**AMAT 363 - Homework 1**
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**PROBLEM #1:**

For each of the following statements, label "T" if it is true, otherwise, label "F".

T a. A statistic is a numerical value that can be computed from sample data. T

T b. The random variables $X_1, \cdots, X_n$ are said to form a random sample of size $n$ if they are i.i.d. T

T c. Let $X_1, \cdots, X_n$ be random variables from a distribution with variance $\sigma^2$. Then $V(\overline{X}) = \sigma^2/n$. T

F d. Let $X_1, \cdots, X_n$ be a random sample. Then, for any $n$, $\overline{X}$ is normally distributed. F

T e. The Central Limit Theorem can be used if the sample size $n > 30$. T

T f. For any $X_1, \cdots, X_n$, $V(a_1X_1 + \cdots + a_nX_n) = \sum_{i=1}^{n}\sum_{j=1}^{n} a_ia_j\text{cov}(X_i, X_j)$. T

T g. Let $X_1$ and $X_2$ be two random variables. We have $\mathbb{E}(X_1 - X_2) = \mathbb{E}(X_1) - \mathbb{E}(X_2)$. T

F h. Let $X$ and $Y$ be two random variables. Then we have $M_{X-Y}(t) = M_X(t) \cdot M_Y(-t)$. F

T i. If $X_1 \sim \chi_1^2$, $X_2 \sim \chi_2^2$, and they are independent, then $X_1 + X_2 \sim \chi_3^2$. T

T j. If $X_1, \cdots, X_n$ are a random sample from a normal distribution, then we have

$$(n-1)S^2/\sigma^2 \sim \chi_{n-1}^2.$$ T

**SOLUTION:**

PROBLEM #2:

Let $X$ be the number of packages being mailed by a randomly selected customer at a shipping facility. Suppose the distribution of $X$ is as follows:

| $x$ | 1 | 2 | 3 | 4 |
|---|---|---|---|---|
| $p(x)$ | .4 | .3 | .2 | .1 |

a. Consider a random sample of size $n = 2$ (two customers), and let $\overline{X}$ be the sample mean number of packages shipped. Obtain the probability distribution of $\overline{X}$.

b. Refer to part (a) and calculate $P(\overline{X} \le 2.5)$. $= \boxed{.85}$ $\boxed{.24}$

c. If a random sample of size $n = 4$ is selected, what is $P(\overline{X} \le 1.5)$? [Hint: You should not have to list all possible outcomes, only those for which $\overline{X} \le 1.5$.]

SOLUTION:

A)

| 1 | 1.5 | 2 | 2.5 | 3 | 3.5 | 4 |
|---|---|---|---|---|---|---|
| (1,1) (.4)(.4) = .16 | (1,2) (.4)(.3) = .12 | (1,3) (.4)(.2) = .08 | (1,4) (.4)(.1) = .04 | (2,4) (.3)(.1) = .03 | (3,4) (.2)(.1) = .02 | (4,4) (.1)(.1) = .01 |
| | | (2,2) (.3)(.3) = .09 | (2,3) (.3)(.2) = .06 | (3,3) (.2)(.2) = .04 | | |

$P(\bar{x}=1) = (.4)(.4) = .16$

$P(\bar{x} = 1.5) = (.12)(.12) = .24$

$P(\bar{x} = 2) = (.08) + .08 + .09 = .25$

$P(\bar{x} = 2.5) = .04 + .04 + .06 + .06 = .20$

$P(\bar{x} = 3) = .03 + .03 + .04 = .10$

$P(\bar{x} = 3.5) = .02 + .02 = .04$

$P(\bar{x} = 4) = (.1)(.1) = .01$

Answer

| $\bar{X}$ | 1 | 1.5 | 2 | 2.5 | 3 | 3.5 | 4 |
|---|---|---|---|---|---|---|---|
| $P(\bar{X})$ | .16 | .24 | .25 | .20 | .10 | .04 | .01 |

B) $P(\bar{X} \le 2.5) = .16 + .24 + .25 + .20 = \boxed{.85} = \boxed{85\%}$

c)

| 1 → 1 | 1.25 → 4 | 1.5 → $(.4)^2(.3)^2 = .0144$ |
|---|---|---|
| (1,1,1,1) (.4)(.4)(.4)(.4) = .0256 | (1,1,1,2) (.4)(.4)(.4)(.3) = .0192 | (1,1,2,2) (2,2,1,1) (2,1,2,1) (1,2,1,2) (2,1,1,2) 2 (1,2,2,1) |
| | (1,1,2,1) (1,2,1,1) (2,1,1,1) | (1,1,1,3) (3,1,1,1) (1,1,3,1) (1,3,1,1) $= (.4)^3(.2) = .0128$ |

$P(\bar{x} \le 1.5) = .0256 + 4(.0192) + 6(.0144) + 4(.0128)$

$\boxed{= .024} = \boxed{24\%}$

PROBLEM #3:

Five automobiles of the same type are to be driven on a 300-mile trip. The first two will use an economy brand of gasoline, and the other three will use a name brand. Let $X_1$, $X_2$, $X_3$, $X_4$, and $X_5$ be the observed fuel efficiencies (mpg) for the five cars. Suppose these variables are independent and normally distributed with $\mu_1 = \mu_2 = 20$, $\mu_3 = \mu_4 = \mu_5 = 21$, and $\sigma^2 = 4$ for the economy brand and 3.5 for the name brand. Define an random variable $Y$ by

$$Y = \frac{X_1 + X_2}{2} - \frac{X_3 + X_4 + X_5}{3}$$

so that $Y$ is a measure of the difference in efficiency between economy gas and name-brand gas. Compute $P(0 \leq Y)$ and $P(-1 \leq Y \leq 1)$.

SOLUTION:

$$y = \frac{X_1 + X_2}{2} - \frac{X_3 + X_4 + X_5}{3}$$

$$y = \frac{(20+20)}{2} - \frac{(21+21+21)}{3} \Rightarrow 20 - 21 = \boxed{-1}$$

$$SD = \sqrt{\left(\frac{4+4}{2^2}\right) + \frac{(3.5)+(3.5)+(3.5)}{3^2}} \approx \boxed{1.77}$$

$$P(0 \leq Y) = P\frac{(0-(-1))}{1.77} = .56$$

used table A.3 in textbook

$$1 - P(.56 < z) = 1 - \Phi(.56)$$

$$= 1 - .712$$

$$= \boxed{.287}$$

$$P(-1 \leq Y \leq 1) = P\frac{(-1-(-1))}{1.77} \leq z \leq \frac{(1-(-1))}{1.77}$$

$$= P(0 \leq z \leq 1.13) = \Phi(1.12) - \Phi(0)$$

$$= .87 - .5 = \boxed{.37}$$

PROBLEM #4:

The inside diameter of a randomly selected piston ring is a random variable with mean value 12 cm and standard deviation 0.04 cm.

a. If $\overline{X}$ is the sample mean diameter for a random sample of $n = 16$ rings, where is the sampling distribution of $\overline{X}$ centered, and what is the standard deviation of the $\overline{X}$ distribution?

b. Answer the questions posed in part (a) for a sample size of $n = 64$ rings.

c. For which of the two random samples, the one of part (a) or the one of part (b), is $\overline{X}$ more likely to be within 0.01 cm of 12 cm? Explain your reasoning.

SOLUTION:

SD $\frac{.04}{\sqrt{16}} = \boxed{.01}$

a) It is centered at 12 cm with SD of .01

b) SD $\frac{.04}{\sqrt{64}} = \boxed{.005}$

Centered at 12 cm with SD of .005

c) $\boxed{\text{Part b}}$ is more likely to be within .01 cm because .005 is closer to .04 cm. This is possible due to the larger sample size of n = 64 compared to n = 16.

PROBLEM #5:

In probability theory and statistics, the moment-generating function of a real-valued random variable is an alternative specification of its probability distribution. Thus, it provides the basis of an alternative route to analytical results compared with working directly with probability density functions or cumulative distribution functions. Answer the following questions related to the moment generating functions.

a. Let X have the pdf $f(x) = \frac{1}{2}e^{-|x|}$, $-\infty < x < +\infty$, find out the moment generating function of X. Show your work.

b. Suppose $X$ and $Y$ are independent Poisson random variables, where $X$ has mean $\lambda$ and $Y$ has mean $\nu$. Find out the moment generating function of $X+Y$ and its mean value. Show your work.

SOLUTION:

$$M_X(t) = E(e^{tX}) = \int_{-\infty}^{\infty} f(x)\,dx$$

$$= \int_{-\infty}^{\infty} e^{tx} f(x)\,dx \leftarrow \text{plug in for } f(x)$$

$$= \int_{-\infty}^{\infty} e^{tx} \cdot \frac{1}{2} e^{-|x|}\,dx$$

$$= \frac{1}{2}\int_{-\infty}^{\infty} e^{tx} \cdot e^{-|x|}\,dx$$

$$= \frac{1}{2}\int_{-\infty}^{0} e^{tx+x}\,dx + \int_{0}^{\infty} e^{tx-x}\,dx$$

$u = (t+1)x$ $\quad$ $u = (t-1)x$

$du = t+1\,dx$ $\quad$ $du = t-1\,dx$

$dx = \frac{du}{t+1}$ $\quad$ $dx = \frac{du}{t-1}$

$$= \frac{1}{2}\left[\int_{-\infty}^{0} e^{u}\frac{1}{t+1} + \int_{0}^{\infty} e^{u}\frac{1}{t-1}\right]$$

$$= \frac{1}{2}\left[\frac{e^{(t+1)x}}{t+1}\Big|_{-\infty}^{0} + \frac{e^{(t-1)x}}{t-1}\Big|_{0}^{-\infty}\right]$$

$$\boxed{= \frac{1}{2}\left[\frac{e^{(t+1)x}}{(t+1)}\Big|_{-\infty}^{0} + \frac{e^{(t-1)x}}{t-1}\Big|_{0}^{-\infty}\right]}$$

b)

$$M_{(X+Y)}T = m_X(t)\cdot m_Y(t) = e^{\lambda(e^t-1)}e^{\nu(e^t-1)} = e^{(\lambda+\nu)(e^t-1)}$$

$$\boxed{= \lambda + \nu}$$