Model-order Reduction for LPTV Systems

and connections with nonlinear oscillator macromodelling

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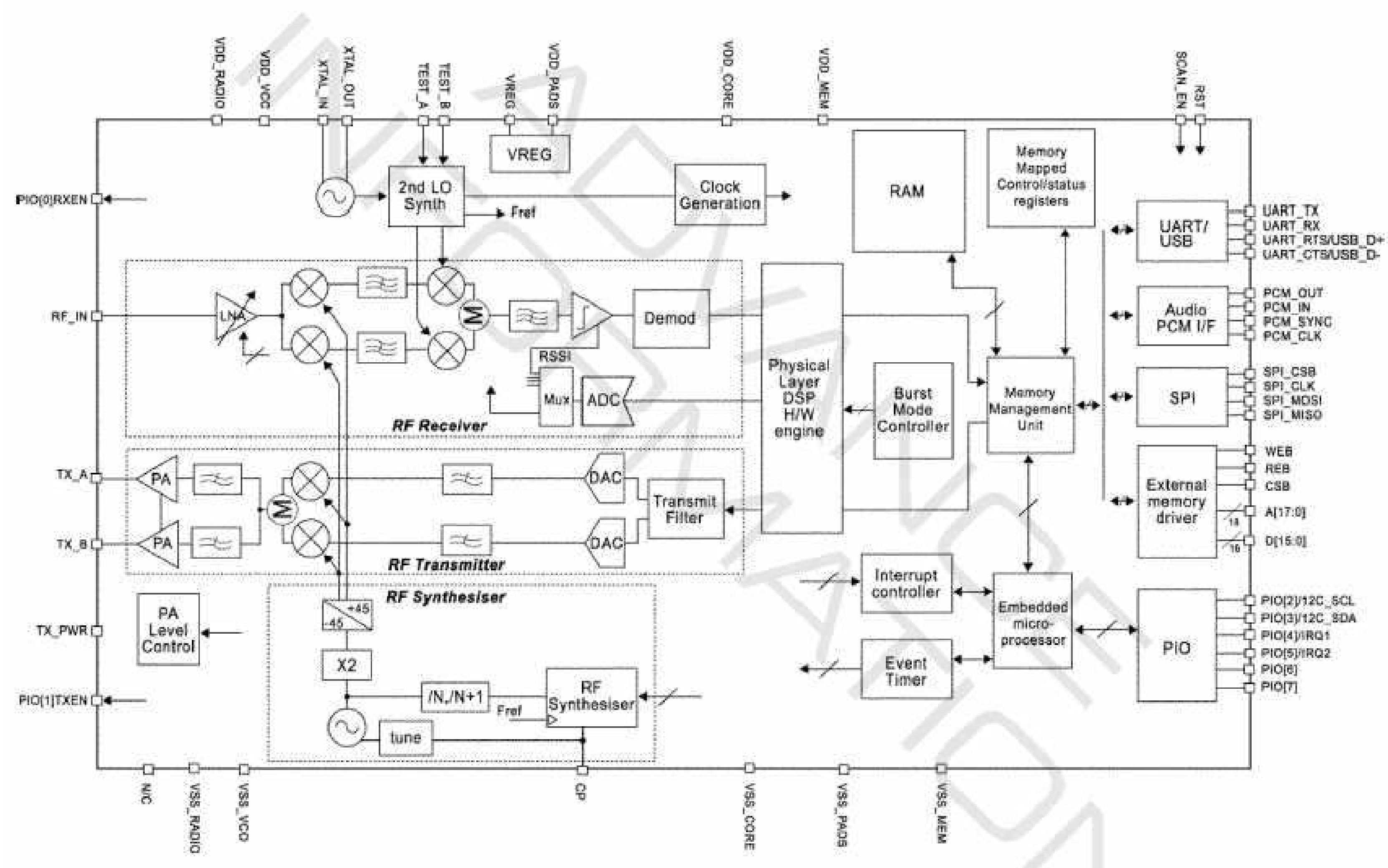
Bluetooth mixed-signal RFIC

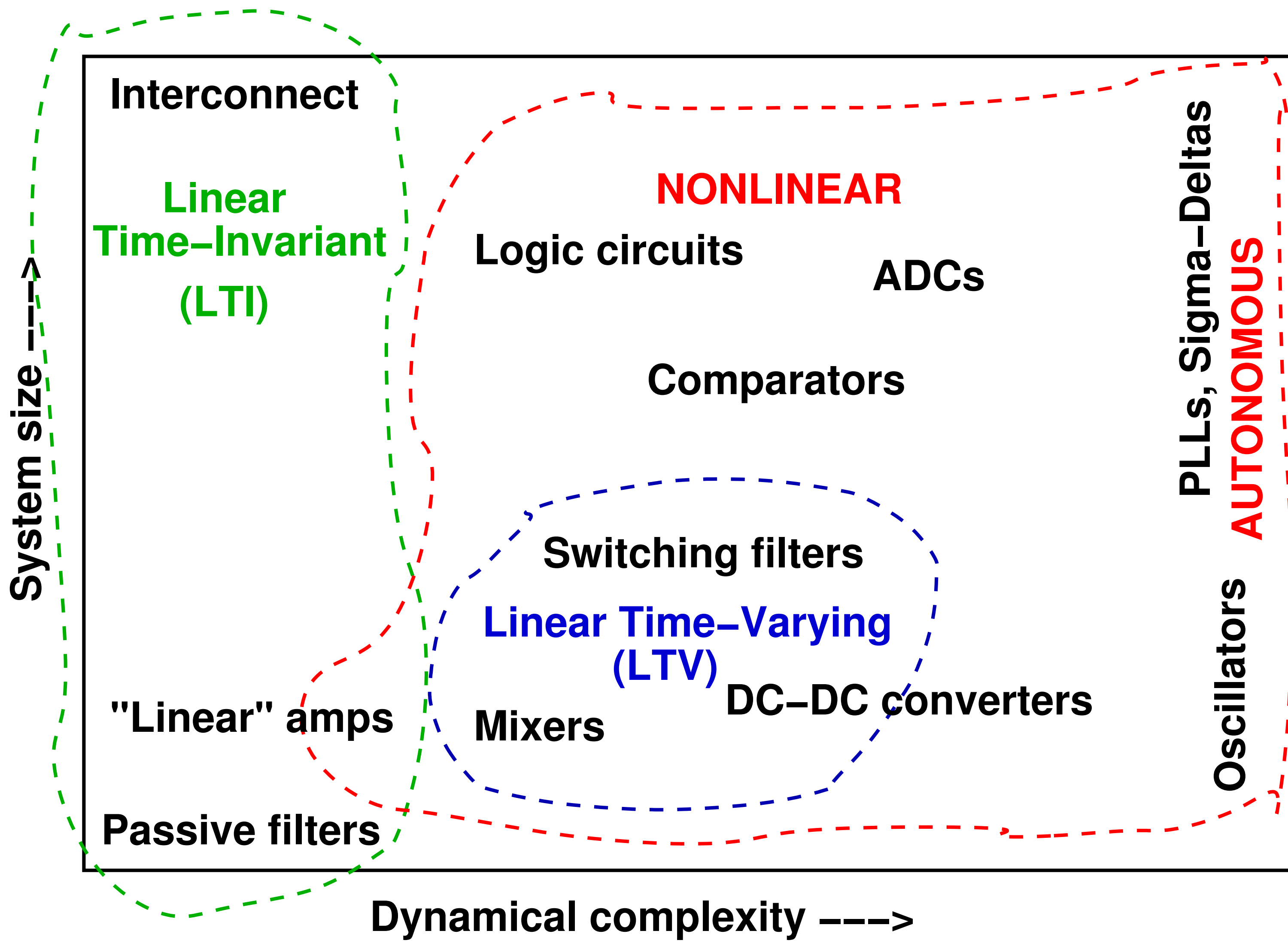


Cambridge Silicon

☛Cheap ☛Low margins☛ **Must work first time**

☛3-5 spins = cutting edge; >10 common

Classification of Circuits



Types of Algorithmic Macromodelling

- Linear Time-Invariant (LTI) macromodelling
  - main application: interconnect networks (delay, crosstalk)
  - AWE, PVL, PRIMA, TBR
- Linear Time-Varying (LTV) macromodelling
  - mixers, sampling/switching circuits: Time-Varying Padé
- Weakly nonlinear (Volterra) macromodelling
  - “linear” amplifiers/mixers: low-order polynomial-based reduction
- Strongly nonlinear macromodelling
  - clipping, slewing, digital, oscillators
  - TPWL, PWP, phase-domain oscillator macromodels

Linear Time Varying (LTV) MM

- LTV: useful abstraction for some nonlinear systems
  - mixers, switching filters, samplers, DC/DC converters, ...
  - frequency translation/TD nonlinear sampling captured
  - signal-path nonlinearities **not** captured
    - but weakly-nonlinear MM can easily be applied
- Overview of LPTV reduction
  - map LPTV to **MIMO** LTI system
  - apply **MIMO LTI MOR** (any, especially Krylov)
  - map back to LPTV reduced model
  - steady-state computations (eg, harmonic balance) used

Time-Varying Linearization of DAEs

- $\frac{\partial}{\partial t} q(x(t)) + f(x(t)) + B(t) = 0, \quad z(t) = l^T x(t)$
  - given a (transient) solution $x^*(t)$ to the above
  - what if $B(t)$ is **perturbed** by a **small** amount $bu(t)$? How much does $x^*(t)$ change?
  - **Assume** that deviation $y(t)$ (from $x^*(t)$) is **small**
  - $\frac{\partial}{\partial t} q(x^*(t) + y(t)) + f(x^*(t) + y(t)) + b(t) + bu(t) = 0$
  - Linearize $q()$ and $f()$, keep only the first-order terms:
    * $\frac{\partial}{\partial t} [C(t) y(t)] + G(t) y(t) + bu(t) = 0$
    * $C(t) = \left.\frac{dq}{dx}\right|_{x^*(t)}, \quad G(t) = \left.\frac{df}{dx}\right|_{x^*(t)}$
  - $x^*(t)$ periodic $\Rightarrow G(t), C(t)$ periodic

The LPTV transfer function

- How to obtain the LPTV transfer function?
- Laplace transforms: interaction of “input” and “system” time variations
  - $\frac{\partial}{\partial t}\left[C(t)y(t)\right]+G(t)y(t)+bu(t)=0$
- One solution: separate them with **<u>multiple time scales</u>**
- $\left(\frac{\partial}{\partial t_1}+\frac{\partial}{\partial t_2}\right)\left[C(t_2)\hat{y}(t_1,t_2)\right]+G(t_2)\hat{y}(t_1,t_2)+bu(t_1)=0$
  - (the Multitime PDE form: MPDE)
  - key property: any solution $\hat{y}(t_1,t_2)$ of the MPDE leads to a solution $y(t)=\hat{y}(t,t)$ of the LPTV DAE
    - $*$ proof: just apply the chain rule

The LPTV transfer function

- Laplace transforms in t_1
  - $\frac{\partial}{\partial t_2}\left[C(t_2)Y(s,t_2)\right] + (G(t_2) + sC(t_2))\,Y(s,t_2) + bU(s) = 0$
- $Y(s,t_2) = H(s,t_2)U(s) = -\left[\frac{\partial}{\partial t_2}\left[C(t_2)\cdot\right] + (G(t_2) + sC(t_2))\right]^{-1} bU(s)$
  - Operator expression for time-varying transfer function
  - $\left(Y(s) = \int_{-\infty}^{\infty} Y(s_2, s - s_2)\, ds_2\right)$
- Solving for solution periodic in t_2
  - "Discretize" equation along t_2 timescale
  - Write all unknowns in a long vector
  - Obtain a larger, **LTI**, matrix equation system

LPTV System: Fourier Analysis

- $\frac{\partial}{\partial t_2}\left[C(t_2)Y(s,t_2)\right] + \left(G(t_2) + sC(t_2)\right)Y(s,t_2) + bU(s) = 0$
- $Z(s,t_2) = l^T Y(s,t_2)$: the output
- Expand t_2 dependence in Fourier series
  - $Y(s,t_2) = \sum_{i=-\infty}^{\infty} Y_i(s)\, e^{J 2\pi i \frac{t}{T}}$ (a vector)
  - $C(t_2) = \sum_{i=-\infty}^{\infty} C_i\, e^{J 2\pi i \frac{t}{T}}$ (a matrix)
  - $G(t_2) = \sum_{i=-\infty}^{\infty} G_i\, e^{J 2\pi i \frac{t}{T}}$ (a matrix)
  - $Z(s,t_2) = \sum_{i=-\infty}^{\infty} l^T Y_i(s)\, e^{J 2\pi i \frac{t}{T}}$ (a vector)

LPTV System: Fourier Analysis

- Some algebra: given any T-periodic size n vector/matrix $A(t)$, T-periodic with F. coeffs $\{A_i\}$, define
  - $\mathbb{V}_{A(t)} = \begin{bmatrix} \vdots \\ A_2 \\ A_1 \\ A_0 \\ A_{-1} \\ A_{-2} \\ \vdots \end{bmatrix}$ (a block-vector of the F. coeffs), and

LPTV System: Fourier Analysis

– $$\mathbb{T}_{A(t)} = \begin{bmatrix} \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \\ \dots & A_0 & A_1 & A_2 & A_3 & A_4 & \dots \\ \dots & A_{-1} & A_0 & A_1 & A_2 & A_3 & \dots \\ \dots & A_{-2} & A_{-1} & A_0 & A_1 & A_2 & \dots \\ \dots & A_{-3} & A_{-2} & A_{-1} & A_0 & A_1 & \dots \\ \dots & A_{-4} & A_{-3} & A_{-2} & A_{-1} & A_0 & \dots \\ & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

(block-Toeplitz matrix of the F. coeffs), and

– $\Omega_{\text{FD}} = \jmath \frac{2\pi}{T} \text{diag}(\dots, 2I_n, I_n, 0I_n, -I_n, 2I_n, \dots)$

LPTV System: Fourier Analysis

- Then
  - if $X(t)$ and $Y(t)$ are T-periodic and $Z(t) = X(t)Y(t)$,
    * $\mathbb{V}_{Z(t)} = \mathbb{T}_{X(t)}\mathbb{V}_{Y(t)}$
    * $\mathbb{T}_{Z(t)} = \mathbb{T}_{X(t)}\mathbb{T}_{Y(t)}$
    * $\mathbb{V}_{\dot{X}(t)} = \Omega_{\text{FD}}\mathbb{V}_{X(t)}$
    * $\mathbb{T}_{\dot{X}(t)} = \Omega_{\text{FD}}\mathbb{T}_{X(t)} - \mathbb{T}_{X(t)}\Omega_{\text{FD}}$

LPTV System: Fourier Analysis

- Applying to
  $\frac{\partial}{\partial t_2}\left[C(t_2)Y(s,t_2)\right] + \left(G(t_2) + sC(t_2)\right)Y(s,t_2) + bU(s) = 0$, we get
  - $\Omega_{\text{FD}}\mathbb{T}_{C(t)}\mathbb{V}_{Y(s,t)} + \left(\mathbb{T}_{G(t)} + s\mathbb{T}_{C(t)}\right)\mathbb{V}_{Y(s,t)} + \mathbb{V}_b U(s) = 0$
  - $\boxed{\mathbb{V}_{z(s,t)} = -\underbrace{\mathbb{T}_{l^T}}_{l_{FD}^T}\underbrace{\left[s\underbrace{\mathbb{T}_{C(t)}}_{C_{FD}} + \underbrace{\left(\Omega_{\text{FD}}\mathbb{T}_{C(t)} + \mathbb{T}_{G(t)}\right)}_{G_{FD}}\right]^{-1}}_{J_{HB}(s)}\mathbb{V}_b\, U(s)}$

$$J_{HB}(s) = \begin{bmatrix} G_0 + (s+\jmath\omega_0 2)C_0 & G_1 + (s+\jmath\omega_0 2)C_1 & G_2 + (s+\jmath\omega_0 2)C_2 & G_3 + (s+\jmath\omega_0 2)C_3 & G_4 + (s+\jmath\omega_0 2)C_4 \\ G_{-1} + (s+\jmath\omega_0)C_{-1} & G_0 + (s+\jmath\omega_0)C_0 & G_1 + (s+\jmath\omega_0)C_1 & G_2 + (s+\jmath\omega_0)C_2 & G_3 + (s+\jmath\omega_0)C_3 \\ G_{-2} + C_{-2} & G_{-1} + C_{-1} & G_0 + C_0 & G_1 + C_1 & G_2 + C_2 \\ G_{-3} + (s-\jmath\omega_0)C_{-3} & G_{-2} + (s-\jmath\omega_0)C_{-2} & G_{-1} + (s-\jmath\omega_0)C_{-1} & G_0 + (s-\jmath\omega_0)C_0 & G_1 + (s-\jmath\omega_0)C_1 \\ G_{-4} + (s-\jmath\omega_0 2)C_{-4} & G_{-3} + (s-\jmath\omega_0 2)C_{-3} & G_{-2} + (s-\jmath\omega_0 2)C_{-2} & G_{-1} + (s-\jmath\omega_0 2)C_{-1} & G_0 + (s-\jmath\omega_0 2)C_0 \end{bmatrix}$$

LPTV System: Fourier Analysis

- Truncate infinite matrices/vectors to finite no of harmonics N
  - approximate Toeplitz by circulant matrices for efficient calculations
  - closely related to Harmonic Balance Jacobian matrix
    * efficient matrix-vector products: $O(nN\log(N))$
- Pick as outputs as many harmonic components (of $\mathbb{V}_{z(s,t)}$) as relevant
  - all: N times as many outputs as originally
  - pick (eg) only DC or first upconversion transfer function
- This is a SIMO (in general MIMO) LTI system
  - reduce using block-Krylov techniques (or other methods!)

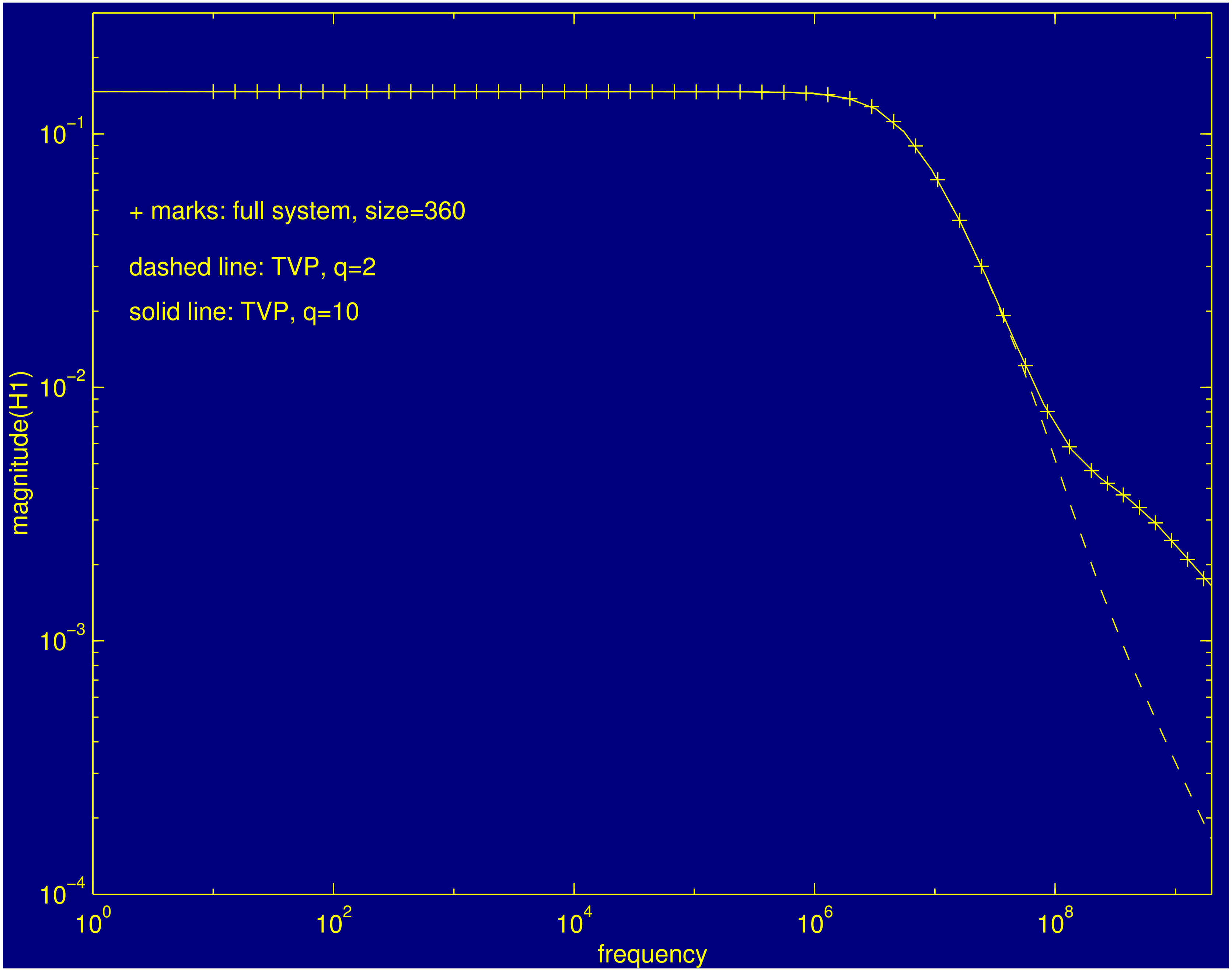
LPTV Reduction

- Using, eg, block-Arnoldi or block-Lanczos, the reduced model has the form
- $\mathbb{V}_{z(s,t)} \simeq L_q^T \left[I - sT_q\right]^{-1} R_q \, U(s)$
  - $L_q^T \in \mathcal{R}^{N \times p}$
  - reduction uses efficient matrix-vector multiplications
- Mapping back to differential equations
  - $-T_q \frac{d\hat{x}}{dt} + \hat{x} = R_q \, u(t), \qquad z(t) \simeq l_q(t)\hat{x}(t)$
  - $l_q(t) = \sum_i L_{q,i} e^{ij\omega_0 t}, \qquad L_{q,i}$ are the columns of L_q
- **LTI + memoryless** form!
  - fits with Floquet theory for LPTV systems

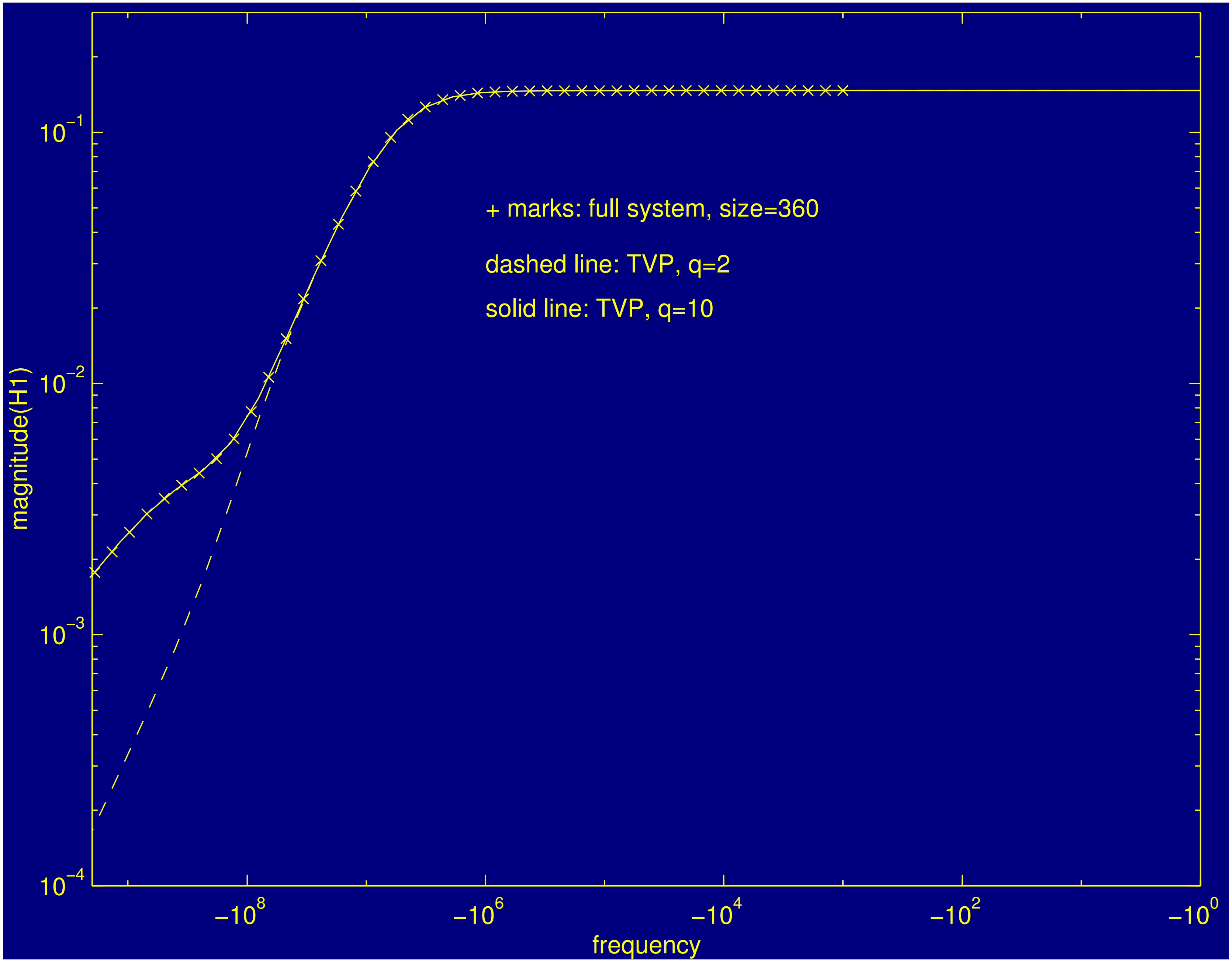
I-Channel Mixer/Buffer Circuit

- 360 nodes, small-signal input ~ 80 kHz, LO=178MHz
- Periodic large-signal steady-state: HB with N=21
- Reduction with block-Lanczos
  - $q = 2$: reasonable macromodel
  - $q = 10$: matches xfer function upto twice LO frequency
  - $\sim$ 500$\times$ evaluation speedup

Upper Sideband Transfer Function



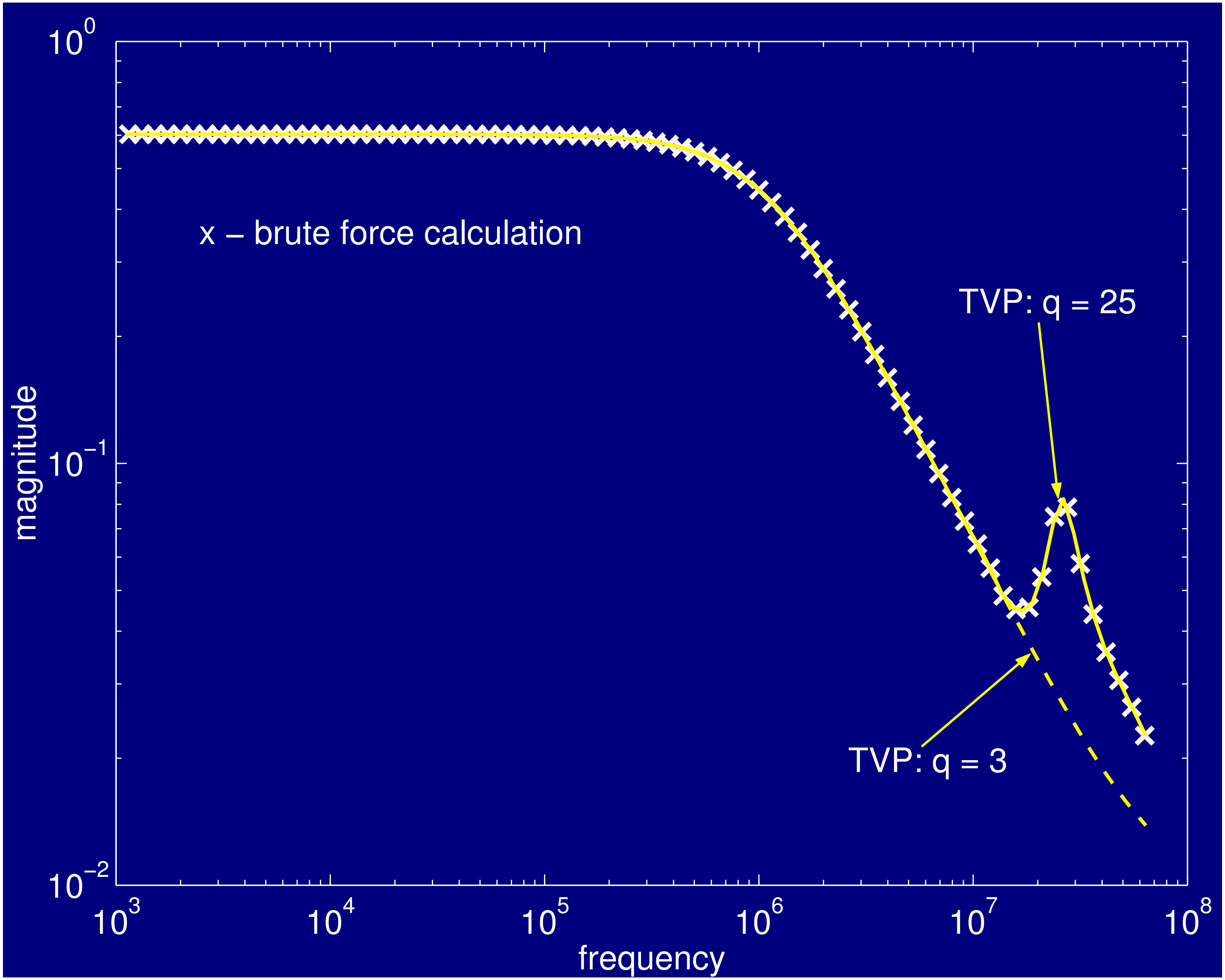
Lower Sideband Transfer Function



Switched-Capacitor Integrator

- ~ 350 MOSFETs
  - switched on and off at 12.8MHz
- Basic active filter building block: sampling action
- Large-signal steady-state: **<u>time-domain discretization</u>**, N$\sim$ 50
- Reduction with block-Lanczos

Zeroth-harmonic transfer function



Floquet Theory for LPTV Systems

- (Floquet theorem): If $\dot{y}(t) = A(t)y(t)$, with $A(t)$ T-periodic, then
  - $$\Phi(t,t_0) = U(t) \begin{bmatrix} e^{\mu_1(t-t_0)} & & & \\ & e^{\mu_2(t-t_0)} & & \\ & & \ddots & \\ & & & e^{\mu_n(t-t_0)} \end{bmatrix} V^T(t_0)$$
  - $V^T(t)U(t) \equiv I, \quad \forall t$; $U(t)$, $V(t)$ are T-periodic $n \times n$ matrices
- Expensive to compute; desirable to find **a few** $u_i(t), v_i(t), \mu_i$
- $y(t) = U(t)D(t-t_0)V^T(t_0)\,y_0 + U(t)\int_{t_0}^{t} D(t-\tau)V^T(\tau)b(\tau)\,d\tau$

$$y(t) = \sum_{i=1}^{n} u_i(t) \underbrace{\int_{t_0}^{t} e^{\mu_i(t-\tau)}\, v_i^T(\tau)\, b(\tau)\, d\tau}_{\text{scalar}}, \qquad \text{if } y_0 = 0$$

Oscillators

- $\dot{x}(t) = f(x)$
  - Periodic solution $x^*(t)$, with period T
  - LPTV linearization $\dot{y}(t) = G(t)y(t) \quad (+b(t),$perturbation$)$
- Can show:
  - $\mu_1 = 0$
  - $u_1(t) = \dot{x}^*(t)$
- Would like to find $v_1(t)$ - the Perturbation Projection Vector (PPV)
  - useful for oscillator perturbation and noise analysis
    * $\boxed{\dot{\alpha} = v_1^T(t + \alpha(t)) \cdot b(t)}$ (**nonlinear phase macromodel**)
  - other dominant $\mu_i, u_i(t), v_i(t)$ also useful

Floquet-HB Relationship

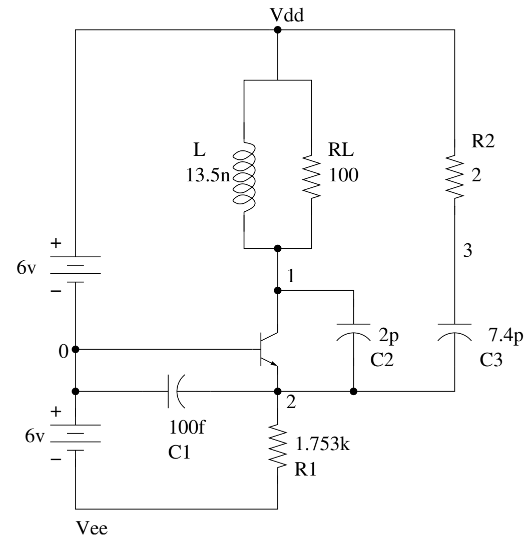
- HB Jacobian and Floquet quantities related by:

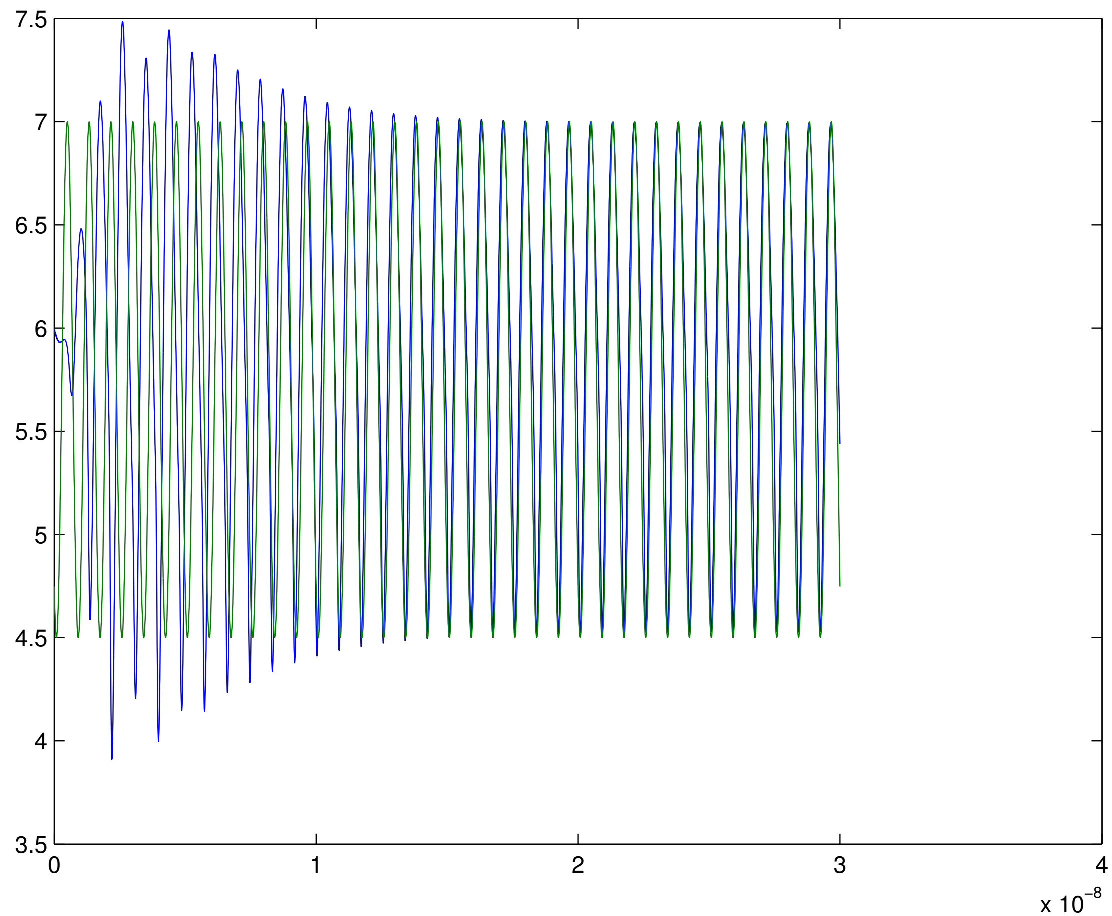
$$J_{HB}(0) = \mathbb{T}_{U(t)} \left(\Omega_{FD} - \mathbb{T}_M\right) \mathbb{T}_{V^T(t)}$$

  – where $M = \begin{bmatrix} \mu_1 & & \\ & \ddots & \\ & & \mu_n \end{bmatrix}$, ie, the Floquet exponents

- Can be applied to find a few Floquet eigenquantities via MOR
- But if $\mu_1 = 0$ (oscillators), then $J_{HB}(0)$ is singular with rank deficiency of 1
- Null space of $J_{HB}(0)$ = Fourier components of PPV $v_1(t)$
  – can solve for iteratively using augmented $J_{HB}(0)$ matrix

Injection Locking: Colpitts Oscillator





injection locking captured by PPV macromodel

89x speedup of MM over original circuit