AMAT 363 - Final Exam
Instructor: Yunlong Feng

Name: Jahnavi Bonagiri
Due Time: 10:00 a.m., Saturday, July 16, 2022

Problem #1:

(20 points) For each of the following statements, determine whether it is true or false. If it is true, please label "T". Otherwise, please label "F" **and explain why.**

1. A statistic is a numerical value that can be computed from sample data.
2. Let X be a random variable with the probability density function $p(x) = ca/(a+x^2)$ where $0 < c, a < 1$. The moment generating function of X is well defined.
3. Point estimators constructed via maximum likelihood estimation are not MVUE.
4. For a distribution with finite mean μ, the point estimator for μ is $\overline{X}$.
5. For a constructed 95% confidence interval for the mean, it can be interpreted as follows: if we take large random samples over and over again from the same population, then at least 95% of the resulting intervals will cover the sample mean.
6. Assuming that we are given a random sample $X_1, \cdots, X_n$ and we are interested in constructing a 90% confidence interval for the population mean, it is necessary that the distribution of the variable of interest follows a normal curve or a t-curve.
7. Confidence intervals are constructed as interval estimators for population mean, population proportion, population variance, or population standard deviation.
8. The null hypothesis is the claim that is true while the alternative hypothesis is the assertion that is contradictory to the null hypothesis. They are two competing hypotheses.
9. For a hypothesis test, we have two results: rejecting H_0 and reject H_a.
10. Let T be a random variable that has a t-distribution and $t_{\alpha,\nu}$ be a t critical value. Then we have $\Pr(T \leq -t_{\alpha,\nu}) = \alpha$.

we can't claim its a numeric value

1) False, a statistic is a quantity whose value is computed from data
2) False because possible set values of X are not given (textbook, 122)
3) True (Textbook 7.2 pg 363)
4) True (Textbook 7.2)
5) True
6) True textbook pg 383 (It is 1 of the CI properties)
7) True Textbook 382

8) True Textbook page 426
9) False because you can fail to reject Ho There are multiple ways (Textbook 426)
10) True Textbook pg 402

Problem #2:

(10 points) Answer the following questions on moment generating functions.

1. Let X be a continuous random variable with density

$$f_X(x) = \frac{1}{2}\exp(-|x-\mu|),$$

where $-\infty < x < \infty$ and μ is a constant. Find the moment generating function for X, and calculate the mean value of X by using the obtained moment generating function.

2. Let X be a discrete random variable with the probability mass function

$$P(X = k) = p(1-p)^k,$$

where $k = 0, 1, \cdots, +\infty$. Compute the moment generating function for X.

2 1)

$$M_x(t) = E(e^{tx}) = \int e^{tx} f_x dx$$

$$= \int_{-\infty}^{\infty} e^{tx} \frac{1}{2} e^{-|x-\mu|} dx$$

$$= \frac{1}{2}\int_{-\infty}^{\infty} e^{-|x-\mu|} e^{tx} dx$$

$\boxed{(x-\mu) = y \quad x = y+\mu \quad dx = dy}$

$$M_x(t) = \frac{1}{2}\int_{-\infty}^{\infty} e^{t(y+\mu)} e^{-|y|} dx$$

$$= \frac{e^{\mu t}}{2}\int_{-\infty}^{\infty} e^{ty-|y|} dy$$

$$\frac{e^{\mu t}}{2}\left[\int_{-\infty}^{0} e^{y(1+t)} dy + \int_{0}^{\infty} e^{-y(1-t)} dy\right]$$

$e^{-\infty} = 0$

$e^{0} = 1$

$$\frac{e^{\mu t}}{2}\left[\left(\frac{1}{1+t} - 0\right) + \left(0 - \frac{1}{-(1-t)}\right)\right]$$

$$M_x(t) \Rightarrow \boxed{= \frac{e^{\mu t}}{2}\left[\frac{1}{1+t} + \frac{1}{1-t}\right] ; |t| < 1}$$

Calculate Mean

$$M_x(t) = e^{\mu t}(1-t^2)^{-1}$$

$$m_x'(t) \; \mu e^{\mu t}(1-t^2)^{-1} + e^{\mu t}[-(1-t^2)^{-2}](-2t) \Rightarrow m'(0) = \mu e^0 (1-0)^{-1} + e^0[-(1-0)^{-2}]$$

(-20)

$$= \mu + 0 = \mu$$

$\boxed{E(X) \; \mu}$

2b) $P(X=K) = p(1-p)^K$ (Textbook 3.97)

$M_X(t) = E(e^{tX}) = \sum_{x \in D} e^{tx} p(x)$

$M_X(t) = \sum_{k=0} e^{tx} (1-p)^k p$

$= M_X(t) = p[e^0 (1-p)^0 + e^t (1-p) \ldots]$

$+1$

$$\boxed{= M_X(t) = p\left[\frac{1}{1-(1-p)e^t}\right]}$$

Problem #3:

(20 points) Let $X_1, \cdots, X_n$ be a random sample from a distribution with density

$$f_\theta(x) = \frac{2\theta^2}{x^3} \text{ for } x > \theta \text{ and } \theta > 0.$$

Find the estimator for θ by using the method of moments.

If the probability density function is given as

$$f_\theta(x) = \theta^2 x e^{-\theta x} \text{ for } x \in [0, \infty) \quad \text{and} \quad \theta > 0.$$

Find the maximum likelihood estimator for θ.

3)

$f_\theta(x) = \frac{2\theta^2}{x^3}$ for $x > \theta$ and $\theta > 0$.

$E(x) = \int x f x \, dx$

1) $E(X) = \int_\theta^\infty \frac{2\theta^2}{x^3} dx$

$2\theta^2 \int_\theta^\infty \frac{dx}{x^3} \rightarrow 2\theta^2 \left. \frac{-1}{x} \right|_\theta^\infty \rightarrow 2\theta^2 \cdot \frac{1}{\theta} = 2\theta$

$\frac{2\theta}{2} = \frac{\bar{X}}{2} = \boxed{\theta = \frac{\bar{X}}{2}}$

2) $f\theta(x) = \theta^2 x e^{-\theta x}$ $x \in [0, \infty)$ $\theta > 0$ $\quad u = \frac{2}{\theta}$ $\quad \sigma^2 = \frac{2}{\theta^2}$

$\theta^{2n} \left(\prod_{i=1} x_i \right) \exp\left(-\theta \sum_{i=1}^{n} x_i \right)$

$\ln(f(x_1 \ldots x_n|\theta)) = 2n \ln(\theta) + \sum_{i=1}^{n} \log(x_i) - \theta \sum_{i=1}^{n} x_i$

$\frac{d}{d\theta} \Rightarrow \frac{2n}{\theta} - \sum_{i=1}^{n} x_i = 0$

$\boxed{\theta = \frac{2n}{\sum_{i=1}^{n} x_i}}$ MLE

Problem #4:

(20 points) Let $X_1, \cdots, X_n$ be drawn independently from a distribution with density

1. $f(x|\theta) = 2/\beta, \theta - 0.25\beta < x < \theta + 0.25\beta$, where β is a positive constant.
2. $f(x|\theta) = 2x/\theta^2, 0 < x < \theta, \theta > 0$.
3. $f(x|\theta) = e^{\theta - x}$ for $x > \theta$.

For each of the above three cases, construct a 95% confidence interval for θ.

4)

1.) $f(x|\theta) = \frac{2}{\beta} \quad \theta - .25\beta < x < \theta + .25\beta$

$$F(t) = \int_{\theta - .25\beta}^{\theta + x} \frac{2}{\beta}\, dx$$

$$= \frac{2}{\beta}[x] \Big|_{\theta - 0.25\beta}^{\theta + x} = \frac{2}{\beta}[\cancel{\theta} + x - \cancel{\theta} + .25\beta]$$

$$= \frac{2}{\beta}[x + .25\beta]$$

PDF of T: $f(t) = \frac{d}{dx} = \frac{2}{\beta}[x + .25\beta]$

$= \frac{2}{\beta}$

95% so $\frac{.05}{2} = .025$

$a = -.25\beta + .025$

$b = .25\beta - .025$

$$\boxed{\text{Thus} \quad CI = (x - .25\beta + .025,\ x + .25\beta - .025)}$$

4)

a) $f(x|\theta) = \frac{2x}{\theta^2} \quad 0 < x < \theta, \theta > 0$

PDF of T

$f(x) = \frac{2x}{\theta^2} \quad F(x) = \int_0^x \frac{2t}{\theta^2}\,dt = \frac{1}{\theta^2} \cdot t^2 \Big]_0^x$

$= \frac{x^2}{\theta^2}$

$f_T(x) = n\,(F(x))^{n-1} f(x) = n \cdot \left(\frac{x^2}{\theta^2}\right)^{n-1} \cdot \frac{2x}{\theta^2}$

$= \frac{n \cdot x^{2n-2} \cdot 2x}{\theta^{2n-2} \cdot \theta^2} = \frac{2n \cdot x^{2n-1}}{\theta^{2n}}$

$\Rightarrow f_T(x) = 2n \cdot \frac{x^{2n-1}}{\theta^{2n}} \quad 0 < x < \theta$

$y = \frac{T}{\theta}$

$\Rightarrow f_y(y) = 2n y^{2n-1} \longrightarrow P(a \le y \le b) = 1 - \alpha$

$= \int_a^b 2n y^{2n-1}\,dy = 1 - \alpha$

$= y^{2n} \Big]_a^b = 1 - \alpha = b^{2n} - a^{2n} = 1 - \alpha$

$a \le \frac{T}{\theta} \le b \Rightarrow \begin{cases} \theta \ge \frac{T}{b} \\ \theta \le \frac{T}{a} \end{cases} \Rightarrow \left[\frac{T}{b}, \frac{T}{a}\right]$

for 95% $= P\left(\frac{t}{b} \le \theta \frac{t}{a}\right) = 1 - \alpha = .95$

4)

3) $f(x|\theta) = e^{\theta - x}$ for $x > 0$

pdf

$F(x) = \int_0^x e^{-(t-\theta)} dx$

$= (e^{-(t-\theta)}) \Big|_0^x$

$= 1 - e^{-(x-\theta)}$

$f(x_{1n}) = n\exp[-n(x-\theta)]$

$= \int_0^{t/2} \exp(-y)\,dy$

$x = \frac{t}{2n}$

$= [-e^{-y}]_0^{t/2}$

$y = \frac{t}{2}$

$= [-e^{-t/2} + e^{-0}]$

$= 1 - e^{-t/2}$

$\frac{d}{dt}(1 - e^{-t/2})$

$\boxed{= 1/2\, e^{t/2}}$

$P(a \le T \le b) = 1 - \alpha$

$P\left(\frac{a}{2n} - x_1 \le -\theta \le \frac{b}{2n} - x_1\right)$

$P\left(x_{(1)} - \frac{b}{2n} \le \theta \le x_{(1)} - \frac{a}{2n}\right) = .25$

$\boxed{CI\ \left(x_{(1)} - \frac{b}{2n},\ x_{(1)} - \frac{a}{2n}\right)}$

5)

MLE

① $L(\mu) = \prod_{i=1}^{u} f_y(y_i)$

$= \prod_{i=1}^{u} e^{y_i - u}$

$= e^{\sum_{i=1}^{u}(y-u)}$

$\boxed{= \hat{u} = y(n)}$

MLE for u

moments estimator

② $E(y) = \int_u^\infty y f_y \, dy$

$= \int_u^\infty y e^{-(y-u)} dy$

$t = y - u \qquad dy = dt$

$= \int_u^\infty (t+u) e^{-t} dt$

$= \int_0^\infty t e^{-t} dt + u \int_0^\infty e^{-t} dt$

$= u+1 \qquad \cancel{u[e^{-t}]_0^\infty}$

$y = u+1$

$\boxed{\tilde{u} = \bar{y} - 1}$

---

③ MSE $= V(\hat{\theta}) + E[(\hat{\theta} - \theta)]^2$ → 7.4 Textbook

variance of estimator + bias

MLE is preffered because it gives a more concise accurate answer with small estimation errors compared to moments estimator

$\boxed{\text{MLE is preffered}}$ Proved below

$E(y^2) = \int_u^\infty y^2 f_y(y)\, dy \rightarrow \int_u^\infty y^2 e^{-(y-u)} dy$

$E(y^2) = \int_0^\infty (t+u)^2 e^{-t} dt$ foil

$= \int_0^\infty (t^2 + u^2 + 2ut) e^{-t} dt \rightarrow \int_0^\infty t^2 e^{-t} dt + u^2 \int_0^\infty e^{-t} dt + 2u \int_0^\infty t e^{-t} dt$

$\boxed{2 + u^2 + 2u}$

$\text{var}(y) = E(y^2) - [E(y)^2]$

$= (2 + u^2 + 2u) - (u+1)^2$

$= 2 + \cancel{u^2} + \cancel{2u} - \cancel{u^2} - 1 - \cancel{2u}$

$= 2 - 1 = \boxed{1}$

$E(\tilde{u}) = E(\bar{y} - 1)$

$= E\left(\frac{1}{n} \sum_{i=1}^{n} y_i - 1\right) \Rightarrow \frac{1}{n} \sum_{i=1}^{n} E(y_i) - 1 = \frac{n(u+1)}{n} + 1 = \boxed{u}$

5

3) continued

$\text{bias}(\tilde{u}) = E(\tilde{u}) - \mu = \boxed{0}$

$\text{var}(\tilde{u}) = \text{var}(\bar{y}-1) = \frac{1}{n}\sum_{i=1}^{n} y_i$

$= \frac{\cancel{n}}{n^2}$

$\boxed{= \frac{1}{n}}$

$MSE = \text{var} + \text{bias}^2$

$= \frac{1}{n} + 0^2 = \boxed{\frac{1}{n}}$

Pdf of $y(n)$

$= n[1-1+e^{-(y-u)}]^{n-1} e^{-(y-u)}$

$= n\, e^{-(n-1)(y-u)-(y-\mu)}$

$= n e^{-n(y-\mu)}$

$E[y_{(n)}] = n\int_{\mu}^{\infty} y\, e^{-n(y-u)}\, dy$

$= n\left(\int_0^{\infty} t e^{-nt} dt + \mu\int_0^{\infty} e^{-nt} dt\right)$

$= \frac{1}{n} + \mu$

$E(y(n))^2 = \int_{\mu}^{\infty} y^2 f_y(y)\, dy$

$= n\int_{\mu}^{\infty} y^2 e^{-n(y-u)}\, dy \qquad t = y-\mu$

$= n\int_{\mu}^{\infty} (t+u)^2 e^{-nt}\, dt$

$= n\int_{\mu}^{\infty} t^2 e^{-nt} dt + \mu^2\int_0^{\infty} e^{-nt} dt + 2u\int_0^{\infty} t e^{-nt} dt$

$= \frac{2}{n^2} + \mu^2 + \frac{2u}{n}$

$\text{var} = E[(y(n))^2] - (E(y_{(n)}))^2$

$= \frac{2}{n^2} + \mu^2 + \frac{2u}{n} - \left(\frac{1}{n} + u\right)^2$

$= \frac{1}{n^2}$

$\text{Bias} = E(\hat{u}) - \mu$

$= \frac{1}{n} + u - \mu$

$= 1/n$

$MSE =$

$\text{var}(\hat{u}) + \text{bias}^2$

$= \frac{1}{n^2} + \frac{1}{n^2} = \boxed{\frac{2}{n^2}}$

MLE is better

6

1) $\frac{\partial f(b_0, b_1)}{\partial b_0} = \Sigma (y_1 - b_0 - b_1 x_1)$

$S = \sum_{i=1}^{n} (y_i - \beta_0 - \beta_1 x_1)^2$

$\frac{\partial S}{\partial \beta_0} = -2 \sum_{i=1}^{n} (y_1 - \beta_0 - \beta_1 x_1)$; $\frac{\partial S}{\partial \beta_1} = -2 \sum_{i=1}^{n} (y_i - \beta_0 - \beta_1 x_i) x_i$

$\frac{\partial S}{\partial \beta_0} = 0$ ↑ $\frac{\partial S}{\partial \beta_1} = 0$ ↑

$= -2 \sum_{i=1}^{n} (y_1 - \beta_0 - \beta_1 x_1) = 0$ TEXTBOOK PG 626

1 $= \sum_{i=1}^{n} y_i - n\beta_0 - \beta_1 \sum_{i=1}^{n} x_1 = 0 \Longrightarrow (\bar{y} - \beta_0 - \beta_1 \bar{x} = 0)\bar{x}$

$\hookrightarrow \bar{y}\bar{x} - \beta_0 \bar{x} - \beta \bar{x}^2$

$-2 \sum_{i=1}^{n} (y_i - \beta_0 - \beta_1 x_i) x_i = 0$

2 $= \sum_{i=1}^{n} (y_1 x_i - \beta_0 x_i - \beta_1 x_i^2) = 0 \Rightarrow \sum_{i=1}^{n} y_1 x_i - \beta \sum_{i=1}^{n} x_i - \beta \sum_{i=1}^{n} x_i^2 = 0$

$\sum_{i=1}^{n} y_i x_1 - n\bar{y}\bar{x}$

$= \beta_0 \sum_{i=1}^{n} x_i - \beta_0 \sum_{i=1}^{n} x_i + \beta_1 \sum_{i=1}^{n} x_i^2 - n\bar{x}^2$

$$\Rightarrow \boxed{\hat{\beta}_1 = \frac{\sum_{i=1}^{n} (y_1 - \bar{y})(x_1 - \bar{x})}{\sum_{i=1}^{n} (x_i - \bar{x})^2} \qquad \beta_0 = \bar{y} - \hat{\beta}_1 \bar{X}}$$

(2) $\sum_{i=1}^{n} (y - \hat{\beta}_1 x_1 - \hat{\beta}_0)(\hat{\beta}_1 x_1 + \hat{\beta}_0)$

$\sum_{i=1}^{n} (y_i - \hat{\beta}_1 x_i - (\bar{y} - \hat{\beta}_1 x_i))(\hat{\beta}_1 x_1 + \bar{y} - \hat{\beta}_1 x_i)$

$= \sum_{i=1}^{n} (y - \hat{\beta}_1 x_i - \bar{y} + \hat{\beta}_1 x_i)\bar{y}$

$= \sum_{i=1}^{n} (y_i - \bar{y})\bar{y} \Rightarrow \bar{y} \sum_{i=1}^{n} (y_i - \bar{y})$

$= \bar{y}\left(\sum_{i=1}^{n} y_i - n\bar{y}\right) \Rightarrow \bar{y}(n\bar{y} - n\bar{y}) \Rightarrow \boxed{0}$

On line? $\sum_{i=1}^{n} (y_i - \hat{y}_i) = 0$

$\sum_{i=1}^{n} y_i = \sum_{i=1}^{n} \hat{y}_i \Rightarrow \bar{y} = \frac{1}{n} \sum_{i=1}^{n} \hat{y}_i$

$\hat{y} = \hat{\beta}_0 + \hat{\beta}_1 x$

$\hat{y} = \bar{y}$ thus $(\bar{x}, \bar{y})$ is on line $\hat{y} = \hat{\beta}_0 + \hat{\beta}_1 x$

(3) $\left(\frac{1}{2\pi\sigma^2}\right)^{n/2} e^{-\sum_{i=1}^{n} \frac{(y_i - \beta_0 - \beta_1 x_i)^2}{2\sigma^2}}$

$\log L(\beta_0 \beta_1)$

$= -\frac{n}{2} \log 2\pi - \frac{n}{2} \log \sigma^2 - \sum_{i=1}^{n} \frac{(y_i - \beta_0 - \beta_1 x_i)^2}{2\sigma^2}$

$\frac{\partial \log L(\beta_0 \beta_1)}{\partial \beta_0} = \sum_{i=1}^{n} \frac{(y_i - \beta_0 - \beta_1 x_i)}{\sigma^2}$

set to 0 $\sum_{i=1}^{n} \frac{(y - \beta_0 - \beta_1 x_i)}{\sigma^2} = 0$

$= \sum_{i=1}^{n} y_i - n\beta_0 - \beta_1 \sum_{i=1}^{n} x_i = 0$

$\frac{\partial \log L}{\partial \beta_1} = \sum_{i=1}^{n} \frac{(y_i - \beta_0 - \beta_1 x_i)x_i}{\sigma^2}$

$\sum_{i=1}^{n} \frac{(y_i - \beta_0 - \beta_1 x_i)x_i}{\sigma^2} = 0$

$\sum_{i=1}^{n} y_i x_i - \beta_0 \sum_{i=1}^{n} x_i - \beta_1 \sum_{i=1}^{n} x_i^2 = 0$

$\tilde{\beta}_1 = \frac{\sum_{i=1}^{n} (y_i - \bar{y})(x_i - \bar{x})}{\sum_{i=1}^{n} (x_i - \bar{x})^2}$

$\beta_0 = \bar{y} - \tilde{\beta}_1 \bar{x}$

$y_i = \beta_0 + \beta_1 x_1 + \epsilon_1$ $\quad \bar{\epsilon} = \frac{1}{n} \sum_{i=1}^{n} \epsilon_i$

$\bar{y} = \beta_0 + \beta_1 \bar{x} + \bar{\epsilon}$

(6)

continued

3)
$$\beta_1 = \frac{\sum_{i=1}^{n} (\beta_0 + \beta_1 x_i + \epsilon_1 - \beta_0 - \beta_1 \bar{x} - \bar{\epsilon})}{\sum_{i=1}^{n} (x_i - \bar{x})^2}$$

$$= \sum_{i=1}^{n} \frac{((\beta_1(x_1 - \bar{x}) + (\epsilon_1 - \bar{\epsilon}))(x_i - \bar{x})}{\sum_{i=1}^{n} (x_1 - \bar{x})^2}$$

$$= \beta_1 + \sum_{i=1}^{n} \frac{(\epsilon_1 - \bar{\epsilon})(x_i - \bar{x})}{\sum_{i=1}^{n} (x_i - \bar{x})^2}$$

$$E(\hat{\beta}_1) = \beta_1 + \sum_{i=1}^{n} \frac{\{E(\epsilon_1) - E(\bar{\epsilon})\}(x_i - \bar{x})}{\sum_{i=1}^{n} (x_i - \bar{x})^2}$$

$\rightarrow \beta_1 + 0$
$= \beta_1$
Hence β_1 is unbiased estimator of β_1

$$\tilde{\beta}_0 = \bar{y} - \hat{\beta}_1 \bar{x}$$
$$= \beta_0 + \beta_1 \bar{x} + \bar{\epsilon} - \hat{\beta}_1 \bar{x} \Rightarrow$$

$$E(\tilde{\beta}_0) = \beta_0 + \beta_1 \bar{x} + E(\epsilon) - \bar{x} E(\tilde{\beta}_1)$$
$$= \beta_0 + \beta_1 \bar{x} - \beta_1 \bar{x}$$
$$= \boxed{\beta_0}$$

Hence $\tilde{\beta}_0$ is also an unbiased estimator of β_0

6)

$$\sigma^2 = s^2 = \frac{SSE}{n-2} = \frac{\Sigma(y_i - \hat{y}_i)^2}{n-2}$$

5)

$SSE = \Sigma y_i^2 - \hat{\beta}_0 \Sigma y_i - \hat{\beta}_1 \Sigma x_i y_i$ Text book Chapter 12

$$\frac{\Sigma(x_i - \bar{x})(y_i - \bar{y})}{\Sigma(x_i - \bar{x})^2} = \frac{S_{xy}}{S_{xx}}$$

The point estimator is not biased

$$\hat{B}_1 = \frac{\Sigma(x_i - \bar{x}) y_i}{\Sigma(x_i - \bar{x})^2}$$

$$= \frac{\Sigma(x_i - \bar{x}) B_0 + B_1 (x_i - \bar{x}) x_i + (x_i - \bar{x})}{\Sigma(x_i - \bar{x})^2}$$

$$= \beta_1 \frac{\Sigma(x_i - \bar{x}) x_i'}{\Sigma(x_i - \bar{x})^2} + \frac{\Sigma(x_i - \bar{x})}{\sum_{i=1}^{n} (x_i - \bar{x})^2}$$

$$\boxed{E(\hat{B}_1) = B_1}$$