Basic Discrete Structure : Set

Lecture 4

Basic Discrete Structure

Discrete math =
– study of the discrete structures used to represent discrete objects

Many discrete structures are built using **sets**
– **Sets = collection of objects**

Examples of discrete structures built with the help of sets:
- **Combinations**
- **Relations**
- **Graphs**

Set

Definition:

A set is a (unordered) collection of objects.

These objects are sometimes called **elements or members of the set.**

- **Examples:**

– **Vowels in the English alphabet**

$V = \{ a, e, i, o, u \}$

– **First seven prime numbers.**

$X = \{ 2, 3, 5, 7, 11, 13, 17 \}$

Representing Set

Representing a set by:

1) Listing (enumerating) the members of the set.
2) Definition by property, using the set builder notation
   {x| x has property P}.

Example:

- Even integers between 50 and 63.

1) E = {50, 52, 54, 56, 58, 60, 62}
2) E = {x| 50 <= x < 63, x is an even integer}

If enumeration of the members is hard we often use ellipses.
Example: a set of integers between 1 and 100

- A= {1,2,3 …, 100}

Important set in discrete math

- Natural numbers
  - N = {0,1,2,3, …}
- Integers
  - Z = {…, -2,-1,0,1,2, …}
- Positive integers
  - Z+ = {1,2, 3.…}
- Rational numbers
  - $Q = \{p / q \mid p \in Z, q \in Z, q \neq 0\}$
- Real numbers
  - R
- Positive Real numbers
  - R+

Equality of Set

Definition: Two sets are equal if and only if they have the same elements.

Example:
- {1,2,3} = {3,1,2} = {1,2,1,3,2}

Note: Duplicates don't contribute anything new to a set, so remove them. The order of the elements in a set doesn't contribute anything new.

Example: Are {1,2,3,4} and {1,2,2,4} equal?
No!

Universal set

Special sets:

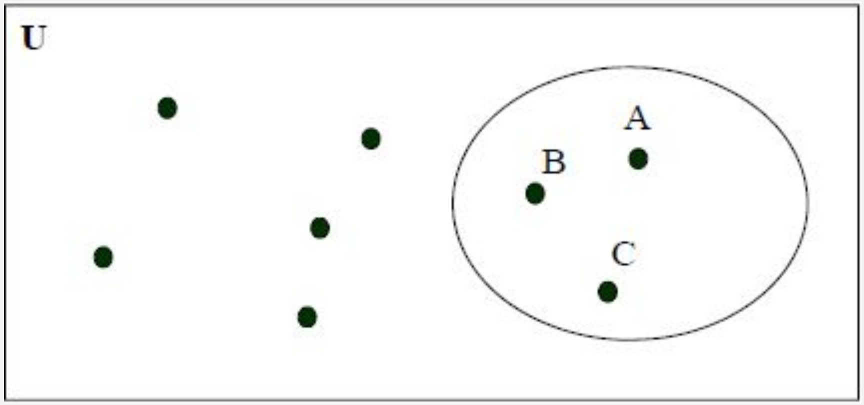
– The universal set is denoted by U: the set of all objects under the consideration.

– The empty set is denoted as $\emptyset$ or $\{\ \}$.

Venn Diagram

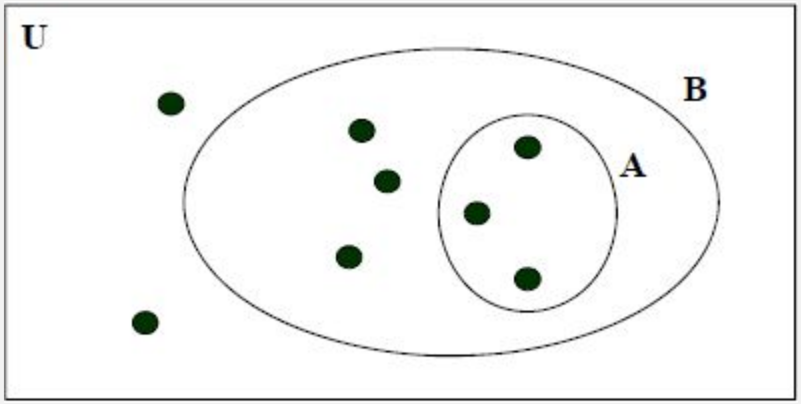
A set can be visualized using **Venn Diagrams:**

– V={ A, B, C }



A subset

Definition: A set A is said to be a subset of B if and only if every element of A is also an element of B. We use $A \subseteq B$ **to** indicate **A is a subset of B.**



Alternate way to define A is a subset of B:

$$\forall x (x \in A) \rightarrow (x \in B)$$

Empty set/subset property

Theorem : $\phi \in S$

- Empty set is a subset of any set.

Proof:

- Recall the definition of a subset: all elements of a set A must be also elements of B: $\forall x(x \in A \rightarrow x \in B)$
- We must show the following implication holds for any

$$\forall x(x \in \phi \rightarrow x \in S)$$

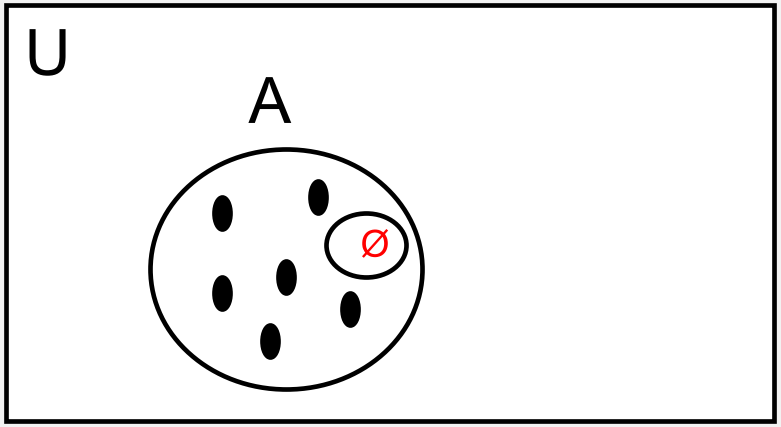
- Since the empty set does not contain any element, $x \in \phi$ is **always False**
- Then the implication is **always True**. **(F → T/F =T)**

End of proof

Venn diagram of Empty set

Theorem : $\phi \in S$

- Empty set is a subset of any set.



Subset property

Theorem: $S \subseteq S$

- Any set S is a subset of itself

Proof:

- the definition of a subset says: all elements of a set A must be also elements of B: $\forall x(x \in A \rightarrow x \in B)$

- Applying this to S we get:
- $\forall x(x \in S \rightarrow x \in S)$ which is trivially **True**

- End of proof

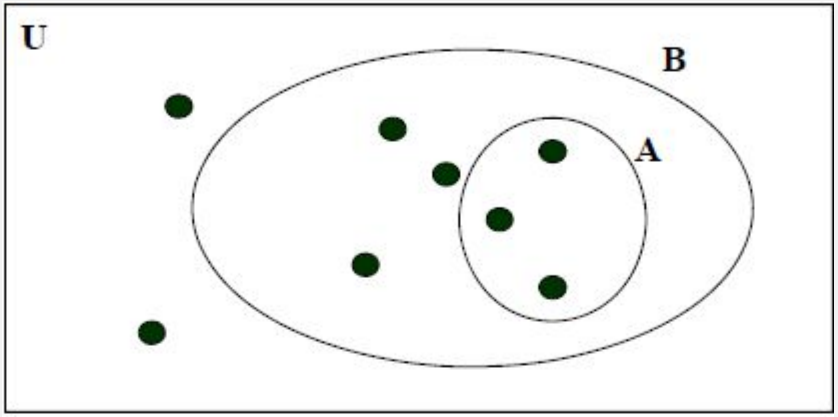
Note on equivalence:

- Two sets are equal if each is a subset of the other set.

A proper Subset

Definition:

A set A is said to be a proper subset of B if and only if $A \subseteq B$ **and** $A \neq B$ **. We denote that A is a proper subset of B** with the notation $A \subset B$.



Example: A={1,2,3} B ={1,2,3,4,5}

Is: $A \subset B$?

A proper Subset

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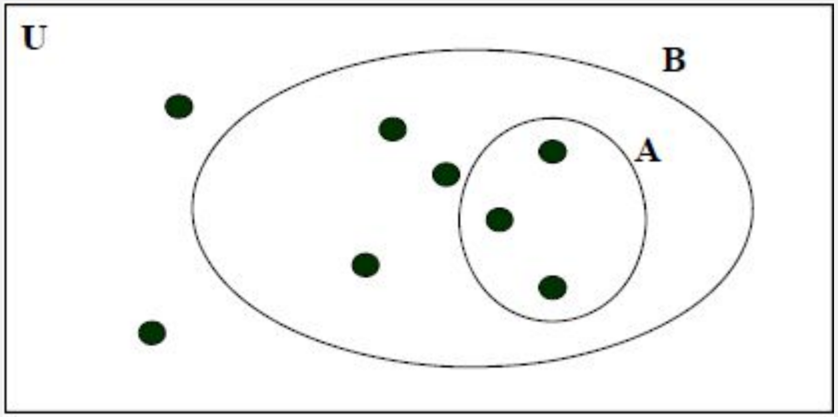


Example: A={1,2,3} B ={1,2,3,4,5}

Is: $A \subset B$? Yes.

Cardinality

Definition: Let S be a set. If there are exactly n distinct elements in S, where n is a nonnegative integer, we say S is a finite set and that n is the cardinality of S. The cardinality of S is denoted by $|S|$.

Examples:

- V={1, 2, 3, 4, 5}
  $|V| = 5$

- A={1,2,3,4, …, 20}
  $|A| = 20$

- $|\varnothing| = 0$

Infinite set

Definition: A set is infinite if it is not finite.

Examples:

- The set of natural numbers is an infinite set.
- N = {1, 2, 3, ... }

- The set of real numbers is an infinite set.

Power set

Definition: Given a set S, the power set of S is the set of all subsets of S. The power set is denoted by P(S).

Example

- What is the power set of Ø ? P(Ø) = {Ø}
- What is the cardinality of P(Ø) ? | P(Ø) | = 1.

Assume {1,2,3}

- P({1,2,3}) = {Ø, {1}, {2}, {3}, {1,2}, {1,3}, {2,3}, {1,2,3} }
- |P({1,2,3} | = 8

If S is a set with |S| = n then | P(S) | = ? **2^n**

N-tuple

Sets are used to represent unordered collections.

- Ordered-n tuples are used to represent an ordered collection.

Definition: An ordered n-tuple (x1, x2, ..., xN) is the ordered collection that has x1 as its first element, x2 as its second element, ..., and xN as its N-th element, N >= 2.

Example: Coordinates of a point in the 2-D plane (12, 16)

Cartesian Product

Definition: Let S and T be sets. The Cartesian product of S and T, denoted by S x T, is the set of all ordered pairs (s,t), where s $\in$ S and t $\in$ T. Hence,

- $S \times T = \{ (s,t) \mid s \in S \wedge t \in T\}$.

Examples:

- S = {1,2} and T = {a,b,c}

- S x T = { (1,a), (1,b), (1,c), (2,a), (2,b), (2,c) }
- T x S = { (a,1), (a, 2), (b,1), (b,2), (c,1), (c,2) }

- Note: S x T $\neq$ T x S **!!!!**

Cardinality of a Cartesian Product

- $|S \times T| = |S| * |T|$.

Example:

- A= {John, Peter, Mike}
- B ={Jane, Ann, Laura}

- A x B= {(John, Jane),(John, Ann) , (John, Laura), (Peter, Jane), (Peter, Ann) , (Peter, Laura) , (Mike, Jane) , (Mike, Ann) , (Mike, Laura)}

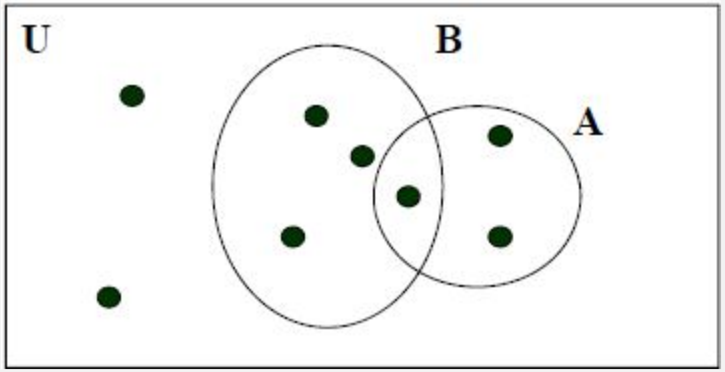
- $|A \times B| = 9$
- $|A|=3$, $|B|=3$ $\rightarrow$ $|A|\ |B|= 9$

Definition: A subset of the Cartesian product A x B is called a relation from the set A to the set B.

Set Operation

Definition: Let A and B be sets. The union of A and B, denoted by A U B, is the set that contains those elements that are in both A andB.

- Alternate: A U B = { x | x ϵ A V x ϵ B }.



Example:

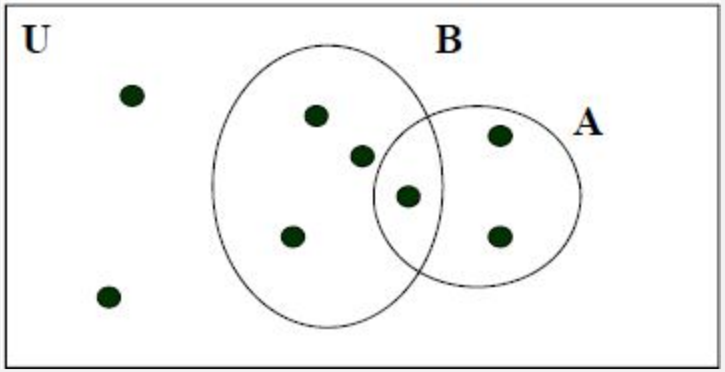
- A = {1,2,3,6} and B = { 2,4,6,9}

- A U B = { 1,2,3,4,6,9 }

Set Operation

Definition: Let A and B be sets. The intersection of A and B, denoted by $A \cap B$, is the set that contains those elements that are in both A andB.

- Alternate: $A \cap B = \{ x \mid x \in A \wedge x \in B \}$.



Example:

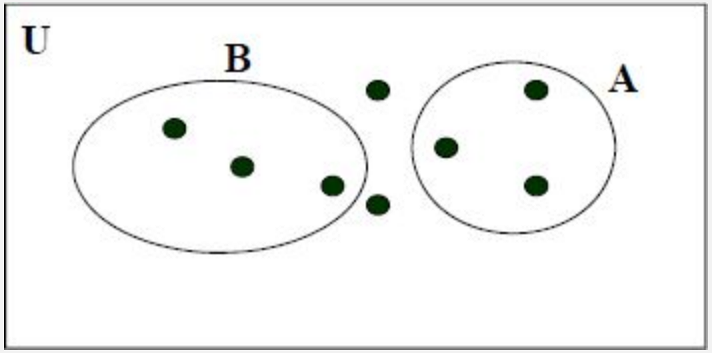
- $A = \{1,2,3\}$ and $B = \{ 2,4,6,9\}$

- $A \cap B = \{ 2 \}$

Disjoin Set

Definition: Two sets are called disjoint if their intersection is empty.

- Alternate: A and B are disjoint **if and only if A ∩ B = Ø.**



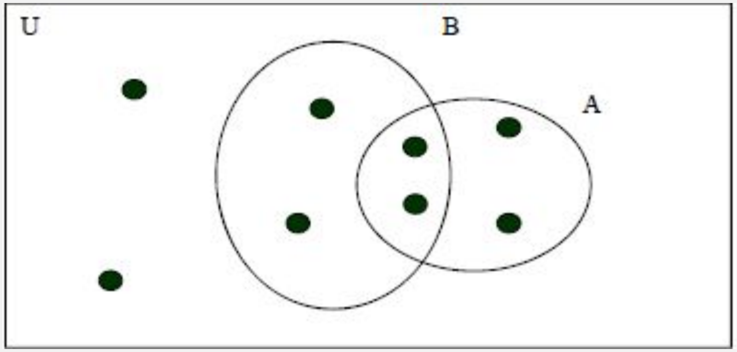
Example:

- A={1,2,3,6} B={4,7,8} Are these disjoint?
- Yes.
- **A ∩ B = Ø**

Cardinality of set union

Cardinality of the set union.

- $|A \cup B| = |A| + |B| - |A \cap B|$

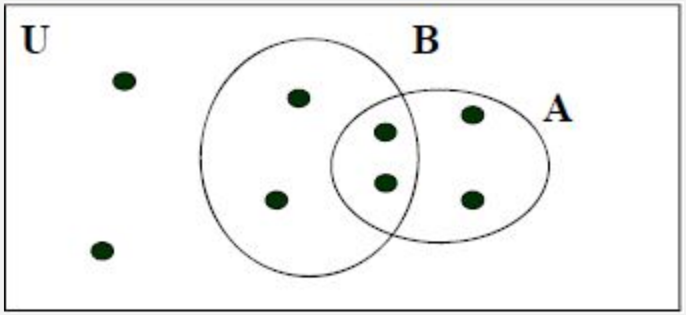


Why this formula? Correct for an over-count.

Set Difference

Definition: Let A and B be sets. The difference of A and B, denoted by A - B, is the set containing those elements that are in but not in B. The difference of A and B is also called the complement of B with respect to A.

- Alternate: $A - B = \{x \mid x \in A \wedge x \notin B\}$



Example: A= {1,2,3,5,7} B = {1,5,6,8}

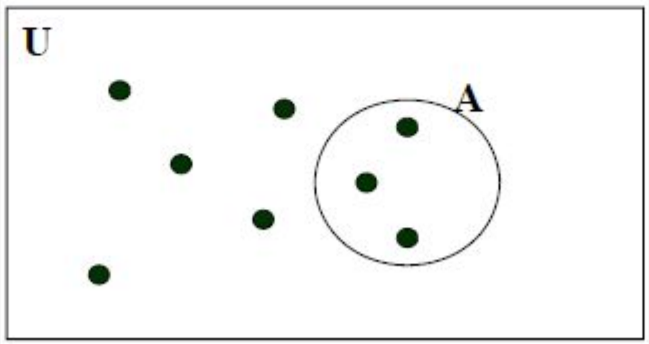
- A - B ={2,3,7}

Complement of a Set

Definition: Let U be the universal set: the set of all objects under the consideration.

Definition: The complement of the set A, denoted by Ã, is the complement of A with respect to U.

- Alternate: Alternate: $\overline{A} = \{x \mid x \notin A\}$



Example: U={1,2,3,4,5,6,7,8} A ={1,3,5}

- Ã={2,4,6,7,8}

Generalized union

Definition: The union of a collection of sets is the set that contains those elements that are members of at least one set in the collection.

$$\bigcup_{i=1}^{n} A_i = \{A_1 \cup A_2 \cup \ldots \cup A_n\}$$

Example:

- Let $A_i = \{1,2,\ldots,i\}$ $\quad i = 1,2,\ldots,n$

$$\bigcup_{i=1}^{n} A_i = \{1,2,\ldots,n\}$$

$A_1 = \{1\}$
$A_2 = \{1,2\}$
$A_3 = \{1,2,3\}$
…………..
…………..
$A_n \{1,2,3,4,\ldots,n\}$

Generalized intersection

Definition: The intersection of a collection of sets is the set that contains those elements that are members of all sets in the collection.

$$\bigcap_{i=1}^{n} A_i = \{A_1 \cap A_2 \cap \ldots \cap A_n\}$$

Example:

- Let $A_i = \{1,2,\ldots,i\}$ $\quad i = 1,2,\ldots,n$

$$\bigcap_{i=1}^{n} A_i = \{1\}$$

$A_1 = \{1\}$
$A_2 = \{1,2\}$
$A_3 = \{1,2,3\}$
…………..
…………..
$A_n\ \{1,2,3,4,\ldots,n\}$

Computer representation of set

How to represent sets in the computer?

- One solution: Data structures like a list

- A better solution: Assign a bit in a bit string to each element in the universal set and set the bit to 1 if the element is present otherwise use 0

Example:

All possible elements: U={1 2 3 4 5}

- Assume A={2,5}

– Computer representation: A = 01001

- Assume B={1,5}

– Computer representation: B = 10001

Computer representation of set

Example:
- A = 01001
- B = 10001

- The union is modeled with a bitwise or
- $A \cup B$ = 11001

- The intersection is modeled with a bitwise and
- $A \cap B$ = 00001

- The complement **is modeled with a bitwise negation**
- $\tilde{A}$ =10110

Basic Discrete Structure :
Set and Function

Quiz Problem 1

1. Suppose the following two statements are true.

I love Dad or I love Mum

If I love Dad then I love Mum

Does it necessarily follow that I love Dad? Does it necessarily follow that I love Mum? Use propositional logic to answer the questions.

P (I love dad)	**Q** (I love mum)	**P∨Q**	**P→Q**
F	F	F	T
F	T	T	T
T	F	T	F
T	T	T	T

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P (I love dad)	**Q (I love mum)**	**$P \vee Q$**	**$P \rightarrow Q$**
F	F	F	T
F	T	T	T
T	F	T	F
T	T	T	T

Quiz Problem 2

2. Consider the following specification with

two predicate symbols $p_1 \equiv (<)$, $p_2 \equiv (=)$

two function symbols $f_1 \equiv (+)$, $f_2 \equiv (\times)$

two constant symbols $c_1 \equiv (0)$, $c_2 \equiv (1)$

Let domain of discourse be $<Z_+ \cup \{0\}>$ where $Z_+ = \{1,2,....\}$

What are the truth values of the following statements?

i. $\forall x\, p_1(c_1, x)$

ii. $\forall x\, \forall y\, \exists z\, (p_1(x,z) \wedge p_1(z,y))$

iii. $\exists x\, \forall y p_1(x,y)$

iv. $\forall x\, \forall y\, p_2(f_2(f_1(x,c_2),y), f_1(f_2(x,y),y))$

i.	False	$0<x$

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i.	False	$0<x$
ii.	False	$x< z \wedge z<y$

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i.	False	$0<x$
ii.	False	$x< z \wedge z<y$
iii.	False	$x<y$

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iv. $\forall x\, \forall y\, p_2(f_2(f_1(x,c_2),y), f_1(f_2(x,y),y))$

i.	False	$0<x$
ii.	False	$x< z \wedge z<y$
iii.	False	$x<y$
iv	True	$xy + y = xy+y$

Quiz Problem 3

3. Let A be the set of English words that contains x, and B be the set of English words that contain the letter q. Express each of these sets as a combination of A and B.

i. The set of English words that do not contain the letter x.
ii. The set of English words that contain an x but not a q.
iii. The set of English words that do not contain either an x or a q.

i.	**U – A**

Quiz Problem 3

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i.	**U – A**
ii.	A –B

Quiz Problem 3

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i. The set of English words that do not contain the letter x.
ii. The set of English words that contain an x but not a q.
iii. The set of English words that do not contain either an x or a q.

i.	**$U - A$**
ii.	$A - B$
iii.	$U - (A \cap B)$

Quiz Problem 4

4. Let A = {a, b, c}, B = {x, y} and C= {0, 1}. Find C × B × A.

C×B	{(0,x),(0,y),(1,x),(1,y)}

Quiz Problem 4

4. Let A = {a, b, c}, B = {x, y} and C= {0, 1}. Find C × B × A.

C×B	{(0,x),(0,y),(1,x),(1,y)}
C × B × A	{(0,x,a), (0,x,b), (0,x,c),(0,y,a), (0,y,b), (0,y,c),(1,x,a), (1,x,b), (1,x,c),(1,y,a), (1,y,b), (1,y,c)}

Quiz Problem 5

5. Let *f* be the function from **R** to **R** defined by `y=f(m,n)=2m-n`. Write a method signature in C with appropriate return type and parameter list that could be used to realize the function.

```
double f(double m, double n);
```

Definitions and notation

Check these:

Is $\{x\} \subseteq \{x\}$? Yes

Is $\{x\} \in \{x,\{x\}\}$? Yes

Is $\{x\} \subseteq \{x,\{x\}\}$? Yes

Is $\{x\} \in \{x\}$? No

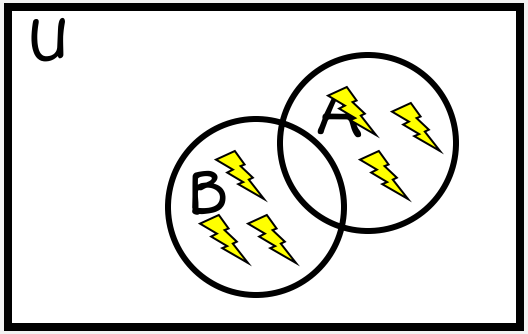
Operators

like
"exclusive or"

The symmetric difference, $A \oplus B$, is:

$$A \oplus B = \{ x : (x \in A \wedge x \notin B) \vee (x \in B \wedge x \notin A)\}$$

$$= (A - B) \cup (B - A)$$



Operators

Proof: $\{ x : (x \in A \land x \notin B) \lor (x \in B \land x \notin A)\}$
$= (A - B) \cup (B - A)$

$A \oplus B = \{ x : (x \in A \land x \notin B) \lor (x \in B \land x \notin A)\}$
$= \{ x : (x \in A - B) \lor (x \in B - A)\}$
$= \{ x : x \in ((A - B) \cup (B - A))\}$
$= (A - B) \cup (B - A)$

Famous Identities

- Identity
  $$A \cap U = A$$
  $$A \cup \varnothing = A$$
- Domination
  $$A \cup U = U$$
  $$A \cap \varnothing = \varnothing$$
- Idempotent
  $$A \cup A = A$$
  $$A \cap A = A$$

Famous Identities

- Excluded Middle $A \cup \bar{A} = U$
- Uniqueness $A \cap \bar{A} = \varnothing$
- Double complement $\bar{\bar{A}} = A$

Famous Identities

- Commutativity

$$A \cup B = B \cup A$$
$$A \cap B = B \cap A$$

- Associativity

$$(A \cup B) \cup C = A \cup (B \cup C)$$
$$(A \cap B) \cap C = A \cap (B \cap C)$$

- Distributivity

$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$$
$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$$

Famous Identities

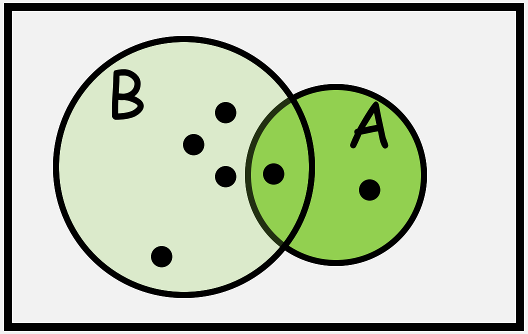
- DeMorgan's I $\overline{(A \cup B)} = \overline{A} \cap \overline{B}$
- DeMorgan's II $\overline{(A \cap B)} = \overline{A} \cup \overline{B}$

Inclusion/Exclusion

Example:
How many people are wearing a watch?
How many people are wearing sneakers?

How many people are wearing
a watch OR sneakers?

What's wrong?

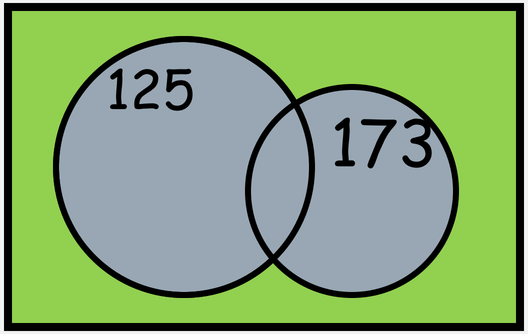


$|A \cup B| = |A| + |B|$ 7

$|A \cup B| = |A| + |B| - |A \cap B|$ 6

Inclusion/Exclusion

Example:
There are 217 cs majors.
157 are taking cs125.
145 are taking cs173.
98 are taking both.

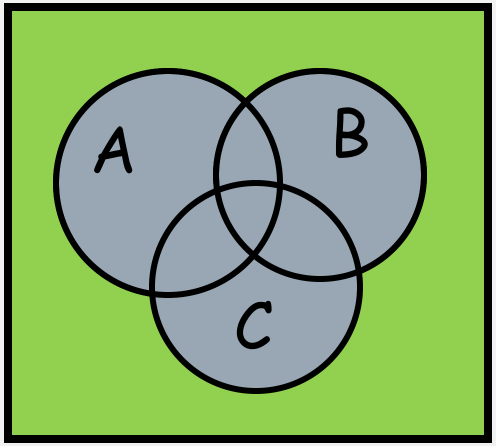


How many are taking neither?

$$217 - (157 + 145 - 98) = 13$$

Generalized Inclusion/Exclusion

Suppose we have:



A = {0, 2, 4, 6, 8},
B = {0, 1, 2, 3, 4},
C = {0, 3, 6, 9}.

And I want to know |A U B U C|

|A U B U C| = |A| + |B| + |C|

- |A ∩ B| - |A ∩ C| - |B ∩ C|

+ |A ∩ B ∩ C|

|A U B U C| = 5+5+4-3-2-2+1 ≡ 8 ≡{0, 1, 2, 3, 4, 6, 8, 9}.

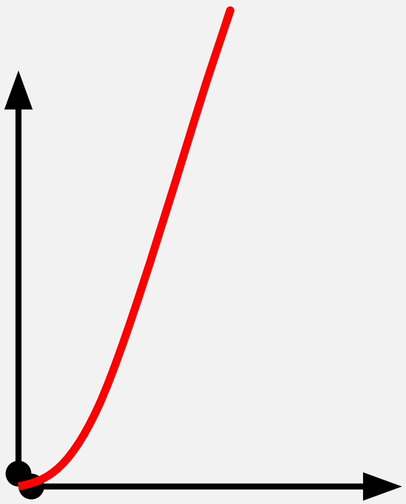
Functions - examples

Suppose f: $R^+ \rightarrow R^+$, $f(x) = x^2$.

Is f one-to-one? yes

Is f onto? yes

Is f bijective? yes



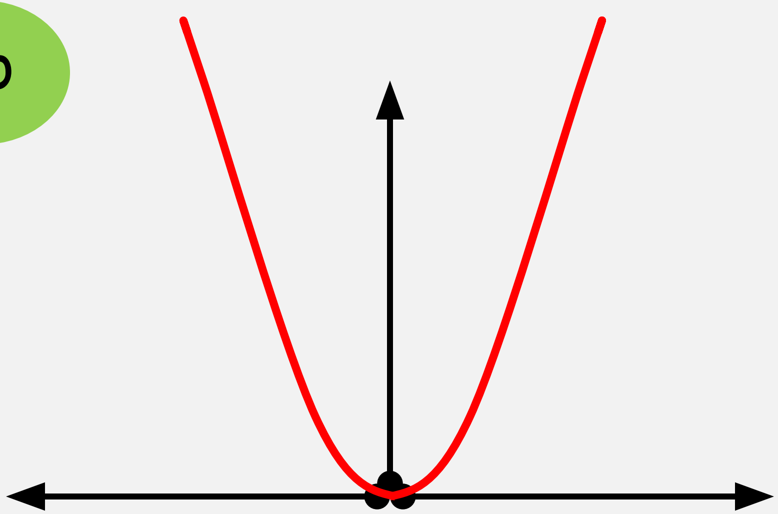
Functions - examples

Suppose f: R $\rightarrow$ R^+, $f(x) = x^2$.

Is f one-to-one? no

Is f onto? yes

Is f bijective? no



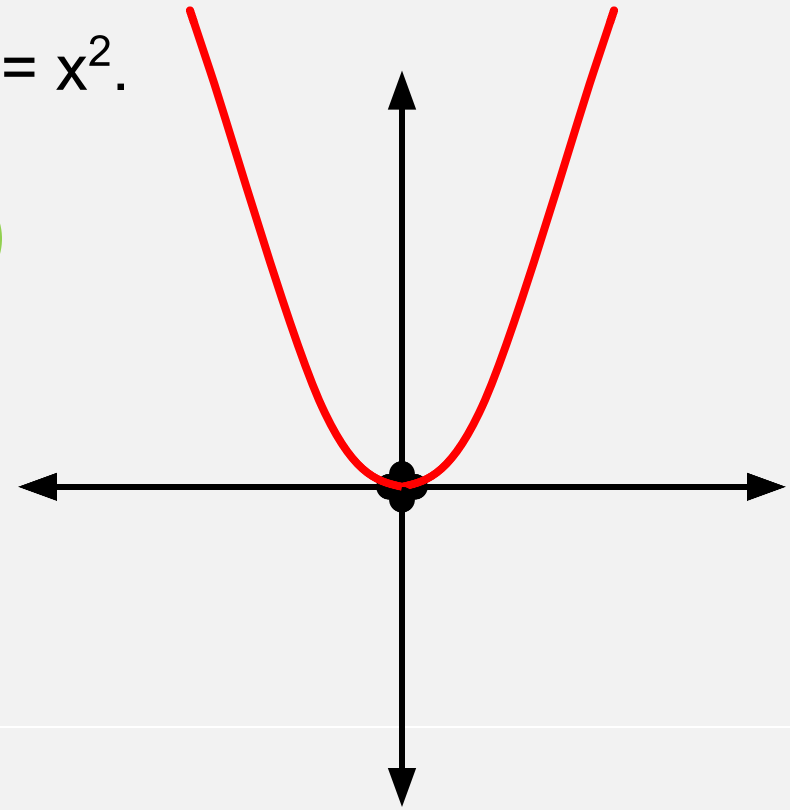
Functions - examples

Suppose f: R → R, $f(x) = x^2$.

Is f one-to-one? no

Is f onto? no

Is f bijective? no



Thank You