Name: ______________________

AMAT 367: Discrete Probability

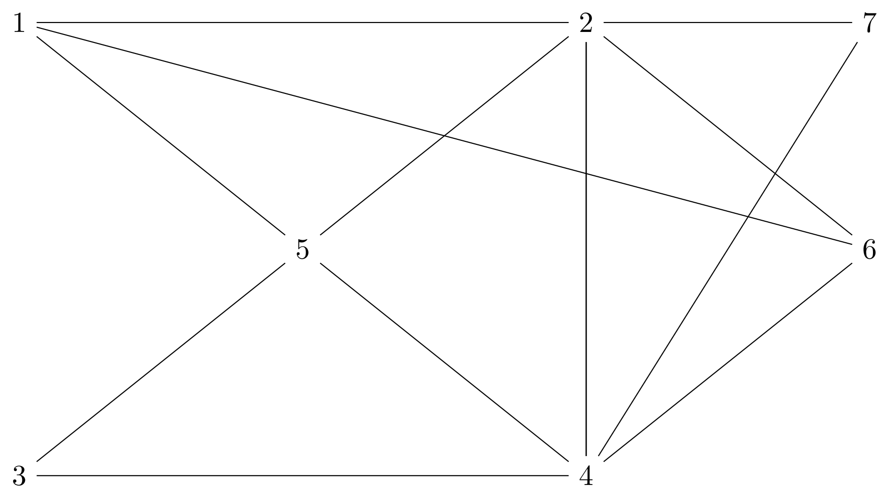
Final Exam

May, 2022

Show all work for each problem in the space provided. If you run out of room for an answer, continue on the back of the page.

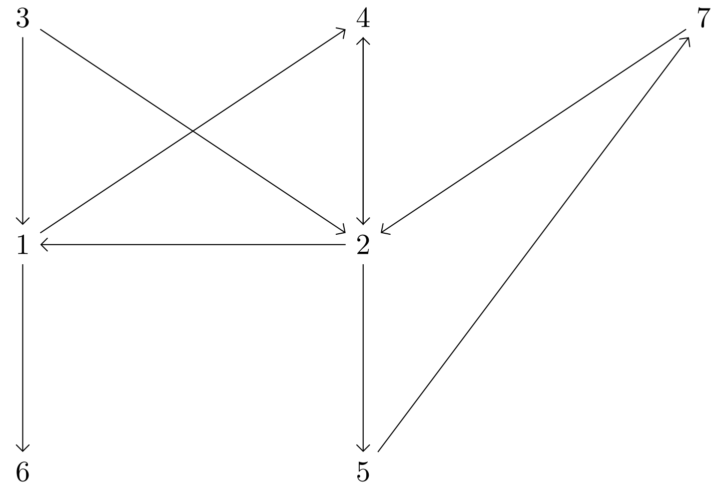
Question	Points	Bonus Points	Score
1	0	0	
2	0	0	
3	0	0	
4	0	0	
5	0	0	
6	0	0	
7	0	0	
Total:	0	0	

1. Consider the undirected graph G below with vertex set $V = \{1, 2, 3, 4, 5, 6, 7\}$,



Compute the stochastic matrix M for the random walk on this undirected graph.

2. Consider the directed graph G with vertex set $V = \{1, 2, 3, 4, 5, 6, 7\}$



Compute the stochastic matrix M for the random walk on this directed graph.

3. Given the Markov Chain (S, M, x_0) where $S = \{s_1, s_2, s_3\}$, $x_0 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$, and

$$M = \begin{bmatrix} 0 & .3 & .2 \\ 1 & 0 & .4 \\ 0 & .7 & .4 \end{bmatrix} \quad M^2 = \begin{bmatrix} .3 & .14 & .2 \\ 0 & .58 & .36 \\ .7 & .28 & .44 \end{bmatrix} \quad M^3 = \begin{bmatrix} .14 & .23 & .196 \\ .58 & .252 & .376 \\ .28 & .518 & .428 \end{bmatrix}$$

$$M^4 = \begin{bmatrix} .23 & .1792 & .1984 \\ .252 & .4372 & .3692 \\ .518 & .3836 & .4344 \end{bmatrix} \quad M^5 = \begin{bmatrix} .1792 & .20788 & .19704 \\ .4372 & .33264 & .37218 \\ .3836 & .45948 & .4308 \end{bmatrix} \quad M^6 = \begin{bmatrix} .20788 & .191688 & .197808 \\ .33264 & .391672 & .36936 \\ .45948 & .41664 & .432832 \end{bmatrix}$$

compute the probability the Markov Chain is in state s_1 at time 6.

4. Consider a simple random walk on the set $S = \{1, 2, 3, 4\}$, where $p = \dfrac{1}{2}$ is one of the transition probabilities. What is the probability of moving from state 3 to state 1 in exactly two steps if the random walk has reflecting boundaries?

5. Given the directed graph below

$$\begin{tikzcd}
1 \arrow[r, leftrightarrow] \arrow[dr] & 2 \arrow[d] & \\
 & 3 \arrow[r] & 4
\end{tikzcd}$$

respond to the following questions:

a.) Compute the stochastic matrix M associated to the random walk on this directed graph.

b.) Compute the modified matrix $M^\star$ of the Page Ranking algorithm.

c.) Compute the regularization matrix M_R for $p = \dfrac{1}{2}$

d.) Does a steady-state vector σ exist for the regular matrix, M_R? If so, why?

6. Let $M = \begin{bmatrix} .6 & .3 \\ .4 & .7 \end{bmatrix}$ be a stochastic matrix. Which one of the following vectors

$$\sigma = \begin{bmatrix} \frac{3}{7} \\ \\ \frac{4}{7} \end{bmatrix}$$

$$\sigma = \begin{bmatrix} 3 \\ \\ 4 \end{bmatrix}$$

$$\sigma = \begin{bmatrix} \frac{3}{4} \\ \\ 1 \end{bmatrix}$$

$$\sigma = \begin{bmatrix} \frac{6}{10} \\ \\ \frac{4}{10} \end{bmatrix}$$

is the steady-state vector for this transition matrix? Justify your response by demonstrating it is the steady-state vector by a computation that verifies the definition of steady-state vector.

7. Let us consider the population of people living in a city and its suburb and the migration within this population from the city and the suburbs to the city and the suburbs. The migration of these populations from and to each other is given by a stochastic matrix

$$M = \begin{bmatrix} .95 & .03 \\ .05 & .97 \end{bmatrix}$$

The entries in this matrix were obtained from collected data that demonstrates individuals are 95 % likely to remain in the city, 5 % likely to move from the city to the suburbs, 3 % likely to move from the suburbs to the city, and 97 % likely to remain in the suburbs.

Now suppose in the year 2000 60 % or .6 percent of people live in the city and 40 % or .4 percent of people live in the suburbs. What will be the percentage of people living in the city be in the year 2001? What will be the percentage of people living in the suburbs be in 2002?

Hint: Recall, for a general Markov Chain (S, Mx_0) *the initial vector* x_0 *is required to merely be a probability vector, that is, a vector whose entries add up to 1. In most of our examples, however, we have considered random walks where the initial vector is all 0's and a single 1. That is not the case in this problem. Use the information in the problem to identify an initial vector* x_0 *so that you can find the vectors* $Mx_0 = x_1$ *and* $Mx_1 = x_2$ *whose entries contain the probabilities the population lives in the city and in the suburbs. Measure time in years so that* x_1 *will contain the probability required to answer the problem in 2001 and* x_2 *will contain the probability required to answer the problem in 2002.*

extra paper