Computing Phase Noise Eigenfunctions Directly from Steady-State Jacobian Matrices

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Abstract

The main effort in oscillator phase noise calculation lies in computing a vector function called the Perturbation Projection Vector (PPV). Current techniques for PPV calculation use time domain numerics to generate the system's monodromy matrix, followed by full or partial eigenanalysis. We present a superior method that finds the PPV using only a single linear solution of the oscillator's time- or frequency-domain steady-state Jacobian matrix. The new method is better suited for existing tools with fast harmonic balance or shooting capabilities, and also more accurate than explicit eigenanalysis. A key advantage is that it dispenses with the need to select the correct one-eigenfunction from amongst a potentially large set of choices, an issue that explicit eigencalculation based methods have to face.

1 Introduction

An important consideration during the design of RF communication circuits is the phase noise performance of oscillators. Phase noise corrupts spectral purity and generates large power content in a continuous spread of frequencies around the desired oscillator tone, thus contributing to adjacent channel interference. For clocked digital circuits, which are also of increasing importance in RF design, phase noise manifests itself as timing jitter and degrades system performance.

Recently, we presented a rigorous general theory of phase noise [, ,][1]. One of the main results of this theory is that phase noise in an oscillator is completely characterized by a single scalar constant c, which, moreover, has the physical interpretation of jitter per second. This constant c depends on the noise generators in the oscillator and also on a periodic vector function $v_1(t)$, termed the *Perturbation Projection Vector* or *PPV*. The PPV is the periodic eigenfunction corresponding to the unity eigenvalue of the linearized oscillator's adjoint equations, hence it can be computed directly by eigendecomposition of the monodromy matrix of the adjoint system. Although eigencalculations are expensive for large systems, iterative linear methods are used to compute only a few eigenpairs.

In this paper, we provide a new computational procedure for the PPV that does not require the monodromy matrix. Instead, the method uses only a *single* linear solution of the steady-state Jacobian matrix of the oscillator, obtained from either harmonic balance or time-domain methods like shooting. The method is especially advantageous for high-Q oscillators, monodromy matrices of which often have many eigenvalues close to 1 that are numerically indistinguishable from the oscillatory mode. In such situations, explicit eigendecomposition methods need to find a potentially large number of candidate PPVs and choose one from amongst them using heuristics. In contrast, the new method directly locates the correct PPV by embedding exact periodicity and an orthonormality condition implicitly into the one-step calculation. Since time-domain integration in generating or applying the monodromy matrix is not involved, the method fully exploits the inherent accuracy advantages of frequency-domain calculations using harmonic balance. The method is easy to implement in existing steady-state codes, since it requires only an extra adjoint solution of the same Jacobian matrix that is generated and solved at each Newton step of the steady-state calculation. This computation is negligible compared to that for obtaining the steady state.

Proof of correctness of the new method relies on a link, established in Section 2, between the PPV and the null space of the oscillator's frequency- and time-domain steady-state Jacobians. In Section 3, we compare the new technique with monodromy-matrix eigendecomposition.

[1] A varied and extensive literature, developed over many decades, exists on the phase noise problem. We do not provide a review of prior work here as it is not central to our contribution, but we refer the interested reader to, *e.g.*, [,].

2 Relationship of the PPV $v_1(t)$ to the oscillator's steady-state Jacobian

We consider an orbitally stable oscillator with a single oscillation mode, described by the DAE system:

$$\dot{q}(x) + f(x) = 0. \tag{1}$$

We assume that this sytem has a known periodic solution $x_s(t)$. The linearization of (1) around the solution $x_s(t)$ is (*e.g.*, [,]):

$$\frac{d}{dt}\left(C(t)y(t)\right) + G(t)y(t) = 0. \tag{2}$$

$C(t)$ and $G(t)$ are periodic matrices. The rank m of $C(t)$ can be less than the system size n; m is assumed independent of t.

It can be shown (*e.g.*, [ ]) that the state-transition matrix

$$\begin{aligned} &\Phi(t,s) = U(t)D(t-s)V^T(s)C(s), \\ &\text{with } D(t) = \text{diag}(e^{\mu_1 t}, \cdots, e^{\mu_m t}, \underbrace{0, \cdots, 0}_{k=n-m}) \end{aligned} \tag{3}$$

satisfies (2). $U(t)$ and $V(t)$ are periodic matrices of full rank satisfying the biorthogonality condition

$$V^T(t)C(t)U(t) = I_m = \begin{pmatrix} I_m & 0 \\ 0 & 0_k \end{pmatrix}. \tag{4}$$

$\mu_1, \cdots, \mu_m$ are the Floquet eigenvalues. Since the system has an oscillatory mode, one of these is zero, say $\mu_1 = 0$. The Floquet eigenvectors corresponding to this mode are the first

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columns of $U(t)$ and $V(t)$, denoted by $u_1(t)$ and $v_1(t)$, respectively. It can be shown that $u_1(t)$ can be taken equal to $\dot{x}_s(t)$ without loss of generality [1] and computed easily from the known large-signal periodic solution. Our goal is to calculate the other oscillatory-mode Floquet eigenvector, $v_1(t)$.

It can be easily verified that the adjoint of (2), defined by

$$C^T(t)\frac{d}{dt}y(t) \quad G^T(t)y(t) = 0, \tag{5}$$

has the state-transition matrix

$$\Psi(t,s) \quad V(t)D(s-t)U^T(s)C^T(s), \tag{6}$$

which satisfies (5).

Before proceeding to connections with steady state matrices, we establish the following two results (proofs are omitted for brevity):

Lemma 2.1

$$\begin{aligned} u_1(t) \quad U(t)e_1 \text{ satisfies (2)} \\ v_1(t) = V(t)e_1 \text{ satisfies (5)} \end{aligned} \tag{7}$$

where e_1 is the first unit vector, corresponding to $\mu_1 = 0$.

Lemma 2.2

$$\begin{aligned} M = \; & \dot{V}^T(t)C(t) - V^T(t)G(t) \;\; U(t)\, I_m \\ = \; & I_m \;\; \dot{V}^T(t)C(t) \quad V^T(t)G(t) \;\; U(t) \\ & \text{where } M = diag(\mu_1, \cdots, \mu_m, 0, \cdots \; 0)\cdot, \end{aligned} \tag{8}$$

2.1 Frequency-domain computations

Frequency domain computations are natural for many applications, *e.g.*, mildly nonlinear RF system components. We cast and apply (8) using frequency domain quantities to establish a connection with harmonic balance.

We first develop some useful algebra involving Toeplitz matrices of Fourier components.

Definition 2.1 *Given any T-periodic vector or matrix $A(t)$, we denote its Fourier components by A_i, i.e.,*

$$A(t) = \sum_i A_i e^{ji\omega_0 t}, \quad \omega_0 = \frac{2\pi}{T} \tag{9}$$

Definition 2.2 *Given any vector or matrix $A(t)$, define the block-vector of its Fourier components to be*

$$\overset{FD}{A}_t = \begin{matrix} \vdots \\ A_2 \\ A_1 \\ A_0 \\ A_{-1} \\ A_{-2} \\ \vdots \end{matrix} . \tag{10}$$

Definition 2.3 *Given any matrix or vector $A(t)$, define the block-Toeplitz matrix of its Fourier components to be*

$$A_t = \begin{matrix} \vdots & \vdots & \vdots & \vdots & \vdots & \\ \cdots & A_0 & A_1 & A_2 & A_3 & A_4 & \cdots \\ \cdots & A_{-1} & A_0 & A_1 & A_2 & A_3 & \cdots \\ \cdots & A_{-2} & A_{-1} & A_0 & A_1 & A_2 & \cdots \\ \cdots & A_{-3} & A_{-2} & A_{-1} & A_0 & A_1 & \cdots \\ \cdots & A_{-4} & A_{-3} & A_{-2} & A_{-1} & A_0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \end{matrix} . \tag{11}$$

Lemma 2.3 *If $X(t)$ and $Y(t)$ are T-periodic vectors or matrices, and $Z(t) = X(t)Y(t)$, then*

$$\overset{FD}{Z}_t = X_t \overset{FD}{Y}_t \tag{12}$$

$$Z_t = X_t \; Y_t \tag{13}$$

Lemma 2.4 *If $X(t)$ is a T-periodic vector or matrix, then*

$$\overset{FD}{\dot{X}}_t = \overset{FD}{\Omega} X_t \tag{14}$$

$$X_t = \overset{FD}{\Omega} X_t - X_t \overset{FD}{\Omega} \tag{15}$$

where

$$\overset{FD}{\Omega} = j\omega_0 \begin{matrix} \ddots & & & & & & \\ & 2I_n & & & & & \\ & & I_n & & & & \\ & & & 0 & & & \\ & & & & -I_n & & \\ & & & & & -2I_n & \\ & & & & & & \ddots \end{matrix} \tag{16}$$

We are now in a position to use the above definitions and lemmas. Applying (12) and (14) to (2), we obtain the linearized harmonic balance equations:

$$\underbrace{\overset{FD}{\Omega} C_t + G_t}_{HB} \overset{FD}{y}_t = 0. \tag{17}$$

Next, we state an important intermediate result:

Lemma 2.5

$$\left(\overset{HB}{V^T} U + M - \overset{FD}{\Omega} \right)_m = 0, \text{ and} \tag{18}$$

$$_m\left(\overset{HB}{V^T} U + M - \overset{FD}{\Omega} \right) = 0. \tag{19}$$

We concentrate on a single row of (19) by premultiplying by $_{FD}{}^T e_1$, where e_1 is a unit vector (of size $m+k$) chosen to correspond to the oscillatory mode ($\mu_1 = 0$) of the system.

Theorem 2.1

$$\overset{FD}{e_1}{}^T \overset{HB}{V^T} = 0 \tag{20}$$

Remark 2.1 *From (20), we observe that* $V^T \; {}^{FD}e_1$ *(i.e., the vector of Fourier components of* $v_1(t)$*) is in the null space of* J_{HB}^T*, and that* ^{HB} *is singular.*

Next, consider the augmented harmonic balance matrix:

Definition 2.4 (Augmented HB matrix for oscillators)

$$ {}^{HB} = \begin{matrix} {}^{HB} & p \\ q^T & r \end{matrix}, \quad with \; {}^{HB^{-1}} = \begin{matrix} A & b \\ l^T & d \end{matrix}, \tag{21} $$

where p, q, b and l are column vectors, and r and d scalars.

^{HB} is the harmonic balance matrix augmented with a row and column, which are chosen to make it nonsingular. The following theorem establishes a simple means of computing the last row of its inverse.

Theorem 2.2 *If* $p = C \; U \; {}^{FD}e_1$ *and* ^{HB} *is nonsingular, then*

$$ l^T = {}^{FD^T}_{e_1} \; V^T. \tag{22} $$

Remark 2.2 $p = C \; U \; {}^{FD}e_1$ *is the vector of the Fourier coefficients of* $C(t)\dot{x}_s(t)$*, i.e.,* $p(t) = {}^{FD}_{C\,t\,\dot{x}_s\,t}$.

Remark 2.3 $l = V^T \; {}^{FD}e_1$ *is the conjugated vector of the Fourier components of* $v_1(t)$*, i.e.,* $\bar{l} = {}^{FD}_{v_1\,t}$.

Corollary 2.1 *From (21), l is the solution of the system*

$$ {}^{HB^T}x = \quad 0, \cdots \; 0, \cdots, \cdots 1^T, \tag{23} $$

hence $\bar{l} = {}^{FD}_{v_1\,t}$ *(i.e., the Fourier coefficients of* $v_1(t)$*) is the solution of*

$$ {}^{HB}x = \quad 0, \cdots, 0; \cdots \; 1^T, \tag{24} $$

Remark 2.4 *The augmented harmonic balance matrix* ^{HB}*, with* $p = C \; U \; {}^{FD}e_1$*, arises naturally as the Jacobian matrix of the oscillator's steady-state equations augmented by a phase condition, with the frequency of oscillation as an additional unknown (e.g., [7, 6]). Hence, from (24), the Fourier coefficients of* $v_1(t)$ *can be obtained from a single solution of the hermitian of the augmented harmonic balance Jacobian of the oscillator, with right-hand-side equal to a unit vector with value 1 in the phase condition equation. By exploiting circulant approximations to* ^{HB} *and applying iterative linear methods to solve (23) (e.g., [5, 9]), this computation becomes approximately linear in the system size.*

We note that the accuracy of the calculation (24) is dominated primarily by the smallest of the non-oscillatory eigenvalues $\mu_2, \cdots, \mu_m{}^2$. For high-Q oscillators, some of these eigenvalues can be very close to zero themselves. Since finding the steady-state solution of the oscillator is itself dependent on accurate solutions with the augmented HB matrix, it is to be expected that $v_1(t)$ will also be found to a similarly acceptable accuracy. This indicates that the main issue in calculating $v_1(t)$ by (24) is the accurate formation of, and solution with, the augmented HB matrix – a task that has already been accomplished during steady-state solution.

Direct approaches to calculating $v_1(t)$, based on finding the 1-eigenpair of the system's state-transition or monodromy matrix, do not exploit the accuracy of the steady-state calculation to the same extent as (24). In the absence of a periodicity condition, transient integration errors can accumulate in computing the monodromy matrix, causing the oscillatory eigenvalue to become numerically indistinguishable from other eigenvalues close to 1. Hence several eigenvectors corresponding to multiple eigenvalues close to 1 often need to be found, followed by subsequent selection of v_1 using the criterion of orthogonality with $C(t)\dot{x}_s(t)$. It is interesting to note that this orthogonality criterion is effectively embedded into (24), due to augmentation with p; as a result, calculation of multiple eigenvectors and subsequent selection is eliminated.

2.2 Time-domain computations

Time-domain computations are useful for systems with strong nonlinearities, such as ring oscillators. We first establish some notation.

Definition 2.5 *Denote by* $t_0, \cdots, t_{N-1}$ *a set of N ordered sample points of the interval* $0, T)$.

Definition 2.6 *Given any T-periodic vector or matrix* $A(t)$*, define*

$$ {}^{TD}_{A\,t} = \begin{matrix} A(t_0) \\ A(t_1) \\ \vdots \\ A(t_{N-1}) \end{matrix} \tag{25} $$

Definition 2.7 *given any T-periodic matrix or vector* $A(t)$*, define the block-diagonal matrix of its samples to be*

$$ {}_{A\,t} = \begin{matrix} A(t_0) & & & \\ & A(t_1) & & \\ & & \ddots & \\ & & & A(t_{N-1}) \end{matrix} \tag{26} $$

Lemma 2.6 *If* $X(t)$ *and* $Y(t)$ *are T-periodic vectors or matrices, and* $Z(t) = X(t)Y(t)$*, then*

$$ {}^{TD}_{Z\,t} = {}_{X\,t} \; {}^{TD}_{Y\,t} \tag{27} $$

$$ {}_{Z\,t} = {}_{X\,t} \; {}_{Y\,t} \tag{28} $$

[2] The $\mu_1 \quad 0$ eigenvalue of the non-augmented HB Jacobian is shifted to a non-zero value by the augmentation, resulting in a non-singular augmented HB Jacobian.

Lemma 2.7 *If $X(t)$ is a T-periodic vector or matrix, then*

$$\dot{X}(t) = \Omega_{TD} X(t) \tag{29}$$

where Ω_{TD} is a time-domain differentiation matrix corresponding to a linear multistep formula. For example, Ω_{TD} for the Backward Euler method is:

$$\Omega_{BE} = \begin{bmatrix} \frac{1}{T-t_{N-1}} I_n & & & -\frac{1}{T-t_{N-1}} I_n \\ -\frac{1}{t_1-t_0} I_n & \frac{1}{t_1-t_0} I_n & & \\ & \ddots & \ddots & \\ & & -\frac{1}{t_{N-1}-t_{N-2}} I_n & \frac{1}{t_{N-1}-t_{N-2}} I_n \end{bmatrix}$$

We now establish the time-domain analogue of (20):

Theorem 2.3

$$\underbrace{\Omega_{TD} C(t) + G(t)}_{J_{TD}} u_1(t) = 0 \tag{30}$$

$$\underbrace{C^T(t) \Omega_{TD} - G^T(t)}_{J_{TD}} v_1(t) = 0 \tag{31}$$

J_{TD} and J_{TD} are the forward and reverse time-domain Jacobian matrices, respectively. Next, consider the augmented Jacobian:

Definition 2.8

$$J_{TD} = \begin{bmatrix} J_{TD} & p \\ q^T & r \end{bmatrix} \tag{32}$$

where p and q are column vectors and r is a scalar. J_{TD} is the reverse time-domain Jacobian matrix augmented with a row and column, chosen to make it nonsingular.

Solving the following augmented Jacobian system results directly in the PPV $v_1(t)$:

Theorem 2.4 *If $q = C(t) u_1(t) = C(0) U^T e_1$ and J_{TD} is nonsingular,*

$$J_{TD} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ N \end{pmatrix} \tag{33}$$

has solution $x = v_1(t), y = 0$.

As in the frequency-domain case, the equation system (33) can be solved efficiently with iterative methods, as a final step after solving the time-domain steady-state equations of the system.

3 Evaluation of the new technique

To evaluate the new method, it was compared against the established method that uses monodromy matrix eigendecomposition. The steady-state of a tank-circuit-based oscillator was computed using Harmonic Balance with $m = 31$ harmonics, resulting in $N = 63$ distinct frequency components. The frequency of oscillation f_0 was 159154.853364298Hz. The time-domain voltage waveform at the tank capacitor, and the currents through the inductor and power supply are shown in Figure 1, Figure 2, and Figure 3 respectively.

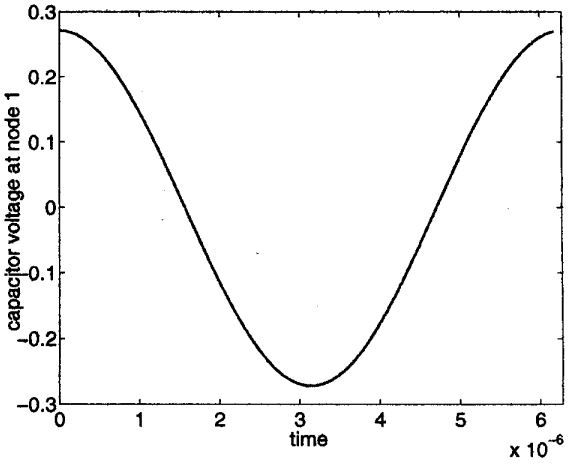

Figure 1: Oscillator steady-state: voltage at capacitor

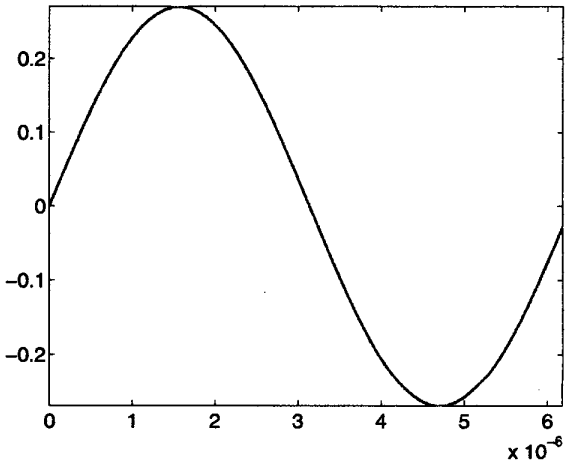

Figure 2: Oscillator steady-state: current through inductor

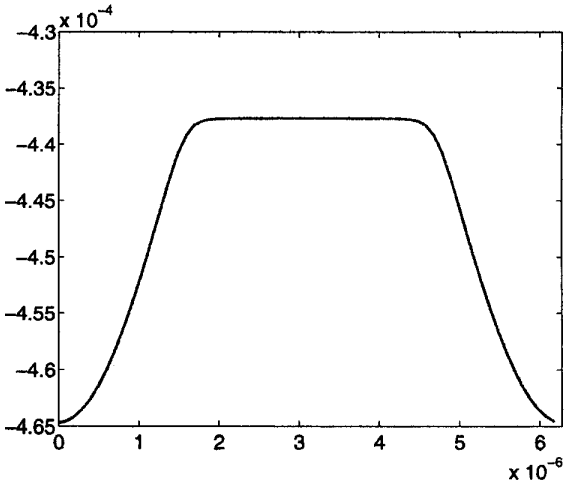

Figure 3: Oscillator steady-state: current through vdd

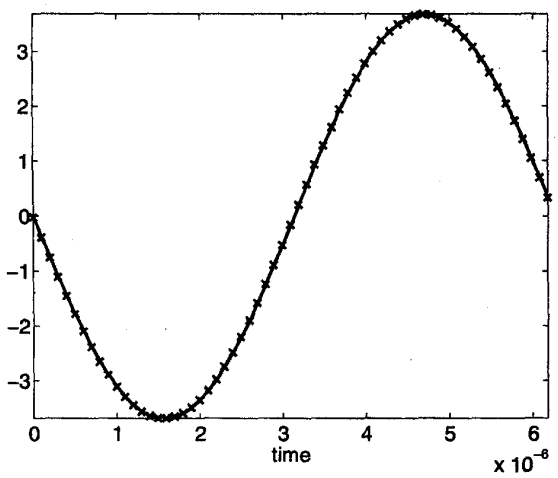

Figure 4: Capacitor node of PPVs v_{1d} and v_{1m}

The PPV $v_1(t)$ was first determined through the time-domain monodromy matrix by computing its 1-eigenpair using iterative linear methods followed by manual selection from among candidate eigenpairs. The eigenvector thus obtained was then used as an initial condition for a transient simulation of the adjoint system, using a time-step corresponding to an oversampling factor of 4 (*i.e.*, $4N$ timepoints) to limit accuracy loss from linear multistep formulae for DAE solution. The result of this transient simulation, after normalization, is the conventionally computed PPV. We refer to as $v_{1m}(t)$.

The new method described above simply computes the system (24) directly from the oscillator's harmonic balance Jacobian, with a single iterative linear solve. No oversampling is used by the method. The PPV obtained in this manner is denoted by $v_{1d}(t)$.

Figure 4, Figure 5, and Figure 6 depict the components of $v_{1d}(t)$ (solid red line) and $v_{1m}(t)$ (blue marks) corresponding to the capacitor node, the inductor current and the power supply current, respectively. It can be seen that the PPV waveforms produced by the two methods are visually indistinguishable from each other.

A more critical assessment of the two methods can be made using the fact that $u_1^T(t)v_1(t)$ 1[3]. We plot the error $\varepsilon_d(t)$ $u_1^T(t)v_{1d}(t)$ 1 vs $\varepsilon_m(t) = u_1^T(t)v_{1m}(t)$ 1 in Figure 7. The solid red line indicates $\varepsilon_d(t)$, the error of the new method, while the blue marks indicate $\varepsilon_m(t)$. The new method is about 2 orders of magnitude more accurate than monodromy matrix eigendecomposition, despite the 4 oversampling used by the latter method.

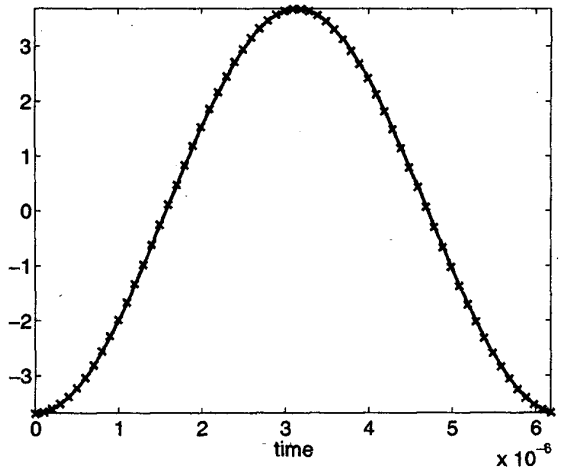

Figure 5: Inductor current component of PPVs v_{1d} and v_{1m}

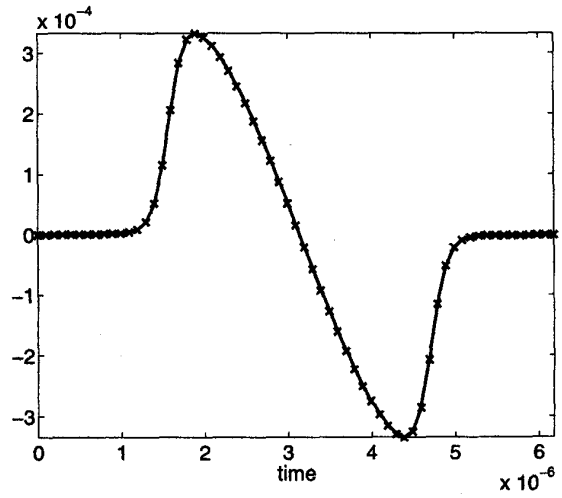

Figure 6: Power supply current component of PPVs v_{1d} and v_{1m}

Acknowledgments

We acknowledge Kiran Gullapalli of Motorola for providing the test oscillator circuit, its steady state solution, and $v_1(t)$ computed using monodromy matrix methods.

References

[3] See, *e.g.*, [4]. Note that u_1 t defined in (7) is identical to p t ^{FD} $C\,t\,\dot{x}_s\,t$.

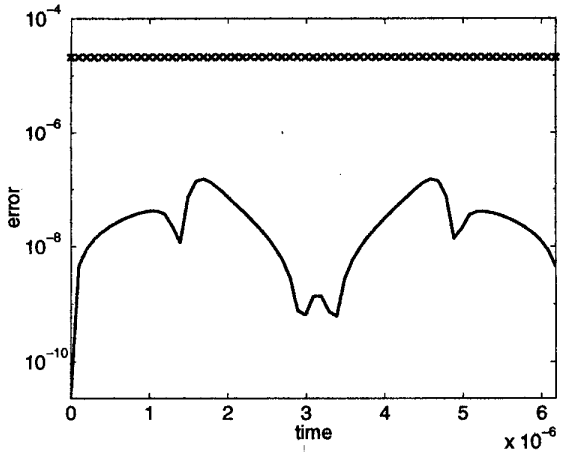

Figure 7: Errors in the PPV obtained using the monodromy and new methods

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A Matrix notation for Jacobians

Lemma 2.3 *If $X(t)$ and $Y(t)$ are T-periodic vectors or matrices, and $Z(t) = X(t)Y(t)$, then*

$$Z^{FD} = X^{FD} Y \tag{34}$$

$$Z = X\ Y \tag{35}$$

Proof: From definition,

$$\sum_i Z_i e^{ji\omega_0 t} = \sum_{k\,l} X_k Y_l e^{j\,k\;l\,\omega_0 t}$$

$$Z_i = \sum_l X_{i-l} Y_l. \tag{36}$$

(12) follows directly.

Next, we rewrite (36) as

$$\begin{aligned} Z_{i\;\;k} &= \sum_l X_{i-\;l-k}\, Y_l \\ &= \sum_{l=l-k} X_{i-l}\, Y_{l\;\;k}. \end{aligned} \tag{37}$$

(37) yields the k^{th} block-column of Z, hence (13) follows. ■

Lemma 2.4 *If $X(t)$ is a T-periodic vector or matrix, then*

$$\dot{X}^{FD} = \Omega^{FD\,FD} X \tag{38}$$

$$\dot{X} = \Omega^{FD} X - X\,\Omega^{FD} \tag{39}$$

where

$$\Omega^{FD} = j\omega_0 \begin{bmatrix} \ddots & & & & & \\ & 2I_n & & & & \\ & & I_n & & & \\ & & & 0 & & \\ & & & & -I_n & \\ & & & & & -2I_n \\ & & & & & & \ddots \end{bmatrix} \tag{40}$$

Proof: Using

$$\dot{X}(t) = \sum_i ji\omega_0 X_i e^{j\omega_0 it},$$

(14) and (15) may be verified directly. ■

Lemma 2.6 *If $X(t)$ and $Y(t)$ are T-periodic vectors or matrices, and $Z(t) = X(t)Y(t)$, then*

$$Z^{TD} = X^{TD}\ Y \tag{41}$$

$$Z = X\ Y^{TD} \tag{42}$$

Proof: Follows directly from definition of ^{TD} and . ■

Lemma 2.7 *If $X(t)$ is a T-periodic vector or matrix, then*

$$\dot{X}^{TD} = \Omega^{TD\,TD} X \tag{43}$$

where Ω^{TD} is a time-domain differentiation matrix corresponding to a linear multistep formula. For example, Ω^{TD} for the Backward Euler method is:

$$\Omega^{TD}_{BE} = \begin{bmatrix} \frac{1}{T-t_{N-1}} I_n & & & & -I_n \\ -\frac{1}{t_1-t_0} I_n & \frac{1}{t_1-t_0} I_n & & & \\ & -\frac{1}{t_2-t_1} I_n & \frac{1}{t_2-t_1} I_n & & \\ & & \ddots & \ddots & \\ & & & -\frac{1}{t_{N-1}-t_{N-2}} I_n & \frac{1}{t_{N-1}-t_{N-2}} I_n \end{bmatrix}$$

Proof: Follows directly from the definitions of ^{TD} and Ω^{TD} and linear multistep formulae for differentiation. ■