Logic

Lecture 2

Limitation of Propositional Logic

Propositional logic: the world is described in terms of elementary propositions and their logical combinations

Elementary statements:

•Typically refer to objects, their properties and relations.

•But these are not explicitly represented in the propositional logic

• Hasan is a GUB student

Hasan - object
A GUB student - a property

Objects and properties are hidden in the statement, it is not possible to reason about them

Limitation of Propositional Logic

Statements that must be repeated for many objects

Example
If Hasan is a CSE GUB graduate then Hasan has passed CSE101

Translation
Hasan is a CSE GUB graduate → Hasan has passed CSE101

Solution: make statements with variables
If x is a CSE GUB graduate then x has passed CSE101
x a CSE GUB graduate → x has passed CSE101

Limitation of Propositional Logic

Statements that define the property of the group of objects

Example

Some of the CSE graduates graduate with honors.

Solution: make statements with **quantifiers**

- Universal quantifier –the property is satisfied by all members of the group
– Existential quantifier – at least one member of the group satisfy the property

Predicate Logic

Remedies the limitations of the propositional logic

- Explicitly models objects and their properties
- Allows to make statements with variables and quantify them

Basic building blocks of the predicate logic:

- Constant –models a specific object

Examples: “Hasan”, “Khulna”, “7”

- Variable – represents object of specific type
  (defined by the *universe of discourse*)

Examples: x, y
(universe of discourse can be people, students, numbers)

- Predicate - Represents properties of objects

Examples: Red(car23), student(x), married(John,Ann)

Assigning value to Predicate

Predicate - Represents properties of objects
A predicate P(x) assigns a value true or false to each x depending on whether the property holds or not for x.

x is a prime number (universe of discourse is integers)

What are the truth values of:

- P(2) T
- P(3) T
- P(4) F
- P(5) T

All statements P(2), P(3), P(4), P(5) are propositions.

Is P(x) a proposition? No. Many substitutions are possible.

Assigning value to Predicate

Predicates can have more arguments which represent the relations between objects

Example:

- Let $Q(x,y)$ denote '$x+5 > y$'

– Is $Q(x,y)$ a proposition? **No!**
– Is $Q(3,7)$ a proposition? **Yes**

– What is the truth value of:
– $Q(3,7)$ T
– $Q(1,6)$ F

- $Q(3,y)$ may be T or F. not a proposition.

Predicate is not proposition

Important:

• statement **P(x) is not a proposition** since there are many objects it can be applied to

This is the same as in propositional logic …

… But the difference is:

• predicate logic allows us to explicitly manipulate and substitute for the objects

• Predicate logic permits quantified sentences where variables are substituted for statements about the group of objects

Quantified Statements

Predicate logic lets us to make statements about groups of objects

• To do this we use special quantified expressions

Two types of quantified statements:

• **universal**

Example: ‘ all CSE GUB graduates have to pass CSE 101”

– the statement is true for all graduates

• **existential**

Example: ‘Some CSE GUB students graduate with honor.’

– the statement is true for some people

Universal Quantifier

Defn: The universal quantification of P(x) is the proposition:
"*P(x) is true for all values of x in the domain of discourse*."

The notation $\forall x P(x)$ denotes the universal quantification of P(x), and is expressed as for every x, P(x).

Example:

- Let P(x) denote $x > x - 1$.
- What is the truth value of $\forall x P(x)$?

- Assume the universe of discourse of x is all real numbers.

- Answer: Since every number x is greater than itself minus 1. Therefore, $\forall x P(x)$ is true.

Universally Quantified Statement

Predicate logic lets us make statements about groups of objects

Universally quantified statement

- CSE-major(x) $\rightarrow$ Student(x)
  - Translation: "if x is a CSE-major then x is a student"
  - Proposition: no.

- $\forall x \; \text{CSE-major}(x) \rightarrow \text{Student}(x)$
  - Translation: "(For all people it holds that) if a person is a CSE-major then s/he is a student."
  - Proposition: yes.

Existential Quantifier

Definition: The existential quantification of P(x) is the proposition
"*There exists an element in the domain (universe) of discourse such that P(x) is true.*"

The notation: $\exists x P(x)$

Example 1:

- Let T(x) denote x > 5 and x is from Real numbers.
- What is the truth value of $\exists x T(x)$?

- **Answer:**
- Since 10 > 5 is true. Therefore, it is true that $\exists x T(x)$

Quantified statements

Statements about groups of objects

Example:

• CSE-GUB-graduate (x) ∧ Honor-student(x)

– **Translation: "x is a CSE-GUB-graduate and x is an honor** student"

– **Proposition: no.**

• $\exists x \text{ CSE - NSU - graduate}(x) \wedge \text{Honor - student}(x)$

– **Translation:** "There is a person who is a CSE-GUB-graduate and who is also an honor student."

– **Proposition: ? yes**

Translation with Quantifier

Sentence:

- All GUB students are smart.

- Assume: the domain of discourse of x are GUB students
- Translation:

$$\forall x\, smart(x)$$

- Assume: the universe of discourse are students (all students):
- $\forall x\, at(x, GUB) \rightarrow smart(x)$

- Assume: the universe of discourse are people:
- $\forall x\, student(x) \wedge at(x, GUB) \rightarrow smart(x)$

Translation with Quantifier

Sentence:

- Someone at GUB is smart.

- Assume: the domain of discourse are all GUB affiliates
- Translation:

$$\exists x\, smart(x)$$

- Assume: the universe of discourse are people:
- $$\exists x\, at(x, GUB) \wedge smart(x)$$

Translation with Quantifier

Assume the following statement

Every student in this class has studied calculus

Consider the domain is the set of all people.

"For every person x, if person x is a student in this class then x has studied calculus."

-C(x) = x has studied calculus.

-S(x) = Person x is in this class.

-Can be written as $\forall x(S(x) \rightarrow C(x))$.

-Can not written as $\forall x(S(x) \land C(x))$. ***WHY?***

Translation with Quantifier

Assume the following statements
"Some student in this class has visited Mexico"

Consider the domain is the set of all people.

There is a person *x* having the properties that *x* is a student in this class and *x* has visited Mexico."

-M(x) = *x* has visited Mexico
-S(x) = Person *x* is in this class.
-Can be written as $\exists x(S(x) \land M(x))$
- *Can not written as* $\exists x(S(x) \rightarrow M(x))$. ***WHY?***

F ---> F = T
For This reason.

Translation with Quantifier

Assume two predicates S(x) and P(x)

Universal statements typically tie with implications

- All S(x) is P(x)
  - $\forall x(S(x) \to P(x))$

- All S(x) is not P(x)
  - $\forall x(S(x) \to \neg P(x))$

Existential statements typically tie with conjunctions

- Some S(x) is P(x)
  - $\exists x(S(x) \land P(x))$

- Some S(x) is not P(x)
  - $\exists x(S(x) \land \neg P(x))$

Translation with Quantifier

Exercise

Consider the following statements.

"All lions are fierce."

"Some lions do not drink coffee."

"Some fierce creatures do not drink coffee."

Let,

P(x) = x is a lion

Q(x) = x is fierce

R(x) = x drinks coffee

$$\forall x(P(x) \to Q(x)).$$
$$\exists x(P(x) \land \neg R(x)).$$
$$\exists x(Q(x) \land \neg R(x)).$$

Nested Quantifier

More than one quantifier may be necessary to capture the meaning of a statement in the predicate logic.

Example:

- Every real number has its corresponding negative.

- **Translation:**

– Assume:

- a real number is denoted as x and its negative as y
- A predicate P(x,y) denotes: “x + y =0”

- Then we can write:

$$\forall x \exists y P(x, y)$$

Nested Quantifier

Translate the following statement

$\forall x(C(x) \vee \exists y(C(y) \wedge F(x, y)))$

$C(x)$ = x has a computer
$F(x, y)$ = x and y are friends
The domain for both x and y consists of all students in your school.

For every student x in your school, x has a computer or there is a student y such that y has a computer and x and y are friends.
In other words, every student in your school has a computer or has a friend who has a computer.

Order of Quantifier

The order of nested quantifiers matters if quantifiers are of different type

Example:

- Assume F(x,y) denotes “x is a friend of y”

- Then: $\forall x \exists y F(x, y)$
- Translates to: Everybody is a friend of somebody.

- And: $\exists y \forall x F(x, y)$
- Translates to: There is someone who is friend of everyone.

The meaning of the two is different.

Order of Quantifier

Suppose that x and y are assigned values. Then, there exists a real number z such that $x + y = z$.

Consequently, the quantification

$\forall x \forall y \exists z\, Q(x, y, z)$

For all real numbers x and for all real numbers y there is a real number z such that $x + y = z$.

$\exists z \forall x \forall y\, Q(x, y, z)$,

There is a real number z such that for all real numbers x and for all real numbers y it is true that $x + y = z$.

The meaning of the two is different.

Order of Quantifier

The order of nested quantifiers does not matter if quantifiers are of the same type

Example:

- For all x and y, if x is a parent of y then y is a child of x
- Assume:

– Parent(x,y) denotes "x is a parent of y"

– Child(x,y) denotes "x is a child of y"

- Two equivalent ways to represent the statement:

$$\forall x \forall y Parent(x, y) \rightarrow child(y, x)$$

$$\forall y \forall x Parent(x, y) \rightarrow child(y, x)$$

Negation of quantifiers

English statement:

- Nothing is perfect.
- Translation: $\neg \exists x Perfect(x)$

Another way to express the same meaning:

- Everything is imperfect.
- Translation: $\forall x \neg Perfect(x)$

Conclusion: $\neg \exists x P(x)$ is equivalent to $\forall x \neg P(x)$

Negation of quantifiers

DeMorgan Laws for quantifiers

Negation	Equivalent
$\neg \exists x P(x)$	$\forall x \neg P(x)$
$\neg \forall x P(x)$	$\exists x \neg P(x)$

Rules of Inference

Rules of inference: logically valid inference patterns

Modus Ponens, or the Law of Detachment

- Rules of inference

p
$p \rightarrow q$

q

- Given p is true and the implication $p \rightarrow q$ is true then q is true.

Rules of Inference

Addition

$p \rightarrow (p \vee q)$

p

$p \vee q$

Simplification

$p \wedge q \rightarrow p$

$p \wedge q$

p

Rules of Inference

Modus Tollens

$[\neg q \wedge (p \rightarrow q)] \rightarrow \neg p$

Hypothetical Syllogism

$[(p \rightarrow q) \wedge (q \rightarrow r)] \rightarrow (p \rightarrow r)$

Disjunctive Syllogism

$[(p \vee q) \wedge \neg p] \rightarrow q$

Logical Equivalence

$p \Leftrightarrow q$

$p \rightarrow q$ is a tautology

DeMorgans law

$\neg(p \vee q) \Leftrightarrow \neg p \wedge \neg q$

Applying rule of Inference

Text:

(1) It is not sunny this afternoon and it is colder than yesterday.
(2) We will go swimming only if it is sunny.
(3) If we do not go swimming then we will take a canoe trip.
(4) If we take a canoe trip, then we will be home by sunset.

Propositions:

1. p = It is sunny this afternoon,
2. q = it is colder than yesterday,
3. r = We will go swimming ,
4. s= we will take a canoe trip
5. t= We will be home by sunset

Applying rule of Inference

Propositions:

p = It is sunny this afternoon,

q = it is colder than yesterday,

r = We will go swimming ,

s= we will take a canoe trip

t= We will be home by sunset

Translation:

(1) $\neg p \wedge q$,

(2) $r \rightarrow p$,

(3) $\neg r \rightarrow s$,

(4) $s \rightarrow t$

(1) It is not sunny this afternoon and it is colder than yesterday.

(2) We will go swimming only if it is sunny.

(3) If we do not go swimming then we will take a canoe trip.

(4) If we take a canoe trip, then we will be home by sunset.

We want to show: t

Applying rule of Inference

Proof:

1. $\neg p \wedge q$	Hypothesis
2. $\neg p$	Simplification
3. $r \to p$	Hypothesis
4. $\neg r$	Modus tollens (step 2 and 3)
5. $\neg r \to s$	Hypothesis
6. s	Modus ponens (steps 4 and 5)
7. $s \to t$	Hypothesis
8. t	Modus ponens (steps 6 and 7)

end of proof

Thank You