

# PROBABILITY

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LAXMINARAYEN



# WHAT IS PROBABILITY?

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- Probability is a value between 0 and 1 that a certain event will occur

# EXAMPLE FOR PROBABILITY

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- The probability that a fair coin will come up heads is 0.5
- Mathematically we write:

$$P(E_{heads}) = 0.5$$

# WHAT IS PROBABILITY?

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- In the above “heads” example, the act of flipping a coin is called a **trial**.
- Over very many trials, a fair coin should come up “heads” half of the time.



# TRIALS HAVE NO MEMORY!

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- If a fair coin comes up tails 5 times in a row, the chance it will come up heads is *still* 0.5
- Each trial is independent of all others



# EXPERIMENTS

- Each trial of flipping a coin can be called an experiment
- Each mutually independent outcome is called a simple event

## SAMPLE SPACE

- The sample space is the sum of every possible simple event

# EXAMPLE FOR SAMPLE SPACE

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- Consider rolling a six-sided die
- One roll is an experiment
- The simple events are:

~~$E_4=1$~~

~~$E_2=2$~~

~~$E_3=3$~~

~~$E_4=4$~~

~~$E_5=5$~~

~~$E_6=6$~~



- Therefore, the sample space is:
  - $S = \{E_1, E_2, E_3, E_4, E_5, E_6\}$



## EXPERIMENTS

- The probability that a fair die will roll a six:  
The simple event is:

$$E_6 = 6 \text{ (one event)}$$



**Total sample space:**

$$S = \{E_1, E_2, E_3, E_4, E_5, E_6\} \text{ (six possible outcomes)}$$

**The probability:**

$$P(\text{Roll Six}) = 1/6$$

# INTERSECTIONS, UNIONS & COMPLEMENTS

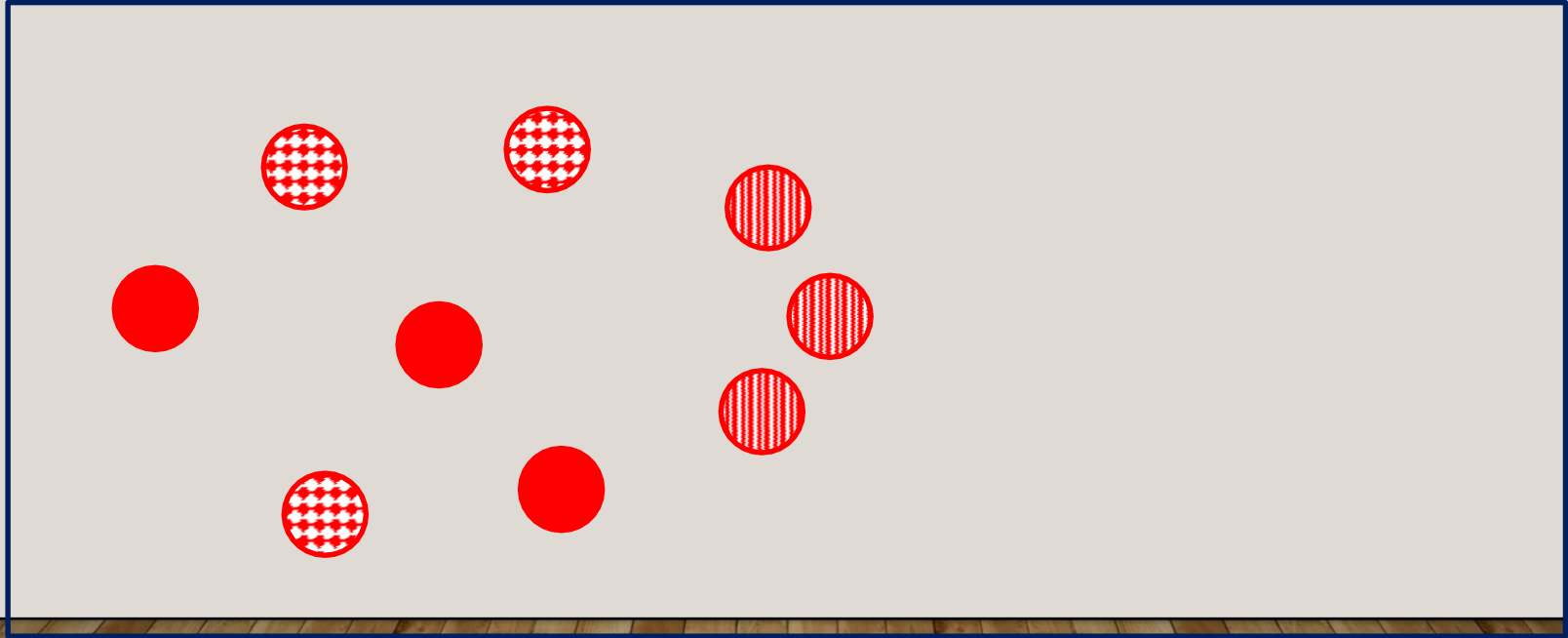
# INTERSECTIONS

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- In probability, an **intersection** describes the sample space where two events *both* occur.

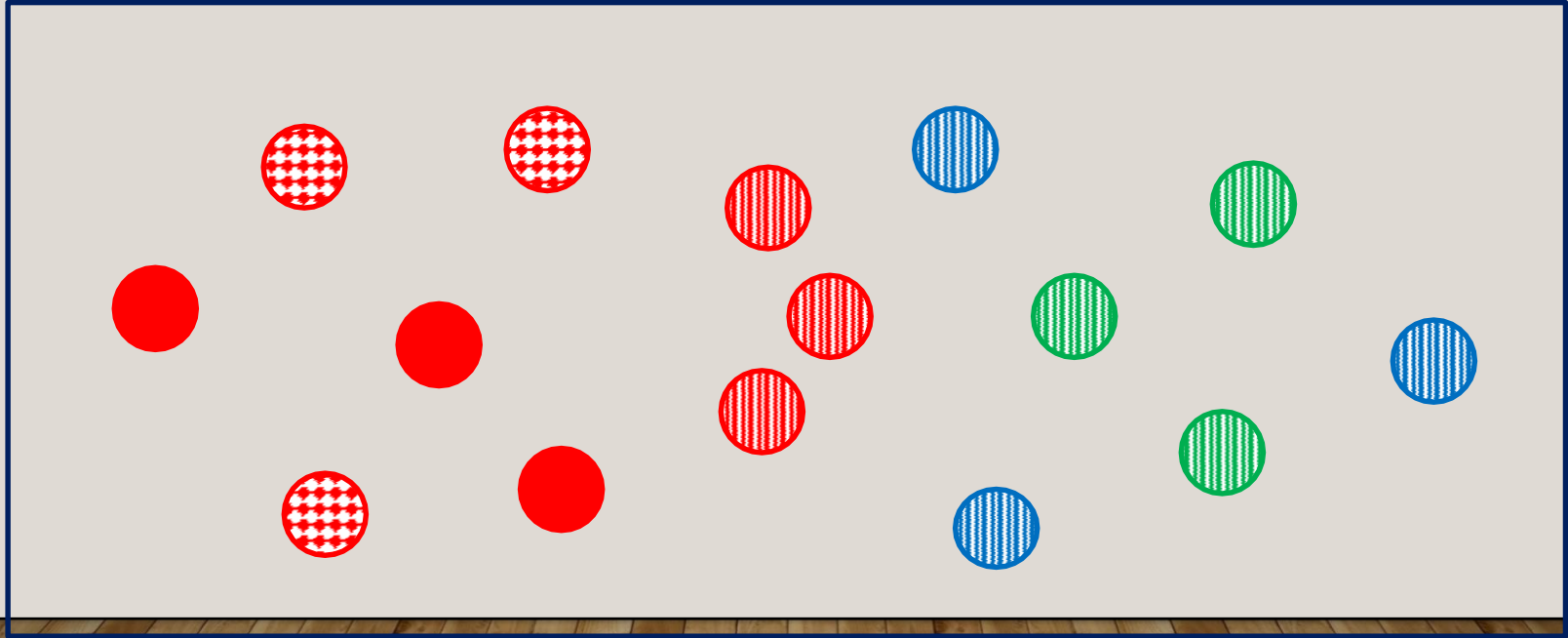
# INTERSECTIONS

- 9 of the balls are red:



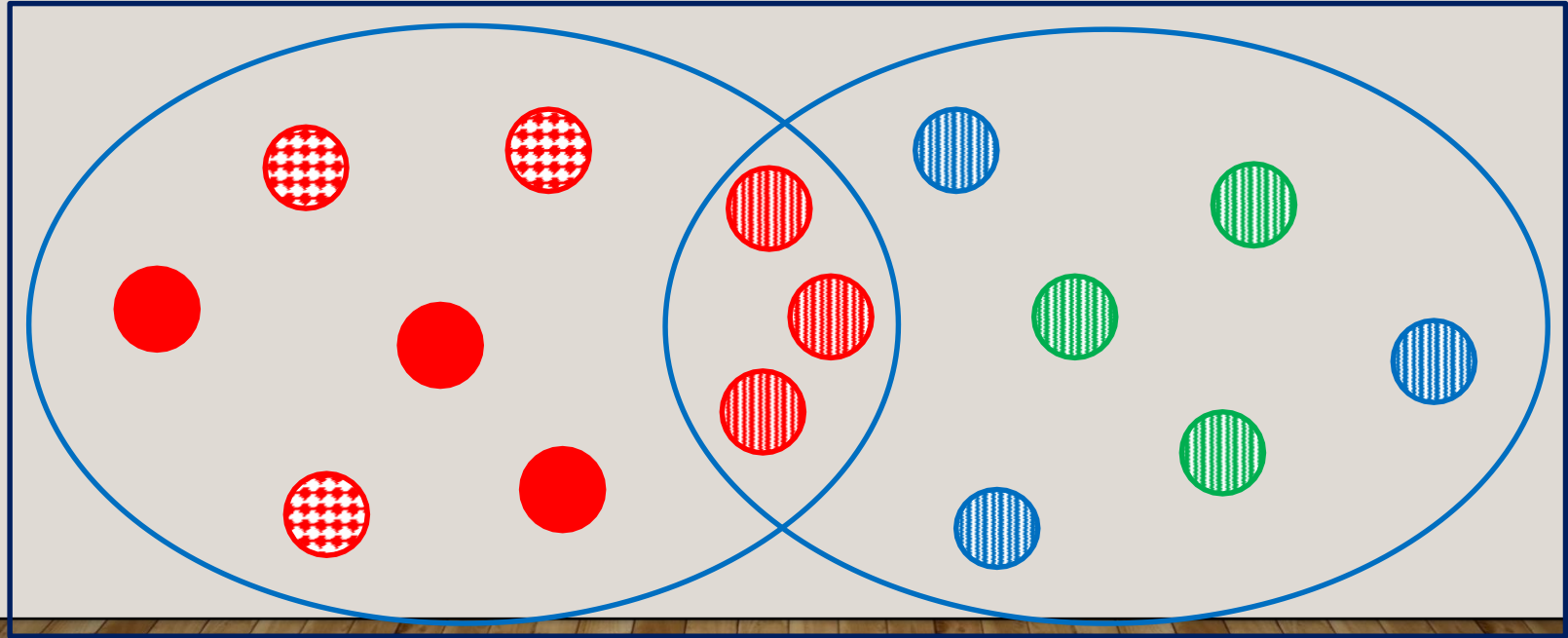
# INTERSECTIONS

- 9 of the balls are striped:
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# INTERSECTIONS

- 3 of the balls are both red and striped:

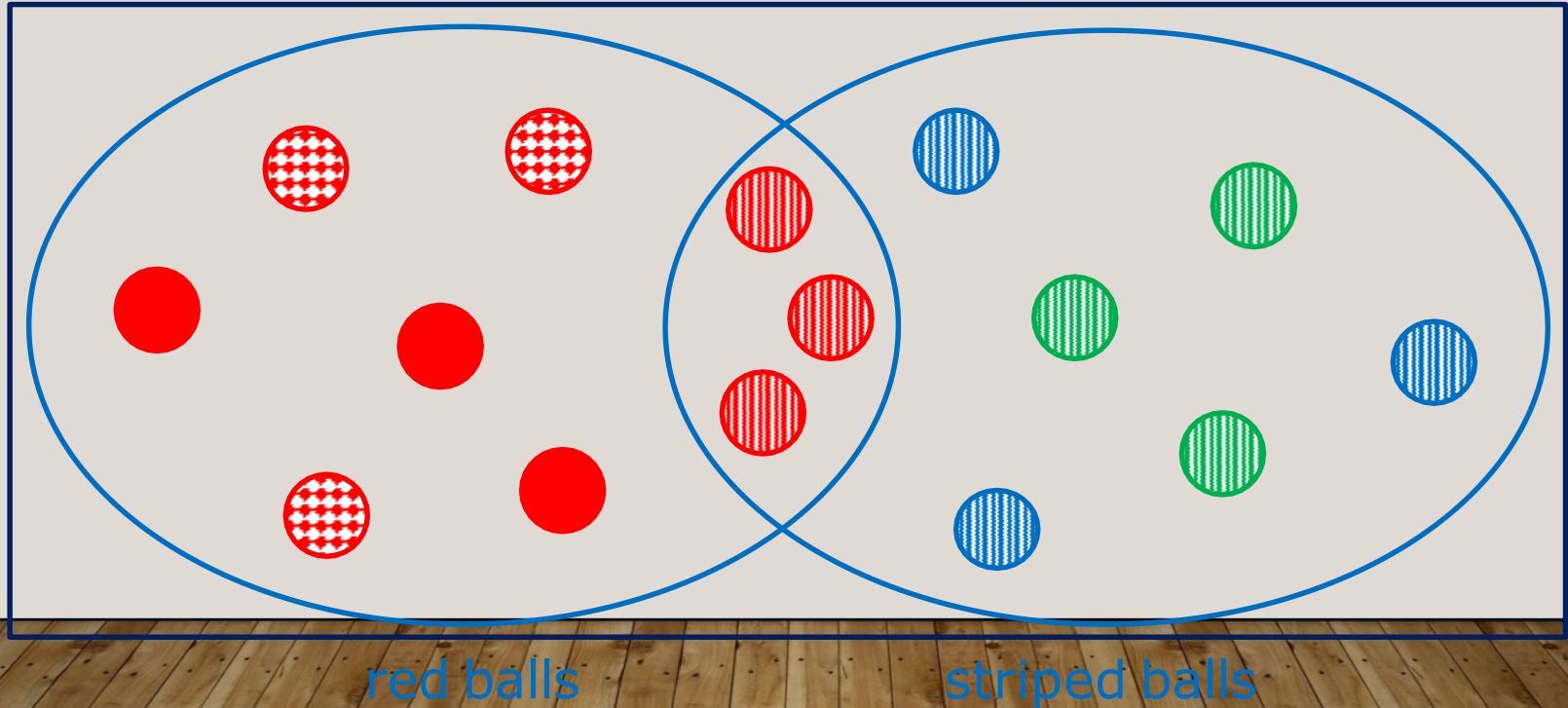


red balls

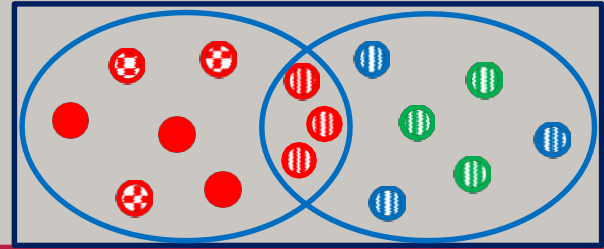
striped balls

# INTERSECTIONS

- What are the odds of a red, striped ball?



# INTERSECTIONS



- If we assign  $A$  as the event of red balls, and  $B$  as the event of striped balls, the intersection of  $A$  and  $B$  is given as:

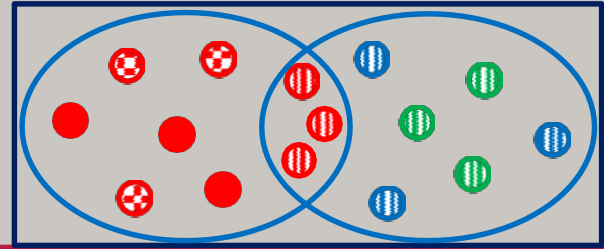
$$A \cap B$$

- Note that order doesn't matter:

$$A \cap B = B \cap A$$



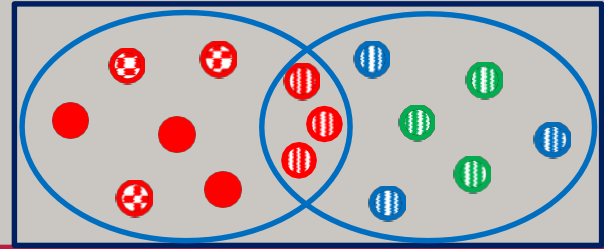
# INTERSECTIONS



- The probability of *A and B* is given as  
 $P(A \cap B)$
- In this case:

$$P(A \cap B) = \frac{3}{15} = 0.2$$

# UNIONS



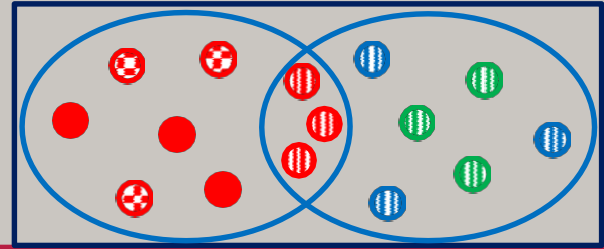
- The **union** of two events considers if **A or B** occurs, and is given as:

$$A \cup B$$

- Note again, order doesn't matter:

$$A \cup B = B \cup A$$

# UNIONS



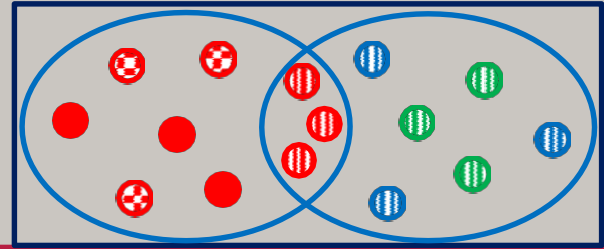
- The probability of *A or B* is given as:

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

- In this case:

$$P(A \cup B) = \frac{9}{15} + \frac{9}{15} - \frac{3}{15} = \frac{15}{15} = \mathbf{1.0}$$

# COMPLEMENTS



- The **complement** of an event considers everything outside of the event, given by:

$$\bar{A}$$

- The probability of *not* A is:

$$P(\bar{A}) = 1 - P(A) = \frac{15}{15} - \frac{9}{15} = \frac{6}{15} = \mathbf{0.4}$$

# INDEPENDENT & DEPENDENT EVENTS

# INDEPENDENT EVENTS

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- An **independent** series of events occur when the outcome of one event has no effect on the outcome of another.

## EXAMPLE FOR INDEPENDENT EVENT

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- An example is flipping a fair coin twice
- The chance of getting heads on the second toss is independent of the result of the first toss.



# INDEPENDENT EVENTS

- The probability of seeing two heads with two flips of a fair coin is:

$$P(H_1H_2) = P(H_1) \times P(H_2)$$

$$= \frac{1}{2} \times \frac{1}{2} = \frac{1}{4}$$

1st Toss	2nd Toss
H	H
H	T
T	H
T	T



# DEPENDENT EVENTS

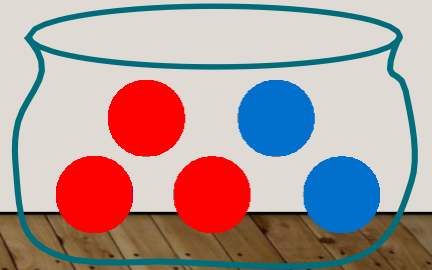
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- A **dependent** event occurs when the outcome of a first event does affect the probability of a second event.
- A common example is to draw colored marbles from a bag *without replacement*.

# EXAMPLE FOR DEPENDENT EVENTS

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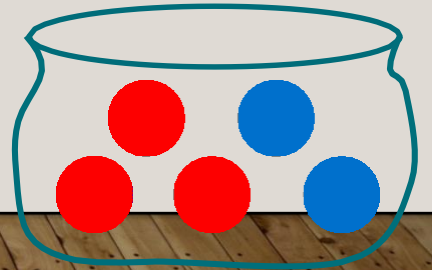
- Imagine a bag contains 2 blue marbles and 3 red marbles.
- If you take two marbles out of the bag, what is the probability that they are both red?



## EXAMPLE FOR DEPENDENT EVENTS

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- Here the color of the first marble affects the probability of drawing a 2<sup>nd</sup> red marble.

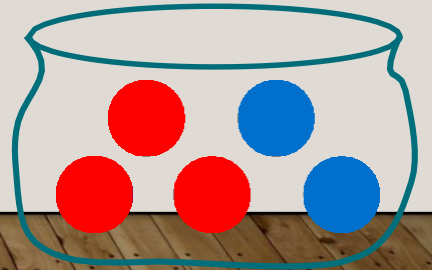


## EXAMPLE FOR DEPENDENT EVENTS

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- The probability of drawing a first red marble is easy:

$$P(R_1) = \frac{3}{5}$$

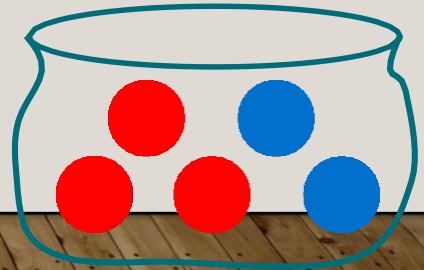


# EXAMPLE FOR DEPENDENT EVENTS

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- The probability of drawing a second red marble *given that* the first marble was red is written as:

$$P(R_2|R_1)$$

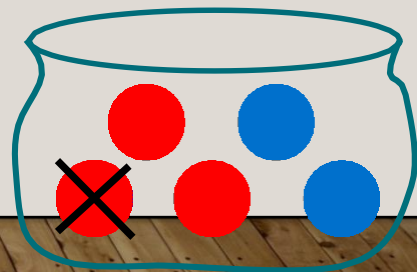


## EXAMPLE DEPENDENT EVENTS

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- After removing a red marble from the sample set this becomes:

$$P(R_2|R_1) = \frac{2}{4}$$

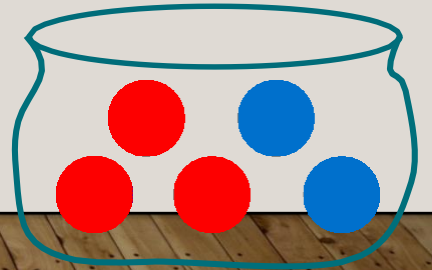


## EXAMPLE FOR DEPENDENT EVENTS

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- So the probability of two red marbles is:

$$P(R_1 \cap R_2) = P(R_1) \cdot P(R_2|R_1)$$
$$= \frac{3}{5} \times \frac{2}{4} = \frac{6}{20} = \mathbf{0.3}$$



# CONDITIONAL PROBABILITY



# CONDITIONAL PROBABILITY

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- The idea that we want to know the probability of event  $A$ , *given* that event  $B$  has occurred, is **conditional probability**.
- This is written as  $P(A|B)$

# CONDITIONAL PROBABILITY

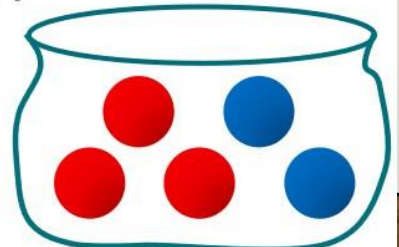
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- Going back to dependent events, the probability of drawing two red marbles is:

$$P(R_1 \cap R_2) = P(R_1) \cdot P(R_2|R_1)$$

- The conditional in this equation is:

$$P(R_2|R_1)$$



# CONDITIONAL PROBABILITY

- Rearranging the formula gives:

$$P(A|B) = \frac{P(A \cap B)}{P(B)}$$

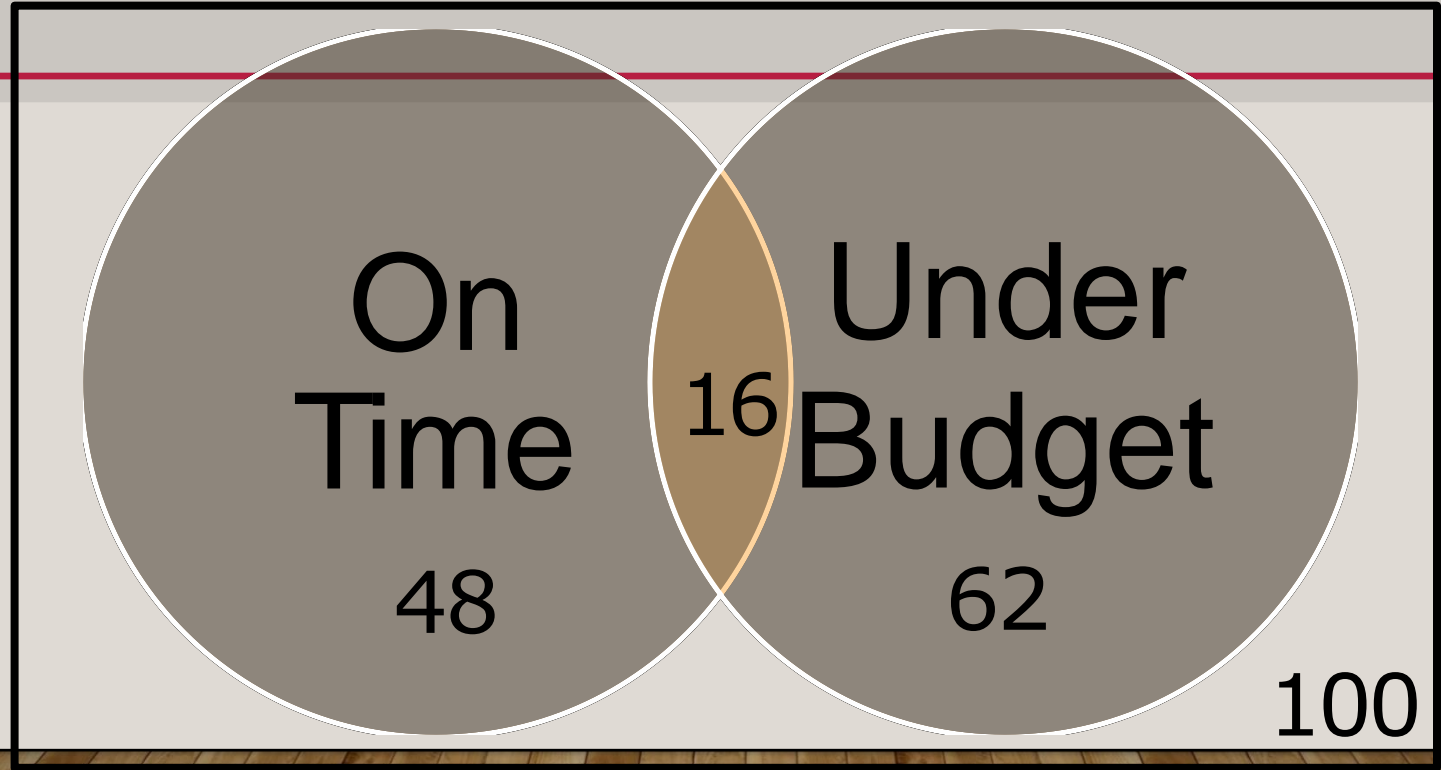
- That is, the probability of **A given B** equals the probability of **A and B** divided by the probability of **B**

# CONDITIONAL PROBABILITY EXERCISE

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- A company finds that out of every 100 projects, 48 are completed on time, 62 are completed under budget, and 16 are completed both on time and under budget.
- Given that a project is completed on time, what is the probability that it is under budget?

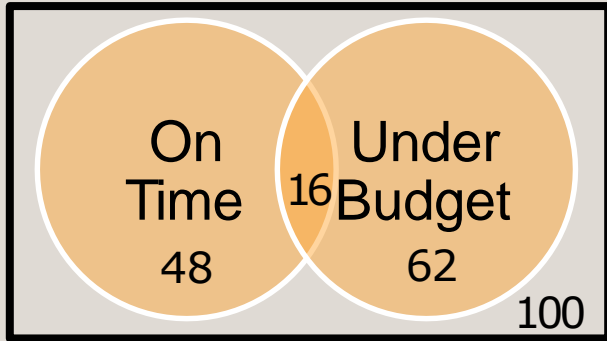
# CONDITIONAL PROBABILITY EXERCISE



# CONDITIONAL PROBABILITY EXERCISE

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Given that a project is completed on time **B**, what is the probability that it is under budget **A**?

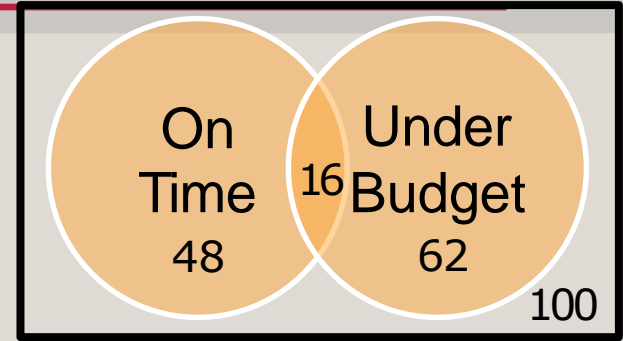


$$\begin{aligned} P(A|B) &= \frac{P(A \cap B)}{P(B)} \\ &= \frac{16}{64} = 0.25 \end{aligned}$$

# ADDITION & MULTIPLICATION RULES

# ADDITION RULE

- From our project example, what is the probability of a project completing on time *or* under budget?



- Recall from the section on unions:

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

- This is the **addition rule**



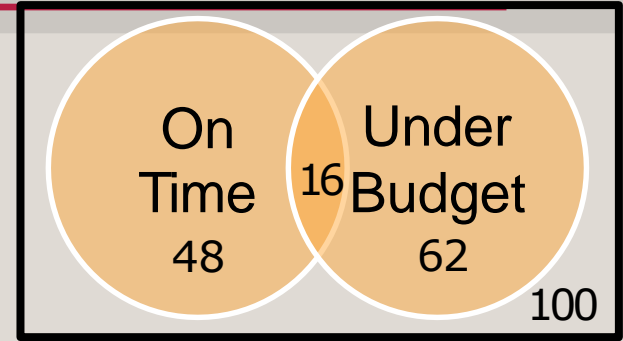
## ADDITION RULE

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

$$= \frac{48}{100} + \frac{62}{100} - \frac{16}{100}$$

$$= 0.48 + 0.62 - 0.16$$

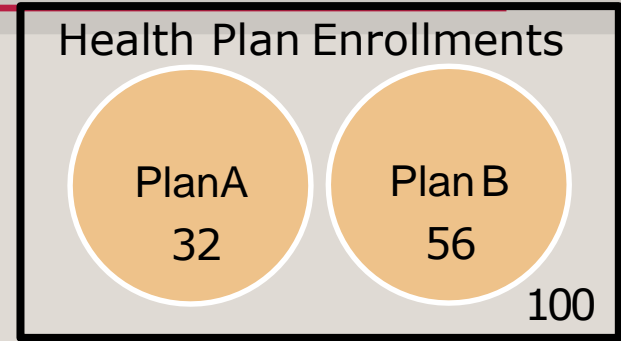
$$= 0.94$$



# ADDITION RULE FOR MUTUALLY EXCLUSIVE EVENTS

- When two events cannot both happen, they are said to be **mutually exclusive**.
- In this case, the addition rule becomes:

$$P(A \cup B) = P(A) + P(B) - \cancel{P(A \cap B)}$$



# MULTIPLICATION RULE

- From the section on dependent events we saw that the probability of A and B is:

$$P(A \cap B) = P(A) \cdot P(B|A)$$

- This is the **multiplication rule**

# BAYESTHEOREM

# BAYESTHEOREM

- We've already seen conditional probability:

$$P(A|B) = \frac{P(A \cap B)}{P(B)} \text{ provided that } P(B) > 0$$

$$P(A \cap B) = P(A) \cdot P(B|A) \text{ provided that } P(A) > 0$$

# BAYESTHEOREM

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- We can then connect the two conditional probability formulas to get Bayes' Theorem:

$$P(A|B) = \frac{P(B|A) \cdot P(A)}{P(B)} \text{ provided that } P(A), P(B) > 0$$

# Discrete Distributions

Relative frequency distributions for “counting” experiments.

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