Approximation theory

Spectral methods
Polynomial interpolation
Truncated and discrete expansions

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Outline

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Hilbert space=Vector space+inner product

David Hilbert (1982-1943)

1. Vector space V (finite or infinite dimensional space)
2. Inner product is a mapping

$$\langle \cdot, \ \cdot \rangle = V \times V \to \mathbb{R}$$

3. It is the generalization of the Euclidean space to an infinite–dimensional space.
4. It is complete with respect to the norm induced by the inner product:

$$\|x\| = \langle x, x \rangle.$$

Every Cauchy sequence converges to some point.

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Set of functions $f \in \mathcal{C}^\infty[a, b]$

1. Inner product

$$\langle f,\ g \rangle_w = \int_a^b f(x)\ g(x)\ w(x)\ dx,$$

where $w(x)$ is the weight function on the interval (a, b) strictly positive and integrable in (a, b).

2. The norm induced by the inner product is:

$$\|f\|^2 = \int_a^b f^2(x)\ w(x)\ dx.$$

3. This set is an infinite-dimensional Hilbert space.

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Orthogonal basis

1. Set of functions $\Phi_0, \Phi_1, \ldots$

$$\langle \Phi_k, \Phi_m \rangle_w = \int_a^b \Phi_k(x)\ \Phi_m(x)\ w(x)\ dx = \gamma_m\ \delta_{km},$$

   where δ_{km} is Kronecker delta.

2. Examples of orthogonal basis: trigonometric functions, Chebyshev polynomials, Legendre polynomials, ...
3. Expansion with this orthogonal basis:

$$f(x) = \sum_{k=0}^{\infty} \hat{c}_k \Phi_k(x), \quad \hat{c}_k = \frac{1}{\gamma_k} \int_a^b f(x)\, \Phi_k(x)\, w(x)\, dx.$$

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Infinite–dimensional space

1. Truncated expansion

$$P_N(x) = \sum_{k=0}^{N} \hat{c}_k \Phi_k(x), \quad \hat{c}_k = \frac{1}{\gamma_k} \int_a^b f(x)\, \Phi_k(x)\, w(x)\, dx,$$

2. Error of the truncated expansion

$$E(x) = \sum_{k=N+1}^{\infty} \hat{c}_k \Phi_k(x).$$

3. Spectral convergence

$$\hat{c}_k \ll O\left(\frac{1}{k^q}\right).$$

   for all $q > 0$.

4. If spectral convergence is assured, few terms are needed to have a very small error.

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Vector space $\mathbb{R}^N$

1. Given a set of collocation points $x_0, \ldots, x_N$ in $[a, b]$, vectors are defined by means: $\mathbf{f} = (f(x_0), \ldots, f(x_N))$.
2. Inner product

$$\langle \mathbf{f},\ \mathbf{g} \rangle_N = \sum_{j=0}^{N} f_j\ g_j\ \alpha_j,$$

   where α_j are the weight coefficients.
3. The norm induced by the inner product is:

$$\| f \|^2 = \sum_{j=0}^{N} f_j^2\ \alpha_j.$$

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Orthonormal basis

1. Finite-dimensional space: Set of vectors $\boldsymbol{\Phi}_0, \ldots, \boldsymbol{\Phi}_N$

$$\langle \boldsymbol{\Phi}_k, \boldsymbol{\Phi}_m \rangle_N = \sum_{j=0}^{N} \Phi_k(x_j)\ \Phi_m(x_j)\ \alpha_j = \gamma_m\ \delta_{km},$$

where δ_{km} is Kronecker delta.

2. Is it possible to find $x_0, \ldots, x_N$ to have an orthogonal basis ?

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Determine $x_0, \ldots, x_N$ to give orthogonal $\mathbf{\Phi_k}$

1. $2N+2$ Equations ($k = 0, \ldots, N \quad m = 0, \ldots, N$)

$$\sum_{j=0}^{N} \Phi_k(x_j)\ \Phi_m(x_j)\ \alpha_j = \gamma_m\ \delta_{km},$$

2. $2N+2$ Unknowns ($x_0, \ldots, x_N \quad \alpha_0, \ldots, \alpha_N$)
3. Solution: Gauss quadrature formula allows to equal

$$\langle \Phi_k,\ \Phi_m \rangle_w \ = \ \langle \mathbf{\Phi_k},\ \mathbf{\Phi_m} \rangle_N$$

for polynomials of degree $2N+1$.

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Gauss quadratura formula

1. Gauss quadrature formula

$$\int_a^b g(x)\ w(x)\ dx = \sum_{j=0}^{N} \alpha_j\ g(x_j).$$

is exact for polynomials $g(x)$ of degree $2N+1$ with $N+1$ nodal points x_j which are zeroes of Φ_{N+1}.

2. This formula allows to equal the infinite-dimensional inner product and the finite-dimensional inner product

$$\int_a^b \Phi_k(x)\ \Phi_m(x)\ w(x)\ dx = \sum_{j=0}^{N} \Phi_k(x_j)\ \Phi_m(x_j)\ \alpha_j,$$

$\forall\ k \leq N$ and $\forall\ m \leq N$.

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Interpolant or discrete expansion

1. Given a set of collocation points: $x_0, x_1, \ldots, x_N$
2. Look for an approximation:

$$I_N(x) = \sum_{k=0}^{N} \tilde{c}_k \Phi_k(x),$$

3. Determine $\tilde{c}_k$ by imposing:

$$f(x_j) = I_N(x_j), \qquad j = 0, \ldots, N.$$

4. Proyecting **f** over $\mathbf{\Phi_k}$

$$\tilde{c}_k = \frac{1}{\gamma_k} \langle \mathbf{f}, \mathbf{\Phi_k} \rangle_N.$$

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Expansions

1. Truncated expansion:

$$P_N(x) = \sum_{k=0}^{N} \hat{c}_k \Phi_k(x), \quad \hat{c}_k = \frac{1}{\gamma_k} \int_a^b f(x)\, \Phi_k(x)\, w(x)\, dx.$$

2. Discrete expansion:

$$I_N(x) = \sum_{k=0}^{N} \tilde{c}_k \Phi_k(x), \qquad \tilde{c}_k = \frac{1}{\gamma_k} \sum_{j=0}^{N} f(x_j)\, \Phi_k(x_j)\, \alpha_j.$$

Relation between $\hat{c}_k$ and $\tilde{c}_k$

1. Expansion of f at x_j

$$f(x_j) = \sum_{k=0}^{\infty} \hat{c}_k \Phi_k(x_j).$$

2. Interpolant at x_j coincides with $f(x_j) = I_N(x_j)$

$$\sum_{k=0}^{N} \tilde{c}_k \Phi_k(x_j) = \sum_{k=0}^{\infty} \hat{c}_k \Phi_k(x_j).$$

3. Multiply by $\Phi_m(x_j)\alpha_j$ and sum from $j=0$ to $j=N$

$$\sum_{k=0}^{N} \sum_{j=0}^{N} \tilde{c}_k \Phi_k(x_j) \Phi_m(x_j) \alpha_j = \sum_{k=0}^{\infty} \sum_{j=0}^{N} \hat{c}_k \Phi_k(x_j) \Phi_m(x_j) \alpha_j.$$

4. Aliasing error

$$\tilde{c}_m = \hat{c}_m + \frac{1}{\gamma_m} \sum_{k=N+1}^{\infty} \hat{c}_k \, \langle \boldsymbol{\Phi}_m, \boldsymbol{\Phi}_k \rangle_N$$

Conclusions

1. Computational treatment requires a finite number of degrees of freedom: coefficients of the truncated series or the collocation of nodal points.
2. If the function is infinitely differentiable, the expansion in some orthogonal basis can show spectral convergence which means that few terms or degrees of freedom are needed to approximate the function properly.
3. If nodal points are given by the Gauss quadrature formula and the truncated expansion has spectral convergence, the error between the coefficients of the discrete expansion and the coefficients of the truncated expansion is very small and equal to the aliasing error.