Mathematical Induction

Lecture 13

Proof Technique

Proof of quantified statements:

- **There exists x with some property P(x).**
  - It is sufficient to find one element for which the property holds.

- **For all x some property P(x) holds.**
  - Proofs of 'For all x some property P(x) holds' must cover all x and can be harder.

- **Mathematical induction is a technique that can be applied to** prove the universal statements for sets of positive integers or their associated sequences.

Mathematical Induction

Used to prove statements of the form $\forall x P(x)$ where x ϵ Z+

Mathematical induction proofs consists of two steps:

1) **Basis: The proposition P(1) is true.**
2) **Inductive Step: The implication**

P(n) → P(n+1), is true for all positive n.

- Therefore we conclude $\forall x P(x)$

Mathematical Induction

- **Example: Prove the sum of first n odd integers is n^2**
  .

- i.e. $1 + 3 + 5 + 7 + ... + (2n - 1) = n^2$ for all positive integers.

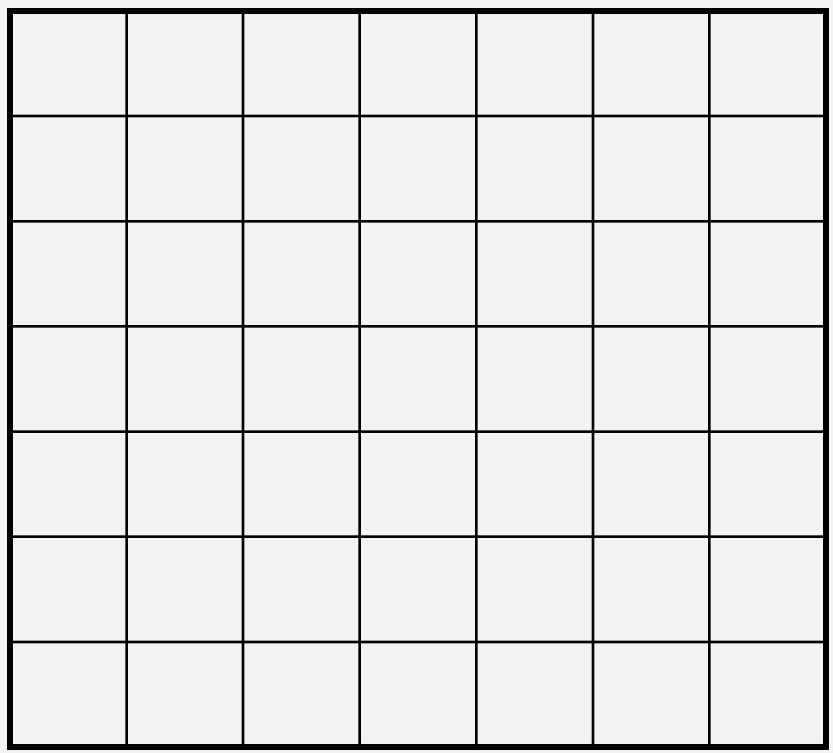
Proof:

- What is P(n)?

$$P(n): 1 + 3 + 5 + 7 + ... + (2n - 1) = n^2$$

Mathematical Induction Example.

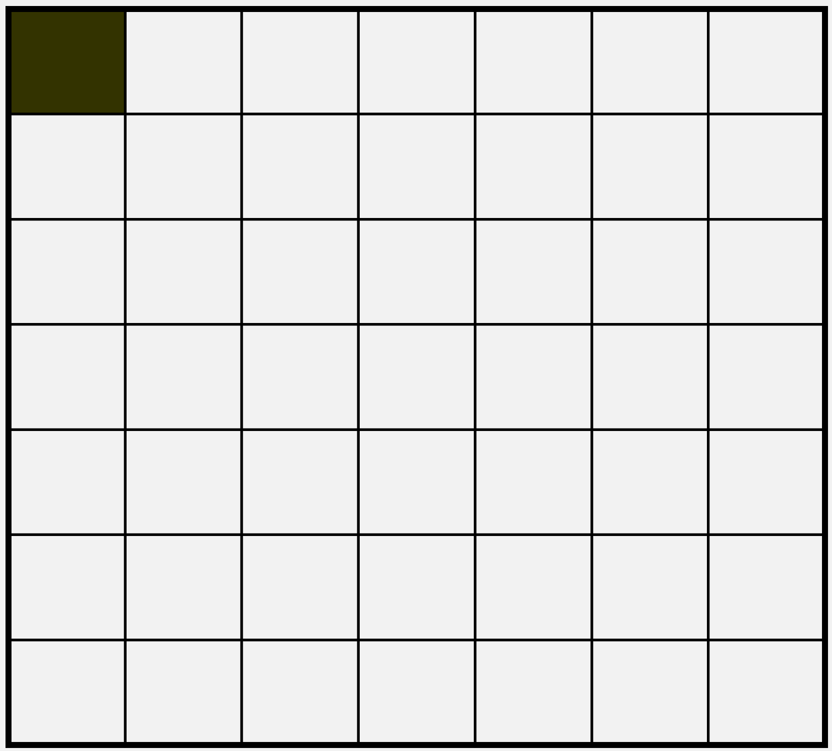
Geometric interpretation. To get next square, need to add next odd number:



Mathematical Induction Example.

Geometric interpretation. To get next square, need to add next odd number:

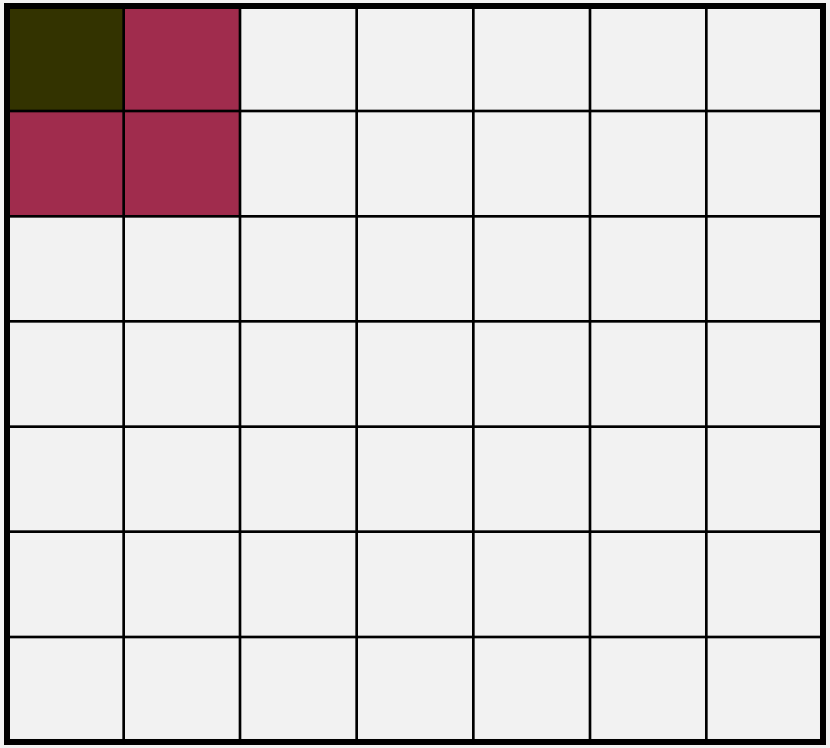
1



Mathematical Induction Example.

Geometric interpretation. To get next square, need to add next odd number:

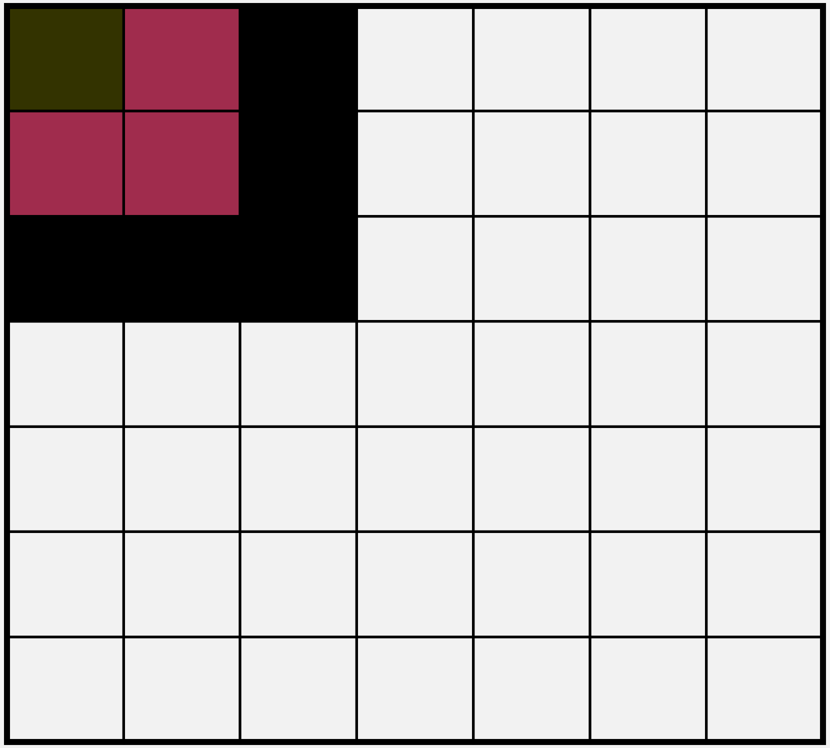
1+3



Mathematical Induction Example.

Geometric interpretation. To get next square, need to add next odd number:

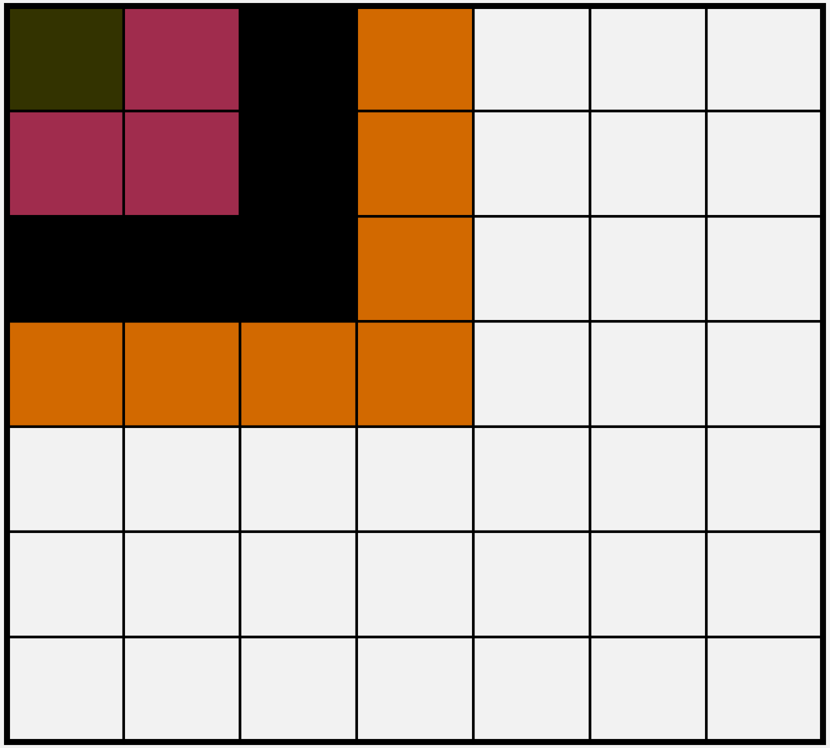
1+3+5



Mathematical Induction Example.

Geometric interpretation. To get next square, need to add next odd number:

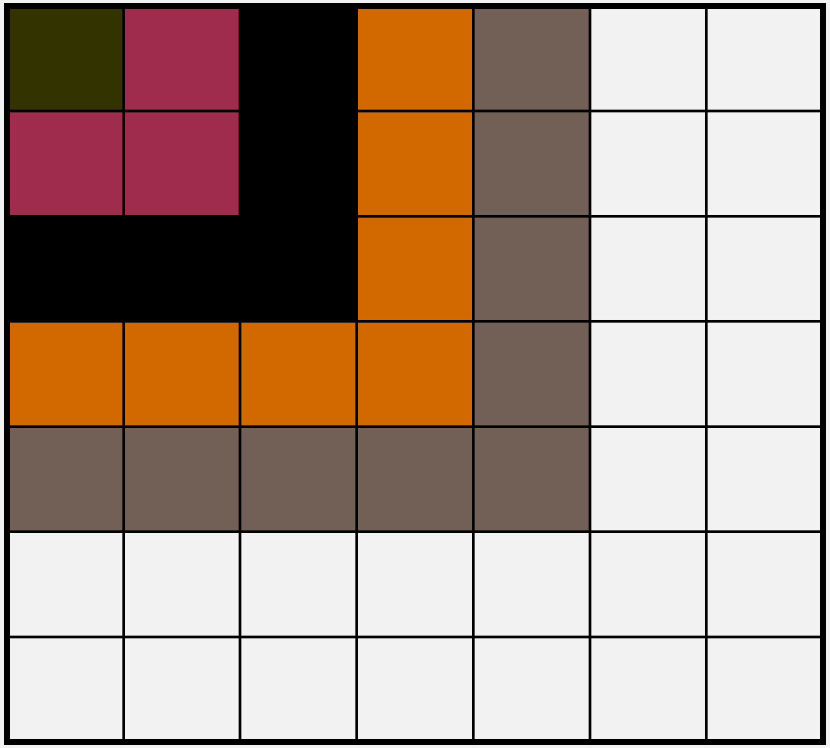
1+3+5+7



Mathematical Induction Example.

Geometric interpretation. To get next square, need to add next odd number:

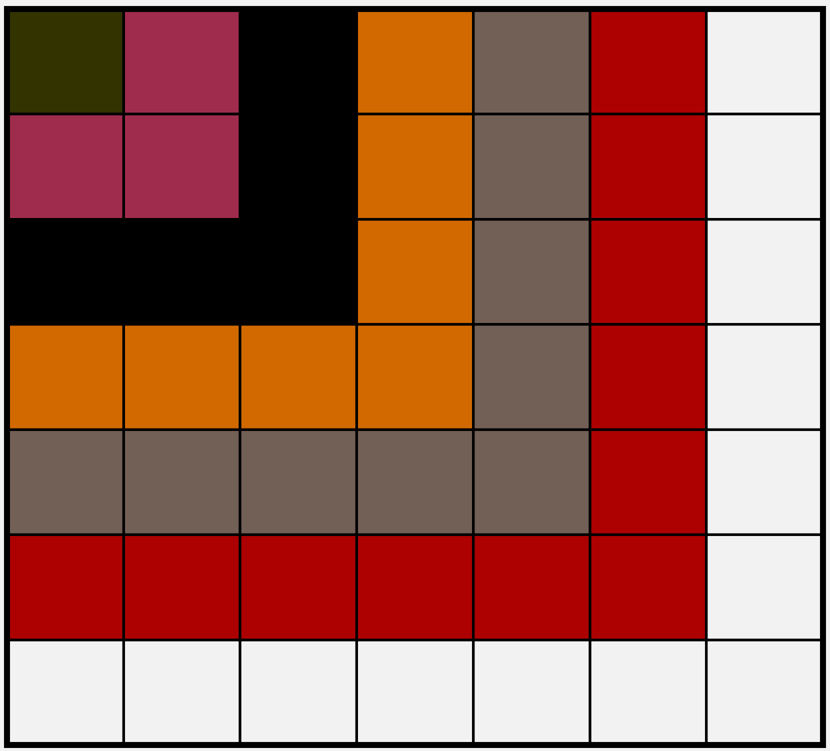
1+3+5+7+9



Mathematical Induction Example.

Geometric interpretation. To get next square, need to add next odd number:

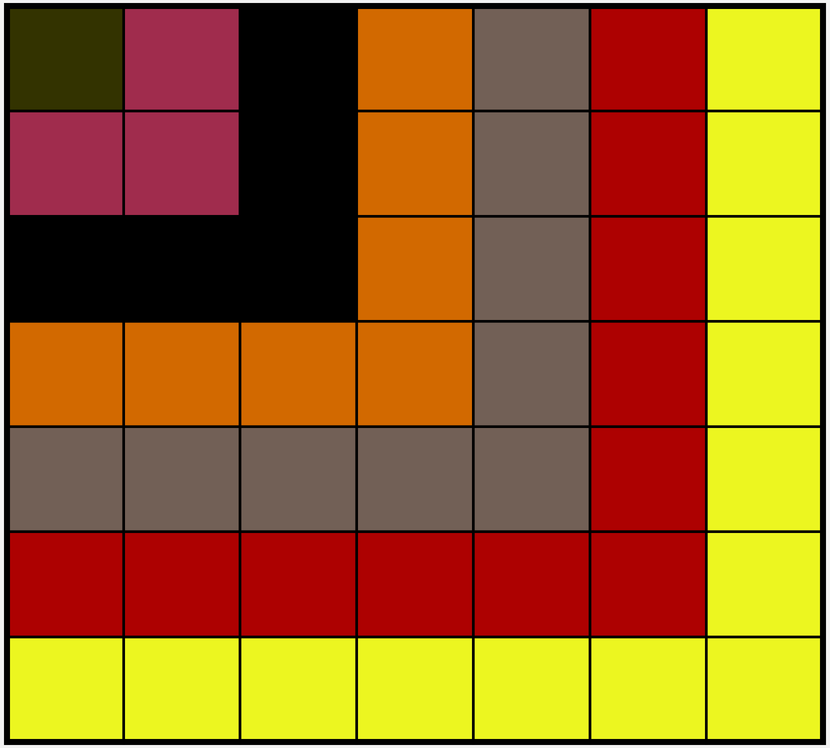
1+3+5+7+9 +11



Mathematical Induction Example.

Geometric interpretation. To get next square, need to add next odd number:

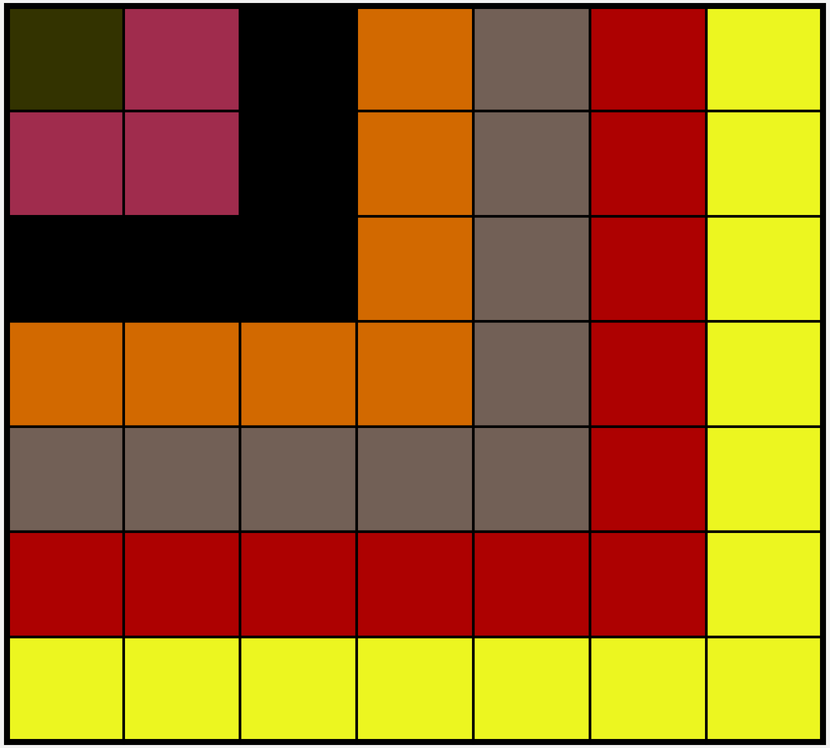
1+3+5+7+9 +11+13



Mathematical Induction Example.

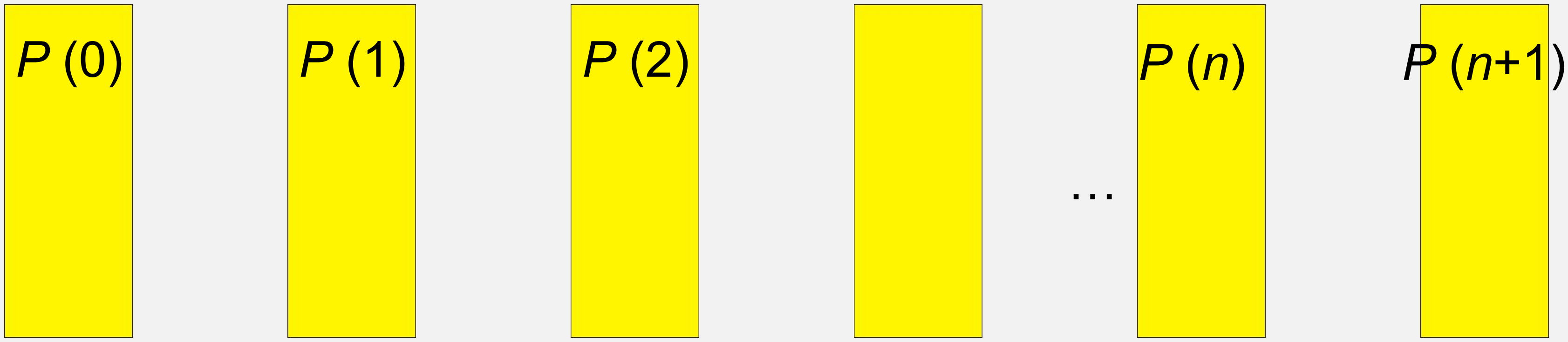
Geometric interpretation. To get next square, need to add next odd number:

$1+3+5+7+9+11+13=7^2$



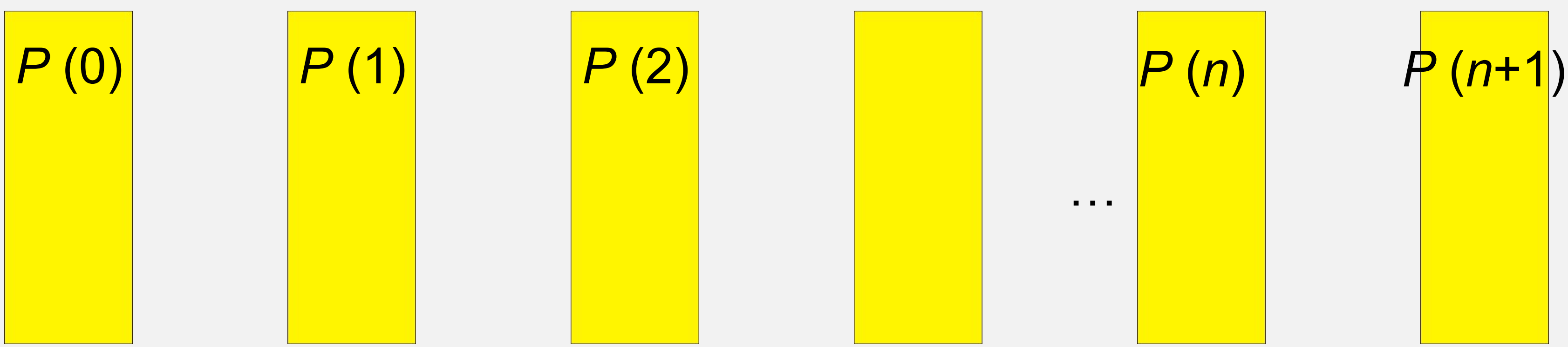
Mathematical Induction

So sequence of propositions is a sequence of dominos.



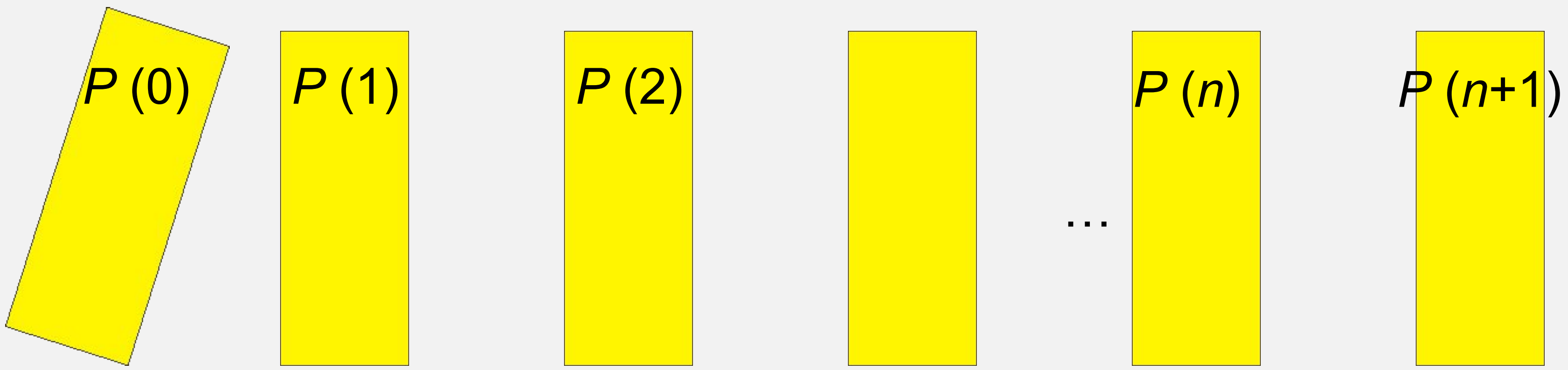
Mathematical Induction

Then can conclude that all the dominos fall!



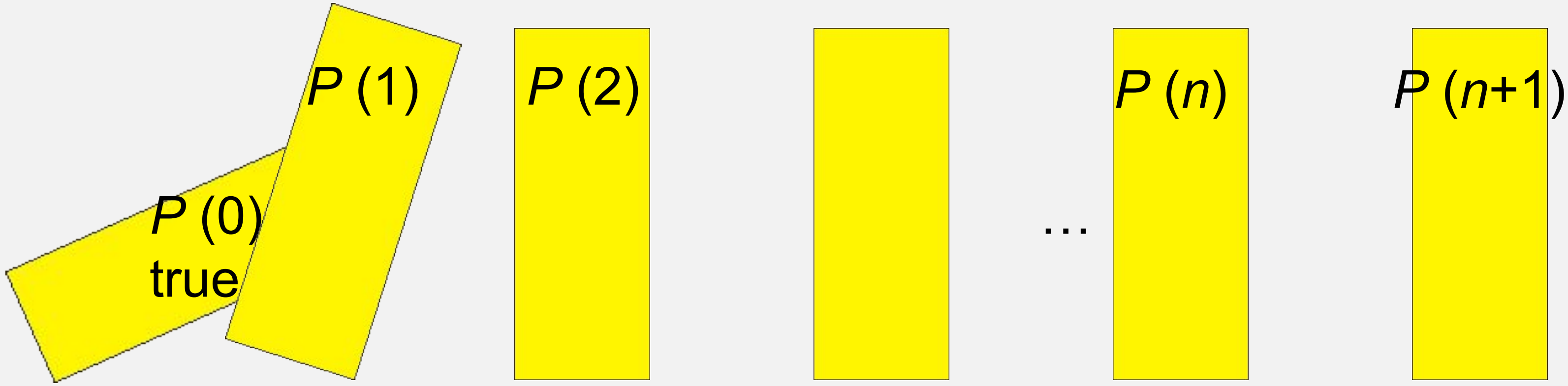
Mathematical Induction

Then can conclude that all the dominos fall!



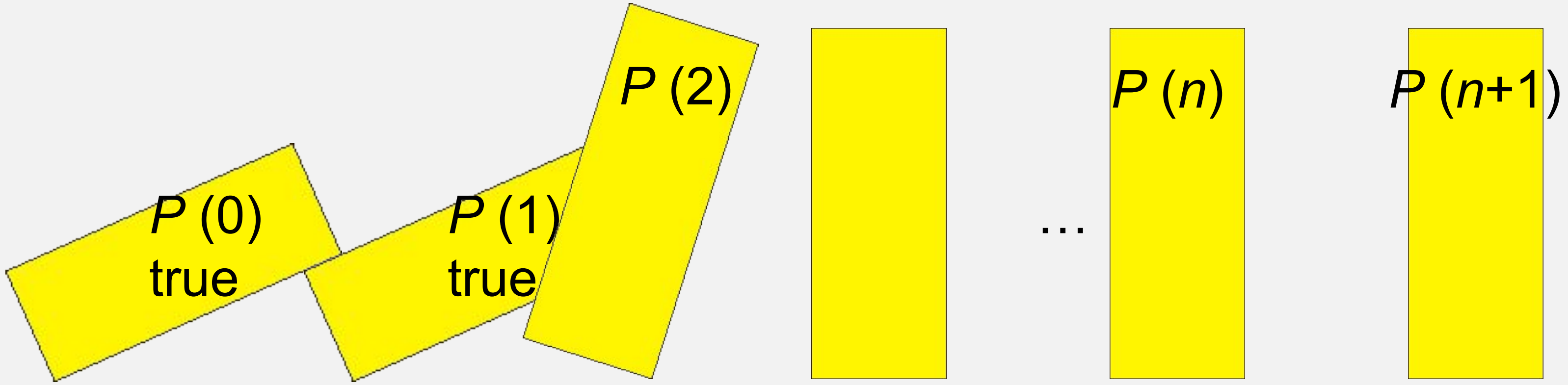
Mathematical Induction

Then can conclude that all the dominos fall!



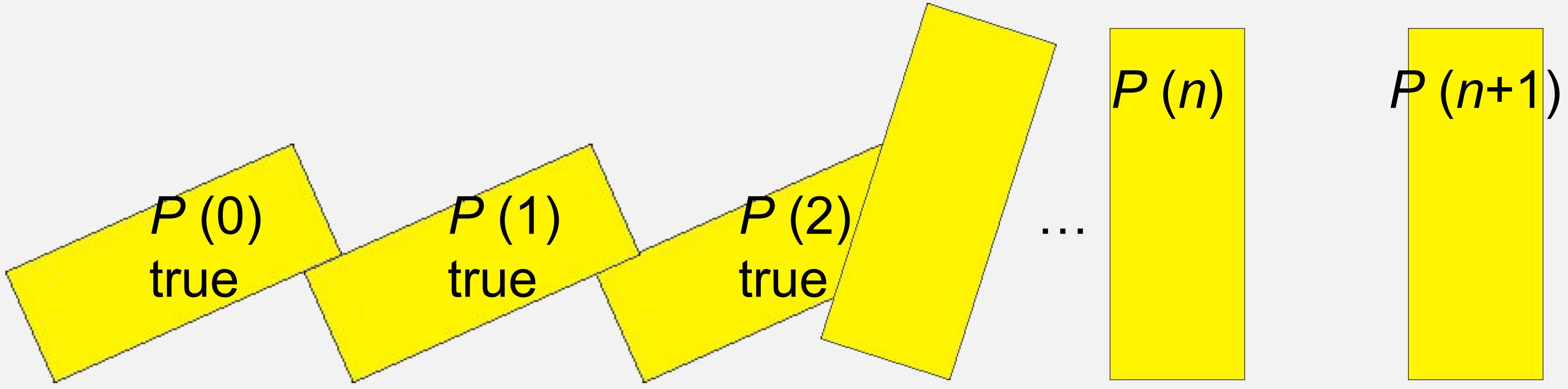
Mathematical Induction

Then can conclude that all the dominos fall!



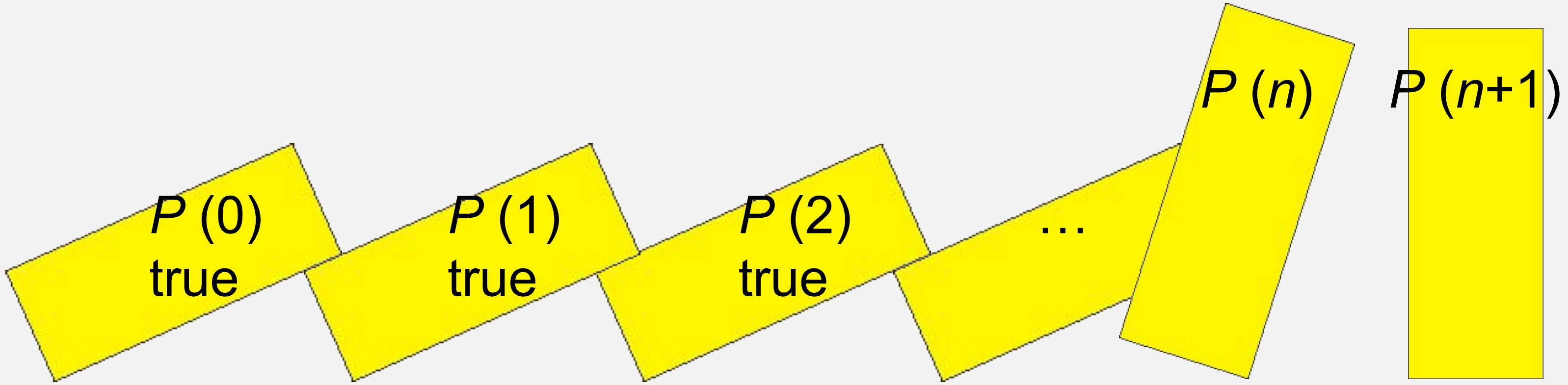
Mathematical Induction

Then can conclude that all the dominos fall!



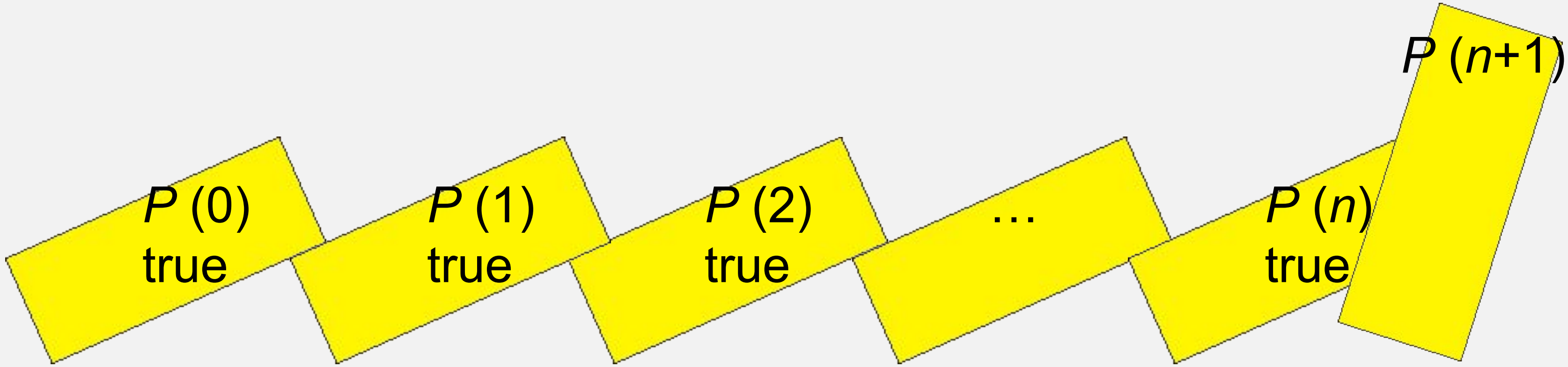
Mathematical Induction

Then can conclude that all the dominos fall!



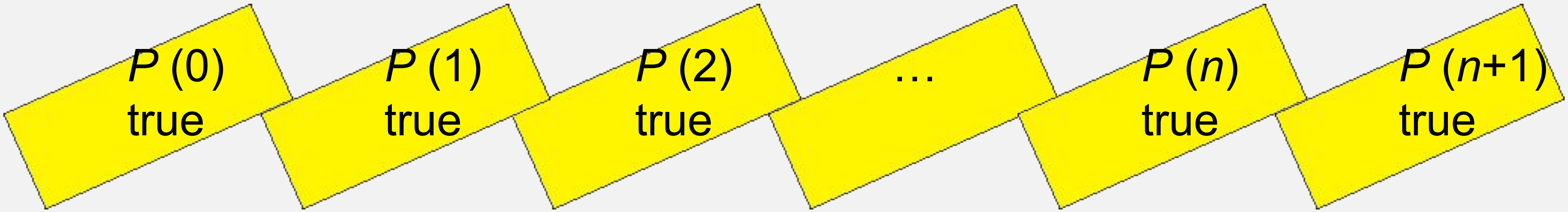
Mathematical Induction

Then can conclude that all the dominos fall!



Mathematical Induction

Then can conclude that all the dominos fall!



Mathematical Induction

- **Basis Step Show P(1) is true**
- Trivial: $1 = 1^2$
- **Inductive Step**

Show if P(n) is true then P(n+1) is true for all n.

- • Suppose P(n) is true,

  that is $1 + 3 + 5 + 7 + \ldots + (2n - 1) = n^2$
- • Show P(n+1):

$$1 + 3 + 5 + 7 + \ldots + (2n - 1) + (2n + 1) = (n+1)^2$$

follows:

- $1 + 3 + 5 + 7 + \ldots + (2n - 1) + (2n + 1) = n^2 + (2n+1)$

$$= (n+1)^2$$

Mathematical Induction

Example: Prove $n^3 - n$ is divisible by 3 for all positive integers.

- P(n): $n^3 - n$ is divisible by 3
- **Basis Step: P(1): $1^3 - 1 = 0$ is divisible by 3 (obvious)**
- **Inductive Step: If P(n) is true then P(n+1) is true for each**
- positive integer.
- Suppose P(n): $n^3 - n$ is divisible by 3 is true.
- Show P(n+1): $(n+1)^3 - (n+1)$ is divisible by 3.

$$(n+1)^3 - (n+1) = n^3 + 3n^2 + 3n + 1 - n - 1$$
$$= (n^3 - n) + 3n^2 + 3n$$
$$= (n^3 - n) + 3(n^2 + n)$$

Divisible by 3

Divisible by 3

Mathematical Induction

Example: Prove than $2^n < n!$ for all $n \geq 4$.

1. Basis Step: $P(4) = 2^4 < 4! = 16 < 24$ (trivial)
2. Inductive Step:

If $P(n)$ is true then $P(n+1)$ is also true

$$2^{n+1} = 2 \,.\, 2^n$$
$$< 2.\, n! \qquad (P(n) \text{ is true})$$
$$< (n+1)n! \qquad (2<n+1)$$
$$= (n+1)!$$

Strong Induction

- **The regular induction:**
  - basis step **P(1) and**
  - inductive step **P(n-1) $\rightarrow$ P(n)**

- **Strong induction uses:**
  - basis step **P(1) and**
  - inductive step **P(1) and P(2) … P(n-2) and P(n-1) $\rightarrow$ P(n)**

- **Example:** Show that a positive integer greater than 1 can be written as a product of primes.

Strong Induction

Example: Show that a positive integer greater than 1 can be written as a product of primes.

1. **Basis step: P(2) is true**
2. **Inductive step: Assume true for P(2), P(3), … P(n)**

Show that P(n+1) is true as well.

2 Cases:

- If $n+1$ is a prime then $P(n+1)$ is trivially true
- If $n+1$ is a composite then it can be written as a product of two integers $(n+1) = a*b$ such that $1 < a, b < n+1$
- From the assumption $P(a)$ and $P(b)$ holds.
- Thus, $n+1$ can be written as a product of primes
- **End of proof**

Strong Induction

- **Example:** Prove that every amount of postage of 12 cents or more can be formed using just 4-cent and 5-cent stamps.
- In this proof, in the basis step we show that $P(12)$, $P(13)$, $P(14)$, and $P(15)$ are true, that is, that postage of 12, 13, 14, or 15 cents can be formed using just 4-cent and 5-cent stamps.

- **Basis step:** We can form postage of
  - 12 cents using three 4-cent stamps,
  - 13 cents using two 4-cent stamps and one 5-cent stamp,
  - 14 cents using one 4-cent stamp, and two 5-cent stamps, and
  - 15 cents using three 5-cent stamps, respectively.

This shows that $P(12)$, $P(13)$, $P(14)$, and $P(15)$ are true. This completes the basis step.

Strong Induction

Example: Prove that every amount of postage of 12 cents or more can be formed using just 4-cent and 5-cent stamps.

- **Inductive step:** The inductive hypothesis is the statement that $P(j)$ is true for $12 \leq j \leq k$, where k is an integer with $k \geq 15$. To complete the inductive step, we assume that we can form postage of j cents, where $12 \leq j \leq k$.
- Using the inductive hypothesis, we can assume that $P(k - 3)$ is true because $k - 3 \geq 12$, that is, we can form postage of $k - 3$ cents using just 4-cent and 5-cent stamps.
- To form postage of $k + 1$ cents, we need only add another 4-cent stamp to the stamps we used to form postage of $k - 3$ cents.
- That is, we have shown that if the inductive hypothesis is true, then $P(k + 1)$ is also true. This completes the inductive step.

Homeworks

Chapter 5, Page 329

Exercises – 3, 4, 5, 7, 12, 13, 15, 16, 17, 18

Thank You