CHAPTER 18

WAVE MOTION

Wave motion appears in almost every branch of physics. Surface waves on bodies of water are commonly observed. Sound waves and light waves are essential to our perception of the environment, because we have receptors (eyes and ears) capable of their detection. In the past century we learned how to produce and use radio waves. The similarity of the physical and mathematical descriptions of these different kinds of waves indicates that wave motion is one of the unifying themes of physics.

In this chapter and the next we develop the verbal and mathematical descriptions of waves. We use the example of mechanical waves, *in part because we have already developed the laws of mechanics in this text. Later in the text we develop the laws that govern other types of waves (light and other electromagnetic waves, for example). For simplicity, we concentrate on the study of harmonic waves (that is, those that can be represented by sine and cosine functions), but the principles that we develop apply to more complex waveforms as well.*

18-1 MECHANICAL WAVES

Waves are a common and essential part of our environment. We are surrounded by sound waves, light waves, water waves, and other kinds of waves, which we can control and use to convey information or transport energy from one location to another.

All types of waves use similar mathematical descriptions. We can therefore learn a great deal about waves in general by making a careful study of one type of wave. In this chapter we consider only *mechanical waves,* which include sound waves and water waves. In particular, we choose one special type of mechanical wave — the oscillation of a stretched string such as might be found on a guitar.

Mechanical waves travel through an elastic medium. They can originate when we cause an initial disturbance at one location in the medium. Because of the elastic properties of the medium, the disturbance travels through the medium.

On a microscopic level, the forces between atoms are responsible for the propagation of mechanical waves. Each atom exerts a force on the atoms that surround it, and through this force the motion of the atom is transmitted to its neighbors. However, the particles of the medium do not experience any net displacement in the direction of the wave — as the wave passes, the particles simply move back and forth through small distances about their equilibrium positions.

For example, a leaf floating on a lake may bob up and down as a wave passes, but after the wave has passed the leaf returns very nearly to its original position. A sound wave can travel through air, but there is no net motion of the air molecules in the direction that the wave is moving. The wave can transport energy and momentum from one location to another without any material particles making that journey. As Leonardo da Vinci observed about water waves in the 15th century: "It often happens that the wave flees the place of its creation, while the water does not; like the waves made in a field of grain by the wind, where we see the waves running across the field while the grain remains in place."

18-2 TYPES OF WAVES

In listing water waves, light waves, and sound waves as examples of wave motion, we are classifying waves according to their broad physical properties. Waves can also be classified in other ways.

1. *Direction of particle motion.* We can classify mechanical waves by considering how the direction of motion of the particles of the medium is related to the direction of propagation of the wave. If the motion of the particles is perpendicular to the direction of propagation of the wave itself, we have a *transverse* wave. For example, when a string under tension is set oscillating back and forth at one end, a transverse wave travels along the string; the disturbance moves along the string but the string particles vibrate at right angles to the direction of propagation of the disturbance (Fig. 18-1*a*). Light waves, although they are not mechanical waves, are also transverse waves.

If, however, the motion of the particles in a mechanical wave is back and forth along the direction of propagation, we have a *longitudinal* wave. For example, when a spring under tension is set oscillating back and forth at one end, a longitudinal wave travels along the spring; the coils vibrate back and forth parallel to the direction in which the disturbance travels along the spring (Fig. 18-1*b*). Sound waves in a gas are longitudinal waves. We discuss them in greater detail in Chapter 19.

Some waves are neither purely longitudinal nor purely transverse. For example, in waves on the surface of water the particles of water move both up and down and back and forth, tracing out elliptical paths as the water waves move by.

2. *Number of dimensions.* Waves can also be classified as propagating in one, two, and three dimensions. Waves moving along the string or spring of Fig. 18-1 are one-dimensional. Surface waves or ripples on water, caused by dropping a pebble into a quiet pond, are two-dimensional (Fig. 18-2). Sound waves and light waves traveling radially outward from a small source are three-dimensional.

3. *Periodicity.* Waves may be classified further according to how the particles of the medium move in time. For example, we can produce a *pulse* traveling down a stretched string by applying a single sidewise movement at its end (Fig. 18-1*c*). Each particle remains at rest until the pulse reaches it, then it moves during a short time, and then it again remains at rest. If we continue to move the end of the string back and forth (Fig. 18-1*a*), we produce a *train of waves* traveling along the string. If our motion is periodic, we produce a *periodic train of waves* in which each particle of the string has a periodic motion. The simplest special case of a periodic wave is a *harmonic wave,* in which each particle undergoes simple harmonic motion.

4. *Shape of wavefronts.* Imagine a stone dropped in a still lake. Circular ripples spread outward from the point where the stone entered the water (Fig. 18-2). Along a given circular ripple, all points are in the same state of motion. Those points define a surface called a *wavefront.* If the medium is of uniform density, the direction of motion of the waves is at right angles to the wavefront. A line normal to the wavefronts, indicating the direction of motion of the waves, is called a *ray.*

Wavefronts can have many shapes. A point source at the surface of water produces two-dimensional waves with circular wavefronts and rays that radiate outward from the point of the disturbance (as in Fig. 18-2). On the other hand, a very long stick dropped horizontally into the water would produce (near its center) disturbances that travel as straight lines, in which the rays are parallel lines. The three-dimensional analogy, in which the disturbances travel in a single direction, is the *plane wave.* At a given instant, conditions are the same everywhere on any plane perpendicular to the direction of propagation. The wavefronts are planes, and the

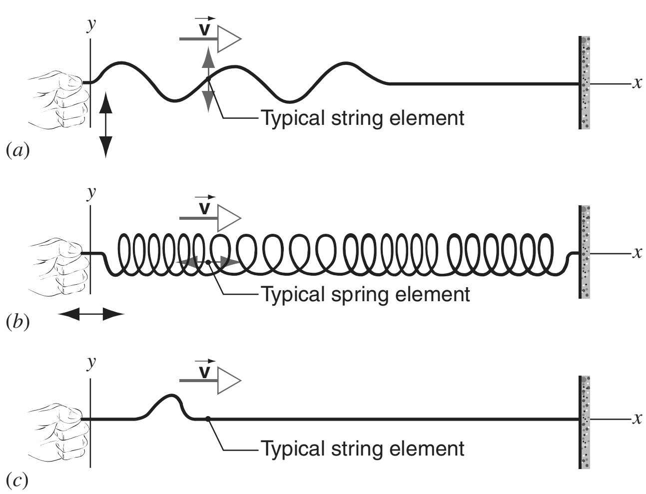

FIGURE 18-1. (*a*) Sending a transverse wave along a string. Each element of the string vibrates at right angles to the direction of propagation of the wave. (*b*) Sending a longitudinal wave along a spring. Each element of the spring vibrates parallel to the direction of propagation of the wave. (*c*) Sending a single transverse pulse along a string.

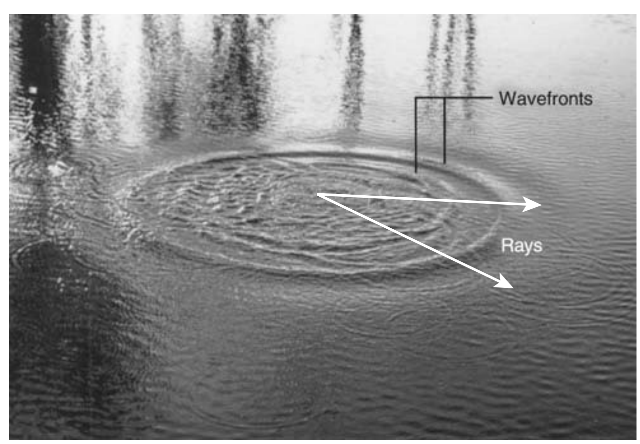

FIGURE 18-2. Waves on the surface of a lake. The circular ripples represent wavefronts. The rays, which are perpendicular to the wavefronts, indicate the direction of motion of the wave.

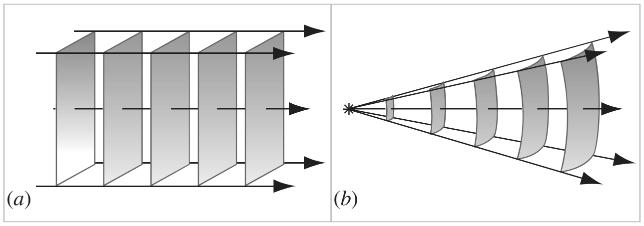

FIGURE 18-3. (*a*) A plane wave. The planes represent wavefronts spaced one wavelength apart, and the arrows represent rays. (*b*) A spherical wave. The wavefronts, spaced one wavelength apart, are spherical surfaces, and the rays are in the radial direction.

rays are parallel straight lines (Fig. 18-3*a*). The three-dimensional analogy of circular waves is spherical waves. Here the disturbance is propagated outward in all directions from a point source of waves. The wavefronts are spherical, and the rays are radial lines leaving the point source in all directions (Fig. 18-3*b*). Far from the source the spherical wavefronts have very small curvature, and over a limited region they can often be regarded as planes. Of course, there are many other possible shapes for wavefronts.

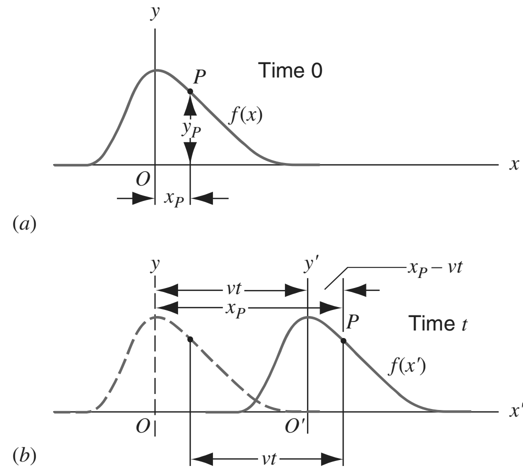

FIGURE 18-4. (*a*) A transverse pulse, shown as a snapshot at time $t = 0$. The point P represents a particular location on the phase of the pulse, *not* a particular point of the medium (the string, for instance). (*b*) At a time t later, the pulse has moved a distance vt in the positive x direction. The point P on the phase has also moved a distance vt. The peak of the pulse defines the origin of the x' coordinate.

18-3 TRAVELING WAVES

As an example of a mechanical wave, we consider a transverse waveform that travels on a long stretched string. We assume an "ideal" string, in which the disturbance, whether it is a pulse or a train of waves, keeps its form as it travels. For this to occur, frictional losses and other means of energy dissipation must be negligibly small. The disturbance lies in the xy plane and travels in the x direction.

Figure 18-4*a* shows an arbitrary waveform at $t = 0$; we can consider this to be a snapshot of the pulse traveling along the string shown in Fig. 18-1*c*. Let the pulse move in the positive x direction with speed v. At a later time t, the pulse has moved a distance vt, as shown in Fig. 18-4*b*. Note that the waveform is the same at $t = 0$ as it is at later times.

The coordinate y indicates the transverse displacement of a particular point on the string. This coordinate depends on both the position x and the time t. We indicate this dependence on two variables as $y(x, t)$.

We can represent the waveform of Fig. 18-4*a* as

$$y(x, 0) = f(x), \tag{18-1}$$

where f is a function that describes the shape of the wave. At time t, the waveform must still be described by the same function f, because we have assumed that the shape does not change as the wave travels. Relative to the origin O' of a reference frame that travels with the pulse, the shape is described by the function $f(x')$, as indicated in Fig. 18-4*b*. The relationship between the x coordinates in the two reference frames is $x' = x - vt$, as you can see from Fig. 18-4*b*. Thus, at time t, the wave is described by

$$y(x, t) = f(x') = f(x - vt). \tag{18-2}$$

That is, the function $f(x - vt)$ has the same shape relative to the point $x = vt$ at time t that the function $f(x)$ has relative to the point $x = 0$ at time $t = 0$.

To describe the wave completely, we must specify the function f. Later we shall consider harmonic waves, for which f is a sine or cosine function.

Equations 18-1 and 18-2 together indicate that we can change a function of any shape into a wave traveling in the positive x direction by merely substituting the quantity $x - vt$ for x everywhere that it appears in $f(x)$. For example, if $f(x) = x^2$, then $f(x - vt) = (x - vt)^2$. Furthermore, a wave traveling in the positive x direction must depend on x and t *only* in the combination $x - vt$; thus $x^2 - (vt)^2$ does not represent such a traveling wave.

Let us follow the motion of a particular part (or *phase*) of the wave, such as that of location P of the waveform of Fig. 18-4. If the wave is to keep its shape as it travels, then the y coordinate y_P of P must not change. We see from Eq. 18-2 that the only way this can happen is for x_P, the x coordinate of P, to increase as t increases in such a way that the quantity $x_P - vt$ keeps a fixed value. That is, evaluating the quantity $x_P - vt$ gives the same result at P in Fig. 18-4*b* and at P in Fig. 18-4*a*. This remains true for any location on the waveform and for all times t. Thus for the motion of any particular phase of the wave we must have

$$x - vt = \text{constant}. \tag{18-3}$$

We can verify that Eq. 18-3 characterizes the motion of the phase of the waveform by differentiating with respect to time, which gives

$$\frac{dx}{dt} - v = 0 \quad \text{or} \quad \frac{dx}{dt} = v. \tag{18-4}$$

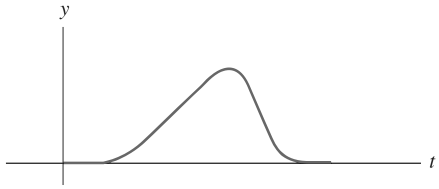

FIGURE 18-5. An observer stationed at a particular point on the x axis would record this y displacement as a function of time as the pulse of Fig. 18-4 passes. Note that the form appears to be reversed, because the leading edge of the traveling pulse arrives at the observer at the earliest times. That is, the displacements recorded by the observer at earlier times are closer to the origin here.

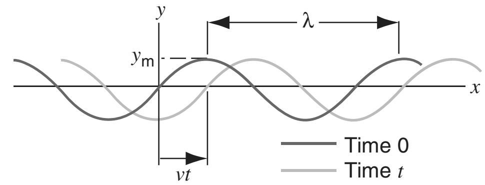

FIGURE 18-6. At $t = 0$ (darker color), the string has the sinusoidal shape given by $y = y_m \sin 2\pi x/\lambda$. At a later time t (lighter color), the wave has moved to the right a distance $x = vt$, and the string has a shape given by $y = y_m \sin 2\pi(x - vt)/\lambda$.

The velocity dx/dt describes the motion of the phase of the wave, and so it is known as the *phase velocity.* We take v to be a positive constant, independent of any property of the wave but possibly (as we shall see) depending on properties of the medium.

If the wave moves in the *negative* x direction, all we need do is replace v by $-v$. In this case we would obtain

$$y(x, t) = f(x + vt), \tag{18-5}$$

where once again $f(x)$ represents the shape at $t = 0$. That is, substituting in $f(x)$ the quantity $x + vt$ in place of x gives a wave that would move to the left in Fig. 18-4. The motion of any phase of the wave would then be characterized by the requirement that $x + vt =$ constant, and by analogy with Eq. 18-4 we can show that $dx/dt = -v$, indicating that the x component of the phase velocity in this case is indeed negative.

The function $y(x, t)$ contains the complete description of the shape of the wave and its motion. At any particular time, say t_1, the function $y(x, t_1)$ gives y as a function of x, which defines a curve; this curve represents the actual shape of the string at that time and can be regarded as a "snapshot" of the wave. On the other hand, we can consider the motion of a particular point on the string, say at the fixed coordinate x_1. The function $y(x_1, t)$ then tells us the y coordinate of that point as a function of the time. Figure 18-5 shows how a point on the x axis might move with time as the pulse of Fig. 18-4 passes, moving in the positive x direction. At times near $t = 0$, the point is not moving at all. It then begins to move gradually, as the leading edge of the pulse of Fig. 18-4 arrives. After the peak of the wave passes, the displacement of the point drops rapidly back to zero as the trailing edge passes.

Sinusoidal Waves

The above description is quite general. It holds for arbitrary wave shapes, and it holds for transverse as well as longitudinal waves. Let us consider, for example, a transverse waveform having a sinusoidal shape, which has particularly important applications. Suppose that at the time $t = 0$ we have a wavetrain along the string given by

$$y(x, 0) = y_m \sin \frac{2\pi}{\lambda} x. \tag{18-6}$$

The wave shape is shown in Fig. 18-6. The maximum displacement y_m is called the *amplitude* of the sine curve. The value of the transverse displacement y is the same at any x as it is at $x + \lambda$, $x + 2\lambda$, and so on. The symbol λ represents the *wavelength* of the wavetrain and indicates the distance between two adjacent points in the wave having the same phase. If the wave travels in the $+x$ direction with phase speed v, then the equation of the wave is

$$y(x, t) = y_m \sin \frac{2\pi}{\lambda}(x - vt). \tag{18-7}$$

Note that this has the form $f(x - vt)$ required for a traveling wave (Eq. 18-2).

The *period* T of the wave is the time necessary for a point at any particular x coordinate to undergo one complete cycle of transverse motion. During this time T, the wave travels a distance vT that must correspond to one wavelength λ, so that

$$\lambda = vT. \tag{18-8}$$

The inverse of the period is called the *frequency* f of the wave: $f = 1/T$. Frequency has units of cycles per second, or hertz (Hz). Period and frequency were previously discussed in Chapter 17.

Putting Eq. 18-8 into Eq. 18-7, we obtain another expression for the wave:

$$y(x, t) = y_m \sin 2\pi\left(\frac{x}{\lambda} - \frac{t}{T}\right). \tag{18-9}$$

From this form it is clear that y, at any given time, has the same value at x, $x + \lambda$, $x + 2\lambda$, and so on, and that y, at any given position, has the same value at the times t, $t + T$, $t + 2T$, and so on.

To reduce Eq. 18-9 to a more compact form, we introduce two quantities, the *wave number* k and the *angular frequency* ω. They are defined by

$$k = \frac{2\pi}{\lambda} \quad \text{and} \quad \omega = \frac{2\pi}{T} = 2\pi f. \tag{18-10}$$

The wave number k is, like ω, an angular quantity, and units for both involve radians. Units for k might be, for instance, rad/m, and for ω, rad/s. In terms of these quantities, the equation of a sine wave traveling in the positive x direction (to the right in Fig. 18-6) is

$$y(x, t) = y_m \sin(kx - \omega t). \tag{18-11}$$

The equation of a sine wave traveling in the negative x direction (to the left in Fig. 18-6) is

$$y(x, t) = y_m \sin (kx + \omega t). \tag{18-12}$$

Comparing Eqs. 18-8 and 18-10, we see that the phase speed v of the wave (which we will often call the *wave speed*) is given by

$$v = \lambda f = \frac{\lambda}{T} = \frac{\omega}{k}. \tag{18-13}$$

Transverse Velocity of a Particle

The motion of a particle in a transverse wave such as that of Fig. 18-6 is in the y direction. The wave speed describes the motion of the wave along the direction of travel (the x direction). The wave speed does *not* characterize the transverse motion of the particles of the string.

To find the transverse velocity of a particle of the string, we must find the change in the y coordinate with time. We focus our attention on a single particle of the string—that is, on a certain coordinate x. We therefore need the derivative of y with respect to t at constant x. This is represented by the symbol $\partial y/\partial t$, which indicates the *partial derivative* of y with respect to t, holding constant all other variables on which y may depend. We represent the particle velocity, which varies with x (the location of the particle) as well as with t, as $u_y(x, t)$. Assuming that we are dealing with a sinusoidal wave of the form of Eq. 18-11, we then have

$$\begin{aligned} u_y(x, t) &= \frac{\partial y}{\partial t} = \frac{\partial}{\partial t}[y_m \sin (kx - \omega t)] \\ &= -y_m \omega \cos (kx - \omega t) \end{aligned} \tag{18-14}$$

Depending on a particle's location and on the time at which it is observed, Eq. 18-14 shows that the transverse velocity can range from $-y_m\omega$ to $+y_m\omega$.

Continuing in this way, we can find the transverse acceleration of the particle at this location x according to

$$\begin{aligned} a_y(x, t) &= \frac{\partial^2 y}{\partial t^2} = \frac{\partial u_y}{\partial t} = -y_m \omega^2 \sin (kx - \omega t) \\ &= -\omega^2 y. \end{aligned} \tag{18-15}$$

Equation 18-15 has the same form as Eq. 17-5; the transverse acceleration of any point is proportional to its transverse displacement, but oppositely directed. This shows that each particle of the string undergoes transverse simple harmonic motion as the sinusoidal wave passes.

Keep in mind the differences between the speed v of the wave and the transverse velocity u_y of a particle. The speed v represents the entire wave; all points on the phase of the wave move in the same direction with the same speed v. However, the transverse velocity u_y of a particle depends on the location of the particle and on the time. At one instant of time, one particle might have $u_y = 0$ while another particle might be moving with the maximum transverse velocity (which is $y_m\omega$ according to Eq. 18-14). At some other instant, these roles might be reversed. It is also important to note that, as we discuss in the next section, the wave speed v depends on the properties of the medium and not on the properties of the wave. The transverse particle velocity, on the other hand, depends on the properties of the wave such as amplitude and frequency, as Eq. 18-14 shows, and not on the properties of the medium.

Phase and Phase Constant

In the traveling waves of Eqs. 18-11 and 18-12 we have assumed that the displacement y is zero at the position $x = 0$ at the time $t = 0$. This, of course, need not be the case. The general expression for a sinusoidal wave traveling in the positive x direction is

$$y(x, t) = y_m \sin (kx - \omega t - \phi). \tag{18-16}$$

The quantity that appears in the argument of the sine, namely, $kx - \omega t - \phi$, is called the *phase* of the wave. Two waves with the same phase (or with phases differing by any integer multiple of 2π) are said to be "in phase"; they execute the same motion at the same time.

The angle ϕ is called the *phase constant.* The phase constant does not affect the shape of the wave; it moves the wave forward or backward in space or time. To see this, we rewrite Eq. 18-16 in two equivalent forms:

$$y(x, t) = y_m \sin \left[k\left(x - \frac{\phi}{k}\right) - \omega t \right] \tag{18-17a}$$

or

$$y(x, t) = y_m \sin \left[kx - \omega\left(t + \frac{\phi}{\omega}\right) \right]. \tag{18-17b}$$

Figure 18-7a shows a "snapshot" at any time t of the two waves represented by Eqs. 18-11 (in which $\phi = 0$) and

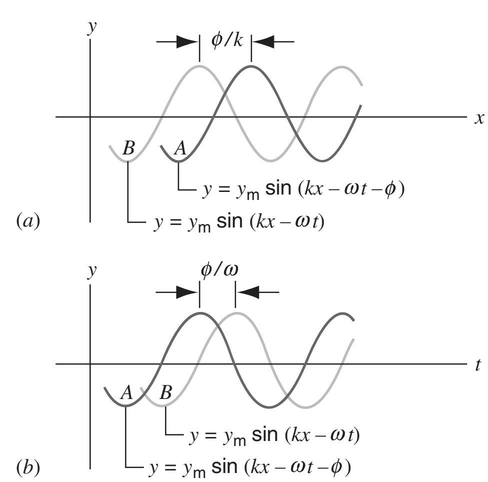

FIGURE 18-7. (a) A snapshot of two sine waves traveling in the positive x direction. Wave A has phase constant ϕ, and wave B has $\phi = 0$. Wave A is a distance of ϕ/k ahead of wave B. (b) The motion of a single point in time due to the same two waves. Wave A is a time ϕ/ω ahead of wave B. Note that, in a graph of y versus t, "ahead of" means "to the left of," whereas in a graph of y versus x, "ahead of" means "to the right of," if the waves travel in the positive x direction.

18-16. Note that any particular point on the wave described by Eq. 18-17*a* (say, a certain wave crest) is a distance ϕ/k *ahead of* the corresponding point in the wave described by Eq. 18-11.

Equivalently, if we were to observe the displacement at a fixed position x resulting from each of the two waves represented by Eqs. 18-11 and 18-16, we would obtain the result indicated by Fig. 18-7*b*. The wave described by Eq. 18-17*b* is similarly *ahead of* the wave having $\phi = 0$, in this case by a time difference ϕ/ω.

When the phase constant in Eq. 18-16 is positive, the corresponding wave is ahead of a wave described by a similar equation having $\phi = 0$. It is for this reason that we introduced the phase constant with a negative sign in Eq. 18-16. When one wave is ahead of another in time or space, it is said to "lead." On the other hand, putting a negative phase constant into Eq. 18-16 moves the corresponding wave behind the one with $\phi = 0$. Such a wave is said to "lag."

If we fix our attention on a particular point of the string, say x_1, the displacement y at that point can be written

$$y(t) = -y_m \sin(\omega t + \phi'),$$

where we have substituted a new phase constant $\phi' = \phi - kx_1$. This expression for $y(t)$ is similar to Eq. 17-6 for simple harmonic motion. Hence any particular element of the string undergoes simple harmonic motion about its equilibrium position as this wavetrain travels along the string.

SAMPLE PROBLEM 18-1. A transverse sinusoidal wave is generated at one end of a long horizontal string by a bar that moves the end up and down through a distance of 1.30 cm. The motion is continuous and is repeated regularly 125 times per second. (*a*) If the distance between adjacent wave crests is observed to be 15.6 cm, find the amplitude, frequency, speed, and wavelength of the wave motion. (*b*) Assuming the wave moves in the $+x$ direction and that, at $t = 0$, the element of the string at $x = 0$ is at its equilibrium position $y = 0$ and moving downward, find the equation of the wave.

Solution (*a*) As the bar moves a total of 1.30 cm, the end of the string moves $\frac{1}{2}(1.30 \text{ cm}) = 0.65$ cm away from the equilibrium position, first above it, then below it; therefore the amplitude y_m is 0.65 cm.

The entire motion is repeated 125 times each second, and thus the frequency is 125 vibrations per second, or $f = 125$ Hz.

The distance between adjacent wave crests, which is given as 15.6 cm, is the wavelength, as Fig. 18-6 shows. Thus $\lambda = 15.6 \text{ cm} = 0.156$ m.

The wave speed is given by Eq. 18-13:

$$v = \lambda f = (0.156 \text{ m})(125 \text{ s}^{-1}) = 19.5 \text{ m/s}.$$

(*b*) The general expression for a transverse sinusoidal wave moving in the $+x$ direction is given by Eq. 18-16,

$$y(x, t) = y_m \sin(kx - \omega t - \phi).$$

Imposing the given initial conditions ($y = 0$ and $\partial y/\partial t < 0$ for $x = 0$ and $t = 0$) yields

$$y_m \sin(-\phi) = 0 \quad \text{and} \quad -y_m\omega\cos(-\phi) < 0,$$

which means that the phase constant ϕ may be taken to be zero (or any integer multiple of 2π). Hence, for this wave

$$y(x, t) = y_m \sin(kx - \omega t),$$

and with the values just found,

$$y_m = 0.65 \text{ cm},$$

$$k = \frac{2\pi}{\lambda} = \frac{2\pi}{0.156 \text{ m}} = 40.3 \text{ rad/m},$$

$$\omega = vk = (19.5 \text{ m/s})(40.3 \text{ rad/m}) = 786 \text{ rad/s},$$

we obtain as the equation for the wave

$$y(x, t) = (0.65 \text{ cm}) \sin[(40.3 \text{ rad/m})x - (786 \text{ rad/s})t].$$

SAMPLE PROBLEM 18-2. As the wave of Sample Problem 18-1 passes along the string, each particle of the string moves up and down at right angles to the direction of the wave motion. (*a*) Find expressions for the velocity and acceleration of a particle P located at $x_P = 0.245$ m. (*b*) Evaluate the transverse displacement, velocity, and acceleration of this particle at $t = 15.0$ ms.

Solution (*a*) For a particle at $x_P = 0.245$ m in the wave of Sample Problem 18-1, we obtain, using Eq. 18-14,

$$\begin{aligned} u_y(x_P, t) &= -(0.65 \text{ cm})(786 \text{ rad/s}) \\ &\quad \times \cos[(40.3 \text{ rad/m})(0.245 \text{ m}) - (786 \text{ rad/s})t] \\ &= -(511 \text{ cm/s}) \cos[9.87 \text{ rad} - (786 \text{ rad/s})t]. \end{aligned}$$

Similarly, using Eq. 18-15, we find the magnitude of the maximum acceleration to be $\omega^2 y_m = 4.02 \times 10^5 \text{ cm/s}^2$, and so

$$a_y(x_P, t) = -(4.02 \times 10^5 \text{ cm/s}^2) \sin[9.87 \text{ rad} - (786 \text{ rad/s})t].$$

(*b*) At $t = 15.0$ ms, we evaluate the expressions for y, u_y, and a_y to give

$$y = -0.61 \text{ cm}, \quad u_y = +173 \text{ cm/s}, \quad a_y = +3.8 \times 10^5 \text{ cm/s}^2.$$

That is, the particle is close to its maximum negative displacement, it is moving in the positive y direction (away from that maximum), and it is accelerating in the positive y direction (its velocity is increasing in magnitude as the particle moves toward its equilibrium position).

18-4 WAVE SPEED ON A STRETCHED STRING

So far we have obtained a general expression for a transverse wave—for example, Eq. 18-16. The phase speed was given in Eq. 18-13: $v = \lambda f = \omega/k$. However, this expression does not tell us about the phase speed itself; it shows only how the wavelength and frequency are related to one another in terms of the wave speed.

The phase speed of a sinusoidal wave can be derived based on the mechanical properties of the medium through which the wave travels, in our case a stretched string. In this section we will obtain the phase speed by applying Newton's laws to the motion of the wave along the string. In other cases, such as sound traveling in a gas, similar methods can be used to find an expression for the wave speed.

The speed of a wave depends on the properties of the medium and is assumed to be independent of frequency and wavelength. (If the speed does depend on the frequency or wavelength of the wave, the medium is said to be *dispersive,* which we discuss later in this section.) Each element of the string pulls on its neighbors with a force given by the tension F in the string. The stronger the tension, the greater the force between neighboring elements and the more rapidly any disturbance will propagate down the string. Thus the wave speed should increase with increasing tension.

On the other hand, the inertia of each element limits how effective the tension will be in accelerating that element to move the wave along the string. Thus, for the same tension, the wave speed will be smaller in strings having more massive elements. The mass of each small element can be given in terms of the *mass density* μ (mass per unit length), which for a uniform string is equal to its mass divided by its length. On the basis of these general principles we therefore expect

$$v \propto \frac{F^a}{\mu^b},$$

where a and b are exponents that must be determined from the analysis.

It turns out that we can deduce the values of a and b based on a *dimensional analysis;* that is, there is only one combination of force and mass density that gives a quantity with the dimensions of velocity. From this type of analysis (see Exercise 5), we deduce $a = \frac{1}{2}$ and $b = \frac{1}{2}$, so that $v \propto \sqrt{F/\mu}$ or, introducing a constant of proportionality C, we have $v = C\sqrt{F/\mu}$. As we see next, the analysis using Newton's laws gives this same result and shows that $C = 1$.

Mechanical Analysis

Now let us derive an expression for the speed of a pulse in a stretched string by a mechanical analysis. In Fig. 18-8 we show a "snapshot" of a wave pulse that is moving from left to right in the string with a speed v. We can imagine instead that the entire string is moved from right to left with this same speed so that the wave pulse remains fixed in space (perhaps by pulling the string through a frictionless tube having the desired shape of the pulse). This simply means that, instead of taking our reference frame to be the walls between which the string is stretched, we choose a reference frame that is in uniform motion with respect to that one. In effect, we observe the pulse while running along the string at the same speed as the pulse. Because Newton's laws involve only accelerations, which are the same in both frames, we can use them in either frame. We just happen to choose a more convenient frame.

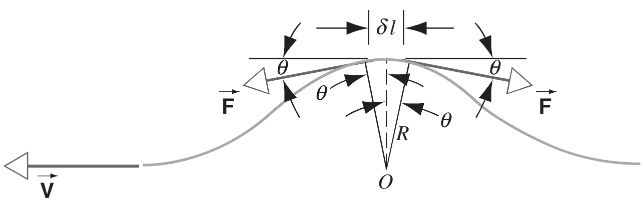

Figure 18-8. A pulse moving to the right on a stationary string is equivalent to a pulse in a fixed position on a string that is moving to the left. We consider the tension forces on a section of string of length δl on the "fixed" pulse.

We consider a small section of the pulse of length δl, as shown in Fig. 18-8. This section approximately forms an arc of a circle of radius R. The mass δm of this element is $\mu\ \delta l$, where μ is the mass density of the string. The tension F in the string is a tangential pull at each end of this small segment of the string. The horizontal components of $\vec{\mathbf{F}}$ cancel, and the vertical components are each equal to $F \sin\theta$. Hence the total vertical force F_y is $2F \sin\theta$. Because θ is small, we can take $\sin\theta \approx \theta$. From Fig. 18-8, we see that $2\theta = \delta l/R$, and so we obtain

$$F_y = 2F\sin\theta \approx 2F\theta = F\frac{\delta l}{R}. \tag{18-18}$$

This gives the force supplying the centripetal acceleration v^2/R of the string particles directed toward O. Note that the tangential velocity v of this mass element along the top of the arc is horizontal and is in magnitude equal to the wave speed. Applying Newton's second law to the string element δm, we have $\Sigma F_y = (\delta m)\, a_y$ or, using Eq. 18-18,

$$F\frac{\delta l}{R} = (\delta m)\, a_y = (\delta m)\frac{v^2}{R} = (\mu\ \delta l)\frac{v^2}{R},$$

where we have used $a_y = v^2/R$ for the centripetal acceleration and $\delta m = \mu\delta l$ for the mass of the string element. From the first and last terms of this equation, we obtain

$$v = \sqrt{\frac{F}{\mu}} \tag{18-19}$$

Equation 18-19 shows from a mechanical analysis that the constant C introduced in the dimensional analysis has the value 1.

If the amplitude of the pulse were very large compared to the length of the string, we would have been unable to use the approximation $\sin\theta \approx \theta$. Furthermore, the tension F in the string would be changed by the presence of the pulse, whereas we assumed F to be unchanged from the original tension in the stretched string. Therefore our result holds only for relatively small transverse displacements of the string, a case that is widely applicable in practice.

A periodic wave that enters a medium usually results from an external influence that disturbs the medium at a certain frequency. The wave that travels through that medium will have the same frequency as the source of the wave. The speed of the wave is determined by the properties of the medium. Given the frequency f of the wave and its speed v in the medium, the wavelength of the periodic wave *in that medium* is determined from Eq. 18-13, $\lambda = v/f$. When a wave passes from one medium to another medium of different wave speed (for example, two strings of different linear mass densities), the frequency in one medium must be the same as the frequency in the other. (Otherwise there would be a discontinuity at the point where the two strings are joined.) The wavelengths, however, will differ from one another. The relationship between the wavelengths follows from the equality of the frequencies f_1 and f_2 in the two media; that is, $f_1 = f_2$ gives

$$\frac{v_1}{\lambda_1} = \frac{v_2}{\lambda_2}. \tag{18-20}$$

Group Speed and Dispersion (Optional)

Pure sinusoidal waves are useful mathematical devices for helping us understand wave motion. In practice, we use other kinds of waves to transport energy and information. These waves may be periodic but nonsinusoidal, such as square waves or "sawtooth" waves, or they may be nonperiodic pulses, such as that of Fig. 18-4.

We have used the phase speed to describe the motion of two kinds of waves: the pulse that preserves its shape as it travels (Fig. 18-4) and the pure sine wave (Fig. 18-6). In other cases, we must use a different speed, called the *group speed,* which is the speed at which energy or information travels in a real wave.

Figure 18-9 shows a pulse traveling through a medium. The shape of the pulse changes as it travels; the pulse spreads out, or *disperses.* (Dispersion is not the same as energy dissipation. The energy content of the pulse in Fig. 18-9 may remain constant as it travels, even though the pulse disperses. We assume that the medium is *dispersive,* but not necessarily *dissipative.*) As we see in Section 18-7, any periodic wave can be regarded as the sum or superposition of a series of sinusoidal waves of different frequencies or wavelengths. The frequencies, amplitudes, and phases of the component sinusoidal waves must be carefully chosen according to a prescribed mathematical procedure, known as *Fourier analysis,* so that the waves add to give the desired waveform. In most real media, the speed of propagation of these component waves (that is, the phase speed) depends on the frequency or wavelength of the particular component. Each component wave may travel with its own unique speed. Thus, as the wave travels, the phase relationships of the components may change, and the waveform of the sum of the components will correspondingly change as the wave travels. This is the origin of dispersion—the component waves travel at different phase speeds. There is no simple relationship between the phase speeds of the components and the group speed of the wave; the relationship depends on the dispersion of the medium.

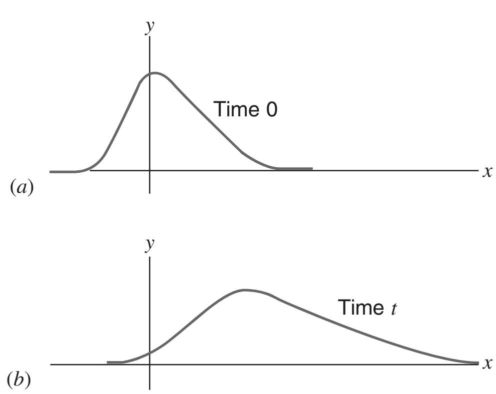

FIGURE 18-9. In a dispersive medium, the waveform changes as the wave travels.

Some real media are approximately nondispersive, in which case the wave keeps its shape, and all component waves travel with the same speed. An example is sound waves in air. If air were strongly dispersive for sound waves, conversation would be impossible, because the waveform produced by your friend's vocal cords would be jumbled by the time it reached your ears. Furthermore, the care taken by the players in an orchestra to play precisely at the same time would be to no avail, because (if air were dispersive for sound) the notes of high frequency would travel to the listener's ear at a speed different from that of the notes of low frequency, and the listener would hear the sounds at different times. Fortunately, this does not occur for sound waves. Light waves in vacuum are perfectly nondispersive; the dispersion of light waves in real media is responsible for such effects as the spectrum of colors in rainbows.

In a nondispersive medium, all the component waves in a complex waveform travel at the same phase speed, and the group speed of the waveform is equal to that common value of the phase speed. Only in this case can we speak of the phase speed of the entire waveform. In this chapter, we assume that we are dealing with mechanical waves that propagate in a nondispersive medium. ■

18-5 THE WAVE EQUATION (Optional)

In Chapter 17 we discussed the commonly encountered phenomenon of oscillation. One reason that this phenomenon is so common is that the basic equation that describes an oscillating system [$x = x_m \cos(\omega t + \phi)$, Eq. 17-6] is a solution of Eq. 17-5,

$$\frac{d^2x}{dt^2} = -\left(\frac{k}{m}\right)x,$$

which is an equation of a general form that can be derived from a mechanical analysis of a variety of physical situations, some of which were discussed in Section 17-5.

The situation is similar in the case of wave motion. As we demonstrate in this section, the mechanical analysis gives an equation of another commonly encountered form, the solution of which is a wave of the form of Eq. 18-2 or 18-5.

Figure 18-10 shows an element of a long string that is under tension F. A passing wave has caused the element to be displaced from its equilibrium position at $y = 0$. We

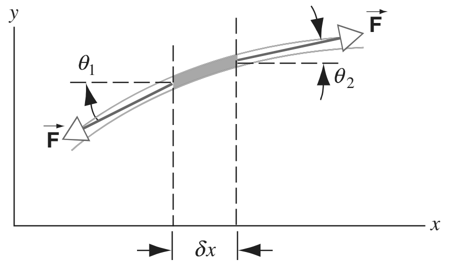

FIGURE 18-10. A small element of length δx of a long string under tension F. The figure represents a snapshot of the element at a particular time during the passage of a wave.

consider the element of the string of length δx, and we apply Newton's second law to analyze *how* this element is made to move.

The element is acted on by two forces, exerted by the portions of the string on either side of the element. These forces have equal magnitudes, because the tension is evenly distributed along the string, but they have slightly different directions, because they act tangent to the string at the endpoints of the element. The y component of the net force is

$$\sum F_y = F \sin \theta_2 - F \sin \theta_1 .$$

We consider only small displacements from equilibrium, so that the angles θ_1 and θ_2 are small, and we can write $\sin \theta \approx \tan \theta$, which gives

$$\sum F_y \approx F \tan \theta_2 - F \tan \theta_1 = F\,\delta(\tan \theta), \quad (18\text{-}21)$$

where $\delta(\tan \theta) = \tan \theta_2 - \tan \theta_1$. This resultant force must be equal to the mass of the element, $\delta m = \mu\,\delta x$, times the y component of the acceleration. If frictional or other dissipative forces can be neglected, Newton's second law gives

$$\sum F_y = \delta m\, a_y$$

$$F\delta(\tan \theta) = \mu\, \delta x\, a_y$$

$$\frac{\delta(\tan \theta)}{\delta x} = \frac{\mu}{F} a_y.$$

For the y component of the acceleration a_y, we use the transverse acceleration for a particle, $\partial^2 y/\partial t^2$. We also replace $\tan \theta$, which is the slope of the string, by the equivalent partial derivative $\partial y/\partial x$. Making these substitutions, we obtain

$$\frac{\delta(\partial y/\partial x)}{\delta x} = \frac{\mu}{F}\frac{\partial^2 y}{\partial t^2}. \quad (18\text{-}22)$$

We now take the limit of Eq. 18-22 as the mass element becomes very small. The left side is in the standard form for expressing the derivative with respect to x as a limit:

$$\lim_{\delta x \to 0} \frac{\delta(\partial y/\partial x)}{\delta x} = \frac{\partial}{\partial x}\left(\frac{\partial y}{\partial x}\right) = \frac{\partial^2 y}{\partial x^2},$$

and Eq. 18-22 becomes

$$\frac{\partial^2 y}{\partial x^2} = \frac{\mu}{F}\frac{\partial^2 y}{\partial t^2}. \quad (18\text{-}23)$$

Using Eq. 18-19 to replace μ/F with $1/v^2$, we obtain

$$\frac{\partial^2 y}{\partial x^2} = \frac{1}{v^2}\frac{\partial^2 y}{\partial t^2}. \quad (18\text{-}24)$$

Equation 18-24 is the general form of equation that describes waves: the second derivative of the wave displacement y with respect to the coordinate x in the direction of propagation is equal to $1/v^2$ times the second derivative with respect to time. This general form of equation is called the *wave equation.* It arises not only in mechanics but in other situations as well. For example, as we discuss in Chapter 38, if we use the equations of electromagnetism instead of the equations of mechanics (Newton's laws), we obtain an equation of exactly the same form as Eq. 18-24, except that the displacement y is replaced by the strength of an electric or magnetic field. The speed of propagation v for electromagnetic waves traveling in a vacuum becomes the speed of light c.

Let us see how our general formula for a traveling wave, $y(x, t) = f(x \pm vt)$, is the solution of Eq. 18-24. We make a simple change of variable and let z represent $x \pm vt$, so that $y = f(z)$. Then, repeatedly using the chain rule of calculus,

$$\frac{\partial y}{\partial x} = \frac{df}{dz}\frac{\partial z}{\partial x} = \frac{df}{dz}$$

$$\frac{\partial^2 y}{\partial x^2} = \frac{d}{dz}\left(\frac{df}{dz}\right)\frac{\partial z}{\partial x} = \frac{d^2 f}{dz^2}$$

$$\frac{\partial y}{\partial t} = \frac{df}{dz}\frac{\partial z}{\partial t} = \pm v\frac{df}{dz}$$

$$\frac{\partial^2 y}{\partial t^2} = \frac{d}{dz}\left(\pm v\frac{df}{dz}\right)\frac{\partial z}{\partial t} = (\pm v)^2\frac{d^2 f}{dz^2} = v^2\frac{d^2 f}{dz^2}.$$

Thus

$$\frac{d^2 f}{dz^2} = \frac{\partial^2 y}{\partial x^2} = \frac{1}{v^2}\frac{\partial^2 y}{\partial t^2}.$$

and Eq. 18-24 is satisfied. It can be shown that *only* the combinations $x \pm vt$ in f satisfy the wave equation, so that all traveling waves must be in the form of Eq. 18-2 or 18-5.

To express these results in another way, Eq. 18-23, which was derived from Newton's laws, represents a traveling wave only when $\mu/F = 1/v^2$. This discussion thus provides an independent derivation of Eq. 18-19 for the velocity of propagation of waves along a stretched string. ■

18-6 ENERGY IN WAVE MOTION

If, as in Fig 18-1, you shake one end of a long string, your hand is doing work on the string. You are thus providing energy to the string. That energy travels along the string as a wave, and a friend at the other end of the string could extract that energy. Energy transport is an important property of waves, and in this section we examine the energy of a wave on a stretched string.

Figure 18-11*a* shows a wave traveling along the string at times t_1 and t_2 (a time $T/4$ later). Consider two elements on the string, each of length dx. The element at A was located on a wave crest at t_1, after which it moved downward and is crossing the axis at t_2. The element at B was crossing the axis at t_1 but is on a wave crest at t_2.

Element A is at rest at t_1, while at t_2 it has the maximum particle speed. This element therefore gains kinetic energy between t_1 and t_2. Element A has very nearly its relaxed length dx at time t_1, but at time t_2 it has been stretched to a greater length by the tension in the string. It thus gains potential energy from t_1 to t_2. On the other hand, element B loses kinetic energy between t_1 and t_2. Moreover, element B is stretched at t_1 but has its relaxed length at t_2, so its potential energy also decreases. We can thus view the travel of a wave along the string in terms of the kinetic and potential energy of each element of the string. By calculating how the energy changes with time, we can determine the power delivered by the wave.

Figure 18-11*b* shows an expanded view of a string element at an arbitrary point in its motion. Its length has been stretched from its relaxed length dx to dl. The element has mass $dm = \mu\, dx$ and is moving with velocity u_y given by Eq. 18-14, so its kinetic energy dK is

$$dK = \tfrac{1}{2}\, dm\, u_y^2 = \tfrac{1}{2}\,(\mu\, dx)[-y_m\omega \cos(kx - \omega t)]^2. \qquad (18\text{-}25)$$

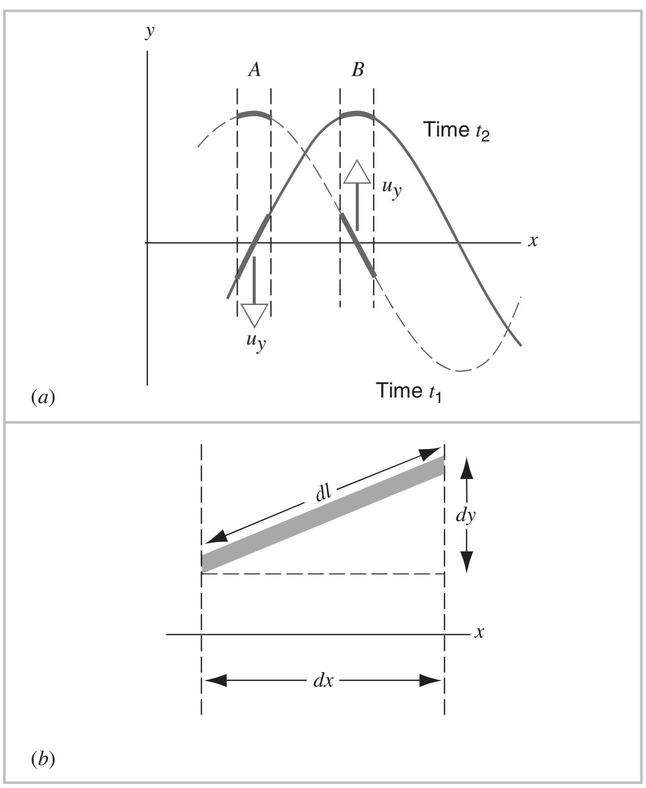

FIGURE 18-11. (*a*) Two small elements of string, labeled A and B, are shown on a wave at time t_1 and again at a time t_2 (one-quarter of a cycle later). The wave is moving to the right (in the direction of increasing x). (*b*) A magnified view of a small element of the string at an arbitrary time.

This change in kinetic energy has occurred in the time dt that it takes for the wave to move a distance along the x axis equal to the x component of the length of the element; that is, $dt = dx/v$, where v is the wave speed. The rate at which kinetic energy is transported by the wave is dK/dt, or

$$\begin{aligned}\frac{dK}{dt} &= \tfrac{1}{2}\mu\omega^2 y_m^2 \frac{dx}{dt}\cos^2(kx - \omega t)\\ &= \tfrac{1}{2}\mu\omega^2 y_m^2 v\cos^2(kx - \omega t). \qquad (18\text{-}26)\end{aligned}$$

To find the potential energy in the element, we must evaluate the work done by the tension force F as it stretches the element from length dx to length dl, or $dU = F(dl - dx)$. Approximating dl as the hypotenuse of a right triangle, as in Fig. 18-11*b*, we have

$$\begin{aligned}dU &= F\left[\sqrt{(dx)^2 + (dy)^2} - dx\right]\\ &= F dx\left[\sqrt{1 + (\partial y/\partial x)^2} - 1\right]. \qquad (18\text{-}27)\end{aligned}$$

The quantity $\partial y/\partial x$ gives the slope of the string, and if the amplitude of the wave is not too large this slope will be small. We can then use the binomial expansion $(1 + z)^n \approx 1 + nz + \cdots$ to write

$$dU = F dx\left[1 + \frac{1}{2}\left(\frac{\partial y}{\partial x}\right)^2 - 1\right] = \tfrac{1}{2}F dx\left(\frac{\partial y}{\partial x}\right)^2. \qquad (18\text{-}28)$$

With $\partial y/\partial x = -y_m k\cos(kx - \omega t)$, we find the rate at which potential energy is transported along the string to be

$$\begin{aligned}\frac{dU}{dt} &= \tfrac{1}{2}F\frac{dx}{dt}[-y_m k\cos(kx - \omega t)]^2\\ &= \tfrac{1}{2}F v y_m^2 k^2\cos^2(kx - \omega t). \qquad (18\text{-}29)\end{aligned}$$

Using Eqs. 18-19 and 18-13 we can write $F = v^2\mu = (\omega/k)^2\mu$; substituting this result into Eq. 18-29 and comparing with Eq. 18-26, it immediately follows that $dU/dt = dK/dt$.

Note that dK and dU are both zero when the element has its maximum displacement (as in the case of element A at time t_1), and both dK and dU have their maximum values when the element crosses the x axis (as in the case of element A at time t_2). Although the motion of an element of the string reminds us of the simple harmonic oscillator, there is an important difference: the mechanical energy $dE = dU + dK$ of the mass element is *not* a constant, but instead it varies from zero at the crests and valleys to a maximum where the string crosses the axis. This should not be a surprise, because the mass element is not an isolated system—neighboring mass elements are doing work on it to change its energy.

Power and Intensity in Wave Motion

Because $dU/dt = dK/dt$, we have

$$\frac{dE}{dt} = \frac{dK}{dt} + \frac{dU}{dt} = 2\frac{dU}{dt} = \mu\omega^2 y_m^2 v\cos^2(kx - \omega t). \qquad (18\text{-}30)$$

The rate at which mechanical energy is transmitted along the string is simply the power: $P = dE/dt$. This quantity varies with location along the string as well as with time. Usually we are more interested in the average power P_{av}:

$$P_{av} = \left(\frac{dE}{dt}\right)_{av} = \mu\omega^2 y_m^2 v[\cos^2(kx - \omega t)]_{av}. \quad (18\text{-}31)$$

Often we observe waves over a time that is very long compared with the period of the wave, so that we take the average over many cycles of the oscillation. The average value of the $\cos^2$ over any number of full cycles is $\frac{1}{2}$, and so

$$P_{av} = \tfrac{1}{2}\mu\omega^2 y_m^2 v. \quad (18\text{-}32)$$

The dependence of the average rate of energy transfer on the *square* of the amplitude and the *square* of the frequency is a general characteristic property of waves.

This calculation assumes that the wave transports energy with no losses due to friction or other dissipative forces. None of the mechanical energy is lost to internal energy of the string or heat transferred to the surroundings.

We have also assumed that the amplitude of the wave remains constant as it travels. This remains true (in the ideal approximation) for waves on a string, and it is strictly true for the ideal plane wave (as in Fig. 18-3*a*). However, for spherical wavefronts (as in Fig. 18-3*b*), the energy content of each wavefront remains the same, but that energy is spread over an increasing area as the wave travels. For such spherical waves, it is often more useful to describe the wave in terms of its *intensity I*, which is defined as *the average power per unit area transmitted across an area A perpendicular to the direction in which the wave is traveling,* or

$$I = \frac{P_{av}}{A}. \quad (18\text{-}33)$$

The SI unit of intensity is watts per meter squared (W/m^2).

Just as with the power of a wave, the intensity is always proportional to the square of the amplitude. However, for circular or spherical waves, the amplitude is not constant as the wavefront advances. In a spherical wave, such as might be emitted by a point source of light or sound, the surface area of a wavefront of radius r is $4\pi r^2$, so the intensity is proportional to $1/r^2$. If the distance from a source of spherical waves is doubled, the intensity becomes one-quarter as large while the amplitude of the wave becomes half as large.

18-7 THE PRINCIPLE OF SUPERPOSITION

We often observe two or more waves to travel simultaneously through the same region of space independently of one another. For example, the sound reaching our ears from a symphony orchestra is very complex, but we can pick out the sound made by individual instruments. The electrons in the antennas of our radio and TV sets are set into motion by a whole array of signals from different broadcasting centers, but we can nevertheless tune to any particular station, and the signal we receive from that station is in principle the same as that which we would receive if all other stations were to stop broadcasting.

The above examples illustrate the *principle of superposition,* which asserts that, when several waves combine at a point, the displacement of any particle at any given time is simply the sum of the displacements that each individual wave acting alone would give it. For example, suppose that two waves travel simultaneously along the same stretched string. Let $y_1(x, t)$ and $y_2(x, t)$ be the displacements that the string would experience if each wave acted alone. The displacement of the string when both waves act is then

$$y(x, t) = y_1(x, t) + y_2(x, t). \quad (18\text{-}34)$$

For mechanical waves in elastic media, the superposition principle holds whenever the restoring force varies linearly with the displacement.

Figure 18-12 shows a time sequence of "snapshots" of two pulses traveling in opposite directions in the same stretched string. When the pulses overlap, the displacement of the string is the algebraic sum of the individual displacements of the string caused by each of the two pulses alone, as Eq. 18-34 requires. The pulses simply move through one another, each moving along as if the other were not present.

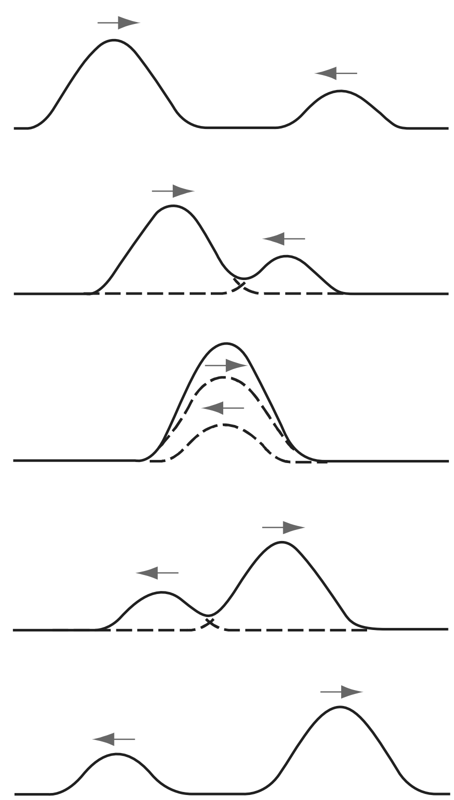

FIGURE 18-12. Two pulses travel in opposite directions along a stretched string. The superposition principle applies as they move through each other.

The superposition principle may seem to be an obvious result, but there are instances in which it does not hold. Suppose, for instance, that one of the waves has such a large amplitude that the elastic limit of the medium is exceeded. The restoring force is no longer directly proportional to the displacement of a particle in the medium. Then, no matter what the amplitude of the second wave (even if it is very small), its effect at a point is not a linear function of its amplitude. Furthermore, the second wave will be changed by passing through the nonlinear region, and its subsequent behavior will be altered. This situation arises only very rarely, and in most circumstances the principle of superposition is valid (as we assume throughout this text).

Fourier Analysis (Optional)

The importance of the superposition principle physically is that, where it holds, it makes it possible to analyze a complicated wave motion as a combination of simple waves. In fact, as was shown by the French mathematician J. Fourier (1768–1830), all that we need to build up the most general form of periodic wave are simple harmonic waves. Fourier showed that any periodic motion of a particle can be represented as a combination of simple harmonic motions. For example, if $y(x)$ represents the waveform (at a particular time) of a source of waves having a wavelength λ, we can analyze $y(x)$ as follows:

$$y(x) = A_0 + A_1 \sin kx + A_2 \sin 2kx + A_3 \sin 3kx + \cdots + B_1 \cos kx + B_2 \cos 2kx + B_3 \cos 3kx + \cdots, \quad (18\text{-}35)$$

where $k = 2\pi/\lambda$. This expression is called a Fourier series. The coefficients A_n and B_n have definite values for any particular periodic motion $y(x)$. For example, the so-called sawtooth wave of Fig. 18-13*a* can be described by

$$y(x) = -\frac{1}{\pi}\sin kx - \frac{1}{2\pi}\sin 2kx - \frac{1}{3\pi}\sin 3kx - \cdots .$$

If the motion is not periodic, as in the case of a pulse, the sum is replaced by an integral—the Fourier integral. Hence any motion (pulsed or continuous) of a source of waves can be represented in terms of a superposition of simple harmonic motions, and any waveform so generated can be analyzed as a combination of components that are individually simple harmonic waves. This once again illustrates the importance of harmonic motion and harmonic waves.

Only in the case of a nondispersive medium will the waveform maintain its shape as it travels. In a dispersive medium, the waveforms of the sinusoidal component waves do not change, but each may travel at a different speed. In this case, the combined waveform changes as the phase relationship between the components is altered. The wave can also change its shape if it loses mechanical energy to the medium, such as by air resistance, viscosity, or internal friction. Such dissipative forces often depend on the speed, and so the Fourier components most strongly affected are those with higher particle speeds (that is, those with high frequencies, according to Eq. 18-14 in which u_y is seen to depend on ω). Here again the wave shape may change, as the higher frequency components lose amplitude more quickly. The decay with time of the sound of piano strings is an example of this phenomenon. The vibrational motion of a piano string, immediately after it is struck by the hammer, includes a wide range of frequencies, which give it its characteristic tone. The higher frequency components of this complex motion dissipate their energy more rapidly than the lower frequency components, and thus the character of a sustained tone may change with time. ■

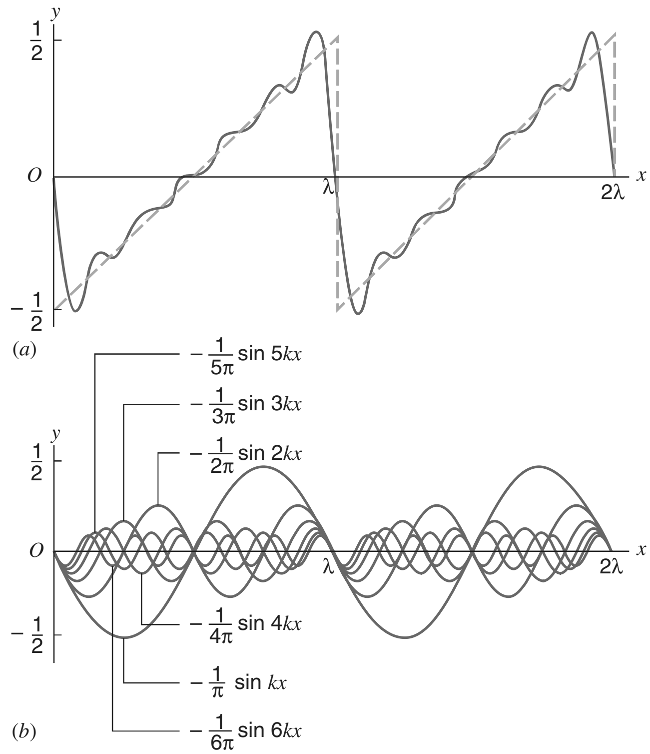

FIGURE 18-13. (*a*) The dashed line is a sawtooth wave commonly encountered in electronics. It can be represented as a Fourier series of sine waves. (*b*) The first six sine waves of the Fourier series that represents the sawtooth wave are shown, and their sum is shown as the solid curve in part (*a*). As more terms are included, the Fourier series becomes a better approximation of the wave.

18-8 INTERFERENCE OF WAVES

When two or more waves combine at a particular point, they are said to *interfere,* and the phenomenon is called *interference.* As we shall see, the resultant waveform is strongly dependent on the relative phases of the interfering waves. Figure 18-14 shows an example of interfering waves.

Let us first consider two transverse sinusoidal waves of equal amplitude and wavelength, which travel in the x direction with the same speed. We take the phase constant of one wave to be ϕ, while the other has $\phi = 0$. Figure 18-15 shows two individual waves y_1 and y_2 and their sum $y_1 + y_2$ at a particular time for the two cases of ϕ nearly 0 (the

FIGURE 18-14. Two wave trains, in this case circular ripples from two different disturbances, interfere where they overlap at particular points. The displacement at any point is the superposition of the individual displacements due to each of the two waves.

waves are nearly in phase) and ϕ nearly 180° (the waves are nearly out of phase). You can see by merely adding the individual displacements at each x that in the first case there is nearly complete reinforcement of the two waves and the resultant has nearly double the amplitude of the individual components, whereas in the second case there is nearly complete cancellation at every point and the resultant amplitude is close to zero. These cases are known, respectively, as *constructive* interference and *destructive* interference.

Let us see how interference arises from the equations for the waves. We consider a general case in which the two waves have phase constants ϕ_1 and ϕ_2, respectively. The equations of the two waves are

$$y_1(x, t) = y_m \sin(kx - \omega t - \phi_1) \tag{18-36}$$

and

$$y_2(x, t) = y_m \sin(kx - \omega t - \phi_2). \tag{18-37}$$

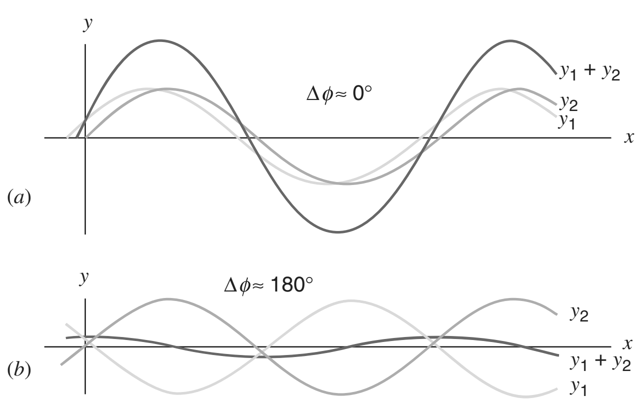

FIGURE 18-15. (*a*) The superposition of two waves of equal wavelength and amplitude that are almost in phase results in a wave of almost twice the amplitude of either component. (*b*) The superposition of two waves of equal wavelength and amplitude that are almost 180° out of phase results in a wave whose amplitude is nearly zero. Note that the wavelength of the resultant is unchanged in either case.

Now let us find the resultant wave. Using the principle of superposition, we take the sum of Eqs. 18-36 and 18-37, which gives

$$\begin{aligned} y(x, t) &= y_1(x, t) + y_2(x, t) \\ &= y_m[\sin(kx - \omega t - \phi_1) \\ &\quad + \sin(kx - \omega t - \phi_2)]. \end{aligned} \tag{18-38}$$

From the trigonometric identity for the sum of the sines of two angles,

$$\sin B + \sin C = 2 \sin \tfrac{1}{2}(B + C) \cos \tfrac{1}{2}(B - C), \tag{18-39}$$

we obtain, after some rearrangement,

$$y(x, t) = [2y_m \cos(\Delta\phi/2)] \sin(kx - \omega t - \phi'), \tag{18-40}$$

where $\phi' = (\phi_1 + \phi_2)/2$. The quantity $\Delta\phi = (\phi_2 - \phi_1)$ is called the *phase difference* between the two waves.

This resultant wave corresponds to a new wave having the same frequency but with an amplitude $2y_m|\cos(\Delta\phi/2)|$. If $\Delta\phi$ is very small (close to 0°), the resultant amplitude is nearly $2y_m$ (as shown in Fig. 18-15*a*). When $\Delta\phi$ is zero, the two waves overlap completely: the crest of one falls on the crest of the other and likewise for the valleys, which gives total constructive interference. The resultant amplitude is just twice that of either wave alone. If $\Delta\phi$ is close to 180°, on the other hand, the resultant amplitude is nearly zero (as shown in Fig. 18-15*b*). When $\Delta\phi$ is exactly 180°, the crest of one wave falls exactly on the valley of the other. The resultant amplitude is zero, corresponding to total destructive interference.

Notice that Eq. 18-40 always has the form of a sinusoidal wave. Thus adding two sine waves of the same wavelength and amplitude always gives a sine wave of the identical wavelength. We can also add components that have the same wavelength but different amplitudes. In this case the resultant again is a sine wave with the identical wavelength, but the resultant amplitude does not have the simple form given by Eq. 18-40. If the individual amplitudes are y_{1m} and y_{2m}, then if the waves are in phase ($\Delta\phi = 0$) the resultant amplitude is $y_{1m} + y_{2m}$ (Fig. 18-16*a*), whereas if they

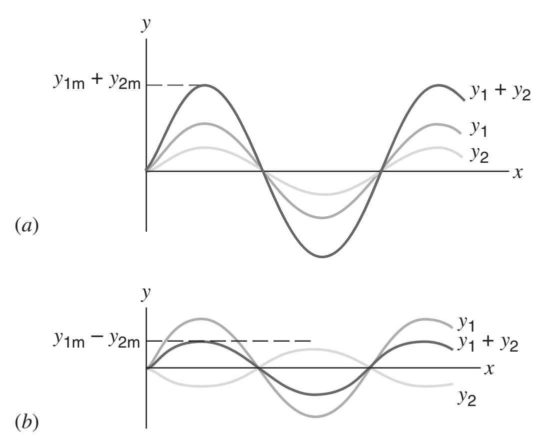

FIGURE 18-16. The addition of two waves of the same wavelength and phase but differing amplitudes (lighter color) gives a resultant of the same wavelength and phase. (*a*) The amplitudes add if the waves are in phase, and (*b*) they subtract if the waves are 180° out of phase.

are out of phase ($\phi = 180°$) the resultant amplitude is $|y_{1m} - y_{2m}|$ (Fig. 18-16*b*). There can be no complete destructive interference in this case, although there is partial destructive interference.

SAMPLE PROBLEM 18-3. Two waves travel in the same direction along a string and interfere. The waves have the same wavelength and travel with the same speed. The amplitude of each wave is 9.7 mm, and there is a phase difference of 110° between them. (*a*) What is the amplitude of the combined wave resulting from the interference of the two waves? (*b*) To what value should the phase difference be changed so that the combined wave will have an amplitude equal to that of one of the original waves?

Solution (*a*) The amplitude of the combined wave (always a positive quantity) was given in Eq. 18-40:

$$2y_m|\cos(\Delta\phi/2)| = 2(9.7\text{ mm})|\cos(110°/2)| = 11.1\text{ mm}.$$

(*b*) If the quantity $2y_m|\cos(\Delta\phi/2)|$ is to equal y_m, then we must have

$$2|\cos(\Delta\phi/2)| = 1,$$

or

$$\Delta\phi = 2\cos^{-1}(\tfrac{1}{2}) = 120° \quad \text{or} \quad -120°.$$

Either wave can be leading the other by 120° (plus or minus any integer multiple of 360°) to produce the desired combination wave.

18-9 STANDING WAVES

In the previous section we considered the effect of superposing two component waves of equal amplitude and frequency moving in the same direction on a string. What is the effect if the waves are moving along the string in *opposite* directions?

Figure 18-17 is a graphical indication of the effect of adding the component waveforms to obtain the resultant. Two traveling waves are shown in the figure, one moving to the left and the other to the right. "Snapshots" are shown of the two component waves and their resultant at intervals of one-quarter period.

One particular feature results from this superposition: there are certain points along the string, called *nodes,* at which the displacement is zero *at all times.* (Figure 18-16 also showed some points in which the resultant had zero displacement, but that figure represented a snapshot of traveling waves *at a particular time.* If we took another snapshot an instant later, we would find that those points no longer had zero displacement, because the wave is traveling. In Fig. 18-17*c*, the zeros remain zeros at all times.) Between the nodes are the *antinodes,* where the displacement oscillates with the largest amplitude. Such a pattern of nodes and antinodes is known as a *standing wave.*

To analyze the standing wave mathematically, we represent the two waves by

$$y_1(x,t) = y_m \sin(kx - \omega t),$$

$$y_2(x,t) = y_m \sin(kx + \omega t).$$

Hence the resultant may be written

$$y(x,t) = y_1(x,t) + y_2(x,t)$$

$$= y_m \sin(kx - \omega t) + y_m \sin(kx + \omega t) \tag{18-41}$$

or, making use of the trigonometric relation of Eq. 18-39,

$$y(x,t) = [2y_m \sin kx]\cos \omega t. \tag{18-42}$$

Equation 18-42 is the equation of a standing wave. It cannot represent a traveling wave, because x and t do *not* appear in the combination $x - vt$ or $x + vt$ required for a traveling wave.

Note that a particle at any particular location x undergoes simple harmonic motion, and that all particles vibrate with the same angular frequency ω. In a traveling wave each particle of the string vibrates with the same amplitude. In a standing wave, however, *the amplitude is not the same for different particles but varies with the location x of the particle.* In fact, the amplitude, $|2y_m \sin kx|$, has a *maximum* value of $2y_m$ at positions where $kx = \frac{1}{2}\pi, \frac{3}{2}\pi, \frac{5}{2}\pi, \ldots$. That is,

$$kx = \left(n + \frac{1}{2}\right)\pi \qquad n = 0, 1, 2, \ldots$$

or, substituting $k = 2\pi/\lambda$,

$$x = \left(n + \frac{1}{2}\right)\frac{\lambda}{2} \qquad n = 0, 1, 2, \ldots \tag{18-43}$$

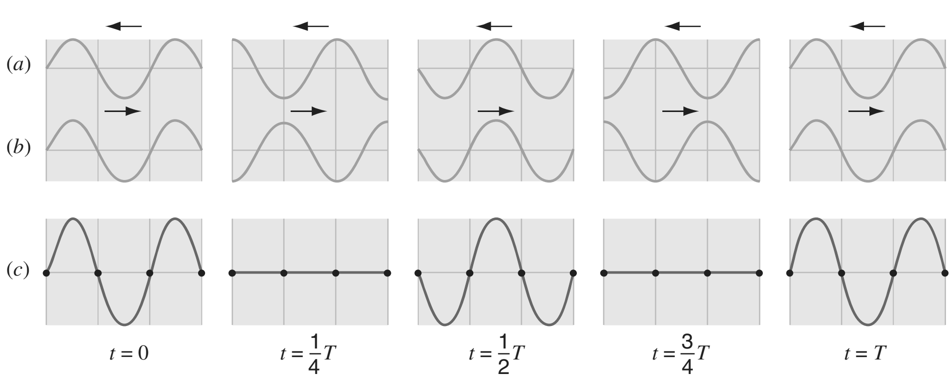

FIGURE 18-17. (*a*, *b*) Two traveling waves of the same wavelength and amplitude, moving in opposite directions. (*c*) The superposition of the two waves at different instants of time. The nodes in the standing wave pattern are indicated by dots. Note that the traveling waves have no nodes.

These points are the antinodes and are spaced one-half wavelength apart.

The amplitude has a *minimum* value of zero at positions where $kx = 0, \pi, 2\pi, 3\pi, \ldots$, so

$$kx = n\pi \qquad n = 0, 1, 2, \ldots$$

or

$$x = n\frac{\lambda}{2} \qquad n = 0, 1, 2, \ldots. \tag{18-44}$$

These points are the nodes and are also spaced one-half wavelength apart. The separation between a node and an adjacent antinode is one-quarter wavelength.

It is clear that energy is not transported along the string to the right or to the left, for energy cannot flow past the nodes in the string, which are permanently at rest. Hence the energy remains "standing" in the string, although it alternates between vibrational kinetic energy and elastic potential energy. When the antinodes are all at their maximum displacements, the energy is stored entirely as potential energy, in particular as the elastic potential energy associated with the stretching of the string. When all parts of the string are simultaneously passing through equilibrium (as in the second and fourth snapshots of Fig. 18-17*c*), the energy is stored entirely as kinetic energy. Figure 18-18 shows a more detailed description of the shifting of energy between kinetic and potential forms during one cycle of oscillation. Compare Fig. 18-18 with Fig. 12-5 for the oscillating block–spring system. How are these systems similar?

We can equally well regard the motion as an oscillation of the string as a whole, each particle undergoing simple harmonic motion of angular frequency ω and with an amplitude that depends on its location. Each small part of the string has inertia and elasticity, and the string as a whole can be thought of as a collection of coupled oscillators. Hence the vibrating string is the same in principle as the block–spring system, except that the block–spring system has only one natural frequency, and a vibrating string has a large number of natural frequencies (see Section 18-10.)

Reflection at a Boundary

To set up a standing wave in a string, we want to superimpose two waves traveling in opposite directions. One way to achieve this is to send a wave along a string so that it meets its reflection coming back. Here we consider the reflection process in more detail.

By way of illustration we consider a pulse rather than a sinusoidal wave. Suppose a pulse travels along a string that is fixed at one end, as shown in Fig. 18-19*a*. When the

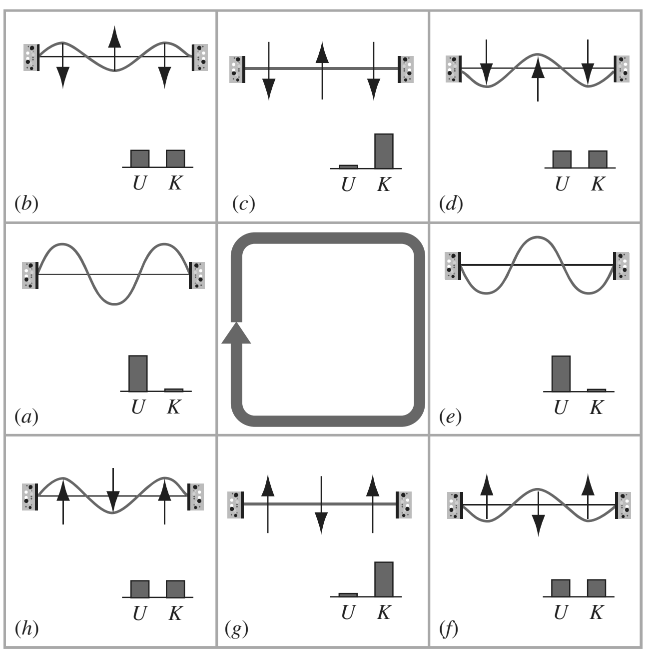

Figure 18-18. A standing wave on a stretched string, showing one cycle of oscillation. At (*a*) the string is momentarily at rest with the antinodes at their maximum displacement. The energy of the string is all elastic potential energy. (*b*) One-eighth of a cycle later, the displacement is reduced and the energy is partly potential and partly kinetic. The vectors show the instantaneous velocities of particles of the string at certain locations. (*c*) The displacement is zero; there is no potential energy, and the kinetic energy is maximum. The particles of the string have their maximum velocities. (*d–h*) The motion continues through the remainder of the cycle, with the energy being continually exchanged between potential and kinetic forms.

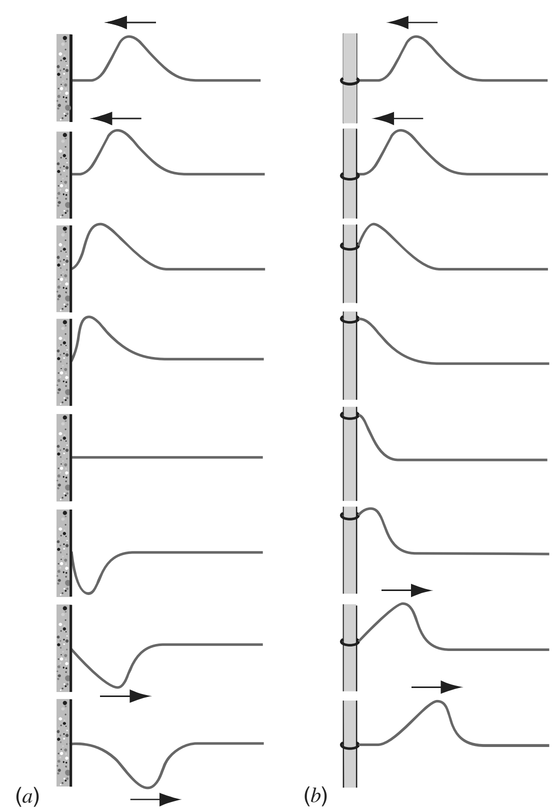

Figure 18-19. (*a*) A transverse pulse incident from the right is reflected by a rigid wall. Note that the phase of the reflected pulse is inverted, or changed by 180°. (*b*) Here the end of the string is free to move, the string being attached to a loop that can slide freely along the rod. The phase of the reflected pulse is unchanged.

pulse arrives at that end, it exerts an upward force on the support. Because the support is rigid, it does not move, and by Newton's third law it must exert an equal but oppositely directed force on the string. That force would be downward in Fig. 18-19*a* and causes an inverted pulse to travel in the opposite direction along the string. The incident and reflected pulses must tend to produce opposite displacements at the fixed end of the string, in order to keep that point fixed. We can consider this to be a situation of total destructive interference—the incident and reflected waves must be 180° out of phase. *On reflection from a fixed end, a transverse wave undergoes a phase change of 180°.*

The reflection of a pulse at a free end of a stretched string—that is, at an end that is free to move transversely—is represented in Fig. 18-19*b*. The end of the string is attached to a very light ring that is free to slide without friction along a transverse rod. When the pulse arrives at the free end, it exerts a force on the element of string there. This element is accelerated, and (as in the case of a pendulum) its motion carries it past the equilibrium point; it "overshoots" and exerts a reaction force on the string. This generates a pulse that travels back along the string in a direction opposite to that of the incident pulse. Once again we get reflection, but now at a free end. The free end will obviously suffer the maximum displacement of the particles on the string; an incident and a reflected wavetrain must interfere constructively at that point if we are to have a maximum there. Hence the reflected wave is always in phase with the incident wave at that point. *At a free end, a transverse wave is reflected without change of phase.*

So far we have assumed that the wave reflects at the boundary with no loss of intensity. In practice, we always find that at any boundary between two media there is partial reflection and partial transmission; for example, looking at a piece of ordinary window glass, you can see some light reflected back toward you and some transmitted through the glass. We can demonstrate this effect with transverse waves on strings by tying together two strings of different mass densities. When a wave traveling along one of the strings reaches the point where the strings are joined, part of the wave energy is transmitted to the other string and part is reflected back. The amplitude of the reflected wave is less than the amplitude of the original incident wave, because the wave transmitted to the second string carries away some of the incident energy.

If the second string has a greater mass density than the first, the wave reflected back into the first string still suffers a phase shift of 180° on reflection. However, because its amplitude is less than the incident wave, the boundary point is not a node and moves. Thus a net energy transfer occurs along the first string into the second. If the second string has a smaller mass density than the first, partial reflection occurs without change of phase, but once again energy is transmitted to the second string. In practice, the best way to realize a "free end" for a string is to attach it to a long and very much lighter string. The energy transmitted is negligible, and the second string serves to maintain the tension in the first one.

Note that the transmitted wave travels with a speed different from that of the incident and reflected waves. The wave speed is determined by the relation $v = \sqrt{F/\mu}$; the tension is the same in both strings, but their densities are different. Hence the wave travels more slowly in the denser string. The frequency of the transmitted wave is the same as that of the incident and reflected waves. (If this were not true, there would be a discontinuity at the point where the strings are joined.) Waves having the same frequency but traveling with different speeds have different wavelengths. From the relation $\lambda = v/f$, we conclude that in the denser string, where v is smaller, the wavelength is shorter. This phenomenon of change of wavelength as a wave passes from one medium to another will be encountered frequently in our study of light waves. It also occurs for sound waves: a string, such as on a guitar, vibrates with a certain frequency and wavelength; the wave transmitted to the air has the same frequency as that of the string, but a different wavelength, because the speed of waves on the string differs from their speed in air.

18-10 STANDING WAVES AND RESONANCE

Consider a string of length L that is fixed at both ends, such as we might find on a guitar or a violin. If we pluck the string near the middle and then examine its motion, we might find that it looks like Fig. 18-20*a*. A standing wave is established with a node at each end and an antinode in the middle.

How is it possible that by plucking the string we set up standing waves? The initial shape of the string, just as we release it, might have a triangular shape that we can analyze as a sum of sine and cosine terms using the method of

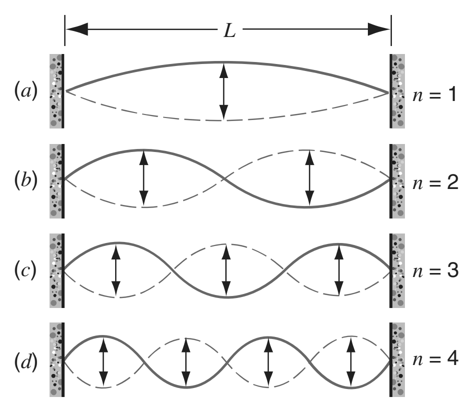

FIGURE 18-20. Standing wave patterns on a string of length L stretched between two fixed supports. Four different patterns are shown, corresponding to different wavelengths and frequencies.

Fourier analysis described in Section 18-7. Each of these waves travels along the string, reflects from the ends, and interferes with all of the waves traveling on the string. The higher frequencies tend to damp out more rapidly, leaving us with only the standing wave corresponding to the lowest possible frequency, which is shown in Fig. 18-20*a*. The spacing between nodes is always $\lambda/2$, so for the standing wave pattern shown in Fig. 18-20*a* we have $L = \lambda/2$.

We can produce a different standing wave on the string by placing a finger lightly near the center to keep it from moving and plucking about 1/4 of the way from either end. This procedure will produce a standing wave that looks like Fig. 18-20*b*. For this wave $L = \lambda$. By damping and plucking the string in suitably chosen locations, we can produce the standing wave patterns shown in Figs. 18-20*c* and 18-20*d*, for which $L = 3\lambda/2$ and $L = 2\lambda$, respectively.

You can see that the condition for a standing wave to be set up in a string of length L fixed at both ends is

$$L = n\frac{\lambda}{2} \qquad (n = 1, 2, 3, \ldots)$$

or

$$\lambda_n = \frac{2L}{n} \qquad (n = 1, 2, 3, \ldots), \tag{18-45}$$

where λ_n is the nth wavelength in this infinite series. Note that n is the number of half-wavelengths or "loops" that appear in the patterns of Fig. 18-20. Using Eq. 18-13 ($v = \lambda f$), we can write Eq. 18-45 as

$$f_n = \frac{v}{\lambda_n} = n\frac{v}{2L} \qquad (n = 1, 2, 3, \ldots). \tag{18-46}$$

These are the allowed frequencies of the standing waves on the string.

If we consider the similarity between a vibrating spring and a simple harmonic oscillator, we may wonder why the simple oscillator such as the block–spring system has only one allowed frequency whereas the string has an infinite number. In the block–spring system, the inertia is concentrated ("lumped") in a single element of the system (the block), but in the string the inertia is distributed throughout the system. Similarly, the elasticity of the block–spring system is lumped in one element (the spring) but in the string it is distributed throughout the system. Although there is only one way for the block–spring system to store kinetic and potential energy, the vibrating spring has an infinite number of ways to store its energy.

In general, a lumped system of N elements has N different oscillating frequencies, each of which corresponds to a different pattern of oscillation. Figure 18-21 shows an example of a lumped system with one, two, or three elements. The limit as N tends to infinity leads us to the completely distributed system of the stretched string, with its infinite number of vibrational frequencies.

Resonance in the Stretched String

Figure 18-22 shows time exposures of a student shaking one end of a string that is fixed at the other end. The resulting patterns of oscillation look just like the standing waves of Fig. 18-20. Careful examination would show that the student's hand is moving back and forth at small amplitude with one of the frequencies given by Eq. 18-46. We can regard those frequencies as the natural frequencies of the vibrating system. The student's hand is the driving force that sets the string into oscillation, and when the driving frequency matches one of the natural frequencies we get an oscillation at large amplitude, in exact analogy with our discussion of resonance of the forced oscillator in Section 17-8.

As the student shakes the string, his hand is doing work on it to pump energy into the vibrating system. Energy is lost by the system, perhaps to internal energy of the string, to air resistance, or to the support at the fixed end. As in the case of the forced oscillator, eventually a steady state is reached in which the energy supplied by the student exactly balances the energy lost by the string to dissipative forces.

If the student shakes the string at a frequency that differs from one of the natural frequencies, the reflected wave returns to the student's hand out of phase with the motion of the hand. In this case the string does work on the hand, in addition to the hand doing work on the string. No fixed standing wave pattern is produced; the amplitude of the resulting motion of the string is small and not much different

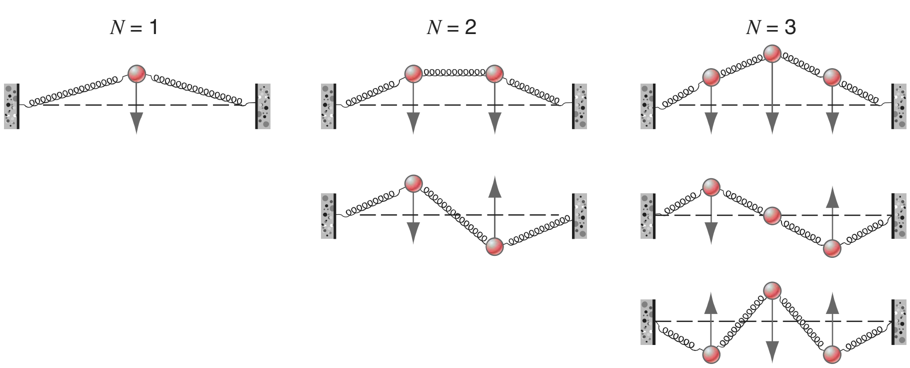

Figure 18-21. Some patterns of oscillation of an oscillator having lumped elements—in this case oscillating bodies connected by springs of negligible mass. Each different pattern of motion has a different natural frequency, the number of natural frequencies being equal to the number of oscillating bodies.

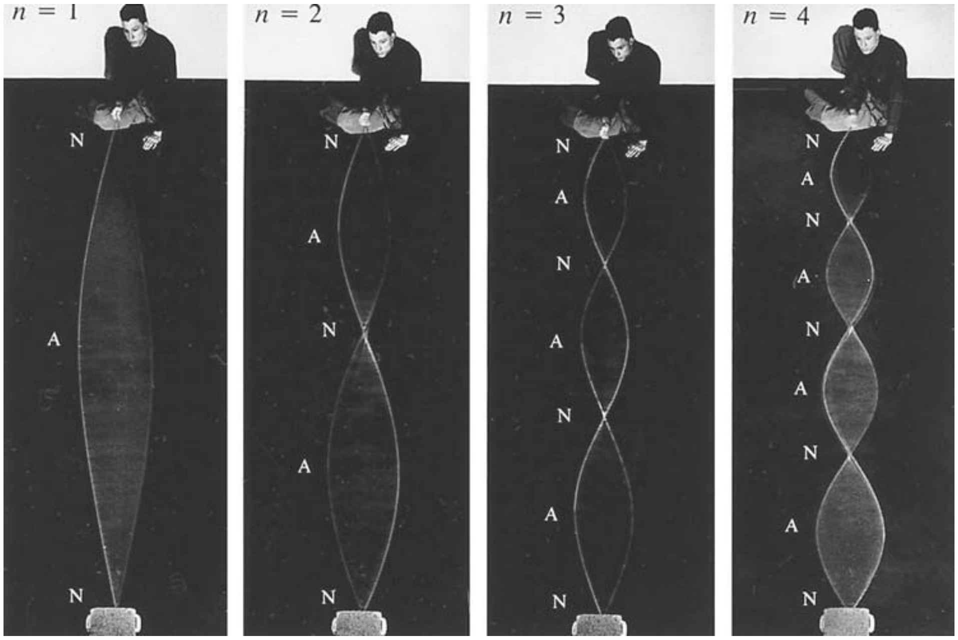

FIGURE 18-22. A student shakes a stretched string (actually a rubber tube) at four resonant frequencies, producing four different patterns of standing waves. The letters N and A indicate the nodes and antinodes, respectively.

from the motion of the student's hand. This situation is analogous to the erratic, small-amplitude motion of a swing being pushed with a frequency other than its natural one. At resonance, the motion of the student's hand is in phase with that of the string, so no energy is lost by the string through work done on the student's hand.

In actuality, the motion of the string is a very good approximation to the standing wave patterns of Fig. 18-20, but not quite an exact one. The resonant frequency is almost, but not exactly, a natural frequency of the system. The apparent nodes are not true nodes, because some energy must be flowing past them along the string to compensate for losses due to damping. If there were no damping, the resonant frequency would be exactly a natural frequency, and the amplitude would increase without limit as energy continued to be supplied to the string by the student's hand. Eventually the elastic limit would be exceeded and the string would break.

If it were possible to shake the string with an assortment of frequencies, the motion of the string would select those frequencies that were equal to its natural frequencies. Motion at those frequencies would be reinforced and would occur at large amplitude, whereas motion at the other frequencies would be damped or suppressed. This principle governs the production of sound by musical instruments, as we discuss in the next chapter.

SAMPLE PROBLEM 18-4. In the arrangement of Fig. 18-23, a motor sets the string into motion at a frequency of 120 Hz. The string has a length of $L = 1.2$ m, and its linear mass density is 1.6 g/m. To what value must the tension be adjusted (by increasing the hanging weight) to obtain the pattern of motion having four loops?

Solution To find the tension, we can substitute Eq. 18-19 into Eq. 18-46 and obtain

$$F = \frac{4L^2 f_n^2 \mu}{n^2}.$$

The tension corresponding to $n = 4$ (for 4 loops) is found to be

$$F = \frac{4(1.2\text{ m})^2(120\text{ Hz})^2(0.0016\text{ kg/m})}{4^2} = 8.3\text{ N}.$$

This corresponds to a hanging weight of about 2 lb.

SAMPLE PROBLEM 18-5. A violin string tuned to concert A (440 Hz) has a length of 0.34 m. (*a*) What are the three longest wavelengths of the resonances of the string? (*b*) What are the corresponding wavelengths that reach the ear of the listener?

Solution (*a*) The resonant wavelengths of a string of length $L =$ 0.34 m can be found directly from Eq. 18-45:

$$\lambda_1 = 2L/1 = 2(0.34\text{ m}) = 0.68\text{ m},$$

$$\lambda_2 = 2L/2 = 0.34\text{ m},$$

$$\lambda_3 = 2L/3 = 0.23\text{ m}.$$

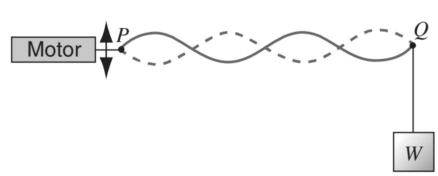

FIGURE 18-23. Sample Problem 18-4. A string under tension is connected to a vibrator. For a fixed vibrator frequency, standing wave patterns will occur for certain discrete values of the tension in the string.

(*b*) When a wave passes from one medium (the string) to another (the air) of differing wave speed, the frequency remains the same, but the wavelength changes. Equation 18-20 gives the relationship between the wavelengths. To find the wave speed on the string, we note that in the lowest resonant mode $f = 440$ Hz and $\lambda = 0.68$ m, so that

$$v = f\lambda = (440\text{ Hz})(0.68\text{ m}) = 299\text{ m/s}.$$

In air, the wave speed is 343 m/s, and from Eq. 18-20 we obtain

$$\lambda_{\text{air}} = \lambda_{\text{string}} \frac{v_{\text{air}}}{v_{\text{string}}} = \lambda_{\text{string}} \frac{343\text{ m/s}}{299\text{ m/s}} = 1.15\lambda_{\text{string}}$$

We thus find the wavelengths in air:

$$\lambda_1 = 0.78\text{ m}, \qquad \lambda_2 = 0.39\text{ m}, \qquad \lambda_3 = 0.26\text{ m}.$$

MULTIPLE CHOICE

18-1 Mechanical Waves

18-2 Types of Waves

18-3 Traveling Waves

1. A disturbance can be written

$$y(x, t) = (e^{-(x/b)^2} e^{2xt/b} e^{-t^2})^a.$$

This disturbance is

(A) not a traveling wave.
(B) a traveling wave with speed $v = a$.
(C) a traveling wave with speed $v = a/b$.
(D) a traveling wave with speed $v = b$.

2. A traveling wave is of the form

$$y(x, t) = A\cos(kx - \omega t) + B\sin(kx - \omega t),$$

which can also be written as

$$y(x, t) = D\sin(kx - \omega t - \phi),$$

where

(*a*)

(A) $D = A + B$ (B) $D = |A| + |B|$
(C) $D^2 = A^2 + B^2$ (D) $D = A - B$

and

(*b*)

(A) $\phi = \tan^{-1}(A/B)$. (B) $\phi = \tan^{-1}(B/A)$.
(C) $\phi = \tan^{-1}(-A/B)$. (D) $\phi = 0$.

3. Consider the maximum transverse speed $u_{\max}$ of a particle in a wave and the wave speed v. Which of the following statements is most true?

(A) $u_{\max}$ is always greater than v.
(B) $u_{\max}$ is always equal to v.
(C) $u_{\max}$ is always less than v.
(D) $u_{\max}$ is unrelated to v.

18-4 Wave Speed on a Stretched Spring

4. A string is stretched horizontally between a fixed point and a frictionless pulley; the string passes over the pulley and an object of mass m is hanging from the end of the string. The tension in this string is T_0; the speed of a wave on this string is v_0. A second string is connected beside the first, passes over the same pulley, and then is attached to the same object. Assuming both strings support the object equally,

(*a*) the tension in the first string is now

(A) $T_0/2$. (B) T_0 (C) $2T_0$.

(*b*) The speed of a wave on the first string would now be

(A) $\sqrt{v_0/2}$. (B) $v_0/\sqrt{2}$. (C) $v_0/2$.
(D) v_0. (E) $\sqrt{2}v_0$.

The two strings are now twisted together to make one string with twice the mass density. This new string is still attached to the same hanging object.

(*c*) The speed of a wave on this new string would now be

(A) $\sqrt{v_0/2}$. (B) $v_0/\sqrt{2}$. (C) $v_0/2$.
(D) v_0. (E) $\sqrt{2}v_0$.

5. Dispersion happens as a wave pulse travels through a medium because

(A) different wave frequencies lose energy at different rates.
(B) different wave amplitudes lose energy at different rates.
(C) different wave frequencies travel through the medium at different wave speeds.
(D) different wave amplitudes travel through the medium with different wavelengths.

18-5 The Wave Equation

6. Which of the following functions is *not* a solution to the wave equation (Eq. 18-24)?

(A) $y = \sin x \cos t$ (B) $y = \tan(x + t)$
(C) $y = x^3 - 6x^2t + 12xt^2 - 8t^3$
(D) $y = \sin(x + t)\cos(x - t)$

7. Which of the following functions *is* a solution to the wave equation (Eq. 18-24)?

(A) $y = x^2 - t^2$ (B) $y = \sin x^2 \sin t$
(C) $y = \log(x^2 - t^2) - \log(x - t)$
(D) $y = e^x \sin t$

18-6 Energy in Wave Motion

8. A certain wave on a string with amplitude A_0 and frequency f_0 transfers energy at an average rate of P_0. If the amplitude and frequency are both doubled, the new wave would transfer energy at an average rate of

(A) P_0. (B) $4P_0$. (C) $\pi^2 P_0$.
(D) $4\pi^2 P_0$. (E) $16P_0$.

9. A wave on a string passes the point $x = 0$ with amplitude A_0, angular frequency ω_0, and average rate of energy transfer P_0. As the wave travels down the string it gradually loses energy; at the point $x = l$ the average rate of energy transfer is now $P_0/2$.

(*a*) At the point $x = l$ the angular frequency of the wave

(A) is still ω_0.
(B) can be less than ω_0 but is more than $\omega_0/\sqrt{2}$.
(C) can be less than ω_0 but is more than $\omega_0/2$.
(D) is equal to $\omega_0/\sqrt{2}$.
(E) is equal to $\omega_0/2$.

(b) At the point $x = l$ the amplitude of the wave

(A) is still A_0.
(B) can be less than A_0 but is more than $A_0/\sqrt{2}$.
(C) can be less than A_0 but is more than $A_0/2$.
(D) is equal to $A_0/\sqrt{2}$.
(E) is equal to $A_0/2$.

18-7 The Principle of Superposition

18-8 Interference of Waves

10. Two waves travel down the same string. The waves have the same velocity, frequency (f_0), and wavelength but different phase constants ($\phi_1 > \phi_2$) and amplitudes ($A_1 > A_2$).
(a) According to the principle of superposition, the resultant wave has an amplitude A such that

(A) $A = A_1 + A_2$. (B) $A = A_1 - A_2$.
(C) $A_2 \le A \le A_1$. (D) $A_1 - A_2 \le A \le A_1 + A_2$.

(b) According to the principle of superposition, the resultant wave has a frequency f such that

(A) $f = f_0$. (B) $f_0/2 < f < f_0$.
(C) $0 < f < f_0$. (D) $f = 2f_0$.

11. Two waves moving along the same string are defined by $y_1 = 2 \sin(kx - \omega t + 0)$ and $y_2 = 2 \sin(kx - \omega t + 2\pi)$. The amplitude of the resultant wave is

(A) 0. (B) 2. (C) $2\sqrt{2}$. (D) 4.

18-9 Standing Waves

12. In the equation for the standing wave (Eq. 18-42), what does the quantity ω/k represent?

(A) The transverse speed of the particles of the string.
(B) The speed of either of the component waves.
(C) The speed of the standing wave.
(D) A quantity that is independent of the properties of the string.

13. A standing wave occurs on a string when two waves of equal amplitude, frequency, and wavelength move in opposite directions on a string. If the wavelength of the two waves is decreased to one-half the original length while the wave speed remains unchanged, then the angular frequency of oscillation of the standing wave will

(A) decrease to one-half. (B) remain the same.
(C) double.

14. Assume that one of the components of the standing wave (as written in Eq. 18-41) has an additional phase constant $\Delta\phi$. How will this affect the standing wave?

(A) The standing wave will have a different frequency.
(B) The standing wave will have a different amplitude.
(C) The standing wave will have a different spacing between the nodes.
(D) None of these things will happen.

15. In a standing wave on a string, the spacing between nodes is Δx. If the tension in the string is doubled but the frequency of the standing waves is fixed, then the spacing between the nodes will change to

(A) $2\Delta x$. (B) $\sqrt{2}\Delta x$.
(C) $\Delta x/2$. (D) $\Delta x/\sqrt{2}$.

18-10 Standing Waves and Resonance

16. A string is stretched between fixed points. The string has a mass density μ, is under a tension F, and has a length L. The string is vibrating at the lowest allowed frequency.
(a) The wave speed on this string is a function of

(A) μ. (B) F. (C) L.
(D) μ and F. (E) μ, F, and L.

(b) The lowest allowed standing wave frequency is a function of

(A) μ. (B) F. (C) L.
(D) μ and F. (E) μ, F, and L.

(c) The lowest allowed standing wave wavelength is a function of

(A) μ. (B) F. (C) L.
(D) μ and F. (E) μ, F, and L.

17. A 10-cm-long rubber band obeys Hooke's law. When the rubber band is stretched to a total length of 12 cm the lowest resonant frequency is f_0. The rubber band is then stretched to a length of 13 cm. The lowest resonant frequency will now be

(A) higher than f_0.
(B) the same as f_0.
(C) lower than f_0.
(D) changed, but the direction of the change depends on the elastic constant and the original tension.

QUESTIONS

1. How could you prove experimentally that energy is associated with a wave?

2. Energy can be transferred by particles as well as by waves. How can we experimentally distinguish between these methods of energy transfer?

3. Can a wave motion be generated in which the particles of the medium vibrate with angular simple harmonic motion? If so, explain how and describe the wave.

4. In analyzing the motion of an elastic wave through a material medium, we often ignore the molecular structure of matter. When is this justified and when is it not?

5. How do the amplitude and the intensity of surface water waves vary with the distance from the source?

6. How can one create plane waves? Spherical waves?

7. A passing motor boat creates a wake that causes waves to wash ashore. As time goes on, the period of the arriving waves grows shorter and shorter. Why?

8. The following functions in which A is a constant are of the form $y = f(x \pm vt)$:

$$y = A(x - vt), \qquad y = A\sqrt{x - vt},$$
$$y = A(x + vt)^2, \qquad y = A \ln(x + vt).$$

Explain why these functions are not useful in wave motion.

9. Can one produce on a string a waveform that has a discontinuity in slope at a point—that is, a sharp corner? Explain.

10. The inverse-square law does not apply exactly to the decrease in intensity of sounds with distance. Why not?

11. When two waves interfere, does one alter the progress of the other?

12. When waves interfere, is there a loss of energy? Explain your answer.

13. Why do we not observe interference effects between the light beams emitted from two flashlights or between the sound waves emitted by two violins?

14. As Fig. 18-17 shows, twice during the cycle the configuration of standing waves in a stretched string is a straight line, exactly what it would be if the string were not vibrating at all. Discuss from the point of view of energy conservation.

15. Two waves of the same amplitude and frequency are traveling on the same string. At a certain instant the string looks like a straight line. Are the two waves necessarily traveling in the same direction? What is the phase relationship between the two waves?

16. If two waves differ only in amplitude and are propagated in opposite directions through a medium, will they produce standing waves? Is energy transported? Are there any nodes?

17. The partial reflection of wave energy by discontinuities in the path of transmission is usually wasteful and can be minimized by insertion of "impedance matching" devices between sections of the path bordering on the discontinuity. For example, a megaphone helps match the air column of mouth and throat to the air outside the mouth. Give other examples and explain qualitatively how such devices minimize reflection losses.

18. Consider the standing waves in a string to be a superposition of traveling waves and explain, using superposition ideas, why there are no true nodes in the resonating string of Fig. 18-23, even at the "fixed" end. (Hint: Consider damping effects.)

19. Standing waves in a string are demonstrated by an arrangement such as that of Fig. 18-23. The string is illuminated by a fluorescent light and the vibrator is driven by the same electric outlet that powers the light. The string exhibits a curious color variation in the transverse direction. Explain.

20. In the discussion of transverse waves on a string, we have dealt only with displacements in a single plane, the xy plane. If all displacements lie in one plane, the wave is said to be plane polarized. Can there be displacements in a plane other than the plane dealt with? If so, can two different plane polarized waves be combined? What appearance would such a combined wave have?

21. A wave transmits energy. Does it transfer momentum? Can it transfer angular momentum? (See "Energy and Momentum Transport in String Waves," by D. W. Juenker, *American Journal of Physics,* January 1976, p. 94.)

22. In the Mexico City earthquake of September 19, 1985, areas with high damage alternated with areas of low damage. Also, buildings between 5 and 15 stories high sustained the most damage. Discuss these effects in terms of standing waves and resonance.

EXERCISES

18-1 Mechanical Waves

18-2 Types of Waves

18-3 Traveling Waves

1. A wave has a wave speed of 243 m/s and a wavelength of 3.27 cm. Calculate (*a*) the frequency and (*b*) the period of the wave.

2. By rocking a boat, a child produces surface water waves on a previously quiet lake. It is observed that the boat performs 12 oscillations in 30 s and also that a given wave crest reaches shore 15 m away in 5.0 s. Find (*a*) the frequency, (*b*) the speed, and (*c*) the wavelength of the waves.

3. A sinusoidal wave travels along a string. The time for a particular point to move from maximum displacement to zero displacement is 178 ms. The wavelength of the wave is 1.38 m. Find (*a*) the period, (*b*) the frequency, and (*c*) the speed of the wave.

4. Write an expression describing a transverse wave traveling along a string in the $+x$ direction with wavelength 11.4 cm, frequency 385 Hz, and amplitude 2.13 cm.

18-4 Wave Speed on a Stretched Spring

5. Assuming that the wave speed on a stretched string depends on the tension F and linear mass density μ as $v \propto F^a/\mu^b$, use dimensional analysis to show that $a = \frac{1}{2}$ and $b = \frac{1}{2}$.

6. The equation of a transverse wave traveling along a string is given by

$$y = (2.30 \text{ mm}) \sin [(1822 \text{ rad/m})x - (588 \text{ rad/s})t].$$

Find (*a*) the amplitude, (*b*) the frequency, (*c*) the velocity, (*d*) the wavelength of the wave, and (*e*) the maximum transverse speed of a particle in the string.

7. The equation of a transverse wave traveling along a very long string is given by

$$y = (6.0 \text{ cm}) \sin [(2.0\pi \text{ rad/m})x + (4.0\pi \text{ rad/s})t].$$

Calculate (*a*) the amplitude, (*b*) the wavelength, (*c*) the frequency, (*d*) the speed, (*e*) the direction of propagation of the wave, and (*f*) the maximum transverse speed of a particle in the string.

8. Calculate the speed of a transverse wave in a string of length 2.15 m and mass 62.5 g under a tension of 487 N.

9. The speed of a wave on a string is 172 m/s when the tension is 123 N. To what value must the tension be increased in order to raise the wave speed to 180 m/s?

10. The equation of a particular transverse wave on a string is

$$y = (1.8 \text{ mm}) \sin [(23.8 \text{ rad/m})x + (317 \text{ rad/s})t].$$

The string is under a tension of 16.3 N. Find the linear mass density of the string.

11. A simple harmonic transverse wave is propagating along a string toward the left (or $-x$) direction. Figure 18-24 shows a plot of the displacement as a function of position at time $t = 0$. The string tension is 3.6 N and its linear density is 25 g/m. Calculate (*a*) the amplitude, (*b*) the wavelength, (*c*) the wave speed, (*d*) the period, and (*e*) the maximum speed of a particle in the string. (*f*) Write an equation describing the traveling wave.

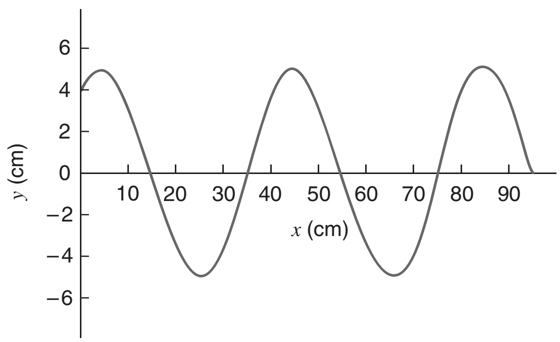

FIGURE 18-24. Exercise 11.

12. In Fig. 18-25*a*, string 1 has a linear mass density of 3.31 g/m, and string 2 has a linear mass density of 4.87 g/m. They are under tension due to the hanging block of mass $M = 511$ g. (*a*) Calculate the wave speed in each string. (*b*) The block is now divided into two blocks (with $M_1 + M_2 = M$) and the apparatus is rearranged as shown in Fig. 18-25*b*. Find M_1 and M_2 such that the wave speeds in the two strings are equal.

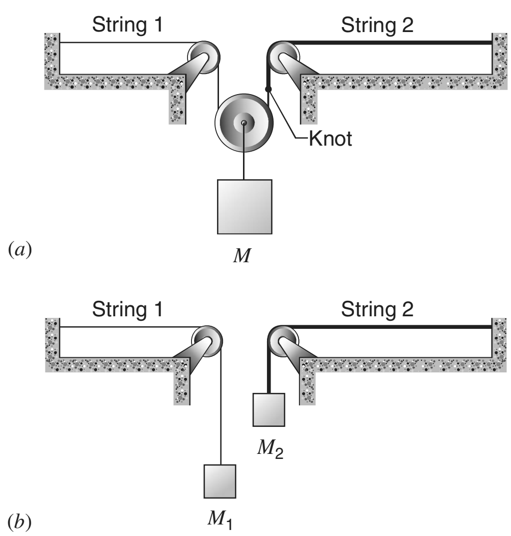

FIGURE 18-25. Exercise 12.

13. A wire 10.3 m long and having a mass of 97.8 g is stretched under a tension of 248 N. If two pulses, separated in time by 29.6 ms, are generated one at each end of the wire, where will the pulses meet?

18-5 The Wave Equation

14. In a spherically symmetric system, the three-dimensional wave equation is given by

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial y}{\partial r}\right) = \frac{1}{v^2}\frac{\partial^2 y}{\partial t^2}.$$

(*a*) Show that

$$y(r, t) = \frac{A}{r}\sin(kr - \omega t)$$

is a solution to this wave equation. (*b*) What are the dimensions of the constant A?

18-6 Energy in Wave Motion

15. A string 2.72 m long has a mass of 263 g. The tension in the string is 36.1 N. What must be the frequency of traveling waves of amplitude 7.70 mm in order that the average transmitted power be 85.5 W?

16. A line source emits a cylindrical expanding wave. Assuming that the medium absorbs no energy, find how (*a*) the intensity and (*b*) the amplitude of the wave depend on the distance from the source.

17. An observer measures an intensity of 1.13 W/m^2 at an unknown distance from a source of spherical waves whose power output is also unknown. The observer walks 5.30 m closer to the source and measures an intensity of 2.41 W/m^2 at this new location. Calculate the power output of the source.

18. (*a*) Show that the intensity I is the product of the energy density u (energy per unit volume) and the speed of propagation v of a wave disturbance; that is, show that $I = uv$. (*b*) Calculate the energy density in a sound wave 4.82 km from a 47.5-kW siren, assuming the waves to be spherical, the propagation isotropic with no atmospheric absorption, and the speed of sound to be 343 m/s.

18-7 The Principle of Superposition

18-8 Interference of Waves

19. What phase difference between two otherwise identical traveling waves, moving in the same direction along a stretched string, will result in the combined wave having an amplitude 1.65 times that of the common amplitude of the two combining waves? Express your answer in both degrees and radians.

20. Determine the amplitude of the resultant wave when two sinusoidal waves having the same frequency and traveling in the same direction are combined, if their amplitudes are 3.20 cm and 4.19 cm and they differ in phase by $\pi/2$ rad.

21. For the case in which the component waves in Eq. 18-38 have different amplitudes y_{m1} and y_{m2}, show that the quantity in square brackets in Eq. 18-40 becomes $[y_{m1}^2 + y_{m2}^2 + 2y_{m1}y_{m2}\cos\Delta\phi]^{1/2}$ and the phase constant ϕ' becomes

$$\phi' = \sin^{-1}\left[\frac{y_{m1}\sin\phi_1 + y_{m2}\sin\phi_2}{(y_{m1}^2 + y_{m2}^2 + 2y_{m1}y_{m2}\cos\Delta\phi)^{1/2}}\right].$$

Check that both expressions reduce to the expected results when $y_{m1} = y_{m2} = y_m$.

22. Two pulses are traveling along a string in opposite directions, as shown in Fig. 18-26. (*a*) If the wave speed is 2.0 m/s and the pulses are 6.0 cm apart, sketch the patterns after 5.0, 10, 15, 20, and 25 ms. (*b*) What has happened to the energy at $t = 15$ ms?

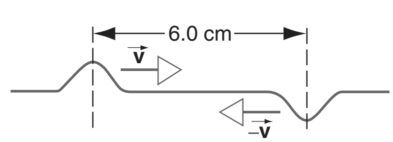

FIGURE 18-26. Exercise 22.