Name: ______________________________

AMAT 367: Discrete Probability

Homework 6

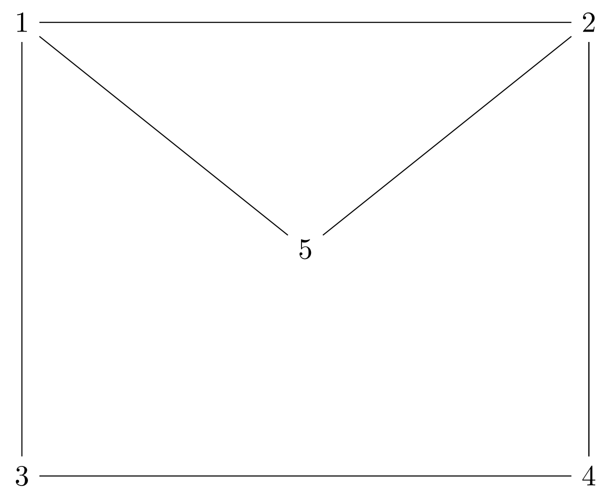
May, 2021

Show all work for each problem in the space provided. If you run out of room for an answer, continue on the back of the page. You may NOT use a calculator

Question	Points	Bonus Points	Score
1	0	0	
2	0	0	
3	0	0	
4	0	0	
5	0	0	
6	0	0	
Total:	0	0	

Opening Remarks Before I delve into the solutions, allow to me describe a little bit of what shall follow. Along with your homework submissions, I've received several inquiries besides on definitions, which rule to use to compute the transition matrix, etc. So rather than re-type the notes, I'll use each of the questions to recall these definitions and formation rules so that you can commit them to memory for the test on Monday.Throughout the solution manual, let $\langle S, x_0, \mathcal{P} \rangle$ be a Markov chain. Recall that this means S is an ordered set, i.e. $S = (s_1, s_2, \ldots, s_m)$ an initial vector x_0 and a transition matrix P whose i, j entries, below denoted p_{ij} record the transition probabilities, i.e. the probability that the Markov chain transitions from state s_j or j for short to state s_i or i for short. I must further emphasize that the columns of P are probability vectors, so a check on whether you're doing the work correctly is to make sure the entries *of the column* add up to 1. The key point of this last remark is throughout all of our Markov Chain exercises, for the most part, to solve them, you must compute p_{ij} according to some formation rule we give you. These different formation rules will distinguish the diferent problems, because abstractly, any Markov chain is the triple of data given above together with one key axiom: $P^n x_0 = x_n$ I mention this last in the preamble because understanding the meaning of the m-entries in x_n is another way in which you'll be able to solve problems. The entry a_j is the probability that chain is in state j at time n. As far as steady state vectors go, remember, there is no subscript n so they occur as $n \to \infty$. We interpret these as the occupation times of the chain, i.e. the entry a_j is the average amount of time the chain occupies state j.

1. Consider the undirected graph with vertex set $V = \{1, 2, 3, 4, 5\}$ and edge set E as indicated in the diagram below:



Compute the transition matrix P for the random walk (S, x_0, P) on this undirected graph.

Solution: First, to clarify, a **random walk on an undirected graph** is a Markov Chain whose state space S consists of the vertices of the graph, e.g. $S = \{1, 2, 3, 4, 5\}$, above, and whose transition matrix is given by a formation rule below. To answer this problem, we only have to compute P, the transition matrix. Let us recall from lecture the formation rule we use to find the transition probabilities p_{ij}, which are the i, j entries of the 5 X 5 transition matrix P. The rule is,

$$\text{if state } j \text{ is connected to } n \text{ distinct states and } j \text{ is connected to } i \text{ by an edge, then } p_{ij} = \frac{1}{n}, \text{ 0 otherwise}$$

Equivalently, n is the valence of state j, cf. lecture, so the rule becomes

$$\text{if } n = val(j) \text{ is the valence of state } j \text{ and } j \text{ is connected to } i \text{ by an edge, then } p_{ij} = \frac{1}{val(j)}, \text{ 0 otherwise.}$$

Please notice that this rule will be different in subsequent walks on *directed* graphs. We shall review the formation rule in those questions later. So, to finish this problem, proceed by the above formation rule by computing the stochastic matrix P as follows:

$$P = \begin{bmatrix} 0 & \frac{1}{3} & \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{3} & 0 & 0 & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{3} & 0 & 0 & \frac{1}{2} & 0 \\ 0 & \frac{1}{3} & \frac{1}{2} & 0 & 0 \\ \frac{1}{3} & \frac{1}{3} & 0 & 0 & 0 \end{bmatrix}$$

A good check on our work is to make sure the entries in each column add up to 1. In terms of our lecture work, this means the columns of P are probability vectors. Furthermore, you should recognize this computation from the lecture as the "FROM-TO" matrix, a more pedestrian way of describing the stochastic matrix. This is because if one visualizes the states along the top of the matrix, reading from left to right, and along the right side, reading from top to bottom, then the i, j-entry, namely p_{ij} is the probability the Markov chain transitions FROM state j TO state i.

2. Consider the directed graph with vertex set $V = \{1, 2, 3, 4\}$ and edge set E as indicated in the diagram below:

$$\begin{tikzcd}
1 \arrow[r] \arrow[d] \arrow[dr] & 2 \arrow[d, leftrightarrow] \\
3 & 4 \arrow[l]
\end{tikzcd}$$

Compute the transition matrix P for the random walk on this directed graph.

Solution: First, a **random walk on an directed graph** is a Markov Chain whose state space S are the vertices of the graph, e.g. $S = \{1, 2, 3, 4\}$, above, and a formation rule for the transition probabilities as follows. Recall, a directed edge connecting state j to state i is a vector with tail j and tip i. We define the directed valence of a state j, denoted $dval(j)$, to be the number of directed edges connecting j to distinct states i such that j is the tail of the directed edge. The formation rule for the entries p_{ij} in the stochastic matrix P is as follows, for $j \in S$

First, assume its directed valence is non-zero and $dval(j) = n$ Then $p_{ij} = \dfrac{1}{n}$ if there exists a directed edge from j to i with j as its tail, 0 otherwise. Equivalently, the reciprocal of the directed valence. Second, assume the directed valence is zero. Then $p_{jj} = 1$ and $p_{ij} = 0$ for all $i \neq j$.

As I mentioned in the previous problem, the key difference here in the formation rule is we only count directed edges. So, for example, in the graph, state 1 is connected to state 2 with 1 as the tail of the directed edge $1 \to 2$ in the graph. This edge adds to state 1's directed valence. Conversely, the same edge does not contribute to 2's directed valence, as there is no *directed* edge connecting 2 . Proceeding by the above rule, the entries of P are as follows:

$$P = \begin{bmatrix} 0 & 0 & 0 & 0 \\ \frac{1}{3} & 0 & 0 & \frac{1}{2} \\ \frac{1}{3} & 0 & 1 & \frac{1}{2} \\ \frac{1}{3} & 1 & 0 & 0 \end{bmatrix}$$

Please notice in both problems you can check that you answer is correct by checking whether the column entries all add up to 1-after all, the transition matrix is a stochastic matrix, and this is the defining feature of such a matrix.

3. Given a Markov Chain (S, x_0, P) where $S = \{s_1, s_2, s_3\}$, $x_0 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$, and

$$P = \begin{bmatrix} .1 & .2 & .3 \\ .2 & .3 & .4 \\ .7 & .5 & .3 \end{bmatrix} \quad P^2 = \begin{bmatrix} .26 & .23 & .2 \\ .36 & .33 & .3 \\ .38 & .44 & .5 \end{bmatrix} \quad P^3 = \begin{bmatrix} .21 & .22 & .23 \\ .31 & .32 & .33 \\ .47 & .45 & .44 \end{bmatrix}$$

$$P^4 = \begin{bmatrix} .22 & .22 & .22 \\ .32 & .32 & .32 \\ .44 & .45 & .45 \end{bmatrix} \quad P^5 = \begin{bmatrix} .22 & .22 & .22 \\ .32 & .32 & .32 \\ .45 & .45 & .45 \end{bmatrix} \quad P^6 = \begin{bmatrix} .22 & .22 & .22 \\ .32 & .32 & .32 \\ .45 & .45 & .45 \end{bmatrix}$$

compute the probability the chain is in state s_3 at time 4.

rmk 1: One may notice that the powers of the transition matrix are not true stoachastic matrices. This is so because I have rounded the entries to two decimal places. One may ignore this simplification when computing their answer.
rmk 2: One may have noticed that successive powers of the transition matrix look ever more similar. This is because the transition matrix is regular, and by class work, it will converge to a matrix Π such that all columns of Π are the steady-state vector q for the Markov chain. That detail is not a part of this question, but it's enlightening to see this principle in action.
Solution: So this question amounts to understanding what the entries of x_n mean, for $n \geq 0$. We reviewed that in the preamble to this solution manual. Nonetheless, I shall repeat the interpretation here. So, to find the probability the chain is in state s_3 at time 4, we need to compute the third entry of x_4. That can be done by using the main axiom of a Markov chain by computing

$$P^4 x_0 = x_4 = \begin{bmatrix} .22 \\ .32 \\ .45 \end{bmatrix}$$

So, the probability the chain is in state s_3 in $x_4 = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}$ at time 4 is the entry $a_3 = .45$, or 45 % Additionally, this may be computing in wolfram alpha,a freely available online calculator with the syntax

{{.22,.22,.22},{.32,.32,.32},{.44,.45,.45}}*{{0},{1},{0}} MULTIPLY

Notice in this syntax you input the row entries in the braces and separate rows by commas. This is why inputting x_0 seems a little awkward. It is o.k. to use computer algebra to compute this product if you do not known elementary linear algebra.

4. Consider a random walk on the set $S = \{1, 2, 3, 4, 5\}$. What is the probability of moving from state 2 to state 3 in exactly three steps if:

   a.) the walk has reflecting boundaries?
   b.) the walk has absorbing boundaries?

   Solution: First, the walk is on a state space with $m = 5$ states, then the $m \times (m-2)$ or 5 X 3 minor of the transition matrix P that excludes the first and last columns in the transition matrix is the same whether the walk is reflecting or absorbing. The formula for this minor is given in the lecture notes.

   Second, once we specify whether the boundaries i.e. the first and last states with respect to the ordering i.e. 1 and 5, are either reflecting or absorbing, we can determine the first and last columns of the transition matrix P, respectively. The formation rules for the transition probabilities in P according to the given hypothesis are as follows.

   The formation rule for the 5 X 3 minor excluding the first and last column is as follows:

   Let j and i be non-boundary states, then $p_{ij} = 0$ unless j and i are adjacent states, i.e $i = j-1$ or $i = j+1$. If $i = j+1$, then $p_{ij} = p$ and if $i = j-1$, then $p_{ij} = 1-p$

   Since this walk is unbiased (we did not specify p) this means we take $p = q = \dfrac{1}{2}$. In general, however, if $p \neq q$, one could say the random walk is biased.

   Next, suppose first the walk is reflecting. Then the way we determine the first and last column of the transition matrix P is given by

   Reflecting Suppose $j = 1$ OR 5 i.e. a boundary state. Then the random walk has reflecting boundaries if we insist $p_{ij} = 0$ unless either $i = 2$ or $i = 5-1 = 4$, respectively. In these cases, $p_{ij} = 1$.

   In terms of the actual vectors, the first column is therefore $\begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$ and the fifth or last column is $\begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$

   Next, to answer part b.) we need to determine the first and last columns when the walk is absorbing. For this hypothesis, we have the following rule:

   Absorption Suppose $j = 1$ OR 5 i.e. a boundary state. Then the random walk has absorbing boundaries if we insist $p_{ij} = 0$ unless $i = j$. In this case, $p_{ij} = 1$.

   In terms of the actual vectors, the first column is therefore $\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$ and the fifth or last column is $\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$ So, now you can form the full transition matrix because the absorption rules explains how to create the first and last columns that we initially left blank. Now I shall finish part a.) and leave part b.) to the reader.

   So, the random walk with reflecting boundaries has transition matrix

   $P = \begin{bmatrix} 0 & \frac{1}{2} & 0 & 0 & 0 \\ 1 & 0 & \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{2} & 0 & 1 \\ 0 & 0 & 0 & \frac{1}{2} & 0 \end{bmatrix}$ by the above formation rule for a reflecting walk. Next, by hypothesis, the walk begins in state 2, so this determines the initial vector $x_0 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$. Notice the second entry being 1 can be interpreted as a 100 % chance the Markov Chain/Random Walk is in state 2 at time 0. Last, to compute the probability that the walk is in state 3 at time three, that is, in three steps, we must compute the third entry of x_3. This is given by the main axiom of Markov Chains, that is, we compute $P^3 x_0 = x_3 = \begin{bmatrix} 0.375 \\ 0 \\ 0.5 \\ 0 \\ 0.125 \end{bmatrix}$ The third entry of this vector is the probability the random walk is in the third state at time three, so therefore, the answer is .5 or 50 %.

My input into wolfram alpha is $\{\{0,.5,0,0,0\},\{1,0,.5,0,0\},\{0,.5,0,.5,0\},\{0,0,.5,0,1\},\{0,0,0,.5,0\}\}^3 * \{\{0\},\{1\},\{0\},\{0\},\{0\}$

5. Given the directed graph

$$\begin{tikzcd}
1 \arrow[r] \arrow[d, leftrightarrow] \arrow[dr] & 2 \arrow[r] \arrow[d, leftrightarrow] & 5 \\
3 & 4 \arrow[l]
\end{tikzcd}$$

respond to the following questions:

a.) Compute the transition matrix P associated to the random walk on this directed graph.
b.) Compute the modified matrix P_*
c.) Given your response in b.) compute the Google matrix G for $p = .85$.
d.) Compute the steady-state vector q associated to G, that is, q such that $(G - I)q = 0$. You may use a computer algebra application in order to do this and write the solution.

Solution: First, let me start off by saying I will not answer question d) because it involves more Linear Algebra than the class in general seems to understand. Moreover, such a question, means to solve a system of homogeneous equations, will not be on the test in this form. However, comparing this to question 6 part d.), there the answer would be, "a steady state vector exists because G is a regular matrix."
a.) We proceed by the example set in question 2. The directed valences of states 1,2,3,4,5 are 3,2,1,2, and 0, respectively. Therefore, following the FROM j TO i formalism to compute transition matrices, we have

$$P = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 \\ \frac{1}{3} & 0 & 0 & \frac{1}{2} & 0 \\ \frac{1}{3} & 0 & 0 & \frac{1}{2} & 0 \\ \frac{1}{3} & \frac{1}{2} & 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & 0 & 1 \end{bmatrix}$$

b.) To compute P_* we must replace the columns corresponding to dangling nodes by a column of reciprocals of the cardinality of the state space, namely $\frac{1}{5}$ since there are 5 states in this problem. Accordingly,

$$P_* = \begin{bmatrix} 0 & 0 & 1 & 0 & \frac{1}{5} \\ \frac{1}{3} & 0 & 0 & \frac{1}{2} & \frac{1}{5} \\ \frac{1}{3} & 0 & 0 & \frac{1}{2} & \frac{1}{5} \\ \frac{1}{3} & \frac{1}{2} & 0 & 0 & \frac{1}{5} \\ 0 & \frac{1}{2} & 0 & 0 & \frac{1}{5} \end{bmatrix}$$

c.) Let K be the 5×5 matrix consisting of entries $a_{ij} = \dfrac{1}{5}$ for all $1 \leq i \leq 5$ and $1 \leq j \leq 5$, where 5 is as above. Then G is given by the following formula, for $p = .85$ and $r = 1 - p = .15$

$$G = pP_* + rK = P_* = \begin{bmatrix} 0.03 & 0.03 & 0.88 & 0.03 & 0.02 \\ 0.3105 & 0.03 & 0.03 & 0.455 & 0.2 \\ 0.3105 & 0.03 & 0.03 & 0.455 & 0.2 \\ 0.3105 & 0.455 & 0.03 & 0.03 & 0.2 \\ 0.03 & 0.455 & 0.03 & 0.03 & 0.2 \end{bmatrix}$$

The syntax in wolfram alpha is (.85){{0,0,1,0,.2},{.33,0,0,.5,.2},{.33,0,0,.5,.2},{.33,.5,0,0,.2},{0,.5,0,0,.2}} +(.15) {{.2,.2,.2,.2,.2},{.2,.2,.2,.2,.2},{.2,.2,.2,.2,.2},{.2,.2,.2,.2,.2},{.2,.2,.2,.2,.2}}
.)

d.) To solve this question, the syntax for wolfram alpha I used is "SOLVE {(.85){{0,0,1,0,.2},{.33,0,0,.5,.2},{.33,0,0,.5,.2},{.
+(.15) {{.2,.2,.2,.2,.2},{.2,.2,.2,.2,.2},{.2,.2,.2,.2,.2},{.2,.2,.2,.2,.2},{.2,.2,.2,.2,.2}}
-{{1,0,0,0,0},{0,1,0,0,0},{0,0,1,0,0},{0,0,0,1,0},{0,0,0,0,1}}}q={{0},{0},{0},{0},{0}}

This computation computes the nullspace of the operator $G - I$. Remember, in the end, a steady state vector is nothing more than an eigenvector for eigenvalue 1. Therefore, you may compute such a vector by any means you know. Just remember to scale your answer so it is a probability vector.

6. Given the directed graph below

$$
\begin{tikzcd}
 & 1 \arrow[d] & \\
2 \arrow[ur] \arrow[d, leftrightarrow] & 3 \arrow[l] \arrow[r] \arrow[d, leftrightarrow] & 4 \\
5 \arrow[r, leftrightarrow] & 6 \arrow[d] & \\
 & 7 &
\end{tikzcd}
$$

respond to the following questions:

a.) Compute the transition matrix $\mathcal{P}$ associated to the random walk on this graph.
b.) Compute the modified matrix $\mathcal{P}_*$.
c.) Compute the Google matrix $\mathcal{G}$ for $p = .45$.
d.) Does a steady-state vector q exist for the Google matrix? If so, why?

Solution: This is just another example of question 5. Please notice part d.) is different, but I address this difference in the remark preceding the solution to question 5. Or, to spell out the solution, a steady state vector exists because G is regular which, by a theorem in class, is guaranteed to have a steady state vector. Indeed, the entire reason to regularize stochastic matrices is to assert the existence of a steady state vector.