Hierarchical Abstraction of Phase Response Curves of Synchronized Systems of Coupled Oscillators

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***Abstract*—We prove that a group of injection-locked oscillators, each modelled using a nonlinear phase macromodel, responds as a single oscillator to small external perturbations. More precisely, we show that any group of injection-locked oscillators has a single effective PRC [1] or PPV [2], [3] that characterises its phase/timing response to small external perturbations. This result constitutes a foundation for understanding and predicting synchronization/timing hierarchically in large, complex systems that arise in nature and engineering.**

I. PRC/PPV Phase Macromodels

Given an ODE or DAE description

$$\frac{d}{dt}\,\vec{q}(\vec{x}(t)) + \vec{f}(\vec{x}) + \vec{b}(t) = \vec{0} \tag{1}$$

of an oscillator with an orbitally stable T-periodic autonomous solution $\vec{x}_s(t)$, it can be shown [2], [4] that the timing jitter or phase characteristics of the oscillator, under the influence of small perturbations $\vec{b}(t)$, can be captured by the nonlinear scalar differential equation

$$\frac{d}{dt}\,\alpha(t) = \vec{v}_1^T(t+\alpha(t))\cdot\vec{b}(t), \tag{2}$$

where the quantity $\vec{v}_1(\cdot)$, a T-periodic function of time, is known as the *Phase Response Curve* (PRC) [1] or *Perturbation Projection Vector* (PPV) [2], [3].

For convenience, we scale the time axis to normalize all periods to 1. Define a 1-periodic version of the steady state solution to be

$$\vec{x}_p(t) = \vec{x}_s(tT), \tag{3}$$

and a 1-periodic version of the PPV to be

$$\vec{p}(t) = \vec{v}_1(tT). \tag{4}$$

Using these 1-periodic quantities and defining $f \triangleq \frac{1}{T}$, (2) can be expressed as

$$\frac{d}{dt}\,\alpha(t) = \vec{p}^T\left(ft + f\alpha(t)\right)\cdot\vec{b}(t). \tag{5}$$

Defining *phase* to be

$$\phi(t) = ft + f\alpha(t), \tag{6}$$

(5) becomes

$$\frac{d}{dt}\,\phi(t) = f + f\vec{p}^T(\phi(t))\cdot\vec{b}(t). \tag{7}$$

$x(t)$, the solution of (1), can often be approximated usefully by a phase-shifted version of its unperturbed periodic solution, *i.e.*,

$$\vec{x}(t) \simeq \vec{x}_s(T\phi(t)) = \vec{x}_p(\phi(t)). \tag{8}$$

(2) (or equivalently, (7)) is termed the *PPV equation* or *PPV phase macromodel*. In the absence of any perturbation $\vec{b}(t)$, note that $\alpha(t) \equiv 0$ (w.l.o.g), $\vec{x}(t) = \vec{x}_s(t) = \vec{x}_p(ft)$ and $\phi(t) = ft$. We will call the latter the *phase of natural oscillation* and denote it by $\phi^\diamond(t) \triangleq ft$.

II. Derivation of Hierarchical PPV Macromodel

A. Coupled Phase System and its Properties

1) Coupled system of PPV phase macromodels: Consider a group of $N \geq 2$ coupled oscillators (Figure 1). We model each oscillator by its PPV equation (7):

$$\frac{d}{dt}\,\phi_i(t) = f_i + f_i\vec{p}_i^T\left(\phi_i(t)\right)\cdot\vec{b}_i(t), \quad i = 1,\cdots,N, \tag{9}$$

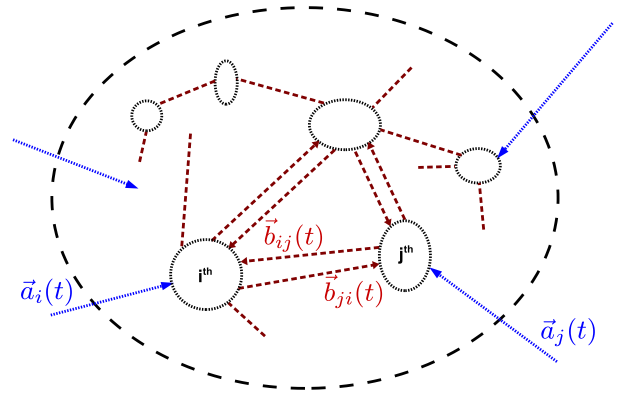

Fig. 1. Oscillator system with internal coupling and external inputs.

where i-subscripted quantities refer to the i^{th} oscillator. Inputs to each oscillator are drawn from two sources (as depicted in Figure 1): 1) internal couplings with other oscillators, and 2) external sources. $\vec{b}_i(t)$ can therefore be written as

$$\vec{b}_i(t) = \vec{a}_i(t) + \sum_{\substack{j\neq i\\ j=1}}^{N} \vec{b}_{ij}\left(\phi_j(t)\right), \tag{10}$$

where $\vec{a}_i(t)$ is the external input (*i.e.*, from outside the group of N oscillators) to the i^{th} oscillator, and $\vec{b}_{ij}\left(\phi_j(t)\right)$ represents the influence of the j^{th} oscillator on the i^{th}.

We make the natural assumption that the $\vec{b}_{ij}(\cdot)$ are 1-periodic — *i.e.*, that each oscillator generates outputs that follow its own phase and timing properties; it is these outputs that couple internally to the inputs of other oscillators. Note that as i varies, the dimensions of $\vec{p}_i(t)$, $\vec{a}_i(t)$ and $\vec{b}_{ij}$ can differ, since they depend on the size of the i^{th} oscillator's differential equations.

The system of N equations (9) can be written in vector ODE form as

$$\frac{d}{dt}\,\vec{\phi}(t) = \vec{g}_\phi\left(\vec{\phi}(t)\right) + \vec{b}_\phi\left(\vec{\phi}(t),t\right), \tag{11}$$

where

$$\vec{\phi}(t) \triangleq \begin{bmatrix}\phi_1(t)\\ \vdots\\ \phi_N(t)\end{bmatrix}, \tag{12}$$

$$\vec{g}_\phi(\vec{\phi}) \triangleq \begin{bmatrix} f_1 + f_1\vec{p}_1^T(\phi_1)\cdot\sum_{j=2}^{N}\vec{b}_{1j}\left(\phi_j(t)\right)\\ \vdots\\ f_N + f_N\vec{p}_N^T(\phi_N)\cdot\sum_{j=1}^{N-1}\vec{b}_{Nj}\left(\phi_j(t)\right)\end{bmatrix}, \tag{13}$$

$$\text{and } \vec{b}_\phi(\vec{\phi},t) \triangleq \begin{bmatrix} f_1\vec{p}_1^T(\phi_1)\cdot\vec{a}_1(t)\\ \vdots\\ f_N\vec{p}_N^T(\phi_N)\cdot\vec{a}_N(t)\end{bmatrix}. \tag{14}$$

We will refer to (11) as the *Coupled Phase System (CPS)*.

Note that $\vec{b}_\phi(\vec{\phi},t) \equiv \vec{0}$ in the absence of inputs external to the group of oscillators, *i.e.*, when $a_i(t) \equiv \vec{0}$. Note also that $\vec{g}_\phi(\vec{\phi})$ is 1-periodic in each component of $\vec{\phi}$, *i.e.*, it is 1-periodic in each ϕ_i. Such functions are termed *cylindrical* [5, page 236].

2) Locking in the absence of external inputs:

Assumption 2.1: In the absence of external inputs (*i.e.*, if $\vec{a}_i(t) \equiv \vec{0}, \forall i$), assume that the group of N oscillators locks stably[1] to a common frequency f^* (equivalently, with a common period $T^* = \frac{1}{f^*}$). Denote the phase of the i^{th} oscillator, when locked as in Assumption 2.1 to the other oscillators in the group, by $\phi_i^*(t)$. Note that this phase will typically be different from the oscillator's phase of natural oscillation $\phi_i^\diamond(t) = f_i t$, on account of inputs via coupling from other oscillators in the group.

Denoting

$$\vec{\phi}^*(t) \triangleq \begin{bmatrix} \phi_1^*(t) \\ \vdots \\ \phi_N^*(t) \end{bmatrix}, \tag{15}$$

note that $\vec{\phi}^*(t)$ satisfies (11) with no external inputs, *i.e.*,

$$\frac{d}{dt}\,\vec{\phi}^*(t) = \vec{g}_\phi\left(\vec{\phi}^*(t)\right). \tag{16}$$

We term $\vec{\phi}^*(t)$ the *system phase during externally-unperturbed lock*.

3) D-periodicity of $\vec{\phi}^*(t)$: T^*-periodicity of each oscillator locked at frequency f^* implies that

$$\vec{x}_i(t) = \vec{x}_{p,i}(\phi_i^*(t)) \tag{17}$$

is T^*-periodic $\forall i$; *i.e.*,

$$\begin{aligned} \vec{x}_i(t+T^*) &= \vec{x}_i(t), \quad \text{or} \\ \vec{x}_{p,i}(\phi_i^*(t+T^*)) &= \vec{x}_{p,i}(\phi_i^*(t)). \end{aligned} \tag{18}$$

From definition, $\vec{x}_{p,i}(\cdot)$ is 1-periodic. (18) is satisfied for *arbitrary* 1-periodic $\vec{x}_{p,i}(\cdot)$ iff

$$\phi_i^*(t+T^*) = n + \phi_i^*(t), \quad \forall t, \quad n \in \mathbb{Z}. \tag{19}$$

Define the *ideal phase of oscillation at frequency f^** to be

$$\phi_{\text{ideal}}^*(t) = f^* t. \tag{20}$$

$\phi_{\text{ideal}}^*(t)$ satisfies (19) (with $n=1$), but note that the phase of each locked oscillator in the system need not necessarily equal $\phi_{\text{ideal}}^*(t)$. A more general form for $\phi_i^*(t)$ that also satisfies (19) is

$$\phi_i^*(t) = \phi_{\text{ideal}}^*(t) + \Delta\phi_i^*(t) = f^* t + \Delta\phi_i^*(t), \tag{21}$$

where $\Delta\phi_i^*(t)$ is itself T^*-periodic[2]. Equivalently[3],

$$\vec{\phi}^*(t) = \phi_{\text{ideal}}^*(t) + \overrightarrow{\Delta\phi}^*(t) = f^* t + \overrightarrow{\Delta\phi}^*(t), \tag{22}$$

where

$$\overrightarrow{\Delta\phi}^*(t) \triangleq \begin{bmatrix} \Delta\phi_1^*(t) \\ \vdots \\ \Delta\phi_N^*(t) \end{bmatrix} \tag{23}$$

is T^*-periodic. Functions of the form (22) are termed *D-periodic* or *derivo-periodic* with period T^* [5]. $\{\Delta\phi_i^*(t)\}$ represent short-term phase changes within each cycle that do not affect the long-term frequency of the oscillator.

The above considerations motivate:

Assumption 2.2: $\vec{\phi}^*(t)$, the phase of the CPS during externally-unperturbed lock, is T^*–D-periodic.

4) Arbitrary time shifts of $\vec{\phi}^*(t)$ are also solutions of the CPS:

Lemma 2.1: The phase during externally-unperturbed lock, $\vec{\phi}^*(t)$, is not unique. Indeed, for any arbitrary time-shift τ,

$$\vec{\phi}^*(t-\tau) \tag{24}$$

solves (16).

Proof: Follows directly from substituting (24) in the autonomous system (16) and using the facts that 1) $\vec{g}_\phi(\cdot)$ in (13) is cylindrical with period 1, and 2) $\vec{\phi}^*(t)$ is T^*–D-periodic (Assumption 2.2). ■

B. Periodic time-varying linearization of the CPS

1) Linearization under small-deviation assumption: If the external inputs $\{\vec{a}_i(t)\}$ are small, then $\vec{b}_\phi(\vec{\phi},t)$ is small and (11) constitutes a small perturbation of (16). We express $\vec{\phi}(t)$, the solution of (11), as a deviation from $\vec{\phi}^*(t)$, the solution of (16):

$$\vec{\phi}(t) = \vec{\phi}^*(t) + \overrightarrow{\delta\phi}(t). \tag{25}$$

We term $\overrightarrow{\delta\phi}(t)$ the *orbital deviation*. Using (25), we now attempt to solve (11) via linearization.

Assumption 2.3: $\overrightarrow{\delta\phi}(t)$ remains small for all t, provided the external input $\vec{b}_\phi(\cdot,\cdot)$ is small enough for all t.[4]

Applying Assumption 2.3, we start the process of linearizing (11):

$$\begin{aligned} \frac{d}{dt}\,\vec{\phi}^*(t) + \frac{d}{dt}\,\overrightarrow{\delta\phi}(t) \simeq \vec{g}_\phi\left(\vec{\phi}^*(t)\right) &+ \frac{\partial \vec{g}_\phi}{\partial \vec{\phi}}\left(\vec{\phi}^*(t)\right)\overrightarrow{\delta\phi}(t) \\ &+ \vec{b}_\phi\left(\vec{\phi}(t),t\right). \end{aligned} \tag{26}$$

Using (16), we obtain

$$\begin{aligned} \frac{d}{dt}\,\overrightarrow{\delta\phi}(t) &\simeq \frac{\partial \vec{g}_\phi}{\partial \vec{\phi}}\left(\vec{\phi}^*(t)\right)\overrightarrow{\delta\phi}(t) + \vec{b}_\phi\left(\vec{\phi}(t),t\right) \\ &\simeq \frac{\partial \vec{g}_\phi}{\partial \vec{\phi}}\left(\vec{\phi}^*(t)\right)\overrightarrow{\delta\phi}(t) + \vec{b}_\phi\left(\vec{\phi}^*(t),t\right) \\ &\qquad + \frac{\partial \vec{b}_\phi}{\partial \vec{\phi}}\left(\vec{\phi}^*(t),t\right)\overrightarrow{\delta\phi}(t) \\ &= \left(\frac{\partial \vec{g}_\phi}{\partial \vec{\phi}}\left(\vec{\phi}^*(t)\right) + \frac{\partial \vec{b}_\phi}{\partial \vec{\phi}}\left(\vec{\phi}^*(t),t\right)\right)\overrightarrow{\delta\phi}(t) \\ &\qquad + \vec{b}_\phi\left(\vec{\phi}^*(t),t\right). \end{aligned} \tag{27}$$

From the definition of $\vec{b}_\phi(\cdot,\cdot)$ (14), observe that $\frac{\partial \vec{b}_\phi}{\partial \vec{\phi}}\left(\vec{\phi}^*(t),t\right)$ is a diagonal matrix with entries

$$f_i \vec{p}_i'^{T}(\phi_i^*(t)) \cdot \vec{a}_i(t),$$

i.e., it is directly proportional to the external inputs $\{\vec{a}_i(t)\}$, which are small by assumption. Therefore, the product term $\frac{\partial \vec{b}_\phi}{\partial \vec{\phi}}\left(\vec{\phi}^*(t),t\right)\overrightarrow{\delta\phi}(t)$ in (27) is of second order and can be dropped from the linearization. Applying this observation and denoting

$$J_\phi^*(t) \triangleq \frac{\partial \vec{g}_\phi}{\partial \vec{\phi}}\left(\vec{\phi}^*(t),t\right), \text{ and} \tag{28}$$

$$\vec{b}_{\text{ext}}(t) \triangleq \vec{b}_\phi\left(\vec{\phi}^*(t),t\right), \tag{29}$$

(27) becomes

$$\boxed{\frac{d}{dt}\,\overrightarrow{\delta\phi}(t) \simeq J_\phi^*(t)\overrightarrow{\delta\phi}(t) + \vec{b}_{\text{ext}}(t)}. \tag{30}$$

(30) is the linearization of the CPS (11) around its externally-unperturbed solution $\vec{\phi}^*(t)$.

2) T^*-periodicity of $J_\phi^*(t)$: From (13), we can obtain expressions for the entries of $J_\phi^*(t)$. The diagonal entries of J_ϕ^* are

$$J_{\phi\ i,i}^*(t) = f_i \vec{p}_i'^{T}(\phi_i^*(t)) \cdot \sum_{\substack{j \neq i \\ j=1}}^{N} \vec{b}_{ij}\left(\phi_j^*(t)\right), \tag{31}$$

[1] Section II-E will expand on the notion of lock stability.

[2] $\Delta\phi_i^*(t)$ can in fact itself satisfy (19) with arbitrary n, but $n \neq 0$ would make the long-term frequency of $x_{p,i}(\phi_i(t))$ different from f^*, violating Assumption 2.1.

[3] We use the notation, borrowed from MATLAB, that the sum of a scalar and a vector means that the scalar is added to each element of the vector.

[4] *i.e.*, $\|\overrightarrow{\delta\phi}(t)\| < M\|\vec{b}_\phi(\cdot,\cdot)\|$ for some finite constant $M > 0$.

while the off-diagonal entries are

$$J^*_{\phi\ i,j}(t) = f_i \vec{p}_i^{\,T}(\phi_i^*(t)) \cdot \vec{b}_{ij}'\left(\phi_j^*(t)\right). \tag{32}$$

Because of the 1-periodicity of $\vec{p}_i(\cdot)$ and $\vec{b}_{ij}(\cdot)$, and the T^*–D-periodicity of $\phi_i^*(t)$, each entry of J_ϕ^* is T^*-periodic, hence the entire matrix function $J_\phi^*(t)$ is T^*-periodic. The linearized CPS (30) is therefore periodically time varying, *i.e.*, it is a *linear periodically time varying (LPTV)* system.

C. T^*-periodic homogeneous solution of the linearized CPS

Lemma 2.2: The homogeneous part of the linearized CPS (30), *i.e.*,

$$\frac{d}{dt}\,\overrightarrow{\delta\phi}(t) = J_\phi^*(t)\,\overrightarrow{\delta\phi}(t), \tag{33}$$

has the T^*-periodic solution

$$\overrightarrow{\delta\phi}^*(t) \triangleq \frac{d}{dt}\,\vec{\phi}^*(t). \tag{34}$$

Proof: Follows immediately from differentiating (16). Note that T^*–D-periodicity of $\vec{\phi}^*(t)$ immediately implies that $\overrightarrow{\delta\phi}^*(t)$ in (34) is T^*-periodic, since

$$\overrightarrow{\delta\phi}^*(t) = f^* + \frac{d}{dt}\,\overrightarrow{\Delta\phi}^*(t), \tag{35}$$

with the latter term T^*-periodic. ■

D. Floquet-theoretic solution of the linearized CPS

Floquet theory [5] provides an analytical form[5] for the solution of (30):

$$\begin{aligned}\overrightarrow{\delta\phi}(t) = U(t)D(t-t_0)V^T(t_0)\overrightarrow{\delta\phi}_0 + \\ U(t)\int_{t_0}^t D(t-\tau)V^T(\tau)\vec{b}_{\text{ext}}(\tau)\,d\tau.\end{aligned} \tag{36}$$

$U(t)$ and $V^T(t)$ are T^*-periodic matrix functions, of size $N\times N$, that satisfy

$$U(t)V^T(t) = V^T(t)U(t) = I_{N\times N}. \tag{37}$$

(37) implies that the columns of U and V are bi-orthogonal, *i.e.*,

$$\vec{v}_i^T(t)\cdot\vec{u}_j(t) = \delta_{ij}, \quad i,j = 1,\cdots,N. \tag{38}$$

(37) can be written more explicitly, showing $\vec{v}_i$ and $\vec{u}_j$, as

$$\begin{pmatrix} \cdots\vec{v}_1^T(t)\cdots \\ \vdots \\ \cdots\vec{v}_N^T(t)\cdots \end{pmatrix}\begin{pmatrix} \vdots & & \vdots \\ \vec{u}_1(t) & \cdots & \vec{u}_N(t) \\ \vdots & & \vdots \end{pmatrix} \equiv \begin{pmatrix} 1 & & \\ & \ddots & \\ & & 1 \end{pmatrix}. \tag{39}$$

Note that, in particular,

$$\vec{v}_1^T(t)\cdot\vec{u}_1(t) \equiv 1, \forall t, \tag{40}$$

a relation we will rely on later.

$D(\cdot)$ in (36) is a *diagonal* matrix of the form

$$D(t) = \begin{pmatrix} e^{\mu_1 t} & & \\ & \ddots & \\ & & e^{\mu_N t} \end{pmatrix}, \tag{41}$$

where $\{\mu_i\}$ are termed *Floquet* or *characteristic exponents*, and

$$\rho_i \triangleq e^{\mu_i T^*}, \quad i = 1,\cdots,N, \tag{42}$$

are known as *Floquet* or *characteristic multipliers*. Note that $D(t)$ is not periodic.

[5](36) holds for the case where the Floquet multipliers (41) are distinct, but subject to Assumption 2.4, all subsequent results in this section remain valid for the case of repeated Floquet multipliers.

(36) can be rewritten using $\vec{v}_i(\cdot)$ and $\vec{u}_i(\cdot)$ as

$$\begin{aligned}\overrightarrow{\delta\phi}(t) = \sum_{i=1}^N \vec{u}_i(t)\underbrace{e^{\mu_i(t-t_0)}\vec{v}_i^T(t_0)\cdot\overrightarrow{\delta\phi}_0}_{\text{scalar}} + \\ \sum_{i=1}^N \vec{u}_i(t)\underbrace{\int_{t_0}^t e^{\mu_i(t-\tau)}\vec{v}_i^T(\tau)\cdot\vec{b}_{\text{ext}}(\tau)\,d\tau}_{\text{scalar}}.\end{aligned} \tag{43}$$

E. Conditions on Floquet multipliers; stability and isolation of $\vec{\phi}^*(t)$

Lemma 2.1 suggests that the stability of the CPS (11) around its locked solution (15) in the absence of external inputs is of an orbital nature [5, Definition 5.1.1] and not, *e.g.*, of a Lyapunov one [5, Definition 1.4.1].

Lemma 2.3: At least one of the Floquet multipliers $\{\rho_i\}$ (42) is 1 (equivalently, at least one of the Floquet exponent $\{\mu_i\}$ is 0).

Proof: Follows from the existence of the T^*-periodic homogeneous solution of the linearized CPS (35), established in Lemma 2.2, and [5, Corollary 2.2.3]. ■

Lemma 2.4: $|\rho_i| \le 1, \forall i$ (equivalently, $\mathbb{R}(\mu_i) \le 0$).

Proof: Follows from Assumption 2.1 (*i.e.*, that the externally-unperturbed oscillator system is mutually injection locked in a *stable* orbit) and [5, Theorem 5.1.3]. ■

We now make an additional technical assumption regarding the Floquet multipliers:

Assumption 2.4: Only one Floquet multiplier (ρ_1, without loss of generality) is 1 (equivalently, w.l.o.g, $\mu_1 = 0$).

Note that Assumption 2.4 strengthens Lemma 2.3. There are several factors that motivate this assumption:

1) Assumption 2.4, together with Lemma 2.4, constitute *sufficient* conditions for the Andronov-Witt theorem [5, Theorems 5.3.8 and 5.1.2] to hold. The Andronov-Witt theorem guarantees that the CPS (11) is *asymptotically orbitally stable (a.o.s)* [5, Definition 5.1.2] with the *asymptotic phase property (a.o.p)* [5, Definition 5.1.3]. These properties are central to the intuitive concept of stable lock, assumed in Assumption 2.1 and typically valid in most applications.
2) Assumption 2.4 also constitutes a sufficient condition for the orbit $\vec{\phi}^*(t)$ (15) to be *isolated*, *i.e.*, not embedded in a continuum of orbits with continuously-varying periods [5, Theorems 5.3.9 and 5.2.3]. Isolation is also central to the intuitive notion of stable lock.
3) Although Assumption 2.4 is not a *necessary* condition for asymptotic orbital stability with the asymptotic phase property, or for isolation, oscillators that do not satisfy the assumption while still being a.o.s + a.o.p tend to be “unnatural”. For example, [5, Example 5.2.2] features an orbit that is a.o.s + a.o.p but has three repeated Floquet multipliers that equal 1; however, this orbit is not periodic.

Thus, in most practical situations, Assumption 2.4 is essentially equivalent to the oscillator group’s being “stably locked”.

F. Assumption 2.3 (deviations are small) is invalid

Lemma 2.5: Assumption 2.3 is invalid.

Proof: Using Assumption 2.4, the second summation term of (43), which captures the linearized system’s response to external perturbations $\vec{b}_{\text{ext}}(t)$, can be written as

$$\begin{aligned}\vec{u}_1(t)\overbrace{\int_{t_0}^t \vec{v}_1^T(\tau)\cdot\vec{b}_{\text{ext}}(\tau)\,d\tau}^{\text{scalar } c_1(t)} \\ + \sum_{i=2}^N \vec{u}_i(t)\underbrace{\int_{t_0}^t e^{\mu_i(t-\tau)}\vec{v}_i^T(\tau)\cdot\vec{b}_{\text{ext}}(\tau)\,d\tau}_{\text{scalar } c_i(t)}.\end{aligned} \tag{44}$$

The first term is of the form $c_1(t)\vec{u}_1(t)$, where $c_1(t)$ is the scalar

$$c_1(t) = \int_{t_0}^{t} \vec{v}_1^T(\tau) \cdot \vec{b}_{\text{ext}}(\tau)\, d\tau. \tag{45}$$

Because $\vec{v}_1(\tau)$ is periodic, there exist many possibilities for small $\vec{b}_{\text{ext}}(t)$ that make $c_1(t)$ increase indefinitely and without bound as t increases. For example, if $\vec{b}_{\text{ext}}(t) = \varepsilon\vec{u}_1(t)$, with $\varepsilon \neq 0$ being any constant, then, from (40), $c_1(t) = (t-t_0)\varepsilon$; *i.e.*, $c_1(t)$ increases without bound. The remaining terms in (44) are bounded because $\mathbb{R}(\mu_i) < 0, \quad \forall i > 1$, hence cannot cancel the unbounded increase of the first term.

In other words, Assumption 2.3, upon which the linearized system (30), its solution (36), and indeed, the expression for the unbounded term $c_1(t)$ in (45) all depend, is violated. Thus we have arrived at a contradiction, implying that the original premise Assumption 2.3 must be invalid (subject to the other assumptions' validity). ■

That deviations can grow to be large even when external inputs remain small is a manifestation of the inherently marginal nature of orbital stability, *i.e.*, that $\mu_1 = 0$.

G. Time-shifted perturbed response assumption

Lemma 2.6: Without loss of generality,

$$\vec{u}_1(t) = \overrightarrow{\delta\phi}^*(t) = \frac{d}{dt}\,\vec{\phi}^*(t)\,. \tag{46}$$

Proof: The first summation term in (43), *i.e.*,

$$\sum_{i=1}^{N} \vec{u}_i(t) \underbrace{e^{\mu_i(t-t_0)} \vec{v}_i^T(t_0) \cdot \overrightarrow{\delta\phi}_0}_{\text{scalar}}, \tag{47}$$

represents a general solution of (33). We already know that $\overrightarrow{\delta\phi}^*(t)$ (34) is a nontrivial *periodic* solution of (33). Using $\rho_1 = 1$ from Assumption 2.4, this periodic solution must equal the $i = 1$ term in (47), since (also from Assumption 2.4) the remaining terms for $i = 2, \cdots, N$ are not periodic and indeed, decay to 0 as $t \to \infty$. Hence we have

$$\overrightarrow{\delta\phi}^*(t) = k_2\vec{u}_1(t) \underbrace{\vec{v}_1^T(t_0) \cdot \overrightarrow{\delta\phi}^*(t_0)}_{\text{scalar constant } k_1}. \tag{48}$$

where k_2 is an arbitrary scalar constant. Note that $k_1 \neq 0$, otherwise (47) would be identically zero, hence would not match any nontrivial $\overrightarrow{\delta\phi}^*(t)$. Choosing $k_2 = \frac{1}{k_1}$ (without loss of generality, since $\vec{v}_1(t)$ can be scaled to satisfy (40)) results in (46). ■

Geometrically, $\frac{d}{dt}\vec{\phi}^*(t)$ is the tangent to the externally-unperturbed orbit of the locked system in phase space; Lemma 2.6 thus justifies the terminology *tangent vector* for $\vec{u}_1(t)$.

Attempting to restore validity to the failed linearization procedure above, observe that using (46), the unbounded term in (45) can be written as

$$c_1(t)\vec{u}_1(t) = c_1(t)\frac{d}{dt}\,\vec{\phi}^*(t)\,. \tag{49}$$

Observe also that if $c_1(t)$ *were* bounded and small, then

$$\vec{\phi}^*(t) + c_1(t)\frac{d}{dt}\,\vec{\phi}^*(t) \simeq \vec{\phi}^*\big(t + c_1(t)\big), \tag{50}$$

to first order. This suggests that the unboundedly growing component of $\overrightarrow{\delta\phi}(t)$ in (43) may be the manifestation of a time-shift to the unperturbed solution $\vec{\phi}^*(t)$. A time shift along the orbit is also suggested by the definition of orbital stability [5, Definition 5.1.1] and by the physical intuition that autonomous oscillators, having no intrinsic "time reference", can "slip in phase", *i.e.*, they cannot correct errors in phase. Accordingly, we modify the assumed form of the perturbed solution (25) to

Assumption 2.5:

$$\vec{\phi}(t) = \vec{\phi}^*\big(t + \alpha_g(t)\big) + \overrightarrow{\delta\phi}(t), \tag{51}$$

where $\overrightarrow{\delta\phi}(t)$ remains small for all time (*i.e.*, $\|\overrightarrow{\delta\phi}(t)\| < M\|\vec{b}_{\text{ext}}(t)\|$ for some finite $M > 0$).

$\alpha_g(t)$ is a (yet-to-be-determined) time shift that can depend on the input perturbation $\vec{b}_{\text{ext}}(t)$ and can grow unboundedly with time. Importantly, we have retained Assumption 2.3, *i.e.*, that $\overrightarrow{\delta\phi}(t)$ in (51) remains bounded and small for all time.

We shall prove that unlike (25), the time-shifted deviation form (51) will allow $\overrightarrow{\delta\phi}(t)$ to remain bounded and small, providing the time-shift $\alpha_g(t)$ is chosen appropriately.

H. Base for time-shifted linearization

In Section II-B, the CPS was linearized around the unperturbed orbit $\vec{\phi}^*(t)$. The process of linearization relied on the fact that $\vec{\phi}^*(t)$ satisfied (16). We would like to find a replacement for (16) that is satisfied by

$$\vec{\phi}(t) = \vec{\phi}^*(t + \alpha_g(t)) \tag{52}$$

instead.

Lemma 2.7: Given any scalar, differentiable function $\alpha_g(t)$, the CPS (11) is solved exactly by (52) for perturbations of the form

$$\vec{b}_\phi(\vec{\phi}(t),t) \triangleq K(t)\vec{u}_1(t + \alpha_g(t)), \tag{53}$$

where $K(t) \equiv \frac{d}{dt}\alpha_g(t)$.

Proof: Denoting "shifted time" to be

$$t^\dagger \triangleq t + \alpha_g(t), \tag{54}$$

substituting (52) and (53) into (11) and simplifying using (16) and (46), we obtain

$$\begin{aligned}
&(1+\dot{\alpha}_g(t))\frac{d}{dt^\dagger}\,\vec{\phi}^*(t^\dagger) = \vec{g}_\phi\big(\vec{\phi}^*(t^\dagger)\big) + K(t)\vec{u}_1(t^\dagger) \\
\Rightarrow\, &\dot{\alpha}_g(t)\frac{d}{dt^\dagger}\,\vec{\phi}^*(t^\dagger) = K(t)\vec{u}_1(t^\dagger) \\
\Rightarrow\, &\dot{\alpha}_g(t)\vec{u}_1(t^\dagger) = K(t)\vec{u}_1(t^\dagger) \\
\Rightarrow\, &\dot{\alpha}_g(t)\vec{u}_1(t+\alpha_g(t)) = K(t)\vec{u}_1(t+\alpha_g(t)).
\end{aligned} \tag{55}$$

(55) is always satisfied if $\alpha_g(t)$ and $K(t)$ are related by

$$\dot{\alpha}_g(t) = K(t). \tag{56}$$

■

I. Time-shifted linearization

We proceed to linearize the CPS (11) around solutions of the form (52). To this end, we split the external input $\vec{b}_\phi(\vec{\phi},t)$ (14) into two parts:

$$\vec{b}_\phi(\vec{\phi},t) = \vec{b}_{\phi_1}(t) + \vec{b}_{\phi_2}(\vec{\phi},t), \tag{57}$$

with the intent that if only the first component $\vec{b}_{\phi_1}(t)$ is retained, then (52) should solve the CPS (11) *exactly*, *i.e.*,

$$\frac{d}{dt}\,\vec{\phi}^*\big(t+\alpha_g(t)\big) = \vec{g}_\phi\big(\vec{\phi}^*(t+\alpha_g(t))\big) + \vec{b}_{\phi_1}(t). \tag{58}$$

Motivated by Lemma 2.7, we explore perturbations of the form (53) along the tangent vector, *i.e.*, of the form

$$\vec{b}_{\phi_1}(t) = K(t)\vec{u}_1\big(t+\alpha_g(t)\big). \tag{59}$$

Given any small external perturbation $\vec{b}_\phi\big(\vec{\phi}(t),t\big)$ (14), our goal is to find such an $\alpha_g(t)$ (and, using Lemma 2.7, its derivative $K(t)$) that $\vec{b}_{\phi_2}(\cdot,\cdot)$ in (57), as well as the orbital deviation $\overrightarrow{\delta\phi}(t)$ in (51), both remain small.

The flow of the time-shifted linearization procedure is:

1) Start by assuming *any* scalar function $K(t)$;
2) Define $\alpha_g(t)$ using (56), *i.e.*, $\frac{d}{dt}\,\alpha_g(t) = K(t)$;
3) Define $\vec{b}_{\phi_1}(t)$ using (59);
4) Incorporate the split-up form (57) of the external input perturbation in the CPS (11);

5) Assuming a solution of the form (51) in Assumption 2.5, linearize (11) using (58) as the base case; and
6) Using the solution of the above linearization, obtain a constraint on $\alpha_g(t)$ (equivalently, on $K(t)$) under which Assumption 2.5 holds with $\overrightarrow{\delta\phi}(t)$ bounded and small for all time. The equation specifying this constraint will turn out to be of central importance, in that it governs the phase/timing responses of the injection-locked system of oscillators to external perturbations.
7) When $\alpha_g(t)$ (equivalently, its derivative $K(t)$) is chosen to satisfy the above constraint, show that the phase deviation $\overrightarrow{\delta\phi}(t)$ in (51) always remains small, thus validating Assumption 2.5 and the entire time-shifted linearization procedure.

Starting from Step 4, write the CPS (11) as

$$\frac{d}{dt}\,\vec{\phi}(t) = \vec{g}_\phi\left(\vec{\phi}(t)\right) + \vec{b}_{\phi_1}(t) + \vec{b}_{\phi_2}\left(\vec{\phi}(t),t\right). \tag{60}$$

Incorporating (51), (56) and (59) in (60), we obtain

$$\begin{aligned}\frac{d}{dt}\left[\vec{\phi}^*(t+\alpha_g(t)) + \overrightarrow{\delta\phi}(t)\right] &= \vec{g}_\phi\left(\vec{\phi}^*(t+\alpha_g(t)) + \overrightarrow{\delta\phi}(t)\right)\\ &\quad + K(t)\vec{u}_1(t+\alpha_g(t)) + \vec{b}_{\phi_2}\left(\vec{\phi}^*(t+\alpha_g(t)) + \overrightarrow{\delta\phi}(t),t\right).\end{aligned} \tag{61}$$

Linearizing $\vec{g}_\phi(\cdot)$ in (61), we obtain

$$\begin{aligned}\frac{d}{dt}\,\vec{\phi}^*(t+\alpha_g(t)) + \frac{d}{dt}\,\overrightarrow{\delta\phi}(t) &= \vec{g}_\phi\left(\vec{\phi}^*(t+\alpha_g(t))\right)\\ &\quad + J_\phi^*\left(t+\alpha_g(t)\right)\overrightarrow{\delta\phi}(t) + K(t)\vec{u}_1(t+\alpha_g(t))\\ &\quad + \vec{b}_{\phi_2}\left(\vec{\phi}^*(t+\alpha_g(t)) + \overrightarrow{\delta\phi}(t),t\right).\end{aligned} \tag{62}$$

Applying the base for time-shifted linearization (58) and our proposed form (59) for $\vec{b}_{\phi_1}(t)$, (62) can be simplified to

$$\frac{d}{dt}\,\overrightarrow{\delta\phi}(t) = J_\phi^*\left(t+\alpha_g(t)\right)\overrightarrow{\delta\phi}(t) + \vec{b}_{\phi_2}\left(\vec{\phi}^*(t+\alpha_g(t)) + \overrightarrow{\delta\phi}(t),t\right). \tag{63}$$

Observe that from definition (57), (59),

$$\begin{aligned}\vec{b}_{\phi_2}\left(\vec{\phi}^*(t+\alpha_g(t)) + \overrightarrow{\delta\phi}(t),t\right) &= \vec{b}_\phi(\vec{\phi}^*(t+\alpha_g(t)) + \overrightarrow{\delta\phi}(t),t)\\ &\quad - K(t)\vec{u}_1(t+\alpha_g(t)),\end{aligned} \tag{64}$$

hence (63) can be written as

$$\begin{aligned}\frac{d}{dt}\,\overrightarrow{\delta\phi}(t) &= J_\phi^*\left(t+\alpha_g(t)\right)\overrightarrow{\delta\phi}(t) - K(t)\vec{u}_1(t+\alpha_g(t))\\ &\quad + \vec{b}_\phi(\vec{\phi}^*(t+\alpha_g(t)) + \overrightarrow{\delta\phi}(t),t).\end{aligned} \tag{65}$$

Using the same reasoning as for (27) in Section II-B1, $\overrightarrow{\delta\phi}(t)$ in the last term of (65) can be dropped because it contributes only a second-order term to the linearization. Hence (65) becomes

$$\begin{aligned}\frac{d}{dt}\,\overrightarrow{\delta\phi}(t) &= J_\phi^*\left(t+\alpha_g(t)\right)\overrightarrow{\delta\phi}(t) - K(t)\vec{u}_1(t+\alpha_g(t))\\ &\quad + \vec{b}_\phi(\vec{\phi}^*(t+\alpha_g(t)),t)\\ &= J_\phi^*\left(t^\dagger\right)\overrightarrow{\delta\phi}(t) + \vec{b}_\phi(\vec{\phi}^*(t^\dagger),t) - K(t)\vec{u}_1(t^\dagger)\\ &= J_\phi^*\left(t^\dagger\right)\overrightarrow{\delta\phi}(t) + \vec{b}_{\phi_2}(\vec{\phi}^*(t^\dagger),t),\end{aligned} \tag{66}$$

where we have used the notation $t^\dagger$, defined in (54), for shifted time.

J. Recasting time-shifted linearization in LPTV form

We would now like to obtain an analytical solution of (66) and use it to validate that $\overrightarrow{\delta\phi}(t)$ remains small for all time. However, two differences between (30) and (66) make this more involved than for (30) in Section II-D:

1) The input to (66) is $\vec{b}_{\phi_2}(\cdot,\cdot)$, not $\vec{b}_\phi(\cdot,\cdot)$ as in (30). Whereas the latter is known small (due to the assumption of small external perturbations $\{a_i(t)\}$ in (14)), there is no guarantee that $\vec{b}_{\phi_2}(\cdot,\cdot)$ is also small. Ensuring that $\vec{b}_{\phi_2}(\cdot,\cdot)$ is small is important: if even the input to (66) cannot be guaranteed small, it is unreasonable to expect that its solution will remain small for all time.
2) Unlike (30), which is LPTV, (66) is *not* LPTV because though $J_\phi^*(t)$ is T^*-periodic, $J_\phi^*\left(t+\alpha_g(t)\right)$ is not, except for special choices such as $\alpha_g(t) \equiv 0$. We are interested in a solution of (66) that is valid for any $\alpha_g(t)$ (equivalently, any $K(t)$), if possible. Because (66) is not LPTV, the Floquet expressions in Section II-D do not apply directly.

Both issues can be addressed by restricting $\dot{\alpha}_g(t) \equiv K(t)$ to be small. We state this as an assumption for the moment[6]:

Assumption 2.6:

$$K(t) \triangleq \dot{\alpha}_g(t) \tag{67}$$

is small and bounded with respect to $\vec{b}_\phi(\cdot,\cdot)$ (14) for all time. In particular, $|K(t)| \ll 1$.

The first consequence of Assumption 2.6 is that it becomes possible to guarantee that $\vec{b}_{\phi_2}(\cdot,\cdot)$, the input to the time-shifted linearization (66), is small:

Lemma 2.8: $\vec{b}_{\phi_2}(\vec{\phi}^*(t^\dagger),t)$ is small for all $t^\dagger,t$.

Proof: From definition (57),

$$\vec{b}_{\phi_2}(\vec{\phi}^*(t^\dagger),t) = \vec{b}_\phi(\phi^*(t^\dagger),t) - K(t)\vec{u}_1(t^\dagger). \tag{68}$$

The first term is small from our underlying assumption of small external perturbations. The tangent vector $\vec{u}_1(t)$ is a periodic, bounded quantity, hence under Assumption 2.6, the second term is also small. ■

Another important consequence of Assumption 2.6 is

Lemma 2.9: The mapping (54)

$$t \mapsto t^\dagger, \text{ i.e., } t^\dagger(t) \triangleq t + \alpha_g(t)$$

is invertible.

Proof: It suffices to show that the mapping is a monotonically increasing one, *i.e.*, its derivative is always positive. We have

$$\frac{d}{dt}\,t^\dagger(t) = 1 + \dot{\alpha}_g(t) = 1 + K(t).$$

From Assumption 2.6, $|K(t)| < 1$, hence $\frac{d}{dt}\,t^\dagger(t) > 0$, *i.e.*, $t^\dagger(t)$ is monotonically increasing. ■

We now make the following definitions:

$$\overrightarrow{\delta\phi}^\dagger(t^\dagger) \triangleq \overrightarrow{\delta\phi}(t), \tag{69}$$

$$\vec{b}_{\phi_2}^\dagger(a,t^\dagger) \triangleq \vec{b}_{\phi_2}(a,t), \text{ and} \tag{70}$$

$$\vec{b}_\phi^\dagger(a,t^\dagger) \triangleq \vec{b}_\phi(a,t). \tag{71}$$

The significance of the invertibility of shifted time $t^\dagger$, as established by Lemma 2.9, lies in that the above definitions become possible: given any $t^\dagger$, a unique t is available for use in the right hand sides of the above definitions.

Using the above definitions, (66) can be expressed using $t^\dagger$ as

$$\left(1+\dot{\alpha}_g(t)\right)\frac{d}{dt^\dagger}\,\overrightarrow{\delta\phi}^\dagger(t^\dagger) = J_\phi^*(t^\dagger)\,\overrightarrow{\delta\phi}^\dagger(t^\dagger) + \vec{b}_{\phi_2}(\vec{\phi}^*(t^\dagger),t^\dagger). \tag{72}$$

We now make a technical assumption for the moment, the validity of which will be demonstrated later:

Assumption 2.7: $\|\frac{d}{dt}\,\overrightarrow{\delta\phi}(t)\| < M\|\overrightarrow{\delta\phi}(t)\|$ for some $0 < M < \infty$.

Assumption 2.7 implies that the magnitude of $\frac{d}{dt}\,\overrightarrow{\delta\phi}(t)$ is within a constant factor of the magnitude of $\overrightarrow{\delta\phi}(t)$, *i.e.*, the two are of the same order of magnitude. Intuitively, it implies that $\overrightarrow{\delta\phi}(t)$ has a bounded rate of change. It follows that $\overrightarrow{\delta\phi}^\dagger(t^\dagger)$ also has a bounded rate of change:

Lemma 2.10: $\|\frac{d}{dt^\dagger}\,\overrightarrow{\delta\phi}^\dagger(t^\dagger)\| < M_2\|\overrightarrow{\delta\phi}^\dagger(t^\dagger)\|$ for some $0 < M_2 < \infty$.

[6] We will establish later that this assumption is in fact a consequence of the external inputs $\vec{b}_\phi(\cdot,\cdot)$ being small.

Proof: Using (69), (54) and (56), we have

$$\frac{d}{dt^\dagger}\,\overrightarrow{\delta\phi}^\dagger(t^\dagger) = \frac{d}{dt^\dagger}\,\overrightarrow{\delta\phi}(t) = \frac{1}{1+K(t)}\frac{d}{dt}\,\overrightarrow{\delta\phi}(t).$$

Since $|K(t)| \ll 1$ from Assumption 2.6, we have $\|\frac{d}{dt^\dagger}\,\overrightarrow{\delta\phi}^\dagger(t^\dagger)\| < m\|\frac{d}{dt}\,\overrightarrow{\delta\phi}(t)\|$ for some $m<\infty$. Using Lemma 2.10, the result follows. ■

From Lemma 2.10, it is apparent that the term $\dot{\alpha}_g\frac{d}{dt^\dagger}\,\overrightarrow{\delta\phi}^\dagger(t^\dagger)$ in (72) is of second order, hence can be dropped from the linearization. As a result, (72) becomes

$$\frac{d}{dt^\dagger}\,\overrightarrow{\delta\phi}^\dagger(t^\dagger) = J^*_\phi(t^\dagger)\,\overrightarrow{\delta\phi}^\dagger(t^\dagger) + \vec{b}_{\phi_2}(\vec{\phi}^*(t^\dagger),t^\dagger). \tag{73}$$

Note that (73) is an LPTV system with period T^*.

K. Floquet solution of time-shifted LPTV system

Since (73) is LPTV, the Floquet expressions in Section II-D apply, with t and $\overrightarrow{\delta\phi}(t)$ replaced by $t^\dagger$ and $\overrightarrow{\delta\phi}^\dagger(t^\dagger)$, respectively, and $\vec{b}_{\text{ext}}(t)$ replaced by $\vec{b}^\dagger_{\phi_2}(\vec{\phi}^*(t^\dagger),t^\dagger)$. (43) becomes

$$\begin{aligned}\overrightarrow{\delta\phi}^\dagger(t^\dagger) = &\sum_{i=1}^{N}\vec{u}_i(t^\dagger)\underbrace{e^{\mu_i(t^\dagger-t_0^\dagger)}\vec{v}_i^T(t_0^\dagger)\cdot\overrightarrow{\delta\phi}^\dagger(t_0^\dagger)}_{\text{scalar}} + \\ &\sum_{i=1}^{N}\vec{u}_i(t^\dagger)\underbrace{\int_{t_0^\dagger}^{t^\dagger} e^{\mu_i(t^\dagger-\tau)}\vec{v}_i^T(\tau)\cdot\vec{b}^\dagger_{\phi_2}(\vec{\phi}^*(\tau),\tau)\,d\tau}_{\text{scalar}},\end{aligned} \tag{74}$$

while the term $c_1(t)$ in (45), which causes unbounded growth and resulting breakdown of linearization, becomes

$$c_1^\dagger(t^\dagger) \triangleq \int_{t_0^\dagger}^{t^\dagger}\vec{v}_1^T(\tau)\cdot\vec{b}^\dagger_{\phi_2}(\vec{\phi}^*(\tau),\tau)\,d\tau. \tag{75}$$

L. Choosing $\alpha_g(t)$ to circumvent breakdown of linearization

To avoid unbounded growth of $\overrightarrow{\delta\phi}^\dagger(t^\dagger)$, which would invalidate the present time-shifted linearization procedure in the same manner as Section II-F and Lemma 2.5 previously, it is imperative that $c_1^\dagger(t^\dagger)$ in (75) remain small and bounded (with respect to $\vec{b}^\dagger_{\phi_2}(\cdot,\cdot)$). This can be achieved by the simple expedient of requiring that the integrand in (75) vanish, *i.e.*,

$$\vec{v}_1^T(\tau)\cdot\vec{b}^\dagger_{\phi_2}(\vec{\phi}^*(\tau),\tau) \equiv 0, \forall\tau. \tag{76}$$

Substituting $t^\dagger$ for τ in (76), and applying (70), (68), (40), (67) and (54), we obtain:

$$\begin{aligned}&0 = \vec{v}_1^T(\tau)\cdot\vec{b}^\dagger_{\phi_2}(\vec{\phi}^*(\tau),\tau)\\ &\Rightarrow 0 = \vec{v}_1^T(t^\dagger)\cdot\vec{b}^\dagger_{\phi_2}(\vec{\phi}^*(t^\dagger),t^\dagger)\\ &\Rightarrow 0 = \vec{v}_1^T(t^\dagger)\cdot\left[\vec{b}_\phi(\phi^*(t^\dagger),t) - K(t)\vec{u}_1(t^\dagger)\right]\\ &\Rightarrow K(t)\vec{v}_1^T(t^\dagger)\cdot\vec{u}_1(t^\dagger) = \vec{v}_1^T(t^\dagger)\cdot\vec{b}_\phi(\phi^*(t^\dagger),t)\\ &\Rightarrow K(t) = \vec{v}_1^T(t^\dagger)\cdot\vec{b}_\phi(\phi^*(t^\dagger),t)\\ &\Rightarrow \dot{\alpha}_g(t) = \vec{v}_1^T(t^\dagger)\cdot\vec{b}_\phi(\phi^*(t^\dagger),t)\\ &\Rightarrow \boxed{\frac{d}{dt}\,\alpha_g(t) = \vec{v}_1^T(t+\alpha_g(t))\cdot\vec{b}_\phi\left(\vec{\phi}^*(t+\alpha_g(t)),t\right)}.\end{aligned} \tag{77}$$

From the considerations of Section II-I through (77), we are able to prove the following Theorem:

Theorem 1: Given a system of N coupled oscillators modelled in the phase domain by the CPS equations (11) and mutually injection locked, satisfying Assumption 2.1 and Assumption 2.4. If the external perturbations to the system $\{a_i(t)\}$ (10) (equivalently, $\vec{b}_\phi(\cdot,\cdot)$ in (11)) are small, and if $\alpha_g(t)$ is chosen to satisfy (77), *i.e.*,

$$\frac{d}{dt}\,\alpha_g(t) = \vec{v}_1^T(t+\alpha_g(t))\cdot\vec{b}_\phi\left(\vec{\phi}^*(t+\alpha_g(t)),t\right),$$

then the solution of the CPS can be expressed as in (51), *i.e.*, as

$$\vec{\phi}(t) = \vec{\phi}^*\big(t+\alpha_g(t)\big) + \overrightarrow{\delta\phi}(t),$$

where $\vec{\phi}^*(t)$ is the periodic, synchronized solution of the externally-unperturbed system of oscillators. $\overrightarrow{\delta\phi}(t)$, **the deviations from the orbit of the externally-unperturbed system, remain small and bounded for all** t (with respect to the external perturbations $\{a_i(t)\}$).

Proof: Subject to Assumption 2.5 and Assumption 2.6, Lemma 2.8 establishes that $\vec{b}_{\phi_2}(\vec{\phi}^*(t^\dagger),t)$ is small; applying (70) shows that $\vec{b}^\dagger_{\phi_2}(\cdot,\cdot)$, which appears in (74), is also small.

Choosing $\alpha_g(t)$ to satisfy (77) ensures that $c_1^\dagger(t^\dagger)$ (75) vanishes, as demonstrated above. As a result, the $i=1$ terms in (74) (which correspond to Floquet multiplier $\rho_1 = 1$ or equivalently, Floquet exponent $\mu_1 = 1$) remain bounded and small. From Assumption 2.4 and Lemma 2.4, the remaining Floquet exponents $\mu_2,\cdots,\mu_N$ all have strictly negative real parts. With $\vec{b}^\dagger_{\phi_2}(\cdot,\cdot)$ small as noted above, this implies that the terms corresponding to $i = 2,\cdots,N$ in (74) also remain bounded and small for all t. As a result, $\overrightarrow{\delta\phi}^\dagger(t^\dagger)$ remains bounded and small for all t. Applying (69), $\overrightarrow{\delta\phi}(t)$ also remains small and bounded for all time. This immediately validates Assumption 2.5. That $\vec{b}_\phi(\cdot,\cdot)$ is small (by assumption) and (77) holds also validates Assumption 2.6.

Differentiating (74) and proceeding in a similar manner, Assumption 2.7 can also be shown to be valid. ■

III. Conclusion

(77) and Theorem 1 establish that $\alpha_g(t)$, the time shift (or phase) of the coupled PPV system, obeys a relationship identical in form to the PPV equation (2) for individual oscillators. In other words, groups of synchronized oscillators may be abstracted by a single "effective PRC/PPV" function that dictates the group's "effective phase response" $\alpha_g(t)$ to external perturbations via the *single, scalar* differential equation (77). As such, it provides a rigorous basis for the empirical practice of measuring PRCs of complex oscillatory systems that are synchronized (*e.g.*, [6]).

(77) may be used to analyze the dynamics (*e.g.*, noise and locking/pulling behaviour) of groups of synchronized oscillators, just as (2) is used for individual oscillators. Moreover, Theorem 1 may be applied repeatedly to abstract the effective PRC/PPV of groups of synchronized oscillators that are organized hierarchically over multiple levels, enabling the development of computationally efficient and scalable methods for analysing and abstracting the phase dynamics of large systems of coupled oscillators.

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