

Module - I

Vector differentiation

Important result :-

- 1] The position Vector defined on xy plane, yz plane, xz plane is $\vec{r} = \vec{x}\hat{i} + \vec{y}\hat{j} + \vec{z}\hat{k}$ and its d can be defined as $d\vec{r} = d\vec{x}\hat{i} + d\vec{y}\hat{j} + d\vec{z}\hat{k}$ in other words if x, y, z are in function of t, then the position Vector $\vec{r} = x(t)\hat{i} + y(t)\hat{j} + z(t)\hat{k}$ and its derivative is

$$\frac{d\vec{r}}{dt} = \frac{dx}{dt}\hat{i} + \frac{dy}{dt}\hat{j} + \frac{dz}{dt}\hat{k}$$

- 2] for any $F_1 = F_1(x, y, z)$, $F_2 = F_2(x, y, z)$, $F_3 = F_3(x, y, z)$ Then the fun $\vec{F} = \vec{F}_1\hat{i} + \vec{F}_2\hat{j} + \vec{F}_3\hat{k}$ is called the Vector point

- 3] Suppose the two Vectors point fun $\vec{A} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$, $\vec{B} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$. Then $\vec{A} \cdot \vec{B} = (a_1\hat{i} + a_2\hat{j} + a_3\hat{k}) \cdot (b_1\hat{i} + b_2\hat{j} + b_3\hat{k})$
 $a_1b_1 + a_2b_2 + a_3b_3$ is a scalar

$$(ii) \vec{A} \times \vec{B} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

$$= (a_2b_3 - a_3b_2)\hat{i} - (b_3a_1 - a_3b_1)\hat{j} + (a_1b_2 - a_2b_1)\hat{k} \text{ is a Vector point function}$$

- 4] The gradient of a Vector can be denoted by the symbol ∇ instead of del \equiv grad and which will be denoted as

$$\nabla = \frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}$$

Suppose $f(x, y, z) = c$ be a scalar point function. Then the gradient of f can be defined as

$$\nabla f = \text{grad } f = \left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k} \right) f$$

$$\nabla \phi = \frac{\partial \phi}{\partial x} \vec{i} + \frac{\partial \phi}{\partial y} \vec{j} + \frac{\partial \phi}{\partial z} \vec{k}$$

5] Suppose $\phi_1(x, y, z) = c_1$, $\phi_2(x, y, z) = c_2$ are the two scalar functions then

$$(i) \nabla(\lambda_1 \phi_1 + \lambda_2 \phi_2) = \lambda_1 \nabla \phi_1 + \lambda_2 \nabla \phi_2$$

$$(ii) \nabla(\phi_1 \phi_2) = \phi_1 \nabla \phi_2 + \phi_2 \nabla \phi_1$$

(iii) The angle between the given two surfaces can be evaluated by using $\cos \theta = \frac{\nabla \phi_1 \cdot \nabla \phi_2}{|\nabla \phi_1| |\nabla \phi_2|}$

6] Suppose $\vec{F} = F_1 \vec{i} + F_2 \vec{j} + F_3 \vec{k}$ be a vector point function for which F_1, F_2, F_3 are the functions of x, y, z then

(i) $\text{div } \vec{F}$ can be defined as

$$\text{div } \vec{F} = \nabla \cdot \vec{F}$$

$$= \left(\frac{\partial}{\partial x} \vec{i} + \frac{\partial}{\partial y} \vec{j} + \frac{\partial}{\partial z} \vec{k} \right) \cdot (F_1 \vec{i} + F_2 \vec{j} + F_3 \vec{k})$$

$\text{div } \vec{F} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}$ is a scalar point function

$$(ii) \vec{F} = \nabla \times \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix}$$

$$\text{curl } \vec{F} = \left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z} \right) \vec{i} - \left(\frac{\partial F_3}{\partial x} - \frac{\partial F_1}{\partial z} \right) \vec{j} + \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right) \vec{k}$$

is a vector point function

7] If $\text{div } \vec{F} = 0$ then we say that \vec{F} is a solenoidal and if $\vec{F} = \vec{0} + \vec{0} + \vec{0} = \vec{0}$ then we say that the given \vec{F} is a irrotational at any point $P(x_0, y_0, z_0)$

1] Find the angle between the surface $x^2 + y^2 + z^2 = 9$, $x^2 + y^2 - 3z = 0$
at the point $(2, -1, 2)$

\Rightarrow

$$\phi_1 = x^2 + y^2 + z^2 - 9 = 0$$

$$\phi_2 = x^2 + y^2 - 3z = 0$$

$$\nabla \phi_1 = \frac{\partial \phi_1}{\partial x} \vec{i} + \frac{\partial \phi_1}{\partial y} \vec{j} + \frac{\partial \phi_1}{\partial z} \vec{k}$$

$$\nabla \phi_1 = 2x \vec{i} + 2y \vec{j} + 2z \vec{k}$$

$$\nabla \phi_1 |_{(2, -1, 2)} = 2(2) \vec{i} + 2(-1) \vec{j} + 2(2) \vec{k}$$

$$\Rightarrow \nabla \phi_1 |_{(2, -1, 2)} = 4 \vec{i} - 2 \vec{j} + 4 \vec{k}$$

$$\phi_2 = x^2 + y^2 - 3z = 0$$

$$\nabla \phi_2 = \frac{\partial \phi_2}{\partial x} \vec{i} + \frac{\partial \phi_2}{\partial y} \vec{j} + \frac{\partial \phi_2}{\partial z} \vec{k}$$

$$\nabla \phi_2 = 2x \vec{i} + 2y \vec{j} - 3 \vec{k}$$

$$\nabla \phi_2 |_{(2, -1, 2)} = 2(2) \vec{i} + 2(-1) \vec{j} - 3 \vec{k}$$

$$\Rightarrow \nabla \phi_2 |_{(2, -1, 2)} = 4 \vec{i} - 2 \vec{j} - 3 \vec{k}$$

$$\nabla \phi_1 = 4 \vec{i} - 2 \vec{j} + 4 \vec{k} = \vec{A}$$

$$\nabla \phi_2 = 4 \vec{i} - 2 \vec{j} - 3 \vec{k} = \vec{B}$$

$$\cos \theta = \frac{\vec{A} \cdot \vec{B}}{|\vec{A}| |\vec{B}|}$$

$$|\vec{A}| |\vec{B}|$$

$$\cos \theta = \frac{(4 \vec{i} - 2 \vec{j} + 4 \vec{k}) \cdot (4 \vec{i} - 2 \vec{j} - 3 \vec{k})}{\sqrt{(4)^2 + (-2)^2 + (4)^2} \sqrt{(4)^2 + (-2)^2 + (-1)^2}}$$

$$\cos \theta = \frac{16 + 4 - 4}{6\sqrt{21}}$$

$$\cos \theta = \frac{16}{6\sqrt{21}}$$

$$\boxed{\Theta = \cos^{-1} \left(\frac{8}{3\sqrt{21}} \right)}$$

2) Find the angle between the surfaces $x^2 + y^2 - z^2 = 4$ and $x^2 + y^2 - 13 = z$ at the point $(2, 1, 2)$

\Rightarrow

$$\phi_1 = x^2 + y^2 - z^2 - 4 = 0$$

$$\phi_2 = x^2 + y^2 - 13 - z = 0$$

$$\nabla \phi_1 = \frac{\partial \phi_1}{\partial x} \vec{i} + \frac{\partial \phi_1}{\partial y} \vec{j} + \frac{\partial \phi_1}{\partial z} \vec{k}$$

$$\nabla \phi_1 = 2x\vec{i} + 2y\vec{j} - 2z\vec{k}$$

$$\nabla \phi_1|_{(2,1,2)} = 2(2)\vec{i} + 2(1)\vec{j} - 2(2)\vec{k}$$

$$\Rightarrow \nabla \phi_1|_{(2,1,2)} = 4\vec{i} + 2\vec{j} - 4\vec{k}$$

$$\nabla \phi_2 = \frac{\partial \phi_2}{\partial x} \vec{i} + \frac{\partial \phi_2}{\partial y} \vec{j} + \frac{\partial \phi_2}{\partial z} \vec{k}$$

$$\nabla \phi_2 = 2x\vec{i} + 2y\vec{j} + 1\vec{k}$$

$$\nabla \phi_2|_{(2,-1,2)} = 2(2)\vec{i} + 2(-1)\vec{j} - 1\vec{k}$$

$$\Rightarrow \nabla \phi_2|_{(2,-1,2)} = 4\vec{i} - 2\vec{j} - \vec{k}$$

$$\nabla \phi_1 = 4\vec{i} + 2\vec{j} - 4\vec{k} \Rightarrow \vec{A}$$

$$\nabla \phi_2 = 4\vec{i} + 2\vec{j} - \vec{k} \Rightarrow \vec{B}$$

$$\cos \theta = \frac{\vec{A} \cdot \vec{B}}{|\vec{A}| |\vec{B}|}$$

$$\cos \theta = \frac{(4\vec{i} + 2\vec{j} - 4\vec{k}) \cdot (4\vec{i} + 2\vec{j} - \vec{k})}{\sqrt{(4)^2 + (2)^2 + (4)^2} \sqrt{(4)^2 + (2)^2 + (-1)^2}}$$

$$\cos \theta = \frac{16 + 4 + 4}{6\sqrt{21}}$$

$$\cos \theta = \frac{24}{6\sqrt{21}}$$

$$\boxed{\theta = \cos^{-1} \left(\frac{4}{\sqrt{21}} \right)}$$

3] Find the values of a and b such that surfaces $ax^2 - byz = (a+2)x$ and $4x^2y + z^3 = 4$ are orthogonal at the point $(1, -1, 2)$

\Rightarrow

$$\phi_1 = ax^2 - byz - (a+2)z = 0 \rightarrow ①$$

$$\phi_2 = 4x^2y + z^3 - 4 = 0 \rightarrow ②$$

The point $p(1, -1, 2)$ is on both the surfaces

$$① \Rightarrow a(1)^2 - b(-1)(2) = (a+2)(1)$$

$$\Rightarrow a+2b = a+2$$

$$\therefore b = 1$$

$$① \Rightarrow \phi_1 = ax^2 - yz - (a+2)z = 0$$

$$\nabla \phi_1 = \frac{\partial \phi_1}{\partial x} \vec{i} + \frac{\partial \phi_1}{\partial y} \vec{j} + \frac{\partial \phi_1}{\partial z} \vec{k}$$

$$\nabla \phi_1 = (2ax - a-2)\vec{i} - z\vec{j} - y\vec{k}$$

$$\nabla \phi_1|_{p(1, -1, 2)} = 2(a(1) - a-2)\vec{i} - 2\vec{j} + \vec{k}$$

$$\nabla \phi_1 = (a-2)\vec{i} - 2\vec{j} + \vec{k}$$

$$\nabla \phi_2 = \frac{\partial \phi_2}{\partial x} \vec{i} + \frac{\partial \phi_2}{\partial y} \vec{j} + \frac{\partial \phi_2}{\partial z} \vec{k}$$

$$\nabla \phi_2 = 8x^2y\vec{i} + 4x^2\vec{j} + 3z^2\vec{k}$$

$$\nabla \phi_2|_{p(1, -1, 2)} = -8\vec{i} + 4\vec{j} + 12\vec{k}$$

Given that ϕ_1 and ϕ_2 orthogonally intersect

$$\nabla \phi_1 \cdot \nabla \phi_2 = 0$$

$$|(a-2)\vec{i} - 2\vec{j} + \vec{k}| \cdot |-8\vec{i} + 4\vec{j} + 12\vec{k}| = 0$$

$$\Rightarrow -8(a-2) - 8 + 12 = 0$$

$$\Rightarrow -8a + 16 - 8 + 12 = 0$$

$$\Rightarrow 80 - 8a = 0$$

$$\Rightarrow 80 = 8a$$

$$\Rightarrow a = \frac{20}{8}$$

$$\boxed{a = \frac{5}{2}}$$

5 Find the angle between the Normals to the Surface $xy = z^2$
at the points $(4, 1, 2)$ and $(3, 3, -3)$



$$xy = z^2$$

$$xy - z^2 = 0$$

$$\phi = xy - z^2$$

$$P = (4, 1, 2) \quad Q = (3, 3, -3)$$

$$\nabla \phi = \frac{\partial \phi}{\partial x} \vec{i} + \frac{\partial \phi}{\partial y} \vec{j} + \frac{\partial \phi}{\partial z} \vec{k}$$

$$\nabla \phi = y \vec{i} + x \vec{j} - 2z \vec{k}$$

$$(\nabla \phi)_P = \vec{i} + 4\vec{j} - 4\vec{k} = \vec{A}$$

$$(\nabla \phi)_Q = 3\vec{i} + 3\vec{j} + 6\vec{k} = \vec{B}$$

to find $\cos \theta$

$$\cos \theta = \frac{\vec{A} \cdot \vec{B}}{|\vec{A}| |\vec{B}|}$$

$$\cos \theta = \frac{(\vec{i} + 4\vec{j} - 4\vec{k}) \cdot (3\vec{i} + 3\vec{j} + 6\vec{k})}{\sqrt{1^2 + 4^2 + (-4)^2} \sqrt{3^2 + 3^2 + 6^2}}$$

$$\cos \theta = \frac{3 + 12 - 24}{\sqrt{33} \sqrt{54}}$$

$$\cos \theta = \frac{-9}{\sqrt{33} \sqrt{54}}$$

$$\cos \theta = \frac{-9}{\sqrt{33} \sqrt{54}}$$

$$\cos \theta = \frac{-3}{\sqrt{198}}$$

$$\theta = \cos^{-1} \left(\frac{-3}{\sqrt{198}} \right)$$

Q If $\vec{F} = (xy^3z^2)$ (or) $\vec{F} = \text{grad}(xy^3z^2)$ find $\text{div } \vec{F}$ and $\text{curl } \vec{F}$
at the point $(1, -1, 1)$

\Rightarrow

$$\vec{F} = \nabla(xy^3z^2)$$

$$\vec{F} = \nabla\phi$$

$$\therefore \phi = xy^3z^2$$

$$\frac{\partial \phi}{\partial x} = y^3z^2, \quad \frac{\partial \phi}{\partial y} = 3y^2xz^2, \quad \frac{\partial \phi}{\partial z} = 2xyz^3$$

$$\vec{F} = \nabla\phi$$

$$\vec{F} = \frac{\partial \phi}{\partial x}\vec{i} + \frac{\partial \phi}{\partial y}\vec{j} + \frac{\partial \phi}{\partial z}\vec{k}$$

$$\vec{F} = y^3z^2\vec{i} + 3y^2xz^2\vec{j} + 2xyz^3\vec{k}$$

$$(i) \text{ div } \vec{F} = \nabla \cdot \vec{F}$$

$$\text{div } \vec{F} = \left| \frac{\partial}{\partial x}\vec{i} + \frac{\partial}{\partial y}\vec{j} + \frac{\partial}{\partial z}\vec{k} \right| \cdot \left| y^3z^2\vec{i} + 3y^2xz^2\vec{j} + 2xyz^3\vec{k} \right|$$

$$\text{div } \vec{F} = \frac{\partial(y^3z^2)}{\partial x} + \frac{\partial(3y^2xz^2)}{\partial y} + \frac{\partial(2xyz^3)}{\partial z}$$

$$\text{div } \vec{F} = 0 + 6xyz^2 + 2xy^3$$

$$\text{div } \vec{F} = 6xyz^2 + 2xy^3$$

$$\text{div } \vec{F}_{P(1,-1,1)} = 6(1)(-1)(1)^2 + 2(1)(-1)^3$$

$$\text{div } \vec{F} = -6 - 2$$

$$\text{div } \vec{F} = -8$$

$$(ii) \text{ curl } \vec{F} = \nabla \times \vec{F}$$

$$\text{curl } \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y^3z^2 & 3y^2xz^2 & 2xyz^3 \end{vmatrix}$$

$$\text{curl } \vec{F} = (6yz^2 - 6xy^2z) \hat{i} - (2y^3z - y^3z^2) \hat{j} + (3y^2z^2 - 3y^2z^2) \hat{k}$$

$$\text{curl } \vec{F} = \left[[6(1)(-1)^2(1) - 6(1)(-1)^2(1)] \hat{i} \right] - \left[[2(-1)^3(1) - (-1)^3 2(-1)] \hat{j} \right]$$

$$+ \left[[3(-1)^2(1) - 3(-1)^2(1)] \hat{j} \right]$$

$$\text{curl } \vec{F} = [6 - 6] \hat{i} - [2 - 2] \hat{j} + [3 - 3] \hat{k}$$

$$\text{curl } \vec{F} = 0\hat{i} - 0\hat{j} + 0\hat{k}$$

~~curl \vec{F} = 0; i.e. irrotational~~

Q) If $\vec{F} = \nabla(\phi) = \nabla(x^3 + y^3 + z^3 - 3xyz)$ find div \vec{F} and curl \vec{F}

$$\Rightarrow \vec{F} = \nabla(x^3 + y^3 + z^3 - 3xyz) \rightarrow ①$$

$$\phi = x^3 + y^3 + z^3 - 3xyz$$

$$\frac{\partial \phi}{\partial x} = 3x^2 - 3yz \Rightarrow 3(x^2 - yz)$$

$$\frac{\partial \phi}{\partial y} = 3y^2 - 3xz \Rightarrow 3(y^2 - xz)$$

$$\frac{\partial \phi}{\partial z} = 3z^2 - 3xy \Rightarrow 3(z^2 - xy)$$

$$① \Rightarrow \vec{F} = \nabla \phi$$

$$\vec{F} = \frac{\partial \phi}{\partial x} \hat{i} + \frac{\partial \phi}{\partial y} \hat{j} + \frac{\partial \phi}{\partial z} \hat{k}$$

$$\vec{F} = 3(x^2 - yz) \hat{i} + 3(y^2 - xz) \hat{j} + 3(z^2 - xy) \hat{k}$$

$$(i) \text{ div } \vec{F} = \nabla \cdot \vec{F}$$

$$\text{div } \vec{F} = \left| \frac{\partial}{\partial x} \hat{i} + \frac{\partial}{\partial y} \hat{j} + \frac{\partial}{\partial z} \hat{k} \right| \cdot [3(x^2 - yz) \hat{i} + 3(y^2 - xz) \hat{j} + 3(z^2 - xy) \hat{k}]$$

$$\text{div } \vec{F} = 3 \frac{\partial}{\partial x} (x^2 - yz) + 3 \frac{\partial}{\partial y} (y^2 - xz) + 3 \frac{\partial}{\partial z} (z^2 - xy)$$

$$\text{div } \vec{F} = 6x + 6y + 6z$$

$$\text{div } \vec{F} = 6(x + y + z)$$

$$(ii) \text{ curl } \vec{F} = \nabla \times \vec{F}$$

$$\text{curl } \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 3(x^2 - y^2) & 3(y^2 - zx) & 3(z^2 - xy) \end{vmatrix}$$

$$\text{curl } \vec{F} = (-3z + 3x)\vec{i} - (-3y + 3y)\vec{j} + (-3x + 3z)\vec{k}$$

$$\text{curl } \vec{F} = \vec{0}$$

8] Find 'a' for which $\vec{F} = (x+3y)\vec{i} + (y-2z)\vec{j} + (x+az)\vec{k}$ is solenoidal

$$\Rightarrow \vec{F} = (x+3y)\vec{i} + (y-2z)\vec{j} + (x+az)\vec{k}$$

\vec{F} is a solenoidal

$$\Rightarrow \text{div } \vec{F} = 0$$

$$\Rightarrow \nabla \cdot \vec{F} = 0$$

$$\Rightarrow \left[\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z} \right] \cdot [(x+3y)\vec{i} + (y-2z)\vec{j} + (x+az)\vec{k}] = 0$$

$$\Rightarrow 1 + 1 + a = 0$$

$$a+2=0$$

$$\boxed{a=-2}$$

9] Find the constants a, b and c such that

$\vec{F} = (axy + bz^3)\vec{i} + (3x^2 - cz)\vec{j} + (3xz^2 - y)\vec{k}$ is irrotational
also find the scalar potential such that $\vec{F} = \nabla \phi$

$$\Rightarrow \vec{F} = (axy + bz^3)\vec{i} + (3x^2 - cz)\vec{j} + (3xz^2 - y)\vec{k} \rightarrow ①$$

Given that $\text{curl } \vec{F} = \vec{0}$

$$\nabla \times \vec{F} = \vec{0}$$

$$\begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ (axy + bz^3) & (3x^2 - cz) & (3xz^2 - y) \end{vmatrix} = \vec{0} + \vec{0} + \vec{0}$$

$$\Rightarrow (-1+0)\vec{i} + (3z^2 - 3bz^2)\vec{j} + (6x - ax)\vec{k}$$

$$\Rightarrow -\vec{i} - 3z^2(1-b)\vec{j} + x(6-a)\vec{k} = \vec{0} + \vec{0} + \vec{0}$$

$$3z^2(1-b) = 0 \quad x(6-a) = 0 \quad -1+c=0$$

$$1-b=0$$

$$6-a=0$$

$$c=1$$

$$b=1$$

$$6=a$$

突破口方程 ①

$$\vec{F} = (6xy + z^3)\vec{i} + (3x^2 - z)\vec{j} + (3xz^2 - y)\vec{k}$$

$$\vec{F} = \nabla \phi$$

$$(6xy + z^3)\vec{i} + (3x^2 - z)\vec{j} + (3xz^2 - y)\vec{k} = \frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y} + \frac{\partial \phi}{\partial z}$$

$$\frac{\partial \phi}{\partial x} = 6xy + z^3$$

$$\Rightarrow \int \partial_x \phi = \int (6xy + z^3) dx$$

$$\phi = 3x^2y + xz^3 + f_1(y, z) \rightarrow ②$$

$$\frac{\partial \phi}{\partial y} = 3x^2 - z$$

$$\Rightarrow \int \partial_y \phi = \int 3x^2 - z dy$$

$$\phi = 3x^2y - yz + f_2(x, z) \rightarrow ③$$

$$\frac{\partial \phi}{\partial z} = 3xz^2 - y$$

$$\Rightarrow \int \partial_z \phi = \int 3xz^2 - y dz$$

$$\phi = 3xz^3 - yz + f_3(x, y) \rightarrow ④$$

from ② ③ and ④

$$\boxed{\phi = 3x^2y - yz + xz^3 + c}$$

10) Show that $\vec{F} = \frac{x\vec{i} + y\vec{j}}{x^2 + y^2}$ is solenoidal and irrotational

$$\Rightarrow \vec{F} = \frac{x\vec{i} + y\vec{j}}{x^2 + y^2}$$

$$\vec{F} = \frac{x\vec{i}}{x^2 + y^2} + \frac{y\vec{j}}{x^2 + y^2} + \vec{0k}$$

$$(i) \operatorname{div} \vec{F} = \nabla \cdot \vec{F}$$

$$= \left[\frac{\partial \vec{i}}{\partial x} + \frac{\partial \vec{j}}{\partial y} + \frac{\partial \vec{k}}{\partial z} \right] \cdot \left[\frac{x}{x^2 + y^2} \vec{i} + \frac{y}{x^2 + y^2} \vec{j} + \vec{0k} \right]$$

$$= \frac{\partial}{\partial z} \left(\frac{x}{x^2 + y^2} \right) + \frac{\partial}{\partial y} \left(\frac{y}{x^2 + y^2} \right) + 0$$

$$= \frac{(x^2 + y^2) - x(2x)}{(x^2 + y^2)^2} + \frac{(x^2 + y^2) - y(2y)}{(x^2 + y^2)^2}$$

$$= \frac{y^2 - x^2}{(x^2 + y^2)^2} + y^2 - \frac{x^2 - y^2}{(x^2 + y^2)^2}$$

$$= \frac{y^2 - x^2 + x^2 - y^2}{(x^2 + y^2)^2}$$

$$\operatorname{div} \vec{F} = 0$$

\vec{F} is a solenoidal

$$(ii) \operatorname{curl} \vec{F} = \nabla \times \vec{F}$$

$$\operatorname{curl} \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{x}{x^2 + y^2} & \frac{y}{x^2 + y^2} & 0 \end{vmatrix}$$

$$= (0-0)\bar{i} - (0-0)\bar{j} + \left[\frac{-2xy}{(x^2+y^2)^2} + \frac{2xy}{(x^2+y^2)^2} \right] \bar{k}$$

$$= \bar{0}\bar{i} + \bar{0}\bar{j} + \bar{0}\bar{k}$$

$$= \vec{0}$$

$\text{curl } \vec{F} = \vec{0}$ \therefore \vec{F} is a irrotational

11) Show that the vector field $\vec{F} = (x^2-yz)\bar{i} + (y^2-zx)\bar{j} + (z^2-xy)\bar{k}$ is an irrotational

$$\Rightarrow \vec{F} = (x^2-yz)\bar{i} + (y^2-zx)\bar{j} + (z^2-xy)\bar{k}$$

$$\text{curl } \vec{F} = \nabla \times \vec{F}$$

$$\text{curl } \vec{F} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ (x^2-yz) & (y^2-zx) & (z^2-xy) \end{vmatrix}$$

$$\text{curl } \vec{F} = (-1+1)\bar{i} - (-1+1)\bar{j} + (-1+1)\bar{k}$$

$$\text{curl } \vec{F} = \bar{0}\bar{i} + (1-1)\bar{j} + \bar{0}\bar{k}$$

$$\text{curl } \vec{F} = \bar{0}\bar{i} + \bar{0}\bar{j} + \bar{0}\bar{k}$$

$$\text{curl } \vec{F} = \vec{0}$$

$\therefore \vec{F}$ is a irrotational ~~/~~

12) Find the constant a, b and c such that

$$\vec{F} = (x+y+az)\bar{i} + (bx+2y-z)\bar{j} + (x+cy+2z)\bar{k}$$

is an irrotational also find the scalar potential ϕ for which $\vec{F} = \nabla \phi$

$$\Rightarrow \vec{F} = (x+y+az)\bar{i} + (bx+2y-z)\bar{j} + (x+cy+2z)\bar{k} \rightarrow (1)$$

\vec{F} is an irrotational

$$\Rightarrow \nabla \times \vec{F} = \vec{0}$$

$$\Rightarrow \nabla \times \vec{F} = \vec{0}$$

$$\begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ (x+y+az) & (bx+2y+z) & (x+cy+2z) \end{vmatrix} = \vec{0i} + \vec{0j} + \vec{0k}$$

$$\Rightarrow (c+1)\vec{i} - (1-a)\vec{j} + (b-1)\vec{k} = \vec{0i} + \vec{0j} + \vec{0k}$$

$$\Rightarrow (c+1)\vec{i} + (a-1)\vec{j} + (b-1)\vec{k} = \vec{0i} + \vec{0j} + \vec{0k}$$

$$c=-1 \quad a=1 \quad b=1$$

$$\Leftrightarrow (a, b, c) \Rightarrow (-1, 1, 1)$$

By eliminating ① we get

$$\textcircled{1} \Rightarrow \vec{F} = (x+y+z)\vec{i} + (x+2y-z)\vec{j} + (x-y+2z)\vec{k}$$

Given that $\vec{F} = \nabla \phi$

$$\Rightarrow (x+y+z)\vec{i} + (x+2y-z)\vec{j} + (x-y+2z)\vec{k} = \frac{\partial \phi}{\partial x}\vec{i} + \frac{\partial \phi}{\partial y}\vec{j} + \frac{\partial \phi}{\partial z}\vec{k}$$

$$\frac{\partial \phi}{\partial x} = x+y+z$$

$$\Rightarrow \rho \phi = \int x+y+z \, dx$$

$$\phi = \frac{x^2}{2} + xy + zx + f_1(y, z) \rightarrow \textcircled{2}$$

$$\frac{\partial \phi}{\partial y} = x+2y-z$$

$$\Rightarrow \rho \phi = \int x+2y-z \, dy$$

$$\phi = xy + y^2 - zy + f_2(x, z) \rightarrow \textcircled{3}$$

$$\frac{\partial \phi}{\partial z} = x-y+2z$$

$$\Rightarrow \rho \phi = \int x-y+2z \, dz$$

$$\phi = xy - yz + z^2 + f_3(x, y) \rightarrow \textcircled{4}$$

from ② ③ and ④

$$\boxed{\phi = \frac{x^2}{2} + y^2 + z^2 + xy - Ix - Iy + C}$$

- 13) Find the constant a, b, c so that the vector field $\vec{F} = (x+2y+az)\vec{i} + (bx-3y-Iz)\vec{j} + (4x+cy+2z)\vec{k}$ is irrotational, also find the scalar potential ϕ for which $\vec{F} = \nabla\phi$

$$\Rightarrow \vec{F} = (x+2y+az)\vec{i} + (bx-3y-Iz)\vec{j} + (4x+cy+2z)\vec{k} \rightarrow ①$$

Given that

\vec{F} is an irrotational

$$\Rightarrow \text{curl } \vec{F} = \vec{0}$$

$$\Rightarrow \nabla \times \vec{F} = \vec{0}$$

$$\Rightarrow \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ (x+2y+az) & (bx-3y-Iz) & (4x+cy+2z) \end{vmatrix} = \vec{0} + \vec{0} + \vec{0}$$

$$\Rightarrow (c+1)\vec{i} - (4-a)\vec{j} + (b+2)\vec{k} = \vec{0} + \vec{0} + \vec{0}$$

$$c+1=0 \quad 4-a=0 \quad b+2=0$$

$$c=-1 \quad a=4 \quad b=-2$$

\Rightarrow ① Simplify as

$$\vec{F} = (x+2y+4z)\vec{i} + (-2x-3y-Iz)\vec{j} + (4x-y+2z)\vec{k}$$

Given that $\vec{F} = \nabla\phi$

$$\Rightarrow (x+2y+4z)\vec{i} + (-2x-3y-Iz)\vec{j} + (4x-y+2z)\vec{k} = \frac{\partial\phi}{\partial x}\vec{i} + \frac{\partial\phi}{\partial y}\vec{j} + \frac{\partial\phi}{\partial z}\vec{k}$$

$$\frac{\partial \phi}{\partial x} = x - 2y + 4z$$

$$\Rightarrow \rho \circ \phi = \int (x - 2y + 4z) dz$$

$$\phi = \frac{x^2}{2} - 2xy + 4xz + f_1(y, z) \rightarrow ②$$

$$\frac{\partial \phi}{\partial y} = -2x - 3y - z$$

$$\Rightarrow \int \rho \circ \phi = \int (-2x - 3y - z) dy$$

$$\phi = -2yx - \frac{3y^2}{2} - zy + f_2(x, z) \rightarrow ③$$

$$\frac{\partial \phi}{\partial z} = 4x - y + 2z$$

$$\Rightarrow \int \rho \circ \phi = \int (4x - y + 2z) dz$$

$$\phi = 4xz - yz + \frac{z^2}{2} + f_3(x, y) \rightarrow ④$$

from ② ③ and ④

$$\boxed{\vec{F} = \frac{x^2}{2} - \frac{3y^2}{2} + z^2 - 2xy + 4xz - 2y + C}$$

- 14] Show that $\vec{F} = (2xy + z^3)\vec{i} + x^2\vec{j} + (3xz^2)\vec{k}$ is irrotational
 (or) force field (or) conservative force (or) potential field
 find the scalar potential ϕ such that $\vec{F} = \nabla \phi$

$$\Rightarrow \text{curl } \vec{F} = \nabla \times \vec{F}$$

$$\text{curl } \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2xy + z^3 & x^2 & 3xz^2 \end{vmatrix}$$

$$\begin{aligned}
 &= (0-0)\vec{i} - (3x^2 - 3z^2)\vec{j} + (2z - 2x)\vec{k} \\
 &= 0\vec{i} + 0\vec{j} + 0\vec{k} \\
 &= \vec{0} \text{ a } \text{Irrotational}
 \end{aligned}$$

$$\vec{F} = \nabla \phi$$

$$\Rightarrow (2xy + z^3)\vec{i} + x^2\vec{j} + (3xz^2)\vec{k} = \frac{\partial \phi}{\partial x}\vec{i} + \frac{\partial \phi}{\partial y}\vec{j} + \frac{\partial \phi}{\partial z}\vec{k}$$

$$\frac{\partial \phi}{\partial x} = 2xy + z^3$$

$$\Rightarrow \int \partial \phi = \int 2xy + z^3$$

$$\phi = x^2y + xz^3 + f_1(y, z) \rightarrow ①$$

$$\frac{\partial \phi}{\partial y} = x^2$$

$$\Rightarrow \int \partial \phi = \int x^2 dy$$

$$\phi = x^2y + f_2(x, z) \rightarrow ②$$

$$\frac{\partial \phi}{\partial z} = 3xz^2$$

$$\Rightarrow \int \partial \phi = \int 3xz^2 dz$$

$$\phi = xz^3 + f_3(x, y) \rightarrow ③$$

from ① ② and ③

$$\boxed{\phi = x^2y + xz^3 + c}$$

15) Find the constants a & b such that $\vec{F} = (axy + z^3)\vec{i} + (3x^2 - z)\vec{j} + (bxz^2 - y)\vec{k}$ is an irrotational and also find the scalar potential for which

$$\vec{F} = \nabla\phi$$

$$\Rightarrow \vec{F} = (axy + z^3)\vec{i} + (3x^2 - z)\vec{j} + (bxz^2 - y)\vec{k}$$

from given

$$\Rightarrow \text{curl } \vec{F} = \vec{0}$$

$$\Rightarrow \nabla \times \vec{F} = \vec{0}$$

$$\Rightarrow \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ (axy + z^3) & (3x^2 - z) & (bxz^2 - y) \end{vmatrix} = \vec{0}\vec{i} + \vec{0}\vec{j} + \vec{0}\vec{k}$$

$$\Rightarrow (-1+1)\vec{i} - (bz^2 - 3z^2)\vec{j} + (6z - ax)\vec{k} = \vec{0}\vec{i} + \vec{0}\vec{j} + \vec{0}\vec{k}$$

$$\Rightarrow \vec{0}\vec{i} + (3-b)z^2\vec{j} + (6-a)x\vec{k} = \vec{0}\vec{i} + \vec{0}\vec{j} + \vec{0}\vec{k}$$

$$(3-b)z^2 = 0 \quad (6-a)x = 0$$

$$3-b=0 \quad 6-a=0$$

$$3=b \quad a=6$$

Original equation ①

$$\vec{F} = (6xy + z^3)\vec{i} + (3x^2 - z)\vec{j} + (3xy^2 - y)\vec{k}$$

$$\vec{F} = \nabla\phi$$

$$(6xy + z^3)\vec{i} + (3x^2 - z)\vec{j} + (3xy^2 - y)\vec{k} = \frac{\partial\phi}{\partial x}\vec{i} + \frac{\partial\phi}{\partial y}\vec{j} + \frac{\partial\phi}{\partial z}\vec{k}$$

$$\frac{\partial\phi}{\partial x} = 6xy + z^3$$

$$\Rightarrow \frac{\partial\phi}{\partial x} = 6xy + z^3 \quad \text{eq 2}$$

$$\phi = 3x^2y + xz^3 + f_1(y, z) \rightarrow ②$$

$$\frac{\partial\phi}{\partial y} = 3x^2 - z$$

$$\Rightarrow \frac{\partial\phi}{\partial y} = 3x^2 - z \quad \text{eq 3}$$

$$\phi = 3x^2y - yz + f_2(z) \rightarrow ③$$

$$\frac{\partial \phi}{\partial z} = 3xz^2 - y$$

$$\Rightarrow \rho \circ \phi = \rho(3xz^2 - y) \circ z \\ \phi = xz^3 - yz + f(x, y) \rightarrow ④$$

from ② ③ and ④

$$\boxed{\phi = 3x^2y - yz + xz^3 + c}$$

Directional derivatives

16 Find the directional derivatives of $\phi = 4xz^3 - 3x^2y^2z$ and $(2, -1, 2)$ along $\vec{d} = 2\vec{i} - 3\vec{j} + 6\vec{k}$

$$\Rightarrow \phi = 4xz^3 - 3x^2y^2z$$

$$\nabla \phi = \frac{\partial \phi}{\partial x} \vec{i} + \frac{\partial \phi}{\partial y} \vec{j} + \frac{\partial \phi}{\partial z} \vec{k}$$

$$\nabla \phi = (4z^3 - 6xy^2z) \vec{i} + (-6x^2yz) \vec{j} + (12xz^2 - 3x^2y^2) \vec{k}$$

$$\nabla \phi_{\rho(2, -1, 2)} = (32 - 24) \vec{i} + 48 \vec{j} + (96 - 12) \vec{k}$$

$$\nabla \phi = 8\vec{i} + 48\vec{j} + 84\vec{k}$$

and given that $\vec{d} = 2\vec{i} - 3\vec{j} + 6\vec{k}$

$$\hat{n} = \frac{\vec{d}}{|\vec{d}|}$$

$$\hat{n} = \frac{2\vec{i} - 3\vec{j} + 6\vec{k}}{\sqrt{4+9+36}}$$

$$\hat{n} = \frac{2\vec{i} - 3\vec{j} + 6\vec{k}}{\sqrt{4+9+36}}$$

$$DD = \nabla \phi \cdot \hat{n}$$

$$DD = (8\vec{i} + 48\vec{j} + 84\vec{k}) \cdot \left(\frac{2\vec{i} - 3\vec{j} + 6\vec{k}}{\sqrt{4+9+36}} \right)$$

$$DD = \frac{1}{7} [16 - 144 + 504]$$

$$DD = \frac{376}{7}$$

$$\boxed{DD = \frac{376}{7}}$$

17 Find the directional derivative of $\phi = x^2 + y^2 + 3z^2$ at $(1, 2, 3)$ along the direction of line $\overrightarrow{PQ} = 4\vec{i} - 2\vec{j} + \vec{k}$



$$\nabla \phi = \frac{\partial \phi}{\partial x} \vec{i} + \frac{\partial \phi}{\partial y} \vec{j} + \frac{\partial \phi}{\partial z} \vec{k}$$

$$\nabla \phi = 2x\vec{i} + 2y\vec{j} + 6z\vec{k}$$

$$\nabla \phi |_{(1, 2, 3)} = 2\vec{i} + 4\vec{j} + 12\vec{k}$$

and given that $\vec{d} = 4\vec{i} - 2\vec{j} + \vec{k}$

$$\hat{n} = \frac{\vec{d}}{|\vec{d}|}$$

$$\hat{n} = \frac{4\vec{i} - 2\vec{j} + \vec{k}}{\sqrt{(4)^2 + (-2)^2 + (1)^2}}$$

$$\hat{n} = \frac{4\vec{i} - 2\vec{j} + \vec{k}}{\sqrt{16 + 4 + 1}}$$

$$\hat{n} = \frac{4\vec{i} - 2\vec{j} + \vec{k}}{\sqrt{12}}$$

$$D \cdot D = \nabla \phi \cdot \hat{n}$$

$$= (2\vec{i} + 4\vec{j} + 12\vec{k}) \cdot \left(\frac{4\vec{i} - 2\vec{j} + \vec{k}}{\sqrt{12}} \right)$$

$$= \frac{1}{\sqrt{12}} (8 - 8 + 12)$$

$$\boxed{DD = \frac{12}{\sqrt{12}}}$$

18] Find the DD of $xy^3 + yz^3$ at $(2, -1, 1)$ in the direction of the vector $\vec{i} + 2\vec{j} + 2\vec{k}$

$$\Rightarrow f = xy^3 + yz^3$$

$$\nabla f = \frac{\partial f}{\partial x} \vec{i} + \frac{\partial f}{\partial y} \vec{j} + \frac{\partial f}{\partial z} \vec{k}$$

$$\nabla f = (y^3)\vec{i} + (3xy^2 + z^3)\vec{j} + (3yz^2)\vec{k}$$

$$\nabla f|_P = (-1)^3\vec{i} + (6+1)\vec{j} + (-3)\vec{k}$$

$$\nabla f|_P = \vec{i} + 7\vec{j} - 3\vec{k}$$

$$\vec{d} = \vec{i} + 2\vec{j} + 2\vec{k}$$

$$\hat{u} = \frac{\vec{d}}{|\vec{d}|}$$

$$\hat{u} = \frac{\vec{i} + 2\vec{j} + 2\vec{k}}{\sqrt{(1)^2 + (2)^2 + (2)^2}}$$

$$\hat{u} = \frac{\vec{i} + 2\vec{j} + 2\vec{k}}{3}$$

$$DD = \nabla f \cdot \hat{u}$$

$$= (-\vec{i} + 7\vec{j} - 3\vec{k}) \cdot \left(\frac{\vec{i} + 2\vec{j} + 2\vec{k}}{3} \right)$$

$$= \frac{1}{3} (-1 + 14 - 6)$$

$DD = \frac{14}{3}$

Q] If $\vec{F} = (x+y+1)\vec{i} + \vec{j} - (x+y)\vec{k}$ show that $\vec{F} \cdot \text{curl } \vec{F} = 0$

$$\Rightarrow \vec{F} = (x+y+1)\vec{i} + \vec{j} - (x+y)\vec{k}$$

$$\text{curl } \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x+y+1 & 1 & -x-y \end{vmatrix}$$

$$\text{curl } \vec{F} = (-1-0)\vec{i} - (-1-0)\vec{j} + (0-1)\vec{k}$$

$$\text{curl } \vec{F} = -\vec{i} - \vec{j} - \vec{k}$$

$$\therefore \vec{F} \cdot \text{curl } \vec{F} = [(x+y+1)\vec{i} + \vec{j} - (x+y)\vec{k}] \cdot (-\vec{i} - \vec{j} - \vec{k}) \\ = (-x+y+1) + (1+x+y)$$

$$\boxed{\vec{F} \cdot \text{curl } \vec{F} = 0}$$

Vector Integration

Line Integral :- Any integral which is to be evaluated along the curve C is called a Line Integral. Suppose

$\vec{F} = f_1\vec{i} + f_2\vec{j} + f_3\vec{k}$ be a Vector point function in (x, y, z) and $\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$ a position Vector, then the Line Integral on C can defined as

$$\int_C \vec{F} \cdot d\vec{r} = \int_C [f_1 dx + f_2 dy + f_3 dz] \cdot [dx\vec{i} + dy\vec{j} + dz\vec{k}]$$

$$\boxed{\int_C \vec{F} \cdot d\vec{r} = \int_C (f_1 dx + f_2 dy + f_3 dz)}$$

Q) If $\vec{F} = 3xy\vec{i} - y^2\vec{j}$ evaluate $\int_C \vec{F} \cdot d\vec{r}$ where c is the curve
 $y = 2x^2$ from point (0,0) to (1,2)

$$\Rightarrow \vec{F} = 3xy\vec{i} - y^2\vec{j} \rightarrow ①$$

$$\text{Let } \vec{r} = \vec{x}\vec{i} + \vec{y}\vec{j} + \vec{z}\vec{k}$$

$$\Rightarrow d\vec{r} = dx\vec{i} + dy\vec{j} + dz\vec{k}$$

$$\therefore \vec{F} \cdot d\vec{r} = (3xy\vec{i} - y^2\vec{j}) \cdot (dx\vec{i} + dy\vec{j} + dz\vec{k})$$

$$\Rightarrow \vec{F} \cdot d\vec{r} = 3xy \, dx - y^2 \, dy \rightarrow ②$$

$$\text{and given } y = 2x^2$$

diff w.r.t 'x'

$$\Rightarrow \frac{dy}{dx} = 4x$$

$$\Rightarrow dy = 4x \, dx$$

$$② \Rightarrow \vec{F} \cdot d\vec{r} = 3x(2x^2) \, dx - (2x^2)^2 4x \, dx \\ = 6x^3 \, dx - 16x^5 \, dx$$

$$\Rightarrow \vec{F} \cdot d\vec{r} = (6x^3 - 16x^5) \, dx$$

$$\therefore \int_C \vec{F} \cdot d\vec{r} = \int_C (6x^3 - 16x^5) \, dx$$

$$= \int_{x=0}^1 (6x^3 - 16x^5) \, dx$$

$$= \left[\frac{6x^4}{4} - 16 \frac{x^6}{6} \right]_0^1$$

$$= \frac{6}{4} - \frac{16}{6}$$

$$= \frac{3}{2} - \frac{8}{3}$$

$$= \frac{9 - 16}{6}$$

$$\boxed{\int_C \vec{F} \cdot d\vec{r} = -\frac{7}{6}}$$

$\int_C \vec{F} = x^2\vec{i} + xy\vec{j}$ Evaluate $\int \vec{F} \cdot d\vec{s}$ when from (0,0) to (1,1)
along with the curves i) $y=x$
ii) $y=\sqrt{x}$

$$\Rightarrow \vec{F} = x^2\vec{i} + xy\vec{j} \rightarrow ①$$

$$\text{Let } \vec{r} = \vec{x}\vec{i} + \vec{y}\vec{j} + \vec{z}\vec{k}$$

$$\Rightarrow d\vec{r} = dx\vec{i} + dy\vec{j} + dz\vec{k}$$

$$\therefore \vec{F} \cdot d\vec{r} = (x^2\vec{i} + xy\vec{j}) \cdot (dx\vec{i} + dy\vec{j} + dz\vec{k})$$

$$\Rightarrow \vec{F} \cdot d\vec{r} = x^2 dx + xy dy \rightarrow ②$$

and given i) Towards the curve $C, y=x$

$$\frac{dy}{dx} = 1 \\ dy = dx$$

$$② \Rightarrow \vec{F} \cdot d\vec{r} = x^2 dx + x \cdot x dx$$

$$\vec{F} \cdot d\vec{r} = 2x^2 dx$$

$$\int_C \vec{F} \cdot d\vec{r} = \int_0^1 (2x^2) dx$$

$$= \int_0^1 (2x^2) dx$$

$$= \left. \frac{2x^3}{3} \right|_0^1 dx$$

$$\int_C \vec{F} \cdot d\vec{r} = \frac{2}{3}$$

ii) Towards the curve $y=\sqrt{x}$

$$\Rightarrow y^2 = x$$

$$2y = \frac{dx}{dy}$$

$$\Rightarrow dx = 2y dy$$

$$\frac{x}{2} = \frac{y}{1} = \frac{z}{3} = t$$

$$x = 2t, y = t, z = 3t$$

$$dx = 2dt, dy = dt, dz = 3dt$$

$$\begin{aligned}\textcircled{1} \Rightarrow \vec{F} \cdot d\vec{s} &= 3(2t)^2 dt + 2(3t)(2t) - t dt + 3t \rightarrow 3dt \\ &= 24t^2 dt + 12t^2 dt - t dt + 3t dt \\ &= (36t^2 + 8t) dt\end{aligned}$$

$$\begin{aligned}\therefore \int_C \vec{F} \cdot d\vec{s} &= \int_0^1 (36t^2 + 8t) dt \\ &= \left[\frac{36t^3}{3} + \frac{8t^2}{2} \right]_0^1 \\ &= \frac{36}{6} + \frac{8}{2} \\ &= 12 + 4\end{aligned}$$

$$\int_C \vec{F} \cdot d\vec{s} = 16$$

$$\textcircled{ii}) \quad \vec{F} \cdot d\vec{s} = 3x^2 dx + (2xz - y) dy + z dz \rightarrow \textcircled{1}$$

$$x^2 - 4y \rightarrow \textcircled{2}$$

$$3x^2 - 8z \rightarrow \textcircled{3}$$

$$x = 0, z = 2$$

$$\text{Let } x = t$$

$$dx = dt$$

$$\textcircled{2} \Rightarrow y = \frac{x^2}{4}$$

$$y = \frac{t^2}{4}$$

$$dy = \frac{2t}{4} dt$$

$$dy = \frac{1}{2} t dt$$

$$\textcircled{2} \Rightarrow \vec{F} \cdot d\vec{s} = (y^2) 2y dy + y(y^2) dy \\ = 2y^5 dy + y^3 dy \\ \vec{F} \cdot d\vec{s} = dy(2y^5 + y^3)$$

$$\int_C \vec{F} \cdot d\vec{s} = \int_C (2y^5 + y^3) dy \\ = \int_0^1 (2y^5 + y^3) dy \\ = \left[\frac{2y^6}{6} + \frac{y^4}{4} \right]_0^1 \\ = \frac{2}{6} + \frac{1}{4} \\ = \frac{1}{3} + \frac{1}{4}$$

$$\boxed{\int_C \vec{F} \cdot d\vec{s} = \frac{7}{12}}$$

Q1 Find the work done is moving a particle the force
filled $\vec{F} = 3x^2 \vec{i} + (2xz - y) \vec{j} + z \vec{k}$ along

- i) A straight line from $(0,0,0)$ to $(2,1,3)$
- ii) The curve defined by $x^2 = 4y$ & $3x^2 = 8z$
from $x=0$ to $x=2$

$$\Rightarrow \vec{F} = 3x^2 \vec{i} + (2xz - y) \vec{j} + z \vec{k}$$

$$\text{let } \vec{ds} = \vec{xi} + \vec{yj} + \vec{zk}$$

$$\Rightarrow d\vec{s} = dx\vec{i} + dy\vec{j} + dz\vec{k}$$

$$\vec{F} \cdot d\vec{s} = [3x^2 \vec{i} + (2xz - y) \vec{j} + z \vec{k}] \cdot [dx\vec{i} + dy\vec{j} + dz\vec{k}]$$

$$\vec{F} \cdot d\vec{s} = 3x^2 dx + (2xz - y) dy + z dz \longrightarrow \textcircled{1}$$

i] Given The curve along the straight line passing through $(0,0,0)$ to $(1,2,3)$

$$\frac{x-0}{1-0} = \frac{y-0}{1-0} = \frac{z-0}{3-0}$$

$$\frac{x}{2} = \frac{y}{1} = \frac{z}{3} = t$$

$$x = 2t, y = t, z = 3t$$

$$dx = 2dt, dy = dt, dz = 3dt$$

$$\begin{aligned} \textcircled{1} \Rightarrow \vec{F} \cdot d\vec{s} &= 3(2t)^2 dt + 2(3t)(2t) - t dt + 3t dt \rightarrow 3dt \\ &= 24t^2 dt + 12t^2 dt - t dt + 3t dt \\ &= (36t^2 + 8t) dt \end{aligned}$$

$$\begin{aligned} \therefore \int_C \vec{F} \cdot d\vec{s} &= \int_0^1 (36t^2 + 8t) dt \\ &= \left[\frac{36t^3}{3} + \frac{8t^2}{2} \right]_0^1 \\ &= \frac{36}{6} + \frac{8}{2} \\ &= 12 + 4 \end{aligned}$$

$$\int_C \vec{F} \cdot d\vec{s} = 16$$

$$\textcircled{ii}) \quad \vec{F} \cdot d\vec{s} = 3x^2 dx + (2xz - y) dy + z dz \rightarrow \textcircled{1}$$

$$x^2 - 4y \rightarrow \textcircled{2}$$

$$2x^2 - 8z \rightarrow \textcircled{3}$$

$$x=0, z=2$$

$$\text{Let } x=t$$

$$dx = dt$$

$$\textcircled{2} \Rightarrow y = \frac{x^2}{4}$$

$$y = \frac{t^2}{4}$$

$$dy = \frac{2t}{4} dt$$

$$dy = \frac{1}{2}t dt$$

$$\textcircled{3} \Rightarrow z_1 = \frac{3x^2}{8}$$

$$z_1 = \frac{3t^2}{8}$$

$$dz_1 = \frac{9t^2}{8} dt$$

$$\begin{aligned}\textcircled{1} \Rightarrow \vec{F} \cdot d\vec{r} &= (3t^2)(dt) - \left[2 \times \frac{3t^2}{8} \times t - \frac{1}{4} \right] \left[\frac{t^2}{2} dt \right] + \frac{3t^3}{8} \times \frac{9t^2}{8} dt \\ &= \left[3t^2 + \left(\frac{3t^4}{4} - \frac{t^2}{4} \right) \frac{t}{2} + \frac{27t^5}{64} \right] dt \\ &= \left[3t^2 + \frac{3t^5}{8} - \frac{t^3}{8} + \frac{27t^5}{64} \right] dt \\ &= \left[3t^2 - \frac{t^3}{8} + \frac{51t^5}{64} \right] dt\end{aligned}$$

$$\int_C \vec{F} \cdot d\vec{r} = \int_0^2 \left[3t^2 - \frac{t^3}{8} + \frac{51t^5}{64} \right] dt$$

$$= \left[t^3 - \frac{t^4}{32} + \frac{51}{64} \frac{t^6}{6} \right]_0^2$$

$$= 8 - \frac{16}{32} + \frac{51}{384} (64)$$

$$= 8 - \frac{1}{2} + \frac{51}{6}$$

$$= \frac{48 - 3 + 51}{6}$$

$$= \frac{96}{6}$$

$$\boxed{\int_C \vec{F} \cdot d\vec{r} = 16}$$

22) If $\vec{F} = (3x^2 + 6y)\vec{i} - 14yz\vec{j} + 20xz^2\vec{k}$ Evaluate $\int_C \vec{F} \cdot d\vec{s}$ from (0,0,0)
to (1,1,1) along the curve given by $x=t$, $y=t^2$, $z=t^3$

$$\Rightarrow \vec{F} = (3x^2 + 6y)\vec{i} - 14yz\vec{j} + 20xz^2\vec{k}$$

$$d\vec{s} = \vec{x}\vec{i} + \vec{y}\vec{j} + \vec{z}\vec{k}$$

$$d\vec{s} = dx\vec{i} + dy\vec{j} + dz\vec{k}$$

$$\vec{F} \cdot d\vec{s} = [(3x^2 + 6y)\vec{i} - 14yz\vec{j} + 20xz^2\vec{k}] \cdot [dx\vec{i} + dy\vec{j} + dz\vec{k}]$$

$$\vec{F} \cdot d\vec{s} = (3x^2 + 6y)dx - 14yzdy + 20xz^2dz$$

$$\vec{F} \cdot d\vec{s} = 3x^2y + 6ydx - 14yzdy + 20xz^2dz \rightarrow (1)$$

$$\text{Also } x=t \rightarrow (1) \quad z=t^3$$

$$(2) \Rightarrow dx=dt$$

$$(4) \Rightarrow dz=3t^2dt$$

$$y=t^2 \rightarrow (2)$$

$$(3) \Rightarrow dy=2t dt$$

$$(1) \Rightarrow \vec{F} \cdot d\vec{s} = (3t^2 + 6t^2)dt - 14t^2 \cdot t^3 \cdot 2t dt + 20t \cdot t^6 \cdot 3t^2 dt$$

$$\int_C \vec{F} \cdot d\vec{s} = \int_C (9t^2 - 28t^6 + 60t^9) dt$$

$$= \int_0^1 (9t^2 - 28t^6 + 60t^9) dt$$

$$= \left[3t^3 - 28 \frac{t^7}{7} + 60 \frac{t^{10}}{10} \right]_0^1$$

$$= 3 - \frac{28}{7} + 60$$

$$= \frac{21 - 28 + 420}{7}$$

$$= \frac{355}{7}$$

$$\int_C \vec{F} \cdot d\vec{s} = 50$$

Gjreen's theorem :-

Statement :- If $M(x,y)$ & $N(x,y)$ be the two continuous functions of x & y having continuous partial Integratives $\frac{\partial M}{\partial y}$ & $\frac{\partial N}{\partial x}$ in the region R of the xy -plane bounded by a closed curve $\int_R M(x,y) dx + N(x,y) dy = \iint_R \left(\frac{\partial M}{\partial x} - \frac{\partial N}{\partial y} \right) dx dy$

Q3 Evaluate $\oint_C (3x^2 - 8y^2) dx + (4y - 6xy) dy$ where C is the boundary of the region Enclosed by $y = \sqrt{x}$ & $y = x^2$

$$\Rightarrow \oint_C M dx + N dy = \oint_C (3x^2 - 8y^2) dx + (4y - 6xy) dy$$

$$M = 3x^2 - 8y^2 \quad N = 4y - 6xy$$

$$\frac{\partial M}{\partial y} = -16y \quad \frac{\partial N}{\partial x} = -6y$$

$$\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} = -6y + 16y = 10y$$

The curve bounded by $y = \sqrt{x}$ and $y = x^2$

∴ By the Gjreen's theorem we get

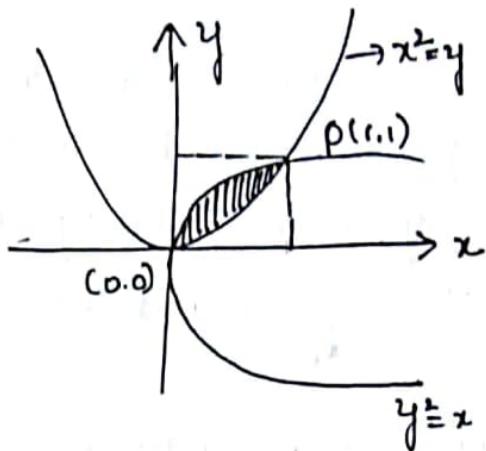
$$\oint_C M dx + N dy = \iint_R \left(\frac{\partial M}{\partial x} - \frac{\partial N}{\partial y} \right) dx dy$$

$$= \iint_R (10y) dx dy$$

$$= 10 \int_{x=0}^1 \int_{y=\sqrt{x}}^{x^2} y dx dy$$

$$= 10 \int_0^1 \left[\frac{y^2}{2} \right]_{\sqrt{x}}^{x^2} dx$$

$$\begin{aligned}
 &= 5 \int_0^1 (x^4 - x) dx \\
 &= 5 \left[\frac{x^5}{5} - \frac{x^2}{2} \right]_0^1 \\
 &= 5 \left[\frac{-3}{10} \right] \\
 &= \left| \frac{-3}{5} \right| \\
 &= \frac{3}{5}
 \end{aligned}$$



Q4 Evaluate $\oint_C (xy + y^2) dx + x^2 dy$, where C is the closed curve in the region bounded by $y=x$ and $y=x^2$ using Green's theorem

$$\Rightarrow \oint_C M dx + N dy = \oint_C (xy + y^2) dx + x^2 dy$$

$$M = xy + y^2 \quad N = x^2$$

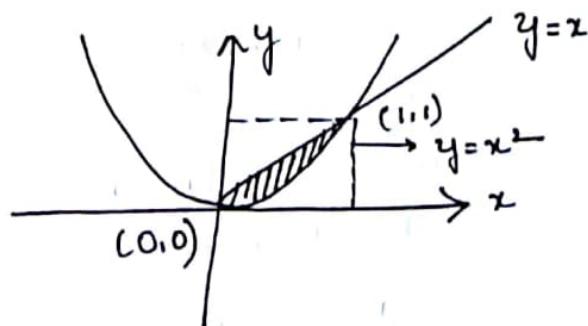
$$\frac{\partial M}{\partial y} = x + 2y \quad \frac{\partial N}{\partial x} = 2x$$

$$\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} = 2x - x - 2y$$

$$\frac{\partial M}{\partial x} - \frac{\partial N}{\partial y} = x - 2y$$

The curve C is bounded by $y=x$ and $y=x^2$

\therefore By the Green's theorem. L.H.T

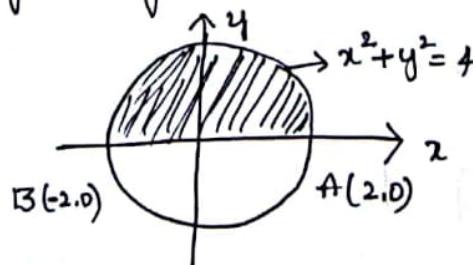


$$\begin{aligned}
 \oint_C M dx + N dy &= \iint_R \left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x} \right) dx dy \\
 \Rightarrow I &= \int_{x=0}^1 \int_{y=x^2}^x (x - xy) dy dx \\
 &= \int_{x=0}^1 [xy - \frac{y^2}{2}]_{x^2}^x dx \\
 &= \int_{x=0}^1 [(x^2 - x^2) - (x^3 - x^4)] dx \\
 &= \int_0^1 (x^4 - x^3) dx \\
 &= \left[\frac{x^5}{5} - \frac{x^4}{4} \right]_0^1 \\
 &= \frac{1}{5} - \frac{1}{4} \\
 I &= \frac{-1}{20}
 \end{aligned}$$

Q5 Use Green's theorem to Evaluate $\oint_C (x^2+y^2) dx + 3x^2y dy$
 where C is the circle $x^2+y^2=4$ traced in the positive
 Signs of the circle

$$\begin{aligned}
 \Rightarrow \oint_C M dx + N dy &= \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dx dy \\
 M &= x^2+y^2 \quad N = 3x^2y \\
 \frac{\partial M}{\partial y} &= 2y \quad \frac{\partial N}{\partial x} = 6xy \\
 \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} &= 6xy - 2y \\
 \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} &= 2y(3x-1)
 \end{aligned}$$

∴ Try the Green's theorem



$$\oint_C M dx + N dy = \iint_{R_1} \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dx dy$$

$$I = \int_{x=-2}^2 \int_{y=0}^{\sqrt{4-x^2}} R_1 (3x-1) dy dx$$

$$= \int_{x=-2}^2 (3x-1)(4-x^2) dx$$

$$= \int_{x=-2}^2 (12x - 3x^2 - 4 + x^3) dx$$

$$= \int_{-2}^2 (-3x^3 + x^2 + 12x - 4) dx$$

$$= \left[-\frac{3}{4}x^4 + \frac{x^3}{3} + 6x^2 - 4x \right]_{-2}^2$$

$$= \left[-16 + \frac{8}{3} + 24 - 8 \right] - \left[-16 - \frac{8}{3} + 24 + 8 \right]$$

$$= -16 + \frac{8}{3} + 24 - 8 + 16 + \frac{8}{3} - 24 - 8$$

$$I = -\frac{32}{3}$$

26 Find the area between the parabolas $y^2 = 4x$ & $x^2 = 4y$
by using Green's theorem

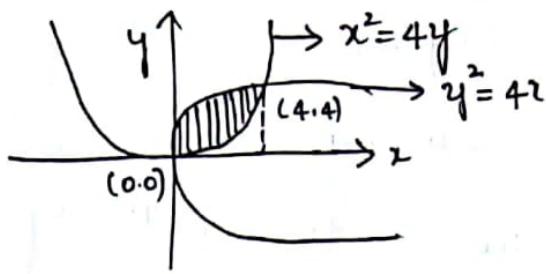
$$\Rightarrow \oint_C M dx + N dy = \iint_{R_1} \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dx dy \rightarrow ①$$

$$\oint_C M dx + N dy = \frac{1}{2} \int (-y dx + x dy)$$

$$M = \frac{-y}{2}, \quad N = \frac{x}{2}$$

$$\frac{\partial M}{\partial y} = -\frac{1}{2} \quad \frac{\partial N}{\partial x} = \frac{1}{2}$$

$$\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x} = \frac{1}{2} - \left(-\frac{1}{2}\right) = 1$$



$$\textcircled{1} \Rightarrow \int_M dx + N dy = \iint_{R_1} 1 \, dx \, dy$$

$$= \int_{x=0}^4 \int_{y=x^2/4}^{2\sqrt{x}} 1 \, dy \, dx$$

$$= \int_{x=0}^4 [y]_{x^2/4}^{2\sqrt{x}} \, dx$$

$$= \int_{x=0}^4 [2\sqrt{x} - \frac{x^2}{4}] \, dx$$

$$= 2 \int_0^4 x^{1/2} - \frac{1}{4} \int_0^4 x^2 \, dx$$

$$= \left[\frac{2x^{3/2}}{\frac{3}{2}} \right]_0^4 - \frac{1}{4} \left[\left[\frac{x^3}{3} \right] \right]_0^4$$

$$= \frac{4}{3}(4^{3/2}) - \frac{1}{12}(4^3)$$

$$= \frac{4 \times 8}{3} - \frac{64}{12}$$

$$= \frac{32}{3} - \frac{16}{3}$$

$$= \underline{\underline{\frac{16}{3} \text{ sq. units}}}$$

Ex Evaluate $\int_C (x^2 + y^2) dx + 3xy dy$, where C is the circle $x^2 + y^2 = 4$ traced in the positive signs

$$\Rightarrow \int_M dx + N dy = \int_C (x^2 + y^2) dx + 3xy dy$$

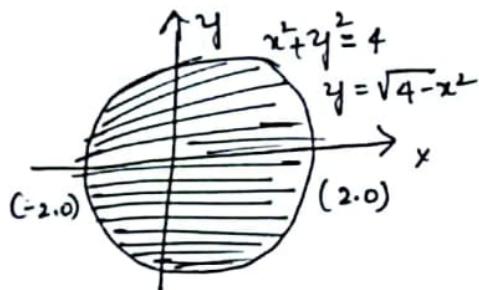
$$M = x^2 + y^2 \quad N = 3xy$$

$$\frac{\partial M}{\partial y} = 2y \quad \frac{\partial N}{\partial x} = 6xy$$

$$\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) = 6xy - 2y$$

$$\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} = 2y(3x-1)$$

And given that the circle having the radius 2 and
 $x^2 + y^2 = 4$ in the +ve sense
 \therefore By Green's theorem



$$\begin{aligned}
 \int_C M dx + N dy &= \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dx dy \\
 \Rightarrow \int_C (x^2 + y^2) dx + 3x^2 y dy &= \int_{x=-2}^2 \int_{y=\sqrt{4-x^2}}^{\sqrt{4-x^2}} 2y(3x-1) dy dx \\
 &= \int_{-2}^2 2(3x-1) \int_{y=\sqrt{4-x^2}}^{\sqrt{4-x^2}} y dy dx \\
 &= \int_{-2}^2 2(3x-1) \left[\frac{y^2}{2} \right]_{\sqrt{4-x^2}}^{\sqrt{4-x^2}} dx \\
 &= \int_{-2}^2 (3x-1) [(4-x^2) - (4-x^2)] dx \\
 &= \underline{\underline{0}}
 \end{aligned}$$

Stoke's theorem

Suppose S be an open surface bounded by a closed curve C .
 & if $\vec{F} = F_1\vec{i} + F_2\vec{j} + F_3\vec{k}$ be any vector point function having
 the continuous fun then $\oint_C \vec{F} \cdot d\vec{s} = \iint_S \text{curl } \vec{F} \cdot \hat{n} ds$, where \hat{n} is
 the draw Unit Normal Vector to the Surface S (or)

$$\hat{n} ds = dy dz \vec{i} + dz dx \vec{i} + dx dy \vec{k}$$

28] Using Stoke's theorem for $\vec{F} = y\vec{i} + z\vec{j} + x\vec{k}$. where S is a
 upper half of the sphere $x^2 + y^2 + z^2 = 1$ and C is its boundary

$$\Rightarrow \vec{F} = y\vec{i} + z\vec{j} + x\vec{k}$$

$$\text{curl } \vec{F} = \nabla \times \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y & z & x \end{vmatrix}$$

$$= (0-1)\vec{i} - (1-0)\vec{j} + (0-1)\vec{k}$$

$$\text{curl } \vec{F} = -\vec{i} - \vec{j} - \vec{k}$$

Given that Surface ' S ' is upper half of the sphere

$x^2 + y^2 + z^2 = 1$ its Normal $\hat{n} = \vec{k}$

$$\therefore \text{curl } \vec{F} \cdot \hat{n} = (-\vec{i} - \vec{j} - \vec{k}) \cdot \vec{k} = -1$$

By the Stoke's Theorem L.H.T

$$\oint_C \vec{F} \cdot d\vec{s} = \iint_S \text{curl } \vec{F} \cdot \hat{n} ds$$

$$= \iint_S -1 ds$$

$$= - \iint_S 1 ds$$

$$= -A$$

where A is the Area of the Circle $x^2 + y^2 = 1$ for
 which $z=0$

$$A = \pi r^2$$

$$= \pi x l$$

$$A = \pi l$$

$$\therefore \int_C \vec{F} \cdot d\vec{s} = -\pi l$$

Gauss divergence theorem :-

If $\vec{F} = F_1 \vec{i} + F_2 \vec{j} + F_3 \vec{k}$ is a vector point function having continuous function in the region & bounded by a closed surface S, then $\iiint_V \nabla \cdot \vec{F} dV = \iint_S \vec{F} \cdot \hat{n} dS$ where \hat{n} is the unit normal vector to the surface S.

Q9 Evaluate $\vec{F} = (x^2 - yz) \vec{i} + (y^2 - zx) \vec{j} + (z^2 - xy) \vec{k}$ taken over the rectangular parallelepiped $0 \leq x \leq a, 0 \leq y \leq b, 0 \leq z \leq c$
find $\iiint_V \nabla \cdot \vec{F} dV$

$$\Rightarrow \vec{F} = (x^2 - yz) \vec{i} + (y^2 - zx) \vec{j} + (z^2 - xy) \vec{k}$$

$$\therefore \operatorname{div} \vec{F} = \nabla \cdot \vec{F}$$

$$= \left[\frac{\partial}{\partial x} \vec{i} + \frac{\partial}{\partial y} \vec{j} + \frac{\partial}{\partial z} \vec{k} \right] \cdot [(x^2 - yz) \vec{i} + (y^2 - zx) \vec{j} + (z^2 - xy) \vec{k}]$$

$$= 2x + 2y + 2z$$

$$\operatorname{div} \vec{F} = 2(x+y+z)$$

also given that given surface is parallel to the axes bounded by $0 \leq x \leq a, 0 \leq y \leq b, 0 \leq z \leq c$

$$\iiint_V \operatorname{div} \vec{F} dV = \iiint_V 2(x+y+z) dx dy dz$$

$$= \int_{x=0}^a \int_{y=0}^b \int_{z=0}^c (x+y+z) dz dy dx$$

$$\begin{aligned}
 &= \int_0^a \int_{y=0}^b \left(cx + cy + \frac{c^2}{2} z^2 \right) dy dz \\
 &= \int_0^a \int_{y=0}^b \left[cy + cy^2 + \frac{c^2}{2} z^2 \right]_0^b dy dz \\
 &= \int_0^a \left[bc \frac{z^2}{2} + \frac{bc^2}{2} z + \frac{bc^2}{2} z^3 \right]_0^a dz \\
 &= \int_0^a \left[\frac{a^2 bc}{2} + \frac{abc^2}{2} + \frac{abc^2}{2} a^2 \right] dz \\
 &= abc [a+b+c]
 \end{aligned}$$

30 By using divergence theorem evaluate $\iint_S \vec{F} \cdot \hat{n} ds$ where $\vec{F} = 4\vec{x} - 2y\vec{j} + \vec{z}^2\vec{k}$ if S is the surface enclosing the region for which $x^2 + y^2 \leq 4$, $0 \leq z \leq 3$,

$$\Rightarrow \vec{F} = 4\vec{x} - 2y\vec{j} + \vec{z}^2\vec{k}$$

$$\text{div } \vec{F} = \nabla \cdot \vec{F}$$

$$\text{div } \vec{F} = \left(\frac{\partial}{\partial x} \vec{i} + \frac{\partial}{\partial y} \vec{j} + \frac{\partial}{\partial z} \vec{k} \right) \cdot (4\vec{x} - 2y\vec{j} + \vec{z}^2\vec{k})$$

$$\Rightarrow \text{div } \vec{F} = 4 - 4y + 2z$$

z varies from 0 to 3

$$\text{by using } x^2 + y^2 = 4$$

$$\Rightarrow y^2 = 4 - z^2$$

$$y = \pm \sqrt{4 - z^2}$$

and x varies from -2 to 2

$$\iint_S \vec{F} \cdot \hat{n} ds = \iiint_V \text{div } \vec{F} \cdot dV$$

$$\Rightarrow I = \iiint_V (4 - 4y + 2z) dV$$

$$\begin{aligned}
 I &= 2 \int_{x=-2}^2 \int_{y=\sqrt{4-x^2}}^{\sqrt{4-x^2}} \int_{z=0}^3 (2 - 2y + z) dz dy dx \\
 &= 2 \int_{x=-2}^2 \int_{y=-\sqrt{4-x^2}}^{\sqrt{4-x^2}} \left[2z - 2yz + \frac{z^2}{2} \right]_0^3 dy dx \\
 &= 2 \int_{x=-2}^2 \int_{y=-\sqrt{4-x^2}}^{\sqrt{4-x^2}} (6 - 6y + \frac{9}{2}) dy dx \\
 &= 2 \int_{x=-2}^2 \left(6y - 3y^2 + \frac{9}{2}y \right) \Big|_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} dx \\
 &= \int_{x=-2}^2 \left[21y - 6y^2 \right]_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} dy \\
 &= \int_{x=-2}^2 \left[21\sqrt{4-x^2} - 6(4-x^2) \right] - \left[-21\sqrt{4-x^2} - 6(4-x^2) \right] dx \\
 &= 42 \int_{x=-2}^2 \sqrt{4-x^2} dx \\
 &= 42 * 2 \int_0^2 \sqrt{4-x^2} dx \\
 &= 84 \int_0^2 \sqrt{4-x^2} dx \\
 &= 84 \cdot 2 \cdot \frac{\pi}{4} \\
 &= 84\pi
 \end{aligned}$$

3) $\vec{F} = (2x^2 - 3z)\vec{i} - 2xy\vec{j} - 4x\vec{k}$. Evaluate $\iiint_V \nabla \cdot \vec{F} dV$ where V is the region bounded by the planes $x=0, y=0, z=0$
 $2x+2y+z=4$

$$\Rightarrow \vec{F} = (2x^2 - 3z)\vec{i} - 2xy\vec{j} - 4x\vec{k}$$

$$\begin{aligned} \text{div } \vec{F} &= \left[\frac{\partial}{\partial x} \vec{i} + \frac{\partial}{\partial y} \vec{j} + \frac{\partial}{\partial z} \vec{k} \right] \cdot [(2x^2 - 3z)\vec{i} - 2xy\vec{j} - 4x\vec{k}] \\ &= 4x - 2x + 0 \end{aligned}$$

$$\text{div } \vec{F} = 2x$$

The planes $x=0, y=0, z=0$ and $2x+2y+z=4$

\therefore By the Gauss theorem we have

$$\iint_S \vec{F} \cdot \hat{n} dS = \iiint_V \text{div } \vec{F} dV$$

$$\begin{aligned} \Rightarrow 1 &= \iiint_V 2x dV \\ &= \int_{x=0}^2 \int_{y=0}^{2-x} \int_{z=0}^{4-2x-2y} x dz dy dx \\ &= \int_{x=0}^2 \int_{y=0}^{2-x} x [z]_0^{4-2x-2y} dy dx \\ &= \int_{x=0}^2 \int_{y=0}^{2-x} (4x - 2x^2 - 2xy) dy dx \\ &= \int_{x=0}^2 (4xy - 2x^2y - xy^2) \Big|_0^{2-x} dx \\ &= \int_{x=0}^2 4x(2-x) - 2x^2(2-x) - x^2(2-x)^2 dx \\ &= \int_{x=0}^2 (8x - 4x^2 - 4x^2 + 2x^3 - 4x + x^2 + 4x^2) dx \end{aligned}$$

$$\begin{aligned}
 &= \int_0^2 (x^3 - 4x^2 + 4x) dx \\
 &= \left[\frac{x^4}{4} - 4 \frac{x^3}{3} + 4x^2 \right]_0^2 \\
 &= \left[4 - \frac{32}{3} + 8 \right] \\
 &= \left[12 - \frac{32}{3} \right] \\
 &= \left[\frac{4}{3} \right]
 \end{aligned}$$

$$I = \frac{8}{3}$$

32 Use divergence theorem to evaluate $\iint_S \vec{F} \cdot \hat{n} ds$ over the entire surface of the region above $xy bounded by the cone $z^2 = x^2 + y^2$ and the plane $z=4$,$

$$\vec{F} = 4xz\vec{i} + xy\vec{z}\vec{j} + 3z\vec{k}$$

$$\Rightarrow \vec{F} = 4xz\vec{i} + xy\vec{z}\vec{j} + 3z\vec{k}$$

$$\therefore \operatorname{div} \vec{F} = \nabla \cdot \vec{F}$$

$$= \left(\frac{\partial}{\partial x} \vec{i} + \frac{\partial}{\partial y} \vec{j} + \frac{\partial}{\partial z} \vec{k} \right) \cdot (4xz\vec{i} + xy\vec{z}\vec{j} + 3z\vec{k})$$

$$\operatorname{div} \vec{F} = 4z + xz^2 + 3$$

$$x^2 + y^2 = z^2 \rightarrow \textcircled{1} \text{ the plane } z = 4$$

$\therefore z$ varies from 0 to 4

$$\textcircled{1} \Rightarrow x^2 + y^2 = 4^2$$

$$x^2 + y^2 = 16 \rightarrow \textcircled{2}$$

When $y=0$

$$\textcircled{2} \Rightarrow x^2 = 16$$

$$\Rightarrow x = \pm 4$$

$$x = -4, x = 4$$

$$\text{and } y^2 = 16 - x^2$$

$$y = \pm \sqrt{16 - x^2}$$

$$y = +\sqrt{16 - x^2} \quad y = -\sqrt{16 - x^2}$$

$$h, K, I = \iint_S \vec{F} \cdot \hat{n} \, dS = \iiint_V \operatorname{div} \vec{F} \, dV$$

$$\implies \iint_V \operatorname{div} \vec{F} \, dV = \iiint_V (4z + xz^3 + 3) \, dV$$

$$= \int_{x=-4}^4 \int_{y=-\sqrt{16-x^2}}^{\sqrt{16-x^2}} \int_{z=0}^4 (4z + xz^2 + 3) \, dz \, dy \, dx$$

$$= \int_{x=-4}^4 \int_{y=-\sqrt{16-x^2}}^{\sqrt{16-x^2}} \left[2z^2 + 3\frac{z^3}{3} + 3z \right]_0^4 \, dy \, dx$$

$$= \int_{x=-4}^4 \int_{y=-\sqrt{16-x^2}}^{\sqrt{16-x^2}} \left(32 + \frac{64}{3}x + 12 \right) \, dy \, dx$$

$$= \int_{x=-4}^4 \int_{y=-\sqrt{16-x^2}}^{\sqrt{16-x^2}} \left(48 + \frac{64}{3}x \right) \, dy \, dx$$

$$= \frac{1}{3} \int_{x=-4}^4 \int_{y=-\sqrt{16-x^2}}^{\sqrt{16-x^2}} (132 + 64x) \, dy \, dx$$

$$= \frac{4}{3} \int_{x=-4}^4 \int_{y=-\sqrt{16-x^2}}^{\sqrt{16-x^2}} (33 + 16x) \, dy \, dx$$

$$= \frac{4}{3} \int_{x=-4}^4 \int_{y=-\sqrt{16-x^2}}^{\sqrt{16-x^2}} (33 + 16x) \, dy \, dx$$

$$\begin{aligned}
 &= \frac{4}{3} \int_{x=-4}^4 (33+16x)[y] \sqrt{16-x^2} dx \\
 &= \frac{4}{3} \int_{x=-4}^4 (33+16x)(\sqrt{16-x^2} + \sqrt{16-x^2}) dx \\
 &= \frac{8}{3} \int_{x=-4}^4 (16x+33)\sqrt{16-x^2} dx \\
 &= \frac{8}{3} * 16 \int_{x=-4}^4 x\sqrt{16-x^2} dx + \frac{88}{3} * 33 \int_{x=-4}^4 \sqrt{16-x^2} dx \\
 &= \frac{128}{3}(0) + 88 \int_{x=-4}^4 \sqrt{16-x^2} dx \\
 &= 176 \int_0^4 \sqrt{16-x^2} dx
 \end{aligned}$$

put $x = 4 \sin \theta$

$$dx = 4 \cos \theta d\theta$$

$$\Rightarrow \sin \theta = \frac{x}{4} \Rightarrow \theta = \sin^{-1}(x/4)$$

$$x=4 \Rightarrow \theta=\pi/2$$

$$x=0 \Rightarrow \theta=0$$

$$I = 176 \int_0^{\pi/2} \sqrt{16 - 16 \sin^2 \theta} , (4 \cos \theta) d\theta$$

$$= 176 \int_0^{\pi/2} 4^2 \cos^2 \theta d\theta$$

$$= 2816 \int_0^{\pi/2} \cos^2 \theta d\theta$$

$$= 2816 * \frac{r_0 - 1}{2} * \pi/2$$

$I = 704\pi$

33 If $\vec{F} = (2x-y)\vec{i} - yz^2\vec{j} - yz\vec{k}$, where S is the upper half surface of the sphere $x^2 + y^2 + z^2 = 1$ is its boundary. Use Stokes theorem to find $\int_C \vec{F} \cdot d\vec{s}$

$$\Rightarrow \vec{F} = (2x-y)\vec{i} - yz^2\vec{j} - yz\vec{k}$$

$$x^2 + y^2 + z^2 = 1$$

$$\therefore \text{curl } \vec{F} = \nabla \times \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ (2x-y) & (-yz^2) & -yz \end{vmatrix}$$

$$= \begin{matrix} (-2yz + 2yz)\vec{i} - (0-0)\vec{j} + (0+1)\vec{k} \\ \vec{k} \end{matrix}$$

i.e. \vec{n}

$$\hat{n} ds = dy dz \vec{i} + dz dz \vec{j} + dz dy \vec{k}$$

$$\therefore \text{curl } \vec{F} \cdot \hat{n} ds = [\vec{k}] \cdot [dy dz \vec{i} + dz dz \vec{j} + dz dy \vec{k}]$$

$$= dz dy$$

Area of a circle $x^2 + y^2 = 1$

$$= \pi r^2$$

$$= \pi(1)$$

$$= \pi$$

34 A vector field is given by $\vec{F} = 8\sin y \vec{i} + x(1+\cos y) \vec{j}$
 Evaluate the line integral over a circular path given
 by $x^2 + y^2 = a^2$, $z=0$

$$\Rightarrow \vec{F} = 8\sin y \vec{i} + x(1+\cos y) \vec{j}$$

$$\vec{r} = xi + yj + zk$$

$$d\vec{r} = dx \vec{i} + dy \vec{j} + dz \vec{k}$$

$$\begin{aligned}\vec{F} \cdot d\vec{s} &= [\sin y \vec{i} + x(1+\cos y) \vec{j} + 0 \vec{k}] \cdot [dx \vec{i} + dy \vec{j} + dz \vec{k}] \\ &= \sin y dx + x(1+\cos y) dy \\ &= \sin y dx + x dy + x \cos y dy \\ &= [\sin y dx + x \cos y dy] + x dy\end{aligned}$$

$$\Rightarrow \vec{F} \cdot d\vec{s} = d[x \sin y] + x dy \quad \text{--- (1)}$$

and given the curve $x^2 + y^2 = a^2 \rightarrow (2)$

$$\begin{aligned}\text{Let } x &= a \cos \theta & y &= a \sin \theta \\ \Rightarrow dx &= -a \sin \theta & dy &= a \cos \theta\end{aligned}$$

θ varies from 0 to 2π

$$\begin{aligned}\therefore \int_C \vec{F} \cdot d\vec{s} &= \int_C [x \sin y] + x dy \\ &= \int_0^{2\pi} a [\cos \theta \cdot \sin(\sin \theta)] + \int_0^{2\pi} a \cos \theta \cdot a \cos \theta d\theta \\ &= [a \cos \theta \sin(\sin \theta)]_0^{2\pi} + a^2 \int_0^{2\pi} \cos^2 \theta d\theta \\ &= (0 - 0) + a^2 \int_0^{2\pi} \cos^2 \theta d\theta \\ &= a^2 \int_0^{2\pi} \left[\frac{1 + \cos 2\theta}{2} \right] d\theta \\ &= \frac{a^2}{2} \int_0^{2\pi} (1 + \cos 2\theta) d\theta \\ &= \frac{a^2}{2} \left[0 + \frac{\sin 2\theta}{2} \right]_0^{2\pi} \\ &= \frac{a^2}{2} \times 2\pi \\ \boxed{\int_C \vec{F} \cdot d\vec{s}} &= \pi a^2\end{aligned}$$