Counting

Lecture 15-16

Counting

Assume we have a set of **objects with certain properties**

- **Counting is used to determine the number of these objects**

Examples:

- Number of available phone numbers with 11 digits in the local calling area
- Number of possible match starters (football, cricket) given the number of team members and their positions

Basic Counting Rules

Counting problems may be very hard, not obvious

- **Solution:**
  – **simplify the solution by decomposing the problem**

- **Two basic decomposition rules:**
  – **Product rule**
    - A count decomposes into a sequence of dependent counts ("each element in the first count is associated with all elements of the second count")

  – **Sum rule**
    - A count decomposes into a set of independent counts ("elements of counts are alternatives")

Product Rule

A count can be broken down into a sequence of dependent counts

- **“each element in the first count is associated with all elements of the second count”**

Example:

- Assume an auditorium with a seat labeled with a letter and numbers between 1 to 50 (e.g. A23). We want the total number of seats in the auditorium.

- 26 letters and 50 numbers
- How to count?

Product Rule

Example:

• Assume an auditorium with a seat labeled by a letter and numbers in between 1 to 50 (e.g. A23). We want the total number of seats in the auditorium.

• 26 letters and 50 numbers

• How to count?

• **One solution: write down all seats (objects) and count them**

A-1 A-2 A-3 …A-50 B-1… Z-49 Z-50

1 2 3 50 51 … (n-1) n ← eventually we get it

Product Rule

- 26 letters and 50 numbers
- How to count?

- **One solution: write down all seats (objects) and count them**

A-1 A-2 A-3 …A-50 B-1… Z-49 Z-50
1 2 3 50 51 … (n-1) n ← eventually we get it

- **A better solution?**
- For each letter there are 50 numbers
- So the number of seats is 26*50 = 1300
- **Product rule: number of letters * number of integers in [1,50]**

Product Rule

A count can be broken down into a sequence of dependent counts

- **“each element in the first count is associated with all elements of the second count”**

Product rule: If a count of elements can be broken down into a sequence of dependent counts where the first count yields $n1$ elements, the second $n2$ *elements, and kth count* nk *elements, by* the product rule the total number of elements is:

- $n = n1*n2* \ldots * nk$

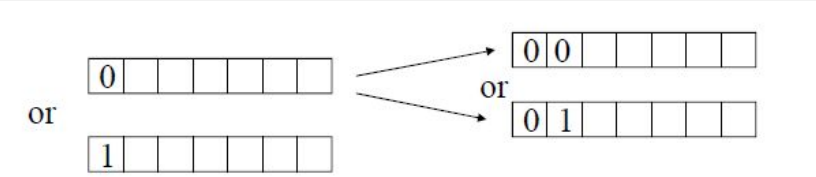
Product Rule

Example:

- How many different bit strings of length 7 are there?

- E.g. 1011010

- Is it possible to decompose the count problem and if yes how?



- **Yes.**

– Count the number of possible assignments to bit 1
– For the specific first-bit count possible assignments to bit 2
– For the specific first two bits count assignments to bit 3

– Number of assignments to the first 3 bits: 2*2*2=8

Product Rule

Example:

• How many different bit strings of length 7 are there?

• E.g. 1011010

• Is it possible to decompose the count problem and if yes how?

• **Yes.**

– Count the number of possible assignments to bit 1
– For the specific first bit count possible assignments to bit 2
– For the specific first two bits count assignments to bit 3
– Gives a sequence of n dependent counts and by the product rule we have: $\mathbf{n = 2 * 2 * 2 * 2 * 2 * 2 * 2 = 2^7}$

Product Rule

Example:

The number of subsets of a set S with k elements.

- How to count them?

- Hint: think in terms of bitstring representation of a set?

- Assume each element in S is assigned a bit position.
- If A is a subset it can be encoded as a bitstring: if an element is in A then use 1 else put 0

- How many different bitstrings are there?

– $n = \underbrace{2 * 2 * \ldots 2}_{\text{k-element}} = 2^k$

Sum Rule

A count decomposes into a set of independent counts

- “elements of counts are alternatives”, they do not depend on each other.

Example:

- You need to travel in between city A and B. You can either fly, take a train, or a bus. There are 3 different flights in between A and B, 2 different trains and 10 buses. How many options do you have to get from A to B?

- We can take only one type of transportation and for each only one option. The number of options:
- n = 3+2+10

Sum rule:

- n = number of flights + number of trains + number of buses

Sum Rule

Example:

Suppose that either a member of the mathematics faculty or a student who is a mathematics major is chosen as a representative of a university committee. How many different choices are there for this representative if there are 37 members of the mathematics faculty and 83 mathematics majors and no one is both a faculty member and a student?

Sum rule:

n = number of mathematics faculty + number of mathematics majors

$n = 37 + 83 = 120$ possible ways.

Sum Rule

A count decomposes into a set of independent counts

- "elements of counts are alternatives"

- **Sum rule**: If a count of elements can be broken down into a set of independent counts where the first count yields *n1 elements,* the second *n2 elements, and kth count nk elements, by the sum* rule the total number of elements is:

- $n = n1+n2+ \ldots + nk$

Inclusion Exclusion Principle

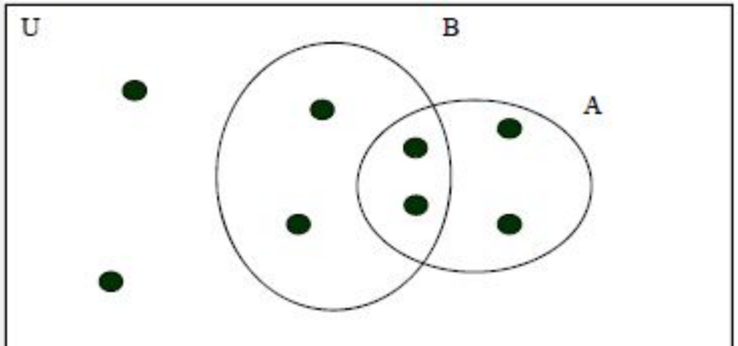
Used in counts where the decomposition yields two dependent count tasks with overlapping elements

- If we used the sum rule some elements would be counted twice

Inclusion-exclusion principle: uses a sum rule and then corrects for the overlapping elements.

We used the principle for the cardinality of the set union.

- $|A \cup B| = |A| + |B| - |A \cap B|$



Inclusion Exclusion Principle

Example: How many bitstrings of length 8 start either with a bit 1 or end with 00?

- It is easy to count **strings that start with 1:**
- How many are there? 2^7
- It is easy to count the **strings that end with 00.**
- How many are there? 2^6
- Is it OK to add the two numbers to get the answer? $2^7 + 2^6$
- **No. Overcount.** There are some strings that can both start with 1 and end with 00. These strings are counted in twice.
- How to deal with it? How to correct for overlap?
- How many of strings were counted twice? 2^5 (1 xxxxx 00)
- Thus we can correct for the overlap simply by using:
- $2^7 + 2^6 – 2^5 = 128 + 64 - 32 =$ **160**

Inclusion Exclusion Principle

Exercise

How many positive integers between 100 and 999 inclusive

a) are divisible by 7? 128

b) are odd? 450

c) have the same three decimal digits?

d) are not divisible by 4? 675

e) are divisible by 3 or 4? $300 + 225 - 75 = 450$

f) are not divisible by either 3 or 4? $900 - 450 = 450$

g) are divisible by 3 but not by 4? $300 - 75 = 225$

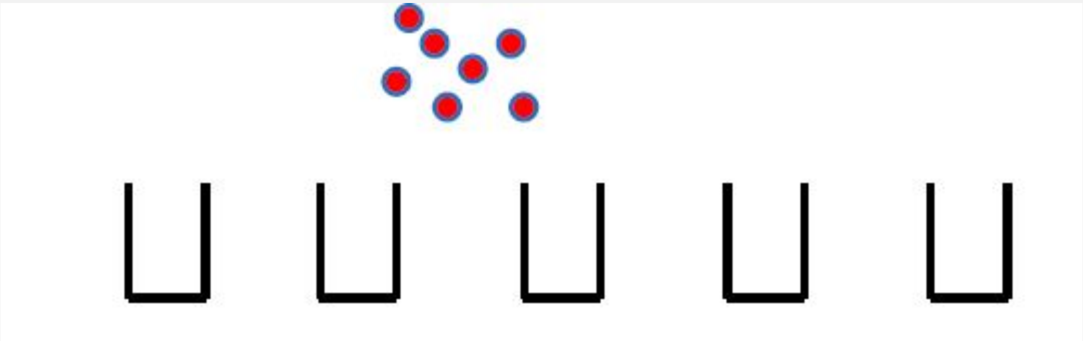
h) are divisible by 3 and 4? 75

Pigeonhole principle

Assume you have a set of objects and a set of bins used to store objects.

- **The pigeonhole principle states that if there are more objects** than bins then there is at least one bin with more than one object.

- **Example: 7 balls and 5 bins to store them**



Pigeonhole principle

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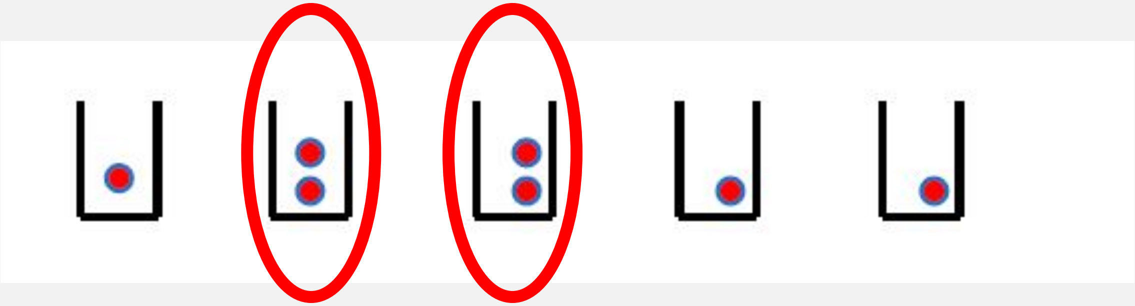
- **The pigeonhole principle states that if there are more objects** than bins then there is at least one bin with more than one object.

- **Example: 7 balls and 5 bins to store them**

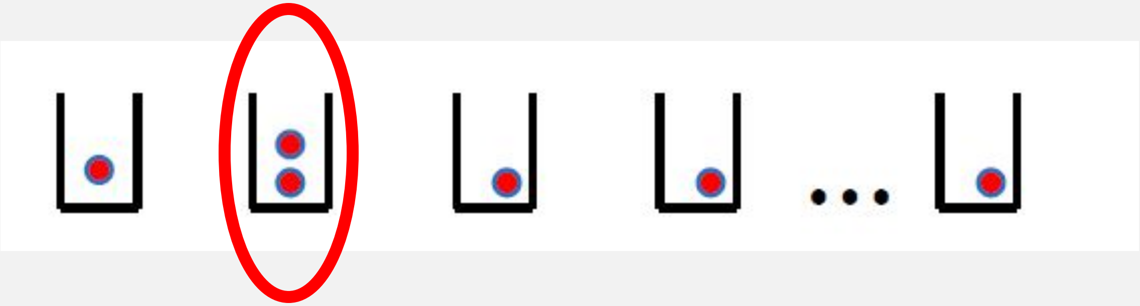
At least one bin with more than 1 ball exists.



Pigeonhole principle

Assume you have a set of objects and a set of bins used to store objects. The pigeonhole principle states that if there are more objects than bins then there is at least one bin with more than one object.

- **Theorem. If there are k+1 objects and k bins. Then there is at least one bin with two or more objects.**



Pigeonhole principle

Assume you have a set of objects and a set of bins used to store objects. The pigeonhole principle states that if there are more objects than bins then there is at least one bin with more than one object.

- **Theorem. If there are k+1 objects and k bins. Then there is at least one bin with two or more objects.**

Proof. (by contradiction)

- Assume that we have k + 1 objects and every bin has at most one element. Then the total number of elements is k which is a contradiction.
- End of proof

Pigeonhole principle

Example:

- Assume 367 people. Are there any two people who have the same birthday?
- How many days are in the year? 365.
- Then there must be at least two people with the same birthday.

Example:

How many students must be in a class to guarantee that at least two students receive the same score on the final exam if the exam is graded on a scale from 0 to 100 points?

Solution

There are 101 possible scores on the final. The pigeonhole principle shows that among any 102 students there must be at least 2 students with the same score.

Generalized Pigeonhole principle

We can often say more about the number of objects.

• Say we have 5 bins and 12 objects. What is it we can say about the bins and number of elements they hold?

• There must be **a bin with at least 3 elements.**
• Why?
• Assume there is no bin with more than 2 elements. Then the max number of elements we can have in 5 bins is 10. We need to place 12 so at least one bin should have at least 3 elements.

Generalized Pigeonhole principle

Theorem. **If *N objects are placed into k bins then there is at least*** one bin containing at least $\lceil N/K \rceil$ *objects.*

Example. Assume 100 people. Can you tell something about the number of people born in the same month.

- Yes. There exists a month in which at least

  = $\lceil 100/12 \rceil = \lceil 8.3 \rceil = 9$ *people were born.*

Generalized Pigeonhole principle

Theorem. If *N objects are placed into k bins then there is at least* one bin containing at least $\lceil N/K \rceil$ *objects.*

Example: What is the minimum number of students required in a discrete mathematics class to be sure that at least six will receive the same grades, (A, B, C, D, and F)?

Solution: 6 = ceiling(N/5)

Formula: N = k (r - 1) + 1
= 5 (6 - 1) + 1
= 26

N = total number of objects.
k = number of bins.
r = number of at least objects.

Permutations

A permutation of a set of distinct objects is an ordered arrangement of the objects.

Since the objects are distinct, they cannot be selected more than once. Furthermore, the order of the arrangement matters.

Example:

- Assume we have a set S with n elements. S={a,b,c}.

- **Permutations of S:**
- **a b c   a c b   b a c   b c a   c a b   c b a**

Number of Permutations

Assume we have a set S with n elements. $S=\{a_1\ a_2\ \ldots\ a_n\}$.

- **Question: How many different permutations are there?**

- In how many different ways we can choose the first element of the permutation? **n (either a_1 or a_2 … or a_n)**
- Assume we picked a_2.
- In how many different ways we can choose the remaining elements? **n-1 (either a_1 or a_3 … or a_n but not a_2)**
- **Assume we picked a_j.**
- In how many different ways we can choose the remaining elements? **n-2 (either a_1 or a_3 … or a_n but not a_2 and not a_j)**

$P(n,n) = n.(n-1)(n-2)\ldots1 = n!$

Number of Permutations

Example 1.

- How many permutations of letters {a,b,c} are there?

- Number of permutations is:

$P(n,n) = P(3,3) = 3! = 6$

- Verify:

abc   acb   bac   bca   cab   cba

Number of Permutations

Example 2

• How many permutations of letters A B C D E F G H contain a substring ABC.

Idea: consider ABC as one element and D,E,F,G,H as other 5 elements for the total of 6 elements.

Then we need to count the number of permutation of these elements.

$6! = 720$

K-Permutations

k-permutation is an ordered arrangement of k elements of a set.

• The number of k-permutations of a set with n distinct elements is:

$P(n,k) = n(n-1)(n-2)\ldots(n-k+1) = n!/(n-k)!$

K-Permutations

***k-permutation* is an *ordered* arrangement of *k elements* of a set.**

- The number of *k-permutations of a set with n distinct elements* is:

$P(n,k) = n(n-1)(n-2)\ldots(n-k+1)= n!/(n-k)!$

Explanation:

- Assume we have a set S with n elements. $S=\{a_1\ a_2\ \ldots\ a_n\}$.
- The 1st element of the *k-permutation may be any of the n* elements in the set.
- The 2nd element of the *k-permutation may be any of the n-1* remaining elements of the set.
- And so on. For last element of the *k-permutation, there are n-k+1 elements remaining to choose from*

K-Permutations

Example:

The 2-permutations of set {*a*,*b*,*c*} *are:*

ab, ac, ba, bc, ca, cb.

The number of 2-permutations of this 3-element set is

$P(n,k) = P(3,2) = 3!/\ (3-2)! = 6.$

K-Permutations

Example:
Suppose that there are eight runners in a race. The winner receives a gold medal, the second-place finisher receives a silver medal, and the third-place finisher receives a bronze medal.
How many different ways are there to award these medals if all possible outcomes of the race can occur and there are no ties?

Solution:
The number of different ways to award the medals is the number of 3-permutations of a set with eight elements.

Hence, there are $P(8, 3) = 8 \cdot 7 \cdot 6 = 336$ possible ways to award the medals.

Combination

A *k-combination* *of elements of a set is an* *unordered* *selection* of *k elements from the set.*

Thus, a k-combination is simply a subset of the set with *k* *elements.*

Example:

- 2-combinations of the set {a,b,c}

a b   a c   b c

a b covers 2-permutations: **a b and b a**

Combination

Theorem: The number of *k-combinations of a set with n distinct* elements, where *n is a positive integer and k is an integer with* $0 \leq k \leq n$

$$C(n,k) = \frac{n!}{k!(n-k)!}$$

Combination

Theorem: The number of *k-combinations of a set with n distinct* elements, where *n is a positive integer and k is an integer with* $0 \leq k \leq n$

$$C(n,k) = \frac{n!}{k!(n-k)!}$$

Proof: The *k-permutations of the set can be obtained by first* forming the *C(n,k) k-combinations of the set, and then ordering* the elements in each *k-combination, which can be done in P(k,k)* ways. Consequently,

*P(n,k) = C(n,k) * P(k,k).*

This implies that

C(n,k) = P(n,k) / P(k,k) = P(n,k) / k! = n! / (k! (n-k)!)

Combination

Intuition (example): Assume elements A1, A2, A3, A4 and A5 in
the set. All 3-combinations of elements are:

- A1 A2 A3
- A1 A2 A4
- A1 A2 A5
- A1 A3 A4
- A1 A3 A5
- A1 A4 A5
- A2 A3 A4
- A2 A3 A5
- A2 A4 A5
- A3 A4 A5

Total of 10.

Each combination cover many 3-permutations

A1 A2 A3
A1 A3 A2
A2 A1 A3
A2 A3 A1
A3 A1 A2
A3 A2 A1

Combination

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So: $P(5,3) = C(5,3)\ P(3,3)$

Combination

Intution (example): Assume elements A1, A2, A3, A4 and A5 in
the set. All 3-combinations of elements are:

- A1 A2 A3
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Total of 10.

Each combination cover many 3-permutations

A1 A2 A3
A1 A3 A2
A2 A1 A3
A2 A3 A1
A3 A1 A2
A3 A2 A1

So: P(5,3) = C(5,3) * P(3,3)
and: C(5,3) = P(5,3)/P(3,3)

Combination

Example:

• We need to create a team of 5 players for the competition out of 10 team members. How many different teams is it possible to create?

Answer:

• When creating a team we do not care about the order in which players were picked. It is important that the player is in.

Because of that, we need to consider unordered sets of combinations.

• C(10,5) = 10!/(10-5)!5! = (10.9.8.7.6) / (5 4 3 2 1)
= 2.3.2.7.3
= 6.14.3 = 6.42 = **252**

Combination

Example:

- Suppose that there are 9 faculty members in the mathematics department and 11 in the computer science department. How many ways are there to select a committee to develop a discrete mathematics course at a school if the committee is to consist of three faculty members from the mathematics department and four from the computer science department?

Answer:

- By the product rule, the answer is the product of the number of 3-combinations of a set with nine elements and the number of 4-combinations of a set with 11 elements.

$$C(9, 3) \cdot C(11, 4) = 9!/(3!*6!) \cdot 11!/(4!*7!)$$
$$= 84 \cdot 330$$
$$= 27{,}720.$$

Combination

Corrolary:

- **$C(n,k) = C(n,n-k)$**

Proof.

- $C(n,k) = n! / (n-k)!\, k!$

  $= n!/ (n-k)!\, (n - (n-k))!$

  $= C(n,n-k)$

Combination

Example:

- How many poker hands of five cards can be dealt from a standard deck of 52 cards? Also, how many ways are there to select 47 cards from a standard deck of 52 cards?

Answer:

- Because the order in which the five cards are dealt from a deck of 52 cards does not matter.

  $C(52, 5) = 52!/(5!*47!) = (52\cdot 51\cdot 50\cdot 49\cdot 48) / (5\cdot 4\cdot 3\cdot 2\cdot 1)$

  $= 2{,}598{,}960$

Different ways to select 47 cards from a standard deck of 52 cards is $C(52, 47)$

We Know $C(52, 47) = C(52, 5) = 2{,}598{,}960$

Thank You

Extra Topic for Students' reading

Beyond basic counting rules

More complex counting problems typically require a combination of the sum and product rules.

Example: A login password:

- The minimum password length is 6 and the maximum is 8. The password can consist of either an uppercase letter or a digit. There must be at least one digit in the password.

- How many different passwords are there?

Beyond basic counting rules

Example: A login password:

- The minimum password length is 6 and the maximum is 8. The password can consist of either an uppercase letter or a digit. There must be at least one digit in the password.

- How many different passwords are there?

Step 1:

- The password we select has either 6,7 or 8 characters. So the total number of valid passwords is by the sum rule:

- P = P6+P7+P8

The number of passwords of length 6,7 and 8 respectively

Beyond basic counting rules

Step 1:

- The password we select has either 6,7 or 8 characters. So the total number of valid passwords is by the sum rule:

- P = P6+P7+P8

The number of passwords of length 6,7 and 8 respectively

Step 2:

- Assume passwords with 6 characters (upper-case letters):
- How many are there?
- If we let each character to be at any position we have:

– P6-nodigits = 26^6 different passwords of length 6

Beyond basic counting rules

Step 1:

- The password we select has either 6,7 or 8 characters. So the total number of valid passwords is by the sum rule:

- P = P6+P7+P8

The number of passwords of length 6,7 and 8 respectively

Step 2:

- Assume passwords with 6 characters
  (either digits + upper case letters):
- How many are there?
- If we let each character to be at any position we have:

– P6-all = $(26+10)^6 = (36)^6$ different passwords of length 6

Beyond basic counting rules

Step 2

But we must have a password with at least one digit. How to account for it?

A trick. Split the count of all passwords of length 6 into to two mutually exclusive groups:

- **P6-all = P6-digits + P6-nodigits**

1. P6-digits – count when the password has one or more digits
2. P6-nodigits – count when the password has no digits

- We know how to easily compute P6-all and P6-nodigits

– **P6-all = 36^6 and P6-nodigits = 26^6**

– Then **P6-digits = P6-all – P6-nodigits**

Beyond basic counting rules

Step 1:

the total number of valid passwords is by the sum rule:

- **P = P6+P7+P8**
- The number of passwords of length 6,7 and 8 respectively

Step 2

The number of valid passwords of length 6:

P6= P6-digits = P6-all – P6-nodigits

$= 36^6 - 26^6$

Analogically:

P7= P7-digits = P7-all – P7-nodigits

$= 36^7 - 26^7$

P8= P8-digits = P8-all – P8-nodigits

$= 36^8 - 26^8$