Katholieke
Universiteit
Leuven

**Department of
Physics and Astronomy**

1D RIEMANN SOLVER FOR EULER EQUATIONS

Computational Methods for Astrophysical Applications

Jai Mirchandani(r0915812)

Academic year 2024–2025

Contents

1 Introduction

The flows that are recognized by high variations in density under the influence of pressure and temperature are called compressible flows. studying the dynamics of these flows is very important to understand many astrophysical phenomena such as supernovas, interstellar medium, plasma dynamics, etc. These flows are governed by complex set of equations requiring the use of computational methods to solve them. The entire discussion below is primarily based on Toro [2009] and Keppens and Sundqvist [2024]

1.1 Euler equations

The Euler equations are set of Partial differential equations(PDEs) that govern the dynamics of compressible flows. The Euler equations are given as:

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\mathbf{v}\rho) = 0 \tag{1}$$

$$\frac{\partial (m)}{\partial t} + \nabla \cdot (m\mathbf{v}) + \nabla p = 0 \tag{2}$$

$$\frac{\partial e}{\partial t} + \nabla \cdot (\mathbf{v}e + \mathbf{v}p) = 0 \tag{3}$$

The equation 1 is the conservation of mass, the equation 2 is the conservation of momentum and the equation 3 is the conservation of energy. In these equations the mass density(ρ), the momentum density(m= ρv) and energy density(e) are conserved and denoted as U ϵ (ρ, m, e). Another form of such equations that uses the primitive variables instead of conserved ones can be written as:

$$\partial_t \begin{pmatrix} \rho \\ v \\ p \end{pmatrix} + \begin{pmatrix} v & \rho & 0 \\ 0 & v & \frac{1}{\rho} \\ 0 & \gamma p & v \end{pmatrix} \partial_x \begin{pmatrix} \rho \\ v \\ p \end{pmatrix} = 0. \tag{4}$$

where U ϵ (ρ, v, p) are the primitive variables.

1.2 Riemann problem for Euler

The Riemann problem for Euler is an initial value problem characterized by set of Euler equations with known initial values of conservative/primitive variables. These initial values remain constant in the left and right side of an arbitrary intermediate region, also known as the contact discontinuity. The variables on the left and right side of the contact discontinuity are referred as $U_L \epsilon(\rho_L, p_L, v_L)$ and $U_R \epsilon(\rho_R, p_R, v_R)$. The contact discontinuity, as the name suggests, is a region where the consistency in the values of the variables from the left or right side cease to exist. The variables in the intermediate region may have a different value referred as $U_\star$ ϵ ($\rho_{\star L}$,$\rho_{\star R}$,$p_\star$, $v_\star$) and are not known at the initial state. The value of density is different in the left and the right half of the contact discontinuity. Hence, we must first solve for the $U_\star$ to solve the Riemann problem for Euler. To solve this Riemann Problem, we consider two waves, each going to the left and right direction of the intermediate region connecting it to the left and right constant states respectively. The job of the solver is to determine the evolution of the primitive variables throughout the system over time which primarily depends on the initial conditions and consequently on the nature of the waves originating from the conditions. Based on the initial conditions, we can have three conditions- a shockwave ($p_\star > p_k$), a rarefaction($p_\star \leq p_k$) or no wave. Different combinations of these waves in left and right direction can lead to 9 different possibilities with each possibility occurring either in the presence or absence of contact discontinuity. This makes the total number of configurations for the Riemann problem as 18. Additionally, there are some scalar quantities in the problem that remain constant in specific regions and do not vary with time. Such quantities are called Riemann invariants. We shall discuss the Riemann invariants as we discuss the solution for one of the Riemann problem configurations.

2 This project: 2 - Rarefaction Case

As mentioned in the previous section, there are various configurations for the Riemann problem depending on the nature of the left wave, the right wave and intermediate region. In this project, a case where both the left and right states consist of a rarefaction in the presence of a contact discontinuity referred as the 2-rarefaction case in the rest of the project, has been considered. The goal of this project is to demonstrate a solution for the Riemann problem for Euler for this configuration. Initially, the primitive variables in the left and right state- U_L and U_R are known and are mentioned in the Table 1 taken from the 'TEST 2' as mentioned in Toro [2009]. The simulation has been setup for the domain of x lying from -1 to 1 with contact discontinuity at $x = 0$.

Variable	initial value
p_L	0.4
p_R	0.4
ρ_L	1
ρ_R	1
v_L	-2
V_R	2

Table 1: initial values of the primitive variables in the left and right state of the equation

2.1 Solution for primitive variables in the intermediate region.

To formulate the entire solution for the Riemann problem, we start by finding the values of pressure($p_\star$), velocity($v_\star$) and density ($\rho_\star$) in the central region. The $p_\star$ can be computed by finding the roots of the equation:

$$f(p, U_L, U_R) = f_L(p, U_L) + f_R(p, U_R) + v_R - v_L \tag{5}$$

For the rarefaction case , the function f_l and f_R can be computed as:

$$f_k(p, U_k) = \frac{2c_k}{\gamma - 1}[(\frac{p}{p_k})^{\frac{\gamma-1}{2\gamma}} - 1] \tag{6}$$

where γ is the ratio of specific heats and k ϵ (L,R). The c_k is the speed of the sound in the left or right state and given as

$$c_k = \sqrt{\frac{\gamma p_k}{\rho_k}} \tag{7}$$

2.1.1 Pressure in the intermediate region

We use the Newton-Raphson iterative scheme to find the zero of the equation 5. This scheme iteratively solves for $p_\star$ as:

$$p_{(n)} = p_{(n-1)} - \frac{f(p_{(n-1)})}{f'(p_{(n-1)})} \tag{8}$$

where, n is the number of iterations. The derivative term $f'(p_{(k-1)})$ for the rarefaction case can be computed as:

$$f'(p, U_k) = \frac{1}{\rho_k c_k}(\frac{p}{p_k})^{\frac{-\gamma+1}{2\gamma}} \tag{9}$$

After every iteration we check for the value of relative pressure change which is written as,

$$CHA = \frac{|p_{(n)} - p_{(n-1)}|}{\frac{1}{2}|p_{(n)} + p_{(n-1)}|} \tag{10}$$

The iterations are stopped when this change becomes smaller than a threshold value called tolerance(TOL). The typical tolerance value as given by Toro [2009] is 10^{-6}.

As visible from the equations above, to find the root of the equation 5 through this scheme, an initial guess for $p_\star$ is required. There are various ways to guesstimate the initial pressure value as given by Toro [2009]. There are three ways in which we tried to solve for $p_\star$ out of which two correspond to providing an initial guess for iterative solution for Newton - Raphson scheme. These two guesses are computed as follows:

1.

$$p_0 = 0.5(p_L + p_R) \tag{11}$$

2.

$$\begin{aligned} p_{PV} &= \frac{1}{2}(p_L + p_R) - \frac{1}{8}(v_R - v_L)(\rho_L + \rho_R)(c_L + c_R), \\ p_0 &= \max(\mathrm{TOL}, p_{PV}). \end{aligned} \tag{12}$$

The first guess is the average value of left and right pressure values. This guess wasn't successful in converging to a solution as it resulted in the negative values for $p_\star$ and hence, was rejected from the algorithm. The second guess results from the linearized solution from the primitive variables. If this value is smaller than TOL, then the tolerance value proves to be a good initial guess for the scheme. This method proves to be a good guess for the initial pressure value giving us the value of $p_\star$ as 0.001893(when γ =1.4) which is in alignment with the solution of Toro [2009]. The third alternative used to find $p_\star$ is only applicable in the case of 2 rarefactions(our case) and provides an exact solution without the implementation of Newton-Raphson iterative scheme. The solution is computed as:

$$p_\star = \left[\frac{c_L + c_R - \frac{1}{2}(\gamma - 1)(v_R - v_L)}{\frac{\gamma-1}{2\gamma}\frac{c_L}{\rho_L^\gamma} + \frac{\gamma-1}{2\gamma}\frac{c_R}{\rho_R^\gamma}}\right]^{\frac{2\gamma}{\gamma-1}} \tag{13}$$

This equation results in value of $p_\star = 0.001893$ which is same as from the iterative procedure.

2.1.2 Velocity in the intermediate region

once, the value of $p_\star$ has been determined through any of the procedures discussed above, the velocity in the intermediate region ($v_\star$) can be computed as:

$$v_\star = 0.5(f_R(p_\star) - f_L(p_\star) + v_R + v_L) \tag{14}$$

from the equation, we get $v_\star = 0$

2.1.3 Densities in the intermediate region

In the previous section we discussed that the density doesn't remain constant in the intermediate region but has different values in the left and right half- ($\rho_{\star L}$ and $\rho_{\star R}$). For the case of rarefaction, these densities can be computed from the entropy, which remains constant in the left/right state and the left/right side of the intermediate region:

$$S_L = S_{\star L}; S_R = S_{\star R} \tag{15}$$

where entropy at any point is given as,

$$S = \frac{p}{\rho^\gamma} \tag{16}$$

From equation 15 and 16 we get,

$$\rho_{\star k} = \rho_k (\frac{p_\star}{p_k})^{\frac{1}{\gamma}} \tag{17}$$

From this expression and $\gamma = 1.4$ we get, $\rho_{\star L} = \rho_{\star R} = 0.0218$

Similarly, as the speed of sound is dependent on the density of the medium, it can be calculated in the intermediate region from the equation 7. For $\gamma = 1.4$, we get $c_{\star L} = 0.348$ and $c_{\star R} = 0.348$

2.2 The entire solution

With the knowledge of initial values of primitive variables in the intermediate region, we will discuss the development of solver to understand the evolution of the entire system over time from t=0 to t= 0.3 units. A rarefaction wave will have a head and tail with characteristic speed given as:

$$\begin{aligned} s_{headL} &= v_L - c_L \\ s_{headR} &= v_R + c_R \\ s_{tailL} &= v_\star - a_{\star L} \\ s_{tailR} &= v_\star + a_{\star R} \end{aligned} \tag{18}$$

where, s_{headL} - Speed at the head of left refraction wave, s_{headR}- Speed at the head of Right refraction wave, s_{tailL}- Speed at the tail of left refraction wave, s_{tailR}-Speed at the tail of Right refraction wave.

Once these values have been computed, we can divide the entire domain into 6 parts based on x/t as shown in the table 2

Number	condition	primitive variables
1	$x/t \le s_{headL}$	$U \epsilon U_L$
2	$s_{headL} \le x/t \le s_{tailL}$	$U \epsilon U_{Lfan}$
3	$s_{tailL} \le x/t \le v_\star$	$U \epsilon U_\star$
4	$v_\star \le x/t \le s_{tailR}$	$U \epsilon U_\star$
5	$s_{tailR} \le x/t \le s_{headR}$	$U \epsilon U_{Rfan}$
6	$s_{headR} \le x/t$	$U \epsilon U_R$

Table 2: Solution of primitive variables for 2 rarefaction case throughout the system

The primitive variables inside the rarefaction fan U_{Lfan} and U_{Rfan} can be computed as :

$$\mathbf{U}_{L\text{fan}} = \begin{cases} \rho = \rho_L \left[\frac{2}{\gamma+1} + \frac{\gamma-1}{\gamma+1}\frac{c_L}{v_L - \frac{x}{t}}\right]^{\frac{2}{\gamma-1}}, \\ v = \frac{2}{\gamma+1}\left[c_L + \frac{\gamma-1}{2}v_L + \frac{x}{t}\right], \\ p = p_L \left[\frac{2}{\gamma+1} + \frac{\gamma-1}{\gamma+1}\frac{c_L}{v_L - \frac{x}{t}}\right]^{\frac{2\gamma}{\gamma-1}}. \end{cases} \tag{19}$$

$$\mathbf{U}_{R\text{fan}} = \begin{cases} \rho = \rho_R \left[\frac{2}{\gamma+1} - \frac{\gamma-1}{\gamma+1}\frac{c_R}{v_R - \frac{x}{t}}\right]^{\frac{2}{\gamma-1}}, \\ v = \frac{2}{\gamma+1}\left[-c_R + \frac{\gamma-1}{2}v_R + \frac{x}{t}\right], \\ p = p_R \left[\frac{2}{\gamma+1} - \frac{\gamma-1}{\gamma+1}\frac{c_R}{v_R - \frac{x}{t}}\right]^{\frac{2\gamma}{\gamma-1}}. \end{cases} \tag{20}$$

Additionally, there are 3 Riemann invariants for the 2-rarefaction case as given in Keppens and Sundqvist [2024] can be written as:

$$\mathcal{R}^i = \begin{pmatrix} v - \frac{2c_k}{\gamma-1} \\ S \\ v + \frac{2c_k}{\gamma-1} \end{pmatrix} \tag{21}$$

With knowledge of the solution of the primitive variables, the Riemann invariants for the entire domain can be estimated from equations 21 and 16. With this we have the entire solution of Riemann problem for the 2 rarefaction case.

2.3 Algorithm

In this section, the entire algorithm in which the numerical code has been written is discussed as follows:

- importing the required modules
- assigning the range for x and t
- assign values for U_L and U_R
- Assign different gamma values for which the solution is demonstrated
- For all the gamma values:
  - compute speed of sound in left and right state using equation 7
  - define functions for f(k), f'(k) and Newton Raphson scheme using equations 6, 9 and 8 respectively.
  - initial guess for $p_\star$
  - compute the $p_\star$ values using both Newton- Raphson scheme and exact solution from equation 13
  - calculate $v_\star$ using equation 14
  - calculate $\rho_{\star L}$ and $\rho_{\star R}$ using equation 17
  - calculate $c_{\star L}$ and $c_{\star R}$
  - calculate $S_{(head)L}$, $_{(head)R}$, $S_{(tail)L}$, $S_{(tail)R}$ using equation 18
  - for time from 0 to 0.3

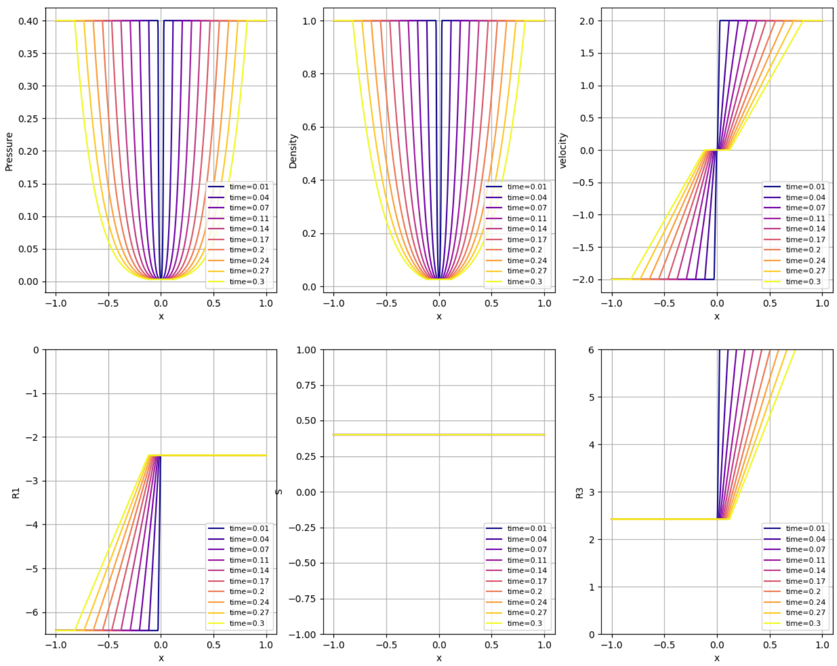

Figure 1: The solution from our Riemann solver for Primitive Variables and 3 Riemann invariants for γ =1.33

- ∗ initialize array for primitive variables and Riemann invariants:
- ∗ for x from -1 to 1
  - · if x is less than 0
  - · calculate U_L and Riemann invariants based on x/t values using first 3 rows of table 2 and equation 21
  - · else
  - · calculate U_R based on x/t and Riemann invariants values using last 3 rows of table 2 and equation 21
- ∗ plot the primitive variable and Riemann variations as function of x

– show and save the plots

3 Results

Based on the algorithm shown in the previous section, the solver was tested for different γ values. The initial primitive variables in the intermediate region for all the values of γ is shown in Table 3 . We can see that that pressure and density keep on reducing with increasing γ values. The time- evolution of 3 primitive variables and 3 Riemann invariants as function of x for the gamma values 1.33,1.4 and 1.66 is shown in the Figure 1, 2 and 3 respectively. In all 3 cases, Riemann invariant 1 is constant on the right side of contact discontinuity, Riemann invariant 2 i.e. the entropy is constant throughout the system and Riemann invariant 3 is constant on the left side of the contact discontinuity.

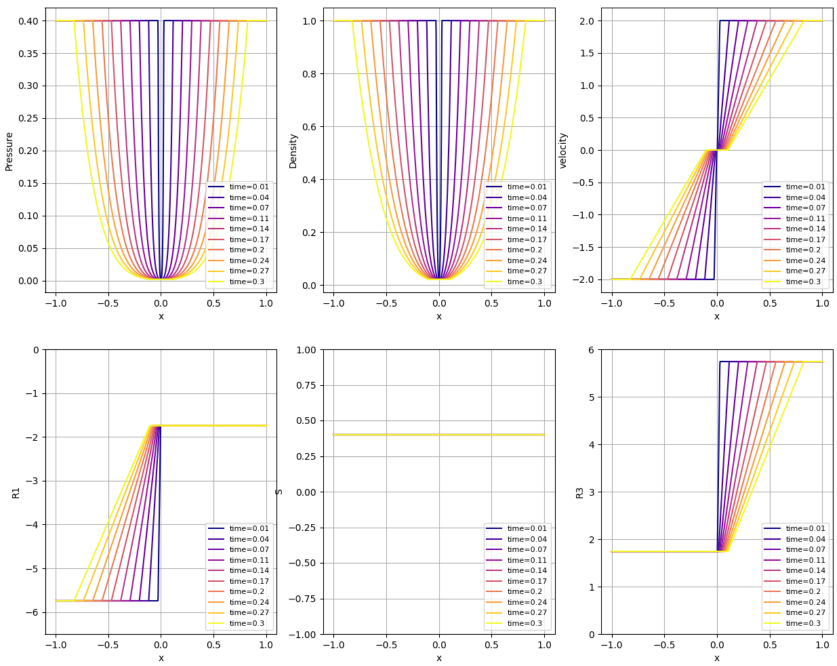

Figure 2: The solution from our Riemann solver for Primitive Variables and 3 Riemann invariants for γ =1.4

Quantity	γ= 1.33	γ= 1.4	γ= 1.66
$P_\star$	0.0031	0.0189	9.43e-05
$v_\star$	0	0	0
$\rho_{\star L}$	0.0259	0.0218	0.0065
$\rho_{\star R}$	0.0259	0.0218	0.0065

Table 3: Primitive variables in the intermediate region

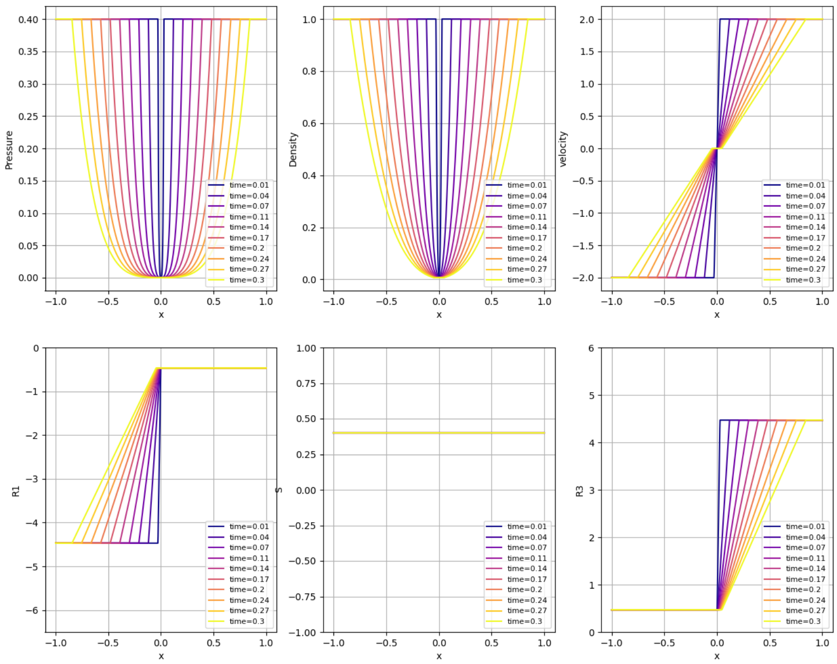

Figure 3: The solution from our Riemann solver for Primitive Variables and 3 Riemann invariants for γ =1.66

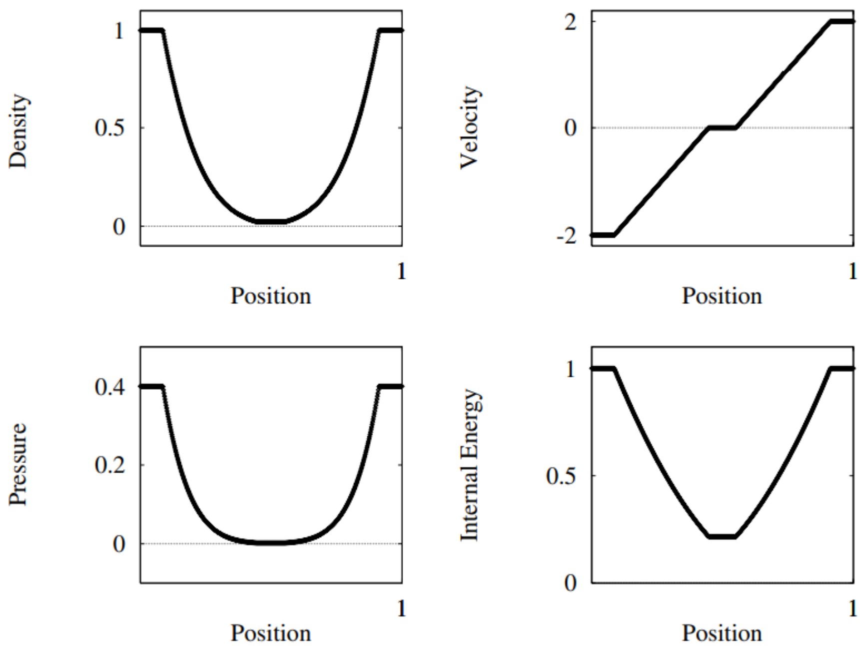

Figure 4: The solution from Toro [2009] for Primitive Variables and internal energy for γ =1.4

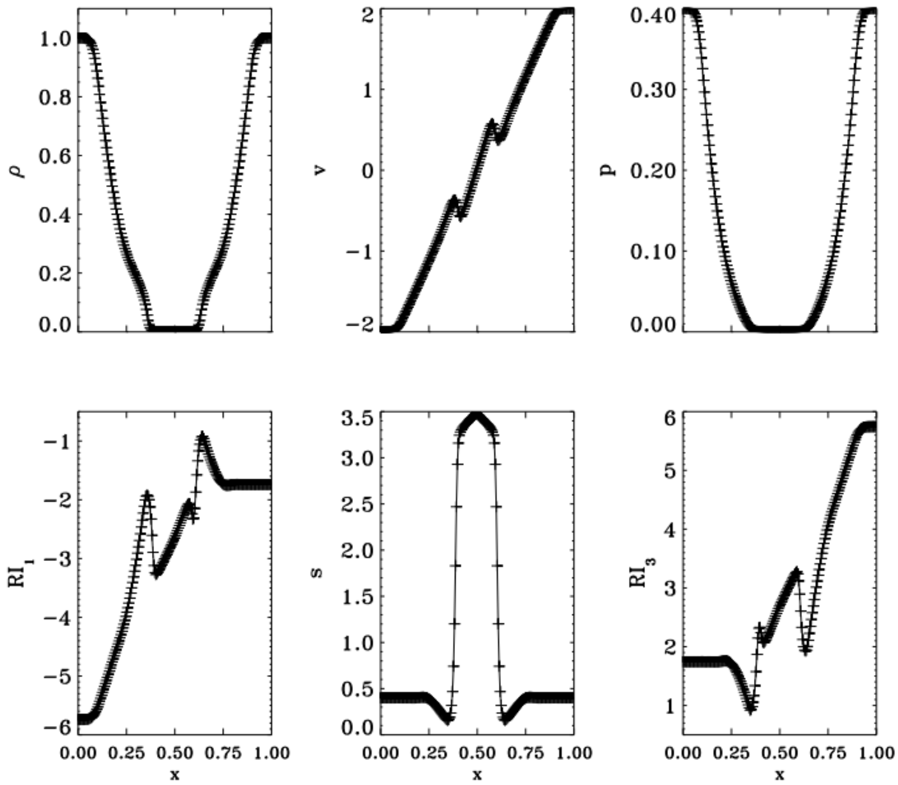

Figure 5: The solution from TVDLF for Primitive Variables and 3 Riemann invariants for γ =1.4 at t=0.15

Next, we compare our solution to the solver from Toro [2009] as seen in Figure 4 and solution from the TVDLF scheme shown in the Keppens and Sundqvist [2024] as seen in Figure 5. Our solver is in perfect agreement with the solution by Toro [2009]. The primitive variables in TVDLF scheme yield similar result to solver developed in this project but a visible advantage that our Riemann solver has over the TVDLF scheme is the error- free solution of Riemann invariants which behave abnormally in the intermediate region for the TVDLF solution. For e.g. the entropy must be constant as the two rarefaction case is isentropic as seen in our solution but a deviation is observed in the solution of TVDLF scheme.

4 Conclusion and future prospects

In this project, we designed a Riemann solver for the Euler equation focused on the 2 rarefaction case. Our solver was successful in replicating the results from Toro [2009] and also demonstrated a significant advantage in solving the Riemann invariants, delivering flawless results. There are various applications for this 2 rarefaction case for example, supernova remnants, which can sometimes have a rarefaction wave on both the ejecta side and ISM side.

This solver, can easily be adapted for all the different configurations of Riemann problem by including the similar equations for shocks as described by Toro [2009] and including the pressure conditions for shocks and rarefactions as discussed in Section 1.2 to automatically determine whether the wave is shock or rarefaction and consequently proceed with the solution.

References

Eleuterio F. Toro. *The Riemann Problem for the Euler Equations*, pages 115–162. Springer Berlin Heidelberg, Berlin, Heidelberg, 2009. ISBN 978-3-540-49834-6. doi: 10.1007/b79761_4. URL https://doi.org/10.1007/b79761_4.

Rony Keppens and Jon Sundqvist. *Computational Methods for Astrophysical Applications*. KU Leuven, 2024.