STATS 363 MIDTERM Jahnavi Bonagiri

1)

A)

1	1.5	2	2.5	3	3.5	4
(1,1) $(.3)^2$ =.09	(1,2) (.3)(.4) =.12	(1,3) (.3)(.2) =.06	(1,4) (.3)(.1)=.03	(2,4) (.4)(.1)=.04	(3,4) (.2)(.1) =.02	(4,4) $(.1)^2$ =.01
		(2,2) $(.4)^2$=.16	(2,3) (.4)(.2)=.08	(3,3) $(.2)^2$=.04		

$P(\bar{X}=1) = (.3)^2 = .09$ $P(\bar{X}=1.5) = (.12)^2 = .24$ $P(\bar{X}=2) = .06+.06+.16 = .28$

$P(\bar{X}=2.5) = .03+.03+.08+.08 = .22$

$P(\bar{X}=3) = .04+.04+.04 = .12$

$P(\bar{X}=3.5) = .02+.02 = .04$

$P(\bar{X}=4) = (.1)^2 = .01$

$\bar{X}$	1	1.5	2	2.5	3	3.5	4
P(x)	.09	.24	.28	.22	.12	.04	.01

$P(\bar{X} < 2.5) = .09 + .24 + .28 + .22 = .83$

$\boxed{= 83\%}$

B) mean $= \frac{21+21+21}{3} + \frac{(21+21)}{2} = 0$ $SD = \sqrt{\frac{(4+4+4)}{3^2} + \frac{(3+3)}{(2)^2}} = 1.68$

$P(-1 \le y \le 1) = P\frac{(-1-0)}{1.68} \le Z \le P\frac{(1-0)}{1.68} \Rightarrow P(-.59 \le Z \le .59)$ ← use table A.3

$= \Phi(.59) - \Phi(-.59) = .7224 - .2776$

$\boxed{= .4448}$

C)

$X = \frac{T_0}{n}$ $(\bar{x})n = T_0$

$\frac{dx}{dy} = \frac{d\bar{x}}{T_0} = \frac{d(\frac{T_0}{n})}{d(T_0)} = \frac{1}{n} \Rightarrow \boxed{f_{\bar{x}}(t) = \frac{1}{n} f_{\bar{x}}(t)}$

D) $X = f_X(x)$ $Y = f_Y(y)$ $Z = X - Y$

$= \int f_X(x) f_Y(y)\, dy\, dx$ $y = x - z$

$= \int_{-\infty}^{\infty} f_X(x) f_Y(x-z)\, dx \Rightarrow$ $x = z + y$

$\boxed{f(z) = \int_{-\infty}^{\infty} f_Y(y) f_X(z+y)\, dy}$

②

A) $M_X(t) = Ee^{xt} = \int_{-\infty}^{\infty} e^{xt} f(x)\,dx$

$= \int_{-\infty}^{\infty} e^{xt} \cdot \frac{1}{2} e^{|x|}\,dx$

$= \frac{1}{2}\int_{-\infty}^{\infty} e^{xt} \cdot e^{|x|}\,dx$

$= \frac{1}{2}\int_{0}^{\infty} e^{xt} \cdot e^{x}\,dx +$

$= \frac{1}{2}\int_{-\infty}^{0} e^{xt} \cdot e^{-x}\,dx$

$\int_{-\infty}^{\infty} e^{xt} \cdot e^{-|x|}\,dx$

$= \frac{1}{2}\int_0^{\infty} e^{xt} \cdot e^{-x}\,dx + \frac{1}{2}\int_{-\infty}^{0} e^{xt} e^{x}\,dx.$

I $\qquad$ II

I $\int_0^{\infty} e^{x(t-1)}\,dx = \frac{1}{t-1} e^{x(t-1)}\Big|_0^{\infty}$

II $\int_{-\infty}^{0} e^{x(t+1)}\,dx = \frac{1}{t+1} e^{x(t+1)}\Big|_{\infty}^{0}$

$= \frac{1}{2}\left(\frac{1}{t-1}\right)(-1) + \frac{1}{2}\left(\frac{1}{t+1}\right)(1)$

$= \boxed{\frac{1}{1-t^2} \quad |t| < 1}$

B) $M_X(t)$. Let $Y = aX + b$

$ax + b = Ee^{(ax+b)t}$

$= E(e^{axt} \cdot e^{bt}) \Rightarrow E(e^{axt}) \cdot (e^{bt})$ → textbook 3.4

$\boxed{M_X(t) = e^{bt} M_X(at)}$

c) $f(x) = \{ 2 \quad 0 < x < \frac{1}{2}$ $\qquad M_X(t) = E(e^{tX})$

$F(x) = \int_0^{1/2} e^{tX} \cdot 2\,dx$

$= 2\left[\frac{e^{tx}}{t}\right]\Big|_0^{1/2}$

$M_X(t) = \left\{ \frac{2}{t}(e^{t/2} - 1) \atop 1 \right\}$

$E(X) = M'_X(0) = \left(\frac{2}{t}\left(e^{t/2} - 1\right)\right)\Big|_{t=0}$

$= .25$

$E(X^2) = M''_X(0) = 1/12$

$V(X) = \sigma^2 = E(X^2) - [E(X)]^2 = \left(\frac{1}{12}\right) - \left(\frac{1}{4}\right)^2 = \boxed{\frac{1}{48}}$

D) According to the textbook, $f(x) = \frac{1}{\pi\, 1+x^2}$ is a cauchy distribution and it fails to have a moment generating function

(3)

A) $P(X=2)\,P(X=0)\,P(X=0)\,P(X=0)\,P(X=2)\,P(X=3)\,P(X=1)\,P(X=1)\,P(X=2)\,P(X=3)$

$$L(\theta)=\prod_{i=1}^{n} P(x|\theta) = \left(\frac{\theta}{3}\right)^3 \cdot \left(\frac{2\theta}{3}\right)^2 \cdot \left(\frac{2(1-\theta)}{3}\right)^3 \cdot \left(\frac{1-\theta}{3}\right)^2$$

$$= \frac{\theta^3}{27} \cdot \frac{4\theta^2}{9} \cdot \frac{8(1-\theta)^3}{27} \cdot \frac{(1-\theta)^2}{9}$$

$$\boxed{= \frac{32(1-\theta)^5\theta^5}{(27)^2(9)^2}} \Rightarrow \frac{d}{dx}(\theta) = 0 - \frac{5}{1-\theta} + \frac{5}{\theta} = \frac{5}{\theta} - \frac{5}{1-\theta}$$

$$\theta = 1-\theta \qquad \boxed{\theta = 1/2}$$

3B) $\boxed{7.17}$ textbook

$f(x; x_0, r) = r x_0^r x^{-r-1} \quad x \geq x_0,\ r>1$ let $x_0>0$

The ln(likelihood is) $= \ln[f(x_1 \ldots x_n; r)] = n\ln(r) - r\sum x_i$

$$\ln = n\log(r) + nr\log(x_0) - (r+1)\sum_{i=1}^{n}\log x_i$$

$$\frac{\partial}{\partial r} = \frac{n}{r} + n\log x_0 - \sum_{i=1}^{n}\log x_i \Rightarrow 0$$

$$\frac{n}{r} = \sum_{i=1}^{n}\log x_i - n\log x_0$$

$$= \frac{1}{r} = \frac{1}{n}\sum_{i=1}^{n}\log x_i - \log x_0$$

$$r = \frac{1}{\frac{1}{n}\sum_{i=1}^{n}\log x_i - \log x_0}$$

$$\boxed{MLE = \hat{r} = \frac{1}{\frac{1}{n}\sum_{i=1}^{n}\log x_i - \log x_0}}$$

3C) $P_\theta(X=x) = \theta^x(1-\theta)^{1-x}, \quad x=0, \text{ or } 1, \quad 0 \leq \theta \leq 0.5$

$$f(x_1, x_2 \ldots x_n; \theta) = \theta^{x_1+x_2\ldots x_n}(1-p)^{n-(x_1+x_2\ldots x_n)}$$

$$\ln[f(x_1, x_2 \ldots x_n; \theta)] = \sum x_i \ln(\theta) + (n - \sum x_i)\ln(1-\theta)$$

$$\frac{\partial}{\partial\theta}\ln[f(x_1, x_2 \ldots x_n; p)] = \frac{\sum x_i}{\theta} - \frac{n-\sum x_i}{1-p} = \frac{\sum x_i - np}{p(1-p)}$$

$$\frac{n\bar{x}}{\theta} - \frac{n - n\bar{x}}{1-\theta} = 0$$

$$\frac{\bar{x}}{\theta} = \frac{1-\bar{x}}{1-\theta}$$

$$\bar{x}(1-\theta) = (1-\bar{x})(\theta)$$

$$\bar{x} - \cancel{\bar{x}\theta} = \theta - \cancel{\bar{x}\theta}$$

$$\boxed{\bar{x}=\theta}$$

$$\boxed{\hat{\theta} = \bar{x} = \frac{\sum x_i}{n}} \Rightarrow \text{MLE (maximum likelehood)}$$

$\boxed{\text{Mean Squared Error}}$ $\hat{\theta}$ is $E[(\hat{\theta}-\theta^2)]$

$E(X) = np \quad V(X) = np(1-p)$

$E(\hat{p}) = E\left(\frac{X}{n}\right) = \frac{1}{n}E(X) = \frac{1}{n}np = p$

$$= E\left[\frac{X^2}{n^2} - \frac{2pX}{n} + p^2\right] = \frac{1}{n^2}E[X^2] - \frac{2p}{n}E[X] + p^2$$

$$= \frac{np}{n^2}((n-1)p+1) - \frac{2}{n}np^2 + p^2 \Rightarrow \boxed{\frac{p(1-p)}{n}} = \boxed{\frac{\theta(1-\theta)}{n}}$$

Textbook 7.4

Answer

(4)

A) $f(x|\theta) = \frac{\theta}{x^2}$ $0 < \theta \leq x < \infty$

$E(x) = \int_0^\infty x f x \, dx$

$E(x) = \int_0^\infty x f\left(\frac{\theta}{x^2}\right) dx$

$= \theta \int_0^\infty \frac{x}{x^2} dx$

$= \theta \int_0^\infty \frac{1}{x} dx$

$= \theta [\log x] \Big|_0^\infty$

$\boxed{E(x) = DNE}$ → no method of moments exists

7.4 textbook

B) $E(x) = \theta$ $V(x) = \theta(1-\theta)$

$MSE = V(\hat{\theta}) + E(\hat{\theta} - \theta)]^2$

variance of estimator + $(\text{bias})^2$

$\frac{\partial}{\partial\theta} \log L\left(\frac{\theta}{x}\right) = \frac{\sum_{i=1}^{n} x_i}{\theta} - \frac{\left(n - \sum_{i=1}^{n} x_i\right)}{1-\theta}$

$\frac{\partial}{\partial\theta} \log L\left(\frac{\theta}{x}\right) = 0$

$\frac{\sum_{i=1}^{n} x_i}{\theta} - \frac{(n - n\bar{x})}{1-\theta} = 0$

$\frac{\bar{x}}{\theta} - \frac{1-\bar{x}}{1-\theta} = 0$

$(1-\theta)\bar{x} - (1-\bar{x})\theta = 0$

$\bar{x} - \cancel{\theta\bar{x}} - \theta + \cancel{\theta\bar{x}} = 0 = 0$

$\bar{x} = \hat{\theta}$

$\bar{x} = \frac{1}{n} \sum_{i=1}^{n} x$

$\underline{\text{Bias}(\hat{\theta}) = E(\bar{x}) \cdot \theta}$

$E\left(\frac{X}{n}\right) - \theta = \frac{1}{n} E(x)$

$= \frac{1}{n} np = p$

$\frac{1}{n} \sum_{i=1}^{n} E(x_i) - \theta$

$\theta - \theta = 0$ $\boxed{(p - p = 0)}$

$\underline{Var(\hat{\theta}) = var\left(\frac{X}{n}\right) = \frac{1}{n^2} V(x)}$

$\frac{1}{n^2} \sum_{i=1}^{n} var(x_i)$

$\frac{\cancel{n}\theta(1-\theta)}{\cancel{n^2}} = \boxed{\frac{\theta(1-\theta)}{n}}$

textbook 7.3

$\hat{\theta} = V(\hat{\theta}) + (\text{bias})^2 \Rightarrow \boxed{\frac{\theta(1-\theta)}{n} + 0}$

$\hat{\theta} = \theta$ $\hat{\theta} - \theta$ error of estimation

c) 3c is preffered because it gives a more concise, accurate answer with small estimation errors compared to 4c, $\hat{\theta} = \bar{X}$ is more precise than a method of moment that deals with bias and variance