Exercise 1 Equilibrium in the Solow-Swan model is characterized by the market clearing condition $S = I$. With a fixed savings rate of s, total savings are $S = sY$. Investment are given by the increase in the capital stock plus capital depreciation $I = \dot{K} + \delta K$. Thus market clearing implies

$$\dot{K} + \delta K = sY \Rightarrow \dot{K} = sF(K, L) - \delta K$$

Now transform this equation into per capita terms by diving through the total labour force L

$$\frac{\dot{K}}{L} = sF\left(\frac{K}{L}, 1\right) - \delta\frac{K}{L} = sf(k) - \delta k$$

and note that

$$\dot{k} = \left(\frac{\dot{K}}{L}\right) = \frac{\dot{K}L - K\dot{L}}{L^2} = \frac{\dot{K}}{L} - k\eta \Rightarrow \frac{\dot{K}}{L} = \dot{k} + \eta k$$

from where it follows that

$$\dot{k} = sf(k) - (\eta + \delta)k \tag{1}$$

(a) In steady state $\dot{k} = 0$, thus

$$s\frac{f(k^*)}{k^*} = \eta + \delta \tag{2}$$

Now we are given a Cobb-Douglas production function of the form

$$Y = F(K, L) = K^{1/3}L^{2/3}$$

The intensive form of this function is obtained by diving through L

$$f(k) = F\left(\frac{K}{L}, 1\right) = k^{1/3}$$

Plugging this functional form into the above definition of the steady state yields

$$s(k^*)^{\alpha-1} = \eta + \delta \Rightarrow k^* = \left(\frac{s}{\eta + \delta}\right)^{\frac{1}{1-\alpha}}$$

Per capita output in steady state is

$$y^* = f(k^*) = (k^*)^\alpha = \left(\frac{s}{\eta+\delta}\right)^{\frac{\alpha}{1-\alpha}}$$

(b) Profit maximisation by firms' leads to the first order condition (see p. 23)

$$f'(k) = r + \delta$$

so the real interest rate is given by

$$r = \alpha k^{\alpha-1} - \delta$$

In steady state the real interest rate is

$$r^* = \frac{\alpha(\eta+\delta)}{s} - \delta$$

(c) Consumption is equal total production minus savings

$$C = Y - sY = (1-s)Y.$$

So per capital consumption is $c = (1-s)y = (1-s)f(k)$. In steady state

$$c^* = (1-s)y^* = (1-s)\left(\frac{s}{\eta+\delta}\right)^{\frac{\alpha}{1-\alpha}}$$

and steady state per capita savings are

$$sy^* = s\left(\frac{s}{\eta+\delta}\right)^{\frac{\alpha}{1-\alpha}}$$

Exercise 2 The golden rule savings rate, is the savings rate that maximises steady state per capita consumption over the steady state per capita capital input. Steady state consumption is given by

$$c^*(k^*) = (1-s)f(k^*) = f(k^*) - sf(k^*) = f(k^*) - (\eta+\delta)k^*$$

using equation (2) for the last step. If we maximise c^* w.r.t. s we get the following first order condition

$$\frac{dc^*}{dk^*} = f'(k^*) - (\eta + \delta) = 0.$$

This gives the maximum as c^* is globally concave in k^*: $c^{*''}(k^*) = f''(k^*) < 0$. Therefore, the golden-rule level of per capita capital k_g is implicitly defined by

$$f'(k_g) = \eta + \delta \tag{3}$$

(a) The golden-rule optimal level of per capita output is

$$\alpha k_g^{\alpha-1} = \eta + \delta$$

The steady state level of per capita capital is

$$k^* = \left(\frac{s}{\eta + \delta}\right)^{\frac{1}{1-\alpha}}$$

If $s = \alpha$, k^* and k_g coincide.

(b) If $s = 2\alpha$, $k^* > k_g$ and thus the economy is dynamically inefficient. To see this note that as per capita consumption is maximised at the golden-rule optimal level of per capita capital, any level of k that exceed k_g leads to a drop in steady state per capital consumption (recall that $c^{*''}(k^*) < 0$ and $c^{*'}(k_g) = 0$). We could therefore decrease savings and increase consumption at the same time. This could for example be achieved by "eating" $k^* - k_g$ of the capital stock.

(c) Recall the definition of steady state consumption as

$$c^* = f(k^*) - (\eta + \delta)k^* = (k^*)^\alpha - (\eta + \delta)k^*$$

Now for the golden rule to hold

$$\alpha k_g^{\alpha-1} = \eta + \delta \Rightarrow k_g^\alpha = \left(\frac{\eta + \delta}{\alpha}\right) k_g$$

So

$$c_g = \left[\frac{\eta + \delta}{\alpha} - (\eta + \delta)\right] k_g = \left(\frac{1-\alpha}{\alpha}\right)(\eta + \delta)k_g$$

Now

$$k_g = \left(\frac{\alpha}{\eta+\delta}\right)^{\frac{1}{1-\alpha}}$$

from where we get

$$c_g = \left(\frac{1-\alpha}{\alpha}\right)(\eta+\delta)\left(\frac{\alpha}{\eta+\delta}\right)^{\frac{1}{1-\alpha}} = (\eta+\delta)\left(\frac{1}{2(\eta+\delta)}\right)^2 = \frac{1}{4}\left(\frac{1}{\eta+\delta}\right)$$

for $\alpha = 0.5$. So

$$\frac{\partial c_g}{\partial \eta} = -\frac{1}{4}\left(\frac{1}{\eta+\delta}\right)^2 < 0$$

Exercise 3 (a)and(b) (see exercise 1(a))

$$s\frac{log_{10}k^*}{k^*} = \eta + \delta$$

(c) (see exercise 4(d))

Exercise 4 (a) Again the steady state per capital capital stock is characterised by equation (2).

$$s\frac{f(k^*)}{k^*} = \eta + \delta \Rightarrow s(k^*)^{-1/3} = \eta + \delta$$

so the steady state capital stock is

$$k^* = \left(\frac{s}{\eta+\delta}\right)^{3/2}.$$

(b) Firms' profits are given by

$$\text{pr} = [(1-\delta)k + f(k)] - (1+r)k$$

Profit maximisation implies to following first order condition

$$f'(k) = r + \delta = \pi \Rightarrow \pi = \frac{2}{3}k^{-1/3}.$$

So in the steady state profit rate is

$$\pi^* = \frac{2}{3}(k*)^{-1/3} = \frac{2}{3}\left(\frac{\eta+\delta}{s}\right)^{1/2}$$

The wage rate is the marginal product of labour

$$w = \frac{\partial}{\partial L}F(K,L) = \frac{\partial}{\partial L}[Lf(k)] = f(k) - kf'(k)$$

So the steady state wage rate w^* is

$$w^* = (k^*)^{2/3} - \frac{2}{3}(k^*)^{2/3} = \frac{1}{3}(k^*)^{2/3} = \frac{1}{3}\left(\frac{s}{\eta+\delta}\right)$$

(c) Per capita savings are $sy^* = sf(k^*) = s(k^*)^{2/3}$ and per capital consumption is $c^* = (1-s)(k^*)^{2/3}$.

(d) Initially the economy is in steady state, with zero growth. Let's call this steady state k_1^*:

$$\gamma_k = s(k_1^*)^{2/3} - (\eta+\delta)(k_1^*) = 0$$

For this capital stock, if the savings rate rises to $s' > s$, the growth rate of the capital stock γ_k will become positive

$$\gamma_k = s'(k_1^*)^{2/3} - (\eta+\delta)(k_1^*) > 0$$

The economy will start growing, both per capital capital and output go up. This will continue until the economy reaches its new steady state $k_2^* > k_1^*$

$$s'(k_2^*)^{2/3} - (\eta+\delta)(k_2^*) = 0$$

at which both per capita capital and output are higher than in the previous steady state. Per capita growth rates are however again zero.

(e) The golden-rule optimal per capita capital stock is given by

$$f'(k_g) = \eta+\delta \Rightarrow \frac{2}{3}k_g^{-1/3} = \eta+\delta \Rightarrow k_g = \left(\frac{2}{3(\eta+\delta)}\right)^{\frac{3}{2}}$$

The steady state capital stock is given by

$$k^* = \left(\frac{s}{\eta + \delta}\right)^{3/2}$$

The economy is dynamically inefficient if $k^* > k_g$, i.e.

$$\left(\frac{s}{\eta + \delta}\right)^{3/2} > \left(\frac{2}{3(\eta + \delta)}\right)^{3/2} \Rightarrow s > \frac{2}{3}$$

irrespective of the value of δ and η.

Exercise 5 (a) (see exercise 4(a))

$$k^* = \left(\frac{s}{\eta + \delta}\right)^{2}$$

(b) (see exercises 4(b) and (c))

$$\begin{aligned}
\pi^* &= \frac{1}{2}(k^*)^{-1/2} \\
w^* &= \frac{1}{2}(k^*)^{1/2} \\
sy^* &= s(k^*)^{1/2} \\
c^* &= (1 - s)(k^*)^{1/2}
\end{aligned}$$

As can easily be seen, the steady state capital stock declines as η increases. Thus wages, pc savings, and pc consumption fall when η increases. The profit rate however rises, as capital becomes relatively scarce as compared to labour.

(c) (see exercise 4(e)and 2(a)) The economy is dynamically inefficient as long as $s > 1/2$ irrespective of η.

Exercise 6 We will consider the fixed proportion (Leontieff) production function

$$Q = F(K, L) = \min\left\{\frac{K}{a}, \frac{L}{b}\right\}$$

with $a > 0$ and $b > 0$.

a. 1) Calculate the marginal product:

Let us first calculate the marginal product of capital. We will have to consider 2 cases
i) $K/a < L/b$ and ii) $K/a \geq L/b$

i) for $K/a < L/b$ the production function will be

$$Q = \frac{K}{a}$$

and thus the marginal product of capital will be

$$\frac{\partial Q}{\partial K} = \frac{1}{a} > 0$$

ii) for $K/a \geq L/b$ the production function is

$$Q = \frac{L}{b}$$

and the marginal product

$$\frac{\partial Q}{\partial K} = 0$$

Same procedure for the marginal product of labour
i) for $K/a < L/b$

$$\frac{\partial Q}{\partial L} = \frac{1}{\partial L}\left(\frac{K}{a}\right) = 0$$

ii) for $K/a \geq L/b$

$$\frac{\partial Q}{\partial L} = \frac{1}{\partial L}\left(\frac{L}{b}\right) = \frac{1}{b} > 0$$

2) Average products:
The average product of capital is defined as Q/K, thus
i) for $K/a < L/b$, $Q/K = 1/a$
ii) for $K/a \geq L/b$, $Q/K = L/(bK)$

The average product of labour is Q/L, thus
i) for $K/a < L/b$, $Q/L = K/(bL)$
ii) for for $K/a \geq L/b$, $Q/L = 1/b$.

graph for unit isoquant here

b. 1) **Definition**: A product function has *constant returns to scale* (CRS), if for all $\lambda > 0$

$$F(\lambda K, \lambda L) = \lambda F(K, L)$$

Let's verify this for the fixed proportions production function

$$F(\lambda K, \lambda L) = \min\left\{\frac{\lambda K}{a}, \frac{\lambda L}{b}\right\} = \lambda \min\left\{\frac{K}{a}, \frac{L}{b}\right\} = \lambda F(K, L)$$

So this production function indeed has constant returns to scale.

2) **Definition**: A function $f : X \to \mathbb{R}$ is *quasi-concave* if for $x, y \in X$ we have

$$f(\lambda x + (1-\lambda)y) \geq \min\{f(x), f(y)\}$$

for all $0 < \lambda < 1$. (Note: Any concave function is also quasi-concave. The reverse is not necessarily true.)

So let's check this for our fixed proportions production function. Take two admissible values (K_1, L_1) and (K_2, L_2). Then

$$F\left(\frac{\lambda K_1 + (1-\lambda)K_2}{a}, \frac{\lambda L_1 + (1-\lambda)L_2}{b}\right) = \min\left\{\frac{\lambda K_1 + (1-\lambda)K_2}{a}, \frac{\lambda L_1 + (1-\lambda)L_2}{b}\right\}$$

$$\geq \min\left\{\min\left\{\frac{K_1}{a}, \frac{L_1}{b}\right\}, \min\left\{\frac{K_2}{a}, \frac{L_2}{b}\right\}\right\} = \min\{F(K_1, L_1), F(K_2, L_2)\}$$

So this function is indeed quasi-concave.

3) To see that both factors are essential, simply note that $F(0, L) = 0$, and $F(K, 0) = 0$.

c. To solve for Solow's basic differential equation, let us first transform the production function into intensive form. As we have shown that the production function has constant returns to scale, we can simply divide it by L to get

$$f(k) = \min\left\{\frac{k}{a}, \frac{1}{b}\right\}$$

For all $k/a \leq 1/b$, the production function will be $f(k) = k/a$, for all $k/a > 1/b$ it will be $f(k) = 1/b$.

So we get two basic differential equations depending on these two cases:

i) for $k/a \leq 1/b$:

$$\frac{\dot{k}}{k} = \frac{s}{a} - (\eta + \delta)$$

ii) for $k/a > 1/b$:

$$\frac{\dot{k}}{k} = \frac{s}{bk} - (\eta + \delta)$$

Let us assume that $s/a > \eta + \delta$, for otherwise k would always decrease, and the only possible steady state would be $k^* = 0$. With this assumption, the capital stock k will eventually exceed the value of $1/b$ and thus the steady state is defined by

$$\frac{s}{bk^*} = (\eta + \delta) \Rightarrow k^* = \frac{s}{b(\eta + \delta)}$$

The real rental rate π is given by

$$\pi = f'(k)$$

Thus, for $k/a < 1/b$, $\pi = 1/a$, and for $k/a > 1/b$, $\pi = 0$. (Note that there is a kink in the production function at k = a/b, so the derivative at this point is not well defined.)

Real wages w are given by

$$w = f(k) - kf'(k)$$

Thus for $k/a < 1/b$

$$w = \frac{k}{a} - k\left(\frac{1}{a}\right) = 0$$

and for $k/a > 1/b$, $w = 1/b$.

So suppose we start with an initial level of per capita capital below a/b. Then initially capital is relatively rare and labour is relatively abundant. The wage rate will be zero, the rental rate will be positive. The capital stock will start growing until it reaches its steady state value of $k^* > a/b$. Once k exceeds a/b labour will now be relatively scarce and the factor capital abundant, thus the rental rate will drop to zero, the wage rate will suddenly become positive. At $k = a/b$, the exact rental and wage rate cannot be pinned down. Any values for π and w such that $f(a/b) = w + \pi(a/b)$ are possible. (Remember that the isoquants of this production function have a kink at $K/a = L/b$.)

The output - capital ratio is given by Q/K. We can divide both numerator and denominator by L to see that this is equivalent to $f(k)/k$. So the capital - output ratio is

i) for $k/a \leq 1/b$, $f(k)/k = 1/a$

ii) for $k/a > 1/b$, $f(k)/k = 1/(bk)$.