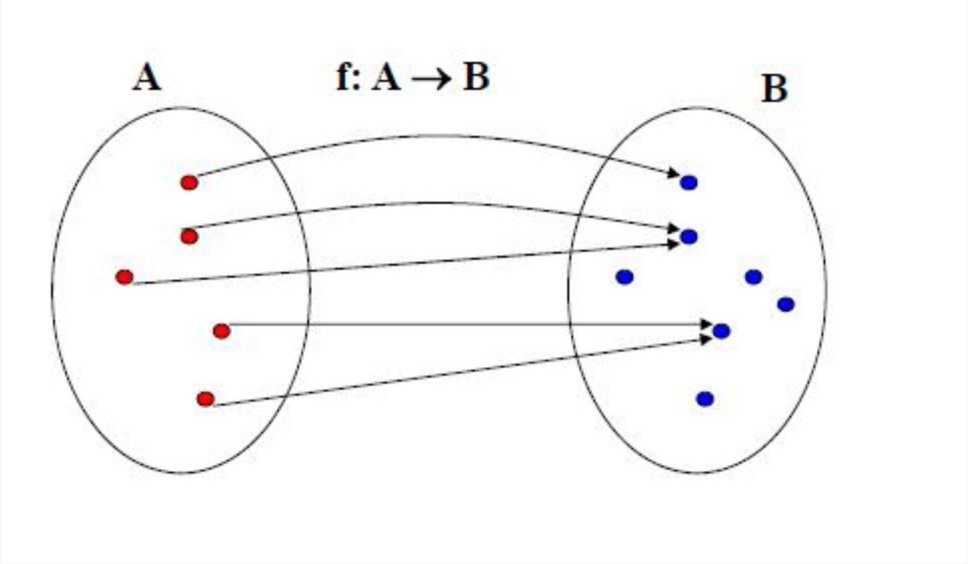
Basic Discrete Structure : Function

Lecture 7-8

Function

Definition: Let A and B be two sets. A function from A to B, denoted $f : A \to B$, is an assignment of exactly one element of B to each element of A. We write $f(a) = b$ to denote the assignment of b to an element a of A by the function f.



Function

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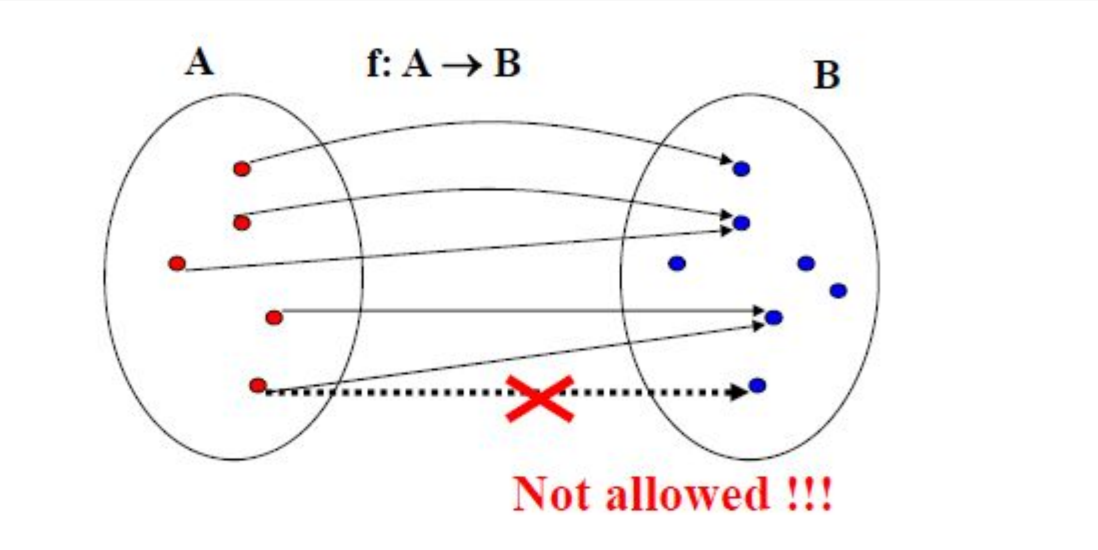

Representing Function

Representations of functions:

1. Explicitly state the assignments in between elements of the two sets
2. Compactly by a formula. (using ‘standard’ functions)

Example1:

- Let A = {1,2,3} and B = {a,b,c}
- Assume f is defined as:
- $1 \rightarrow c$
- $2 \rightarrow a$
- $3 \rightarrow c$
- Is f a function ?
- Yes. since f(1)=c, f(2)=a, f(3)=c. each element of A is assigned an element from B

Representing Function

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Example 2:

- Let A = {1,2,3} and B = {a,b,c}
- Assume g is defined as:
- 1 → c
- 1 → b
- 2 → a
- 3 → c
- Is g a function ?

Representing Function

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Example 2:

- Let A = {1,2,3} and B = {a,b,c}
- Assume g is defined as:
- 1 → c
- 1 → b
- 2 → a
- 3 → c
- Is g a function ?
- No. since g(1) is assigned both c and b.

Representing Function

Representations of functions:

1. Explicitly state the assignments in between elements of the two sets
2. Compactly by a formula. (using ‘standard’ functions)

Example 3:

- $A = \{0,1,2,3,4,5,6,7,8,9\}$, $B = \{0,1,2\}$
- Define $h: A \to B$ as:

$$h(x) = x \bmod 3.$$

- (the result is the remainder after the division by 3)
- Assignments:
- $0 \to 0 \quad 3 \to 0$
- $1 \to 1 \quad 4 \to 1$
- $2 \to 2 \quad \ldots$

Notation of Set

Definitions: Let f be a function from A to B.

- We say that A is the domain of f and B is the codomain of f.

- If $f(a) = b$, b is the image of a and a is a pre-image of b.

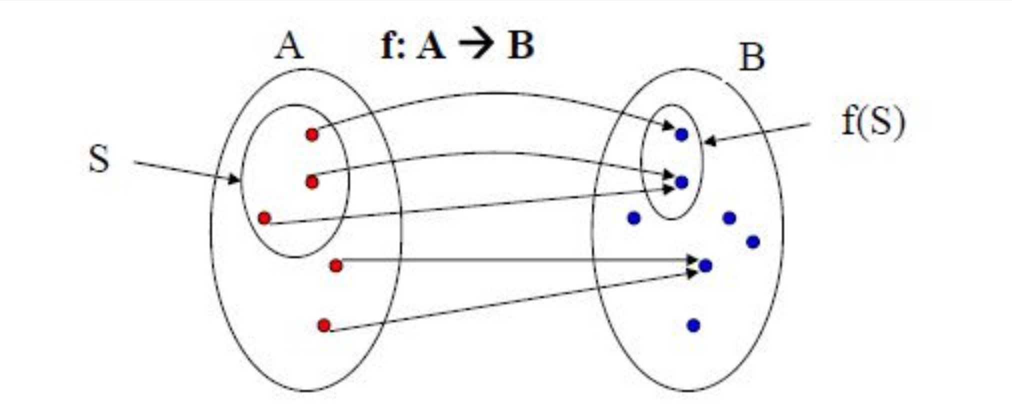
- The range of f is the set of all images of elements of A. Also, if f is a function from A to B, we say f maps A to B.

Example: Let A = {1,2,3} and B = {a,b,c}

- Assume f is defined as: $1 \rightarrow c$, $2 \rightarrow a$, $3 \rightarrow c$
- What is the image of 1?
- $1 \rightarrow c$ c is the image of 1
- What is the pre-image of a?
- $2 \rightarrow a$ 2 is a pre-image of a.
- Domain of f ? {1,2,3}
- Codomain of f ? {a,b,c}
- Range of f ? {a,c}

Image of subset

Definition: Let f be a function from set A to set B and let S be a subset of A. The image of S is a subset of B that consists of the images of the elements of S. We denote the image of S by f(S), so that $f(S) = \{ f(s) \mid s \in S \}$.

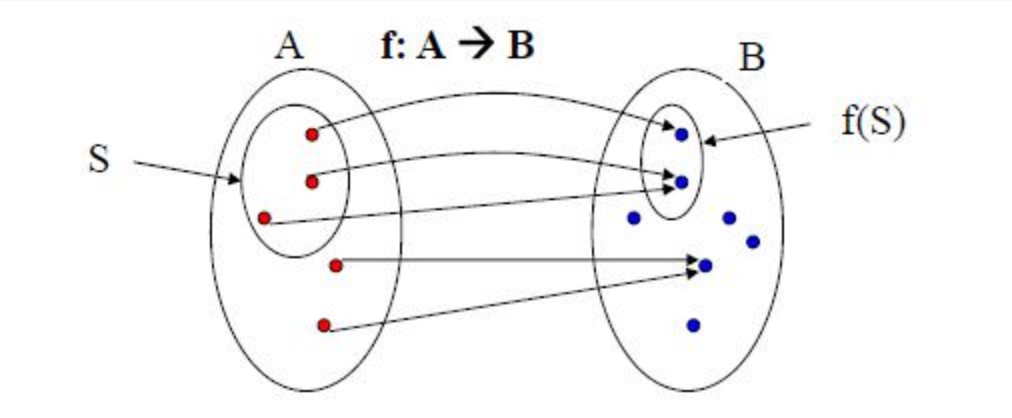


Example:

- Let $A = \{1,2,3\}$ and $B = \{a,b,c\}$ and f: $1 \to c$, $2 \to a$, $3 \to c$
- Let $S = \{1,3\}$ then image $f(S) =$ **?**

Image of subset

Definition: Let f be a function from set A to set B and let S be a subset of A. The image of S is a subset of B that consists of the images of the elements of S. We denote the image of S by f(S), so that $f(S) = \{ f(s) \mid s \in S \}$.



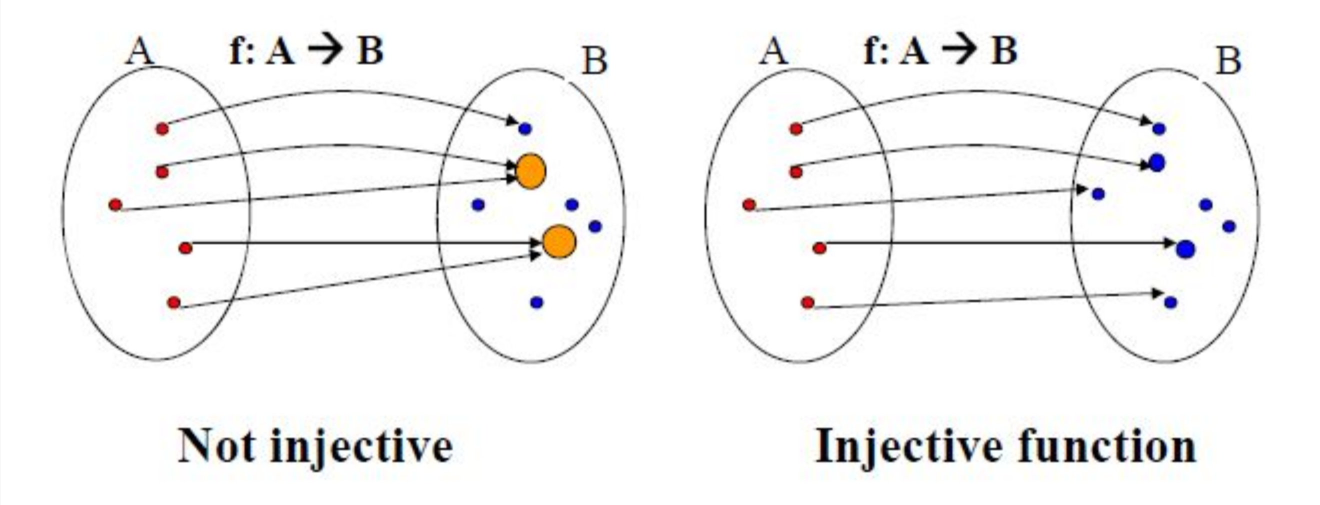
Example:

- Let A = {1,2,3} and B = {a,b,c} and f: 1 → c, 2 → a, 3 → c
- Let S = {1,3} then image f(S) = **{c}**

Injective function

Definition: A function f is said to be one-to-one, or injective, if and only if f(x) = f(y) implies x = y for all x, y in the domain of f. A function is said to be an injection if it is one-to-one.

A **function** for which every element of the range of the **function** corresponds to exactly **one** element of the domain.



Injective function

Example 1: Let $A = \{1,2,3\}$ and $B = \{a,b,c\}$

- Define f as
  - $1 \rightarrow c$
  - $2 \rightarrow a$
  - $3 \rightarrow c$
- Is f one to one?

Injective function

Example 1: Let $A = \{1,2,3\}$ and $B = \{a,b,c\}$

- Define f as
  - $1 \to c$
  - $2 \to a$
  - $3 \to c$
- Is f one to one?

- No, it is not one-to-one
- since $f(1) = f(3) = c$, and $1 \neq 3$.

Example 2: Let $g : Z \to Z$, where $g(x) = 2x - 1$.

- Is g one-to-one?

Injective function

Example 1: Let $A = \{1,2,3\}$ and $B = \{a,b,c\}$

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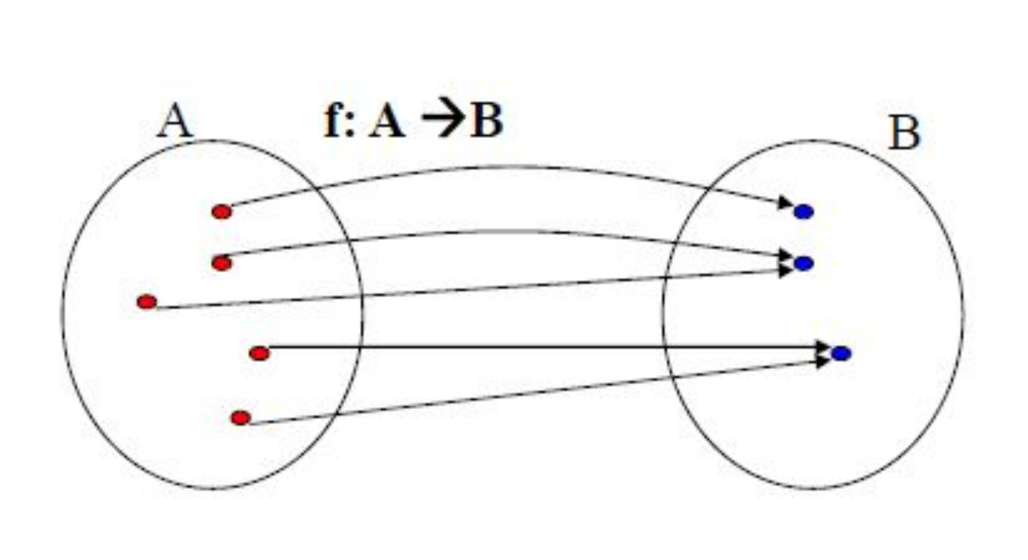
Example 2: Let $g : Z \rightarrow Z$, where $g(x) = 2x - 1$.

- Is g is one-to-one?
- Yes.
- Why? $g(a) = g(b)$, i.e., $2a - 1 = 2b - 1 \Rightarrow 2a = 2b \Rightarrow a = b$.

Surjective function

Definition: A function f from A to B is called onto, or surjective, if and only if for every b ϵ B there is an element a ϵ A such that f(a) = b.

Alternative: all co-domain elements are covered



Surjective function

Example 1: Let A = {1,2,3} and B = {a,b,c}

– Define f as

- 1 $\rightarrow$ c
- 2 $\rightarrow$ a
- 3 $\rightarrow$ c
- Is f an onto?

Surjective function

Example 1: Let A = {1,2,3} and B = {a,b,c}

– Define f as

- 1 → c
- 2 → a
- 3 → c
- Is f an onto?
- No. f is not onto, since b ϵ B has no pre-image.

Example 2: A = {0,1,2,3,4,5,6,7,8,9}, B = {0,1,2}

– Define h: A → B as h(x) = x mod 3.

- Is h an onto function?

Surjective function

Example 1: Let A = {1,2,3} and B = {a,b,c}

– Define f as

- 1 → c
- 2 → a
- 3 → c
- Is f an onto?
- No. f is not onto, since b ϵ B has no pre-image.

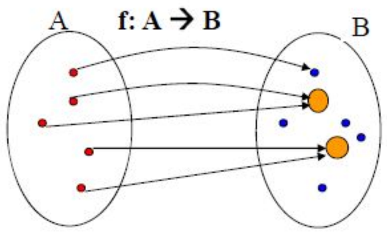
Example 2: A = {0,1,2,3,4,5,6,7,8,9}, B = {0,1,2}

– Define h: A → B as h(x) = x mod 3.

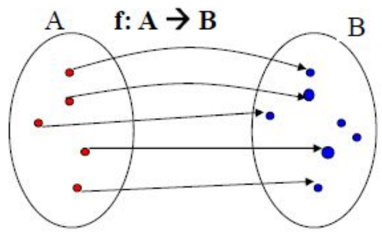
- Is h an onto function?
- Yes. h is onto since a pre-image of 0 is 6, a pre-image of 1 is 4, a pre-image of 2 is 8.

Bijective function

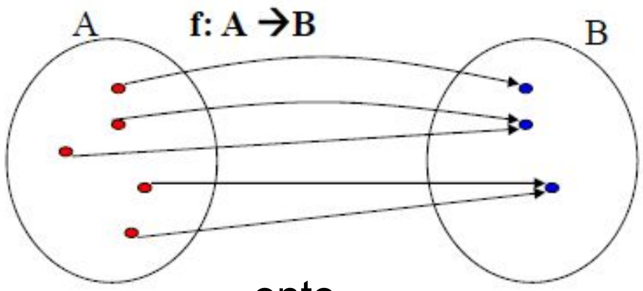
Definition: A function f is called a bijection if it is both one-to one and onto.



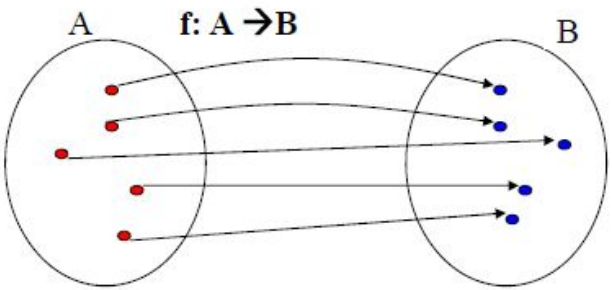
Not injective



Injective function



onto



One to one and onto

Bijective function

Example 1:

- Let A = {1,2,3} and B = {a,b,c}

– Define f as

- $1 \rightarrow c$
- $2 \rightarrow a$
- $3 \rightarrow b$
- Is f a bijection?

Bijective function

Example 1:

- Let A = {1,2,3} and B = {a,b,c}

– Define f as

- 1 → c
- 2 → a
- 3 → b
- Is f is a bijection? Yes. It is both one-to-one and onto.

- Note: Let f be a function from a set A to itself, where A is finite. f is one-to-one if and only if f is onto.

- This is not true if A an infinite set. Define f : Z → Z, where f(z) = 2 * z. f is one-to-one but not onto (3 has no pre-image).

Bijective function

Example 2:

- Define $g : W \rightarrow W$ (whole numbers), where

$g(n) = [n/2]$ (floor function).

- $0 \rightarrow [0/2] = [0] = 0$
- $1 \rightarrow [1/2] = [1/2] = 0$
- $2 \rightarrow [2/2] = [1] = 1$
- $3 \rightarrow [3/2] = [3/2] = 1$
- ...
- Is g a bijection?

– No. g is onto but not 1-1 ($g(0) = g(1) = 0$ however $0 \neq 1$.

Bijective function

Theorem: Let f be a function $f: A \to A$ from a set A to itself, where A is finite. Then f is one-to-one if and only if f is onto.

Assume

- A is finite and f is one-to-one (injective)
- Is f an onto function (surjection)?.

Bijective function

Theorem: Let f be a function $f: A \to A$ from a set A to itself, where A is finite. Then f is one-to-one if and only if f is onto.

Proof:

$\to$ A is finite and f is one-to-one (injective)

- Is f an onto function (surjection)?
- Yes. Every element points to exactly one element. Injection assures they are different. So we have $|A|$ different elements A points to. Since $f: A \to A$ the co-domain is covered thus the function is also a surjection (and a bijection)

$\leftarrow$ A is finite and f is an onto function

- Is the function one-to-one?

Bijective function

Theorem: Let f be a function f: A → A from a set A to itself, where A is finite. Then f is one-to-one if and only if f is onto.

← A is finite and f is an onto function

- Is the function one-to-one?

Yes. Every element maps to exactly one element and all elements in A are covered. Thus the mapping must be one-to one

Bijective function

Theorem. Let f be a function from a set A to itself, where A is finite. Then f is one-to-one if and only if f is onto.

Please note the above is not true when A is an infinite set.

- Example:

– $f : Z \rightarrow Z$, where $f(z) = 2 * z$.

– f is one-to-one but not onto.

- $1 \rightarrow 2$
- $2 \rightarrow 4$
- $3 \rightarrow 6$

– 3 has no pre-image.

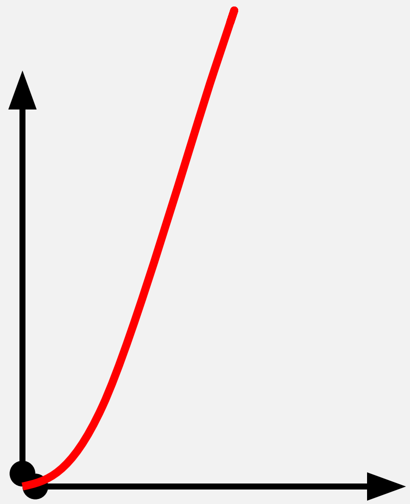
Function on Real Number

Suppose f: $R^+ \rightarrow R^+$, $f(x) = x^2$.

Is f one-to-one? yes

Is f onto? yes

Is f bijective? yes



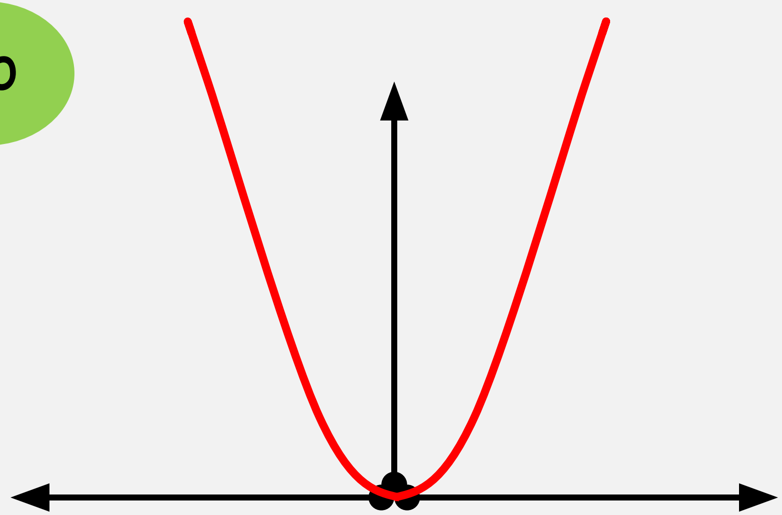
Function on Real Number

Suppose f: R $\rightarrow$ R^+, $f(x) = x^2$.

Is f one-to-one? no

Is f onto? yes

Is f bijective? no



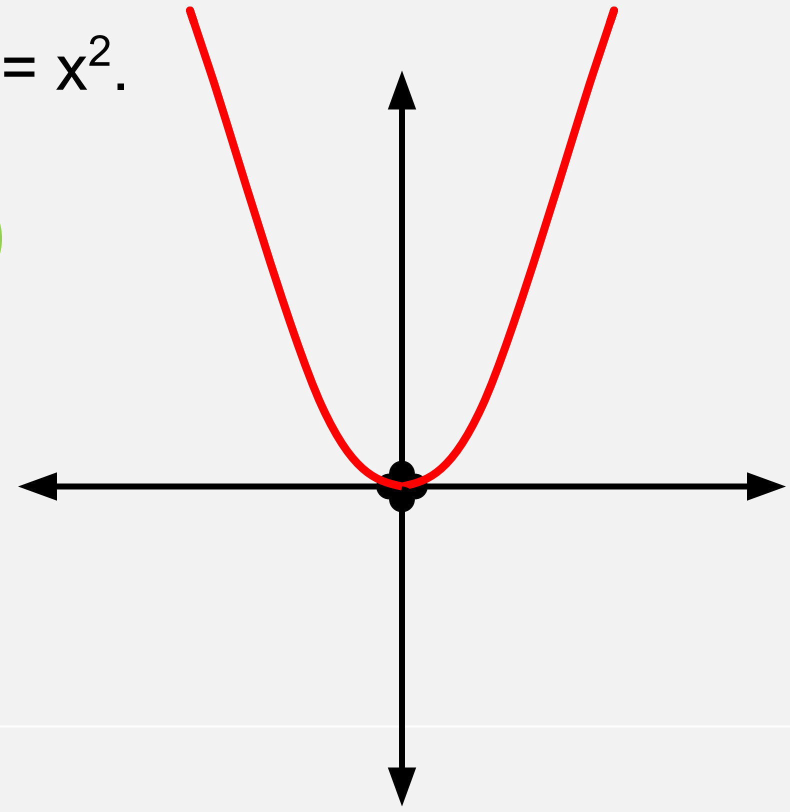
Function on Real Number

Suppose f: R → R, $f(x) = x^2$.

Is f one-to-one? no

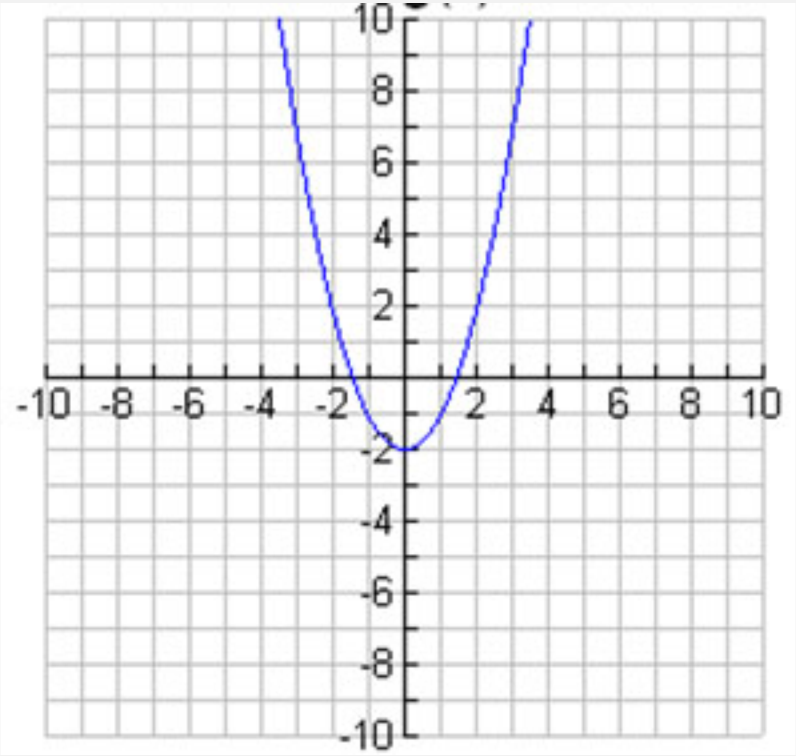
Is f onto? no

Is f bijective? no



Function on Real Number

Is $g(x) = x^2 - 2$ onto where $R \rightarrow R$?



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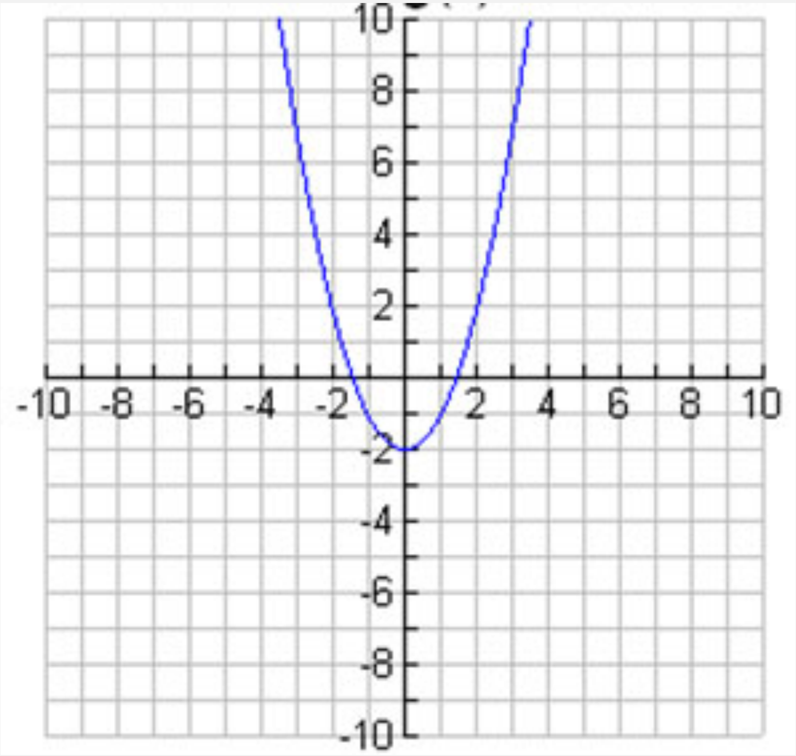
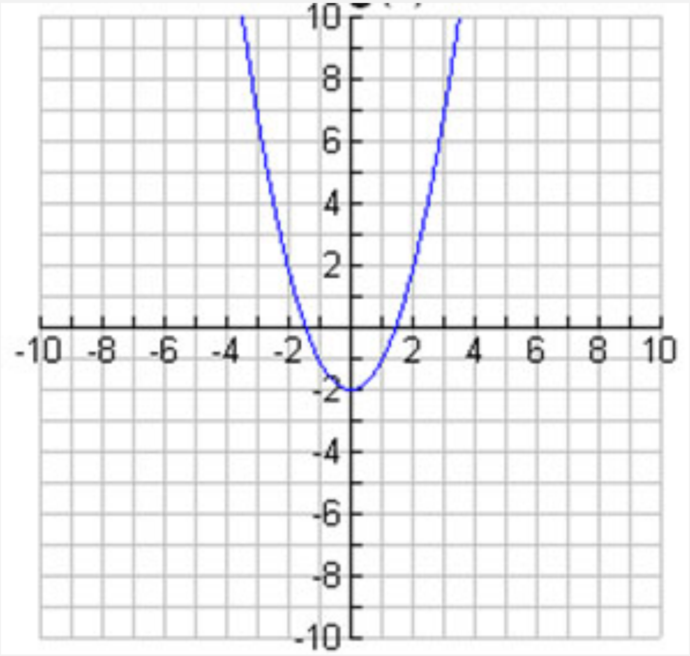


No, Values less than -2 on the *y*-axis are never used.

Is it one to one?

Function on Real Number

Is *g* (*x*) = x^2 - 2 onto where R → R?



No, Values less than -2 on the *y*-axis are never used.

Is it one to one?

No, pair value of x for each y value

Function on Real Number

Definition: Let f1 and f2 be functions from A to R (reals). Then f1 + f2 and f1 * f2 are also functions from A to R defined by

- (f1 + f2)(x) = f1(x) + f2(x)
- (f1 * f2)(x) = f1(x) * f2(x).

Examples:
- Assume
- f1(x) = x - 1
- f2(x) = x^3 + 1

then
- (f1 + f2)(x) = $x^3 + x$
- (f1 * f2)(x) = $x^4 - x^3 + x - 1$

Increasing and Decreasing Function

Definition: A function f whose domain and codomain are subsets of real numbers is strictly increasing if $f(x) > f(y)$ whenever $x > y$ and x and y are in the domain of f.

Similarly, f is called strictly decreasing if $f(x) < f(y)$ whenever $x > y$ and x and y are in the domain of f.

Example:

- Let $g : R \rightarrow R$, where $g(x) = 2x - 1$. Is it increasing ?

Increasing and Decreasing Function

Definition: A function f whose domain and codomain are subsets of real numbers is strictly increasing if $f(x) > f(y)$ whenever $x > y$ and x and y are in the domain of f.

Similarly, f is called strictly decreasing if $f(x) < f(y)$ whenever $x > y$ and x and y are in the domain of f.

Example:

- Let $g : R \rightarrow R$, where $g(x) = 2x - 1$. Is it increasing ?

Proof .

For $x>y$ holds $2x > 2y$ and subsequently $2x-1 > 2y-1$

Thus g is strictly increasing.

Increasing and Decreasing Function

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Similarly, f is called strictly decreasing if $f(x) < f(y)$ whenever $x > y$ and x and y are in the domain of f.

Note: Strictly increasing and strictly decreasing functions are one-to-one.

Why?

Increasing and Decreasing Function

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Similarly, f is called strictly decreasing if $f(x) < f(y)$ whenever $x > y$ and x and y are in the domain of f.

Note: Strictly increasing and strictly decreasing functions are one-to-one.

Why?

One-to-one function: A function is one-to-one if and only if $f(x) \neq f(y)$, whenever $x \neq y$.

Identity function

Definition: Let A be a set. The identity function on A is the function $i_A: A \rightarrow A$ where $i_A(x) = x$.

Example:

- Let A = {1,2,3}

Then:

- $i_A(1) = ?$

Identity function

Definition: Let A be a set. The identity function on A is the function $i_A: A \rightarrow A$ where $i_A(x) = x$.

Example:

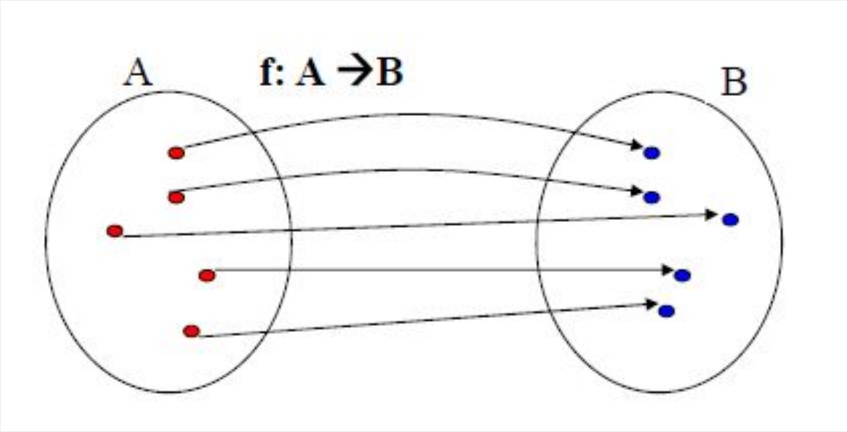
- Let A = {1,2,3}

Then:

- $i_A(1) = 1$
- $i_A(2) = 2$
- $i_A(3) = 3$.

Bijective Function

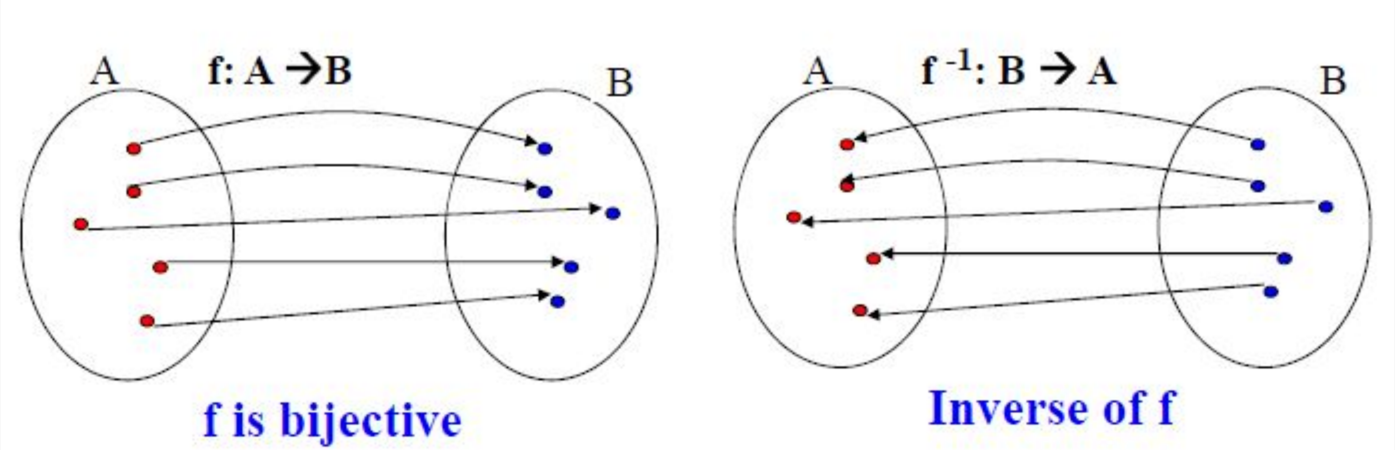
Definition: A function f is called a bijection if it is both one-to one and onto.



Inverse Function

Definition: Let f be a bijection from set A to set B. The inverse function of f is the function that assigns to an element b from B the unique element a in A such that f(a) = b.

The inverse function of f is denoted by f^{-1}. Hence, $f^{-1}(b) = a$, when $f(a) = b$. If the inverse function of f exists, f is called invertible.

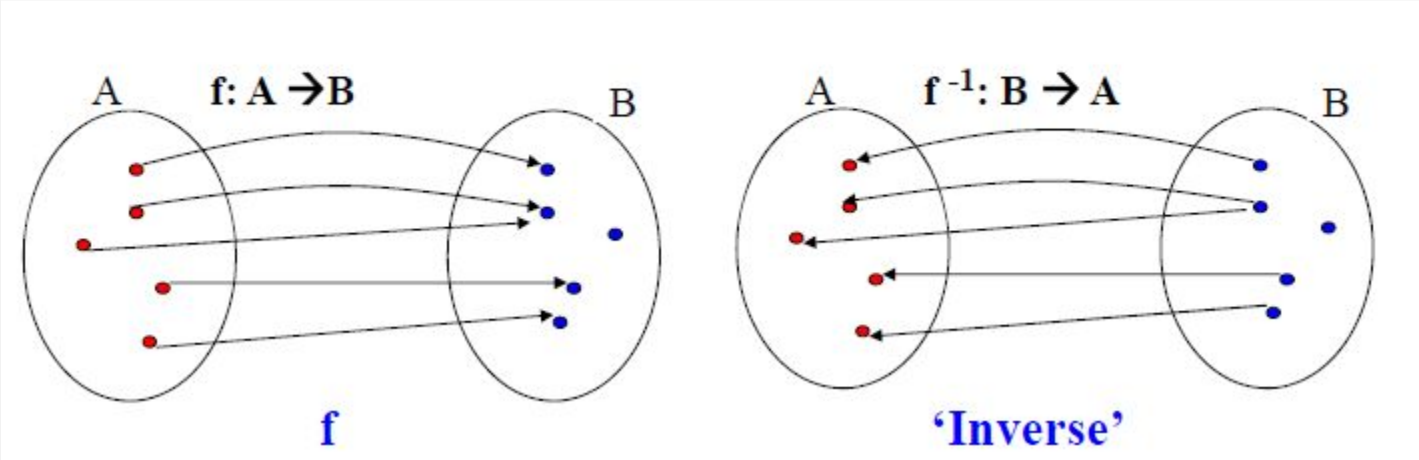


Inverse Function

Note: if f is not a bijection then it is not possible to define the inverse function of f. Why?

Assume f is not one-to-one:

?

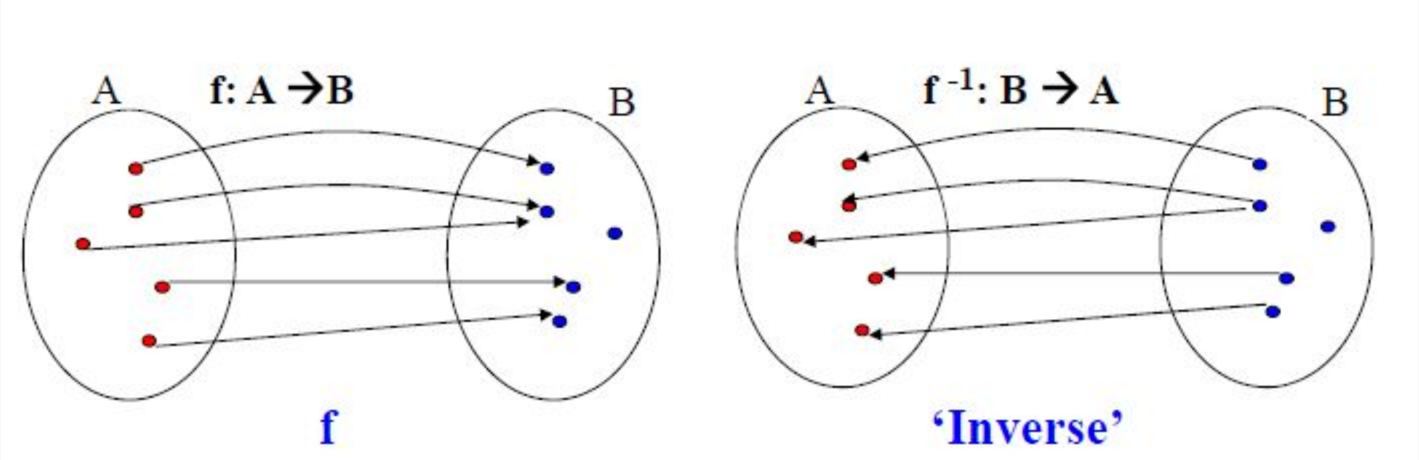


Inverse Function

Note: if f is not a bijection then it is not possible to define the inverse function of f. Why?

Assume f is not one-to-one:

Inverse is not a function. One element of B is mapped to two different elements.

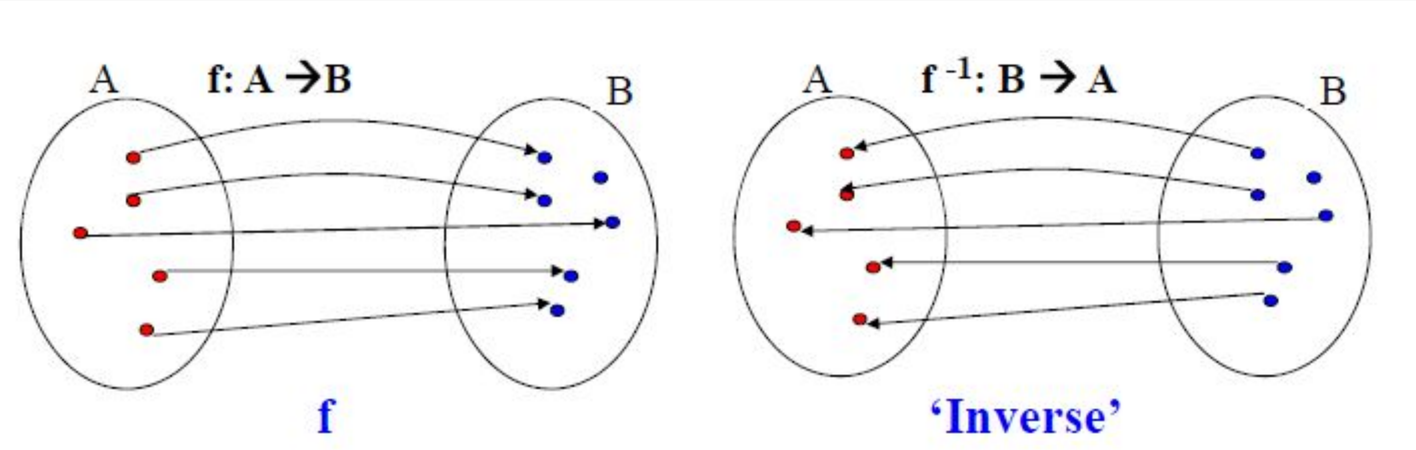


Inverse Function

Note: if f is not a bijection then it is not possible to define the inverse function of f. Why?

Assume f is not onto:

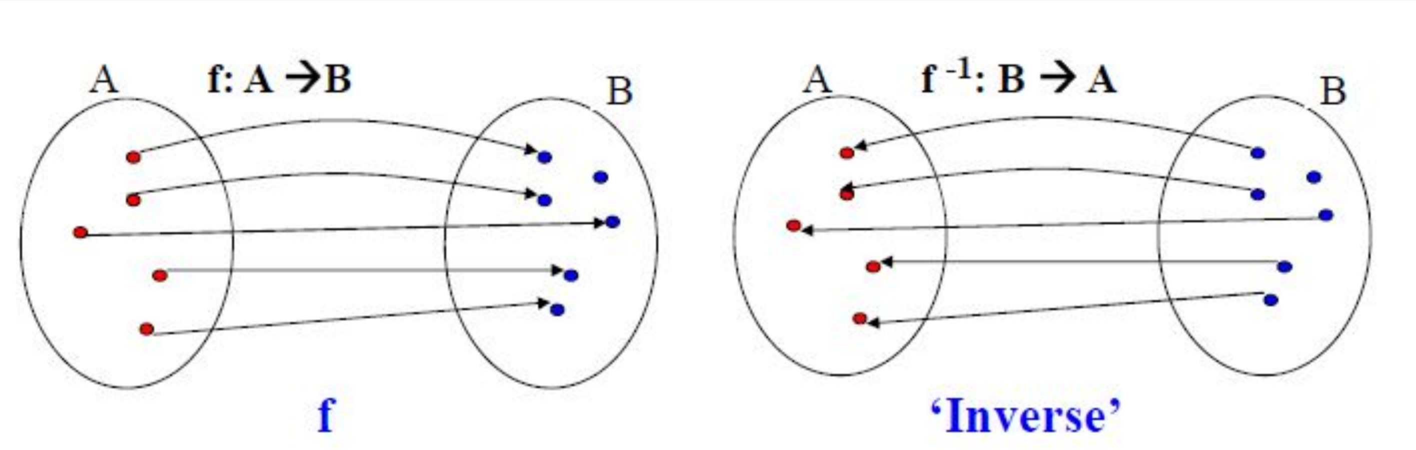
?



Inverse Function

Note: if f is not a bijection then it is not possible to define the inverse function of f. Why?

Assume f is not onto:
Inverse is not a function. One element of B is not assigned any value in A.



Inverse Function

Example 1:

- Let $A = \{1,2,3\}$ and i_A be the identity function

- $i_A(1) = 1$ $\quad$ $\mathbf{i_A^{-1}(1) = 1}$
- $i_A(2) = 2$ $\quad$ $\mathbf{i_A^{-1}(2) = 2}$
- $i_A(3) = 3$ $\quad$ $\mathbf{i_A^{-1}(3) = 3}$

- Therefore, the inverse function of i_A is i_A.

Inverse Function

Example 2:

- Let $g : R \rightarrow R$, where $g(x) = 2x - 1$.

- What is the inverse function g^{-1} ?

Inverse Function

Example 2:

- Let $g : R \rightarrow R$, where $g(x) = 2x - 1$.

- What is the inverse function g^{-1} ?

Approach to determine the inverse:

$y = 2x - 1 \Rightarrow y + 1 = 2x$

$\Rightarrow (y+1)/2 = x$

- Define $g^{-1}(y) = x = (y+1)/2$

Test the correctness of inverse:

- $g(3) =$ ..

Inverse Function

Example 2:

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Test the correctness of inverse:

- $g(3) = 2*3 - 1 = 5$
- $g^{-1}(5) =$

Inverse Function

Example 2:

- Let g : R → R, where $g(x) = 2x - 1$.

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Approach to determine the inverse:

$y = 2x - 1 \Rightarrow y + 1 = 2x$

$\Rightarrow (y+1)/2 = x$

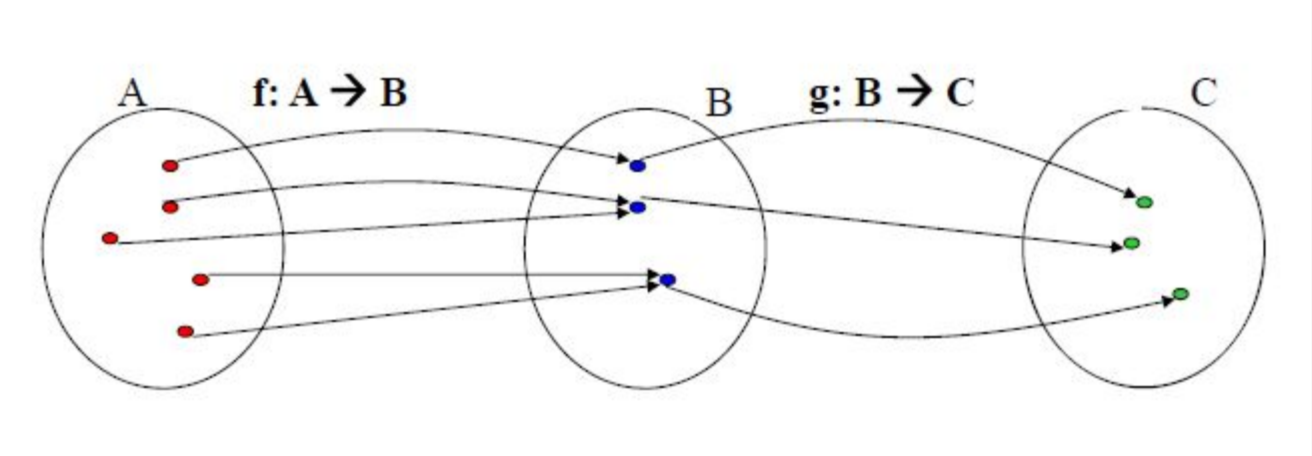
- Define $g^{-1}(y) = x = (y+1)/2$

Test the correctness of inverse:

- $g(3) = 2*3 - 1 = 5$
- $g^{-1}(5) = (5+1)/2 = 3$

Composition of Function

Definition: Let f be a function from set A to set B and let g be a function from set B to set C. The composition of the functions g and f, denoted by g O f is defined by (g O f)(a) = g(f(a)).



Composition of Function

Example 1:

- Let A = {1,2,3} and B = {a,b,c,d}

$g : A \rightarrow A,$	$f: A \rightarrow B$
$1 \rightarrow 3$	$1 \rightarrow b$
$2 \rightarrow 1$	$2 \rightarrow a$
$3 \rightarrow 2$	$3 \rightarrow d$

Composition of Function

Example 1:

- Let A = {1,2,3} and B = {a,b,c,d}

$g : A \rightarrow A,$ $f: A \rightarrow B$

$1 \rightarrow 3$	$1 \rightarrow b$
$2 \rightarrow 1$	$2 \rightarrow a$
$3 \rightarrow 2$	$3 \rightarrow d$

g O f : $A \rightarrow B$:

- $1 \rightarrow d$
- $2 \rightarrow b$
- $3 \rightarrow a$

Composition of Function

Example 2:

- Let f and g be two functions from Z to Z, where

- $f(x) = 2x$ and $g(x) = x^2$.
- $f \circ g : Z \rightarrow Z$

- $(f \circ g)(x) = f(g(x))$

  $= f(x^2)$

  $= 2(x^2)$

- $g \circ f : Z \rightarrow Z$
- $(g \circ f)(x) =$?

Composition of Function

Example 2:

- Let f and g be two functions from Z to Z, where

- $f(x) = 2x$ and $g(x) = x^2$.
- $f \circ g : Z \rightarrow Z$

- $(f \circ g)(x) = f(g(x))$

  $= f(x^2)$

  $= 2(x^2)$

- $g \circ f : Z \rightarrow Z$
- $(g \circ f)(x) = g(f(x))$

  $= g(2x)$

  $= 4x^2$

Composition of Function

Example 3:

- $(f \circ f^{-1})(x) = x$ and $(f^{-1} \circ f)(x) = x$, for all x.
- Let $f : R \rightarrow R$, where $f(x) = 2x – 1$ and $f^{-1}(x) = (x+1)/2$.
- $(f \circ f^{-1})(x)$

$$
\begin{aligned}
(f \circ f^{-1})(x) &= f(f^{-1}(x)) \\
&= f(\ (x+1)/2\ ) \\
&= 2(\ (x+1)/2\ ) - 1 \\
&= (x+1) - 1 \\
&= x
\end{aligned}
$$

Composition of Function

Example 3:

- **$(f \circ f^{-1})(x) = x$ and $(f^{-1} \circ f)(x) = x$, for all x.**
- Let f : **R** → **R, where $f(x) = 2x – 1$ and $f^{-1}(x) = (x+1)/2$.**

- $(f \circ f^{-1})(x) = f(f^{-1}(x))$
$$= f(\ (x+1)/2\ )$$
$$= 2(\ (x+1)/2\ ) - 1$$
$$= (x+1) - 1$$
$$= x$$

- $(f^{-1} \circ f)(x) = f^{-1}(f(x))$
$$= f^{-1}(\ 2x - 1\ )$$
$$= (2x)/2$$
$$= x$$

Some Function

Definitions:

- The floor function assigns a real number x the largest integer that is less than or equal to x. The floor function is denoted by $\lfloor x \rfloor$
- The ceiling function assigns to the real number x the smallest integer that is greater than or equal to x. The ceiling function is denoted by $\lceil x \rceil$.

Other important functions:

- Factorials: $n! = n(n-1)$ such that $1! = 1$

Thank You