AMAT 363 - Homework II
Instructor: Yunlong Feng

Name: Jahnavi Bonagiri
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PROBLEM #1:

For each of the following statements, determine whether it is true or false. Label "T" if it is true, otherwise label "F".

a. If $X_1, \cdots, X_n$ are independent and are from a distribution with mean μ and variance σ^2, then the sample variance S^2 is an unbiased estimator of the variance σ^2 where

$$S^2 = \frac{1}{n-1}\left(\sum_{i=1}^{n}(X_i - \overline{X})\right).$$ F

b. If $X_1, \cdots, X_n$ are from a distribution with mean μ and variance σ^2, then the sample mean $\overline{X}$ is an unbiased estimator of μ. T

c. If $\hat{\theta}$ is a point estimator of θ with $\mathbb{E}(\hat{\theta}) = \theta$, then $\hat{\theta}$ is an unbiased estimator of θ. T

d. If $X_1, \cdots, X_n$ is a random sample from a normal distribution with mean μ and variance σ^2, then $\overline{X}$ is the MVUE for μ. T

e. The likelihood function tells us how likely the observed sample is as a function of the possible parameter values. T

f. S^2 defined above as a point estimator of the variance σ^2 can also be deduced from maximum likelihood estimation. F

g. Let $\hat{\theta}$ be the maximum likelihood estimator of the parameter θ. Then the maximum likelihood estimator of $\sqrt{\theta}$ is $\sqrt{\hat{\theta}}$. F

h. The point estimator of the population proportion p is the sample proportion $\hat{p} = X/n$ where n is the sample size and X is the number of "successes" in the sample. F

i. The maximum likelihood estimator $\hat{\theta}$ of a population parameter θ is its MVUE. T

j. If $X_1, \cdots, X_n$ is a random sample from a normal distribution with mean μ and variance σ^2, then the maximum likelihood estimator of μ is $\overline{X}$ which is the same as the one obtained from the method of moments. T

SOLUTION:

a) F
b) T
c) T
d) T
e) T
f) F
g) F
h) F

i) T
j) T

PROBLEM #2:

Consider a random sample $X_1, \cdots, X_n$ from the shifted exponential pdf

$$f(x;\lambda,\theta) = \begin{cases} \lambda e^{-\lambda(x-\theta)}, & x \geq \theta, \\ 0, & \text{otherwise.} \end{cases}$$

Obtain the maximum likelihood estimators of θ and λ. Show your work.

SOLUTION:

Example 7.17 Text book

$f(x_1 \dots x_n; \lambda) = (\lambda e^{-\lambda x_1}) \dots (\lambda e^{-\lambda x_n}) = \lambda^n e^{-\lambda \sum x_i}$

$\ln[f(x_1 \dots x_n; \lambda)] = n\ln(\lambda) - \lambda \sum x_i$

$\frac{n}{\lambda} - \sum x_i = 0$ or $\lambda = \frac{n}{\sum x_i} = \frac{1}{\bar{x}} \Rightarrow \boxed{\hat{\lambda} = \frac{1}{\bar{X}}}$

$\underline{\lambda}$

$= \ln L = n \ln \lambda - \lambda \sum_{i=1}^{n} (x_i - \theta)$

$\frac{\partial}{\partial \lambda} \ln L = \frac{n}{\lambda} - \sum x_i$

$\frac{\partial}{\partial \lambda} \ln L = 0$

$= \frac{n}{\lambda} = \sum x_i \rightarrow \lambda = \frac{n}{\sum x_i} = \boxed{\frac{1}{\bar{X}} = \hat{\lambda}}$

$\boxed{\theta}$

$\frac{\partial}{\partial \theta} \ln L = \lambda \Rightarrow 0$

$x \geq \theta$

Thus $\boxed{\hat{\theta} = x_{(1)}}$

PROBLEM #3:

Let X denote the proportion of allotted time that a randomly selected student spends working on a certain aptitude test. Suppose the pdf of X is

$$f(x;\theta) = \begin{cases} (\theta+1)x^{\theta}, & 0 \le x \le 1 \\ 0, & \text{otherwise} \end{cases}$$

where $-1 < \theta$. A random sample of ten students yields data $x_1 = 0.92$, $x_2 = 0.79$, $x_3 = 0.90$, $x_4 = 0.65$, $x_5 = 0.86$, $x_6 = 0.47$, $x_7 = 0.73$, $x_8 = 0.97$, $x_9 = 0.94$, $x_{10} = 0.77$.

- Use the method of moments to obtain an estimator of θ, and then compute the estimate for this data. Show your work. $\theta = .6/.20$
- Obtain the maximum likelihood estimator of θ, and then compute the estimate for the given data. Show your work. $\hat{\theta} = 8.4$

SOLUTION:

$E(x) = \int x f x \, dx$

$E(x) = \int_0^1 x(\theta+1)x^{\theta}$

$= \int_0^1 (\theta+1)x^{\theta+1} \, dx$

$= (\theta+1)\left[\frac{x^{\theta+2}}{\theta+2}\right]_0^1$

$= (\theta+1)\left[\frac{1^{\theta+2}}{\theta+2} - \cancel{\frac{0^{\theta+2}}{\theta+1}}\right]$ 0

$= \frac{\theta+1}{\theta+2} = \bar{X} \Rightarrow \theta+1 = \bar{X}(\theta+2)$

$\theta+1 = \bar{X}\theta + \bar{X}2$

$\theta - \bar{X}\theta = 2\bar{X} - 1$

$\theta(1-\bar{X}) = 2\bar{X} - 1$

Part A $\Rightarrow$ $\boxed{\theta = \frac{2\bar{X}-1}{1-\bar{X}}}$

Part A DATA

$\bar{X} = \frac{\text{Sum}}{\text{total}}$

$\bar{X} = \frac{.92 + .79 + .90 + .65 + .86 + .47 + .73 + .97 + .94 + .77}{10}$

$\bar{X} = .80$

$\theta = \frac{2(.80)-1}{1-.80} = \boxed{\frac{.6}{.20}}$

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B) Example 7.17 Textbook

$n \ln(\theta+1) + \theta \ln(\prod x_i)$

$\frac{\partial}{\partial\theta} = \frac{n}{\theta+1} + \theta \sum \ln x_i = 0$

$\frac{n}{\theta+1} + \theta\sum \ln x_i \Rightarrow \frac{n}{\theta+1} = \theta - \sum \ln x_i$

$\theta + 1 = \frac{-n}{\sum \ln x_i}$

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$\boxed{\hat{\theta} = \frac{-n}{\sum \ln x_i} - 1}$

PART B DATA

$\hat{\theta} = \frac{-10}{-2.4} - 1$

$\hat{\theta} = 3.16$

A

$\boxed{\hat{\theta} = 3.12}$

PROBLEM #4:

Let $X_1, \cdots, X_n$ be a random sample from the probability density function

$$f(x|\theta) = \theta x^{-2}, \quad 0 < \theta \leq x < \infty.$$

Find the MLE of θ and the method of moments estimator of θ, respectively. Show your work.

SOLUTION:

A) $f(x|\theta) = \frac{\theta}{x^2}$

$E(x) = \int_0^\infty x f(x) dx$

$E(x) = \int_0^\infty x f\left(\frac{\theta}{x^2}\right) dx$

$= \theta \int_0^\infty \frac{\cancel{x}}{x^{\cancel{2}}} dx$

$= \theta \int_0^\infty \frac{1}{x} dx$

$= \theta [\log x] \Big|_0^\infty$

$\boxed{E(x) = DNE}$

method of moments doesn't exist

MLE

$L(\theta) = \prod_{i=1}^{n} \frac{\theta}{x_i^2}$

$= \frac{\theta^n}{\prod_{i=1}^{n} x_i^2}$

$\log L(\theta) = n \log \theta - 2 \sum_{i=1}^{n} \log x_i$

$\frac{d}{d\theta} \log L(\theta) = \frac{n}{\theta} > 0$

When not 0

$\theta = \min\{x_i\}$

$\hat{\theta} = \min\{x_i\}$

PROBLEM #5:

Let $X_1, \cdots, X_n$ be a random sample with their density function being one of the following two. If $\theta = 0$, then

$$f(x|\theta) = \begin{cases} 1, & \text{if } 0 < x < 1, \\ 0, & \text{otherwise,} \end{cases}$$

while if $\theta = 1$, then

$$f(x|\theta) = \begin{cases} 1/2\sqrt{x}, & \text{if } 0 < x < 1, \\ 0, & \text{otherwise,} \end{cases}$$

Find the MLE of θ. Show your work.

SOLUTION:

when $\theta = 0$ $\quad L(\theta|x) = \prod_{i=1}^{n} f(x_i;\theta)$

when $\theta = 1$ $\quad L(\theta|x) = \prod_{i=1}^{n} \frac{1}{2\sqrt{x_i}} = \frac{1}{2^n \prod_{i=1}^{n} x_i^{\frac{n}{2}}}$

$$= \frac{\frac{1}{2^n \prod_{i=1}^{n} x_i^{\frac{n}{2}}}}{1} \quad \frac{1}{\left(4^n \prod_{i=1}^{n} x_i^{n}\right)^{1/2}} \Rightarrow \frac{1}{\left(4^n 0^n\right)^{1/2}} \quad 0 = 1/4$$

$$= \frac{1}{(4^n 9^n)^{1/2}} \leq \frac{1}{(4^n 9)^{n/2}} \leq \frac{1}{49} \leq \boxed{\frac{1}{4}}$$

Thus $\quad \hat{\theta}_{MLE} = \begin{cases} 1 & G > 1/4 \\ 0 & G < 1/4 \\ 0 & G = 1/4 \end{cases}$