is suspended from it, its length increases by 7.33 cm. The spring is then mounted horizontally, and a block of mass $m = 2.43$ kg is attached to the spring. The block is free to slide along a frictionless horizontal surface, as in Fig. 17-5. (*a*) What is the force constant k of the spring? (*b*) What is the magnitude of the horizontal force required to stretch the spring by a distance of 11.6 cm? (*c*) When the block is displaced a distance of 11.6 cm and released, with what period will it oscillate?

Solution (*a*) The force constant k is determined from the force Mg necessary to stretch the spring by the measured vertical displacement $y = -7.33$ cm. When the suspended body is hanging at rest, $\Sigma F_y = 0$; the y component of the net force on the body is $\Sigma F_y = -ky - Mg$, so $ky = -Mg$, or

$$k = -Mg/y = -(1.65\ \text{kg})(9.80\ \text{m/s}^2)/(-0.0733\ \text{m}) = 221\ \text{N/m}.$$

(*b*) The magnitude of the horizontal force needed to stretch the spring by 11.6 cm is determined from Hooke's law (Eq. 17-2) using the force constant we found in part (*a*):

$$F = kx = (221\ \text{N/m})(0.116\ \text{m}) = 25.6\ \text{N}.$$

(*c*) The period is independent of the amplitude and depends only on the values of the mass of the block and the force constant. From Eq. 17-8,

$$T = 2\pi\sqrt{\frac{m}{k}} = 2\pi\sqrt{\frac{2.43\ \text{kg}}{221\ \text{N/m}}} = 0.6589\ \text{s} = 659\ \text{ms}.$$

(We display the value of T to four significant figures, more than are justified by the input data, because we shall need this result in the solution of Sample Problem 17-2. To avoid rounding errors in intermediate steps, it is standard practice to carry excess significant figures in this way. The final result, of course, must be properly rounded.)

17-4 ENERGY IN SIMPLE HARMONIC MOTION

In any motion in which no dissipative forces act, the total mechanical energy $E\ (= K + U)$ is conserved (remains constant). We can now study this in more detail for the special case of simple harmonic motion.

The potential energy U at any instant is given by

$$U = \tfrac{1}{2}kx^2 = \tfrac{1}{2}kx_m^2 \cos^2(\omega t + \phi). \qquad (17\text{-}12)$$

where we have used Eq. 17-6 for the displacement x. The potential energy thus oscillates with time and has a maximum value of $\frac{1}{2}kx_m^2$. During the motion, the potential energy varies between zero and this maximum value, as the curves in Figs. 17-8*a* and 17-8*b* show.

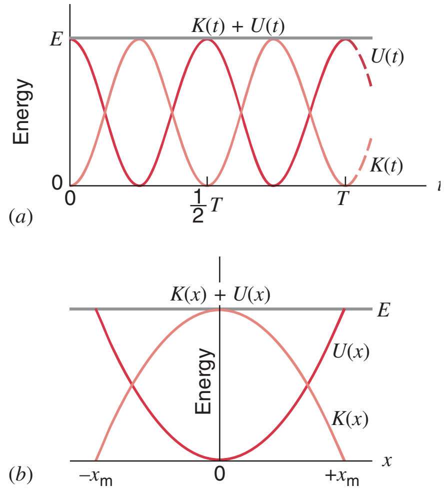

Figure 17-8. The potential energy U, kinetic energy K, and total mechanical energy E of a particle undergoing simple harmonic motion (with $\phi = 0$) are shown as functions of (a) the time and (b) the displacement. Note that in (a) the kinetic and potential energies each reach their maxima twice during each period of the motion. See also Fig. 12-5.

The kinetic energy K at any instant is $\frac{1}{2}mv_x^2$. Using Eq. 17-11 for $v_x(t)$ and Eq. 17-7 for ω^2, we obtain

$$\begin{aligned} K &= \tfrac{1}{2}mv_x^2 \\ &= \tfrac{1}{2}m\omega^2 x_m^2 \sin^2(\omega t + \phi) \\ &= \tfrac{1}{2}kx_m^2 \sin^2(\omega t + \phi). \end{aligned} \tag{17-13}$$

The kinetic energy, like the potential energy, oscillates with time and has a maximum value of $\frac{1}{2}kx_m^2$. During the motion, the kinetic energy varies between zero and this maximum value, as shown by the curves in Figs. 17-8a and 17-8b. Note that the kinetic and potential energies vary with twice the frequency (half the period) of the displacement and velocity. Can you explain this?

The total mechanical energy is the sum of the kinetic energy and the potential energy. Using Eqs. 17-12 and 17-13, we obtain

$$\begin{aligned} E = K + U &= \tfrac{1}{2}kx_m^2 \sin^2(\omega t + \phi) + \tfrac{1}{2}kx_m^2 \cos^2(\omega t + \phi) \\ &= \tfrac{1}{2}kx_m^2. \end{aligned} \tag{17-14}$$

We see that the total mechanical energy is constant, as we expect, and has the value $\frac{1}{2}kx_m^2$. At the maximum displacement the kinetic energy is zero, but the potential energy has the value $\frac{1}{2}kx_m^2$. At the equilibrium position the potential energy is zero, but the kinetic energy has the value $\frac{1}{2}kx_m^2$. At other positions the kinetic and potential energies each contribute terms whose sum is always $\frac{1}{2}kx_m^2$. This constant total energy E is shown in Figs. 17-8a and 17-8b. The total energy of a particle executing simple harmonic motion is proportional to the square of the amplitude of the motion. It can be shown (see Problem 14) that the average kinetic energy for the motion during one period is exactly equal to the average potential energy and that each of these average quantities is half the total energy, or $\frac{1}{4}kx_m^2$.

Equation 17-14 can be written quite generally as

$$K + U = \tfrac{1}{2}mv_x^2 + \tfrac{1}{2}kx^2 = \tfrac{1}{2}kx_m^2. \tag{17-15}$$

From this relation we obtain $v_x^2 = (k/m)(x_m^2 - x^2)$ or

$$v_x = \pm\sqrt{\frac{k}{m}(x_m^2 - x^2)}. \tag{17-16}$$

This relation shows clearly that the speed is a maximum at the equilibrium position ($x = 0$) and is zero at the extreme displacements ($x = \pm x_m$). In fact, we can start from the conservation of energy, Eq. 17-15 (in which $\frac{1}{2}kx_m^2 = E$), and by integration of Eq. 17-16 obtain the displacement as a function of time, as we did in Section 12-5, where we obtained a result identical to Eq. 17-6 with $\phi = 0$.

Sample Problem 17-2. The block–spring combination of Sample Problem 17-1 is stretched in the positive x direction a distance of 11.6 cm from equilibrium and released. (a) What is the total energy stored in the system? (b) What is the maximum speed of the block? (c) What is the magnitude of the maximum acceleration? (d) If the block is released at $t = 0$, what are its position, velocity, and acceleration at $t = 0.215$ s?

Solution (a) The amplitude of the motion is given as $x_m = 0.116$ m. The total energy is given by Eq. 17-14:

$$E = \tfrac{1}{2}kx_m^2 = \tfrac{1}{2}(221\ \text{N/m})(0.116\ \text{m})^2 = 1.49\ \text{J}.$$

(b) The maximum kinetic energy is numerically equal to the total energy; when $U = 0$, $K = K_{max} = E$. The maximum speed is then

$$v_{max} = \sqrt{\frac{2K_{max}}{m}} = \sqrt{\frac{2(1.49\ \text{J})}{2.43\ \text{kg}}} = 1.11\ \text{m/s}.$$

(c) The maximum acceleration occurs just at the instant of release, when the force is greatest:

$$a_{max} = \frac{F_{max}}{m} = \frac{kx_m}{m} = \frac{(221\ \text{N/m})(0.116\ \text{m})}{2.43\ \text{kg}} = 10.6\ \text{m/s}^2.$$

(d) From the period found in Sample Problem 17-1, we can obtain the angular frequency:

$$\omega = \frac{2\pi}{T} = \frac{2\pi}{0.6589\ \text{s}} = 9.536\ \text{rad/s}.$$

Since the block has its maximum displacement of $x_m = 0.116$ m at $t = 0$, its motion can be described by a cosine function:

$$x(t) = x_m \cos \omega t,$$

a result that follows by putting $\phi = 0$ in Eq. 17-6. At $t = 0.215$ s, we find

$$x = (0.116\ \text{m}) \cos (9.536\ \text{rad/s})(0.215\ \text{s}) = -0.0535\ \text{m}.$$

Note that the angle ωt, whose cosine we must find, is expressed in radians. The velocity is given by Eq. 17-11, which, with $\phi = 0$, becomes $v_x(t) = -\omega x_m \sin \omega t$. At 0.215 s, we obtain

$$\begin{aligned} v_x &= -(9.536\ \text{rad/s})(0.116\ \text{m}) \sin (9.536\ \text{rad/s})(0.215\ \text{s}) \\ &= -0.981\ \text{m/s}. \end{aligned}$$

To find the acceleration, we again use Eq. 17-11 and note that, at all times, $a_x = -\omega^2 x$:

$$a_x = -(9.536 \text{ rad/s})^2(-0.0535 \text{ m}) = +4.87 \text{ m/s}^2.$$

Let us examine our results to see whether they are reasonable. The time $t = 0.215$ s is between $T/4 = 0.165$ s and $T/2 = 0.330$ s. If the block begins at $x = x_m = +0.116$ m, then at $T/4$ it will pass through equilibrium, and it is certainly reasonable that at $t = 0.215$ s it is at a negative x coordinate, as we found. Since it is at that time moving toward $x = -x_m$, its velocity must be negative, as we found. However, it has already passed through the point of most negative velocity, and it is slowing as it approaches $x = -x_m$; therefore the acceleration should be positive. We can check the value of the acceleration from $a_x = kx/m$. We can also check the relationship between v_x and x using Eq. 17-16.

SAMPLE PROBLEM 17-3. The block of the block–spring system of Sample Problem 17-1 is pushed from equilibrium by an external force in the positive x direction. At $t = 0$, when the displacement of the block is $x = +0.0624$ m and its velocity is $v_x = +0.847$ m/s, the external force is removed and the block begins to oscillate. Write an equation for $x(t)$ during the oscillation.

Solution Since we have the same mass (2.43 kg) and force constant (221 N/m), the angular frequency is still 9.536 rad/s, as we found in Sample Problem 17-2. The most general equation for $x(t)$ is given by Eq. 17-6,

$$x(t) = x_m \cos(\omega t + \phi),$$

and we must find x_m and ϕ to complete the solution. To find x_m, let us compute the total energy, which at $t = 0$ has both kinetic and potential terms:

$$\begin{aligned} E &= K + U = \tfrac{1}{2}mv_x^2 + \tfrac{1}{2}kx^2 \\ &= \tfrac{1}{2}(2.43 \text{ kg})(0.847 \text{ m/s})^2 + \tfrac{1}{2}(221 \text{ N/m})(0.0624 \text{ m})^2 \\ &= 0.872 \text{ J} + 0.430 \text{ J} = 1.302 \text{ J}. \end{aligned}$$

Setting this equal to $\frac{1}{2}kx_m^2$, as Eq. 17-15 requires, we have

$$x_m = \sqrt{\frac{2E}{k}} = \sqrt{\frac{2(1.302 \text{ J})}{221 \text{ N/m}}} = 0.1085 \text{ m}.$$

To find the phase constant, we use the information given for $t = 0$:

$$x(0) = x_m \cos\phi$$

$$\cos\phi = \frac{x(0)}{x_m} = \frac{+0.0624 \text{ m}}{0.1085 \text{ m}} = +0.5751.$$

In the range of 0 to 2π, there are two values of ϕ whose cosine is $+0.5751$; the possible values are $\phi = 54.9°$ or $\phi = 305.1°$. Either one will satisfy the condition that $x(0)$ have the proper value, but only one will give the correct initial velocity:

$$\begin{aligned} v_x(0) &= -\omega x_m \sin\phi = -(9.536 \text{ rad/s})(0.1085 \text{ m}) \sin\phi \\ &= -(1.035 \text{ m/s}) \sin\phi \\ &= -0.847 \text{ m/s} \quad \text{for } \phi = 54.9° \\ &= +0.847 \text{ m/s} \quad \text{for } \phi = 305.1°. \end{aligned}$$

Obviously the second choice is the one we want, and we therefore take $\phi = 305.1° = 5.33$ radians. We can now write

$$x(t) = (0.109 \text{ m}) \cos[(9.54 \text{ rad/s})t + 5.33 \text{ rad}].$$

See Problem 9 for a derivation of the general relationships that permit x_m and ϕ to be calculated from $x(0)$ and $v_x(0)$.

17-5 APPLICATIONS OF SIMPLE HARMONIC MOTION

A few physical systems that move with simple harmonic motion are considered here. Others are found throughout the text.*

The Torsional Oscillator

Figure 17-9 shows a disk suspended by a wire attached to the center of mass of the disk. The wire is securely fixed to a solid support or clamp and to the disk. With the disk in equilibrium, a radial line is drawn from its center to a point P on its rim, as shown. If the disk is rotated in a horizontal (xy) plane so that the reference line OP moves to the position OQ, the wire will be twisted. The twisted wire will exert a restoring torque on the disk, tending to return the reference line to its equilibrium position. For small twists the restoring torque is found to be proportional to the angular displacement (Hooke's law), so that

$$\tau_z = -\kappa\theta. \tag{17-17}$$

Here κ (the Greek letter kappa) is a constant that depends on the properties of the wire and is called the *torsional* constant. The minus sign shows that the torque is directed op-

* See "A Repertoire of S.H.M.," by Eli Maor, *The Physics Teacher,* October 1972, p. 377, for a full discussion of 16 physical systems that exhibit simple harmonic motion.

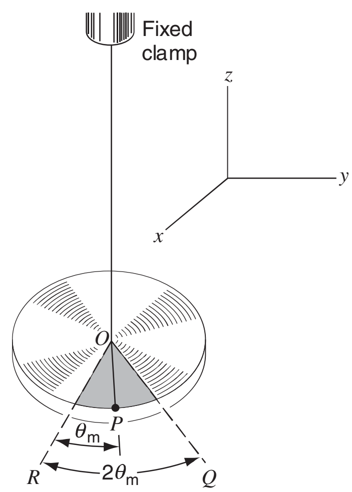

FIGURE 17-9. The torsional oscillator. The line drawn from O to P oscillates between OQ and OR, sweeping out an angle $2\theta_m$, where θ_m is the angular amplitude of the motion. The oscillation takes place in the xy plane; the z axis is along the wire.