

Module-4

Infinite Series :- If u_n is a function of n , defined for all integral values of n , an expression of the form $u_1 + u_2 + u_3 + \dots + u_n + \dots$ containing infinite numbers of terms is called an Infinite Series and is usually denoted by $\sum_{n=1}^{\infty} u_n$ (or) $\sum u_n$ where u_n is the n^{th} term of Series (or) the General term of the Infinite Series.

Suppose S_n is the sum of the first n terms of a Series and is denoted by $S_n = u_1 + u_2 + u_3 + \dots + u_n$

We know that for a geometric Series having the 1st term a and the common ratio r . Then the S_n becomes

$$S_n = \frac{a(1-r^n)}{1-r} \quad r \neq 1 \quad \text{and}$$

$$S_n = \frac{a(r^n - 1)}{r - 1} \quad r > 1$$

Note

1. Suppose S_n be the sum of n numbers of terms of a Series, then if $\lim_{n \rightarrow \infty} S_n = l$, where l is a finite value, then the given Series is called the Convergent and if $\lim_{n \rightarrow \infty} S_n = \infty$. Then we say that the given Series is a Divergent.

2.

If $\sum u_n$ is the sum of the positive terms (or) a Series of positive terms. If $\lim_{n \rightarrow \infty} (u_n)^{1/n} = l$ is a finite and the Series is Convergent if $l < 1$ and Divergent if $l > 1$ and the test fails if $l = 1$

3] a) $\lim_{n \rightarrow \infty} n^{1/n} = 1$

b) $\lim_{n \rightarrow \infty} \left(1 + \frac{1}{n}\right)^n = e = 2.7$

c) $\lim_{n \rightarrow \infty} \left(1 + \frac{x}{n}\right)^n = e^x$

1] Test the Convergence of a Series $1 + \frac{1}{3} + \frac{1}{3^2} + \frac{1}{3^3} + \dots$

Soln :-

$S_n = 1 + \frac{1}{3} + \frac{1}{3^2} + \frac{1}{3^3} + \dots$ is a geometric Series
with common ratio $\frac{1}{3} < 1$

$$S_n = \frac{a(1 - r^n)}{1 - r}$$

$$S_n = \frac{1 \left(1 - \left(\frac{1}{3}\right)^n\right)}{1 - \frac{1}{3}}$$

$$S_n = \frac{\left[1 - \frac{1}{3^n}\right]}{\frac{2}{3}}$$

$$S_n = \frac{3}{2} \left[1 - \frac{1}{3^n}\right]$$

$$\Rightarrow \lim_{n \rightarrow \infty} \frac{3}{2} \left[1 - \frac{1}{3^n}\right]$$

$$\Rightarrow \frac{3}{2} \lim_{n \rightarrow \infty} \left[1 - \frac{1}{3^n}\right]$$

$$\Rightarrow \frac{3}{2} \left[1 - \frac{1}{\infty}\right]$$

$$\Rightarrow \frac{3}{2} (1 - 0)$$

$$\text{As } n \rightarrow \infty, S_n = \frac{3}{2} = \text{finite}$$

So the Series is Convergent

2] Test the convergence of the series $\frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \frac{1}{3 \cdot 4} + \dots$

Soln

$$\text{let } S = \frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \frac{1}{3 \cdot 4} + \dots$$

Here the n^{th} term of a series is $u_n = \frac{1}{n(n+1)} = \frac{1}{n} - \frac{1}{n+1}$

$$S_n = \sum u_n$$

$$\Rightarrow S_n = \sum \left| \frac{1}{n} - \frac{1}{n+1} \right|$$

$$\begin{aligned} \Rightarrow S_n &= \left(1 - \frac{1}{2} \right) + \left(\frac{1}{2} - \frac{1}{3} \right) + \left(\frac{1}{3} - \frac{1}{4} \right) + \dots \\ &\quad + \left(\frac{1}{n-1} - \frac{1}{n} \right) + \left(\frac{1}{n} - \frac{1}{n+1} \right) \end{aligned}$$

$$\Rightarrow S_n = 1 - \frac{1}{n+1}$$

$$\text{lt}_{n \rightarrow \infty} S_n = \text{lt}_{n \rightarrow \infty} \frac{n(n+1)}{2}$$

$$S_n = \frac{1}{2} \text{lt}_{n \rightarrow \infty} n^2 \left[1 + \frac{1}{n} \right] = \infty$$

Does the series is divergent

3] Test the convergence of $1 + 2 + 2^2 + 2^3 + \dots$

Soln let $S = 1 + 2 + 2^2 + 2^3 + \dots$ is an infinite geometric ratio $r > 1$

$$S_n = 1 + 2 + 2^2 + 2^3 + \dots + 2^n$$

$$S_n = \frac{a(r^n - 1)}{r - 1}$$

$$S_n = \frac{2^n - 1}{2 - 1}$$

$$S_n = 2^n - 1$$

$$\text{lt}_{n \rightarrow \infty} S_n = \text{lt}_{S \rightarrow \infty} 2^n - 1 = \infty$$

∴ Does the series is divergent

4] Test The Convergent of $1+2+2^2+2^3+\dots$

soln let $S = 1+2+2^2+2^3+\dots+2^n$ is an infinite Geometric series ratio $r > 1$

$$S_n = 1+2+2^2+2^3+\dots+2^n$$

$$\frac{S_n - 1}{r-1}$$

$$\frac{2^n - 1}{2 - 1}$$

$$= 2^n - 1$$

$$\lim_{n \rightarrow \infty} S_n = \lim_{n \rightarrow \infty} 2^n - 1$$

$$= \infty$$

Does the series is divergent

Cauchy's root test

5] find the nature of the series $\sum_{n=1}^{\infty} a^{n^2} \cdot x^n$. $a < 1$

soln :- $\sum_{n=1}^{\infty} u_n = \sum_{n=1}^{\infty} a^{n^2} \cdot x^n$ $a < 1$

$$u_n = a^{n^2} x^n$$

$$\Rightarrow (u_n)^{1/n} = (a^{n^2} x^n)^{1/n} = (a^{n^2})^{1/2} \cdot (x^n)^{1/n}$$

$$\Rightarrow (u_n)^{1/n} = a^n \cdot x$$

$$\Rightarrow \lim_{n \rightarrow \infty} u_n^{1/n} = \lim_{n \rightarrow \infty} a^n \cdot x$$

$$\lim_{n \rightarrow \infty} u_n^{1/n} = x \lim_{n \rightarrow \infty} a^n = x(0) = 0$$

$$= \lim_{n \rightarrow \infty} u_n = 0 < 1 \quad \therefore \text{Does the series is convergent}$$

Q) Find the nature of the series $\sum_{n=1}^{\infty} \left(1 + \frac{1}{n}\right)^{n^2}$

Soln :-

$$\sum_{n=1}^{\infty} u_n = \sum_{n=1}^{\infty} \left(1 + \frac{1}{n}\right)^{n^2}$$

$$\Rightarrow u_n = \left(1 + \frac{1}{n}\right)^{n^2}$$

$$\Rightarrow (u_n)^{1/n} = \left| \left(1 + \frac{1}{n}\right)^{n^2} \right|^{1/n}$$

$$\Rightarrow (u_n)^{1/n} = \left[1 + \frac{1}{n}\right]^n$$

$$\lim_{n \rightarrow \infty} (u_n)^{1/n} = \lim_{n \rightarrow \infty} \left(1 + \frac{1}{n}\right)^n = e = 2.718 > 1$$

Does the series is Divergent

Q) Test from Convergent $\sum_{n=1}^{\infty} \left(n + \frac{1}{n}\right)^{n^2} \cdot \left(\frac{1}{3^n}\right)$

Soln :-

$$\sum_{n=1}^{\infty} u_n = \sum_{n=1}^{\infty} \left| \frac{n+1}{n} \right|^{n^2} \cdot \frac{1}{3^n}$$

$$\Rightarrow u_n = \left(\frac{n+1}{n}\right)^{n^2} \cdot \frac{1}{3^n}$$

$$\Rightarrow u_n = \left(1 + \frac{1}{n}\right)^{n^2} \cdot \frac{1}{3^n}$$

$$\Rightarrow (u_n)^{1/n} = \left[\left(1 + \frac{1}{n}\right)^{n^2} \cdot \frac{1}{3^n} \right]^{1/n}$$

$$\Rightarrow (u_n)^{1/n} = \left[\left(1 + \frac{1}{n}\right)^{n^2} \right]^{1/n} \cdot \left[\frac{1}{3^n}\right]^{1/n}$$

$$\Rightarrow (u_n)^{1/n} = \left(1 + \frac{1}{n}\right)^n \cdot \frac{1}{3}$$

$$\Rightarrow (u_n)^{1/n} = \frac{1}{3} \left(1 + \frac{1}{n}\right)^n$$

$$\lim_{n \rightarrow \infty} (u_n)^{1/n} = \frac{1}{3} \left(1 + \frac{1}{n}\right)^n = \frac{e}{3} = \frac{2.718}{3} = 0.906$$

Does the Series is Convergent

8] Discuss the Convergent $\sum_{n=1}^{\infty} \frac{(n+1)^n \cdot x^n}{n(n+1)}$

Soln :- $u_n = \frac{(n+1)^n \cdot x^n}{n(n+1)}$

$$\Rightarrow u_n = \frac{n^n \left(1 + \frac{1}{n}\right)^n \cdot x^n}{n^n \cdot n^1}$$

$$\Rightarrow u_n = \frac{\left(1 + \frac{1}{n}\right)^n \cdot x^n}{n}$$

$$\Rightarrow (u_n)^{1/n} = \left[\frac{\left(1 + \frac{1}{n}\right)^n \cdot x^n}{n} \right]^{1/n}$$

$$\Rightarrow u_n^{1/n} = \left[\left(1 + \frac{1}{n}\right)^n \right]^{1/n} \cdot (x^n)^{1/n}$$

$$\Rightarrow u_n^{1/n} = \frac{\left(1 + \frac{1}{n}\right) \cdot x}{n^{1/n}}$$

$$\Rightarrow u_n^{1/n} = \frac{x \left(1 + \frac{1}{n}\right)}{n^{1/n}}$$

$$\underset{n \rightarrow \infty}{\text{Lt}} u_n^{1/n} = x \cdot \underset{n \rightarrow \infty}{\text{Lt}} \frac{1 + 1/n}{n^{1/n}} = x \frac{(1+0)}{1} = x$$

$$\underset{n \rightarrow \infty}{\text{Lt}} u_n^{1/n} = x$$

$$\sum u_n = \begin{cases} \text{Convergent if } x < 1 \\ \text{Divergent if } x > 1 \end{cases}$$

'D' alembert's ratio test

Step 1 :- find the n^{th} term of Series say U_n

Step 2 :- find U_{n+1}

Step 3 :- find $\lim_{n \rightarrow \infty} \frac{U_{n+1}}{U_n} = l$ (say)

Step 4 :- If $l < 1$, then $\sum U_n$ is convergent

If $l > 1$, then $\sum U_n$ is divergent

If $l = 1$ then test fails

Q] Test for Convergence the Series $\frac{1^2}{2^1} + \frac{2^2}{2^2} + \frac{3^2}{2^3} + \frac{4^2}{2^4} + \dots$

$$U_n = \frac{n^2}{2^n}$$

$$U_{n+1} = \frac{(n+1)^2}{2^{n+1}}$$

$$\Rightarrow U_{n+1} = \frac{(n+1)^2}{\frac{n^2}{2^n}}$$

$$\Rightarrow U_{n+1} = \frac{(n+1)^2}{2^{n+1}} \times \frac{2^n}{n^2}$$

$$\Rightarrow \frac{U_{n+1}}{U_n} = \frac{n^2(1+\frac{1}{n})^2}{2^n \cdot 2} \times \frac{2^n}{n^2}$$

$$\Rightarrow \frac{U_{n+1}}{U_n} = \frac{(1+\frac{1}{n})^2}{2} = \frac{1}{2} \left(1+\frac{1}{n}\right)^2$$

$$\lim_{n \rightarrow \infty} \frac{U_{n+1}}{U_n} = \frac{1}{2} \lim_{n \rightarrow \infty} \left(1+\frac{1}{n}\right)^2 = \frac{1}{2}(1+0) = \frac{1}{2} < 1$$

$$\lim_{n \rightarrow \infty} \frac{U_{n+1}}{U_n} = \frac{1}{2} < 1$$

Does the Series is Convergent

10) Test for the convergent (o) divergent the Series $\frac{3}{4+1} + \frac{3^2}{4^2+1} + \frac{3^3}{4^3+1} \dots$

Sohm :-

$$S = \frac{3}{4+1} + \frac{3^2}{4^2+1} + \frac{3^3}{4^3+1} + \dots$$

The n^{th} Term is $U_n = \frac{3^{n+1}}{4^{n+1} + 1}$

$$U_{n+1} = \frac{3^{n+2}}{4^{n+2} + 1}$$

$$\frac{U_{n+1}}{U_n} = \frac{3^{n+2}}{4^{n+2} + 1} \times \frac{4^{n+1} + 1}{3^{n+1}}$$

$$\frac{U_{n+1}}{U_n} = \frac{3 \cdot 3^{n+1}}{4 \cdot 4^{n+1} + 1} \times \frac{4^{n+1} + 1}{3^{n+1}}$$

$$= 3 \left[\frac{4^{n+1} + 1}{4 \cdot 4^{n+1} + 1} \right]$$

$$= 3 \left\{ \frac{4^{n+1} \left[1 + \frac{1}{4^{n+1}} \right]}{4^{n+1} \left[4 + \frac{1}{4^{n+1}} \right]} \right\}$$

$$= 3 \cdot \left[\frac{1 + \frac{1}{4^{n+1}}}{4 + \frac{1}{4^{n+1}}} \right]$$

$$\underset{n \rightarrow \infty}{\text{It}} \quad \frac{U_{n+1}}{U_n} = 3 \cdot \underset{n \rightarrow \infty}{\text{It}} \quad \frac{1 + \frac{1}{4^{n+1}}}{4 + \frac{1}{4^{n+1}}}$$

$$\underset{n \rightarrow \infty}{\text{It}} \quad \frac{U_{n+1}}{U_n} = 3 \left(\frac{1+0}{4+0} \right)$$

$$\underset{n \rightarrow \infty}{\text{It}} \quad \frac{U_{n+1}}{U_n} = \frac{3}{4} < 1 \quad \text{Convergent}$$

Q) Discuss the nature of the series $\sqrt{\frac{1}{2}}x + \sqrt{\frac{2}{3}}x^2 + \sqrt{\frac{3}{4}}x^3 + \dots$

$$\Rightarrow S = \sum_{n=1}^{\infty} \sqrt{\frac{n}{n+1}} x^n$$

$$\therefore u_n = \sqrt{\frac{n}{n+1}} x^n$$

$$u_{n+1} = \sqrt{\frac{n+1}{n+2}} x^{n+1}$$

$$\frac{u_{n+1}}{u_n} = \frac{\sqrt{\frac{n+1}{n+2}} x^{n+1}}{\sqrt{\frac{n}{n+1}} x^n}$$

$$\Rightarrow \frac{u_{n+1}}{u_n} = \frac{\sqrt{n+1}}{\sqrt{n+2}} \cdot x^n \cdot x^1 = \frac{\sqrt{n+1}}{\sqrt{n}} \cdot \frac{1}{x^n}$$

$$\Rightarrow \frac{u_{n+1}}{u_n} = \frac{x^{n+1}}{\sqrt{n(n+2)}}$$

$$= \frac{x \cdot n(1 + \frac{1}{n})}{\sqrt{n} \sqrt{1 + \frac{2}{n}}}$$

$$\frac{u_{n+1}}{u_n} = \frac{x(1 + \frac{1}{n})}{\sqrt{1 + \frac{2}{n}}}$$

$$\text{Ht } \frac{u_{n+1}}{u_n} = \text{Ht}_{n \rightarrow \infty} \frac{1 + \frac{1}{n}}{\sqrt{1 + \frac{2}{n}}} = x$$

$$\therefore \sum u_n = \begin{cases} \text{Converges for } x < 1 \\ \text{Diverges for } x > 1 \\ \text{fail at } x = 1 \end{cases}$$

12] Discuss the nature of the series $\frac{1}{2} + \left(\frac{2}{3}\right)x + \left(\frac{3}{4}\right)x^2 + \left(\frac{4}{5}\right)x^3 + \dots$

$$\Rightarrow S = \frac{1}{2} + \left(\frac{2}{3}\right)x + \left(\frac{3}{4}\right)x^2 + \left(\frac{4}{5}\right)x^3 + \dots$$

$$S = \sum_{n=1}^{\infty} \left(\frac{n}{n+1}\right) x^{n-1}$$

$$U_n = \frac{n}{n+1} x^{n-1}$$

$$U_{n+1} = \frac{n+1}{n+2} x^{(n+1)-1}$$

$$U_{n+1} = \frac{n+1}{n+2} x^n$$

$$\Rightarrow \frac{U_{n+1}}{U_n} = \frac{\frac{n+1}{n+2} x^n}{\frac{n}{n+1} x^{n-1}}$$

$$\Rightarrow \frac{n+1}{n+2} x^n \cdot \frac{n+1}{n} \cdot \frac{1}{x^n} \cdot x^{-1}$$

$$\Rightarrow \frac{x(n+1)^2}{n(n+2)}$$

$$\Rightarrow \frac{x n^2 \left[1 + \frac{1}{n}\right]^2}{n^2 \left[1 + \frac{2}{n}\right]}$$

$$\lim_{n \rightarrow \infty} \frac{U_{n+1}}{U_n} = x \lim_{n \rightarrow \infty} \frac{\left[1 + \frac{1}{n}\right]^2}{\left[1 + \frac{2}{n}\right]}$$

$$\Rightarrow x \left[\frac{(1+0)}{(1+0)} \right] = x$$

$\sum U_n$ = Convergence when $x < 1$

Divergence when $x \geq 1$

(13) Test for convergence (or) divergence of the series

$$\frac{1}{2\sqrt{1}} + \frac{x^2}{3\sqrt{2}} + \frac{x^4}{4\sqrt{3}} + \frac{x^6}{5\sqrt{4}} + \dots$$

$$\Rightarrow S = \sum_{n=1}^{\infty} \frac{x^{2n-2}}{(n+1)\sqrt{n}}$$

$$U_n = \frac{x^{2n-2}}{(n+1)\sqrt{n}}$$

$$U_{n+1} = \frac{x^{2(n+1)-2}}{(n+2)\sqrt{n+1}}$$

$$U_{n+1} = \frac{x^{2n}}{(n+2)\sqrt{n+1}}$$

$$\therefore \frac{U_{n+1}}{U_n} = \frac{\frac{x^{2n}}{(n+2)\sqrt{n+1}}}{\frac{x^{2n-2}}{(n+1)\sqrt{n}}}$$

$$\Rightarrow \frac{U_{n+1}}{U_n} = \frac{x^{2n}}{(n+2)\sqrt{n+1}} \times \frac{(n+1)\sqrt{n}}{x^{2n-2}}$$

$$\Rightarrow \frac{x^2(n+1)\sqrt{n}}{(n+2)\sqrt{n+1}}$$

$$\Rightarrow \frac{x^2\sqrt{n+1}\cancel{\sqrt{n+1}}\sqrt{n}}{(n+2)\cancel{\sqrt{n+1}}}$$

$$\frac{U_{n+1}}{U_n} = \frac{x^2\sqrt{n(n+1)}}{n+2}$$

$$= \frac{x^2 n(\sqrt{1+\frac{1}{n}})}{n(1+\frac{2}{n})}$$

$$= \frac{x^2 \sqrt{1+\frac{1}{n}}}{1+\frac{2}{n}}$$

$$\lim_{n \rightarrow \infty} = x^2 \lim_{n \rightarrow \infty} \frac{\sqrt{1+\frac{1}{n}}}{1+\frac{2}{n}} = x^2$$

$$\sum_n = \begin{cases} \text{Convergence when } x^2 < 1 \\ \text{Divergence when } x^2 > 1 \end{cases}$$

Power Series Solution of Second Order

Consider a 2nd order differential equation in a form

$P_0(x) \frac{d^2y}{dx^2} + P_1(x) \frac{dy}{dx} + P_2(x)y = 0 \rightarrow \textcircled{1}$ where $P_0(x), P_1(x), P_2(x)$ are the polynomial of the x and $P_0(x) \neq 0$ at $x=0$

Step 1 :- Write the Soln of Eqn $\textcircled{1}$ as a Series

$$y = \sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \dots \rightarrow \textcircled{2}$$

Step 2 :- find $y' = \sum_{n=1}^{\infty} n a_n x^{n-1}$, $y'' = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}$

Substitute the y, y', y'' in Eqn $\textcircled{1}$ which results in an infinite Series

Step 3 :- In General when the co-efficient of x^n is Equated to 0, we obtained recurrence relation, which helps us to determine the constants a_2, a_3, a_4, \dots in terms of a_0 & a_1

Step 4 :- thus we get the power Series Solution of the ODE, in the form of $y = a_0 F_1(x) + a_1 F_2(x)$

Note :- Some related Series

$$1) e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots$$

$$2) \bar{e}^x = 1 - \frac{x}{1!} + \frac{x^2}{2!} - \frac{x^3}{3!} + \dots$$

$$3) \cosh x = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \frac{x^6}{6!} + \dots$$

$$4) \sin x = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \frac{x^7}{7!} + \dots$$

$$5) \cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots$$

$$6) \tan x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots$$

14) Obtain Series Solution of $\frac{dy}{dx^2} + y = 0$

$$\text{Given } \frac{dy}{dx^2} + y = 0 \rightarrow ①$$

$$p_0(x) = 1 \neq 0 \text{ for } x=0, p_1(x)=0, p_2(x)=1$$

$$y = \sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + \dots \rightarrow ②$$

$$y' = \sum_{n=1}^{\infty} a_n n x^{n-1} \rightarrow ③$$

$$y'' = \sum_{n=2}^{\infty} a_n n(n-1) x^{n-2} + \sum_{n=0}^{\infty} a_n n x^{n-1} = 0 \rightarrow ④$$

\therefore the co-efficient x^n is $a_{n+2} (n+2)(n+1) + a_n$

$$\Rightarrow a_{n+2} (n+1)(n+2) + a_n = 0$$

$$\Rightarrow (n+1)(n+2) a_{n+2} = -a_n$$

$$a_{n+2} = -\frac{a_n}{(n+1)(n+2)} \rightarrow ⑤$$

When $n=0$

$$a_2 = -\frac{a_0}{2}$$

$$\text{When } n=1 \Rightarrow a_3 = -\frac{a_1}{6}$$

$$\text{When } n=2 \Rightarrow a_4 = -\frac{a_2}{12} = -\frac{(a_1/2)}{12} = \frac{a_1}{24}$$

$$\text{When } n=3 \Rightarrow a_5 = -\frac{a_3}{20} = -\frac{a_1/6}{20} = -\frac{a_1}{120}$$

$$\text{When } n=4 \Rightarrow a_6 = -\frac{a_4}{30} = -\frac{a_1/12}{30} = -\frac{a_1}{360}$$

$$y = a_0 + a_1 x - \frac{a_0 x^2}{2} - \frac{a_1 x^3}{6} + \frac{a_0 x^4}{24} + \frac{a_1 x^5}{120} - \frac{a_0 x^6}{720} + \dots$$

$$y = a_0 \left[1 - \frac{x^2}{2} + \frac{x^4}{24} - \frac{x^6}{720} + \dots \right] + a_1 \left[x - \frac{x^3}{6} + \frac{x^5}{120} \right]$$

15 Let us obtain the power Series Solution of the Eqn $\frac{d^2y}{dx^2} - y = 0$

$$a_{n+2} = \frac{a_n}{(n+2)(n+1)} \quad (n \geq 0)$$

by putting $n = 0, 1, 2, 3, 4, \dots$ we obtain

$$a_2 = \frac{a_0}{2}, \quad a_3 = \frac{a_1}{6}, \quad a_4 = \frac{a_0}{24}, \quad a_5 = \frac{a_1}{120}, \quad a_6 = \frac{a_0}{720}$$

Substituting these values in the expanded

from $y = \sum_{n=0}^{\infty} a_n x^n$ we obtain

$$y = a_0 \left[1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \dots \right] + a_1 \left[x + \frac{x^3}{3!} + \frac{x^5}{5!} + \dots \right] \rightarrow ①$$

we have $(D^2 - 1)y = 0$ where $D = \frac{d}{dx}$

Auxillary Eqn is $m^2 - 1 = 0$
 $m = \pm 1$

$$y = C_1 e^x + C_2 e^{-x} \rightarrow ②$$

$$y = a_0 \left[\frac{e^x + e^{-x}}{2} \right] + a_1 \left[\frac{e^x - e^{-x}}{2} \right]$$

$$(0) \quad y = a_0 \left[\frac{C_1 + C_2}{2} \right] e^x + \left[\frac{C_1 - C_2}{2} \right] e^{-x}$$

$$y = C_1 e^x + C_2 e^{-x} \quad \text{where } C_1 = \frac{a_0 + a_1}{2} \\ C_2 = \frac{a_0 - a_1}{2}$$

[16] Solve $\frac{d^2y}{dx^2} + xy = 0$ by obtaining the solution in the form
 of Bessel's

$$\Rightarrow p_0(x) = 1, \quad p_1(x) = 0, \quad p_2(x) = x$$

$$\text{Let } \sum_{r=0}^{\infty} a_r x^r = a_0 + a_1 x + a_2 x^2 + \dots \quad (2)$$

$$y^1 = \sum_{r=1}^{\infty} a_r r x^{r-1} \rightarrow (3)$$

$$y^{11} = \sum_{r=2}^{\infty} a_r r(r-1) x^{r-2} \rightarrow (4)$$

$$(1) \Rightarrow \sum_{r=2}^{\infty} a_r r(r-1) x^{r-2} + \sum_{r=0}^{\infty} a_r r x^r \cdot x$$

$$\sum_{r=2}^{\infty} a_r r(r-1) x^{r-2} + \sum_{r=0}^{\infty} a_r r x^{r+1} \rightarrow (5)$$

$$\text{The co-efficient of } x^r \text{ is } a_{r+2}(r+2)(r+1) + a_{r-1} \rightarrow (6)$$

$$a_{r+2}(r+2)(r+1) = -a_{r-1}$$

$$a_{r+2} = -\frac{a_{r-1}}{(r+2)(r+1)} \rightarrow (7) \quad x=1, a_2=0$$

$$\text{When } r=1, a_3 = -\frac{a_0}{6}$$

$$\text{When } r=2, a_4 = -\frac{a_1}{12} \dots$$

$$\text{When } r=3, a_5 = -\frac{a_2}{20} = 0$$

$$\text{When } r=4, a_6 = -\frac{a_3}{30} = -\frac{a_0}{30} = -\frac{a_0}{180}$$



Frobenious method

Consider the Second order differential eqn $p_0(x) \frac{d^2y}{dx^2} + p_1(x) \frac{dy}{dx} + p_2(x)y = 0$ for which $p_0(x) = 0$ when $x=0$

Solving procedure

- 1) Find the solution of the given D.E. \rightarrow (1) as $y = \sum_{r=0}^{\infty} a_r x^{r+1}$ (2)
- 2) $y = x^k (a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \dots)$
Find $y' = \sum_{r=0}^{\infty} a_r (k+r)x^{k+r-1}$, $y'' = \sum_{r=0}^{\infty} a_r (k+r)(k+r-1)x^{k+r-2}$
and Substitute the same in the equation (1)
- 3) Equate the co-efficient of x^k [when $r=0$] at 0. We may get the polynomial of second order in k & we get two roots, say k_1 & k_2
- 4) Also Equate the co-efficient of x^{k+1} [when $r=1$] to 0
and find the value of a_1
- 5) On both side Equate co-efficient of x^{k+r} and get the recurrence relation
- 6) find the values a_2, a_3, a_4, \dots by the help of the recurrence relation
- 7) Substitute all the values in the given eqn (1) and they $k=k_1$. we make get $y_1(x)$ at $k=k_1$ and similarly $k=k_2$ we get $y_2(x)$
- 8) finally write the soln of given D.E as $y = C_1 y_1(x) + C_2 y_2(x)$ where C_1 and C_2 are arbitrary constants

Q] Derivation of bessel's function using frobenius method

⇒ Consider a bessel's differential Eqn

$$x^2 \frac{d^2y}{dx^2} + x \frac{dy}{dx} + (x^2 - n^2)y = 0 \rightarrow \textcircled{1}$$

$$y=0$$

Here $P_0(x)=0$, when $n=0$

∴ the soln is $y = \sum_{r=0}^{\infty} a_r x^{k+r} \rightarrow \textcircled{2}$

$$y' = \sum_{r=0}^{\infty} a_r (k+r) x^{k+r-1} \rightarrow \textcircled{3}$$

$$y'' = \sum_{r=0}^{\infty} a_r (k+r)(k+r-1) x^{k+r-2} \rightarrow \textcircled{4}$$

Substitute in Eqn \textcircled{1}

$$\textcircled{1} \Rightarrow x^2 \sum_{r=0}^{\infty} a_r (k+r)(k+r-1) x^{k+r-2} + x \sum_{r=0}^{\infty} a_r (k+r) x^{k+r-1} + (x^2 - n^2) \sum_{r=0}^{\infty} a_r x^{k+r} = 0$$

$$\Rightarrow \sum_{r=0}^{\infty} a_r (k+r)(k+r-1) x^{k+r} + \sum_{r=0}^{\infty} a_r (k+r) x^{k+r} + \sum_{r=0}^{\infty} a_r x^{k+r} - n^2 \sum_{r=0}^{\infty} a_r x^{k+r} = 0$$

$$\Rightarrow \sum_{r=0}^{\infty} [a_r (k+r)(k+r-1) + a_r (k+r) - n^2 a_r] x^{k+r} + \sum_{r=0}^{\infty} a_r x^{k+r+2} = 0$$

$$\Rightarrow \sum_{r=0}^{\infty} a_r (k+r - n^2) x^{k+r} + \sum_{r=0}^{\infty} a_r x^{k+r+2} = 0$$

$$a_0 (k^2 - n^2) = 0$$

$$a_0 \neq 0, k^2 - n^2 = 0$$

$$k = \pm n$$

$$k = -n, k = +n$$

$$a_1[(k+1)^2 - n^2] = 0$$

$$\Rightarrow (k+1)^2 \neq n^2$$

$$a_1 = 0$$

By making $a_r[(k+r)^2 - n^2] + a_{r-2} = 0$

$$\Rightarrow a_r[(k+r)^2 - n^2] = -a_{r-2}$$

$$\Rightarrow a_r = \frac{-a_{r-2}}{(k+r)^2 - n^2}, \forall r \geq 2 \rightarrow \text{Q.E.D}$$

case (ii) :- when $k=n$

$$(i) \Rightarrow a_r = \frac{-a_{r-2}}{(n+r)^2 - n^2}$$

$$a_r = \frac{-a_{r-2}}{n^2 + 2nr + r^2 - n^2}$$

$$a_r = \frac{-a_{r-2}}{2nr + r^2} \rightarrow \text{Q.E.D} \quad \forall r \geq 2$$

$$a_2 = -\frac{a_0}{4^{n+4}} = -\frac{a_0}{4(n+1)}$$

$$a_3 = \frac{-a_1}{6n+9} = 0$$

$$a_4 = \frac{-a_2}{8n+16}$$

$$= \frac{-1}{8n+16} \left[-\frac{a_0}{4(n+1)} \right]$$

$$= \frac{a_0}{8(n+2)(4(n+1))}$$

$$a_4 = \frac{a_0}{32(n+1)(n+2)}$$

$$a_5 = \frac{-a_3}{10n+25} = 0$$

$$a_6 = \frac{-a_4}{12(n+3)}$$

$$= \frac{-1}{12(n+3)} \left[\frac{a_0}{384(n+1)(n+2)} \right]$$

$$a_6 = \frac{-a_6}{384(n+1)(n+2)(n+3)}$$

$$\textcircled{2} \Rightarrow y_1 = x^n \left(a_0 + a_1 x + \frac{-a_0}{4(n+1)} x^2 + a_2 x^3 + \frac{a_0 x^4}{384(n+1)(n+2)} \right. \\ \left. + \dots - \frac{a_0 x^6}{384(n+1)(n+2)(n+3)} + \dots \right)$$

$$y_1 = a_0 x^n \left[1 - \frac{x^2}{4(n+1)} + \frac{x^4}{384(n+1)(n+2)} - \frac{x^6}{384(n+1)(n+2)(n+3)} \right. \\ \left. - \frac{1}{(n+3)} \right] \rightarrow \textcircled{8}$$

case \textcircled{8}

My for $n=-n$ we get

$$y_2 = a_0 x^{-n} \left[1 - \frac{x^2}{4(-n+1)} + \frac{x^4}{384(-n+1)(-n+2)} - \frac{x^6}{384(-n+1)(-n+2)} \right. \\ \left. - \frac{1}{(-n+3)} + \dots \right] \rightarrow \textcircled{9}$$

\therefore the final soln of the given D.E

$$y = C_1 y_1(x) + C_2 y_2(x)$$

for the standardised of beside of 1st
find let $a_0 = \frac{1}{2^n \sqrt{1+t}}$

$$y_1 = a_0 x^n \left[1 - \frac{x^2}{4(n+1)} + \frac{x^4}{38(n+1)(n+2)} - \frac{x^6}{384(n+1)(n+2)(n+3)} + \dots \right]$$

$$\text{Let } y_0 = \frac{1}{\sqrt[n]{n+1}}$$

$$\textcircled{1} \Rightarrow y_1(x) = \overline{J}_n(x) = \frac{x^n}{\sqrt[n]{n+1}} \left[1 - \frac{x^2}{4(n+1)} + \frac{x^4}{38(n+1)(n+2)} - \frac{x^6}{384(n+1)} \right.$$

$$\left. \frac{1}{(n+2)(n+3)} + \dots \right]$$

$$= \left(\frac{x}{2} \right)^n \left[\frac{1}{\sqrt{n+1}} - \frac{x^2}{4(n+1)\sqrt{n+1}} + \frac{x^4}{38(n+1)\sqrt{n+1}(n+2)} \right.$$

$$\left. - \frac{x^6}{384(n+1)(n+2)(n+3)} + \dots \right]$$

$$= \left(\frac{x}{2} \right)^n \left[\frac{1}{\sqrt{n+1}} - \frac{x^2}{\sqrt{n+2}} + \frac{x^4}{38\sqrt{n+3}} - \frac{x^6}{384\sqrt{n+4}} + \dots \right]$$

$$\begin{aligned} \overline{J}_n(x) &= \left(\frac{x}{2} \right)^n \left[\frac{(-1)^0 \left(\frac{x}{2} \right)^{2(0)}}{\sqrt{(n+0+1)_0}} + \frac{(-1)^1 \left(\frac{x}{2} \right)^{2(1)}}{\sqrt{(n+1+1)_1}} + \frac{(-1)^2 \left(\frac{x}{2} \right)^{2(2)}}{\sqrt{(n+2+1)_2}} \right. \\ &\quad \left. + \frac{(-1)^3 \left(\frac{x}{2} \right)^{2(3)}}{\sqrt{(n+3+1)}} + \dots \right] \end{aligned}$$

$$\overline{J}_n(x) = \left(\frac{x}{2} \right)^n \sum_{r=0}^{\infty} \frac{(-1)^r \left(\frac{x}{2} \right)^{2r}}{\sqrt{(n+r+1)_r}}$$

$$\Rightarrow \overline{J}_n(x) = \sum_{r=0}^{\infty} \frac{(-1)^r}{\sqrt{(n+r+1)_r}} \left(\frac{x}{2} \right)^{2r+n}$$

$$\text{My } J_{-n}(x) = \sum_{r=0}^{\infty} \frac{(-1)^r}{(1-n+r+1)r!} \left(\frac{x}{2}\right)^{2r-n}$$

$$y = A J_n(x) + B J_{-n}(x)$$

18 Show that the overall notation i) $J_{1/2}(x) = \sqrt{\frac{2}{\pi x}} \sin x$
ii) $J_{-1/2}(x) = \sqrt{\frac{2}{\pi x}} \cos x$

\Rightarrow 10.5.7

$$J_n(x) = \sum_{r=0}^{\infty} \frac{(-1)^r}{\sqrt{(n+r+1)r!}} \left(\frac{x}{2}\right)^{2r+n} \rightarrow ①$$

$$① \Rightarrow J_{1/2}(x) = \sum_{r=0}^{\infty} \frac{(-1)^r}{\sqrt{(r+\frac{1}{2})r!}} \left(\frac{x}{2}\right)^{2r+1/2}$$

$$\Rightarrow J_{1/2}(x) = \sum_{r=0}^{\infty} \frac{(-1)^r}{\sqrt{(r+\frac{3}{2})r!}} \left(\frac{x}{2}\right)^{2r} \left(\frac{x}{2}\right)^{1/2}$$

$$\Rightarrow J_{1/2}(x) = \sum_{r=0}^{\infty} \frac{(-1)^r}{\sqrt{(r+\frac{3}{2})r!}} \left(\frac{x}{2}\right)^{2r}$$

$$J_{1/2}(x) = \sqrt{\frac{x}{2}} \left[\frac{1}{\sqrt{\frac{3}{2}}(0!)} - \frac{1}{\sqrt{\frac{5}{2}}(1!)} \left(\frac{x}{2}\right)^2 + \right.$$

$$\left. \frac{1}{\sqrt{(-\frac{1}{2})(2!)}} \left(\frac{x}{2}\right)^4 - \frac{1}{\sqrt{\frac{9}{2}}(3!)} \left(\frac{x}{2}\right)^6 + \dots \right]$$

$$\Rightarrow J_{1/2}(x) = \sqrt{\frac{x}{2}} \left[\frac{1}{(\frac{1}{2})(\sqrt{\frac{1}{2}})} - \frac{x^2}{(\frac{3}{2})(\frac{1}{2})(\sqrt{\frac{1}{2}})(4)} + \right]$$

$$\left. \frac{x^4}{(\frac{5}{2})(\frac{3}{2})(\frac{1}{2})\sqrt{\frac{1}{2}}(3!)} - \frac{x^6}{(\frac{7}{2})(\frac{5}{2})(\frac{3}{2})(\frac{1}{2})(\sqrt{\frac{1}{2}})(384)} + \dots \right]$$

$$J_{1/2}(x) = \sqrt{\frac{x}{2}} \left[\frac{2}{\sqrt{\pi}} - \frac{x^2}{3\sqrt{\pi}} + \frac{x^4}{60\sqrt{\pi}} - \frac{x^6}{3520\sqrt{\pi}} + \dots \right]$$

$$J_{1/2}(x) = \sqrt{\frac{x}{2}} \times \frac{2}{\sqrt{\pi}} \left[1 - \frac{x^2}{6} + \frac{x^4}{120} - \frac{x^6}{5040} + \dots \right]$$

$$J_{1/2}(x) = \frac{\sqrt{x}}{\sqrt{2}} \times \frac{\sqrt{2}\sqrt{2}}{\sqrt{\pi}} \times \frac{1}{\sqrt{x}\sqrt{x}} \left[x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots \right]$$

$$J_{1/2}(x) = \sqrt{\frac{2}{\pi x}} \sin x$$

If $n = -1/2$

$$\textcircled{1} \Rightarrow J_{-1/2}(x) = \sum_{r=0}^{\infty} \frac{(-1)^r}{\sqrt{(-1/2)+r+1} r!} \left(\frac{x}{2}\right)^{2r-1/2}$$

$$J_{1/2}(x) = \sum_{r=0}^{\infty} \frac{(-1)^r}{\sqrt{(r+1/2)}(r!)} \left(\frac{x}{2}\right)^{2r} \left(\frac{x}{2}\right)^{-1/2}$$

$$J_{1/2}(x) = \sqrt{\frac{2}{x}} \sum_{r=0}^{\infty} \frac{(-1)^r}{\sqrt{(r+\frac{1}{2})(r!)}} \left(\frac{x}{2}\right)^{2r}$$

$$J_{-1/2}(x) = \frac{\sqrt{2}}{x} \left[\frac{1}{\pi/2} - \frac{(x/2)^2}{(\sqrt{\frac{3}{2}})(1)} + \frac{(x/2)^4}{(\sqrt{\frac{15}{2}})(2)} - \frac{(x/2)^6}{\sqrt{(\frac{3}{2})}(6)} + \dots \right]$$

$$J_{-1/2}(x) = \sqrt{\frac{2}{x}} \left[\frac{1}{\pi/2} - \frac{x^2}{4(1/2)(\pi/2)} + \frac{x^4}{84\sqrt{\pi}} - \dots \right]$$

$$J_{-1/2}(x) = \frac{\sqrt{2}}{\sqrt{x}} \times \frac{1}{\sqrt{\pi}} \left[1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \dots \right]$$

$$J_{-1/2}(x) = \sqrt{\frac{2}{\pi x}} \cos x$$

Orthogonal Property

Theorem

19] Statement :- If α & β are the distinct soln of the Eqn $\sum_{n=0}^{\infty} \frac{x^n}{n!} y^{(n)} = 0$, then $\int_0^x x T_n(\alpha x) \cdot T_m(\beta x) dx = \begin{cases} 0 & \text{If } \alpha \neq \beta \\ \frac{1}{2} [T_n(\alpha)]^2 & \text{If } \alpha = \beta \end{cases}$

Proof :-

W.E.T $T_n(x)$ is the soln of the Eqn

$$x^2 \frac{d^2y}{dx^2} + x \frac{dy}{dx} + (x^2 - n^2)y = 0 \rightarrow ①$$

My $T_n(\lambda x)$ is a soln for

$$x^2 \frac{d^2y}{dx^2} + x \frac{dy}{dx} + \lambda^2 (x^2 - n^2)y = 0 \rightarrow ②$$

$\Rightarrow T_n(\lambda x)$ is a soln for the Eqn

$$x^2 \frac{d^2y}{dx^2} + x \frac{dy}{dx} + (\lambda^2 x^2 - n^2)y = 0 \rightarrow ③$$

& $T_n(\beta x)$ is a soln for the Eqn

$$x^2 \frac{d^2y}{dx^2} + x \frac{dy}{dx} + (\beta^2 x^2 - n^2)y = 0 \rightarrow ④$$

Let $u = T_n(\alpha x)$ and $v = T_n(\beta x)$, then

$$③ \Rightarrow x u'' + u' + (\alpha^2 x^2 - n^2)u = 0 \rightarrow ⑤$$

$$④ \Rightarrow x v'' + v' + (\beta^2 x^2 - n^2)v = 0 \rightarrow ⑥$$

$$⑤ \Rightarrow x u''v + uv' + (\alpha^2 x^2 - n^2)uv = 0 \rightarrow ⑦$$

$$⑥ \Rightarrow x u'v' + uv'' + (\beta^2 x^2 - n^2)uv = 0 \rightarrow ⑧$$

$$\textcircled{6} - \textcircled{8} \Rightarrow x(u''v - uv'') + (vu' - uv') + x(\alpha^2 - \beta^2)uv = 0$$

$$\Rightarrow x(u''v - uv'') + (vu' - uv') = x(\beta^2 - \alpha^2)uv$$

$$\Rightarrow \alpha[x(u''v - uv')] = x(\beta^2 - \alpha^2)uv$$

$$\Rightarrow \int_{x=0}^1 \alpha(x(uv' - uv')) = (\beta^2 - \alpha^2) \int_0^1 xuv dx$$

$$\Rightarrow [x(vu_1 - uv_1)]_0^1 = (\beta^2 - \alpha^2) \int_0^1 xuv dx$$

$$\Rightarrow \int_0^1 x J_n(\alpha x) J_n'(\beta x) dx = \frac{1}{\beta^2 - \alpha^2} [J_n(\beta x) J_n(\alpha x) - \beta J_n(\beta x) J_n(\alpha x)]$$

$$\Rightarrow \int_0^1 x J_n(\alpha x) J_n(\beta x) dx = \frac{1}{\beta^2 - \alpha^2} [\alpha J_n(\alpha) J_n(\beta) - \beta J_n(\alpha) J_n(\beta)]$$

If α & β are distinct we have $J_n(\alpha) = 0, J_n(\beta) = 0$

$$\textcircled{9} \Rightarrow \int_0^1 x J_n(\alpha x) J_n(\beta x) dx = 0 ; \text{ If } \alpha \neq \beta$$

If $\alpha = \beta$ we have $J_n(\alpha) = 0$

$$\begin{aligned} \textcircled{9} \Rightarrow \int_0^1 x J_n(\alpha x) J_n(\beta x) dx &= \frac{2 J_n'(x) J_n(\beta)}{\beta^2 - \alpha^2} \\ &= \lim_{\beta \rightarrow \alpha} \frac{\alpha J_n(\alpha) - J_n(\beta)}{\beta^2 - \alpha^2} - \frac{0}{0} \\ &= \lim_{\beta \rightarrow \alpha} \frac{\alpha J_n'(\alpha) J_n'(\beta)}{\beta - \alpha} \\ &= \frac{J_n'(\alpha) J_n'(\beta)}{2\alpha} \\ &= \frac{1}{\pi} \Gamma J_n(\alpha)^2 \quad \text{if } \alpha = \beta \end{aligned}$$

Legendre's differential Equation

Q) Consider legendre's linear differential eqn of 2nd order

$$(1-x^2) \frac{d^2y}{dx^2} - 2x \frac{dy}{dx} + n(n+1)y = 0 \rightarrow ①$$

$$\text{Let } y = \sum_{r=0}^{\infty} a_r x^r \rightarrow ②$$

$$y' = \sum_{r=0}^{\infty} r a_r x^{r-1} \rightarrow ③$$

$$y'' = \sum_{r=0}^{\infty} r(r-1) a_r x^{r-2} \rightarrow ④$$

$$① \Rightarrow (1-x^2) \sum_{r=0}^{\infty} r(r-1) a_r x^{r-2} - 2x \sum_{r=0}^{\infty} r a_r x^{r-1} + n(n+1) \sum_{r=0}^{\infty} a_r x^r = 0$$

$$\Rightarrow \sum_{r=0}^{\infty} r(r-1) a_r x^{r-2} - \sum_{r=0}^{\infty} r(r-1) a_r x^r - \sum_{r=0}^{\infty} 2r a_r x^r + \sum_{r=0}^{\infty} n(n+1) a_r x^r = 0$$

$$a_0 x^r = 0$$

$$\Rightarrow \sum_{r=0}^{\infty} r(r-1) a_r x^{r-2} - \left\{ \sum_{r=0}^{\infty} \left[n(r-1) + 2r - n+1 \right] \right\} a_r x^r = 0$$

when $r=0$, the co-efficient of x^2 becomes $0(-1)$

$$\therefore 0(-1)a_0 = 0$$

$$0a_0 = 0$$

$$\Rightarrow a_0 \neq 0$$

when $r=1$ the co-efficient of x^1 becomes

$$\therefore 1 \cdot (0)a_1 = 0$$

$$\Rightarrow a_1 \neq 0$$

Compare the co-efficient of x^r on both sides

$$⑤ \Rightarrow [(r+2)(r+1) a_{r+2}] - [r(r-1)a_r + 2r - n(n+1)] a_r = 0$$

$$(\gamma+1)(\gamma+2)a_{\gamma+2} - [\gamma^2 - \gamma + 2\gamma - n(n+1)] a^\gamma = 0$$

$$(\gamma+1)(\gamma+2)a_{\gamma+2} - [\gamma(\gamma+1) - n(n+1)] a_1 = 0$$

$$(\gamma+1)(\gamma+2)a_{\gamma+2} = [\gamma(\gamma+1) - n(n+1)] a_1$$

$$\Rightarrow a_{\gamma+2} = \frac{[\gamma(\gamma+1) - n(n+1)] a_1}{(\gamma+1)(\gamma+2)} \rightarrow \textcircled{6}$$

when $\gamma = 0$

$$\textcircled{6} \Rightarrow a_2 = \frac{[0 - n(n+1)] a_0}{2}$$

$$\Rightarrow a_2 = -\frac{n(n+1)}{2} a_0$$

likewise when $\gamma = 1$

$$a_3 = \frac{[2 - n(n+1)] a_1}{6}$$

$$a_3 = \frac{[-n(n+1) - 2] a_1}{6}$$

$$a_3 = -\frac{[n^2 + n - 2] a_1}{6}$$

$$\Rightarrow a_3 = -\frac{[n^2 - n + 2n - 2] a_1}{6}$$

$$= -\frac{[n(n+1) + 2(n-1)] a_1}{6}$$

$$a_3 = -\frac{(n-1)(n+2) a_1}{6}$$

when $\gamma = 2$

$$\textcircled{6} \Rightarrow a_4 = \frac{[-6 - n^2 - n] a_2}{12}$$

$$a_4 = - \frac{[n^2 - 2n + 3n - 6]}{12} a_2$$

$$= - \frac{[n(n-2) + 3(n-2)]}{12} a_2$$

$$= - \frac{(n-2)(n+3)}{12} a_2$$

$$= - \left[\frac{(n-2)(n+3)}{12} \right] \left[- \frac{n(n+1)}{2} \right] a_0$$

$$a_4 = \frac{n(n+1)(n-2)(n+3)}{24} a_0$$

nehmen $\gamma = 3$

$$a_5 = \frac{[12 - n^2 - n]}{20} a_3$$

$$a_5 = \frac{[-n^2 + n - 12]}{20} a_3$$

$$a_5 = - \frac{(n-3)(n+4)}{20}$$

$$= \left[\frac{(n-3)(n+4)}{20} \right] \left[- \frac{(n-1)(n+2)a_1}{6} \right]$$

$$a_5 = \frac{(n-1)(n-3)(n+2)(n+4)}{120} a_1$$

$$\Rightarrow y = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + \dots$$

$$\Rightarrow y = a_0 + a_1 x - \frac{n(n+1)}{2} a_0 x^2 - \frac{(n-1)(n-2)}{6} a_1 x^3 + \frac{n(n+1)(n-2)(n-3)}{24} a_0 x^4 + \frac{(n-1)(n-3)(n+2)(n+4)}{120} a_1 x^5 + \dots$$

$$\Rightarrow y = a_0 \left[1 - \frac{n(n+1)}{2!} x^2 + \frac{n(n+1)(n-2)(n+3)}{4!} x^4 + \dots \right] + a_1 \left[x - \frac{(n-1)(n+2)}{4!} x^3 + \frac{(n-1)(n-3)(n+2)(n+4)}{8!} x^5 \right]$$

$$y = a_0 y_1(x) + a_1 y_2(x)$$

Legendre's differential equation leading to polynomial

Ques 12.5.7 The legendre's differential eq $(1-x^2) \frac{d^2y}{dx^2} - 2x \frac{dy}{dx} + n(n+1)y = 0$

$y=0 \rightarrow ①$ By taking the soln $y = \sum_{r=0}^{\infty} a_r x^r$ we have the recurrence relation

$$a_{r+2} = \frac{n(n+1)-r(r+1)}{(r+1)(r+2)} a_r \rightarrow ②$$

$y = a_0 y_1(x) + a_1 y_2(x)$ for a_0 It may be observed that the polynomial $y_1(x), y_2(x)$ contained alternative power of x and general form the polynomial representation

either of them increasing or decreasing in the powers

of x and can be represented in the form

$$y = f(x) = a_n x^n + a_{n-2} x^{n-2} + a_{n-4} x^{n-4} - \dots - x \rightarrow ③$$

where $f(0) = a_0$ for n is even

$a_1 x$ for n is odd

When $r=n-2$

$$② \Rightarrow a_n = - \frac{n(n+1) - (n-2)(n-1) a_{n-2}}{n(n-1)}$$

$$a_n = - \frac{(n^2+n-n^2+3n-2) a_{n-2}}{n(n-1)}$$

$$a_n = - \frac{4n-2}{n(n-1)} a_{n-2}$$

$$a_n = - \frac{2(2n-1)}{n(n-1)} a_{n-2}$$

$$\Rightarrow a_{n-2} = - \frac{n(n-1)}{2(2n-1)} a_n$$

When $r=n-4$

$$\textcircled{2} \Rightarrow a_{n-2} = - \left[\frac{(n(n+1) - (n-4)(n-3))}{(n-3)(n-2)} \right] a_{n-4}$$

$$a_{n-2} = - \left[\frac{n^2 + n - n^2 + 7n - 12}{(n-2)(n-3)} \right] a_{n-4}$$

$$\Rightarrow a_{n-2} = \frac{[-7n+12]}{(n-2)(n-3)} a_{n-4}$$

$$\Rightarrow a_{n-2} = -4 \frac{(2n-3)}{(n-2)(n-3)} a_{n-4}$$

$$\Rightarrow a_{n-2} = - \frac{(n-2)(n-3)}{4(2n-3)} a_{n-2}$$

$$a_{n-4} = \left[\frac{-(n-2)(n-3)}{4(2n-3)} \right] \left[\frac{-n(n-1)}{2(2n-1)} a_n \right]$$

$$a_{n-4} = \frac{n(n-1)(n-2)(n-3)a_n}{8(2n-1)(2n-3)}$$

$$\textcircled{3} \Rightarrow y = f(x) = a_n x^n - \frac{n(n-1)}{2(2n-1)} a_n x^{n-2} + \frac{n(n-1)(n-2)(n-3)}{8(2n-1)(2n-3)} a_n x^{n-4} + \dots$$

$$y = f(x) = a_n \left[\frac{x^n - n(n-1)}{2(2n-1)} x^{n-2} + \frac{n(n-1)(n-2)(n-3)}{8(2n-1)(2n-3)} x^{n-4} \right] + \dots$$

$\dots G(x) \rightarrow \textcircled{4}$

hence $g(x) = \begin{cases} a_0 & \text{for } n \text{ is even} \\ a_1(x) & \text{for } n \text{ is odd} \end{cases}$

If the constant small a_n is chosen such that $y=f(x)$ becomes 1 when $x=1$ the polynomial so obtained are called Legendre's polynomial denoted by $P_n(x)$

Let us choose $a_n = \frac{1, 3, 5, 7, \dots, (2n-1)}{n!}$ the Eq ④ becomes

$$P_n(x) = \frac{1, 3, 5, 7, \dots, (2n-1)}{n!} \left[\frac{x^n - n(n-1)x^{n-2} + n(n-1)}{2(2n-1)} \right]$$

$$\frac{(n-2)(n-3)}{8(2n-1)(2n-3)} x^{n-4} + \dots$$

for $n=0, 1, 2, 3, \dots$ we can get the polynomials as

$$P_0(x) = 1 \rightarrow A$$

$$P_1(x) = x \rightarrow B$$

$$P_2(x) = \frac{1}{2}(3x^2 - 1) \rightarrow C$$

$$P_3(x) = \frac{1}{2}(5x^3 - 3x) \rightarrow D$$

$$P_4(x) = \frac{1}{8}(35x^4 - 30x^2 + 3) \rightarrow E$$

In the next part $1 = P_0(x)$

$$x = P_1(x)$$

$$C \Rightarrow 3x^2 - 1 = 2P_2(x) + 1$$

$$\Rightarrow 3x^2 = \frac{1}{3}[(2P_2(x)) + 1]$$

$$\Rightarrow x^2 = \frac{1}{3}[2P_2(x) + P_0(x)]$$

$$D \Rightarrow 5x^3 - 3x = 2P_3(x)$$

$$\Rightarrow 5x^3 = 2P_3(x) + 3x$$

$$\Rightarrow x^3 = \frac{1}{5}[2P_3(x) + 3P_1(x)]$$

$$\begin{aligned} \textcircled{1} \Rightarrow 35x^4 - 30x^2 + 3 &= 8P_4(x) \\ \Rightarrow 35x^4 &= 8P_4(x) + 30x^2 - 3 \\ \Rightarrow x^4 &= \frac{1}{35} [8P_4(x) + 30\frac{1}{3}(2P_2(x) + P_0(x)) - 3P_0(x)] \\ \Rightarrow x^4 &= \frac{1}{35} [8P_4(x) + 20P_2(x) + 1P_0(x)] \end{aligned}$$

Rodrigue's formula

Q22 For positive value of η in the Rodriguez formula for Legendre polynomials can be defined as

$$P_l(n) = \frac{1}{2^n n!} \frac{d^n}{dx^n} (x^2 - 1)^n$$

$$P_0(x) = \frac{1}{1.1} (x^2 - 1) = 0$$

$$P_1(x) = \frac{1}{2} \frac{d}{dx} (x^2 - 1) = \frac{2x}{2} = x$$

Q23 Show that $P_3 \cos \theta = \frac{1}{8} (3 \cos \theta + 5 \cos 3\theta)$

$$\Rightarrow P_3(x) = \frac{1}{8} [5x^3 - 3x] \rightarrow \textcircled{1}$$

$$x = \cos \theta$$

$$\textcircled{1} \Rightarrow P_3(\cos \theta) = \frac{1}{2} [5(\cos \theta)^3 - 3(\cos \theta)] \rightarrow \textcircled{2}$$

$$10.5.7 \quad \cos^3 \theta = 4\cos^3 \theta - 3\cos \theta$$

$$\Rightarrow \cos^3 \theta = \frac{1}{4} [\cos 3\theta + 3\cos \theta]$$

$$\textcircled{2} \Rightarrow P_3(\cos \theta) = \frac{1}{8} \left[\frac{5}{4} \cos 3\theta + 3\cos \theta \right] - 3\cos \theta$$

$$\Rightarrow P_3(\cos \theta) = \frac{1}{8} \left[\frac{5\cos 3\theta + 3\cos \theta}{4} - 12\cos \theta \right]$$

$$P_3 \cos \theta = \frac{1}{8} [5\cos 3\theta + 3\cos \theta]$$

24) Use Lagrange's formula s.t $P_4(\cos\theta) = \frac{1}{64} [35\cos^4\theta + 20\cos^2\theta + 9]$

L.K.T $P_4(x) = \frac{1}{64} [35x^4 - 30x^2 + 3]$

$x = \cos\theta$

$$P_4(\cos\theta) = \frac{1}{8} [35\cos^4\theta - 30\cos^2\theta + 3] \\ = \frac{1}{8} [35(\cos^2\theta)^2 - 30\cos^2\theta + 3]$$

$$\cos^2\theta = \left(\frac{1+\cos 2\theta}{2}\right)$$

$$= \frac{1}{8} \left[35 \left(\frac{1+\cos 2\theta}{2} \right)^2 - 30 \left(\frac{1+\cos 2\theta}{2} \right) + 3 \right] \\ = \frac{1}{8} \left[\frac{35}{4} (1+2\cos 2\theta + \cos^2 2\theta) - \frac{30}{2} (1+\cos 2\theta) + 3 \right] \\ = \frac{1}{32} \left[35 + 70\cos 2\theta + 35\cos^2 2\theta - 60 - 60\cos 2\theta + 12 \right] \\ = \frac{1}{32} \left[35\cos^2 2\theta + 10\cos 2\theta - 13 \right] \\ = \frac{1}{32} \left[35 \left(\frac{1+\cos 4\theta}{2} \right) + 10\cos 2\theta - 13 \right] \\ = \frac{1}{32} \left[35 + \frac{35\cos 4\theta + 20\cos 2\theta}{2} - 26 \right] \\ = \frac{1}{64} \left[35\cos 4\theta + 20\cos 2\theta + 9 \right]$$

=====

Q] Express $f(x) = x^3 + 2x^2 - x - 3$ in terms of Legendre's polynomial

$$\Rightarrow f(x) = x^3 + 2x^2 - x - 3(1)$$

N.K.T

$$1 = P_0(x)$$

$$x = P_1(x)$$

$$x^2 = \frac{1}{3} [2P_2(x) + P_0(x)]$$

$$x^3 = \frac{1}{5} [2P_3(x) + 3P_1(x)]$$

$$f(x) = \frac{1}{5} [2P_3(x) + 3P_1(x)] + \frac{2}{3} [2P_2(x) + P_0(x)] - P_1(x) - 3P_0$$

$$= \frac{1}{5} [3[2P_2(x) + 3P_1(x)] + 10[2P_2(x) + P_0(x)] - 15P_1(x) - 45P_0]$$

$$\Rightarrow f(x) = \frac{1}{15} [6P_3(x) + 9P_1(x) + 20P_2(x) + 10P_0(x) - 15P_1(x) - 45P_0]$$

$$f(x) = \frac{1}{15} [6P_3(x) + 20P_2(x) - 6P_1(x) - 35P_0]$$

$$f(x) = \frac{2}{5} P_3(x) + \frac{4}{3} P_2(x) - \frac{2}{5} P_1(x) - \frac{7}{3} P_0$$

Q] Express $f(x) = x^4 + 3x^3 - x^2 + 5x - 2$ in terms of Legendre's polynomial

$$\Rightarrow f(x) = x^4 + 3x^3 - x^2 + 5x - 2(1) \rightarrow ①$$

N.K.T

$$1 = P_0(x)$$

$$x = P_1(x)$$

$$x^2 = \frac{1}{3} [2P_2(x) + P_0(x)]$$

$$x^3 = \frac{1}{5} [2P_3(x) + 3P_1(x)]$$

$$x^4 = \frac{1}{35} [8P_4(x) + 20P_2(x) + 7P_0(x)]$$

$$\textcircled{1} \Rightarrow f(x) = \frac{1}{35} [8P_4(x) + 20P_2(x) + 7P_0(x)] + \frac{3}{5} [2P_3(x) + 3P_1(x)]$$

$$- \frac{1}{3} [2P_2(x) + P_0(x)] + 5P_1(x) - 2P_0(x)$$

$$= \frac{1}{105} \left\{ 3 [8P_4(x) + 20P_2(x) + 7P_0(x)] + 63 [2P_3(x) + 3P_1(x)] \right. \\ \left. - 35 [2P_2(x) + P_0(x)] + 525P_1(x) - 210P_0(x) \right\}$$

$$= \frac{1}{105} [24P_4(x) + 60P_2(x) + 20P_0(x) + 126P_3(x) + 184P_1(x) \\ - 70P_2(x) - 35P_0(x) + 525P_1(x) - 210P_0(x)]$$

$$\Rightarrow f(x) = \frac{1}{105} [24P_4(x) + 126P_3(x) - 10P_2(x) + 74P_1(x) - 224P_0(x)]$$

$$f(x) = \frac{8}{35} P_4(x) + \frac{6}{5} P_3(x) - \frac{2}{5} P_2(x) + \frac{34}{5} P_1(x) - \frac{32}{5} P_0(x)$$

27 If $x^3 + 2x^2 - x + 1 = aP_0(x) + bP_1(x) + cP_2(x) + dP_3(x)$ find
the values of a, b, c, d

$$\text{let } f(x) = x^3 + 2x^2 - x + 1$$

$$1 = P_0(x)$$

$$x^2 = P_1(x)$$

$$x^3 = \left[\frac{1}{3} \{ (2P_2(x) + P_0(x)) \} \right]$$

$$x^3 = \frac{1}{5} [2P_2(x) + 3P_1(x)]$$

$$\textcircled{1} \Rightarrow f(x) = \frac{1}{5} [2P_3(x) + 3P_1(x)] + \frac{2}{3} [2P_2(x) + P_0(x)] - P_1(x) + P_0(x)$$

$$= \frac{1}{15} \{ 3[2P_3(x) + 3P_1(x)] + 10[2P_2(x) + P_0(x)] - 15P_1(x) +$$

$$15P_0(x) \}$$

$$= \frac{1}{15} [6P_3(x) + 9P_1(x) + 20P_2(x) + 10P_0(x) - 15P_1(x) + 15P_0(x)]$$

$$= \frac{1}{15} [6P_3(x) - 6P_1(x) + 20P_2(x) + 6P_0(x)]$$

$$aP_0(x) + bP_1(x) + cP_2(x) + dP_3(x) = \frac{5}{3}P_0(x) + \left(-\frac{2}{3}\right)P_1(x)$$

$$+ \left(\frac{4}{3}\right)P_2(x) + \left(\frac{-2}{5}\right)P_3(x)$$

$$a = \frac{5}{3}, \quad b = -\frac{2}{3}, \quad c = \frac{4}{3}, \quad d = \frac{2}{5}$$

29 Express $f(x) = x^3 - 5x^2 + 14x + 5$ in terms of Legendre's polynomial

$$f(x) = x^3 - 5x^2 + 14x + 5(1) \rightarrow \textcircled{1}$$

$$1 = P_0(x)$$

$$x = P_1(x)$$

$$x^2 = \frac{1}{3} [2P_2(x) + P_0(x)]$$

$$x^3 = \frac{1}{5} [2P_3(x) + 3P_1(x)]$$

$$\textcircled{1} \Rightarrow f(x) = \frac{1}{5} [2P_3(x) + 3P_1(x)] - \frac{5}{3} [2P_2(x) + P_0(x)] +$$

$$14P_1(x) + 5P_0(x)$$

$$= [6(P_3(x) + 9P_1(x) - 50P_2(x) - 25P_0(x) + 210P_1(x))$$

$$+ 75P_0(x)]$$

$$= \frac{1}{15} [6P_3(x) - 50P_2(x) + 219P_1(x) + 75P_0(x)]$$

$$\frac{2}{5}P_3(x) - \frac{10}{3}P_2(x) + \frac{7}{5}P_1(x) + \frac{10}{3}P_0(x)$$

Express $f(x) = 3x^3 - x^2 + 5x - 2(1)$ in terms of Legendre's polynomials

$$\Rightarrow f(x) = 3x^3 - x^2 + 5x - 2(1)$$

$$1 - P_0(x)$$

$$x = P_1(x)$$

$$x^2 = \frac{1}{3}[2P_2(x) + P_0(x)]$$

$$x^3 = \frac{1}{5}[2P_3(x) + 3P_1(x)]$$

$$f(x) = \frac{3}{5}[2P_3(x) + 3P_1(x)] - \frac{1}{3}[2P_2(x) + P_0(x)] + 5P_1(x) - 2P_0(x)$$

$$= \frac{3}{5}[2P_3(x) + 3P_1(x)] - \frac{1}{3}[2P_2(x) + P_0(x)] + 5P_1(x) - 2P_0(x)$$

$$= \frac{6}{5}P_3(x) + \frac{9}{5}P_1(x) - \frac{2}{3}P_2(x) - \frac{1}{3}P_0(x) + 5P_1(x) - 2P_0(x)$$

$$= \frac{1}{15}[18P_3(x) + 27P_1(x) - 10P_2(x) - 5P_0(x) + 75P_1(x) - 30P_0(x)]$$

$$= \frac{1}{15}[-35P_0(x) + 105P_1(x) - 10P_2(x) + 18P_3(x)]$$

$$= \frac{-35}{15}P_0(x) + \frac{102}{15}P_1(x) - \frac{10}{15}P_2(x) + \frac{18}{15}P_3(x)$$

$$= \frac{-7}{3}P_0(x) + \frac{34}{5}P_1(x) - \frac{2}{3}P_2(x) + \frac{6}{5}P_3(x)$$