CS 522: Programming Language Semantics
Homework-5 Solutions

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1. **Exercise 8, Slide 20**: Prove that $\mathcal{C}(A, B) \simeq \mathcal{C}(\star, B^A)$, for any $A, B \in |\mathcal{C}|$, whenever the exponential of A and B exists in $\mathcal{C}$.

Consider any category $\mathcal{C}$ and any 2 objects $A, B \in |\mathcal{C}|$. Assume that the exponential of A and B, B^A exists in $\mathcal{C}$. So, there is a morphism $app^{A,B} : B^A \times A \to B$, such that for any $C \in \mathcal{C}$, there is a unique $g : C \to B^A$, such that $(g \times 1_A); app^{A,B} = f$. Equivalently, we can say the the following diagram commutes:

$$\begin{tikzcd} C \times A \arrow[rr, "f"] \arrow[dr, "g\times 1_A"'] & & B \\ & B^A \times A \arrow[ur, "app^{A,B}"'] & \end{tikzcd} \tag{1}$$

We now assume that the final object $\star \in |\mathcal{C}|$ exists. We also know that $A \simeq \star \times A$ (proved in question (6)). This can be shown by the following diagram:

$$\begin{tikzcd} \star \times A \arrow[r, bend left, "\pi_A"] & A \arrow[l, bend left, "h"] \end{tikzcd} \tag{2}$$

The morphisms π_A and h are such that $\pi_A; h = 1_{\star\times A}$ and $h; \pi_A = 1_A$ (they are isomorphisms).

Now, to show that $\mathcal{C}(A, B) \simeq \mathcal{C}(\star, B^A)$, we will show that there is a one-to-one correspondence between the sets of morphisms $\mathcal{C}(A, B)$ and $\mathcal{C}(\star, B^A)$. The two components of this bijection, inverse to each other, are shown:

$$\begin{tikzcd} \mathcal{C}(A, B) \arrow[r, bend left, "\textsf{forward}"] & \mathcal{C}(\star, B^A) \arrow[l, bend left, "\textsf{backward}"] \end{tikzcd}$$

Lets first define the function $\textsf{forward}$. For any $m : A \to B$, $\pi_A; m : \star \times A \to B$. We define $\textsf{forward}(m) = g$, where $g : \star \to B^A$ is the unique morphism given by the definition of exponential, with the property that $(g \times 1_A); app^{A,B} = \pi_A; m$. This comes from diagram (1), with $C = \star$ and $f = \pi_A; m$. This can be shown by the following diagram, which commutes:

$$
\begin{tikzcd}
\star \times A \arrow[r, bend left, "\pi_A"] \arrow[rr, bend left=60, "\pi_A;m"] \arrow[dr, bend right, "g\times 1_A"'] & A \arrow[l, bend left, "h"] \arrow[r, "m"] & B \\
& B^A \times A \arrow[ur, bend right, "app^{A,B}"'] &
\end{tikzcd}
\tag{3}
$$

For any $g : \star \to B^A$, define $\mathsf{backward}(g) = h; (g \times 1_A); app^{A,B}$. To prove that these are inverses of each other, we prove these two subproofs:

- For any $m : A \to B$, $\mathsf{backward}(\mathsf{forward}(m)) = m$.

$$
\begin{aligned}
\mathsf{backward}(\mathsf{forward}(m)) &= \mathsf{backward}(g), \ \text{ where } (g \times 1_A); app^{A,B} = \pi_A; m \\
&= h; (g \times 1_A); app^{A,B} \\
&= h; (\pi_A; m) \\
&= (h; \pi_A); m \quad \text{(associativity of morphism composition)} \\
&= 1_A; m \quad (h, \pi_A \text{ are isomorphisms}) \\
&= m
\end{aligned}
$$

- For any $g : \star \to B^A$, $\mathsf{forward}(\mathsf{backward}(g)) = g$.

$$
\begin{aligned}
\mathsf{forward}(\mathsf{backward}(g)) &= \mathsf{forward}(h; (g \times 1_A); app^{A,B}) \\
&= g' \text{ where } g' \text{ is unique and} \\
&\qquad (g' \times 1_A); app^{A,B} = \pi_A; h; (g \times 1_A); app^{A,B} \\
&\qquad\qquad\qquad\qquad\quad = (g \times 1_A); app^{A,B}
\end{aligned}
$$

  $g' = g$ satisifies the last equation, and since g' is unique, we get $\mathsf{forward}(\mathsf{backward}(g)) = g$

This proves that $\mathcal{C}(A, B) \simeq \mathcal{C}(\star, B^A)$.

2. **<u>Property-1 on "category-theory-3.png"</u>**: Given the following diagrams,

$$
D \xrightarrow{h} A \qquad
\begin{tikzcd}
D \times B \arrow[r, "h\times 1_B"] \arrow[dr, bend right, "\mathsf{curry}((h\times 1_B);f)\times 1_B"'] & A \times B \arrow[r, "f"] \arrow[d, "\mathsf{curry}(f)\times 1_B"] & C \\
& C^B \times B \arrow[ur, bend right, "app^{B,C}"'] &
\end{tikzcd}
\tag{4}
$$

We need to prove that $h; \mathsf{curry}(f) = \mathsf{curry}((h \times 1_B); f)$. For this, we will first prove a helper lemma.

Lemma 1. *For a category $\mathcal{C}$ and objects $A_1, A_2, A_3, B \in |\mathcal{C}|$, the following diagram commutes:*

$$
\begin{tikzcd}
 & A_1 \arrow[r, "f_1"] & A_2 \arrow[r, "f_2"] & A_3 \\
A_1 \times B \arrow[ur, "\pi_{A_1}"] \arrow[r, "f_1 \times 1_B"] \arrow[rr, bend right, "(f_1;f_2)\times 1_B"] \arrow[drrr, bend right, "\pi_B^1"'] & A_2 \times B \arrow[ur, "\pi_{A_2}"] \arrow[r, "f_2 \times 1_B"] \arrow[drr, "\pi_B^2"'] & A_3 \times B \arrow[ur, "\pi_{A_3}"'] \arrow[dr, bend left, "\pi_B^3"] & \\
 & & & B
\end{tikzcd}
$$

Proof. Mainly, we need to show that $(f_1 \times 1_B); (f_2 \times 1_B) = (f_1; f_2) \times 1_B$.

We know that $A_3 \times B$ is the limit of the discrete diagram containing A_3 and B. $A_1 \times B$ is a cone for this diagram, so there is a unique morphism $(f_1; f_2) \times 1_B : A_1 \times B \to A_3 \times B$, such that

$$((f_1; f_2) \times 1_B); \pi_{A_3} = \pi_{A_1}; f_1; f_2 \tag{5}$$
$$((f_1; f_2) \times 1_B); \pi_B^3 = \pi_B^1 \tag{6}$$

Similarly, $A_2 \times B$ is also a cone for the discrete diagram containing A_3 and B, so there is a unique morphism $f_2 \times 1_B : A_2 \times B \to A_3 \times B$, such that

$$(f_2 \times 1_B); \pi_{A_3} = \pi_{A_2}; f_2 \tag{7}$$
$$(f_2 \times 1_B); \pi_B^3 = \pi_B^2 \tag{8}$$

Also, $A_1 \times B$ is also a cone for the discrete diagram containing A_2 and B, so there is a unique morphism $f_1 \times 1_B : A_1 \times B \to A_2 \times B$, such that

$$(f_1 \times 1_B); \pi_{A_2} = \pi_{A_1}; f_1 \tag{9}$$
$$(f_1 \times 1_B); \pi_B^2 = \pi_B^1 \tag{10}$$

Right composing equation (9) with f_2, and substituting $\pi_{A_2}; f_2$ from equation (7), we get

$$
\begin{aligned}
& (f_1 \times 1_B); \pi_{A_2}; f_2 = \pi_{A_1}; f_1; f_2 \\
\Rightarrow\ & (f_1 \times 1_B); (f_2 \times 1_B); \pi_{A_3} = \pi_{A_1}; f_1; f_2
\end{aligned}
$$

Similarly, substituting the value of π_B^2 from equation (8) in equation (10), we get

$$
\begin{aligned}
& (f_1 \times 1_B); \pi_B^2 = \pi_B^1 \\
\Rightarrow\ & (f_1 \times 1_B); (f_2 \times 1_B); \pi_B^3 = \pi_B^1
\end{aligned}
$$

We can see that $(f_1 \times 1_B); (f_2 \times 1_B)$ is a morphism from $A_1 \times B$ to $A_3 \times B$ that satisfies equation (5) and equation (6). But since $(f_1; f_2) \times 1_B$ is the unique morphism that satisfies that, we get that $(f_1 \times 1_B); (f_2 \times 1_B) = (f_1; f_2) \times 1_B$.

$\square$

Coming back to the original question, let's see the values of $\mathsf{curry}(f)$ and $\mathsf{curry}((h \times 1_B); f)$.

- $\mathsf{curry}(f)$ is the unique morphism $g : A \to C^B$, where $(g \times 1_B); app^{B,C} = f$
- $\mathsf{curry}((h \times 1_B); f)$ is the unique morphism $g' : D \to C^B$ such that $(g' \times 1_B); app^{B,C} = (h \times 1_B); f$

Left-composing the first equation with $(h \times 1_B)$, and using lemma (1), we get

$$\begin{aligned} & (g \times 1_B); app^{B,C} = f \\ \Rightarrow\ & (h \times 1_B); (g \times 1_B); app^{B,C} = (h \times 1_B); f \\ \Rightarrow\ & ((h; g) \times 1_B); app^{B,C} = (h \times 1_B); f \end{aligned}$$

Since, g' is the unique morphism and $h; g$ also satisfies the equation, we get

$$\begin{aligned} & h; g = g' \\ \Rightarrow\ & h; \mathsf{curry}(f) = \mathsf{curry}((h \times 1_B); f) \end{aligned}$$

Other exercises from slides

1. **Exercise 1, Slide 4**: Prove that **Set**, $\mathbf{Set}^{inj}$ and $\mathbf{Set}^{surj}$ are categories.
   (a) **Set**: it is a category whose objects are sets and whose morphisms are (total) functions. So for any two sets $A, B \in |\mathbf{Set}|$, $\mathbf{Set}(A, B)$ denotes the set of all total functions from A to B.
      - Identity Morphism: for any $A \in |\mathbf{Set}|$, the identity morphism 1_A is simply the identity function on A
      - The composition operator $\_;\_ : \mathbf{Set}(A, B) \times \mathbf{Set}(B, C) \to \mathbf{Set}(A, C)$, can be defined using function composition $\circ$

        $$f; g = g \circ f$$

      - Identity: It is easy to see that the identity morphism is the left and right identity for the composition operator. For any $f \in \mathbf{Set}(A, B)$,

        $$1_A; f = f = f; 1_B$$

      - Associativity: Associativity of the composition operator falls from the associativity of function composition

So, **Set** is indeed a category.

(b) $\mathbf{Set}^{inj}$: it is a category whose objects are sets and whose morphisms are injective functions. So for any two sets $A, B \in |\mathbf{Set}^{inj}|$, $\mathbf{Set}^{inj}(A, B)$ denotes the set of all injective functions from A to B.

- Identity Morphism: for any $A \in |\mathbf{Set}^{inj}|$, the identity morphism 1_A is simply the identity function on A, which is also injective
- The composition operator $\_;\_ : \mathbf{Set}^{inj}(A, B) \times \mathbf{Set}^{inj}(B, C) \rightarrow \mathbf{Set}^{inj}(A, C)$, can be defined using function composition $\circ$

$$f; g = g \circ f$$

  This is valid since composition of injective functions is an injective function
- Identity: It is easy to see that the identity morphism is the left and right identity for the composition operator. For any $f \in \mathbf{Set}^{inj}(A, B)$,

$$1_A; f = f = f; 1_B$$

- Associativity: Associativity of the composition operator falls from the associativity of function composition

So, $\mathbf{Set}^{inj}$ is indeed a category.

(c) $\mathbf{Set}^{surj}$: it is a category whose objects are sets and whose morphisms are surjective functions. So for any two sets $A, B \in |\mathbf{Set}^{surj}|$, $\mathbf{Set}^{surj}(A, B)$ denotes the set of all surjective functions from A to B.

- Identity Morphism: for any $A \in |\mathbf{Set}^{surj}|$, the identity morphism 1_A is simply the identity function on A, which is also surjective
- The composition operator $\_;\_ : \mathbf{Set}^{surj}(A, B) \times \mathbf{Set}^{surj}(B, C) \rightarrow \mathbf{Set}^{surj}(A, C)$, can be defined using function composition $\circ$

$$f; g = g \circ f$$

  This is valid since composition of surjective functions is an surjective function
- Identity: It is easy to see that the identity morphism is the left and right identity for the composition operator. For any $f \in \mathbf{Set}^{surj}(A, B)$,

$$1_A; f = f = f; 1_B$$

- Associativity: Associativity of the composition operator falls from the associativity of function composition

So, $\mathbf{Set}^{surj}$ is indeed a category.

2. **Exercise 3, Slide 13**: Any two limits of a diagram are isomorphic.

Consider any category $\mathcal{C}$ and a diagram $d : (N, E) \rightarrow \mathcal{C}$. Assume that d has two limits, $(L_1, \{\alpha_i\}_{i \in N})$ and $(L_2, \{\beta_i\}_{i \in N})$. The following holds:

- Since L_2 is also a cone, there is a unique $h_2 : L_2 \to L_1$, such that $h_2; \alpha_i = \beta_i$ for all $i \in N$
- Since L_1 is also a cone, there is a unique $h_1 : L_1 \to L_2$, such that $h_1; \beta_i = \alpha_i$ for all $i \in N$

Equivalently, the following diagram commutes:

$$
\begin{tikzcd}
 & d(i_1) \\
L_1 \arrow[ur, "\alpha_{d_1}"] \arrow[ddr, "\alpha_{d_N}"'] \arrow[dd, bend left, "h_1"] & \\
 & \vdots \\
L_2 \arrow[uur, "\beta_{d_1}"] \arrow[dr, "\beta_{d_N}"'] \arrow[uu, bend left, "h_2"] & \\
 & d(i_N)
\end{tikzcd}
\tag{11}
$$

where $N = \{i_1, \ldots, i_N\}$. Substituting for b_i in the second equation, we get

$$h_1; h_2; \alpha_i = \alpha_i, \ \forall i \in N$$

Since L_1 is also a cone and a limit, there exists a unique $g_1 : L_1 \to L_1$, such that $g_1; \alpha_i = \alpha_i, \ \forall i \in N$. $g_1 = 1_{L_1}$ satisfies this, and so does $g_1 = h_1; h_2$. Since g_1 is unique, we get that $h_1; h_2 = 1_{L_1}$.

We can similarly derive that $h_2; h_1 = 1_{L_2}$. So, L_1 and L_2 are isomorphic to each other, and h_1, h_2 are isomorphisms.

3. **Exercise 4, Slide 16**: Explain why in **Set**, the product of an empty set of sets is a one-element set.

   Consider the category **Set**, and the empty diagram (empty set of sets). Then the product of this diagram would be the limit of the diagram. As we know, the limit of an empty diagram is the final object $\star \in |\mathbf{Set}|$, such that there is a unique morphism $!_A : A \to \star$.

   In case of **Set**, any singleton set $\{*\}$ is a final object:

   - for any set $X \in |\mathbf{Set}|$, then we can define $f : X \to \{*\}$, as $f(x) = *$, for all $x \in X$. This is the only function in $\mathbf{Set}(X, \{*\})$, making it the unique morphism

   All other singleton sets are isomorphic to each other.

4. **Exercise 5, Slide 17**: Show that $A \times B \simeq B \times A$ for any $A, B \in |\mathcal{C}|$.

Consider any $A, B \in |\mathcal{C}|$. Consider the following diagram:

$$
\begin{tikzcd}
& A & \\
A \times B \arrow[ur, bend left, "\pi_A^{A\times B}"] \arrow[dr, bend right, "\pi_B^{A\times B}"'] \arrow[rr, bend left, "h_1"] & & B \times A \arrow[ul, bend right, "\pi_A^{B\times A}"'] \arrow[dl, bend left, "\pi_B^{B\times A}"] \arrow[ll, bend left, "h_2"] \\
& B &
\end{tikzcd}
$$

$A \times B$ and $B \times A$ are both products of the discrete diagram containing A and B. They are limits, but they are cones as well.

- Since $A \times B$ is also a cone, there exists a unique $h_1 : A \times B \to B \times A$, such that

$$
\begin{aligned}
h_1; \pi_A^{B\times A} &= \pi_A^{A\times B} \\
h_1; \pi_B^{B\times A} &= \pi_B^{A\times B}
\end{aligned}
$$

- Since $B \times A$ is also a cone, there exists a unique $h_2 : B \times A \to A \times B$, such that

$$
\begin{aligned}
h_2; \pi_A^{A\times B} &= \pi_A^{B\times A} \\
h_2; \pi_B^{A\times B} &= \pi_B^{B\times A}
\end{aligned}
$$

On eliminating $\pi_A^{B\times A}$ and $\pi_B^{A\times B}$, we get

$$
\begin{aligned}
h_1; h_2; \pi_A^{A\times B} &= \pi_A^{A\times B} \\
h_2; h_1; \pi_B^{B\times A} &= \pi_B^{B\times A}
\end{aligned}
$$

It is easy to see that there exist a unique $g_1 : A \times B \to A \times B$, such that $g_1; \pi_A^{A\times B} = \pi_A^{A\times B}$. Since $g_1 = 1_{A\times B}$ and $g_1 = h_1; h_2$ both satisfy this, we get $1_{A\times B} = h_1; h_2$.

Similarly, we can derive that $1_{B\times A} = h_2; h_1$. This means that $A \times B$ is isomorphic to $B \times A$ and h_1, h_2 are isomorphisms.

5. **Exercise 6, Slide 17**: Why the morphism $f_1 \times f_2$ exists and is unique?

Consider a category $\mathcal{C}$ with the following diagram:

$$
\begin{tikzcd}
& A_1 \arrow[rr, "f_1"] & & B_1 \\
A_1 \times A_2 \arrow[ur, "\pi_{A_1}"] \arrow[dr, "\pi_{A_2}"'] \arrow[rr, "f_1 \times f_2"] & & B_1 \times B_2 \arrow[ur, "\pi_{B_1}"'] \arrow[dr, "\pi_{B_2}"] & \\
& A_2 \arrow[rr, "f_2"] & & B_2
\end{tikzcd}
$$

To argue that a unique $f_1 \times f_2 : A_1 \times A_2 \to B_1 \times B_2$ exists such that the diagram commutes, we can see that $A_1 \times A_2$ is a cone for the discrete diagram with B_1 and B_2, and $B_1 \times B_2$ is the limit. There exists a unique $\langle \pi_{A_1}; f_1, \pi_{A_2}; f_2 \rangle : A_1 \times A_2 \to B_1 \times B_2$, such that

$$\langle \pi_{A_1}; f_1, \pi_{A_2}; f_2 \rangle; \pi_{B_1} = \pi_{A_1}; f_1$$
$$\langle \pi_{A_1}; f_1, \pi_{A_2}; f_2 \rangle; \pi_{B_2} = \pi_{A_2}; f_2$$

The function $\langle \pi_{A_1}; f_1, \pi_{A_2}; f_2 \rangle$ is denoted as $f_1 \times f_2$.

6. <u>**Exercise 7, Slide 17**: Show that $A \simeq \star \times A$ for any $A \in |\mathcal{C}|$.</u>

   Consider any category $\mathcal{C}$, with any object $A \in |\mathcal{C}|$, and a final object $\star \in |\mathcal{C}|$. Then there exists a unique morphism $!_A : A \to \star$. Consider the product $\star \times A$ of the discrete diagram containing $\star$ and A. The following diagram holds (may not commute necessarily):

$$\begin{tikzcd} & \star \\ \star \times A \arrow[ur, "\pi_\star"] \arrow[dr, "\pi_A"'] & \\ & A \arrow[uu, "!_A"'] \end{tikzcd}$$

   We can see that $(A, \{1_A, !_A\})$ is a cone of the discrete diagram containing $\star$ and A. So, there exists a unique $h : A \to \star \times A$, such that

$$h; \pi_\star = \ !_A$$
$$h; \pi_A = \ 1_A$$

   We can left compose the last equation with π_A, to get $(\pi_A; h); \pi_A = \pi_A$. Now, $\star \times A$ is a limit, as well as a cone. So, there exists a unique $g : \star \times A \to \star \times A$, such that

$$g; \pi_A = \pi_A$$
$$g; \pi_\star = \pi_\star$$

   $g = 1_{\star \times A}$ and $g = \pi_A; h$ both satisfy the first equation, and since g is unique, we get $\pi_A; h = 1_{\star \times A}$. We can now infer that A and $\star \times A$ are isomorphic, and h, π_A are isomorphisms.