

CALCULUS AND LINEAR ALGEBRA

MODULE - 4

DIFFERENTIAL EQUATION

If an equation contain one dependant variable and then derivative with respect to one or more independent variable is called Differential equation.

There are two types of D.E

1. Ordinary differential equation
2. partial differential equation

⇒ Ordinary differential equation :- If an equation contain one dependant variable and its derivative with respect to one independent variable.

$$\text{ex:- } x^2 \frac{d^2y}{dx^2} - 3x \frac{dy}{dx} = 0$$

⇒ Partial differential equation :- If an equation contain one dependent variable and derivatives with respect to

two or more variable

$$\text{ex:- } x \frac{dz}{dx} + y \frac{dz}{dy} = 2z$$

\Rightarrow Order and degree of differential equation

The highest derivative in a given D.E is called order it's power is called degree

$$\text{ex:- } x^2 \left(\frac{d^3y}{dx^3} \right)^4 - 2x \left(\frac{d^2y}{dx^2} \right)^5 + \left(\frac{dy}{dx} \right)^5 + y = 0$$

The highest derivative is $\frac{d^3y}{dx^3}$ and power 4

order of D.E = 3

degree of D.E = 4

\Rightarrow Solution of 1st order and 1st degree D.E

Generally 1st order and 1st degree D.E is in the form $\frac{dy}{dx} = f(x, y)$

\Rightarrow Exact Differential equation:-

Step 1 :- write the given D.E is in the form $m(x, y)dx + n(x, y)dy = 0$

Step 2 :- Identify the M and N find $\frac{\partial M}{\partial y}, \frac{\partial N}{\partial y}$

Step 3 :- if $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$ we say that given D.E is exact D.E

Step 4 :- Write the solution for E.D.E

$$\int M(x, y) dx + \int (\text{The terms which don't contain } x \text{ in N}) dy = C$$

Problems:-

1. Solve $\frac{dy}{dx} + \frac{2x+3y-1}{3x+4y-2} = 0$

$$(2x+3y-1) dx + (3x+4y+2) dy = 0$$

$$M = (2x+3y-1) \quad N = (3x+4y+2)$$

$$\frac{\partial M}{\partial y} = 3 \quad \frac{\partial N}{\partial x} = 3$$

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} = 3$$

The given D.E is E.D.E

$$\int (xy) dx + \int (\text{the term which don't contain } x) dy = c$$

$$\Rightarrow \int (2x + 3y - 1) dx + \int (4y + 2) dy = c$$

$$\Rightarrow \frac{x^2}{2} + 3yx - x + \frac{4y^2}{2} + 2y = c$$

$$\Rightarrow x^2 + 3yx - x + 2y^2 + 2y = c$$

$$\Rightarrow x^2 + 2y^2 - x + 2y + 3yx = c$$

Q2. Solve $(2x + y + 1) dx + (x + 2y + 1) dy = 0$

$$(2x + y + 1) dx + (x + 2y + 1) dy = 0$$

$$M = (2x + y + 1) \quad N = (x + 2y + 1)$$

$$\frac{\partial M}{\partial y} = 1$$

$$\frac{\partial N}{\partial x} = 1$$

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} = 1$$

\therefore The D.E is E.D.E

$$\Rightarrow \int M(x, y) dx + \int (\text{The term which don't contain } x \text{ in } N) dy = c$$

$$\Rightarrow \int (2x + y + 1) dx + \int (2y + 1) dy = c$$

$$\Rightarrow x^2 + yx + x + y^2 + y = c$$

3. Solve $(y^3 - 3x^2y) dx - (x^3 - 3xy^2) dy = 0$

$$(y^3 - 3x^2y) dx + (-x^3 + 3xy^2) dy = 0$$

$$M = y^3 - 3x^2y$$

$$N = 3xy^2 - x^3$$

$$\frac{\partial M}{\partial y} = 3y^2 - 3x^2$$

$$\frac{\partial N}{\partial x} = 3y^2 - 3x^2$$

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} = 3y^2 - 3x^2$$

\therefore The D.E is E.D.E

$$\Rightarrow \int (y^3 - 3x^2y) dx + \int 0 dy = c \Rightarrow xy^3 - x^3y = c$$

$$xy[x^2 - y^2] = c$$

4. Solve $[5x^4 + 3x^2y^2 - 2xy^3]dx + [2x^3y - 3x^2y^2 - 5y^4]dy$

$$[5x^4 + 3x^2y^2 - 2xy^3]dx + [2x^3y - 3x^2y^2 - 5y^4]dy$$

$$M = 5x^4 + 3x^2y^2 - 2xy^3 \quad N = 2x^3y - 3x^2y^2 - 5y^4$$

$$\frac{\partial M}{\partial y} = 6x^2y - 6xy^2 \quad \frac{\partial N}{\partial x} = 6x^2y - 6xy^2$$

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} = 6x^2y - 6xy^2$$

∴ The given DE is EDE

$$\Rightarrow \int (5x^4 + 3x^2y^2 - 2xy^3)dx + \int -5y^4 dy = c$$

$$\Rightarrow 5\frac{x^5}{5} + 3\frac{x^3}{3}y^2 - 2\frac{x^2}{2}y^3 - \frac{5y^5}{5} = c$$

$$\Rightarrow x^2 - y^5 - x^3y^2 - x^2y^3 = c$$

5. Solve

$$\frac{dy}{dx} + \frac{(x+3y-4)}{(3x+9y-2)} = 0$$

$$\frac{dy}{dx} + \frac{x+3y-4}{(3x+9y-2)} = 0$$

$$(x+3y-4)dx + (3x+9y-2)dy = 0$$

$$M = x + 3y - 4 \quad N = 3x + 9y - 2$$

$$\frac{\partial M}{\partial y} = 3$$

$$\frac{\partial N}{\partial x} = 3$$

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} = 3$$

∴ The given D.E is EDE

$$\Rightarrow \int (x+3y-4)dx + \int (9y-2)dy = c$$

$$\Rightarrow \frac{x^2}{2} + 3yx - 4x + \frac{9y^2}{2} - 2y = c$$

⇒ Reductable exact Differential equation.

Step ① ∵ consider the differential equation

$$M(x,y)dx + N(x,y)dy = 0$$

Step ② ∵ find $\frac{\partial M}{\partial y}$ and $\frac{\partial N}{\partial x}$ and check equality

Step ③ : If it is not equal then find $\left| \frac{\partial M}{\partial y} - \frac{\partial N}{\partial x} \right|$

case ① :- if $\frac{1}{N} \left[\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x} \right] = f(x)$

$$I.F = e^{\int f(x) dx}$$

Multiply the Integration factor to the given DE and follow same procedure

case ② :- if $\frac{1}{M} \left[\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x} \right] = g(y)$

$$I.F = e^{\int g(y) dy}$$

And proceed the same.

1. Solve $(4xy + 3y^2 - x)dx + x^2x + 2yx)dy = 0$

$$(4xy + 3y^2 - x)dx + (x^2 + 2yx)dy = 0 \rightarrow ①$$

$$M = 4xy + 3y^2 - x$$

$$\frac{\partial M}{\partial y} = 4x + 6y$$

$$\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$$

$$N = x^2 + 2yx$$

$$\frac{\partial N}{\partial x} = 2x + 2y$$

$$\Rightarrow \frac{1}{N} \left[\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x} \right] = \frac{(4x + 6y - 2x - 2y)}{x^2 + 2yx} = \frac{2x + 4y}{x^2 + 2yx} = \frac{2(x + 2y)}{x^2 + 2yx}$$

$$\Rightarrow \frac{1}{N} \left[\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x} \right] = \frac{2}{x} = f(x)$$

$$\Rightarrow I.F = e^{\int f(x) dx} = e^{\int 2/x dx} = e^{2 \log x} = e^{\log x^2} = x^2$$

$$I.F \times ① = x^2(4xy + 3y^2 - x)dx + x^2(x^2 + 2yx)dy = 0$$

$$(4x^3y + 3x^2y^2 - x^3)dx + (x^4 + 2yx^3)dy = 0$$

$$M' = 4x^3y + 3x^2y^2 - x^3$$

$$N' = x^4 + 2yx^3$$

$$\frac{\partial M'}{\partial y} = 4x^3 + 6x^2y$$

$$\frac{\partial N'}{\partial x} = 4x^3 + 6yx^2$$

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} = 4x^3 + 6x^2y$$

The given solution is E DE

$$\Rightarrow \int (4x^3y + 3x^2y^2 - x^2) dx - \int 0 dy = c$$

$$\Rightarrow \frac{4x^4y}{4} + \frac{3x^3y^2}{3} - \frac{x^4}{4} = c$$

$$\Rightarrow x^4y + x^3y^2 - \frac{x^4}{4} = c$$

2. Solve $y(2x - y + 1) + x(3x - 4y + 3) dy = 0$

$$(2xy - y^2 + y) dx + (3x^2 - 4yx + 3x) dy = 0 \rightarrow ①$$

$$M = 2xy - y^2 + y$$

$$N = 3x^2 - 4yx + 3x$$

$$\frac{\partial M}{\partial y} = 2x - 2y + 1$$

$$\frac{\partial M}{\partial x} = 6x - 4y + 3$$

$$\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$$

$$\begin{aligned} \frac{1}{M} \left[\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right] &= \frac{6x - 4y + 3 - 2x + 2y - 1}{2xy - y^2 + y} = \frac{4x - 2y + 2}{y(2x - y + 1)} \\ &= \frac{2(2x - y + 1)}{y(2x - y + 1)} \end{aligned}$$

$$\Rightarrow \frac{1}{M} \left[\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right] = \frac{2}{y} = g(y)$$

$$\Rightarrow I \cdot F = e^{\int g(y) dy} = e^{\int \frac{2}{y} dy} = e^{2 \int \frac{1}{y} dy} = e^{\ln y^2} = y^2$$

$$\Rightarrow I \cdot F \times ① \therefore y^2 (2xy - y^2 + y) dx + y^2 (3x^2 - 4yx + 3x) dy = 0$$

$$(2xy^3 - y^4 + y^3) dx + (3x^2y^2 - 4xy^3 + 3xy^2) dy = 0$$

$$M' = (2xy^3 - y^4 + y^3)$$

$$N' = (3x^2y^2 - 4xy^3 + 3xy^2)$$

$$\frac{\partial M'}{\partial y} = 6xy^2 - 4y^3 +$$

$$\frac{\partial N'}{\partial x} = 6x^2y - 4y^3 + 3y^2$$

$$\frac{\partial M'}{\partial y} = \frac{\partial N'}{\partial x} = 6xy^2 - 4y^3 + 3y^2$$

$$\Rightarrow \int (2xy^3 - y^4 + y^3) dx + \int 0 dy = c \Rightarrow x^2y^3 - xy^4 + xy^3 = c$$

$$3. (x^2 + y^2 + x) dx + xy dy = 0$$

$$(x^2 + y^2 + x) dx + xy dy = 0 \rightarrow ①$$

$$M = x^2 + y^2 + x \quad N = xy$$

$$\frac{\partial M}{\partial y} = 2y \quad \frac{\partial N}{\partial y} = x$$

$$\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$$

$$\frac{1}{N} \left[\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x} \right] = \frac{2y - x}{xy} = \frac{y}{x} = \frac{1}{x} = f(x)$$

$$\Rightarrow I.F = e^{\int f(x) dx} = e^{\int \frac{1}{x} dx} = e^{\log x} = x$$

$$\Rightarrow I.F \times ① = (x^3 + xy^2 + x^2) dx + x^2 y dy = 0$$

$$M' = x^3 + xy^2 + x^2$$

$$N' = x^2 y$$

$$\frac{\partial M'}{\partial y} = 2xy$$

$$\frac{\partial N'}{\partial x} = 2xy$$

$$\frac{\partial M'}{\partial y} = \frac{\partial N'}{\partial x} = 2xy$$

$$\Rightarrow \int (x^3 + y^2 x + x^2) dx + \int 0 dy = c$$

$$\Rightarrow \frac{x^4}{4} + \frac{y^2 x^2}{2} + \frac{x^3}{3} = c$$

$$4. \text{ Solve } y(2xy+1) dx - x dy = 0$$

$$(2xy^2 + y) dx - x dy = 0$$

$$M = 2xy^2 + y \quad N = -x$$

$$\frac{\partial M}{\partial y} = 4xy + 1 \quad \frac{\partial N}{\partial x} = -1$$

$$\frac{1}{M} \left[\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right] = \frac{-1 - 4xy - 1}{y(2xy+1)} = -\frac{2(2xy+1)}{y(2xy+1)} = -\frac{2}{y} = g(y)$$

$$\Rightarrow I.F = e^{-\frac{2}{y} dy} = e^{-2 \log y} = e^{\log y^2} = y^2$$

$$\Rightarrow I.F \times ① = \left(\frac{2xy^2}{y^2} + \frac{y}{y^2} \right) dx - \left(\frac{x}{y^2} \right) dy = 0$$

$$(ax + by)dx - x/y^2 dy = 0$$

$$M' = ax + by \quad N' = -x/y^2$$

$$\frac{\partial M'}{\partial y} = -\frac{1}{y^2} \quad \frac{\partial N'}{\partial x} = -1/y^2$$

$$\frac{\partial M'}{\partial y} = -1/y^2 \quad \frac{\partial N'}{\partial x} = -1/y^2$$

$$\Rightarrow \int (ax + by) dx = c$$

$$\Rightarrow \frac{x^2}{2} + \frac{x}{y} = c$$

$$\Rightarrow x^2 + \frac{x}{y} = c$$

5. Solve $y(x+y)dx + (x+ay-1)dy = 0$

$$(xy + y^2)dx + (x + ay - 1)dy = 0 \rightarrow ①$$

$$M = xy + y^2 \quad N = x + ay - 1$$

$$\frac{\partial M}{\partial y} = x + 2y \quad \frac{\partial N}{\partial x} = 1$$

$$\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$$

$$\frac{1}{N} \left[\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x} \right] = \frac{x + 2y - 1}{x + 2y - 1} = 1 - f(x)$$

$$I.F = e^{\int f(x)dx} = e^{\int dx} = e^x$$

$$\Rightarrow I.F \times ① \Rightarrow e^x(xy + y^2)dx + e^x(x + ay - 1)dy = 0$$

$$(xye^x + e^x y^2)dx + (e^x x + aye^x - e^x)dy = 0$$

$$M' = xye^x + e^x y^2$$

$$N' = e^x x + aye^x - e^x$$

$$\frac{\partial M'}{\partial y} = xe^x + e^x y^2$$

$$\frac{\partial N'}{\partial x} = xe^x + aye^x$$

\therefore The given solution is EDE

$$\Rightarrow \int (xye^x + e^x y^2)dx = c$$

$$\Rightarrow y(xe^x - e^x) + y^2(e^x) = c$$

$$\Rightarrow e^x [(xy - y) + y^2] = c$$

6. Solve $(3x^2y^4 + 2xy)dx + (2x^3y^3 - x^2)dy = 0$

$$(3x^2y^4 + 2xy)dx + (2x^3y^3 - x^2)dy = 0$$

$$M = 3x^2y^4 + 2xy \quad N = 2x^3y^3 - x^2$$

$$\frac{\partial M}{\partial y} = 12x^2y^3 + 2x \quad \frac{\partial N}{\partial x} = 6x^2y^3 - 2x$$

$$\frac{1}{M} \left[\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right] = \frac{6x^2y^3 - 2x - 12x^2y^3 - 2x}{(3x^2y^4 + 2xy)} = \frac{2(x^2y^3 + 2x)}{y(3x^2y^3 + 2x)}$$

$$\Rightarrow I.F = e^{\int g(y)dy} = e^{\int 2/y dy} = e^{\log y^2} = y^2$$

$$\textcircled{1} \times I.F = y^2(3x^2y^4 + 2xy)dx + y^2(2x^3y^3 - x^2)dy = 0$$

$$(3x^2y^6 + 2xy^3)dx + 2x^3y^5 - x^2y^2 dy = 0$$

$$M' = 3x^2y^6 + 2xy^3 \quad N' = 2x^3y^5 - x^2y^2$$

$$\frac{\partial M'}{\partial y} = 18x^2y^5 + 6xy^2 \quad \frac{\partial N'}{\partial x} = 6x^2y^5 - 2xy^2$$

\therefore The Reducible D.E is

$$\Rightarrow \int (3x^2y^6 + 2xy^3)dx = C$$

$$\Rightarrow x^3y^6 + x^2y^3 = C$$

7. Solve $(y^4 + 2y)dx + (2y^3 + 2y^4 - 4x)dy = 0$

$$(y^4 + 2y)dx + (2y^3 + 2y^4 - 4x)dy = 0 \rightarrow \textcircled{1}$$

$$M = y^4 + 2y \quad N = 2y^3 + 2y^4 - 4x$$

$$\frac{\partial M}{\partial y} = 4y^3 + 2 \quad \frac{\partial N}{\partial x} = 4y^3 - 4$$

$$\frac{1}{M} \left[\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right] = \frac{(4y^3 - 4 - 4y^3 - 2)}{y^4 + 2y} = \frac{3(y^3 + 2)}{y(y^4 + 2)} \quad (\div \text{ by } y)$$

$$\frac{1}{M} \left[\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right] = \frac{3[y^4 + 2y]}{y(y^4 + 2y)} = \frac{3}{y} = g(y)$$

$$\Rightarrow I.F = e^{\int g(y)dy} = e^{3 \int 1/y dy} = e^{\log y^3} = y^3$$

$$I.F \times ① = y^3(y^4 + 2y)dx + y^3[2xy^3 + 2y^4 - 4x]dy = 0$$

The given Reductable D.E is

$$\Rightarrow \int (y^7 + 2y^4)dx + \int 2y^7 dy = c$$

$$\Rightarrow xy^7 + 2xy^4 + \frac{2y^8}{8} = c$$

$$\Rightarrow xy^7 + 2xy^4 + \frac{y^8}{4} = c$$

Linear differential equation of 1st degree and 1st degree

The general Linear differential equation of 1st degree
and 1st order

$$\frac{dy}{dx} + P(x)y = Q(x)$$

Identify P, Q and also find $I.F = e^{\int P(x)dx}$

Write the solution $y \times I.F = \int Q(x) I.F dx + c$
in other form.

The Solution of $\frac{dy}{dx} + P(y)x = Q(y)$

$$\therefore I.F \times x = \int Q(y) I.F dy + c$$

$$I.F = e^{\int P(y)dy}$$

1. Solve $\frac{dy}{dx} - \frac{y}{x} = 2x^2$

$$\frac{dy}{dx} + \left(-\frac{1}{x}\right)y = 2x^2$$

$$P = -1/x \quad Q = 2x^2$$

$$I.F = \left[e^{\int P(x)dx} \right] = e^{-\int 1/x dx} = e^{\log(1/x)} = \frac{1}{x}$$

$$I.F \times (y) = \int Q(x) I.F dx + c$$

$$\Rightarrow \frac{y}{x} = \int 2x^2 \frac{1}{x} dx + c$$

$$\Rightarrow \frac{y}{x} = \int 2x^2 + \frac{1}{x} dx + c$$

$$\Rightarrow \frac{y}{x} = \int 2x dx + C$$

$$\Rightarrow \frac{y}{x} = \int 2x dx + C$$

$$\Rightarrow \frac{y}{x} = x^2 + C$$

$$\Rightarrow y = x^2 + Cx$$

2. Solve $\frac{dy}{dx} + y \cot x = \cos x$

$$\frac{dy}{dx} + y \cot x = \cos x$$

$$P = \cot x \quad Q = \cos x$$

$$\Rightarrow I.F = e^{\int P(x) dx} = e^{\int \cot x dx} = e^{\log(\sin x)} = \sin x$$

$$\Rightarrow I.F \cdot y = \int Q(x) I.F dx + C$$

$$\Rightarrow y \sin x = \int \cos x \sin x dx + C$$

$$\Rightarrow y \sin x = \frac{1}{2} \int (\sin 2x) dx + C$$

$$\Rightarrow y \sin x = -\frac{\cos 2x}{4} + C$$

Bernoulli's differential equation

The general solution of Bernoulli's differential equation

$$\frac{dy}{dx} + P(x)y = Q(x)y^n \rightarrow ①$$

divide eqⁿ ① by y^n

$$eq^n ① \Rightarrow \frac{1}{y^n} \frac{dy}{dx} + P(x) \frac{1}{y^{n-1}} = Q(x) \rightarrow ②$$

$$\text{let } \frac{1}{y^{n-1}} = u$$

And differentiating w.r.t x we get

$$\Rightarrow (-n+1) y^{-n} \frac{dy}{dx} = \frac{du}{dx}$$

$$\frac{1}{y^n} \frac{dy}{dx} = \frac{1}{(1-n)} \frac{du}{dx}$$

$$\text{then eqn } ② \Rightarrow \frac{1}{1-n} \frac{du}{dx} + P(x)u = Q(x)$$

$$\frac{du}{dx} + (1-n)P(x)u = (1-n)Q(x)$$

$$\frac{du}{dx} + P'(x)u = Q'(x)$$

1. Solve $\frac{dy}{dx} + \frac{y}{x} = y^2 x$

$$\Rightarrow \frac{dy}{dx} + \left(\frac{y}{x}\right) = xy^2$$

$$\frac{1}{y^2} \frac{dy}{dx} + \left(\frac{1}{x}\right)\left(\frac{1}{y}\right)x \rightarrow ①$$

let $u = 1/y \Rightarrow ②$

$$D \rightarrow x$$

$$\Rightarrow \frac{du}{dx} = -\frac{1}{y^2} \left(\frac{dy}{dx} \right)$$

$$\Rightarrow -\frac{du}{dx} = \frac{1}{y^2} \left(\frac{dy}{dx} \right) \rightarrow ③$$

Apply in eqn (a) in eqn ①

$$\Rightarrow -\frac{du}{dx} + \frac{u}{x} = x$$

$$\Rightarrow \frac{du}{dx} + \left(-\frac{1}{x}\right)u = -x$$

$$P = -1/x, Q = -x$$

$$\Rightarrow I.F = e^{\int P(x) dx} = e^{-\int 1/x dx} = e^{\log 1/x} = 1/x$$

$$U \times I.F = \int Q(x) I.F dx + C$$

$$\Rightarrow \frac{U}{x} = \int -\frac{x}{x} dx + C$$

$$\Rightarrow \frac{U}{x} = -\int dx + C$$

$$\Rightarrow \frac{U}{x} = -x + C$$

$$\Rightarrow U = -x^2 + Cx$$

$$\Rightarrow \frac{1}{y} = x^2 + Cx$$

$$2. \text{ Solve } \frac{dy}{dx} - y \tan x = \frac{\sin x \cos^2 x}{y^2}$$

$$\Rightarrow \frac{dy}{dx} - y \tan x = \frac{\sin x \cos^2 x}{y^2}$$

$$\Rightarrow y^2 \frac{dy}{dx} + (-\tan x)y^3 = \sin x \cos^2 x \rightarrow ①$$

$$\text{let } u = y^3$$

$$\frac{du}{dx} = 3y^2 \frac{dy}{dx}$$

$$\frac{1}{3} \frac{du}{dx} = y^2 \frac{dy}{dx} \rightarrow @$$

apply eqn @ in ①

$$\frac{1}{3} \frac{du}{dx} + (-\tan x)u = \sin x \cos^2 x$$

$$\frac{du}{dx} + (-3\tan x)u = 3\sin x \cos^2 x$$

$$\text{let } P = -3\tan x \quad Q = 3\sin x \cos^2 x$$

$$\text{I.F.} = e^{\int P(x) dx} = e^{-\int 3\tan x dx} = e^{-3 \log(\sec x)} = e^{\log(1/\sec^3 x)} = \frac{1}{\sec^3 x} = \cos^3 x$$

$$u \times \text{I.F.} = \int Q(x) \text{ I.F.} dx + C$$

$$u \times \cos^3 x = \int 3\sin x \cos^2 x \cos^3 x dx + C$$

$$u \cos^3 x = 3 \int (\sin x \cos^5 x) dx + C$$

$$\text{let } t = \cos x$$

$$-dt = +\sin x dx$$

$$ut^3 = -3 \int t^3 dt + C$$

$$ut^3 = -\frac{3t^6}{6} + C$$

$$y^3 \cos^3 x = -\frac{\cos^6 x}{2} + C$$

$$3. \text{ Solve } xy(1+xy^2) \frac{dy}{dx} = 1$$

$$xy(1+xy^2) \frac{dy}{dx} = 1$$

$$\frac{dx}{dy} = xy + x^2y^3$$

$$\frac{dx}{dy} - xy = x^2y^3$$

$$\frac{1}{x^2} \frac{dx}{dy} + \left(-\frac{1}{x}\right)y = y^3 \rightarrow ①$$

$$\text{let } u = -\frac{1}{x}$$

$$D-y$$

$$\frac{du}{dy} = \frac{1}{x^2} \frac{dx}{dy}$$

$$\text{eqn } ① \Rightarrow \frac{du}{dy} + uy = y^3$$

$$Q = y^3$$

$$P = y$$

$$I.F = e^{\int P(y) dy} = e^{\int y dy} = e^{y^{2/2}}$$

$$u \times I.F = \int Q(y) I.F dy + C$$

$$ue^{y^{2/2}} = \int y^3 e^{y^{2/2}} dy + C$$

$$\frac{-e^{y^{2/2}}}{x} = \int (y^2 e^{y^{2/2}})(y dy) + C$$

$$\frac{-e^t}{x} = \int (2t e^t) dt + C$$

$$\frac{-e^t}{x} = 2 \int (t-1) dt + C$$

$$\frac{-e^t}{x} = 2(t-1)e^t + C$$

$$\frac{-e^{y^{2/2}}}{x} = 2(y^{2/2}-1) e^{y^{2/2}} + C$$

$$\frac{-e^{y^{2/2}}}{x} = y^2 e^{y^{2/2}} - 2e^{y^{2/2}} + C$$

4. Solve $dy + [x \sin y - x^3 \cos^2 y] dx = 0$

$$dy + [x \sin y - x^3 \cos^2 y] dx = 0$$

$$\frac{dy}{dx} + x \sin y = x^3 \cos^2 y$$

$$\frac{1}{\cos^2 y} \frac{dy}{dx} + \frac{x \sin y \cos y}{\cos^2 y} = x^3$$

$$\sec^2 y \frac{dy}{dx} + \tan y (\ln x) = x^3 \rightarrow ①$$

$$\text{Let } u = 2 \tan y$$

$$D \rightarrow x$$

$$\frac{du}{dx} = 2 \sec^2 y \frac{dy}{dx}$$

$$\frac{1}{2} \frac{du}{dx} = \sec^2 y \frac{dy}{dx} \rightarrow ②$$

Apply eqn ② in ①

$$\frac{1}{2} \frac{du}{dx} + ux = x^3$$

$$P = ux \quad Q = ux^3$$

$$I.F = e^{\int P(x) dx} = e^{\int ux dx} = e^{x^2}$$

$$ux I.F = \int Q(x) I.F dx + C$$

$$ue^{x^2} = \int e^{x^2} ux^3 dx + C$$

$$e^{x^2} x 2 \tan y = \int e^{x^2} x^2 2 \ln x dx + C$$

$$e^t x 2 \tan y = \int e^t (t) dt + C$$

$$= (t-1)e^t + C$$

$$2 \tan y e^{x^2} = (x^2 - 1) e^{x^2} + C$$

$$2 \tan y = (x^2 - 1) + \frac{C}{e^{x^2}}$$

5. Solve $(r \sin \theta - r^2) d\theta - \cos \theta dr = 0$

$$(r \sin \theta - r^2) d\theta - \cos \theta dr = 0$$

$$\cos \theta dr = (r \sin \theta - r^2) d\theta$$

$$\frac{dr}{d\theta} = \frac{r \sin \theta}{\cos \theta} - \frac{r^2}{\cos \theta}$$

$$\frac{dr}{d\theta} - r \tan \theta = -\frac{r^2}{\cos \theta}$$

$$\frac{1}{r^2} \frac{dr}{d\theta} + \left(-\frac{1}{r}\right) \tan \theta = -\frac{1}{\cos \theta} \rightarrow ①$$

$$\text{Let } U = -\frac{1}{r}$$

$$D \rightarrow \theta$$

$$\frac{du}{d\theta} = \frac{1}{r^2} \frac{dr}{d\theta} \rightarrow @$$

$$\text{eqn } ① \Rightarrow \frac{du}{d\theta} + u \tan \theta = -\frac{1}{\cos \theta}$$

$$P = \tan \theta \quad Q = -\frac{1}{\cos \theta}$$

$$I.F = e^{\int P(\theta) d\theta} = e^{\int \tan \theta d\theta} = e^{\log \sec \theta} = \sec \theta$$

The solution is

$$U \times I.F = \int Q(\theta) I.F d\theta + C$$

$$U \sec \theta = \int -\frac{1}{\cos \theta} \sec \theta d\theta + C$$

$$U \sec \theta = - \int \sec^2 \theta d\theta + C$$

$$-\frac{\sec \theta}{\theta} = -\tan \theta + C$$

$$\tan \theta - \frac{\sec \theta}{\theta} = C$$

6. Solve

$$\frac{dx}{dy} = \frac{x + \sqrt{xy}}{y} \quad \text{or} \quad \frac{dy}{dx} = \frac{y}{\sqrt{xy+x}}$$

$$\frac{dx}{dy} = \frac{x}{y} + \frac{\sqrt{xy}}{y}$$

$$\frac{dx}{dy} - \frac{x}{y} = \frac{\sqrt{xy}}{y}$$

$$\frac{dx}{dy} - \frac{x}{y} = \frac{\sqrt{x}}{\sqrt{y}}$$

$$\frac{1}{\sqrt{x}} \frac{dx}{dy} + (\sqrt{x}) \frac{1}{y} = \frac{1}{\sqrt{y}} \rightarrow ①$$

$$U = (-\sqrt{x})$$

$$D \rightarrow y$$

$$\frac{du}{dy} = -\frac{1}{2\sqrt{x}} \frac{dx}{dy}$$

$$\text{Eqn } ① \Rightarrow -\frac{\partial u}{\partial y} + \frac{u}{y} = \frac{1}{\sqrt{y}}$$

$$\frac{du}{dy} + \left(-\frac{1}{ay}\right)u = -\frac{1}{ay\sqrt{y}}$$

$$P = -\frac{1}{ay} \quad Q = -\frac{1}{a\sqrt{y}}$$

$$I.F = e^{\int P(x) dy} = e^{-\int \frac{1}{ay} dy} = \frac{1}{\sqrt{y}}$$

$$P = -\frac{1}{ay} \quad Q = -\frac{1}{a\sqrt{y}}$$

$$I.F = e^{\int Q(y) dy} = e^{-\int \frac{1}{a\sqrt{y}} dy} = \frac{1}{\sqrt{y}}$$

$$U.I.F = \int Q.I.F dy + C$$

$$-\frac{\sqrt{x}}{\sqrt{y}} = \frac{1}{2} \int \frac{1}{y} dy + C$$

$$\frac{\sqrt{x}}{\sqrt{y}} = \frac{1}{2} \log y + C$$

$$\frac{\sqrt{x}}{\sqrt{y}} - \frac{1}{2} \log y = C$$

$$7. x^3 \frac{dy}{dx} - x^2 y = -y^4 \cos x$$

$$x^3 \frac{dy}{dx} - x^2 y = -y^4 \cos x$$

$$\frac{dy}{dx} + \left(-\frac{1}{x^2}\right)y = -\frac{y^4 \cos x}{x^3}$$

$$\frac{1}{y^4} \frac{dy}{dx} + \left(-\frac{1}{x^2}\right) \frac{1}{y^3} = -\frac{\cos x}{x^3} \rightarrow ①$$

$$\text{put } u = \frac{1}{y^3}$$

$D \rightarrow x$

$$\frac{du}{dx} = -\frac{3}{y^4} \frac{dy}{dx}$$

$$-\frac{1}{3} \frac{du}{dx} = \frac{1}{y^4} \frac{dy}{dx}$$

$$-\frac{1}{3} \frac{du}{dx} - \left(\frac{u}{x}\right) = \frac{\cos x}{x^3}$$

$$\frac{dy}{dx} + \frac{3y}{x} = \frac{3\cos x}{x^3}$$

$$\text{Let } P = \frac{3}{x}, Q = \frac{3\cos x}{x^3}$$

$$I.F = e^{\int P dx} = e^{\int \frac{3}{x} dx} = e^{\log x^3} = x^3$$

The solution is given by

$$I.F \times u = \int Q I.F dx + c$$

$$\frac{x^3}{y^3} = \int \frac{3\cos x}{x^3} x^3 dx + c$$

$$\frac{x^3}{y^3} = 3 \int \cos x dx + c$$

$$\frac{x^3}{y^3} = 3 \sin x + c$$

orthogonal trajectory :-

Orthogonal trajectory is a curve which is \perp to the given curve

working procedure :-

Cartesian form :

Step ① :- Consider the given $F(x, y, c) = 0$

where c is the parameter

Step ② :- Construct D E which is free from the parameter

$$f(x, y, dy/dx) = 0$$

Replace $\frac{dy}{dx}$ as $-\frac{dx}{dy}$ in the above we get

$$g(x, y, -dx/dy) = 0$$

Solve the above and we get

$$G(x, y, c') = 0$$

which is O.T of Given Curve

Polar form :

Consider the curve $f(r, \theta, c) = 0$

where c is parameter

consider D.E which is free from parameter

$$f(\tau, \theta, d\tau/d\theta) = 0$$

Replace $\frac{d\tau}{d\theta}$ as $-\tau^2 \frac{d\theta}{d\tau}$ in the above we get

$$g(\tau, \theta, -\tau^2 \frac{d\theta}{d\tau}) = 0$$

Solve the above and get

$$G(\tau, \theta, c') = 0$$

which is require O.T of a given curve

1. Find orthogonal trajectory of parameter $y^2 = 4ax$

Given $y^2 = 4ax \rightarrow ①$

$D \rightarrow x$

$$\frac{\partial y}{\partial x} \frac{dy}{dx} = 4a$$

in eqn ② $y^2 = 2y \left(\frac{dy}{dx} \right) x$

$$y = 2x \left(\frac{dy}{dx} \right) \rightarrow ②$$

$$\frac{dy}{dx} = -\frac{dx}{2y}$$

$$y = 2x \frac{dx}{dy}$$

$$y dy = -2x dx$$

Integration O.B.S

$$\int y dy = -2 \int x dx$$

$$\frac{y^2}{2} = -\frac{2x^2}{2} + C$$

$$y^2 = -2x^2 + 2C$$

$$y^2 + 2x^2 = 2C$$

a. Find the orthogonal trajectory of $\frac{x^2}{a^2} + \frac{y^2}{b^2+\lambda} = 1$ where λ is parameter.

$$\frac{x^2}{a^2} + \frac{y^2}{b^2+\lambda} = 1 \rightarrow ①$$

$$\frac{\partial x}{\partial \lambda} + \frac{\partial y y_1}{b^2+\lambda} = 0 \rightarrow 2$$

$$\frac{x}{a^2} + \frac{\partial y y_1}{b^2+\lambda} = 0$$

$$\frac{y y_1}{b^2+\lambda} = -\frac{x}{a^2}$$

$$\frac{1}{b^2+\lambda} = -\frac{x}{a^2 y y_1}$$

$$b^2 - \lambda = a^2 y y_1 \rightarrow ②$$

$$\text{eqn } ① \Rightarrow \frac{x^2}{a^2} + \frac{y^2}{-a^2 y y_1} = 1$$

$$x^2 - \frac{xy}{y_1} = a^2$$

$$\text{Put } y_1 = -\frac{1}{y_1}$$

$$x^2 - \frac{xy}{(-1/y_1)} = a^2$$

$$x^2 + xy y_1 = a^2$$

$$xy y_1 = a^2 - x^2$$

$$yy_1 = \frac{a^2}{x} - x$$

$$y \frac{dy}{dx} = \frac{a^2}{x} - x$$

Integration obs

$$\int y dy = \int \frac{a^2}{x} dx - \int x dx$$

$$\frac{y^2}{2} = a^2 \log x - \frac{x^2}{2} + C$$

$$\int y dy = \int \frac{a^2}{x} dx - \int x dx$$

$$\frac{y^2}{2} = a^2 \log x - \frac{x^2}{2} + C$$

$$y^2 + x^2 = 2a^2 \log x + 2C$$

3. Show that $\frac{x^2}{a^2+\lambda} + \frac{y^2}{b^2+\lambda} = 1$ is a self orthogonal where λ is a parameter.

$$\frac{x^2}{a^2+\lambda} + \frac{y^2}{b^2+\lambda} = 1 \rightarrow ①$$

$$\frac{\partial x}{a^2+\lambda} + \frac{\partial y y_1}{b^2+\lambda} = 0$$

$$\frac{x}{a^2+\lambda} + \frac{y y_1}{b^2+\lambda} = 0$$

$$\Rightarrow x(b^2+\lambda) + (a^2+\lambda)(y y_1) = 0$$

$$\Rightarrow b^2 x + \lambda x + a^2 y y_1 + \lambda y y_1 = 0$$

$$\Rightarrow (x + y y_1) \lambda = -b^2 x - a^2 y y_1$$

$$\lambda = \frac{-b^2 x - a^2 y y_1}{x + y y_1}$$

$$a^2 + \lambda = a^2 - \frac{b^2 x - a^2 y y_1}{x + y y_1}$$

$$; \quad b^2 + \lambda = b^2 - \frac{b^2 x - a^2 y y_1}{x + y y_1}$$

$$\Rightarrow a^2 + \lambda = \frac{a^2(x + y y_1) - b^2 x - a^2 y y_1}{x + y y_1} ; \quad \Rightarrow b^2 + \lambda = \frac{b^2(x + y y_1) - b^2 x}{x + y y_1} + a^2 y y_1$$

$$\Rightarrow a^2 + \lambda = \frac{a^2 x + a^2 y y_1 - b^2 x - a^2 y y_1}{x + y y_1} ;$$

$$\Rightarrow b^2 + \lambda = -\frac{(a^2 - b^2) y y_1}{x + y y_1}$$

$$\Rightarrow a^2 + \lambda = \frac{(a^2 + b^2)x}{x + y y_1}$$

$$\text{Then eqn } ① \Rightarrow \frac{x^2}{(a^2 - b^2)x} + \frac{y^2}{-(a^2 - b^2)y y_1} = 1$$

$$x(x + y y_1) - y(x + y y_1)/y_1 = a^2 - b^2$$

$$(x - yy_1)(x - \frac{y}{y_1}) = a^2 - b^2 \rightarrow ②$$

$$\text{Put } y_1 = -\frac{1}{y_1}$$

$$(x - y/y_1)(x + yy_1) = a^2 - b^2 \rightarrow ③$$

The eqn ② and ③ are same

∴ The eqn is self orthogonal

4. Show that $y^2 = 4a(x+a)$ is self orthogonal a is a parameter.

$$\text{Consider } y^2 = 4a(x+a) \rightarrow ①$$

$$D \rightarrow x$$

$$2y \frac{dy}{dx} = 4a$$

$$a = \frac{2yy_1}{4}$$

$$a = \frac{yy_1}{2}$$

$$\text{in eqn } ① \quad y^2 = 2yy_1(x + yy_1/2)$$

$$y^2 = y(2xy_1 + yy_1^2)$$

$$y = 2xy_1 + yy_1^2 \rightarrow ②$$

This is D.E of given family

$$\text{Replacing } y_1 = -\frac{1}{y_1}$$

$$y = 2x\left(-\frac{1}{y_1}\right) + y\left(-\frac{1}{y_1}\right)^2$$

$$y = -\frac{2x}{y_1} + \frac{y}{y_1^2}$$

$$y = -\frac{2xy_1 + y}{y_1^2}$$

$$yy_1^2 + 2xy_1 + y = 0 \rightarrow ③$$

This is the D.E of orthogonal family which is same as ② being D.E of the given family

Thus the family of parabola $y^2 = 4a(x+a)$ is orthogonal

5. find the orthogonal trajectory of $r^n = a^n \cos n\theta$ where a is a parameter.

$$r^n = a^n \cos n\theta$$

$$D \rightarrow \theta$$

$$n r^{n-1} \frac{dr}{d\theta} = -a^n \sin n\theta (n)$$

$$\frac{r^n}{r} \frac{dr}{d\theta} = -\frac{a^n \sin n\theta}{r^n}$$

$$\frac{1}{r} \frac{dr}{d\theta} = \frac{-a^n \sin n\theta}{a^n \cos n\theta}$$

$$\frac{1}{r} \frac{dr}{d\theta} = -\tan n\theta$$

Replace

$$\frac{dr}{d\theta} = -r^2 \frac{d\theta}{dr}$$

$$\frac{1}{r} \left[r^2 \frac{d\theta}{dr} \right] = \tan n\theta$$

$$-\frac{1}{r} dr = \frac{1}{\tan n\theta} d\theta$$

Integration OBS

$$\int \frac{1}{r} dr = \int \frac{1}{\tan n\theta} d\theta$$

$$\log r = \int \cot n\theta d\theta$$

$$\log r = \int \cot n\theta d\theta$$

$$\log r = \frac{\log (\sin n\theta)}{n} + \log k$$

$$n \log r^n = \log \sin n\theta + n \log k$$

$$\log r^n = \log (\sin n\theta) + \log k^n$$

$$\log r^n = \log [\sin n\theta k^n]$$

$$r^n = k^n \sin n\theta$$

6. Find the orthogonal trajectory of $r^n = a^n \sin n\theta$ where a is parameter.

$$r^n = a^n \sin n\theta$$

$$\theta \rightarrow \Theta$$

$$\frac{dr}{d\theta} n r^{n-1} = a^n \cos n\theta$$

$$\frac{r^n}{r} \frac{dr}{d\theta} = a^n \cos n\theta$$

$$\frac{1}{r} \frac{dr}{d\theta} = \frac{a^n \cos n\theta}{r^n}$$

$$\frac{1}{r} \frac{dr}{d\theta} = \frac{a^n \cos n\theta}{a^n \sin \theta}$$

$$\frac{1}{r} \frac{dr}{d\theta} = \cot n\theta$$

$$\text{Replace } \left(\frac{dr}{d\theta} \right) = -r^2 \frac{d\theta}{dr}$$

$$\frac{1}{r} \left(-r^2 \frac{d\theta}{dr} \right) = \cot n\theta$$

$$\frac{1}{r} (dr) = \frac{1}{\cot n\theta} d\theta$$

Integration on B.S

$$-\int \frac{1}{r} dr = -\int \tan n\theta d\theta$$

$$\log k - \log r = + \underbrace{\log (\sec n\theta)}_n$$

$$n \log k - n \log r = \log \sec n\theta$$

$$\log k^n - \log r^n = \log \sec n\theta$$

$$\log(r^n) + \log k^n = \log(\sec n\theta)$$

$$\log \left(\frac{k^n}{r^n} \right) = \log(\sec n\theta)$$

$$r^n = k^n \frac{1}{\sec n\theta}$$

$$\Rightarrow r^n = k^n \cos n\theta$$

7. Find the orthogonal trajectory of $r^n \sin n\theta = a^n$ where a is parameter.

$$r^n \sin n\theta = a^n$$

$$D \rightarrow \theta$$

$$n r^{n-1} \frac{dr}{d\theta} \sin n\theta + r^n (\cos n\theta) (n) = 0$$

$$\frac{n}{r} \frac{dr}{d\theta} \sin n\theta = -r^n \cos n\theta$$

$$\frac{1}{r} \frac{dr}{d\theta} = -\frac{\cos n\theta}{\sin n\theta}$$

$$\frac{1}{r} \frac{dr}{d\theta} = -\cot n\theta$$

Replace $\frac{dr}{d\theta} = -r^2 \frac{d\theta}{dr}$

$$\frac{1}{r} - \frac{r^2 d\theta}{dr} = -\cot n\theta$$

$$r d\theta = \cot n\theta$$

$$\frac{1}{r} dr = \frac{1}{\cot n\theta} d\theta$$

Inegration O.B.S

$$\int \frac{1}{r} dr = \int \tan n\theta d\theta$$

$$\log r = \frac{\log (\sin n\theta)}{n} + \log k$$

$$n \log r = -\log (\cos n\theta) + n \log k$$

$$\log r^n + \log (\cos n\theta) = \log k^n$$

$$\log (r^n \cos n\theta) = \log k^n$$

$$r^n (\cos n\theta) = k^n$$

8. find the orthogonal trajectory of family of curve

$$r = 2a \cos \theta \text{ where 'a' is parameter.}$$

$$r = 2a \cos \theta \rightarrow ①$$

$$D \rightarrow \theta$$

$$\frac{dr}{d\theta} = -2a \sin \theta \rightarrow ②$$

divide eq ② and ①

$$r \frac{d\theta}{dr} = -\cot \theta$$

$$\frac{1}{r} \frac{dr}{d\theta} = -\tan \theta$$

$$\text{Replace } \left(\frac{dr}{d\theta} \right) = -r^2 \frac{d\theta}{dr}$$

$$-\frac{1}{r} r^2 \frac{d\theta}{dr} = -\tan \theta$$

$$r \frac{d\theta}{dr} = \tan \theta$$

$$\frac{1}{r} dr = \frac{1}{\tan \theta} d\theta$$

Integrate O BS

$$\int \frac{1}{r} dr = \int \frac{1}{\tan \theta} d\theta$$

$$\log r = \int \cot \theta d\theta$$

$$\log r = \log \sin \theta + \log k$$

$$\log r = \log k \sin \theta$$

$$r = k \sin \theta$$

9. Find the orthogonal trajectory of curve $r = a(1 - \cos \theta)$ where 'a' is parameter.

$$r = a(1 - \cos \theta) \rightarrow ①$$

$$D \rightarrow \theta$$

$$\frac{dr}{d\theta} = a \sin \theta \rightarrow ②$$

eqn ② by ①

$$\frac{1}{r} \frac{dr}{d\theta} = \frac{\sin \theta}{1 - \cos \theta}$$

$$\frac{1}{r} \frac{dr}{d\theta} = \frac{2 \sin \theta/2 \cos \theta/2}{2 \sin^2 \theta/2}$$

$$\frac{1}{r} \frac{dr}{d\theta} = \tan \theta/2$$

$$\text{Replace } \frac{dr}{d\theta} = -r^2 \frac{d\theta}{dr}$$

$$\frac{1}{r} \left(-r^2 \frac{d\theta}{dr} \right) = \tan \theta/2$$

$$r \frac{d\theta}{dr} = -\tan \theta/2$$

$$\frac{1}{r} dr = -\frac{1}{\tan(\theta/2)} d\theta$$

Integrate O.B.S

$$\int \frac{1}{r} dr = - \int \cot(\theta/2) d\theta$$

$$\log r = - \underbrace{\log(\sec(\theta/2))}_{(1/2)} + \log k$$

$$\log r = -2 \log \sec(\theta/2) + \log k$$

$$\log r = \log (\cos^2(\theta/2)) + \log k$$

$$\log r = \log (k \cos^2 \theta/2)$$

$$r = k \cos^2 \theta/2$$

$$r = k \left(\frac{1 + \cos \theta}{2} \right)$$

$$r = \frac{k}{2} [1 + \cos \theta]$$

$$r = b[1 + \cos \theta]$$

10. Find the orthogonal trajectory of $r^n \cos n\theta = a^n$

where 'a' is parameter.

$$r^n \cos n\theta = a^n$$

$$\theta \rightarrow \theta$$

$$n r^{n-1} \frac{dr}{d\theta} (\cos n\theta - r^n \sin n\theta (n)) = 0$$

$$n \frac{r^n}{r} \frac{dr}{d\theta} \cos n\theta = r^n \sin n\theta (n)$$

$$\frac{1}{r} \frac{dr}{d\theta} \cos n\theta = \sin n\theta$$

$$\frac{1}{r} \frac{dr}{d\theta} = \frac{\sin n\theta}{\cos n\theta}$$

$$\frac{1}{r} \frac{dr}{d\theta} = \tan n\theta$$

Replace $\frac{dr}{d\theta} = -\frac{dr}{r^2} \frac{d\theta}{dr}$

$$\frac{1}{r} \left[-r^2 \frac{d\theta}{dr} \right] = \tan n\theta$$

$$\frac{1}{\tan n\theta} d\theta = -\frac{1}{r} dr$$

Integration DBS

$$\int \cot n\theta d\theta = - \int \frac{1}{r} dr$$

$$\log(\sin n\theta) \frac{1}{n} = -\log r + \log k$$

$$\log(\sin n\theta) + n \log r = n \log k$$

$$\log(\sin n\theta r^n) = \log k^n$$

$$r^n \sin n\theta = k^n$$

Newton's law of cooling:-

let 'T' be the temperature of a body, T_0 be the temp. of surrounding medium at any time 't', by the property of the Newton's law of cooling the rate of temperature of body is directly proportional to difference between temperature of the and it's surrounding medium.

$$\frac{dT}{dt} \propto (T - T_0)$$

$$\frac{dT}{dt} = -K [T - T_0]$$

$$dT = -K [T - T_0] dt$$

Integration DBS

$$\int dT = - \int K [T - T_0] dt$$

$$\int \frac{1}{[T - T_0]} dT = -K \int dt$$

$$\log [T - T_0] = -Kt + C$$

$$T - T_b = e^{-kt} + C$$

$$T = T_b + e^{-kt} + C$$

$$T = T_b + e^{-kt} e^C$$

$$\boxed{T = T_b + \lambda e^{-kt}}$$

1. a copper ball originally at 180°C cool down 80°C in 20 min. 40°C what will be the ball after 40 min from the origin.

$$T_A = 40^\circ\text{C}$$

$$t = 0, T = 80^\circ\text{C}$$

$$t = 20, T = 60^\circ\text{C}$$

Suppose T be a temperature of copper ball and T_A is the temperature of the air at any time (t)
w.k.t

$$T = T_b + \lambda e^{-kt} \rightarrow ①$$

given temp. of air $T_b = 40^\circ\text{C}$

$$\text{eqn } ① \Rightarrow T = 40 + \lambda e^{-kt} \rightarrow ②$$

Given $T = 60^\circ\text{C}$ when $t = 0$

$$\text{eqn } ② \Rightarrow 60 = 40 + \lambda e^{-k(0)} \quad O$$

$$\Rightarrow \lambda = 40$$

$$\text{eqn } ③ \Rightarrow T = 40 + 40e^{-kt} \rightarrow ③$$

$$T = 60^\circ\text{C} \quad t = 20^\circ$$

$$60 = 40 + 40e^{-20k}$$

$$e^{-20k} = 0.5$$

$$-20k = \log_e(0.5)$$

$$k = 0.03465$$

$$③ \Rightarrow T = 40 + 40 e^{(-0.0346)t} \rightarrow ④$$

when $t = 40 \text{ min}$

$$\text{eq } ④ \Rightarrow T = 40 + 40 e^{(-0.0346)40}$$

$$T = 40 + 40 e^{-1.384}$$

$$T = 50^\circ\text{C}$$

2. The temperature of air is 30°C a metal ball cools from 100°C to 70°C in 15 min find How many long will it taken for metal to reach the temperature of 40°C

\Rightarrow Suppose T be the temperature of metal ball T_0 is temp. of air at time 't'

$$\text{W.K.T } T = T_0 + \lambda e^{-kt} \rightarrow ①$$

$$\text{given that } T_0 = 30^\circ\text{C}$$

$$T = 30 + \lambda e^{-kt} \rightarrow ②$$

$$\text{Given } T = 100^\circ\text{C} \text{ at } t = 0$$

$$② \Rightarrow 100 = 30 + \lambda e^{-k \cdot 0}$$

$$\lambda = 70$$

$$② \Rightarrow T = 30 + 70 e^{-kt} \rightarrow ③$$

$$T = 70^\circ\text{C}, t = 15 \text{ min}$$

$$③ \Rightarrow 70 = 30 + 70 e^{-(0.0373)t}$$

$$\frac{10}{70} = e^{-0.0373t}$$

$$-0.0373t = \ln(0.1428)$$

$$-0.0373t = -1.9463$$

$$t = 52 \text{ min } 18 \text{ sec}$$

3. A body is of heated 110°C and placed in air at 10°C after 1 hour, its temp. becomes 60°C how much additional time is required to cool.

\Rightarrow Suppose T be the temp. of the body, T_0 is temp. of air

$$T = T_0 + \lambda e^{-Kt} \rightarrow ①$$

$$T_0 = 10$$

$$① \Rightarrow T = 10 + \lambda e^{-Kt} \rightarrow ②$$

$$T = 110^\circ\text{C} \text{ at } t = 0$$

$$② \Rightarrow 110 = 10 + \lambda e^{0(K)}$$

$$\lambda = 100$$

$$T = 10 + 110e^{-60K}$$

$$\text{Given, } T = 60^\circ\text{C}, t = 60$$

$$③ \Rightarrow 60 = 10 + 110e^{-60K}$$

$$0.5 = e^{-60K}$$

$$-60K = -0.69314$$

$$K = 0.01155$$

$$③ \Rightarrow T = 10 + 100e^{-(0.01155t)} \rightarrow ④$$

$$\text{also given } T = 30 \text{, } t = ?$$

$$30 = 10 + 100e^{-(0.01155)t}$$

$$\ln(0.2) = -0.01155t$$

$$t = 139.345$$

$$t = 2 \text{ hours } 19 \text{ min } 34 \text{ sec}$$

4. A bottle of mineral water at a room temp. 72°F is kept in refrigerator where the temp. is 44°F after half of an hour, what are cooled 61°F , what is the temp. of mineral water another half an hour.

\Rightarrow From newton's law

$$T = T_0 + \lambda e^{-Kt} \rightarrow ①$$

$$T_0 = 44^\circ\text{F}$$

$$① \Rightarrow T = 44 + \lambda e^{-Kt} \rightarrow ②$$

$$T = 72^\circ\text{F}, \text{ when } t = 0$$

$$72 = 44 + \lambda e^{-0K}$$

$$\lambda = 28$$

$$\textcircled{2} \Rightarrow T = 44 + 28 e^{-Kt} \rightarrow \textcircled{3}$$

$$T = 61^\circ F, t = 30$$

$$\textcircled{3} \Rightarrow 61 = 44 + 28 e^{-30K}$$

$$K = 0.01663$$

$$\textcircled{3} \Rightarrow T = 44 + 28 e^{-0.01663(60)}$$

$$T = 44 + 10.323$$

$$T = 54.32^\circ F$$

5. A body is at $25^\circ C$ whose temp from $100^\circ C$ to $75^\circ C$ in one minute. find the temperature of a body at 3 minutes.

\Rightarrow From Newton's law

$$T = T_0 + \lambda e^{-Kt} \rightarrow \textcircled{1}$$

Given that air temp $T_0 = 25$

$$\textcircled{1} \Rightarrow T = 25 + \lambda e^{-Kt} \rightarrow \textcircled{2}$$

They given that $T = 100^\circ E$ at $t = 0$

$$\textcircled{2} \Rightarrow 100 = 25 + \lambda e^{-K(0)}$$

$$\lambda = 75$$

$$\textcircled{2} \Rightarrow T = 25 + 75 e^{-Kt} \rightarrow \textcircled{3}$$

$T = 75^\circ F$ at $t = 1$

$$\textcircled{3} \Rightarrow 75 e^{-K} + 25 = 75$$

$$e^{-K} = 0.666$$

$$K = 0.406$$

$$\textcircled{3} \Rightarrow T = 25 + 75 e^{-0.406t} \rightarrow \textcircled{4}$$

given $T = ?$ at $t = 3$

$$T = 25 + 75 e^{(-0.406 \times 3)}$$

$$T = 25 + 22.186$$

$$T = 47.186^\circ C$$

Flow of electricity :- [L-R circuits]

The electrical circuit may have 3 passive elements they are resistance (R), inductance (L), capacitance (C) and active element be voltage source with a emf source with (E) to a current (I) at any time 't'.

① A Series circuit with resistance (R), inductance (L) and emf (E) governed by D.E. $L \frac{di}{dt} + Ri = E$, where L, R constant and initial, the current is zero. find the current at any time (t).

⇒ Given that resistance b/w current, resistance, and inductance, E is $L \frac{di}{dt} + Ri = E$, R, L constant

$$L \frac{di}{dt} + Ri = E \rightarrow ①$$

$$\frac{di}{dt} + \left(\frac{R}{L}\right)i = \frac{E}{L} \rightarrow ②$$

$$P = R/L \quad Q = E/L$$

$$I.F. = e^{\int P dt} = e^{\int R/L dt} = e^{(R/L)t}$$

The Solution is given by $i \times I.F. = \int Q I.F. dt + C$

$$i \times e^{(R/L)t} = \int \frac{E}{L} e^{(R/L)t} dt + C$$

$$i \times e^{(R/L)t} = \frac{E}{L} \frac{e^{(R/L)t}}{R/L} + K$$

$$i \times e^{(R/L)t} = E \frac{e^{(R/L)t}}{R} + K \rightarrow ③$$

$$i(t) = \frac{E}{R} + K e^{-R/L t} \rightarrow ④$$

Given $i=0, t=0$

$$0 = \frac{E}{R} + K e^0$$

$$K = -\frac{E}{R} \rightarrow @$$

Substitute eqⁿ ① in ④

$$i(t) = \frac{E}{R} - \frac{E}{R} e^{-(R/L)t}$$

$$i(t) = \frac{E}{R} [1 - e^{-(R/L)t}]$$

2. An Inductance 2H and resistance 20Ω are connected to a series emf 'E' holds if the current is initial zero when $t=0$ find the current at end 0.01 sec if $E = 100$ volts?

W.K.T the D.E of a L-R circuit is

$$L \frac{di}{dt} + RI = E \rightarrow ①$$

Gives the current at any time 't' is

$$i(t) = \frac{E}{R} [1 - e^{-(R/L)t}] \rightarrow ②$$

Given $E = 100V$, $L = 2H$, $R = 20\Omega$ to find

Current $t = 0.01$ sec

$$i(0.01) = \frac{100}{20} [1 - e^{-(20/2)(0.01)}]$$

$$= 5 [1 - e^{-0.1}]$$

$$= 5 [1 - 0.9048]$$

$$i(0.01) = 5 [0.09516]$$

$$i(0.01) = 0.4758 \text{ Amp}$$

3. L-R Series circuit D.E acted on by an emf 'E' $\sin \omega t$, Satisfy if there is no current in the circuit initial, after the value of current any time (t).

⇒ The given D.E of L-R circuit

$$L \frac{di}{dt} + RI = E \sin \omega t$$

$$\frac{di}{dt} + \frac{R}{L} i = \frac{E}{L} \sin \omega t$$

$$P = R/L, Q = E/L \sin \omega t$$

$$I.F = e^{\int P dt} = e^{\int R/L dt} = e^{(R/L)t}$$

$$i \times I.F = \int Q I.F dt + K$$

$$ie^{(R/L)t} = \int \frac{E}{L} \sin \omega t e^{(R/L)t} dt + K$$

$$ie^{(R/L)t} = \frac{E}{L} \int e^{(R/L)t} \sin \omega t dt + K$$

$$ie^{(R/L)t} = \frac{E}{L} \frac{e^{(R/L)t}}{\sqrt{\frac{R^2}{L^2} + \omega^2}} \sin [\omega t - \tan^{-1}(\omega/R)] + K$$

$$ie^{(R/L)t} = \frac{E}{L} \frac{e^{(R/L)t}}{\sqrt{\frac{R^2}{L^2} + \omega^2 + L^2}} \sin [\omega t - \tan^{-1}(\omega L/R)] + K$$

$$i = \frac{E}{\sqrt{R^2 + \omega^2 L^2}} \sin [\omega t - \tan^{-1}(\omega L/R)] + K e^{(R/L)t} \rightarrow ①$$

$$W.K.T, t=0, i=0$$

$$0 = \frac{E}{\sqrt{R^2 + \omega^2 L^2}} \sin (\omega(0) - \tan^{-1}(\omega L/R)) + K e^{(R/L)0}$$

$$\frac{E}{\sqrt{R^2 + \omega^2 L^2}} \sin (-\phi) + K = 0 \quad \therefore -\phi = -\tan^{-1}(\omega L/R)$$

$$K = \frac{E}{\sqrt{R^2 + \omega^2 L^2}} (\sin \phi) \rightarrow @$$

Substitute eqn @ in ①

$$i = \frac{E}{\sqrt{R^2 + \omega^2 L^2}} \sin [\omega t - \phi] + \frac{E}{\sqrt{R^2 + \omega^2 L^2}} \sin \phi e^{(R/L)t}$$

4. Solve the D.E $L \frac{di}{dt} + Ri = 200 (\sin(300t))$ when $L=0.05$
 and $R=100\Omega$. find the value of current (I) at any time ' t ', initial no current in circuit what values does approach after long time.

$\Rightarrow \text{W.K.T}$

for D.E $L\frac{di}{dt} + Ri = 200 \sin(300t)$

The value of current at any time t in initial current is zero.

$$i(t) = \frac{E}{\sqrt{R^2 + \omega^2 L^2}} \sin(\omega t - \phi) + e^{(R/L)t} \sin \phi \rightarrow ①$$

by comparing given eqⁿ with general

$$R = 100, L = 0.05, E = 200, \omega = 300$$

$$\sqrt{R^2 + \omega^2 L^2} = \sqrt{(100)^2 + (300^2)(0.05)^2} = 101.1187$$

$$\phi = \tan^{-1}(\omega L / R) = \tan^{-1} \left(\frac{300 \times 0.05}{100} \right) = 8.5307$$

$$① \Rightarrow i(t) = \frac{200}{101.1187} \sin(300t - 8.5307) + e^{(100/0.05)t} \sin(8.5307)$$

$$\Rightarrow i(t) = 1.9778 \sin(300t - 8.5307) + e^{2000t} \sin(8.5307) \quad (0.14833)$$

Non Linear differential equation :-

For $P = \frac{dy}{dx}$ the polynomial

$$A_n P^n + A_1 P^{n-1} + A_2 P^{n-2} + A_3 P^{n-3} + \dots + A_{n-1} P^1 + A_0 = 0 \rightarrow ①$$

is called Linear D.E

For which $A_0, A_1, A_2, A_3, \dots, A_n$ are the f^n of x and y then the solution of eqⁿ can be done by the method of solvable P . as follow.

$$① \Rightarrow [P + f_1(x, y)] [P - f_2(x, y)], [P - f_3(x, y)] \dots [P - f_n(x, y)] = 0$$

$$\Rightarrow P - f_1(x, y) = 0 \dots P - f_2(x, y) = 0 \dots \dots [P - f_n(x, y)] = 0$$

$$\Rightarrow P = f_1(x, y) \dots P = f_2(x, y) \dots \dots P = f_n(x, y)$$

$$\Rightarrow F_1(x, y, c_1) = 0, \dots, F_2(x, y, c_2) = 0 \dots \dots F_n(x, y, c_n) = 0$$

$$\therefore F_1(x, y, c_1), F_2(x, y, c_2) \dots \dots . F_n(x, y, c_n) = 0$$

$$1. \left(\frac{dy}{dx} \right)^2 - 7\left(\frac{dy}{dx} \right) + 12 = 0$$

$$\left(\frac{dy}{dx} \right)^2 - 7\left(\frac{dy}{dx} \right) + 12 = 0$$

$$\text{Let } P = \frac{dy}{dx}$$

$$P^2 - 7P + 12 = 0$$

$$P(P-4) - 3(P-4) = 0$$

$$P-4=0, P-3=0$$

$$P=4, P=3$$

$$\frac{dy}{dx} = 4, \frac{dy}{dx} = 3$$

$$dy = 4dx, dy = 3dx$$

Integrate OBS

$$\int dy = \int 4dx$$

$$y = 4x + c_1$$

$$y = 4x - c_1 = 0$$

$$\int dy = \int 3dx$$

$$y = 3x + c_2$$

$$y = 3x - c_2 = 0$$

\therefore The Solution is

$$(y - 4x - c_1)(y - 3x - c_2) = 0$$

$$2. \text{ Solve } y \left(\frac{dy}{dx} \right)^2 + (x-y) \left(\frac{dy}{dx} \right) - x = 0$$

$$y \left(\frac{dy}{dx} \right)^2 + (x-y) \left(\frac{dy}{dx} \right) - x = 0 \rightarrow ①$$

$$\text{Let } P = \frac{dy}{dx}$$

$$\text{in eqn } ① \quad yP^2 + (x-y)P - x = 0$$

$$yP^2 + xP - yP - x = 0$$

$$yP^2 - yP + xP - x = 0$$

$$yP(P-1) + x(P-1) = 0$$

case ① :-

$$P-1=0$$

$$\Rightarrow P=1$$

$$\Rightarrow \frac{dy}{dx} = 1$$

$$\Rightarrow \frac{dy}{dx} = 1$$

$$dy = dx$$

$$\Rightarrow \int dy = \int dx$$

$$\Rightarrow y = x + C_1 = 0$$

case ② :-

$$yP + x = 0$$

$$\Rightarrow yP = -x$$

$$\Rightarrow y \frac{dy}{dx} = -x$$

$$\Rightarrow \int y dy = - \int x dx$$

$$\Rightarrow \frac{y^2}{2} = \frac{-x^2}{2} + C_2$$

$$\Rightarrow \frac{1}{2} [y^2 + x^2] = C_2$$

$$y^2 + x^2 - 2C_2 = 0$$

∴ The solution is given by

$$(y-x-C_1)(y^2+x^2-2C_2)=0$$

3. Solve

$$xy \left(\frac{dy}{dx} \right)^2 - (x^2+y^2) \frac{dy}{dx} + xy = 0$$

$$xy \left(\frac{dy}{dx} \right)^2 - (x^2+y^2) \left(\frac{dy}{dx} \right) + xy = 0 \rightarrow ①$$

$$\text{Let } P = \frac{dy}{dx}$$

$$\text{eqn } ① \Rightarrow xy(P^2) - (x^2+y^2)(P) + xy = 0$$

$$xp[yp-x] - y[yp-x] = 0$$

$$(xp-y)(yp-x) = 0$$

case ① :-

$$Px - y = 0$$

$$Px = y$$

$$x \frac{dx}{dy} = y$$

$$\int \frac{1}{x} dx = \int \frac{1}{y} dy$$

$$\log x = \log y + \log C_1$$

$$\log x = \log(y, C_1)$$

$$x = yC_1$$

$$\therefore \text{The solution is } (x-yC_1)(y^2-x^2-2C_2)=0$$

case ② :-

$$yP - x = 0$$

$$yP = x$$

$$y \frac{dy}{dx} = x$$

$$\int y dy = \int x dx$$

$$\frac{y^2}{2} - \frac{x^2}{2} + C_2$$

$$y^2 - x^2 - 2C_2 = 0$$

$$4. \text{ Solve } \frac{dy}{dx} - \frac{dx}{dy} = \frac{x}{y} - \frac{y}{x}$$

$$\frac{dy}{dx} - \frac{dx}{dy} = \frac{x}{y} - \frac{y}{x} \rightarrow ①$$

$$\text{Let } P = \frac{dy}{dx} \quad \frac{dx}{dy} = \frac{1}{P}$$

$$① \Rightarrow P - \frac{1}{P} = \frac{x}{y} - \frac{y}{x}$$

$$\frac{P^2 - 1}{P} = \frac{x^2 - y^2}{xy}$$

$$(P^2 - 1)xy = P(x^2 - y^2)$$

$$yxp^2 - xy = px^2 - py^2$$

$$xy(P^2 - 1) = P(x^2 - y^2) = 0$$

$$xyp^2 - xy - x^2p + y^2p = 0$$

$$ty[yp - x] + xp[yp - x] = 0$$

$$(xp + y)(yp - x) = 0$$

$$xp + y = 0$$

$$xp = -y$$

$$P = -\frac{y}{x}$$

$$\frac{dy}{dx} = -\frac{y}{x}$$

$$\int \frac{1}{y} dy = - \int \frac{1}{x} dx$$

$$\log y + \log c = \log c$$

$$\log\left(\frac{y}{x}\right) = \log c_1$$

$$\frac{y}{x} = c_1$$

$$y = xc_1$$

$$y = c_1 x = 0$$

$$yp - x = 0$$

$$yp = x$$

$$P = \frac{x}{y}$$

$$\frac{dy}{dx} = \frac{x}{y}$$

$$\int y dy = \int x dx$$

$$\frac{y^2}{2} = \frac{x^2}{2} + C_2$$

$$y^2 - x^2 = 2C_2$$

$$y^2 - x^2 - 2C_2 = 0$$

$$\therefore \text{The solution is } (y - c_1 x)(y^2 - x^2 - 2C_2) = 0$$

$$5. \text{ Solve } p^2 + 2py \cot x = y^2$$

$$p^2 + 2py \cot x - y^2 = 0 \rightarrow ①$$

$$p = -2y \cot x \pm \sqrt{4y^2 \cot^2 x - 4(-y^2)}$$

$$p = -2y \cot x \pm \frac{2y \sqrt{\cot^2 x + 1}}{2}$$

$$p = -y \cot x \pm y \cosec x$$

$$\text{Case ① :- } p = -y \cot x + y \cosec x$$

$$\frac{dy}{dx} = -y [\cosec x - \cot x] dx$$

$$\int \frac{1}{y} dy = \int [\cosec x - \cot x] dx$$

$$\log y = \log \tan(x/2) - \log \sin x + \log C_1$$

$$= \log \left[C_1 \frac{\tan(x/2)}{\sin x} \right]$$

$$= \log \left[C_1 \frac{\sin(x/2)/\cos(x/2)}{2 \sin(x/2) \cos(x/2)} \right]$$

$$= \log \left[\frac{C_1}{2 \cos^2(x/2)} \right]$$

$$= \log \left[\frac{C_1}{1 + \cos x} \right]$$

$$y = \frac{C_1}{(1 + \cos x)}$$

$$y(1 + \cos x) - C_1 = 0$$

$$\text{Case ② :- } p = -y \cot x - y$$

$$p = -y \cot x - y \cosec x$$

$$p = -y [\cot x + \cosec x]$$

$$\frac{dy}{dx} = -y [\cot x + \cosec x]$$

$$-\int \frac{1}{y} dy = \int (\cot x + \cosec x) dx$$

$$-\int \frac{1}{y} dy = \int (\omega t x + \cosec x) dx$$

$$-\log y = \log (\sin x) + \log \tan(x/2) + \log C_2$$

$$\log(\frac{1}{y}) = \log [\sin x \cdot \tan x/2 \cdot C_2]$$

$$\Rightarrow y(1 - \cos x) - C_2 = 0$$

\therefore The Solution is $(y(1 + \cos x) - C_1) \{y(1 - \cos x) - C_2\} = 0$

\Rightarrow Clairaut's equation :-

The D.E of the form $y = P(x) + f(x)$ is said to be Clairaut's form and it's solution is obtained by replacing $P \rightarrow c$ the solution becomes, $y = cx + f(x)$ is called general solution of Clairaut's and D.E differentiate partially w.r.t c, we get function in terms of c and again replace or substitute in given solution it is called singular solution.

①. Solve the Clairaut's equation $y = px + \frac{a}{p}$

$$\text{given } y = px + \frac{a}{p} \rightarrow ①$$

which is in Clairaut's form $y = P(x) + f(x)$

$$\therefore \text{The Solution is } y = cx + \frac{a}{c} \rightarrow ②$$

D \rightarrow c

$$x - \frac{a}{c^2} = 0$$

$$x = \frac{a}{c^2}$$

$$c^2 = \frac{a}{x}$$

$$c = \sqrt{\frac{a}{x}}$$

\therefore The Singular solution is

$$② \Rightarrow y = x \sqrt{\frac{a}{x}} + \frac{a}{\sqrt{x}}$$

$$\Rightarrow y = \sqrt{x} \sqrt{a} + \sqrt{x} \sqrt{a} \Rightarrow y = 2\sqrt{ax} \Rightarrow y^2 = 4ax,$$

a. Show that equation $xp^2 + px - py - y + 1 = 0$ where use Clairaut's equation.

$$xp^2 + px - py + 1 - y = 0$$

$$xp(p+1) - y(p+1) = -1$$

$$(p+1)(xp - y) = -1$$

$$px - y = \frac{-1}{(p+1)}$$

$$-y = -px - \frac{1}{p+1}$$

$$y = px + \frac{1}{p+1} \rightarrow ①$$

eqn ① is Clairaut's equation

$$y = cx + \frac{1}{c+1}$$

$$D \rightarrow x$$

$$0 = x - \frac{1}{(c+1)^2}$$

$$\frac{1}{(c+1)^2} = x$$

$$c+1 = \frac{1}{\sqrt{x}}$$

∴ The solution of the following

$$y = \left(\frac{1}{\sqrt{x}} - 1 \right) x + \frac{1}{\frac{1}{\sqrt{x}} - 1 + 1} \quad ②$$

$$y = \sqrt{x} - x + \sqrt{x}$$

$$y = 2\sqrt{x} - x$$

3. Show that the equation $xp^3 - yp^2 + 1 = 0$

$$xp^3 - yp^2 + 1 = 0$$

$$p^2 [xp - y] = -1$$

$$px - y = \frac{-1}{p^2}$$

$$-y = \frac{-1}{p^2} + px \rightarrow ③$$

① is a clairaut's equation

$$y = cx + \frac{1}{c^2}$$

partial D $\rightarrow c$

$$0 = x - \frac{2}{c^3}$$

$$\frac{2}{c^3} = x$$

$$c^3 = \frac{2}{x}$$

$$c = \sqrt[3]{\frac{2}{x}}$$

The Solution is

$$y = \sqrt[3]{\frac{2}{x}} + (x/2)^{2/3}$$

4. $(Px-y)(Py+x) = 2P$ is reduced to clairaut's form taking
 $x = x^2$ $y = y^2$

\Rightarrow Given that $(Px-y)(Py+x) = 2P \rightarrow ①$

$$x = x^2 \quad y = y^2$$

$$\frac{dx}{dx} = 2x \quad \frac{dy}{dy} = 2y$$

w.k.t

$$P = \frac{dy}{dx}$$

$$P = \frac{dy}{dx} \quad \frac{dy}{dx} \quad \frac{dx}{dx}$$

$$P = \frac{1}{2y} \quad P \neq 1$$

$$P = \frac{x}{y} \quad P$$

$$P = \frac{\sqrt{x}}{\sqrt{y}} \quad P$$

$$① \Rightarrow \left[\frac{\sqrt{x}}{\sqrt{y}} \quad P\sqrt{x} - \sqrt{y} \right] \left[\frac{\sqrt{x}}{\sqrt{y}} \quad P\sqrt{y} + \sqrt{x} \right] = 2\frac{\sqrt{x}}{\sqrt{y}} \quad P$$

$$\left[\frac{P_x - Y}{\sqrt{Y}} \right] \left[(P+1) \right] \sqrt{X} = \frac{2\sqrt{X}P}{\sqrt{Y}}$$

$$(P_x - Y)(P+1) = 2P$$

$$P_x - Y = \frac{2P}{P+1}$$

$$Y = P_x + \left(-\frac{2P}{P+1} \right)$$

The reduced eqn is

$$Y = CX + \left(-\frac{2C}{C+1} \right)$$

5. Solve $e^{4x}(P-1) + e^{2y}P^2 = 0$ by using Substitution $U = 2e^{2x}$
and $V = e^{2y}$

$$\Rightarrow e^{4x}(P-1) + e^{2y}P^2 = 0$$

$$U = e^{2x}, V = e^{2y}$$

$$\frac{dU}{dx} = 2e^{2x}$$

$$\frac{dV}{dy} = 2e^{2y}$$

W.K.T

$$P = \frac{dy}{dx}$$

$$P = \frac{dy}{dx}, \frac{dV}{du}, \frac{dU}{dx}$$

$$P = \frac{1}{2e^{2y}}, P = \frac{2e^{2x}}{2e^{2y}}$$

$$P = \frac{e^{2x}}{e^{2y}} P$$

$$P = \frac{U}{V} P$$

$$\textcircled{1} \Rightarrow e^{4x} \left(\frac{U}{V} P - 1 \right) + e^{2y} \left(\frac{U^2 P^2}{V^2} \right) = 0$$

$$U^2 \left(\frac{PV - V}{V} \right) + V \left(\frac{U^2}{V^2} \right) P^2 = 0$$

$$PU - V = -P^2$$

$$V = P^2 + PU$$

The resultant equation is

$$V = UC + C^2$$

$$e^{2y} = Ce^{2x} + C^2$$

6. Find the general and singular solution $x^2[y - px] = p^2y$ by taking into Clairaut's form using substitution

$$x = x^2, y = y^2$$

$$\Rightarrow x^2[y - px] = p^2y \rightarrow ①$$

$$x = x^2 \quad y = y^2$$

$$\frac{dx}{dx} = 2x \quad \frac{dy}{dy} = 2y$$

$$P = \frac{dy}{dx}$$

$$P = \frac{dy}{dy}$$

$$P = \frac{1}{\frac{dy}{dx}} \quad P \neq x$$

$$P = \frac{\sqrt{x}}{\sqrt{y}} \quad P$$

$$① \Rightarrow x \left[\sqrt{y} - \frac{\sqrt{x}}{\sqrt{y}} P \sqrt{x} \right] = \frac{x}{y} P^2 \sqrt{y}$$

$$x \left[\frac{y - xp}{\sqrt{y}} \right] = \frac{xp^2}{\sqrt{y}} \quad 0$$

$$y - px = p^2$$

∴ The solution is in Clairaut's solution

$$y^2 = cx^2 + c^2$$

$$y = cx + c^2 \rightarrow ②$$

$$D \rightarrow c$$

$$0 = x^2 + 2cx$$

$$2c = x^2$$

$$c = x^2/2$$

∴ The singular solution is

$$y^2 = \left(-\frac{x^2}{2}\right) x^2 + \left(\frac{x^2}{2}\right)^2$$

$$y^2 = -\frac{2x^4 + x^4}{4}$$

$$4y^2 = -x^4$$

$$4y^2 + x^4 = 0$$

7. Solve the $y^2(y - px) = x^4 p^2$ by reducing it is in Clariat's form, taking the substitution $x = \frac{1}{z}$, $y = \frac{1}{z}$
 \Rightarrow Given,

$$y^2(y - px) = x^4 p^2 \rightarrow ①$$

$$x = \frac{1}{z}, y = \frac{1}{z}$$

$$\Rightarrow \frac{dx}{dz} = -\frac{1}{z^2}, \frac{dy}{dz} = -\frac{1}{z^2}$$

$$\frac{dy}{dx} = \frac{dy}{dz} \cdot \frac{dz}{dx}$$

$$\Rightarrow \frac{dy}{dx} = \frac{dy}{dz} \left(\frac{dz}{dx} \right) \left(\frac{dx}{dz} \right)$$

$$\Rightarrow \frac{dy}{dz} = \frac{1}{(-1/z^2)} P \cdot \left(-\frac{1}{z^2}\right)$$

$$P = \frac{Py^2}{x^2}$$

$$\Rightarrow P = \frac{P/y^2}{1/x^2}$$

$$\Rightarrow P = \frac{x^2}{y^2} P$$

$$① \Rightarrow \frac{1}{y^2} \left[\frac{1}{y} - \frac{x^2}{y^2} P \frac{1}{x} \right] = \frac{1}{x^4} \frac{x^4}{y^4} P^2$$

$$\Rightarrow \frac{1}{y^2} \left[\frac{1}{x} - \frac{px}{y^2} \right] = \frac{p^2}{y^4}$$

$$\Rightarrow \frac{[y - px]}{y^4} = \frac{p^2}{y^4}$$

$$\Rightarrow y - px = p^2$$

$$\Rightarrow px + p^2 = y \Rightarrow ②$$

\therefore ② is clariat's equation

\therefore The Solution is,

$$y = cx + c^2$$

$$\frac{1}{y} = \frac{c}{x} + c^2 //$$