Relation

Lecture 17-18

Cartesian product (review)

Let
$A=\{a_1, a_2, ..a_k\}$ and
$B=\{b_1,b_2,..b_m\}$.

The Cartesian product A x B is defined by a set of ordered pairs
$\{(a_1 , b_1), (a_1, b_2), \ldots (a_1, b_m), \ldots, (a_k,b_m)\}$.

Cartesian product defines a product set, or a set of all ordered arrangements of elements in sets in the Cartesian product.

Binary relation

Definition: Let A and B be two sets. A binary relation from A to B is a subset of a Cartesian product A x B.

- Let $R \subseteq A \times B$ means R is a set of ordered pairs of the form (a,b) where a ϵ A and b ϵ B.

- We use the notation **a R b to denote (a,b) ϵ R and a R b to denote (a,b) ϵ R. If a R b, we say a is related to b by R.**

Example: Let A={a,b,c} and B={1,2,3}.

- Is R={(a,1),(b,2),(c,2)} a relation from A to B?

Binary relation

Definition: Let A and B be two sets. A binary relation from A to B is a subset of a Cartesian product A x B.

- Let $R \subseteq A \times B$ means R is a set of ordered pairs of the form (a,b) where a ϵ A and b ϵ B.

- We use the notation **a R b to denote (a,b) ϵ R and a R b to denote (a,b) ϵ R. If a R b, we say a is related to b by R.**

Example: Let A={a,b,c} and B={1,2,3}.

- Is R={(a,1),(b,2),(c,2)} a relation from A to B? **Yes.**
- Is Q={(1,a),(2,b)} a relation from A to B?

Binary relation

Definition: Let A and B be two sets. A binary relation from A to B is a subset of a Cartesian product A x B.

• Let $R \subseteq A \times B$ means R is a set of ordered pairs of the form (a,b) where a ϵ A and b ϵ B.

• We use the notation **a R b to denote (a,b) ϵ R and a R b to denote (a,b) ϵ R. If a R b, we say a is related to b by R.**

Example: Let A={a,b,c} and B={1,2,3}.

• Is R={(a,1),(b,2),(c,2)} a relation from A to B? **Yes.**

• Is Q={(1,a),(2,b)} a relation from A to B? **No.**

• Is P={(a,a),(b,c),(b,a)} a relation from A to A?

Binary relation

Definition: Let A and B be two sets. A binary relation from A to B is a subset of a Cartesian product A x B.

• Let $R \subseteq A \times B$ means R is a set of ordered pairs of the form (a,b) where a ϵ A and b ϵ B.

• We use the notation **a R b to denote (a,b) ϵ R and a R b to denote (a,b) ϵ R. If a R b, we say a is related to b by R.**

Example: Let A={a,b,c} and B={1,2,3}.

• Is R={(a,1),(b,2),(c,2)} a relation from A to B? **Yes.**

• Is Q={(1,a),(2,b)} a relation from A to B? **No.**

• Is P={(a,a),(b,c),(b,a)} a relation from A to A? **Yes**

Representing Binary relation

We can graphically represent a binary relation R as follows:

- if **a R b then draw an arrow from a to b.**

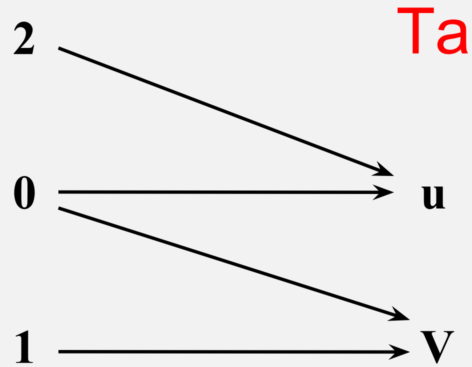
a → b

Example:

- Let A = {0, 1, 2}, B = {u,v} and

  R = { (0,u), (0,v), (1,v), (2,u) }

- Note: $R \subseteq A \times B$
- Graph:



Table

	u	v
0	x	x
1		x
2	x	

Reflexive Relation

Definition (reflexive relation) : A relation R on a set A is called reflexive if (a,a) ϵ R for every element a ϵ A.

Example 1:

- Assume relation R_{div} ={(a b), if a |b} on A = {1,2,3,4}

R_{div} = {(1,1), (1,2), (1,3), (1,4), (2,2), (2,4), (3,3), (4,4)}

- **Is R_{div} reflexive?**

Reflexive Relation

Definition (reflexive relation) : A relation R on a set A is called reflexive if (a,a) ϵ R for every element a ϵ A.

Example 1:

- Assume relation R_{div} ={(a b), if a |b} on A = {1,2,3,4}

R_{div} = {(1,1), (1,2), (1,3), (1,4), (2,2), (2,4), (3,3), (4,4)}

- **Is R_{div} reflexive?**
- **Answer: Yes. (1,1), (2,2), (3,3), and (4,4) ϵ A.**

Reflexive Relation

Definition (reflexive relation) : A relation R on a set A is called reflexive if (a,a) ϵ R for every element a ϵ A.

Example 2:

- Relation R_{fun} on A = {1,2,3,4} defined as:
- R_{fun} = {(1,2),(2,2),(3,3)}.

- **Is R_{fun} reflexive?**

Reflexive Relation

Definition (reflexive relation) : A relation R on a set A is called reflexive if (a,a) ϵ R for every element a ϵ A.

Example 2:

- Relation R_{fun} on A = {1,2,3,4} defined as:
- $R_{fun} = \{(1,2),(2,2),(3,3)\}$.

- **Is R_{fun} reflexive?**

- **No. It is not reflexive since (1,1) ,(4,4) $\notin R_{fun}$.**

Symmetric Relation

Definition (symmetric relation): A relation R on a set A is called symmetric if

$$\forall a, b \in A (a,b) \in R \rightarrow (b,a) \in R$$

Example 1:

- R_{div} ={(a b), if a |b} on A = {1,2,3,4}
- R_{div} = {(1,1), (1,2), (1,3), (1,4), (2,2), (2,4), (3,3), (4,4)}

- **Is R_{div} symmetric?**

Symmetric Relation

Definition (symmetric relation): A relation R on a set A is called symmetric if

. $$(a,b) \in R \rightarrow (b,a) \in R$$

Example 1:

- R_{div} ={(a b), if a |b} on A = {1,2,3,4}
- R_{div} = {(1,1), (1,2), (1,3), (1,4), (2,2), (2,4), (3,3), (4,4)}

- **Is R_{div} symmetric?**

- **Answer: No. It is not symmetric since (1,2) ϵ R but (2,1) $\notin$R.**

Symmetric Relation

Definition (symmetric relation): A relation R on a set A is called symmetric if

$$(a,b) \in R \rightarrow (b,a) \in R$$

Example 2:

- $R_{\neq}$ on A={1,2,3,4}, such that a $R_{\neq}$ b if and only if a ≠ b.

$R_{\neq}$={(1,2),(1,3),(1,4),(2,1),(2,3),(2,4),(3,1),(3,2),(3,4),(4,1),(4,2),(4,3)}

- Is $R_{\neq}$ symmetric ?

Symmetric Relation

Definition (symmetric relation): A relation R on a set A is called symmetric if

$$\forall a,b \in A (a,b) \in R \rightarrow (b,a) \in R$$

Example 2:

- $R_{\neq}$ on A={1,2,3,4}, such that a $R_{\neq}$ b if and only if a ≠ b.

$R_{\neq}$={(1,2),(1,3),(1,4),(2,1),(2,3),(2,4),(3,1),(3,2),(3,4),(4,1),(4,2),(4,3)}

- Is $R_{\neq}$ symmetric ?

Answer: Yes.

Transitive Relation

Definition (transitive relation): A relation R on a set A is called transitive if

$$[(a,b) \in R \; and \; (b,c) \in R] \rightarrow (a,c) \in R \; \forall a,b,c \in A$$

- **Example 1:**
- R_{div} ={(a b), if a |b} on A = {1,2,3,4}
- R_{div} = {(1,1), (1,2), (1,3), (1,4), (2,2), (2,4), (3,3), (4,4)}
- **Is R_{div} transitive?**

Transitive Relation

Definition (transitive relation): A relation R on a set A is called transitive if

$$[(a,b) \in R \ and \ (b,c) \in R] \rightarrow (a,c) \in R \ \forall a,b,c \in A$$

- **Example 1:**

- R_{div} ={(a b), if a |b} on A = {1,2,3,4}

- R_{div} = {(1,1), (1,2), (1,3), (1,4), (2,2), (2,4), (3,3), (4,4)}

- **Is R_{div} transitive?**

- **Answer: Yes.**

Transitive Relation

Definition (transitive relation): A relation R on a set A is called transitive if

$$[(a,b)\in R \text{ and } (b,c)\in R]\rightarrow (a,c)\in R\ \forall a,b,c\in A$$

- Example 2:

- $R_{\neq}$ on A={1,2,3,4}, such that a R≠ b if and only if a ≠ b.

- $R_{\neq}$={(1,2),(1,3),(1,4),(2,1),(2,3),(2,4),(3,1),(3,2),(3,4),(4,1),(4,2),(4,3)}

- Is $R_{\neq}$ transitive ?

Transitive Relation

Definition (transitive relation): A relation R on a set A is calledtransitive if

$$[(a,b) \in R \text{ and } (b,c) \in R] \rightarrow (a,c) \in R \ \forall a,b,c \in A$$

- Example 2:

- $R_{\neq}$ on A={1,2,3,4}, such that a R≠ b if and only if a ≠ b.

•$R_{\neq}$={(1,2),(1,3),(1,4),(2,1),(2,3),(2,4),(3,1),(3,2),(3,4),(4,1),(4,2),(4,3)}

- Is $R_{\neq}$ transitive ? YES

Combining Relation

Definition: Let A and B be sets. A binary relation from A to B is a subset of a Cartesian product A x B.

• Let $R \subseteq A \times B$ means R is a set of ordered pairs of the form (a,b) where a ϵ A and b ϵ B.

Combining Relations

• Relations are sets $\rightarrow$ combinations via set operations

• Set operations of: union, intersection, difference and symmetric difference.

Combining Relation

Example:

- Let A = {1,2,3} and B = {u,v} and
- R1 = {(1,u), (2,u), (2,v), (3,u)}
- R2 = {(1,v),(3,u),(3,v)}

What is:

- R1 U R2 = ?

Combining Relation

Example:

- Let A = {1,2,3} and B = {u,v} and
- R1 = {(1,u), (2,u), (2,v), (3,u)}
- R2 = {(1,v),(3,u),(3,v)}

What is:

- R1 U R2 = {(1,u),(1,v),(2,u),(2,v),(3,u),(3,v)}
- R1 ∩ R2 = ?

Combining Relation

Example:

- Let A = {1,2,3} and B = {u,v} and
- R1 = {(1,u), (2,u), (2,v), (3,u)}
- R2 = {(1,v),(3,u),(3,v)}

What is:

- R1 $\cup$ R2 = {(1,u),(1,v),(2,u),(2,v),(3,u),(3,v)}
- R1 $\cap$ R2 = {(3,u)}
- R1 - R2 = ?

Combining Relation

Example:

- Let A = {1,2,3} and B = {u,v} and
- R1 = {(1,u), (2,u), (2,v), (3,u)}
- R2 = {(1,v),(3,u),(3,v)}

What is:

- R1 ∪ R2 = {(1,u),(1,v),(2,u),(2,v),(3,u),(3,v)}
- R1 ∩ R2 = {(3,u)}
- R1 - R2 = (1,u),(2,u),(2,v)}
- R2 - R1 = ?

Combining Relation

Example:

- Let A = {1,2,3} and B = {u,v} and
- R1 = {(1,u), (2,u), (2,v), (3,u)}
- R2 = {(1,v),(3,u),(3,v)}

What is:

- R1 ∪ R2 = {(1,u),(1,v),(2,u),(2,v),(3,u),(3,v)}
- R1 ∩ R2 = {(3,u)}
- R1 - R2 = (1,u),(2,u),(2,v)}
- R2 - R1 = {(1,v),(3,v)}

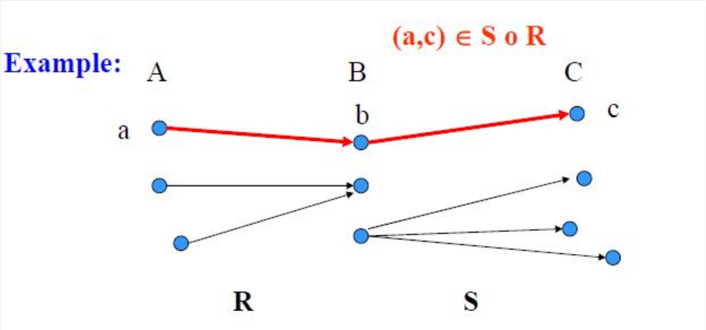
Composite Relation

Definition: Let R be a relation from a set A to a set B and S a relation from B to a set C. The composite of R and S is the relation consisting of the ordered pairs (a,c) where a ϵ A and c ϵ C, and for which there is a b ϵ B such that (a,b) ϵ R and (b,c) ϵ S. We denote the composite of R and S by R o S.

Examples:

- Let A = {1,2,3}, B = {0,1,2} and C = {a,b}.
- R = {(1,0), (1,2), (3,1),(3,2)}
- S = {(0,b),(1,a),(2,b)}

- RoS = ?



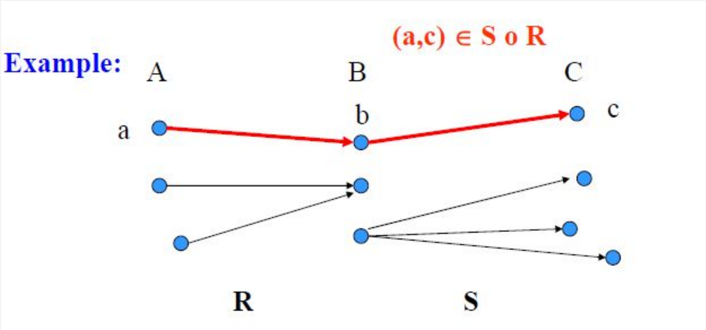
Composite Relation

Definition: Let R be a relation from a set A to a set B and S a relation from B to a set C. The composite of R and S is the relation consisting of the ordered pairs (a,c) where a ϵ A and c ϵ C, and for which there is a b ϵ B such that (a,b) ϵ R and (b,c) ϵ S. We denote the composite of R and S by R o S.

Examples:

- Let A = {1,2,3}, B = {0,1,2} and C = {a,b}.
- R = {(1,0), (1,2), (3,1),(3,2)}
- S = {(0,b),(1,a),(2,b)}

- RoS = {(1,b),(3,a),(3,b)}



Composite Relation

Definition: Let R be a relation on a set A. The powers R^n, n = 1,2,3,... is defined inductively by

- $R^1 = R$ and $R^{n+1} = R^n \circ R$.

Examples

- R = {(1,2),(2,3),(2,4), (3,3)} is a relation on A = {1,2,3,4}.
- $R^1 = ?$

Composite Relation

Definition: Let R be a relation on a set A. The powers R^n, n = 1,2,3,... is defined inductively by

- $R^1 = R$ and $R^{n+1} = R^n \circ R$.

Examples

- R = {(1,2),(2,3),(2,4), (3,3)} is a relation on A = {1,2,3,4}.
- R^1 = {(1,2),(2,3),(2,4), (3,3)}
- R^2 = ?

Composite Relation

Definition: Let R be a relation on a set A. The powers R^n, n = 1,2,3,... is defined inductively by

- $R^1 = R$ and $R^{n+1} = R^n \circ R$.

Examples

- R = {(1,2),(2,3),(2,4), (3,3)} is a relation on A = {1,2,3,4}.
- R^1 = {(1,2),(2,3),(2,4), (3,3)}
- R^2 = {(1,3), (1,4), (2,3), (3,3)}
- R^3 = ?

Composite Relation

Definition: Let R be a relation on a set A. The powers R^n, n = 1,2,3,... is defined inductively by

- $R^1 = R$ and $R^{n+1} = R^n \circ R$.

Examples

- R = {(1,2),(2,3),(2,4), (3,3)} is a relation on A = {1,2,3,4}.
- R^1 = {(1,2),(2,3),(2,4), (3,3)}
- R^2 = {(1,3), (1,4), (2,3), (3,3)}
- R^3 = {(1,3), (2,3), (3,3)}
- R^4 =

Composite Relation

Definition: Let R be a relation on a set A. The powers R^n, n = 1,2,3,... is defined inductively by

- $R^1 = R$ and $R^{n+1} = R^n \circ R$.

Examples

- R = {(1,2),(2,3),(2,4), (3,3)} is a relation on A = {1,2,3,4}.
- R^1 = {(1,2),(2,3),(2,4), (3,3)}
- R^2 = {(1,3), (1,4), (2,3), (3,3)}
- R^3 = {(1,3), (2,3), (3,3)}
- R^4 = {(1,3), (2,3), (3,3)}
- R^k = ? when k>3

Composite Relation

Definition: Let R be a relation on a set A. The powers R^n, n = 1,2,3,... is defined inductively by

- $R^1 = R$ and $R^{n+1} = R^n \circ R$.

Examples

- R = {(1,2),(2,3),(2,4), (3,3)} is a relation on A = {1,2,3,4}.
- R^1 = {(1,2),(2,3),(2,4), (3,3)}
- R^2 = {(1,3), (1,4), (2,3), (3,3)}
- R^3 = {(1,3), (2,3), (3,3)}
- R^4 = {(1,3), (2,3), (3,3)}
- R^k = {(1,3), (2,3), (3,3)}

Transitive Relation

Definition (transitive relation): A relation R on a set A is called transitive if

- $[(a,b) \in R \text{ and } (b,c) \in R] \rightarrow (a,c) \in R$ for all $a, b, c \in A$.

- **Example 1:**

- R_{div} ={(a b), if a |b} on A = {1,2,3,4}

- R_{div} = {(1,1), (1,2), (1,3), (1,4), (2,2), (2, 3), (2,4), (3,3), (4,4)}

- **Is R_{div} transitive?**

Transitive Relation

Definition (transitive relation): A relation R on a set A is called transitive if

- $[(a,b) \in R \text{ and } (b,c) \in R] \rightarrow (a,c) \in R$ for all a, b, c $\in$ A.

- **Example 1:**

- $R_{div} = \{(a\ b), \text{ if } a \mid b\}$ on $A = \{1,2,3,4\}$

- $R_{div} = \{(1,1), (1,2), (1,3), (1,4), (2,2), (2, 3), (2,4), (3,3), (4,4)\}$

- **Is R_{div} transitive?**

- **Answer: Yes.**

Closure of a Relation

Let R={(1,1),(1,2),(2,1),(3,2)} on A ={1 2 3}.

- Is this relation reflexive?
- Answer: ?

Closure of a Relation

Let R={(1,1),(1,2),(2,1),(3,2)} on A ={1 2 3}.

- Is this relation reflexive?

- Answer: No (2,2) and (3,3) is not in R.

The question is what is the minimal relation $S \supseteq R$ that is reflexive?

- How to make R reflexive with minimum number of additions?

- Answer: ?

Closure of a Relation

Let R={(1,1),(1,2),(2,1),(3,2)} on A ={1 2 3}.

- Is this relation reflexive?
- Answer: No (2,2) and (3,3) is not in R.

The question is what is the minimal relation $S \supseteq R$ that is reflexive?

- How to make R reflexive with minimum number of additions?

- Answer: ? Add (2,2) and (3,3)
- Then S= {(1,1),(1,2),(2,1),(3,2),(2,2),(3,3)}
- $R \subseteq S$
- The minimal set $S \supseteq R$ is called **the reflexive closure of R**

Closure of a Relation

Relations can have different properties:

- reflexive,
- symmetric
- transitive

- Because of that we define:
- reflexive,
- symmetric and
- transitive

closures.

Closure of a Relation

Definition: Let R be a relation on a set A. A relation S on A with property P is called the **closure of R with respect to P** if S is a subset of every relation Q ($S \subseteq Q$) with property P that contains R ($R \subseteq Q$).

Closure of a Relation

Definition: Let R be a relation on a set A. A relation S on A with property P is called the closure of R with respect to P if S is a subset of every relation Q ($S \subseteq Q$) with property P that contains R ($R \subseteq Q$).

Example (symmetric closure):

- Assume R={(1,2),(1,3), (2,2)} on A={1,2,3}.
- What is the symmetric closure S of R?
- S=?

Closure of a Relation

Definition: Let R be a relation on a set A. A relation S on A with property P is called the closure of R with respect to P if S is a subset of every relation Q ($S \subseteq Q$) with property P that contains R ($R \subseteq Q$).

Example (symmetric closure):

- Assume R={(1,2),(1,3), (2,2)} on A={1,2,3}.

- What is the symmetric closure S of R?

- S = {(1,2),(1,3), (2,2)} U {(2,1), (3,1)}
  = {(1,2),(1,3), (2,2),(2,1), (3,1)}

Closure of a Relation

Definition: Let R be a relation on a set A. A relation S on A with property P is called the closure of R with respect to P if S is a subset of every relation Q ($S \subseteq Q$) with property P that contains R ($R \subseteq Q$).

Example (transitive closure):

- Assume R={(1,2), (2,2), (2,3)} on A={1,2,3}.

- **Is R transitive?**

Closure of a Relation

Definition: Let R be a relation on a set A. A relation S on A with property P is called the closure of R with respect to P if S is a subset of every relation Q ($S \subseteq Q$) with property P that contains R ($R \subseteq Q$).

Example (transitive closure):

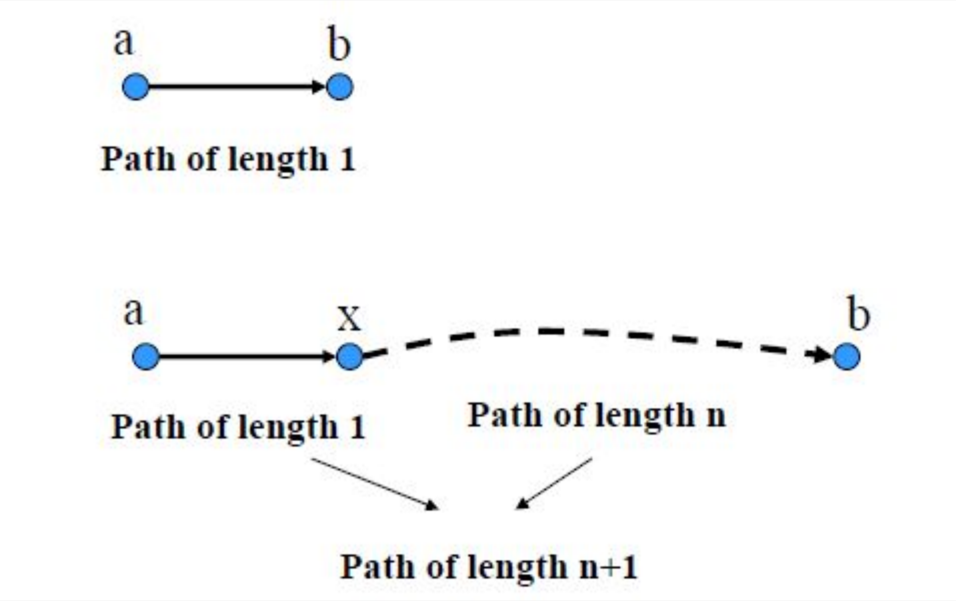
- Assume R={(1,2), (2,2), (2,3)} on A={1,2,3}.

- Is R transitive? No.
- How to make it transitive?
- S = {(1,2), (2,2), (2,3)} U {(1,3)}
  = {(1,2), (2,2), (2,3),(1,3)}
- S is the transitive closure of R

Path length

Theorem: Let R be a relation on a set A. There is a path of length n from a to b if and only if $(a,b) \in R^n$.

Proof (math induction):



Path length

Theorem: Let R be a relation on a set A. There is a path of length n from a to b if and only if $(a,b) \in R^n$.

Proof (math induction):

- P(1): There is a path of length 1 from a to b if and only if $(a,b) \in R^1$, by the definition of R.

Path length

Theorem: Let R be a relation on a set A. There is a path of length n from a to b if and only if $(a,b) \in R^n$.

Proof (math induction):

- P(1): There is a path of length 1 from a to b if and only if $(a,b) \in R^1$, by the definition of R.

Show: $P(n) \rightarrow P(n+1)$: Assume there is a path of length n from a to b if and only if $(a,b) \in R^n$ $\rightarrow$ there is a path of length n+1 from a to b if and only if $(a,b) \in R^{n+1}$.

Path length

Theorem: Let R be a relation on a set A. There is a path of length n from a to b if and only if $(a,b) \in R^n$.

Proof (math induction):

- P(1): There is a path of length 1 from a to b if and only if $(a,b) \in R^1$, by the definition of R.

Show: $P(n) \rightarrow P(n+1)$: Assume there is a path of length n from a to b if and only if $(a,b) \in R^n \rightarrow$ there is a path of length n+1 from a to b if and only if $(a,b) \in R^{n+1}$.

There is a path of length n+1 from a to b if and only if there exists an $x \in A$, such that $(a,x) \in R$ (a path of length 1) and $(x,b) \in R^n$ is a path of length n from x to b.

Path length

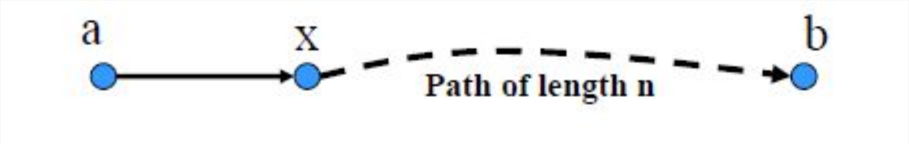
Theorem: Let R be a relation on a set A. There is a path of length n from a to b if and only if $(a,b) \in R^n$.

Proof (math induction):

- P(1): There is a path of length 1 from a to b if and only if $(a,b) \in R^1$, by the definition of R.

Show: $P(n) \rightarrow P(n+1)$: Assume there is a path of length n from a to b if and only if $(a,b) \in R^n \rightarrow$ there is a path of length n+1 from a to b if and only if $(a,b) \in R^{n+1}$.

There is a path of length n+1 from a to b if and only if there exists an $x \in A$, such that $(a,x) \in R$ (a path of length 1) and $(x,b) \in R^n$ is a path of length n from x to b.

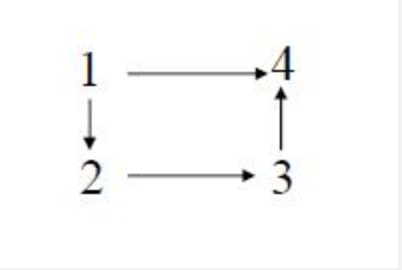


$(x,b) \in R^n$ holds due to P(n). Therefore, there is a path of length n + 1 from a to b. This also implies that $(a,b) \in R^{n+1}$.

Connectivity Relation

Definition: Let R be a relation on a set A. The connectivity relation R* consists of all pairs (a,b) such that there is a path (of any length, ie. 1 or 2 or 3 or ...) between a and b in R.

$$R^{*} = \bigcup_{k=1}^{\infty} R^{k}$$



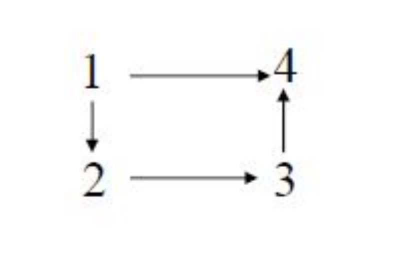
Example:

- A = {1,2,3,4}
- R = {(1,2),(1,4),(2,3),(3,4)}

Connectivity Relation

Definition: Let R be a relation on a set A. The connectivity relation R* consists of all pairs (a,b) such that there is a path (of any length, ie. 1 or 2 or 3 or ...) between a and b in R.

$$R^{*} = \bigcup_{k=1}^{\infty} R^{k}$$



Example:

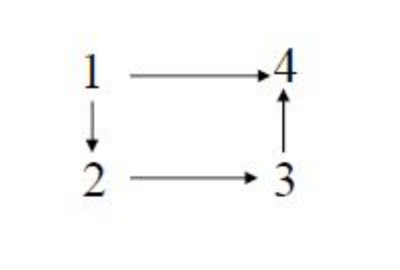
- A = {1,2,3,4}
- R = {(1,2),(1,4),(2,3),(3,4)}

- R^2 =?

Connectivity Relation

Definition: Let R be a relation on a set A. The connectivity relation R* consists of all pairs (a,b) such that there is a path (of any length, ie. 1 or 2 or 3 or ...) between a and b in R.

$$R^{*} = \bigcup_{k=1}^{\infty} R^{k}$$



Example:

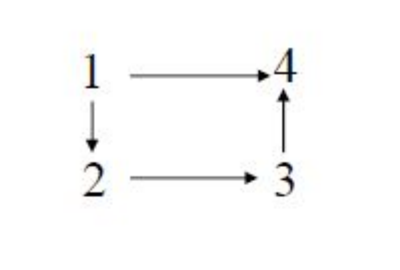
- A = {1,2,3,4}
- R = {(1,2),(1,4),(2,3),(3,4)}

- R^2 = {(1,3),(2,4)}
- R^3 = ?

Connectivity Relation

Definition: Let R be a relation on a set A. The connectivity relation R* consists of all pairs (a,b) such that there is a path (of any length, ie. 1 or 2 or 3 or ...) between a and b in R.

$$R^{*} = \bigcup_{k=1}^{\infty} R^{k}$$



Example:

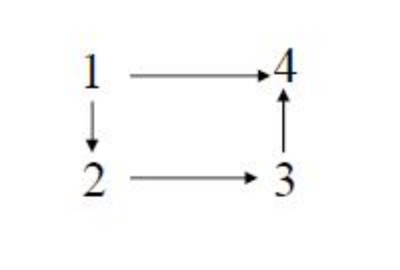
- A = {1,2,3,4}
- R = {(1,2),(1,4),(2,3),(3,4)}

- R^2 = {(1,3),(2,4)}
- R^3 = {(1,4)}
- R^4 = ?

Connectivity Relation

Definition: Let R be a relation on a set A. The connectivity relation R* consists of all pairs (a,b) such that there is a path (of any length, ie. 1 or 2 or 3 or ...) between a and b in R.

$$R^{*} = \bigcup_{k=1}^{\infty} R^{k}$$



Example:

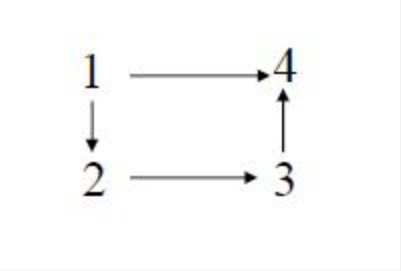
- A = {1,2,3,4}
- R = {(1,2),(1,4),(2,3),(3,4)}

- R^2 = {(1,3),(2,4)}
- R^3 = {(1,4)}
- R^4 = Ø
- R^* = ?

Connectivity Relation

Definition: Let R be a relation on a set A. The connectivity relation R* consists of all pairs (a,b) such that there is a path (of any length, ie. 1 or 2 or 3 or ...) between a and b in R.

$$R^{*} = \bigcup_{k=1}^{\infty} R^{k}$$



Example:

- A = {1,2,3,4}
- R = {(1,2),(1,4),(2,3),(3,4)}

- R^2 = {(1,3),(2,4)}
- R^3 = {(1,4)}
- R^4 = Ø
- R^* = {(1,2),(1,3),(1,4),(2,3),(2,4),(3,4)}

Thank You