Module-3

partial differential Equation

Defination :- If an Equation involves one dependent Variable and its derivative with respect 2 (or) more independent Variable is called partial differential Eqn

Notation :- Suppose $z = f(x,y)$ be a fun the Variables are x and y Then first and Second order partial derivaties can be notated by ~~following~~ Symbols

$$p = \frac{\partial z}{\partial x}, \quad q = \frac{\partial z}{\partial y}, \quad r = \frac{\partial^2 z}{\partial^2 x}$$

$$s = \frac{\partial^2 z}{\partial x \partial y}, \quad t = \frac{\partial z}{\partial y}$$

1 from the partial differential Eqn by Eliminating the orbitary Constants a and b from $z = (x-a)^2 + (y-b)^2$

$\Rightarrow$ Given :- $z = (x-a)^2 + (y-b)^2 \longrightarrow ①$

diff ① w.r.t 'x' partially

$① \Rightarrow \frac{\partial z}{\partial x} = 2(x-a)$

$\Rightarrow p = 2(x-a)$

$\Rightarrow (x-a) = \frac{p}{2} \longrightarrow ②$

diff ① w.r.t 'y' partially

$① \Rightarrow \frac{\partial z}{\partial y} = 2(y-b)$

$\Rightarrow q = 2(y-b)$

$\Rightarrow (y-b) = \frac{q}{2} \longrightarrow ③$

from ② ① and ③

$① \Rightarrow z = \left(\frac{p}{2}\right)^2 + \left(\frac{q}{2}\right)^2$

$$\boxed{4z = p^2 + q^2}$$

2. From the PDE by Elemenating the parameters a and b from the Sphere

$\Rightarrow$ Given :- $(x-a)^2+(y-b)^2+z^2=r^2 \longrightarrow (1)$

Diff w.r.t 'x' partially

$(1) \Rightarrow 2(x-a)+0+2z\frac{\partial z}{\partial x}=0$ $\quad\quad \therefore p=\frac{\partial z}{\partial x}$

$\Rightarrow (x-a)+pz=0$

$\Rightarrow (x-a)=-pz \longrightarrow (2)$

diff (1) w.r.t 'y' partially

$(1) \Rightarrow 0+2(y-b)+2z\,\frac{\partial z}{\partial y}=0$

$\Rightarrow (y-b)+qz=0$ $\quad\quad \therefore q=\frac{\partial z}{\partial y}$

$\Rightarrow y-b=-qz \longrightarrow (3)$

from (1) (2) and (3)

$(1) \Rightarrow (-zp)^2+(-qz)^2+z^2=r^2$

$\Rightarrow p^2z^2+q^2z^2+z^2=r^2$

$$\boxed{(1+p^2+q^2)z^2=r^2}$$

3. from the PDE by Elemenating the arbitary Constants a and b from $(x-a)^2+(y-b)^2=z^2\cot^2\alpha$

$\Longrightarrow$ Given :- $(x-a)^2+(y-b)^2=z^2\cot^2\alpha$

diff (1) w.r.t 'x' partially

$\Rightarrow 2(x-a)+0=2z\frac{\partial z}{\partial x}\cot^2\alpha$

$\Rightarrow (x-a)=zp\cot^2\alpha \longrightarrow (1)$ $\quad\quad \therefore \frac{\partial z}{\partial x}=p$

diff (1) w.r.t 'y' partially

$\Rightarrow 0+2(y-b)=2z\frac{\partial z}{\partial y}\cot^2\alpha$

$\Rightarrow (y-b)=zq\cot^2\alpha \longrightarrow (2)$ $\quad\quad \therefore \frac{\partial z}{\partial y}=q$

from ① ② and ③

$$① \Rightarrow (\cancel{z}p\cot^2\alpha)^2 + (\cancel{z}q\cot^2\alpha)^2 = z^2\cot^2\alpha$$

$$\cancel{z^2}p^2\cot^4\alpha + \cancel{z^2}q^2\cot^4\alpha = \cancel{z^2}\cot^2\alpha$$

$$\cancel{z^2}\cot^4\alpha\,(p^2+q^2) = \cancel{z^2}\cot^2\alpha$$

$$p^2+q^2 = \frac{\cancel{z^2}\cot^2\alpha}{\cancel{z^2}\cot^4\alpha}$$

$$p^2+q^2 = \frac{1}{\cot^2\alpha}$$

$$\boxed{p^2+q^2 = \operatorname{Tan}^2\alpha}$$

4] from the PDE by Eliminating the orbitary Constant a and b from $2z = \frac{x^2}{a^2} + \frac{y^2}{b^2}$

$\Rightarrow$ Given :- $2z = \frac{x^2}{a^2} + \frac{y^2}{b^2} \longrightarrow ①$

diff ① w.r.t to 'x' partially

$$\Rightarrow 2\,\frac{\partial z}{\partial x} = \frac{2x}{a^2}$$

$$\Longrightarrow p = \frac{x}{a^2}$$

$$\Longrightarrow a^2 = \frac{x}{p} \longrightarrow ②$$

diff ① w.r.t to 'y' partially

$$\Rightarrow 2\,\frac{\partial z}{\partial y} = \frac{2y}{b^2}$$

$$\Rightarrow b^2 = \frac{y}{q} \longrightarrow ③$$

from ② and ③

$$① \Rightarrow 2z = \frac{x^2}{x/p} + \frac{y^2}{y/q}$$

$2z =$ $\boxed{2z = px+qy}$

5] from the PDE by Elemenating orbitary constants a,b,c from $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$

<u>Given</u> :- $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1 \longrightarrow (1)$

Diff (1) w.r.t 'x' partially

$$(1) \Rightarrow \frac{2x}{a^2}+0+2z\frac{\frac{\partial z}{\partial x}}{c^2}=0$$

$$\Rightarrow \frac{x}{a^2}+\frac{pz}{c^2}=0 \rightarrow (2)$$

$$\Rightarrow \frac{x}{a^2}=-\frac{pz}{c^2}$$

$$\frac{1}{a^2}=-\frac{pz}{c^2x} \longrightarrow (3)$$

Diff w.r.t 'x' partially

$$(2) \Rightarrow \frac{1}{a^2}+\frac{1}{c^2}\left[p\frac{\partial z}{\partial x}+z\frac{\partial p}{\partial x}\right]=0$$

$$\Rightarrow \frac{1}{a^2}+\frac{1}{c^2}\left[p^2+z\frac{\partial^2 z}{\partial x^2}\right]=0$$

$$\Rightarrow \frac{1}{a^2}+\frac{1}{c^2}\left[p^2+rz\right]=0$$

$$\Rightarrow \frac{-pz}{c^2x}+\frac{1}{c^2}\left[p^2+rz\right]=0$$

$$\Rightarrow \frac{-pz}{x}+\left[p^2+rz\right]=0$$

$$\Rightarrow -pz+x(p^2+rz)=0$$

$$\Rightarrow xp^2+\underline{zxr}=pz$$

$$\Rightarrow x\left(\frac{\partial z}{\partial x}\right)^2+zx\left(\frac{\partial^2 z}{\partial x^2}\right)=z\frac{\partial z}{\partial x}$$

6] From the pde by eliminating the orbitary fu^n $z = f(x^2+y^2)$

$\Rightarrow$ Given :- $z = f(x^2+y^2) \rightarrow ①$

Diff w.r.t 'x' partially

$$① \Rightarrow \frac{\partial z}{\partial x} = f'(x^2+y^2)(2x)$$

$$p = f'(x^2+y^2)2x \longrightarrow ②$$

diff ① w.r.t 'x' partially

$$① \Rightarrow \frac{\partial z}{\partial y} = f'(x^2+y^2)(2y)$$

$$q = 2yf'(x^2+y^2) \longrightarrow ③$$

$$② \div ③ \Rightarrow \frac{p}{q} = \frac{2xf'(x^2+y^2)}{2yf'(x^2+y^2)}$$

$$\frac{p}{q} = \frac{x}{y}$$

$$py = qx$$

$$\boxed{py - qx = 0}$$

7] from the PDE by eliminating the orbitary function $z = y^2 + 2f\left(\frac{1}{x} + \log y\right)$

$$\Rightarrow \quad z = y^2 + 2f\left(\frac{1}{x} + \log y\right) \longrightarrow ①$$

Diff w.r.t 'x' partially

$$① \Rightarrow \frac{\partial z}{\partial x} = 0 + 2f'\left(1/x + \log y\right)\left(-1/x^2\right)$$

$$p = \frac{-2f'}{x^2}\left(\frac{1}{x} + \log y\right) \longrightarrow ②$$

Diff w.r.t 'y' partially

$$\frac{\partial z}{\partial y} = 2y + 2f'\left(\frac{1}{x} + \log y\right)(1/y)$$

$$q-2y = \frac{2f'}{y}\left(\frac{1}{x}+\log y\right)$$

② ÷ ③

$$\frac{p}{q-2y} = \frac{-\frac{2f'}{x^2}\left(\frac{1}{x}+\log y\right)}{\frac{2}{y}f'\left(\frac{1}{x}+\log y\right)}$$

$$\frac{p}{q-2y} = \frac{-y}{x^2}$$

$$\boxed{px^2+(q-2y)y=0}$$

8] $z = f(x+ay)+g(x-ay)$

$\Rightarrow$ Given :- $z = f(x+ay)+g(x-ay) \longrightarrow ①$

$$\frac{\partial z}{\partial x} = f'(x+ay)(1)+g'(x-ay)(1)$$

$$\frac{\partial^2 z}{\partial x^2} = f''(x+ay)+g''(x-ay)$$

diff w.r.t 'y' partialy

$$\frac{\partial z}{\partial y} = f'(x+ay)(a)+g'(x-ay)(-a)$$

$$\frac{\partial z}{\partial y} = af'(x+ay)-ag'(x-ay)$$

$$\frac{\partial^2 z}{\partial y^2} = a^2f''(x+ay)+a^2g''(x-ay)$$

$$\frac{\partial^2 z}{\partial y^2} = a^2[f''(x+ay)+g''(x-ay)]$$

$$\boxed{\frac{\partial^2 z}{\partial y^2} = a^2\frac{\partial z^2}{\partial x^2}}$$

9] from pde by Elemanating $z = yf(x)+x\phi(y)$

$\Rightarrow$ given :- $z = yf(x)+x\phi(y) \longrightarrow ①$

diff w.r.t 'x'

$$\frac{\partial z}{\partial x} = yf'(x)+\phi(y) \longrightarrow ②$$

$$\frac{\partial z}{\partial y} = f(x) + x\phi'(y) \longrightarrow ③$$

$$\frac{\partial^2 z}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right) = yf''(x)$$

$$\Rightarrow \quad r = yf''(x) \rightarrow ④$$

$$\frac{\partial^2 z}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right) = x\phi''(y)$$

$$\Rightarrow \quad t = x\phi''(y) \longrightarrow ⑤$$

$$\frac{\partial^2 z}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right) = f'(x) + \phi'(y)$$

$$\Rightarrow \quad s = f'(x) + \phi'(y)$$

$$s = \left[\frac{p - \phi(y)}{y}\right] + \left[\frac{q - f(x)}{x}\right]$$

$$s = \frac{x[p - \phi(y)] + y[q - f(x)]}{xy}$$

$$xys = [px - x\phi y] + qy - yf(x)$$

$$xys = px + qy - z$$

$$\underline{\underline{px - xys + qy - z = 0}}$$

10 from PDE $z = xf(x+t) + \phi(x+t)$.

$z = xf(x+t) + \phi(x+t) \longrightarrow ①$

$\Rightarrow$ Given :- $z = xf(x+t) + \phi(x+t)$

$$\frac{\partial z}{\partial x} = xf'(x+t) + f(x+t) + \phi_1(x+t) \rightarrow ②$$

$$\frac{\partial z}{\partial t} = xf'(x+t) + \phi'(x+t) \longrightarrow ③$$

$$\frac{\partial^2 z}{\partial x^2} = xf''(x+t) + 2f'(x+t) + \phi''(x+t) \longrightarrow ④$$

$$\frac{\partial^2 z}{\partial t^2} = xf''(x+t) + \phi''(x+t) \rightarrow ⑤$$

$$\frac{\partial^2 z}{\partial x \partial t} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial t}\right)$$

$$\frac{\partial^2 z}{\partial x \partial t} = xf''(x+t) + f'(x+t) + \phi''(x+t) \longrightarrow (6)$$

$$(4)-(6) \Rightarrow \frac{\partial^2 z}{\partial x^2} - \frac{\partial^2 z}{\partial x \partial t} = f'(x+t) \longrightarrow (7)$$

$$(6)-(5) \Rightarrow \frac{\partial^2 z}{\partial x \partial t} - \frac{\partial^2 z}{\partial t^2} = f'(x+t) \longrightarrow (8)$$

from (7) and (8)

$$\frac{\partial^2 z}{\partial x^2} - \frac{\partial^2 z}{\partial x \partial t} = \frac{\partial^2 z}{\partial x \partial t} - \frac{\partial^2 z}{\partial t^2}$$

$$\frac{\partial^2 z}{\partial x^2} - 2\frac{\partial^2 z}{\partial x \partial t} + \frac{\partial^2 z}{\partial t^2} = 0$$

11) From the PDE by elemenating the orbitary function from $lx+my+nz = \phi(x^2+y^2+z^2) \rightarrow$

$$\Rightarrow \quad lx+my+nz = \phi(x^2+y^2+z^2) \rightarrow (1)$$

Diff w.r.t 'x'

$$(1) \Rightarrow l+0+n\frac{\partial z}{\partial x} = \phi'(x^2+y^2+z^2)\left[2x+2z\frac{\partial z}{\partial x}\right]$$

$$\Rightarrow l+np = 2(x+pz)\phi'(x^2+y^2+z^2) \rightarrow (2)$$

Diff w.r.t 'y'

$$(1) \Rightarrow 0+m+n\frac{\partial z}{\partial y} = \phi'(x^2+y^2+z^2)\left[2y+2z\frac{\partial z}{\partial y}\right]$$

$$m+nq = \phi'(x^2+y^2+z^2)\,2[y+qz] \rightarrow (3)$$

$$(2)\div(3) \Rightarrow \frac{l+np}{m+nq} = \frac{x+pz}{y+qz}$$

$$(l+np)(y+qz) - (m+nq)(x+pz) = 0$$

formation of PDE for the type $\phi(u,v)=0$

Step 1 :- Suppose u,v is zero be an fun in the variables $u\,v$ and u,v are the functions of x,y,z and z is a function of x and y

2) Differentiate Equation ① partially w.r.t 'x' $\frac{\partial\phi}{\partial x}\cdot\frac{\partial u}{\partial x}+\frac{\partial\phi}{\partial v}\cdot\frac{\partial v}{\partial x}=0$

$$\Rightarrow \frac{\partial\phi}{\partial u}\cdot\frac{\partial u}{\partial x}=-\frac{\partial\phi}{\partial v}\cdot\frac{\partial v}{\partial x} \longrightarrow ②$$

$$\frac{\partial\phi}{\partial u}\cdot\frac{\partial u}{\partial y}+\frac{\partial\phi}{\partial v}\cdot\frac{\partial v}{\partial y}=0$$

$$\Rightarrow \frac{\partial\phi}{\partial u}\cdot\frac{\partial u}{\partial y}=-\frac{\partial\phi}{\partial v}\cdot\frac{\partial v}{\partial \phi} \longrightarrow ③$$

$2 \div 3$

$$\frac{\frac{\partial\phi}{\partial u}\cdot\frac{\partial u}{\partial x}}{\frac{\partial\phi}{\partial u}\cdot\frac{\partial u}{\partial y}}=\frac{-\frac{\partial\phi}{\partial v}\cdot\frac{\partial v}{\partial x}}{-\frac{\partial\phi}{\partial v}\cdot\frac{\partial v}{\partial y}}$$

$$\frac{\partial u}{\partial x}\cdot\frac{\partial v}{\partial y}-\frac{\partial v}{\partial x}\cdot\frac{\partial u}{\partial y}=0$$

12) from the pde for $f(x+y+z,\ x^2+y^2+z^2)=0$

$\Rightarrow$ Given:- $f(x+y+z,\ x^2+y^2+z^2)=0$

$f(u,v)=0$

$u=x+y+z$

$\frac{\partial u}{\partial x}=1+0+\frac{\partial z}{\partial x}$

$\frac{\partial u}{\partial x}=1+p$

$\frac{\partial u}{\partial y}=0+1+\frac{\partial z}{\partial y}$

$\frac{\partial u}{\partial y}=1+q$

$v=x^2+y^2+z^2$

$\frac{\partial v}{\partial x}=2x+0+2z\,\frac{\partial z}{\partial x}$

$\frac{\partial v}{\partial x}=2(x+zp)$

$\frac{\partial v}{\partial y}=0+2y+2z\,\frac{\partial z}{\partial y}$

$\frac{\partial v}{\partial y}=2(y+zq)$

$\therefore$ the pde is $\frac{\partial u}{\partial x}\frac{\partial v}{\partial y}-\frac{\partial v}{\partial x}\frac{\partial u}{\partial y}=0$

$$\Rightarrow (1+p)\,2(y+pz)-2(x+pz)(1+q)=0$$

$$\Rightarrow (1+p)(y+qz)-(x+pz)(1+q)=0$$

13\] from the pde by elemenating orbitary function $f(x^2+y^2, z-xy)=0$

$\Rightarrow$ Given:- $f(x^2+y^2, z-xy)=0$

$f(u,v)=0$ $\qquad$ $v = z-xy$

$u = x^2+y^2$ $\qquad$ $\frac{\partial v}{\partial x} = \frac{\partial z}{\partial x} - y$

$\frac{\partial u}{\partial x} = 2x$ $\qquad$ $\frac{\partial v}{\partial x} = p-y$

$\frac{\partial u}{\partial y} = 2y$ $\qquad$ $\frac{\partial v}{\partial y} = \frac{\partial z}{\partial y} - x$

$\frac{\partial v}{\partial y} = q-x$

$\therefore$ The pde is $\frac{\partial u}{\partial x}\frac{\partial v}{\partial y} - \frac{\partial v}{\partial x}\frac{\partial u}{\partial y}$

$\Rightarrow 2x(q-x) - (p-y)2y$

$\Rightarrow qx - x^2 - py + y^2 = 0$

$\Rightarrow qx - py - (x^2 - y^2) = 0$

14\] from the pde by elemenating the function from $f\left(\frac{xy}{z}, z\right)=0$

$\Rightarrow$ Given :- $f\left(\frac{xy}{z}, z\right)=0$ $\qquad$ $v = z$

$\Rightarrow f(u,v)=0$ $\qquad$ $\frac{\partial v}{\partial x} = \frac{\partial z}{\partial x} = p$

$u = \frac{xy}{z}$ $\qquad$ $\frac{\partial v}{\partial y} = \frac{\partial z}{\partial y} = q$

$\frac{\partial u}{\partial x} = y\frac{\partial}{\partial x}\left(\frac{x}{z}\right)$ $\qquad$ $\frac{\partial u}{\partial y} = x\cdot\frac{\partial}{\partial y}\left(\frac{y}{z}\right)$

$= y\left[\frac{z(1) - x\frac{\partial z}{\partial x}}{z^2}\right]$ $\qquad$ $= x\left[\frac{z - y\frac{\partial z}{\partial y}}{z^2}\right]$

$\frac{\partial u}{\partial x} = \frac{y}{z^2}(z-px)$ $\qquad$ $\frac{\partial u}{\partial y} = \frac{x}{z^2}(z-qy)$

The pde is $u_x v_y - v_x u_y = 0$

$\Rightarrow q\left[\frac{y}{z^2}(z-px)\right] - p\left[\frac{x}{z^2}(z-py)\right] = 0$

$\Rightarrow py(z-px)-px(z-qy)=0$

Soln of PDE by direct Integration

15. Solve PDE by direct integration method $\frac{\partial^2 z}{\partial x\, \partial t} = e^{-t}\cdot \cos x$ given $z=0$ when $t=0$ & $\frac{\partial z}{\partial t}=0$ when $x=0$

Given :-

$$\frac{\partial^2 z}{\partial x\, \partial t} = e^{-t}\cos x \longrightarrow (1)$$

$$\Rightarrow \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial t}\right) = e^{-t}\cos x$$

$$\Rightarrow \int \partial\left(\frac{\partial z}{\partial t}\right) = \int e^{-t}\cos x\, dx$$

$$\Rightarrow \frac{\partial z}{\partial t} = e^{-t}\int \cos x\, dx$$

$\partial x \mid \partial t$

$$\Rightarrow \frac{\partial z}{\partial t} = e^{-t}\sin x + f(t) \rightarrow (2)$$

$$\Rightarrow \partial z = [e^{-t}\sin x + f(t)]\, dt$$

$$\Rightarrow \int \partial z = \sin x \int e^{-t}\, dt + \int f(t)\, dt$$

$$\Rightarrow z = -e^{-t}\sin x + F(t) + g(x) \longrightarrow (3)$$

where $F(t) = \int f(t)\, dt$

given that when $x=0 \Rightarrow \frac{\partial z}{\partial t}=0$

$(2) \Rightarrow 0 = 0 + f(t)$

$f(t) = 0$

$\therefore \quad F(t) = \int f(t)\, dt = \int 0.\, dt = 0$

$$z = -e^{-t}\sin x + g(x) \longrightarrow (4)$$

where $z=0$

$(4) \Rightarrow \quad 0 = -e^{0}\sin x + g(x)$

$\Rightarrow \quad 0 = -\sin x + g(x)$

$g(x) = \sin x$

$z = -e^{-t}\sin x + \sin x$

$z = (-1-e^{-t})\sin x$

16 Solve $\frac{\partial^3 z}{\partial x^2 \partial y} = \cos(2x+3y)$

$\Rightarrow$ Given :- $\frac{\partial^3 z}{\partial x}\left[\frac{\partial^2 z}{\partial x \partial y}\right] = \cos(2x+3y)$

$$\Rightarrow \frac{\partial}{\partial x}\left[\frac{\partial^2 z}{\partial x \partial y}\right] = \cos(2x+3y)$$

$$\Rightarrow \int \partial\left[\frac{\partial^2 z}{\partial x \partial y}\right] = \int \cos(2x+3y)\,dx$$

$$\Rightarrow \frac{\partial^2 z}{\partial x \partial y} = \frac{\sin(2x+3y)}{2} + f(y)$$

$$\Rightarrow \frac{\partial}{\partial y}\left[\frac{\partial z}{\partial x}\right] = \frac{1}{2}\int \sin(2x+3y)\,dy + \int f(y)\,dy$$

$$\Rightarrow \frac{\partial z}{\partial x} = -\frac{\cos(2x+3y)}{6} + F(y) + g(x)$$

$$\Rightarrow \int \partial z = -\frac{1}{6}\int \cos(2x+3y)\,dx + F_y\int dx + \int g(x)\,dx$$

$$\Rightarrow z = -\frac{\sin(2x+3y)}{12} + xF(y) + G(x) + h(y)$$

where $F(y) = \int f(y)\,dy$, $G(x) = \int g(x)\,dx$

17 Solve $\frac{\partial^2 z}{\partial x^2} = xy$, Subject to the Conditions That $\frac{\partial z}{\partial x} = \log(1+y)$ when $x=1$ and $z=0$ when $x=0$

$\Rightarrow$ Given :- $\frac{\partial^2 z}{\partial x^2} = xy$

$$\Rightarrow \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right) = xy$$

$$\Rightarrow \int \partial\left(\frac{\partial z}{\partial x}\right) = \int xy\,\partial x$$

$$\Rightarrow \frac{\partial z}{\partial x} = \frac{x^2 y}{2} + f(y) \longrightarrow (1)$$

$$\int \partial z = \int \frac{x^2 y}{2}\,dx + \int f(y)\,dx$$

$$z = \frac{x^3 y}{6} + x f(y) + g(y) \longrightarrow (2)$$

$$\frac{\partial z}{\partial x} = \log(1+y) \quad \text{when } x=1$$

$$① \Rightarrow \log(1+y) = \frac{y}{2} + f(y)$$

$$\Rightarrow f(y) = \log(1+y) - \frac{y}{2}$$

$$\therefore z = \frac{x^3 y}{6} + x\log(1+y) - \frac{xy}{2} + g(y) \longrightarrow ③$$

when $x = 0$, $z = 0$

$$③ \Rightarrow g(y) = 0$$

$$z = \frac{x^3 y}{6} + x\log(1+y) - \frac{xy}{2}$$

18 $\frac{\partial^2 z}{\partial x \partial y} = \frac{x}{y}$ Subject to the condition $\frac{\partial z}{\partial x} = \log_e x$ when $y=1$ & $z=0$ when $x=1$

$\Rightarrow$ Given :- $\frac{\partial^2 z}{\partial x \partial y} = \frac{x}{y}$

$$\Rightarrow \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right) = \frac{x}{y}$$

$$\Rightarrow \int \partial\left(\frac{\partial z}{\partial x}\right) = \frac{x}{y}\,\partial y$$

$$\Rightarrow \frac{\partial z}{\partial x} = x\int \frac{1}{y}\,\partial y$$

$$\Rightarrow \frac{\partial z}{\partial x} = x\log y + f(x) \longrightarrow ①$$

$$\Rightarrow \int \partial z = \int x\log y\,\partial x + \int f(x)\,\partial x$$

$$z = \log y \int x\,dx + \int f(x)\,dx$$

$$\Rightarrow z = \frac{x^2}{2}\log y + F(x) + g(y) \rightarrow ②$$

where $F(x) = \int f(x)\,dx$

where $y=1 \Rightarrow \frac{\partial z}{\partial x} = \log_e x$

$$① \Rightarrow \log_e x = x\log 1 + f(x)$$

$$\Rightarrow \log x = 0 + f(x)$$

$\Rightarrow f(x) = \log x$

$f(x) = \int f(x)\,dx$

$= \int \log x\,dx$

$= \int 1.\log x.\,\partial x$

$= \log x.\int 1.dx - \int \frac{1}{x}\cdot\int 1\partial x\,\partial x$

$= x\log x = \left(\frac{1}{x}\cdot x\right)\partial x$

$\Rightarrow f(y) = x\log x - x$

(2) $\Rightarrow z = \frac{x^2}{2}\log y + \log x - x + g(y) \longrightarrow$ (3)

when $x=1 \Rightarrow z=0$

(3) $\Rightarrow 0 = \frac{1}{2}\log y + 0 - 1 + g(y)$

$\Rightarrow g(y) = 1 - \frac{1}{2}\log y$

$\Rightarrow g(y) = 1 - \log\sqrt{y}$

$z = \frac{x^2}{2}\log y + x\log x - x + 1 - \log\sqrt{y}$

19. Solve $\frac{\partial^2 z}{\partial x\partial y} = \sin x.\sin y$. for which $\frac{\partial z}{\partial y} = -2\sin y$ when $x=0$. and $z=0$ if y is an add multiple of $\pi/2$ when $y=(2n+1)\pi/2$ for $n=0,1,2,3$ - - - -]

$\Rightarrow$ Given :- $\frac{\partial^2 z}{\partial x\partial y} = \sin x.\sin y$

$\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right) = \sin x.\sin y$

$\Rightarrow \int \partial\left(\frac{\partial z}{\partial y}\right) = \int \sin x.\sin y\,dx$

$\frac{\partial z}{\partial y} = -\sin y.\cos x + f(y) \longrightarrow$ (1)

$\Rightarrow \int \partial z = -\cos x\int \sin y\,\partial y + \int f(y)\,\partial y$

$\Rightarrow z = \cos x\cos y + f(y) + g(x) \longrightarrow$ (2)

where $x=0 \Rightarrow \frac{\partial z}{\partial y} = -2\sin y$

$$(1) \Rightarrow -2\sin y = -\sin y + f'(y)$$

$$\Rightarrow f'(y) = -\sin y$$

$$f(y) = \int f'(y)\,dy$$

$$\Rightarrow f y = -\int \sin y\,dy = \cos y$$

$$(2) \Rightarrow z = \cos x \cos y + \cos y + g(x) \rightarrow (3)$$

$$y = (2n+1)\frac{\pi}{2} \Rightarrow z = 0$$

$$(3) \Rightarrow 0 = \cos x \cdot \cos(2n+1)\frac{\pi}{2} + \cos(2n+1)\frac{\pi}{2} + g(x)$$

$$\Rightarrow 0 = 0 + 0 + g(x)$$

$$g(x) = 0$$

$$z = \cos x \cos y + \cos y$$

$$z = \cos y(1 + \cos x)$$

Soln of Homogenous partial differential Equ

20. Solve $\frac{\partial^2 z}{\partial y^2} = z$ given that $y=0$. $e^x = z$ and $\frac{\partial z}{\partial y} = e^{-x}$

$\Rightarrow$ Given:- $\frac{\partial^2 z}{\partial y^2} = z$

$$\Rightarrow \frac{\partial^2 z}{\partial y^2} - z = 0$$

$$\Rightarrow (D^2 - 1)z = 0$$

The A.E is $(m^2 - 1) = 0$

$$m = \pm 1$$

$$\therefore z = f(x)e^{-y} + g(x)e^{y} \rightarrow (1)$$

$$\frac{\partial z}{\partial y} = -f(x)e^{-y} + g(x)e^{y} \longrightarrow (2)$$

$$y = 0 \Rightarrow z = e^x$$

$$(1) \Rightarrow e^x = f(x) + g(x) \rightarrow (3)$$

$$\& \; y = 0 \Rightarrow \frac{\partial z}{\partial y} = e^{-x}$$

$(2) \Rightarrow e^{-x} = -f(x) + g(x) \rightarrow (4)$

$(3)+(4) = 2g(x) = e^x + e^{-x}$

$g(x) = \frac{e^x + e^{-x}}{2} = \cosh x$

$(3)-(4) = 2f(x) = e^x - e^{-x}$

$f(x) = \frac{e^x - e^{-x}}{2} = \sinh x$

$z = e^{-y}\sinh x + e^y \underline{\cosh x}$

21. Solve $\frac{\partial^2 z}{\partial x^2} + z = 0$ Given that when $x=0$, $z=e^y$ and $\frac{\partial z}{\partial x} = 1$

$\Rightarrow$ Given :- $\frac{\partial^2 z}{\partial x^2} + z = 0$

$(D^2+1)z = 0$

The A.E is $f(m) = 0$

$m^2+1 = 0$

$m = 0 \pm 1i$

$\therefore z = f(y)\cos x + g(y)\sin x$

$\frac{\partial z}{\partial x} = -f(y)\sin x + g(y)\cos x$

when $x=0 \Rightarrow z = e^y$

$(2) \Rightarrow f(y) = e^y$

where $x=0 \Rightarrow \frac{\partial z}{\partial x} = 1$

$(3) \Rightarrow 1 = g(y) \Rightarrow g(y) = 1$

$\boxed{z = e^y\cos x + \sin x}$

22. Solve $\frac{\partial^2 z}{\partial x^2} = a^2 z$ Given that when $x=0$, $z=0$ and $\frac{\partial z}{\partial x} = a\sin y$

$\Rightarrow$ Given :- $\frac{\partial^2 z}{\partial x^2} = a^2 z \rightarrow (1)$

$\frac{\partial^2 z}{\partial x^2} - a^2 z = 0$

$(D^2 - a^2)z = 0$

$m = \pm a$

$z = f(y)e^{-ax} + g(y)e^{ax} \rightarrow (2)$

$\frac{\partial z}{\partial x} = a f(y) e^{-ax} + a g(y) e^{ax} \rightarrow (3)$

when $x = 0 \Rightarrow z = 0$

$(2) \Rightarrow f(y) + g(y) = 0 \rightarrow (4)$

$x = 0 \Rightarrow \frac{\partial z}{\partial x} = a \sin y$

$(3) \Rightarrow -a f(y) + a g(y) = a \sin y$

$\Rightarrow -f(y) + g(y) = \sin y \rightarrow (5)$

$(4) - (5) \Rightarrow 2g(y) = \sin y$

$g(y) = \frac{\sin y}{2}$

$\Rightarrow f(y) = -g(y) = -\frac{1}{2} \sin y$

$z = \frac{-1}{2}(\sin y) e^{-ax} + \frac{1}{2}(\sin y) e^{ax}$

$z = \sin y \left[\frac{e^{ax} - e^{-ax}}{2} \right]$

$z = \sin y . \sinh(ax)$

23. Solve $\frac{\partial^2 z}{\partial x^2} + 3\frac{\partial z}{\partial x} - 4z = 0$ Subject to the condition that $z = 1$ and $\frac{\partial z}{\partial x} = y$ when $x = 0$

$\Rightarrow$ Given:- $\frac{\partial^2 z}{\partial x^2} + 3\frac{\partial z}{\partial x} - 4z = 0 \longrightarrow (1)$

$(D^2 + 3D - 4)z = 0$

the A.E is $m^2 + 3m - 4 = 0$

$m^2 + 4m - m - 4 = 0$

$m(m+4) - 1(m+4) = 0$

$m = -4, m = 1$

$z = f(y)e^{-4x} + g(x)e^{x} \longrightarrow (3)$

when $x=0 \Rightarrow z=1$

$$② \Rightarrow f(y)+g(y)=1 \longrightarrow ④$$

when $x=0 \Rightarrow \frac{\partial z}{\partial x}=y$

$$③ \Rightarrow -4f(y)+g(y)=y \longrightarrow ⑤$$

$$④-⑤ \Rightarrow 5f(y)=1-y$$

$$f(y)=\frac{1-y}{5}$$

$$\Rightarrow g(y)=1-f(y)$$

$$\Rightarrow g(y)=1-\frac{1}{5}(1-y)$$

$$g(y)=\frac{5-1+y}{5}$$

$$g(y)=\frac{y+4}{5}$$

$$\boxed{z=\left(\frac{1-y}{5}\right)e^{-4x}+\left(\frac{y+4}{5}\right)e^{x}}$$

Lagrange's partial differential Eqn

Step 1 :- The general form of Lagrange's linear P.E can be defined $Pp+Qq=R$ where $p=\frac{\partial z}{\partial x}$, $q=\frac{\partial z}{\partial y}$. P. Q. R are the functions of x, y, z

Step 2 :- Write the Auxillary Equation for the Lagranges Linear P.D.E as $\frac{dx}{P}=\frac{dy}{Q}=\frac{dz}{R}$

Step 3 :- Consider the Suitable pairs and solve the Same, the Soln be the $u(x,y,z)=c_1$

$$v(x,y,z)=c_2$$

Step 4 :- Write the final Solution as $\phi(u,v)=c$

24 Solve $x\frac{\partial z}{\partial x}+y\frac{\partial z}{\partial y}=z$

$\Rightarrow$ Given :- $x\frac{\partial z}{\partial x}+y\frac{\partial z}{\partial y}=z$

$\Rightarrow xp+yq=z$

$\Rightarrow Pp+Qq=R$

$P=x\ ,\ Q=y\ ,\ R=z$

The A.E is $\frac{dx}{P}=\frac{dy}{Q}=\frac{dz}{R}$

$\Rightarrow \frac{dx}{x}=\frac{dy}{y}=\frac{dz}{z}$

case (1) $\Rightarrow \frac{dx}{x}=\frac{dy}{y}$

$\Rightarrow \int\frac{1}{x}dx=\int\frac{1}{y}dy$

$\Rightarrow \log x-\log y=\log c_1$

$\Rightarrow \log(x/y)=\log c_1$

$\Rightarrow \frac{x}{y}=c_1$

case (2) $\Rightarrow \frac{dy}{y}=\frac{dz}{z}$

$\Rightarrow \int\frac{1}{y}dy=\frac{1}{z}dz$

$\Rightarrow \log y=\log z+\log c_2$

$\Rightarrow \log y=\log(c_2z)$

$\Rightarrow y=c_2z$

$\frac{y}{z}=c_2$

The soln is $\phi\left(\frac{x}{y},\frac{y}{z}\right)=c$

25 Solve $(y^2z)p=x^2(zq+y)$

$\Rightarrow$ given :- $(y^2z)p=x^2(zq+y)$

$\Rightarrow y^2zp=x^2zq+x^2y \Rightarrow y^2zp-x^2zq=x^2y$

$\Rightarrow Pp+Qq=R \Rightarrow y^2z=P\ ,\ Q=x^2z\ ,\ R=x^2y$

The A.E

$$\Rightarrow \frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$$

$$\Rightarrow \frac{dx}{y^2 z} = \frac{dy}{-x^2 z} = \frac{dz}{x^2 y}$$

Case ①:- $\dfrac{dx}{y^2 z} = \dfrac{dy}{-x^2 z}$

$$\Rightarrow \frac{dx}{x^2} = \frac{dy}{-x^2}$$

$$\Rightarrow \int x^2\,dx = -\int y^2\,dy$$

$$\Rightarrow \frac{x^3}{3} = -\frac{y^3}{3} + k_1$$

$$\Rightarrow x^3 + y^3 = 3k_1 \longrightarrow ①$$

Case ②:- $\dfrac{dy}{-x^2 z} = \dfrac{dz}{x^2 y}$

$$\Rightarrow \int y\,dy = -\int z\,dz$$

$$\Rightarrow \frac{y^2}{2} = \frac{z^2}{2} + k_2$$

$$\Rightarrow y^2 + z^2 = 2k_2$$

$\therefore$ The Soln is $\phi(x^3+y^3,\ y^2+z^2) = c$

26 Solve $(y-z)p + (z-x)q = (x-y)$

$\Rightarrow$ Given:- $(y-z)p + (z-x)q = (x-y)$

$$\Rightarrow Pp + Qq = R$$

$$P = y-z,\quad Q = z-x,\quad R = x-y$$

The A.E is $\dfrac{dx}{P} = \dfrac{dy}{Q} = \dfrac{dz}{R}$

$$\Rightarrow \frac{dx}{y-z} = \frac{dy}{z-x} = \frac{dz}{x-y} \longrightarrow ①$$

Case ① :- $\frac{dx+dy+dz}{y-z+z-x+x-y} = \frac{dx+dy+dz}{0}$

$\Rightarrow dx+dy+dz=0$

$\Rightarrow \int dx + \int dy + \int dz = 0$

$\Rightarrow x+y+z=c_1$

Case ② :- $\frac{xdx+ydy+zdz}{x(y-z)+y(z-x)+z(x-y)} = \frac{xdx+ydy+zdz}{xy-zx+yz-xy+zx-yz}$

$\Rightarrow \frac{xdx+ydy+zdz}{0}$

$\Rightarrow xdx+ydy+zdz=0$

$\Rightarrow \int x dx + \int y dy + \int z dz = 0$

$\Rightarrow \frac{x^2}{2}+\frac{y^2}{2}+\frac{z^2}{2} = c_2$

$\Rightarrow x^2+y^2+z^2=2c_2$

$\therefore$ the Soln is $\phi(x+y+z,\ x^2+y^2+z^2) = c$

27 Solve $(y^2+z^2)p + xyq = xz$

$\Rightarrow$ given :- $(y^2+z^2)p + xyq = xz$

$Pp + Qq = R$

$\therefore P = y^2+z^2,\ Q = xy,\ R = xz$

the A.E is $\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$

$$\frac{dx}{y^2+z^2} = \frac{dy}{xy} = \frac{dz}{xz}$$

Case① :- $\frac{xdx-ydy-zdz}{x(y^2+z^2)-y(xy)-z(xz)} = \frac{xdx-ydy-zdz}{xy^2+xz^2+xy^2-x^2}$

$$\Rightarrow \frac{x\,dx - y\,dy - z\,dz}{0}$$

$$\Rightarrow \int x\,dx - \int y\,dy - \int z\,dz = 0$$

$$\Rightarrow \frac{x^2}{2} - \frac{y^2}{2} - \frac{z^2}{2} = c_1$$

$$\Rightarrow x^2 - y^2 - z^2 = 2c_1$$

Case (2) :- $\frac{dy}{xy} = \frac{dz}{zx}$

$$\Rightarrow \int \frac{1}{y}\,dy = \int \frac{1}{z}\,dz$$

$$\Rightarrow \log y - \log z = \log c_2$$

$$\Rightarrow \log\left(\frac{y}{z}\right) = \log c_2$$

$$\Rightarrow \frac{y}{z} = c_2$$

$\therefore$ the Soln is $\phi\left(x^2 - y^2 - z^2, \frac{y}{z}\right) = c$

28 Solve $x(y^2 - z^2)p + y(z^2 - x^2)q = z(x^2 - y^2)$

$\Rightarrow$ Given :- $x(y^2 - z^2)p + y(z^2 - x^2)q = z(x^2 - y^2)$

$$Pp + Qq = R_1$$

$$P = xy^2 - xz^2, \quad Q = yz^2 - x^2y, \quad R_1 = z(x^2 - y^2)$$

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R_1}$$

$$\frac{dx}{xy^2 - xz^2} = \frac{dy}{yz^2 - x^2y} = \frac{dz}{z(x^2 - y^2)}$$

Case (1) :- $\frac{x\,dx + y\,dy + z\,dz}{x(xy^2 - xz^2) + y(yz^2 - x^2y) + z^2(x^2 - y^2)}$

$$\Rightarrow \frac{x\,dx + y\,dy + z\,dz}{0} = 0$$

$$\Rightarrow \int x\,dx + \int y\,dy + \int z\,dz = 0$$

$$\Rightarrow \frac{x^2}{2} + \frac{y^2}{2} + \frac{z^2}{2} = c_1$$

$$x^2 + y^2 + z^2 = 2c_1$$

case 2 :- $\dfrac{\frac{1}{x}dx + \frac{1}{y}dy + \frac{1}{z}dz}{y^2 - z^2 + z^2 - x^2 + x^2 - y^2}$

$$\Rightarrow \frac{\frac{1}{x}dx + \frac{1}{y}dy + \frac{1}{z}dz}{0}$$

$$\Rightarrow \int \frac{1}{x}dx + \frac{1}{y}dy + \int \frac{1}{z}dz = 0$$

$$\Rightarrow \log(xyz) = \log c_2$$

$$\Rightarrow xyz = c_2$$

$$\phi = (x^2 + y^2 + z^2, xyz)$$

$\boxed{29}$ Solve $x^2(y-z)p + y^2(z-x)q = z^2(x-y)$

$\Rightarrow$ Given :- $x^2(y-z)p + y^2(z-x)q = z^2(x-y) \longrightarrow (1)$

$$Pp + Qq = R$$

$$P = x^2(y-z), \quad Q = (y^2(z-x)), \quad R = z^2(x-y)$$

$$\Rightarrow \frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$$

$$\Rightarrow \frac{dx}{x^2(y-z)} = \frac{dy}{y^2(z-x)} = \frac{dz}{z^2(x-y)}$$

case(1) :- $\dfrac{\frac{1}{x^2}dx}{y-z} = \dfrac{\frac{1}{y^2}dy}{z-x} = \dfrac{\frac{1}{z^2}dz}{x-y}$

$$\Rightarrow \frac{\frac{1}{x^2}dx + \frac{1}{y^2}dy + \frac{1}{z^2}dz}{y-z+z-x+x-y}$$

$$\Rightarrow \int \frac{1}{x^2}dx + \int \frac{1}{y^2}dy + \int \frac{1}{z^2}dz = c_1$$

$$\Rightarrow \left(\frac{1}{x} + \frac{1}{y} + \frac{1}{z}\right) = c_1$$

Case ② :- $$\frac{\frac{1}{x}dx + \frac{1}{y}dy + \frac{1}{z}dz}{xy - xz + zy - xy + xz - yz}$$

$$\Rightarrow \int \frac{1}{x}dx + \int \frac{1}{y}dy + \int \frac{1}{z}dz = 0$$

$$\Rightarrow \log x + \log y + \log z = \log c_2$$

$$\Rightarrow \log(xyz) = \log c_2$$

$$\Rightarrow (xyz) = c_2$$

$$\phi = \left(\frac{1}{x} + \frac{1}{y} + \frac{1}{z},\ xyz\right) = c$$

30 Solve $x(y^2+z)p - y(x^2+z)q = z(x^2-y^2)$

$\Rightarrow$ Given :- $x(y^2+z)p - y(x^2+z)q = z(x^2-y^2)$

$$Pp - Qq = R$$

$$P = (x(y^2+z)),\quad Q = y(x^2+z),\quad R = z(x^2-y^2)$$

$$\Rightarrow \frac{x\,dx + y\,dy - dz}{x^2(y^2+z) - y^2(x^2+z) - z(x^2-y^2)}$$

$$\Rightarrow \frac{x\,dx + y\,dy - dz}{x^2y^2 + x^2z - x^2y^2 - zy^2 - zx^2 + zy^2}$$

$$\Rightarrow \frac{x\,dx + y\,dy - dz}{x^2y^2 + x^2z - x^2y^2 - zy^2 - zx^2 + zy^2}$$

$$\Rightarrow \int x\,dx + \int y\,dy - \int z\,dz = c_1$$

$$\Rightarrow \frac{x^2}{2} + \frac{y^2}{2} - z = c_1$$

$$x^2 + y^2 - 2z = 2c_1$$

$$\Rightarrow \frac{\frac{1}{x}dx + \frac{1}{y}dy + \frac{1}{z}dz}{y^2 + z - x^2 - z + x^2 - y^2}$$

$$\Rightarrow \int \frac{1}{x}dx + \int \frac{1}{y}dy + \int \frac{1}{z}dz = 0$$

$$\log(xyz) = \log c_2$$

$$xyz = c_2$$

$$\phi = (x^2 + y^2 - 2z,\ xyz) = c$$

31] Solve $(y+z)p + (z+x)q = x+y$

$\Rightarrow$ Given :- $(y+z)p + (z+x)q = x+y$

$$Pp + Qq = R$$

$$P = (y+z),\ Q = (z+x),\ R = (x+y)$$

The A.E $\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{z}$

$$\Rightarrow \frac{dx}{y+z} = \frac{dy}{z+x} = \frac{dz}{x+y}$$

$$\Rightarrow \frac{dx + dy + dz}{2(x+y+z)} = \frac{dx - dy}{y - x} = \frac{dy - dz}{z - y}$$

Case 1 :- $\frac{1}{2}\frac{d(x+y+z)}{(x+y+z)} = -\frac{d(x-y)}{x-y}$

$$\Rightarrow \frac{1}{2}\int \frac{1}{x+y+z}\, d(x+y+z) = -\int \frac{1}{(x-y)}\, d(x-y)$$

$$\Rightarrow \frac{1}{2}\log(x+y+z) = \log(x-y) + \log c_1$$

$$\Rightarrow \log(\sqrt{x+y+z}) = \log(x-y) + \log c_1$$

$$\Rightarrow \log|(x-y)\sqrt{x+y+z}| = \log c_1$$

$$\Rightarrow (x-y)\sqrt{x+y+z} = c_1$$

Case 2 :- $\frac{dx-dy}{y-x} = \frac{dy-dz}{z-y}$

$$\Rightarrow \int \frac{1}{x-y}\, d(x-y) = \int \frac{1}{z-y}\, d(z-y)$$

$$\Rightarrow \log(x-y) = \log(y-z) - \log c_2$$

$$\Rightarrow \log\left(\frac{x-y}{y-z}\right) - \log c_2$$

$$\Rightarrow \frac{x-y}{y-z} = c_2$$

The soln is $\phi\left[(x-y)\sqrt{x+y+z}, \frac{x-y}{y-z}\right] = c$

$\boxed{32}$ Solve $(x^2-y^2-z^2)p + (2xy)q = 2zx$

$\Rightarrow$ Given :- $(x^2-y^2-z^2)p + (2xy)q = 2zx$

$$P = x^2-y^2-z^2, \quad Q = 2xy, \quad R = 2zx$$

the A. E

$$\Rightarrow \frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$$

$$\Rightarrow \frac{dx}{x^2-y^2-z^2} = \frac{dy}{2xy} = \frac{dz}{2zx} \longrightarrow (1)$$

$$\Rightarrow \frac{x\,dx+y\,dy+z\,dz}{x^3-xy^2-xz^2+2xy^2+2xz^2}=\frac{dy}{2xy}=\frac{dz}{2zx}$$

$$\Rightarrow \frac{x\,dx+y\,dy+z\,dz}{x^3-xy^2+xz^2}=\frac{dy}{2xy}=\frac{dz}{2zx}$$

Case ① :- $\frac{x\,dx+y\,dy+z\,dz}{x(x^2+y^2+z^2)}=\frac{dy}{2xy}$

$$\Rightarrow \int\frac{2x\,dx+2y\,dy+2z\,dz}{x^2+y^2+z^2}=\int\frac{dy}{y}$$

$$\Rightarrow \log|x^2+y^2+z^2|=\log y+\log c_1$$

$$\Rightarrow \log\left|\frac{x^2+y^2+z^2}{y}\right|=\log c_1$$

$$\Rightarrow \frac{(x^2+y^2+z^2}{y}=c_1$$

Case ② :- $\frac{dy}{2xy}=\frac{dz}{2zx}$

$$\Rightarrow \int\frac{dy}{y}=\int\frac{dz}{z}$$

$$\Rightarrow \log y=\log z+\log c_2$$

$$\Rightarrow \log\left(\frac{y}{z}\right)=\log c_2$$

$$\left(\frac{y}{z}\right)=c_2$$

The Sol^n is $\phi\left[\frac{x^2+y^2+z^2}{y},\frac{y}{z}\right]=C$

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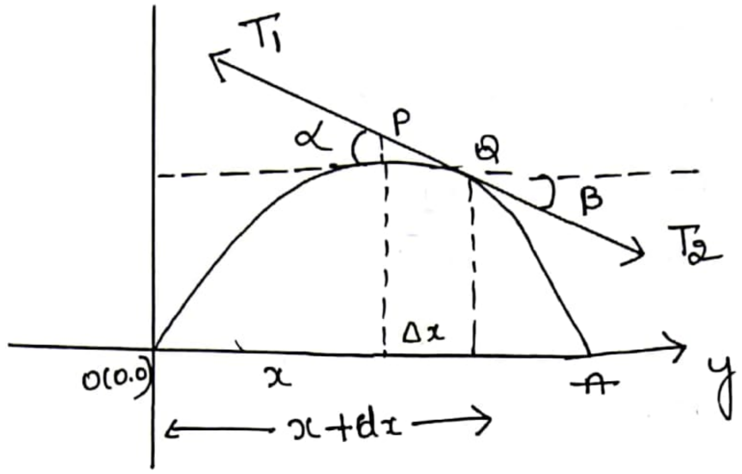
33] One dimensional wave Equation



Consider a Small transverse Vibration of an Elastic string of length l, which is Stretched at the 2 pt origin and A, In the Equilibrium position. OA as the x axis and lined through the origin zero and perpendicular to the x-axis as the y-axis

Let p and q be the 2 points on the string Let (α, β) be the angles at p and q and Let T_1, T_2 be the tension towards the points p and q. Since if There is no motion

$$T_1 \cos\alpha = T_2 \cos\beta = T \longrightarrow (1)$$

Let m be the mass where Unit length of the String the mass of Element of pq is $m\Delta x$

In the Vertical transverse direction, The Components of T_1 & T_2 are $-T_1 \sin\alpha + T_2 \sin\beta$

By the newton's Second Law, we have $F = ma \rightarrow (2)$, Here $F = -T_1 \sin\alpha + T_2 \sin\beta$, The mass $m = m\Delta x$ and a = accelerati[on] $\frac{\partial^2 u}{\partial t^2}$

$$\therefore (1) \Rightarrow T_2 \sin\beta - T_1 \sin\alpha = m\,dx \frac{\partial^2 u}{\partial t^2}$$

$$\Rightarrow \frac{T}{\cos\beta} \sin\beta - \frac{T}{\cos\alpha} \sin\alpha = m\Delta x \frac{\partial^2 u}{\partial t^2}$$

$$\Rightarrow T\tan\beta - T\tan\alpha = m\Delta x \frac{\partial^2 u}{\partial t^2}$$

$$\Rightarrow \tan\beta - \tan\alpha = m\frac{\Delta x}{T}\frac{\partial^2 u}{\partial t^2} \longrightarrow (3)$$

Here $\tan\alpha$ & $\tan\beta$ are the slopes at the point p and q

$$\tan\alpha = \left(\frac{\partial u}{\partial x}\right)_{x=x} \qquad \tan\beta = \left(\frac{\partial u}{\partial x}\right)_{x=x+\Delta x}$$

$$(3) \Rightarrow \left(\frac{\partial u}{\partial x}\right)_{x+\Delta x} - \left(\frac{\partial u}{\partial x}\right)_{x} = m\frac{\Delta x}{T}\frac{\partial^2 u}{\partial t^2}$$

$$\frac{\left(\frac{\partial u}{\partial x}\right)_{x+\Delta x} - \left(\frac{\partial u}{\partial x}\right)_{x}}{\Delta x} = \frac{m}{T}\frac{\partial^2 u}{\partial t^2}$$

$$\Rightarrow \lim_{\Delta x \to 0} \frac{\left(\frac{\partial u}{\partial x}\right)_{x+\Delta x} - \left(\frac{\partial u}{\partial x}\right)_{x}}{\Delta x} = \frac{m}{T}\frac{\partial^2 u}{\partial t^2}$$

$$\Rightarrow \frac{\partial^2 u}{\partial x^2} = \frac{m}{T}\frac{\partial u}{\partial t^2}$$

$$\Rightarrow \frac{\partial^2 u}{\partial t^2} = \frac{T}{m}\frac{\partial^2 u}{\partial x^2}$$

$$\Rightarrow \boxed{\frac{\partial^2 u}{\partial t^2} = c^2\frac{\partial^2 u}{\partial x^2}}$$

$$\Rightarrow c^2 = \frac{T}{m}$$

Solution for one dimention Eqn by method of Separation of Variables:-

34) S.T the wave Equation for 1-D is $\frac{\partial^2 u}{\partial t^2} = c^2\frac{\partial^2 u}{\partial x^2}$

$\Rightarrow$ given :- $\frac{\partial^2 u}{\partial t^2} = c^2\frac{\partial^2 u}{\partial x^2} \longrightarrow (1)$

Let the Soln is $u = XT$ where $X = X(x)$, $T = T(t)$

$$\therefore (1) \Rightarrow \frac{\partial^2}{\partial t^2}(XT) = c^2\frac{\partial^2}{\partial x^2}(XT)$$

$$\Rightarrow X\frac{\partial^2 T}{\partial t^2} = c^2 T\frac{\partial^2 X}{\partial x^2}$$

$$\Rightarrow \frac{1}{T}\frac{\partial^2 T}{\partial t^2} = \frac{c^2}{X}\frac{\partial^2 X}{\partial x^2}$$

$$\Longrightarrow \frac{1}{c^2 T}\frac{\partial^2 T}{\partial t^2} = \frac{1}{X}\frac{\partial x}{\partial x^2} = k$$

$$\frac{1}{c^2 T}\frac{\partial^2 T}{\partial t^2} = k, \quad \frac{1}{X}\frac{\partial^2 X}{\partial x^2} = k$$

$$\frac{\partial^2 T}{\partial t^2} = kc^2 T, \quad \frac{\partial^2 X}{\partial x^2} = kX$$

$$\Longrightarrow \frac{\partial^2 T}{\partial t^2} - kc^2 T = 0 \rightarrow (2) \qquad \frac{\partial^2 X}{\partial x^2} - kX = 0 \longrightarrow (3)$$

case 1) :- If $k=0$

$$(2) \Rightarrow \frac{\partial^2 T}{\partial t^2} = 0$$

$$\Rightarrow \frac{\partial}{\partial t}\left(\frac{\partial T}{\partial t}\right) = 0$$

$$\Rightarrow \int \partial\left(\frac{\partial T}{\partial t}\right) = \int 0\, \partial t$$

$$\Rightarrow \frac{\partial T}{\partial t} = c_1$$

$$\Longrightarrow \int \partial T = c_1 \int \partial t$$

$$\Longrightarrow T = c_1 t + c_2$$

$$(3) \Longrightarrow \frac{\partial^2 X}{\partial x^2} = 0$$

$$\Longrightarrow \frac{\partial}{\partial x}\left(\frac{\partial x}{\partial x}\right) = 0$$

$$\Longrightarrow \int \partial\left(\frac{\partial u}{\partial x}\right) = \int 0\, dx$$

$$\Longrightarrow \frac{\partial x}{\partial x} = c_1$$

$\Rightarrow \int \partial x = c_1 \int \partial x$

$\Rightarrow x = c_1 x + c_2$

$$\boxed{u = (c_1 x + c_2)(c_1' t + c_2')}$$

Case ②:- Let $k = p^2$

② $\Rightarrow \frac{\partial^2 T}{\partial t^2} - p^2 c^2 T = 0$

$\Rightarrow (D^2 - p^2 c^2 T) = 0$, $D = \frac{\partial}{\partial t}$

A.E is $f(m) = 0$

$m^2 - p^2 c^2 = 0$

$m = \pm pc$

$\therefore T = c_1 e^{-pct} + c_2 e^{pct}$

③ $\Rightarrow \frac{\partial^2 X}{\partial x^2} - p^2 X = 0$

$\Rightarrow (D^2 - p^2) X = 0$, $D = \frac{\partial}{\partial x}$

The A.E is $m^2 - p^2 = 0$

$\Longrightarrow m = \pm p$

$X = c_1' e^{-px} + c_2' e^{px}$

$u = (c_1' e^{-px} + c_2' e^{px})$

$(c_1 e^{-pct} + c_2 e^{pct})$

Case ③:- If $k = -p^2$

② $\Rightarrow \frac{\partial^2 T}{\partial t^2} + p^2 c^2 T = 0$

$\Rightarrow (D^2 + p^2 c^2) T = 0$ $\quad D = \frac{\partial}{\partial t}$

The A.E is

$m^2 + p^2 c^2 = 0$

$\Rightarrow m = 0 \pm pi$

$\therefore T = c_1 \cos(pct) + c_2 \sin(pct)$

③ $\Rightarrow \frac{\partial^2 X}{\partial x^2} + p^2 X = 0$

$(D^2+p^2)X = 0 \quad D = \frac{\partial}{\partial x}$

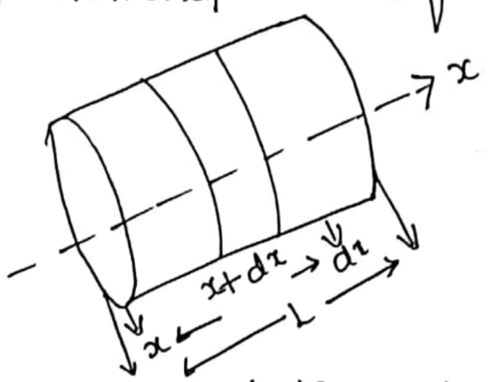
The A.E is $m^2+p^2=0$

$m = 0 \pm pi$

$X = -c_1\cos px + c_2'\sin px$

$u = (c_1'\cos px + c_2\sin px)(c_1\cos(pct) + c_2\sin(pct))$

35 One dimentional Heat Equation



Consider a heat conducting Homogeneous rod of length l placed along the x-axis. One End of rod at $x=0$ (origin) Other End of rod at $x=l$. Assume that the rod as constant density ρ and Uniform cross Section A. also assume that the rod is insulated laterally & therefore heat flows Only in the x-direction, let $u(x,t)$ with the temperature of the cross Section at the point x and at any time t The thermal conductivity of k of the material of the rod an The Temperature gradiant $\frac{\partial u}{\partial x}$

q_1 be the Quantity of heat flowing into the cross Section at a distance x in the Unit time is

$q_1 = -kA\left(\frac{\partial u}{\partial x}\right)_x s^{-1}$ and q_2 is the Quantity of heat flowing out of the cross Section at a distance $x+dx$

$$q_2 = -KA\left(\frac{\partial x}{\partial x}\right)_{x+\partial x}$$

$$q_1 - q_2 = -KA\left(\frac{\partial u}{\partial x}\right)_x + KA\left(\frac{\partial u}{\partial x}\right)_{x+\delta x}$$

$$q_1 - q_2 = KA\left[\left(\frac{\partial u}{\partial x}\right)_{x+\partial x} - \left(\frac{\partial u}{\partial x}\right)_x\right] \longrightarrow (1)$$

But the rate of Increase of heat in the rod is

$$s\rho A\, dx = \frac{\partial u}{\partial t}$$

$$q_1 - q_2 = s\rho A\, \partial x \frac{\partial u}{\partial t} \longrightarrow (2)$$

s is the specific heat

ρ is density of material

$$s\rho A\, \partial x \frac{\partial u}{\partial t} = KA\left[\left(\frac{\partial u}{\partial x}\right)_{x+\delta x} - \left(\frac{\partial u}{\partial x}\right)_x\right]$$

$$\Rightarrow \frac{s\rho}{K}\frac{\delta u}{\partial t} = \frac{\left(\frac{\partial u}{\partial x}\right)_{x+\delta x} - \left(\frac{\partial u}{\partial x}\right)_x}{\delta x}$$

$$\Rightarrow \frac{s\rho}{K}\frac{\delta u}{\partial t} = \lim_{\delta x \to 0} \frac{\left(\frac{\delta u}{\partial x}\right)_{x+\delta x} - \left(\frac{\delta x}{\partial x}\right)_x}{\delta x}$$

$$\Rightarrow \frac{s\rho}{K}\frac{\delta u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$$

$$\Rightarrow \frac{\partial u}{\partial t} = \frac{K}{s.\rho}\frac{\partial^2 u}{\delta x^2}$$

$$\boxed{\frac{\partial u}{\partial t} = c^2\frac{\partial^2 u}{\partial x^2}}$$

where $c^2 = \frac{K}{s.\rho}$

36) Soln for 1 dimension Heat Eqn by Variable Separation

W.K.T, the one dimensional heat equation is $\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2} \rightarrow ①$

where u is function of x and t

Let the soln is $V = XT$

where $X = X(x)$, $T = T(t)$

$① \Rightarrow \frac{\partial}{\partial t}(XT) = \frac{c^2 \partial^2}{\partial x^2}(XT)$

$\Rightarrow X \frac{\partial T}{\partial t} = c^2 T \frac{\partial^2 X}{\partial x^2}$

$\Rightarrow \frac{1}{\cancel{c}T} \frac{\partial T}{\partial t} = \frac{c^2}{X} \frac{\partial^2 X}{\partial x^2}$

$\Rightarrow \frac{1}{c^2 T} \frac{\partial T}{\partial t} = \frac{1}{X} \frac{\partial^2 X}{\partial x^2} = K$

$\Rightarrow \frac{1}{c^2 T} \frac{\partial T}{\partial t} = K_1, \quad \frac{1}{X} \frac{\partial^2 X}{\partial x^2} = K$

$\Rightarrow \frac{\partial T}{\partial t} = Kc^2 T, \quad \frac{\partial^2 X}{\partial x^2} = KX$

$\Rightarrow \frac{\partial T}{\partial t} - Kc^2 T = 0 \longrightarrow ② \qquad \frac{\partial^2 X}{\partial x^2} - KX = 0 \longrightarrow ③$

case 1) If $K = 0$

$② \Rightarrow \frac{\partial T}{\partial t} = 0$

$\Rightarrow \int \partial T = \int 0 \, \partial t$

$\Rightarrow T = c_1$

$③ \Rightarrow \frac{\partial^2 X}{\partial x^2} = 0 \qquad$ Let $D = \frac{d}{dx}$

$\Rightarrow D^2 X = 0$

A.E is $m^2 = 0$

$\Rightarrow m = 0, 0$

$X_1 = c_2 + c_3 x \qquad \Rightarrow \boxed{u = (c_2 + c_3 x) c_1}$

Case 2) If $K = p^2$

$$② \Rightarrow \frac{\partial T}{\partial t} - p^2c^2T = 0$$

$$\Rightarrow (D - p^2c^2)T = 0 \qquad \text{Let } D = \frac{\partial}{\partial t}$$

A.E is $m - p^2c^2 = 0$

$$m = p^2c^2$$

$$T = c_1 e^{p^2c^2t}$$

$$③ \Rightarrow \frac{\partial^2 X}{\partial x^2} - p^2X = 0$$

$$\Rightarrow (D^2 - p^2)X = 0$$

The A.E is $m^2 - p^2 = 0$

$$\Rightarrow m^2 = p^2$$

$$\Rightarrow m = \pm p$$

$$X = c_2 e^{-px} + c_3 e^{px}$$

$$u = \left(c_2 e^{-px} + c_3 e^{px}\right) c_1 e^{p^2c^2t}$$

Case 3) If $K = -p^2$

$$② \Rightarrow \frac{\partial T}{\partial t} + p^2c^2T = 0$$

$$\Rightarrow (D + p^2c^2)T = 0$$

The A.E is $m + p^2c^2 = 0$

$$\Rightarrow m = -p^2c^2$$

$$T = c_1 e^{-p^2c^2t}$$

$$③ \Rightarrow \frac{\partial^2 X}{\partial x^2} + p^2X = 0$$

$$\Rightarrow (D^2 + p^2)X = 0 \qquad D = \frac{\partial}{\partial x}$$

the A.E is $m^2 + p^2 = 0$

$$m^2 = -p^2$$

$$m = 0 + pi$$

$$X = c_2 \cos px + c_3 \sin px$$

$$\boxed{u = c_1 e^{-p^2c^2t}\left[c_2 \cos px + c_3 \sin px\right]}$$