Module-02

Introduction

Suppose $y=f(x)$ be a function in the variable of x then the Equation $\frac{a_0 d^n y}{dx^n}+\frac{a_1 d^{n-1} y}{dx^{n-1}}+\frac{a_2 d^{n-2} y}{dx^{n-2}}+\dots\dots a_n y=Q$

$\Rightarrow a_0 D^n y+a_1 D^{n-1} y+a_2 D^{n-2} y+\dots\dots a_n y=Q$ is called the Linear differential Equation of n^{th} order and 1^{st} degree where D is called differential operation is follows

1. If $a_0, a_1, a_2 \dots\dots a_n$ are the constant and $Q=0$ then Eq (1) called the Linear Homogenous differential Equation of n^{th} order with constant co-efficient

2. If $a_0, a_1, a_2 \dots\dots a_n$ are the constant and $Q\neq 0$ then Equation (1) is called the Linear Non-Homogenous differential Equation of the n^{th} order with constant co-efficient

3. If $a_0, a_1, a_2 \dots\dots a_n$ are the fun in x and $Q=0$ then Eq (1) is called the Linear Homogenous differential Equation of n^{th} order with variable co-efficient

4. If $a_0, a_1, a_2 \dots\dots a_n$ are the function in x and $Q\neq 0$ then Eq (1) is called the Linear non-Homogenous Differential equation of n^{th} order with variable co-efficient

Solution of the Non-Homogenous D.E with variable co-efficient

Step 1 :-

write the given differential Equation

$a_0 D^n+a_1 D^{n-1}+a_2 D^{n-2}+\dots\dots a_n y=Q(x)$

(or) $f(D)y=Q(x)$

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Step 2 :-

Evaluation of complimentary function

a) Identify $f(D)$ in the given d.E

b) Write the Auxilary Equation $f(m)=0$, where this Equation will be polynomially in m of degree n. and It gives n number of Solutions They are $m = m_1, m_2, m_3 ---- m_n$

c) If $m_1, m_2, m_3 ----- m_n$ are real and distinct, then The Soln is $CF = y_c = c_1 e^{m_1 x} + c_2 e^{m_2 x} + c_3 e^{m_3 x} --- + c_n e^{m_n x}$

If first 2 roots are Equal and rest of them are real and distinct, then

$$CF = y_c = (c_1 + c_2 x) e^{m_1 x} + c_3 e^{m_3 x} + c_4 e^{m_4 x} ---- + c_n e^{m_n x}$$

If first 3 roots are Equal and rest of them are real and distinct, then

$$CF = y_c = (c_1 + c_2 x + c_3 x^2) e^{m_1 x} + c_4 e^{m_4 x} + ---- c_n e^{m_n x}$$

If first 2 roots are complex and rest of them real and distinct, then

$$m_1 \pm m_2,\ m_3, m_4 ------ m_n$$

$$CF = y_c = [c_1 \cos m_2 x + c_2 \sin m_2 x] e^{m_1 x} + c_3 e^{m_3 x} + ----- ----- + c_n e^{m_n x}$$

If first 4 complex roots are Equal and rest of them real and distinct, then

$$CF = [(c_1 + c_2 x) \cos m_2 x + (c_3 + c_4 x) \sin m_2 x] e^{m_1 x} + c_3 e^{m_3 x} --- ---------- + c_n e^{m_n x}$$

Evaluation of particular Integral

$\Longrightarrow$ write the given DE in the form of

$$f(D) y = Q(x)$$

⇒ The particular integral Evaluated by writting

$$y_p = \frac{Q(x)}{f(D)}$$

If $f(a)=0$. Then $D-a$ should be factor to the $f(D)$ which will be Evaluated as

$$\frac{e^{ax}}{f(D)} = \frac{e^{ax}}{(D-a)^k \phi(D)} = \frac{x^k}{k!}\frac{e^{ax}}{\phi(a)}$$

If $Q(x) = \cos ax$ (or) $\sin ax$, then y_p becomes $\frac{\cos ax}{f(D)}$ (or) $\frac{\sin ax}{f(D)}$ which can be Evaluated by replacing $f(D)$ $D^2 = -a^2$ when $f(a) \neq 0$ and we have $\frac{\cos ax}{D^2+a^2} = \frac{x}{2a}\sin ax$.

$$\frac{\sin ax}{D^2+a^2} = \frac{-x}{2a}\cos ax$$

1] Solve $(D^3+6D^2+11D+6)y=0$ where $D=\frac{d}{dx}$

⇒ Given $(D^3+6D^2+11D+6)y=0$

$f(D)y=0$

$f(D) = D^3+6D^2+11D+6$

∴ The Auxillary Equation is :

$f(m)=0$

$m=-1$	1	6	11	6
	0	-1	-5	-6
	1	5	6	0

$\Rightarrow m^3+6m^2+11m+6=0$

$\Rightarrow (m+1)(m^2+5m+6)=0$

$\Rightarrow m+1=0 \quad m^2+2m+3m+6=0$

$m=-1, \quad m(m+2)+3(m+2)=0$

$m=-1, \quad m+2=0 \quad m+3=0$

$m=-1, \quad m=-2, \quad m=-3$

$m=-1, -2, -3$

∴ $CF = y_c = c_1 e^{-x} + c_2 e^{-2x} + c_3 e^{-3x}$

2. Solve $\left(4\frac{d^4y}{dx^4} - 4\frac{d^3y}{dx^3} - 23\frac{d^2y}{dx^2} + 12\frac{dy}{dx} + 36y\right) = 0$

$\Rightarrow (4D^4 - 4D^3 - 23D^2 + 12D + 36)y = 0$

$\Rightarrow f(D)y = 0$

where $f(D) = 4D^4 - 4D^3 - 23D^2 + 12D + 36$

the Auxillary Equation is $f(m) = 0$

$\Rightarrow 4m^4 - 4m^3 - 23m^2 + 12m + 36 = 0$

m=2	4	-4	-23	12	26
	0	8	8	-30	-36
+2	4	4	-15	-18	0
	0	8	24	18	
	4	12	9	0	

$\Rightarrow (m-2)(4m^3 + 4m^2 - 15m - 18)$

$\Rightarrow (m-2)(m-2)(4m^2 + 12m + 9) = 0$

$\Rightarrow m = 2, 2, \; 4m^2 + 6m + 6m + 9 = 0$

$\Rightarrow m = 2, 2, \; 2m(2m+3) + 3(2m+3)$

$\Rightarrow m = 2, 2, \; (2m+3)(2m+3)$

$\Rightarrow m = 2, 2 \quad m = \frac{-3}{2}, m = \frac{-3}{2}$

$\therefore y_c = (c_1 + c_2 x)e^{2x} + (c_3 + c_4 x)e^{-3/2x}$

3. Solve $\left(\frac{d^3y}{dx^3} + 6\frac{d^2y}{dx^2} + 11\frac{dy}{dx} + 6\right)y = e^x + 1$

$\Rightarrow$ Given $\left(\frac{d^3y}{dx^3} + 6\frac{d^2y}{dx^2} + 11\frac{dy}{dx} + 6\right)y = e^x + 1 \longrightarrow (1)$

$\Rightarrow (D^3 + 6D^2 + 11D + 6)y = e^x + 1$

$f(D)y = e^x + 1$

When $f(D) = D^3 + 6D^2 + 11D + 6$

-1	1	6	11	6
	0	-1	-5	-6
	1	5	6	0

the A.E $f(m) = 0$

$m^3 + 6m^2 + 11m + 6 = 0$

$\Rightarrow (m+1)(m^2+5m+6)=0$

$\Rightarrow (m+1)(m+2)(m+3)=0$

$\Rightarrow m=-1, \; m=-2, \; m=-3$

$\therefore \; y_c = c_1 e^{-x} + c_2 e^{-2x} + c_3 e^{-3x}$

To find PI

$$PI = y_p = \frac{e^x+1}{f(D)}$$

$$= \frac{e^x}{f(D)} + \frac{1}{f(D)}$$

$$= \frac{e^x}{D^3+6D^2+11D+6} + \frac{e^{0x}}{0+0+0+6}$$

$$= \frac{e^x}{24} + \frac{1}{6}$$

$$y_p = y_c + y_p$$

$$y = c_1 e^{-x} + c_2 e^{-2x} + c_3 e^{-3x} + \frac{e^x}{24} + \frac{1}{6}$$

4 Solve $\frac{d^2y}{dx^2} - 4y = \cosh(2x-1) + 3^x$

$\Rightarrow (D^2-4)y = \cosh(2x-1) + 3^x$

$f(D)y = \cosh(2x-1) + 3^x$

where $f(D) = D^2-4$

To find CF

the AE is $f(m)=0$

$\Rightarrow m^2-4=0$

$\Rightarrow m=\pm 2$

$\therefore \; y_c = c_1 e^{-2x} + c_2 e^{2x}$

To <u>find PI</u>

$$PI = y_p$$

$$y_p = \frac{\cosh(2x-1)}{f(D)} + \frac{3^x}{f(D)}$$

$$y_p = \frac{\frac{e^{(2x-1)} + e^{-(2x-1)}}{2}}{D^2-4} + \frac{e^{(\log_e 3)x}}{D^2-4}$$

$$y_p = \frac{1}{2} \cdot \frac{e^{(2x-1)} + e^{-(2x+1)}}{D^2-4} + \frac{e^{(\log 3)x}}{D^2-4}$$

$$y_p = \frac{1}{2} \cdot \frac{e^{2x-1}}{D^2-4} + \frac{e^{-2x+1}}{D^2-4} + \frac{e^{(\log 3x)}}{D^2-4}$$

$$y_p = \frac{1}{2} \frac{e^{2x-1}}{D^2-4} + \frac{e^{-2x+1}}{D^2-4} + \frac{e^{(\log 3x)}}{D^2-4}$$

$$y_p = \frac{1}{2} \frac{e^{2x-1}}{(D-2)'(D+2)} + \frac{1}{2} \frac{e^{-2x+1}}{(D+2)'(D-2)} + \frac{e^{(\log 3x)}}{D^2-4}$$

$$y_p = \frac{1}{2} \frac{x}{1!} \frac{e^{2x-1}}{2+2} + \frac{1}{2} \frac{e^{-2x+1}}{(-2-2)} + \frac{e^{\log 3x}}{(\log 3)^2-4}$$

$$y_p = \frac{x}{1!} \frac{e^{2x-1}}{2+2} + \frac{1}{2} \frac{e^{-2x+1}}{(-2-2)} + \frac{e^{\log 3x}}{(\log 3)^2-4}$$

$$y_p = \frac{x}{8} e^{2x-1} - \frac{x}{8} e^{-(2x-1)} + \frac{3^x}{(\log 3)^2-4}$$

$$y_p = \frac{x}{4} \left[\frac{e^{2x-1} - e^{-(2x-1)}}{2} \right] + \frac{3^x}{(\log 3)^2-4}$$

$$y_p = \frac{x}{4} \sinh(2x-1) + \frac{3^x}{(\log 3)^2-4}$$

$$y = y_c + y_p$$

$$\boxed{y = c_1 e^{-2x} + c_2 e^{2x} + \frac{x}{4} \sinh(2x-1) + \frac{3^x}{(\log 3)^2-4}}$$

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5] Solve $(D^2+2D+1)y = 2x+x^2$, where $D=\frac{d}{dx}$

$\Longrightarrow$ Given

$$\Rightarrow (D^2+2D+1)y = 2x+x^2$$

$$f(D)y = 2x+x^2$$

where $f(D) = D^2+2D+1 = (D+1)^2$

The A.E is $f(m)=0$

$$\Rightarrow (m+1)^2 = 0$$

$$\Rightarrow m = -1, -1$$

$$\therefore y_c = (c_1 + c_2 x)e^{-x}$$

$$y_p = \frac{x^2+2x}{f(D)}$$

$$y_p = \frac{x^2+2x}{(D+1)^2}$$

$$y_p = (1+D)^{-2}(x^2+2x)$$

$$y_p = (1-2D+3D^2-4D^2+\cdots)(x^2+2x)$$

$$y_p = (x^2+2x) - 2(2x+2) + 3(2)$$

$$y_p = x^2+2x-4x-4+6$$

$$y_p = x^2-2x+2$$

$$\therefore y = y_c + y_p$$

$$\boxed{y = (c_1+c_2x)e^{-x} + x^2-2x+2}$$

6] Solve $(D^2-4D+4)y = 8(e^{2x}+\sin 2x)$, where $D=\frac{d}{dx}$

$\Longrightarrow$ Given

$$(D^2-4D+4)y = 8$$

$$f(D)y = 8$$

where $f(D) = D^2-4D+4$

To find CF

the A.E is $f(m)=0$

$$\Rightarrow m^2-4m+4=0$$

$$\Rightarrow (m-2)^2=0$$

$$\Rightarrow m=+2, -2$$

$$\therefore y_c=(c_1+c_2x)e^{2x}$$

To find PI

$$y_p = \frac{8(e^{2x}+\sin 2x)}{f(D)}$$

$$y_p = \frac{8e^{2x}}{f(D)} + \frac{8\sin 2x}{f(D)}$$

$$y_p = \frac{8e^{2x}}{(D-2)^2} + \frac{8\sin 2x}{D^2-4D+4}$$

$$y_p = 8\frac{x^2}{2!}e^{2x} + 8.\frac{\sin 2x}{(-4-4D+4)}$$

$$y_p = 4x^2e^{2x} - \frac{8}{4}\frac{\sin 2x}{D}$$

$$y_p = 4x^2e^{2x} - 2D\frac{\sin 2x}{D^2}$$

$$y_p = 4x^2e^{2x} - 2D\frac{\sin 2x}{(-4)}$$

$$y_p = 4x^2e^{2x} + \frac{1}{2}(2\cos 2x)$$

$$y_p = 4x^2e^{2x} + \cos 2x$$

$$y = y_c + y_p.$$

$$y = (c_1+c_2x)e^{2x} + 4x^2e^{2x} + \cos 2x$$

7] $(D^3+D^2-4D-4)y = 3e^{-x}-4x-6$ Using Inverse differential Equation

$\Rightarrow$

Given :- $(D^3+D^2-4D-4)y = 3e^{-x}-4x-6$

$f(D)y = 3e^{-x}-4x-6$

when $f(D) = D^3+D^2-4D-4$

To find CF

The A.E is $f(m) = 0$

$\Rightarrow m^3+m^2-4m-4=0$

$\Rightarrow (m+1)(m+2)(m-2)=0$

$\Rightarrow m=-1,\ m=-2,\ m=2$

$\therefore y_c = c_1e^{-x} + c_2e^{-2x} + c_3e^{2x}$

To find PI

$$y_p = \frac{3e^{-x}-4x-6}{f(D)}$$

$$y_p = \frac{3e^{-x}}{D^3+D^2-4D-4} - \frac{6}{D^3+D^2-4D-4} - \frac{4x}{D^3+D^2-4D-4}$$

$$y_p = 3\frac{e^{-x}}{(D+1)(D^2-4)} - 6\frac{e^{0x}}{D^3+D^2-4D-4} - 4\cdot\frac{x}{(-4)\left[1-\frac{(D)^3+(D)^2-4D}{4}\right]}$$

$$y_p = \frac{3x^1}{1!}\frac{e^{x}}{[(-1)^2-4]} - 6\frac{e^{0x}}{(-4)} + \left[1-\left(\frac{D^3+D^2-4D}{4}\right)\right]^{-1}x$$

$$y_p = -xe^{-x} + \frac{3}{2} + \left[1+\left(\frac{D^3+D^2-4D}{4}\right)+----\right]x$$

$$y_p = -xe^{-x} + \frac{3}{2} + x + \frac{1}{4}(-4)$$

$$y_p = -xe^{-x} + \frac{3}{2} + x - 1$$

$$y_p = -xe^{-x} + x + \frac{1}{2}$$

$y = y_c + y_p$

$y = c_1e^{-x} + c_2e^{-x} - xe^{-x} + x + 1/2$

8 Solve $(D^3+8)y = x^4+2x+1$ where $D = \frac{d}{dx}$

$\Rightarrow$ Given :- $(D^3+8)y = x^4+2x+1$

$\Rightarrow f(D)y = x^4+2x+1$

when $f(D) = D^3+8$

To find CF

The A.E is $f(m)=0$

$\Rightarrow m^3+8=0$

$\Rightarrow m^3+2^3=0$

$\Rightarrow (m+2)(m^2-2m+4)=0$

$\Rightarrow m+2=0, \; m^2-2m+4=0$

$m=-2, \quad m = \frac{-(-2)\pm\sqrt{(-2)^2-4(1)(4)}}{2(1)}$

$m = \frac{2\pm\sqrt{4-16}}{2}$

$m = 2\pm\sqrt{-12}$

$m = \frac{2\pm 2\sqrt{3}i}{2}$

$m = 1\pm 1\sqrt{3}$

$\therefore m = -2, \; 1\pm i\sqrt{3}$

$y = c_1e^{-2x} + [c_2\cos\sqrt{3}x + c_3\sin\sqrt{3}x]e^x$

To find PI

$y_p = \frac{x^4+2x+1}{f(D)}$

$y_p = \frac{x^4+2x+1}{D^3+8}$

$y_p = \frac{1}{8} \cdot \frac{x^4+2x+1}{1+\left(\frac{D^3}{8}\right)}$

$$y_p = \frac{1}{8}\left[1 - \frac{D^3}{8} + \frac{D^6}{64} - - - -\right](x^4+2x+1)$$

$$y_p = \frac{1}{8}\left[x^4+2x+1 - \frac{1}{8}(24x)\right]$$

$$y_p = \frac{1}{8}\left[x^4+2x+1 - 3x\right]$$

$$y_p = \frac{1}{8}(x^4 - x + 1)$$

$\therefore$ $y = y_c + y_p$

$$y = c_1 e^{-2x} + \left[c_2 \cos\sqrt{3}x + c_3 \sin\sqrt{3}x\right]e^{x} + \frac{1}{8}(x^4 - x + 1)$$

9) Solve $(D^2+4)y = x^2 + e^{-x}$ using Inverse differential operation method

$\Longrightarrow$ Given :- $(D^2+4)y = x^2 + e^{-x}$

$\Rightarrow f(D)y = x^2 + e^{-x}$

when $f(D) = D^2+4$

To find CF

The A.E is $f(m)=0$

$\Longrightarrow m^2+4=0$

$\Longrightarrow m^2=-4$

$\Longrightarrow m=\pm 2i$

$\Longrightarrow m=0\pm 2i$

$\therefore$ $y_c = c_1 \cos 2x + c_2 \sin 2x$

To find PI

$$y_p = \frac{x^2 + e^{-x}}{D^2+4}$$

$$y_p = \frac{x^2}{D^2+4} + \frac{e^{-x}}{D^2+4}$$

$$y_p = \frac{x^2}{4\left(1+\frac{D^2}{4}\right)} + \frac{e^{-x}}{(-1)^2+4}$$

$$y_p = \frac{1}{4}\left(1+\frac{D^2}{4}\right)^{-1} x^2 + \frac{e^{-x}}{5}$$

$$y_p = \frac{1}{4}\left[1-\frac{D^2}{4}+\frac{D^4}{16}+\cdots\right]x^2+\frac{e^{-x}}{5}$$

$$y_p = \frac{1}{4}\left[x^2-\frac{1}{4}(2)\right]+\frac{e^{-x}}{5}$$

$$y_p = \frac{x^2}{4}+\frac{e^{-x}}{5}-\frac{1}{8}$$

$\therefore\ y = y_c + y_p$

$$y = c_1\cos 2x + c_2 \sin 2x + \frac{x^2}{4} + \frac{e^{-x}}{5} - \frac{1}{8}$$

$\boxed{10}$ Solve $\frac{d^2y}{dx^2} + 3\frac{dy}{dx} + 2y = 1 + 3x + x^2$

Given :- $\frac{d^2y}{dx^2} + 3\frac{dy}{dx} + 2y = 1 + 3x + x^2$

$\Rightarrow (D^2 + 3D + 2)y = x^2 + 3x + 1$

$\Rightarrow f(D)y = x^2 + 3x + 1$

when $f(D) = D^2 + 3D + 2$

The A.E is $f(m) = 0$

$\Rightarrow m^2 + 3m + 2 = 0$

$\Rightarrow m(m+1) + 2(m+1) = 0$

$\Rightarrow m+1 = 0,\ m+2 = 0$

$\Rightarrow m = -1,\ m = -2$

$\therefore\ y = c_1 e^{-x} + c_2 e^{-2x}$

$$y_p = \frac{x^2+3x+1}{D^2+3D+2}$$

$$y_p = \frac{1}{2}\,\frac{x^2+3x+1}{\left[1+\frac{D^2+3D}{2}\right]}$$

$$y_p = \frac{1}{2}\left[1+\left(\frac{D^2+3D}{2}\right)\right]^{-1}(x^2+3x+1)$$

$$y_p = \frac{1}{2}\left[1-\frac{1}{2}(D^2+3D)+\frac{1}{4}(D^2+3D)^2 - - - -\right](x^2+3x+1)$$

$$y_p = \frac{1}{2}\left[1-\frac{1}{2}(D^2+3D)+\frac{1}{4}[D^4+6D^3+9D^2) - - - -\right](x^2+3x+1)$$

$$y_p = \frac{1}{2}\left[(x^2+3x+1)-\frac{1}{2}[2+3(2x+3)]+\frac{1}{4}[0+0+9(2)]\right]$$

$$y_p = \frac{1}{2}\left[x^3+3x+\frac{1}{2}(6x+11)+\frac{9}{2}\right]$$

$$y_p = \frac{1}{2}\left[x^3+3x-3x+1-\frac{11}{2}+\frac{9}{2}\right]$$

$$y_p = \frac{x^2}{2}$$

$\therefore$ $y = y_c + y_p$

$$y = c_1 e^{-x} + c_2 e^{-2x} + \frac{x^2}{2}$$

11. Solve $\frac{d^3y}{dx^3} + 2\frac{d^2y}{dx^2} + \frac{dy}{dx} = x^3$

$\Rightarrow$ Given :- $\frac{d^3y}{dx^3} + 2\frac{d^2y}{dx^2} + \frac{dy}{dx} = x^3$

$\Rightarrow$ $(D^3+2D^2+D)y = x^3$

$\Rightarrow$ $f(D)y = x^3$

When $f(D) = D^3+2D^2+D$

The A.E is $f(m)=0$

$$\Rightarrow m^3+2m^2+m=0$$

$$\Rightarrow m(m^2+2m+1)=0$$

$$\Rightarrow m(m+1)^2=0$$

$$\Rightarrow m=0,\ m=-1,-1$$

$$\therefore y_c = c_1 + (c_2+c_3x)e^{-x}$$

$$y_p = \frac{x^3}{f(D)}$$

$$y_p = \frac{x^3}{D^3+2D^2+D}$$

$$y_p = \frac{x^3}{D(D^2+2D+1)}$$

$$y_p = \frac{\frac{1}{D}x^3}{(D+1)^2}$$

$$y_p = \frac{\int x^3\,dx}{(1+D)^2}$$

$$y_p = \frac{1}{4}\frac{x^4}{(1+D)^2}$$

$$y_p = \frac{1}{4}(1+D)^{-2}x^4$$

$$y_p = \frac{1}{4}\left[1-2D+3D^2-4D^3+5D^4 - - - -\right]x^4$$

$$y_p = \frac{1}{4}\left[x^4-8x^3+3(12x^2)-4(24x)+5(24)\right]$$

$$y_p = \frac{1}{4}\left[x^4-8x^3+36x^2-96x+120\right]$$

$$\therefore y = y_c + y_p$$

$$y = c_1+(c_2+c_3x)e^{-x}+\frac{1}{4}\left[x^4-8x^3+36x^2-96x+120\right]$$

12. Solve $(D^2+4)y = x^2+\cos 2x$. where $D = \frac{d}{dx}$

$\Rightarrow$ Given :- $(D^2+4)y = x^2+\cos 2x$

$$\Rightarrow f(D)y = x^2+\cos 2x$$

where $f(D) = D^2+4$

The A.E is

$$f(m) = 0$$

$$m^2+4 = 0$$

$$m^2 = -4$$

$$m = \pm 2i$$

$$\therefore y_c = c_1\cos 2x + c_2\sin 2x$$

$$y_p = \frac{x^2+\cos 2x}{D^2+4}$$

$$y_p = \frac{x^2}{D^2+4} + \frac{\cos 2x}{D^2+4}$$

$$y_p = \frac{x^2}{4\left(1+\frac{D^2}{4}\right)} + \frac{x}{2\times 2}\sin 2x$$

$$= \frac{1}{4}\left(1+\frac{D^2}{4}\right)^{-1}x^2 + \frac{x}{4}\sin 2x$$

$$= \frac{1}{4}\left(1-\frac{D^2}{4}+\frac{D^4}{8}+ - - - - +\right)x^2 + \frac{x}{4}\sin 2x$$

$$= \frac{1}{4}\left(x^2-\frac{2}{4}\right) + \frac{x}{4}\sin 2x$$

$$= \frac{1}{4}\left(x^2-\frac{1}{2}\right) + \frac{x}{4}\sin 2x$$

$$y_p = \frac{x^2}{4} - \frac{4}{8} + \frac{x}{4}\sin 2x$$

$$\therefore y = y_c + y_p$$

$$y = c_1\cos 2x + c_2\sin 2x + \frac{x^2}{4} - \frac{1}{8} + \frac{x}{4}\sin 2x$$

Method of Variation of parameters

Step 1 :- Write the given DE $p_0 \frac{d^2y}{dx^2} + p_1 \frac{dy}{dx} + p_2 y = Q(x)$ for constant p_0, p_1, p_2

Step 2 :- Write the same in the form of $f(D)y = Q(x)$ and find its Complimentary function as $y_c = c_1 y_1 + c_2 y_2$

Step 3 :- Write the Soln for the given DE by replacing c_1 & c_2 by A & B, we get $y = Ay_1 + By_2$ and where

$$A = -\int \frac{y_2 Q(x)}{W} dx + k_1$$

$$B = \int \frac{y_1 Q(x)}{W} dx + k_2$$

$$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = y_1 y_2' - y_2 y_1'$$

13/ Solve $\frac{d^2y}{dx^2} + y = \sec x . \tan x$ Using the method of Variation of parameter

$\Rightarrow$ Given:- $\frac{d^2y}{dx^2} + y = \sec x . \tan x$

$$\Rightarrow (D^2+1)y = \sec x . \tan x$$

$$\Rightarrow f(D)y = \sec x . \tan x$$

When $f(D) = D^2+1$

the A.E is $f(m) = 0$

$$m^2+1=0$$

$$m^2=-1$$

$$m = 0 \pm 1i$$

$\therefore y_c = c_1\cos x + c_2\sin x$

the soln is $y_c = Ay_1 + By_2$

$y_1 = \cos x \Longrightarrow y_1' = -\sin x$

$y_2 = \sin x \Rightarrow y_2' = \cos x$

$$w = y_1y_2' - y_2y_1'$$
$$w = \cos^2 x + \sin^2 x$$
$$w = 1$$

$A = -\int \frac{y_2\, Q(x)}{w}\, dx + k_1$

$= -\int \frac{\sin x \cdot \sec x \cdot \tan x}{1}\, dx + k_1$

$= -\int \frac{\sin x}{\cos x} \cdot \tan x\, dx + k_1$

$= -\int \tan^2 x\, dx + k_1$

$= -(\sec^2 x - 1)\, dx + k_1$

$A = -\tan x + x + k_1$

$B = \int \frac{y_1\, Q(x)}{w}\, dx + k_2$

$= \int \frac{\cos x \cdot \sec x \cdot \tan x}{1}\, dx + k_2$

$= \int \left(\frac{\cos x}{\cos x} \cdot \tan x\right) dx + k_2$

$= \int ((1) \cdot \tan x)\, dx + k_2$

$B = \log(\sec x) + k_2$

$$y = Ay_1 + By_2$$
$$y = (x - \tan x + k_1)\cos x + [\log(\sec x) + k_2]\sin x$$

14) Solve $\frac{d^2y}{dx^2} + y = \sec x$

$\Longrightarrow$ given :- $\frac{d^2y}{dx^2} + y = \sec x$

$\Rightarrow (D^2+1)y = \sec x$

$\Rightarrow f(D)y = \sec x$

when $f(D) = D^2 + 1$

The A.E is $f(m)=0$

$$m^2+1=0$$

$$m = 0 \pm 1i$$

$$\therefore\ y_c = c_1\cos x + c_2\sin x$$

The sol^n is $y = Ay_1 + By_2$

$$y_1 = \cos x \Rightarrow y_1' = -\sin x$$

$$y_2 = \sin x \Rightarrow y_2' = \cos x$$

$$W = y_1y_2' - y_2y_1'$$

$$W = \cos^2 x + \sin^2 x$$

$$W = 1$$

$$A = -\int \frac{y_2 Q(x)}{W}\,dx + k_1$$

$$= -\int (\sin x \cdot \sec x)\,dx + k_1$$

$$= -\int \frac{\sin x}{\cos x}\,dx + k_1$$

$$= -\int \tan x\,dx + k_1$$

$$= -\log(\sec x) + k_1$$

$$A = \log(\cos x) + k_1$$

$$B = \int \frac{y_1 Q(x)}{W}\,dx + k_2$$

$$= \int \frac{\cos x \cdot \sec x}{1}\,dx + k_2$$

$$= \int \frac{\cos x}{\cos x}\,dx + k_2$$

$$= \int (1)\,dx + k_2$$

$$B = x + k_2$$

$$y = Ay_1 + By_2$$

$$y = (\log(\cos x) + k_1)\cos x + (x + k_2)\sin x$$

$\boxed{15}$ Solve $\frac{d^2y}{dx^2} + y = \tan x$

$\Longrightarrow$ Given :- $\frac{d^2y}{dx^2} + y = \tan x$

$$\Rightarrow (D^2+1)y = \tan x$$

$$\Rightarrow f(D)y = \tan x$$

When $f(D) = D^2+1$

The A.E is $f(m)=0$

$$m^2+1=0$$
$$m^2=-1$$
$$m = 0 \pm 1i$$

$$y_c = c_1 \cos x + c_2 \sin x$$

The soln is $y = Ay_1 + By_2$

$$y_1 = \cos x \Rightarrow y_1' = \sin x$$
$$y_2 = -\sin x \Rightarrow y_2' = \cos x$$

$$W = y_1 y_2' - y_2 y_1'$$
$$W = \cos x(\cos x) - (-\sin x)(\sin x)$$
$$W = \cos^2 x + \sin^2 x$$
$$W = 1$$

$$A = -\int \frac{y_2 \phi(x)}{W} dx + k_1$$
$$A = -\int (-\sin x \cdot \tan x) dx + k_1$$
$$A = -\int \frac{\sin^2 x}{\cos x} dx + k_1$$
$$A = -\int \frac{1 - \cos^2 x}{\cos x} dx + k_1$$
$$A = -\int (\sec x - \cos x) dx + k_1$$
$$A = -\log(\sec x + \tan x) + \sin x + k_1$$

$$B = \int \frac{y_1 \phi(x)}{W} dx + k_2$$
$$B = \int (\cos x \cdot \tan x) dx + k_2$$
$$B = \int \sin x \, dx + k_2$$
$$B = -\cos x + k_2$$

$$y = [-\log(\sec x + \tan x) + \sin x + k_1]\cos x + [-\cos x + k_2]\sin x$$

16 Solve $\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + 2y = e^x \tan x$

$\Rightarrow$ Given:- $\left(\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + 2\right)y = e^x \tan x$

$\Rightarrow (D^2 - 2D + 2)y = e^x \tan x$

$\Rightarrow f(D)y = e^x \tan x$

when $f(D) = D^2 - 2D + 2$

The A.E is $f(m) = 0$

$m^2 - 2m + 2 = 0$

$\Rightarrow m = \frac{2 \pm \sqrt{4 - 4(1)(2)}}{2\times 1}$

$\Rightarrow m = \frac{2 \pm \sqrt{-4}}{2}$

$m = 1 \pm i$

$y_c = [c_1 \cos x + c_2 \sin x] e^x$

$y_c = c_1 e^x \cos x + c_2 e^x \sin x$

$y_c = Ay_1 + By_2$

$y_1 = e^x \cos x \Rightarrow y_1' = e^x \cos x - e^x \sin x$

$y_2 = e^x \sin x \Rightarrow y_2' = e^x \sin x + e^x \cos x$

$W = y_1 y_2' - y_2 y_1'$

$W = [e^x \cos x][e^x \sin x + e^x \cos x] - [e^x \sin x][e^x \cos x - e^x \sin x]$

$W = e^{2x}[\cancel{\cos x \sin x} + \cos^2 x - \cancel{\cos x \sin x} + \sin^2 x]$

$W = e^{2x}$

$$A = -\int \frac{y_2 Q(x)}{w} dx + k_1$$

$$= -\int \left(\frac{e^x \sin x\, e^x \tan x}{e^{2x}}\right) dx + k_1$$

$$= -\int \frac{\sin^2 x}{\cos x} dx + k_1$$

$$= -\int \frac{(1-\cos^2 x)}{\cos x} dx + k_1$$

$$= -\int (\sec x - \cos x) dx + k_1$$

$$A = -\log(\sec x + \tan x) + \sin x + k_1$$

$$B = \int \frac{y_1 Q(x)}{w} dx + k_2$$

$$= \int \frac{e^x \cos x \cdot e^x \tan x}{e^{2x}} dx + k_2$$

$$B = -(\cos x + k_2)$$

$$y = [-\log(\sec x + \tan x) + \sin x + k_1] e^x \sin x - (\cos x + k_2) e^x \cos x$$

17) Solve $\frac{d^2y}{dx^2} + y = \frac{1}{1+\sin x}$ using the method of variation parameters

$\Rightarrow$ Given :- $\frac{d^2y}{dx^2} + y = \frac{1}{1+\sin x}$

$$\Rightarrow (D^2+1)y = \frac{1}{1+\sin x}$$

$$\Rightarrow f(D)y = \frac{1}{1+\sin x}$$

The A.E is $f(m)=0$

$$\Rightarrow m^2+1=0$$

$$\Rightarrow m^2=-1$$

$$m = 0 \pm 1i$$

$$\therefore y_c = c_1 \cos x + c_2 \sin x$$

The soln is

$$y = Ay_1 + By_2$$

$$y_1 = \cos x \Rightarrow y_1' = -\sin x$$

$$y_2 = \sin x \Rightarrow y_2' = \cos x$$

$$W = y_1 y_2' - y_2 y_1'$$
$$W = \cos x(\cos x) - (\sin x)(-\sin x)$$
$$W = \cos^2 x + \sin^2 x$$
$$W = 1$$

$$A = -\int \frac{y_2\, \phi(x)}{W}\, dx + K_1$$
$$= -\int \left(\sin x \cdot \frac{1}{1+\sin x}\right) dx + K_1$$
$$= -\int \frac{\sin x\,(1-\sin x)}{1-\sin^2 x}\, dx + K_1$$
$$= -\int \frac{\sin x - \sin^2 x}{\cos^2 x}\, dx + K_1$$
$$= +\int \frac{\sin^2 x - \sin x}{\cos^2 x}\, dx + K_1$$
$$= \int (\tan^2 x - \tan x \cdot \sec(x))\, dx + K_1$$
$$= \int (\sec^2 x - 1) - \tan x \cdot \sec x\, dx + K_1$$
$$A = \tan x - x - \sec x + K_1$$

$$B = \int y_1 \frac{\phi(x)}{W}\, dx + K_2$$
$$= \int \frac{\cos x}{(1+\sin x)}\, dx + K_2$$
$$= \int \frac{\cos x\,(1-\sin x)}{1-\sin^2 x}\, dx + K_2$$
$$= \int \frac{\cos x - \sin x \cdot \cos x}{\cos^2 x}\, dx + K_2$$
$$= \int (\sec x - \tan x)\, dx + K_2$$
$$= \log(\sec x + \tan x) - \log(\sec x)$$
$$B = \log\left[\frac{\sec x + \tan x}{\sec x}\right] + K_2$$

$$y = [\tan x - \sec x - x + K_2]\cos x + \left[\log\left[\frac{\sec x + \tan x}{\sec x}\right] + K_2\right]\sin x$$

18\. Solve $\frac{d^2y}{dx^2} - y = \frac{2}{1+e^x}$ by the method of Variation of parameter

$\Rightarrow$ Given $\frac{d^2y}{dx^2} - y = \frac{2}{1+e^x}$

$$\Rightarrow (D^2-1)y = \frac{2}{1+e^x}$$

$$\Rightarrow f(D)y = \frac{2}{1+e^x}$$

The A.E is $f(m) = 0$

$$m^2 - 1 = 0$$

$$m^2 = \pm 1$$

$$\therefore \quad y_c = c_1 e^{-x} + c_2 e^{x}$$

$$y = A y_1 + B y_2$$

$$y_1 = e^{-x} \Rightarrow y_1' = -e^{-x}$$

$$y_2 = e^{x} \Rightarrow y_2' = e^{x}$$

$$w = y_1 y_2' - y_2 y_1'$$

$$w = e^{-x}(e^{x}) + e^{x}(e^{-x})$$

$$w = 1 + 1$$

$$w = 2$$

$$A = -\int y_2 \frac{q(x)}{w} dx + k_1$$

$$= -\int \frac{e^{x} \times \frac{2}{1+e^{x}}}{2} dx + k_1$$

$$= -\log[1+e^{x}] + k_1$$

$$A = -\log\left[\frac{1}{1+e^{x}}\right] + k_1$$

$$B = \int \frac{y_1 \, q(x)}{w} + k_2$$

$$= \int e^{-x} \frac{2}{\frac{1+e^{x}}{2}} dx + k_2$$

$$= \int \frac{e^{-x}}{1+e^{x}} dx + k_2$$

$$= \int \frac{1}{e^{x}(1+e^{x})} dx + k_2$$

$$= \int \frac{e^{x}}{(e^{x})^2 (1+e^{x})} dx + k_2$$

$$e^{x} = t$$

$$e^{x} dx = dt$$

$$= \int \frac{1}{t^2(1+t)}\,dt + k_2$$

$$= \int \left(\frac{1}{t^2} - \frac{1}{t} + \frac{1}{t+1}\right) dt + k_2$$

$$= \frac{-1}{t} - \log t + \log(t+1) + k_2$$

$$= \frac{1}{-e^x} - \log e^x + \log(e^x+1) + k_2$$

$$= -e^{-x} - \cancel{\log}\, x + \log(1+e^x) + k_2$$

$$B = \log(1+e^x) - e^{-x} - x + k_2$$

$$y = \left[-\log\left(\frac{1}{1+e^x}\right) + k_1\right] - e^{-x} + \left[\log(1+e^x) - e^{-x} - x + k_2\right] e^x$$

19. Solve by the Variation of parameter $y'' - 6y' + 9y = \frac{e^{3x}}{x^2}$

$\Rightarrow$ Given:- $y'' - 6y' + 9y = \frac{e^{3x}}{x^2}$

$$\Rightarrow (D^2 - 6D + 9)y = \frac{e^{3x}}{x^2}$$

$$\Rightarrow (D-3)^2 y = \frac{e^{3x}}{x^2}$$

The A.E is $f(m) = 0$

$$(m-3)^2 = 0$$

$$m = 3, 3$$

$$y_c = (c_1 + c_2 x)e^{3x}$$

$$y_c = c_1 e^{3x} + c_2 x e^{3x}$$

$$y = Ay_1 + By_2$$

$$y_1 = e^{3x} \Longrightarrow y_1' = 3e^{3x}$$

$$y_2 = xe^{3x} \Longrightarrow y_2' = e^{3x} + 3xe^{3x}$$

$$w = y_1 y_2' - y_2 y_1'$$

$$w = e^{3x}\left(e^{3x} + 3xe^{3x}\right) - xe^{3x} \cdot 3e^{3x}$$

$$w = e^{6x} + 3xe^{6x} - 3xe^{6x}$$

$$w = e^{6x}$$

$$A = -\int \frac{y_2 Q(x)}{w}\,dx + k_1 \qquad B = \int \frac{y_1 Q(x)}{w}\,dx + k_2$$

$$= -\int \frac{xe^{3x}\, e^{3x}/x^2}{e^{6x}}\,dx + k_1 \qquad = \int \frac{e^{3x} \cdot e^{3x}/x^2}{e^{6x}}\,dx + k_2$$

$$= -\int \frac{1}{x}\,dx + k_1 \qquad = \int \frac{1}{x^2}\,dx + k_2$$

$$= -\log x + k_1 \qquad B = -\frac{1}{x} + k_2$$

$$A = -\log x + k_1$$

$$A = -\log(1/x) + k_1$$

$$y = (-\log x + k_1)e^{3x} + \left(\frac{-1}{x} + k_2\right)e^{3x} + 3xe^{3x}$$

Legendre's Linear differential Equation

Let the Equation $a_0(ax+b)^3 \frac{d^3y}{dx^3} + a_1(ax+b)^2 \frac{d^2y}{dx^2} + a_2(ax+b)\frac{dy}{dx} + a_3 y = Q(x) \rightarrow (1)$ is called Legendre's Linear differential Equation for the Constants a_0, a_1, a_2, a_3 a and b of 3rd order

Step 1 :- Take $\log_e(ax+b) = z$

$$\Rightarrow ax + b = e^z$$

$$\Rightarrow ax = e^z - b$$

$$x = \frac{e^z - b}{a}$$

Step 2 :- Write $(ax+b)\frac{dy}{dx} = a_1 Dy$

$$(ax+b)^2\frac{d^2y}{dx^2} = a^2D(D-1)y \text{ and}$$

$$(ax+b)^3\frac{d^3y}{dx^3} = a^3D(D-1)(D-2)y, \text{ where } D=\frac{d}{dz}$$

Simplify the given D.E by substituting the above and solve

Step 3:- finally in the soln replace $z=\log(ax+b)$

Step 4 :- If $a=1$ & $b=0$ in the given legendre's Equation then

Eq ① $\Rightarrow a_0x^3\frac{d^3y}{dx^3}+a_1x^2\frac{d^2y}{dx^2}+a_2x\frac{dy}{dx}+a_3y=a(x)$ is called Cauchy's linear differential eqn of III rd order

20 Solve $(3x+2)^2y''+3(3x+2)y'-36y=8x^2+4x+1$

$\Rightarrow$ Given :- $(3x+2)^2y''+3(3x+2)y'-36y=3x^2+4x+1$

$\log_e(3x+2)=z$

$\Rightarrow 3x+2=e^z$

$x=\frac{e^z-2}{3}$

and $(3x+2)y'=3Dy$

$(3x+2)^2y''=3^2D(D-1)y$ where $D=\frac{d}{dz}$

$$① \Rightarrow 9D(D-1)y+3\cdot3Dy-36y=8\left[\frac{e^z-2}{3}\right]^2+4\left[\frac{e^z-2}{3}\right]+1$$

$$\Rightarrow [9D^2-9Dy+9Dy-36y]=\frac{8}{9}(e^{2z}-4e^z+4)+\frac{4}{3}(e^z-2)+1$$

$$\Rightarrow [9D^2-9D+9D-36]y=\frac{1}{9}[8e^{2z}-32e^z+32+12e^z-24+9]$$

$$\Rightarrow 9(D^2-4)y=\frac{1}{9}[8e^{2z}-20e^z+17]$$

$$\Rightarrow (D^2-4)y=\frac{1}{81}[8e^{2z}-20e^z+17] \longrightarrow ①$$

The AE is $f(m)=0$

$m^2-4=0$

$\Rightarrow (m-2)(m+2)=0$

$\Rightarrow m=-2,2$

$$y_c = c_1 e^{-2z} + c_2 e^{2z}$$

$$y_p = \frac{1}{81}\left[\frac{8e^{2z} - 20e^{z} + 17}{D^2 - 4}\right)$$

$$= \frac{1}{81}\left[\frac{8e^{2z}}{D^2-4} - \frac{20e^{z}}{D^2-4} + \frac{17}{D^2-4}\right]$$

$$= \frac{1}{81}\left[\frac{8e^{2z}}{(D-2)(D+2)} - \frac{20e^{z}}{(1-4)} + \frac{17}{(0-4)}\right]$$

$$y_p = \frac{1}{81}\left[\frac{8z}{1!}\frac{e^{2z}}{4} - \frac{20e^{z}}{3} - \frac{17}{4}\right]$$

$$\therefore \quad y = y_c + y_p$$

$$y = c_1 e^{-2z} + c_2 e^{2z} + \frac{1}{81}\left[2ze^{2z} + \frac{20}{3}e^{z} - \frac{17}{4}\right]$$

$$y = \frac{c_1}{(e^{z})^2} + c_2(e^{z})^2 + \frac{1}{81}\left[2z(e^{z})^2 + \frac{20}{3}e^{z} - \frac{17}{4}\right]$$

$$y = \frac{c_1}{(3x+2)^2} + c_2(3x+2)^2 + \frac{1}{81}\left[2(3x+2)^2 \cdot \log(3x+2) + \frac{20}{3}(3x+2) - \frac{17}{4}\right]$$

21 Solve $(1+x)^2\frac{d^2y}{dx^2} - (1+x)\frac{dy}{dx} + y = 2\sin[\log(1+x)]$

$\Rightarrow$ Given :- $(1+x)^2\frac{d^2y}{dx^2} - (1+x)\frac{dy}{dx} + y = 2\sin[\log(1+x)]$

Let $\log(x+1) = z$

$$(x+1)y' = D.y$$

$$(x+1)^2 y'' = D(D-1)y \quad \text{where } D = \frac{d}{dz}$$

$$① \Rightarrow [D(D-1)y - Dy + y] = 2\sin z$$

$$\Rightarrow [D^2 - D - D + 1]y = 2\sin z$$

$$\Rightarrow [D^2 - 2D + 1]y = 2\sin z$$

The A.E is $f(m)=0$

$$m^2-2m+1=0$$

$$\Rightarrow (m-1)^2=0$$

$$\Rightarrow m=1,1$$

$$y_c=(c_1+c_2z)e^z$$

$$y_p=\frac{2\sin z}{D^2-2D+1}$$

$$=\frac{2\sin z}{-1^2-2D+1}$$

$$=\frac{2\sin z}{-2D}$$

$$=-\frac{\sin z}{D}$$

$$=-\int \sin z\, dz$$

$$y_p=\cos z$$

$$y=y_c+y_p$$

$$y=(c_1+c_2z)e^x+\cos z$$

$$y=[c_1+c_2\log(x+1)](x+1)+\cos[\log(x+1)]$$

$\boxed{22}$ Solve $(1+x)^2\frac{d^2y}{dx^2}+(1+x)\frac{dy}{dx}+y=\sin[2\log(1+x)]$

$\Rightarrow$ Given:- $(1+x)^2\frac{d^2y}{dx^2}+(1+x)\frac{dy}{dx}+y=\sin[2\log(1+x)]$

Let $\log(1+x)=z$

and $(1+x)\frac{dy}{dx}=Dy$

$(1+x)^2\frac{d^2y}{dx^2}=D(D-1)y$ where $D=\frac{d}{dz}$

$(1)\Rightarrow D(D-1)y+Dy+y=\sin 2z$

$\Rightarrow [D(D-1)+D+1]y=\sin 2z$

$$\Rightarrow (D^2-D+D+1)y = \sin 2z$$

$$\Rightarrow f(D)y = \sin 2z$$

The A.E is $f(m)=0$

$$m^2+1=0$$

$$m = 0 \pm i$$

$$\therefore y = c_1 \cos z + c_2 \sin z$$

$$y_p = \frac{\sin 2z}{f(D)}$$

$$y_p = \frac{\sin 2z}{D^2+1}$$

$$y_p = \frac{\sin 2z}{(-2)^2+1}$$

$$y_p = -\frac{1}{3}\sin 2z$$

$$\therefore y = y_c + y_p$$

$$y = c_1 \cos z + c_2 \sin z - \frac{1}{3}\sin 2z$$

$$y = c_1 \cos[\log(1+x)] + c_2 \sin[\log(1+x)] - \frac{1}{3}\sin[2\log(1+x)]$$

23] $(2x+3)^2 y'' - (2x+3)y' - 12y = 6x$

$\Rightarrow$ Let $\log(2x+3) = z$

$$2x+3 = e^z$$

$$x = \frac{e^z-3}{2}$$

and $(2x+3)y' = 2 \cdot Dy$

$$(2x+3)^2 y'' = 4D(D-1)y \quad \text{where } D = \frac{d}{dz}$$

$$(1) \Rightarrow 4D(D-1)y - 2Dy - 12y = 6\left(\frac{e^z-3}{2}\right)$$

$$\Rightarrow (4D^2 - 4D - 2D - 12)y = 3(e^z-3)$$

$$\Rightarrow (4D^2 - 6D - 12)y = 3(e^z - 3)$$

$$\Rightarrow (2D^2 - 3D - 6)y = \frac{3}{2}(e^z - 3)$$

$$f(D)y = \frac{3}{2}(e^z - 3) \longrightarrow ②$$

The A.E is $f(m) = 0$

$$2m^2 - 3m - 6 = 0$$

$$m = \frac{3 \pm \sqrt{9 - 4(2)(-6)}}{2(2)}$$

$$m = \frac{3 \pm \sqrt{57}}{4}$$

$$m = \frac{3+\sqrt{57}}{4}, \quad m = \frac{+3+\sqrt{57}}{4}$$

$$\therefore \; y_c = c_1 e^{\left(\frac{3-\sqrt{57}}{4}\right)z} + c_2 e^{\left(\frac{3+\sqrt{57}}{4}\right)z}$$

$$y_p = \frac{3}{2}\left[\frac{e^z - 3}{2D^2 - 3D - 6}\right]$$

$$y_p = \frac{3}{2}\left[\frac{e^z}{2D^2 - 3D - 6} - \frac{3e^0 z}{2D^2 - 3D - 6}\right]$$

$$y_p = \frac{3}{2}\left[\frac{e^z}{2 - 3 - 6} - \frac{3}{(-6)}\right]$$

$$y_p = \frac{3}{2}\left[\frac{1}{2} - \frac{e^z}{7}\right]$$

$$y = y_c + y_p$$

$$y = c_1(2x+3)^{\frac{3-\sqrt{57}}{4}} + c_2(2x+3)^{\frac{3+\sqrt{57}}{4}} + \frac{3}{2}\left[\frac{1}{2} - \frac{e^z}{7}(2x+3)\right]$$

24) Solve $x^2\left(\frac{d^2y}{dx^2}\right)-x\left(\frac{dy}{dx}\right)+y=\log x$

$\Rightarrow$ Given:- $x^2\left(\frac{d^2y}{dx^2}\right)-x\left(\frac{dy}{dx}\right)+y=\log x$

Let $\log x = z$

$x = e^z$

and W.K.T

$x^2y'' = D(D-1)y$

$xy' = Dy \;,\; D=\frac{d}{dz}$

$(1) \Rightarrow D(D-1)y - Dy + y = z$

$[D^2 - D - D + 1]y = z$

$(D-1)^2 y = z$

$f(D)y = z$

The A.E is $f(m)=0$

$(m-1)^2=0$

$m=1,1$

$\therefore y_c = (c_1+c_2 z)e^z$

$y_p = \frac{z}{(D-1)^2}$

$y_p = \frac{z}{(1-D)^2}$

$= (1-D)^{-2}$

$= (1+2D+3D^2+ - - - - -)z$

$y_p = z+2$ where $D=\frac{d}{z}$

$\Rightarrow y = y_c + y_p$

$\Longrightarrow y = (c_1+c_2 z)e^z + z + 2$

$\Longrightarrow y = [c_1 + c_2 \log x]x + (\log x) + 2$ //

25 Solve $x^2\frac{d^2y}{dx^2}-3x\frac{dy}{dx}+4y=(1+x)^2$

$\Rightarrow$ Given :- $x^2\frac{d^2y}{dx^2}-3x\frac{dy}{dx}+4y=(1+x)^2$

Let $\log x = z$

$x=e^z$

$xy' = D.y$

$x^2y'' = D(D-1)y$ where $D=\frac{d}{dz}$

① $\Rightarrow$ $D(D-1)y-3Dy+4y=(1+x)^2$

$\Rightarrow (D^2-D)y-3Dy+4y=(1+x)^2$

$\Rightarrow (D^2-D-3D+4)y=(1+x)^2$

$\Rightarrow (D^2-2D+4)y=(1+x)^2$

$f(D)y=(1+x)^2$

A.E is $f(m)=0$

$(m-2)^2=0$

$m=2,2$

$\therefore y_c=(c_1+c_2z)e^{2z}$

$$y_p=\frac{(1+e^z)^2}{(D-2)^2}$$

$$y_p=\frac{1+e^{2z}+2e^{2z}}{(D-2)^2}$$

$$y_p=\frac{1}{(D-2)^2}+\frac{e^{2z}}{(D-2)^2}+\frac{2e^{2z}}{(D-2)^2}$$

$$y_p=\frac{e^{2z}}{(D-2)(D+2)}+\frac{2e^{2z}}{(D-2)4}+\frac{1}{(D-2)^2}$$

$$W = y_1 y_2' - y_2 y_1'$$

$$W = e^{-z}(-2e^{-z}) - e^{-2z}(-e^{-z})$$

$$W = -2e^{-3z} + e^{-3z}$$

$$W = e^{-3z}$$

$$A = -\int \frac{y_2 Q(x)}{W} dz + k_1$$

$$A = -\int \frac{e^{-2z} e^{e^z}}{e^{-z} \cdot e^{-2z}} dz + k_1$$

$$A = -\int \frac{-e^{-2z} \cdot e^{e^z}}{e^{-z} \cdot e^{-2z}} dz + k_1$$

$$A = \int \frac{e^{e^z}}{e^{-z}} dz + k_1$$

$$A = \int e^{z} e^{e^z} dz + k_1$$

$$A = e^{e^z} + k$$

$$B = \int y_1 \frac{Q(x)}{W} dz + k_2$$

$$B = \int \frac{e^{-z} e^{e^z}}{-e^{-3z}} dz + k_2$$

$$B = -\int \frac{e^{-z} \cdot e^{e^z}}{e^{-z} \cdot e^{-2z}} dz + k_2$$

$$B = -\int e^{2z} e^{e^z} dz + k_2$$

$$B = -\int e^{z} e^{e^z} (e^{z} dz) + k_2$$

let $e^z = t$

$e^z dz = dt$

$$B = -\int t e^{t} dt + k_2$$

$$B = -((t-1)e^{t} + k_2$$

$$B = (1-t)e^{t} + k_2$$

$$B = (1-e^z) e^{e^z} + k_2$$

$$y = [e^{e^z} + k_1] e^{-z} + [(1-e^z) e^{e^z} + k_2] e^{-2z}$$

$$y = \left(\frac{e^{e^z} + k_1}{e^z}\right) + \frac{(1-e^z) e^{e^z} + k_2}{(e^z)^2}$$

$$y = \left[\frac{e^x + k_1}{x}\right] + \left[\frac{(1-x)e^x + k_2}{x^2}\right]$$

$$= \frac{ze^{2z}}{2!} + \frac{2ze^{z}}{1!} + \frac{1}{4}$$

$$y_p = \frac{\log x e^{2z}}{2} + 2\log x e^{z} + \frac{1}{4}$$

$$y = y_c + y_p$$

$$y = (c_1 + c_2 z)e^{2z} + \frac{\log x e^{2z}}{2} + 2\log x e^{z} + \frac{1}{4}$$

$\boxed{26}$ Solve $x^2y'' + 4xy' + 2y = e^x$

$\Rightarrow$ Given :- $x^2y'' + 4xy' + 2y = e^x \longrightarrow (1)$

Let $\log x = z$

$x = e^z$

and w.k.T $xy' = D.y$

$x^2y'' = D(D-1)y$ where $D = \frac{d}{dz}$

$(1) \Rightarrow D(D-1)y + 4Dy + 2y = e^{e^z}$

$\Rightarrow (D^2 - D + 4D + 2)y = e^{e^z}$

$\Rightarrow (D^2 + 3D + 2)y = e^{e^z}$

- The A.E is $f(m) = 0$

$m^2 + 3m + 2 = 0$

$\Rightarrow (m+1)(m+2) = 0$

$\Rightarrow m = -1, -2$

$$y_c = c_1 e^{-z} + c_2 e^{-2z}$$

$$y = Ay_1 + By_2$$

$y_1 = e^{-z} \Longrightarrow y_1' = -e^{-z}$

$y_2 = e^{-2z} \Longrightarrow y_2' = -2e^{-z}$

$$W = y_1 y_2' - y_2 y_1'$$
$$= e^{-z}(-2e^{-z}) - e^{-2z}(-e^{-z})$$
$$W = e^{-3z}$$

$$A = -\int \frac{y_2\, Q(x)}{W} dz + k_1$$
$$= -\int \frac{e^{-2x} \cdot e^{e^z}}{e^{-3z}} dz + k_1$$
$$= -\int \frac{-e^{-2z} \cdot e^{e^z}}{e^{-z} \cdot e^{-2z}} dz + k_1$$
$$= \int \frac{e^{e^z}}{e^{-z}} dz + k_1$$
$$= \int e^{e^z} \cdot e^{e^z} dz + k_1$$
$$A = e^{e^z} + k_1$$

$$B = \int \frac{y_1\, Q(x)}{W} dz + k_2$$
$$= \int \frac{e^{-z} \cdot e^{e^z}}{-e^{-3z}} dz + k_2$$
$$= -\int \frac{e^{-z} \cdot e^{e^z}}{e^{-z} \cdot e^{-2z}} dz + k_2$$
$$= -\int e^{2z} \cdot e^{e^z} dz + k_2$$
$$= -\int e^z \cdot e^{e^z} (e^z dz) + k_2$$
Let $e^z = t$
$$\Rightarrow e^z dz = dt$$
$$= -\int t e^t dt + k_2$$
$$= -(t-1)e^t + k_2$$
$$= (1-t)e^t + k_2$$
$$B = (1-e^z)e^{e^z} + k_2$$

$$y = [e^{e^z} + k_1]e^{-z} + [(1-e^z)e^{e^z} + k_2]e^{-2z}$$
$$y = \left[\frac{e^{e^z} + k_1}{e^z} + \frac{(1-e^z)e^{e^z} + k_2}{(e^z)^2}\right]$$
$$y = \left[\frac{e^x + k_1}{x}\right] + \left[\frac{(1-x)e^x + k_2}{x^2}\right]$$

27 Solve $x\frac{d^2y}{dx^2} - 2\frac{dy}{dx} = x + \frac{1}{x^2}$

$\Rightarrow$ Given :- $x^2\frac{d^2y}{dx^2} - 2y = x^2 + \frac{1}{x} \rightarrow (1)$

Let $\log x = z$
$x = e^z$

$x^2\frac{d^2y}{dx^2} = D(D-1)y$ where $D = \frac{d}{dz}$

$(1) \Rightarrow D(D-1)y - 2y = e^{2z} + \frac{1}{e^z}$

$\Rightarrow (D^2 - D)y - 2y = e^{2z} + e^{-z}$

$\Rightarrow (D^2 - D - 2)y = e^{2z} + e^{-z}$

$\Rightarrow f(D)y = e^{2z} + e^{-z}$

The A.E $f(m) = 0$

$\Rightarrow m^2 - m - 2 = 0$

$\Rightarrow m^2 - m - 2m - 2 = 0$

$\Rightarrow m(m+1) - 2(m+1) = 0$

$\Rightarrow (m+1)(m+2) = 0$

$m = -1, \; m = 2$

$\therefore y_c = c_1 e^{-z} + c_2 e^{2z}$

$$y_p = \frac{e^{2z} + e^{-z}}{D^2 - D - 2}$$

$$y_p = \frac{e^{2z}}{(D-2)(D+1)} + \frac{e^{-z}}{(D+1)(D-2)}$$

$$y_p = \frac{z^1}{(D-2)(D+1)} + \frac{e^{-z}}{(D-2)(D+1)}$$

$$y_p = \frac{z^1}{1!}\frac{e^{2z}}{3} - \frac{z}{3}e^{-z}$$

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$$y_p = \frac{z}{3}(e^{2z} - e^{-z})$$

$$y = y_c + y_p$$

$$y = c_1 e^{-z} + c_2 e^{2z} + \frac{z}{3}[e^{2z} - e^{-z}]$$

$$y = \frac{c_1}{x} + c_2 x^2 + \frac{\log x}{3}\left[x^2 - \frac{1}{x}\right]$$

$\boxed{28}$ Solve $(x^2D^2 + xD + 9)y = 3x^2 + \sin(3\log x)$, where $D = \frac{d}{dz}$

$\Rightarrow$ Given :- $(x^2D^2 + xD + 9)y = 3x^2 + \sin(3\log x) \longrightarrow (1)$

$$f(D)y = 3x^2 + \sin(3\log x)$$

When $f(D) = x^2D^2 + xD + 9$

$$\log x = z$$
$$x = e^z$$

the. W.K.T

$$x^2D^2 = D(D-1)y$$
$$xD = Dy$$

$(1) \Rightarrow D(D-1)y + Dy + 9y = 3e^{2z} + \sin(3\log x)$

$\Rightarrow (D^2 - D + D + 9)y = 3e^{2z} + \sin(3z)$

$\Rightarrow (D^2 + 9)y = 3e^{2z} + \sin(3z)$

The A.E $f(m) = 0$

$$m^2 + 9 = 0$$
$$m^2 = -9$$
$$m = 0 \pm 3i$$

$$\therefore y_c = (c_1 \cos 3z + c_2 \sin 3z)$$

$$y_p = \frac{3e^{2z}}{D^2+9} + \frac{\sin 3z}{D^2+9}$$

$$= \frac{3e^{2z}}{4+9} - \frac{z}{2 \times 3}\cos 3z$$

$$y_p = \frac{3e^{2z}}{13} - \frac{z}{6}\cos 3z$$

$$y = y_c + y_p$$

$$y = (c_1 \cos 3z + c_2 \sin 3z) + \frac{3e^{2z}}{13} - \frac{z}{6}\cos 3z$$

$$y = c_1 \cos 3(\log x) + c_2 \sin 3(\log x) + \frac{3x^2}{13} - \frac{\log x}{6}\cos 3(\log x)$$

Application of differential Equation

29] The differential eqn of the displacement $x(t)$ of a spring fixed at the upper end a weight at its lower end is given by $10\frac{d^2x}{dt^2} + \frac{dx}{dt} + 200x = 0$ the weight is pulled down 0.25cm below the Equilibrium position and then released. find the expression displacement of the weight from its Equilibrium position at any time during its Ist upward motion

$\Rightarrow$ Given :- the displacement x of a spring oscillations and at any time (t)

The relation of a displacement x with a time t as given $10\frac{d^2x}{dt^2} + \frac{dx}{dt} + 200x = 0$

It is Second order D.E for damped oscillations of an oscillator for any $T = \frac{d}{dt}$. we have

$$(1) \Rightarrow (10D^2 + D + 200)x = 0$$

$$f(D)x = 0$$

where $f(D) = 10D^2 + D + 200$

the A.E is $f(m) = 0$

$$\Rightarrow 10m^2 + m + 200 = 0$$

$$m = \frac{-1 \pm \sqrt{1 - 4(10)(200)}}{2(10)}$$

$$m = \frac{-1 \pm \sqrt{1 - 8000}}{20}$$

$$m = \frac{-1 \pm \sqrt{-7999}}{20}$$

$$m = \frac{-1 \pm i\,89.4371}{20}$$

$$m = \frac{-1}{20} \pm i\,\frac{89.4371}{20}$$

$$m = -0.05 \pm i\,4.4718$$

$$x(t) = [c_1 \cos(4.4718)t + c_2 \sin(4.4718)t]\, e^{-0.05t} \longrightarrow (2)$$

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When time $t = 0 \Rightarrow x = 0$

$(2) \Rightarrow \quad 0 = c_1(1) + c_2(0)$

$c_1 = 0$

$(2) \Rightarrow \quad x(t) = c_2 e^{-0.05t} \sin(4.4718)t$

put the amplitude $c_2 = 0.25\,cm$

$$x(t) = (0.25)\, e^{-0.05t} \sin(4.4718)t$$

Be the displacement of spring travelled at any time (t)

30 If an LCR - circuit. The charge on a plate of a condenser is given by $L\frac{d^2q}{dt^2} + R\frac{dq}{dt} + \frac{q}{c} = E\sin pt$, the circuit is tuned to resistance so that $p^2 = \frac{1}{LC}$. If Initially the current (i) and the charge (q) be zero. Show that for small values of $\frac{R}{L}$, the current is in the circuit at time (t) is given by $\frac{Et}{2L}\sin pt$

$\Rightarrow$ Given :- Inductance (L), resistance (R), capacitance (c) and charge of the battery (q) and given for any $p^2 = \frac{1}{LC} \Rightarrow p = \frac{1}{\sqrt{LC}}$, where, L, R, C, E are constants

$$L\frac{d^2q}{dt^2} + R\frac{dq}{dt} + \frac{q}{c} = E\sin pt \longrightarrow (1)$$

Let $D = \frac{d}{dt}$

$\therefore (1) \Rightarrow (LD^2 + RD + \frac{1}{c})q = E\sin pt$

$\Rightarrow f(D)q = E\sin pt$

The A.E $f(m) = 0$

$\Rightarrow Lm^2 + Rm + \frac{1}{c} = 0$

$\Rightarrow CLm^2 + CRm + 1 = 0$

$$m = \frac{-Rc \pm \sqrt{(Rc)^2 - (4)(cL)(1)}}{2(cL)}$$

$$m = \frac{-Rc \pm \sqrt{R^2c^2 - 4cL}}{2cL}$$

$$m = -\frac{Rc}{2cL} \pm \frac{\sqrt{R^2c^2 - 4cL}}{2cL}$$

$$m = -\frac{R}{2L} \pm \sqrt{\frac{R^2c^2}{4c^2L^2} - \frac{4cL}{4c^2L^2}}$$

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$$m = \frac{-R}{2L} \pm \sqrt{\frac{R^2}{4L^2} - \frac{1}{CL}}$$

$$m = \frac{-R}{2L} \pm \sqrt{\frac{1}{4}\left(\frac{R}{L}\right)^2 - p^2}$$

$\therefore \left(\frac{R}{L} \text{ is Small}\right)$

$$m = \frac{-R}{2L} \pm \sqrt{0 - p^2}$$

$$-m = \frac{-R}{2L} \pm pi$$

$$\therefore \quad q_c = (c_1 \cos pt + c_2 \sin pt) e^{-\frac{Rt}{2L}} \longrightarrow (2)$$

$$PI = \frac{E \sin pt}{LD^2 + RD + \frac{1}{C}}$$

$$PI = \frac{E \sin pt}{L(-p^2) + RD + \frac{1}{C}}$$

$$= \frac{E \sin pt}{-L\left(\frac{1}{LC}\right) + RD + \frac{1}{C}}$$

$$= \frac{E}{R} \frac{\sin pt}{D}$$

$$= \frac{E}{R} \int \sin pt \, dt$$

$$= \frac{-E}{RR} \cos pt$$

$$\therefore q = CF + PI$$

$$q = [c_1 \cos pt + c_2 \sin pt] e^{-\frac{RT}{2L}} - \frac{E}{Rp} \cos pt \longrightarrow (3)$$

$$\Longrightarrow q - \left[1 - \frac{Rt}{2L}\right][c_1 \cos pt + c_2 \sin pt] - \frac{E}{Rp} \cos pt \longrightarrow (4)$$

therfore $i(t) = \frac{dq}{dt}$

$$\Rightarrow i(t) = \left[1 - \frac{Rt}{2L}\right]\left[-c_1 p \sin pt + c_2 p \cos pt\right] - \frac{R}{2L}\left[c_1 \cos pt + c_2 \sin pt\right] + \frac{E}{R} \sin pt \longrightarrow (5)$$

when $t=0 \Rightarrow q=0$

$$\therefore (4) \Rightarrow 0 = c_1 - \frac{E}{Rp} \Rightarrow c_1 = \frac{E}{Rp}$$

when $t=0 \Rightarrow i=0$

$$\therefore (5) \Rightarrow 0 = c_2 p - \frac{Rc_1}{2L}$$

$$\Rightarrow c_2 p = \frac{Rc_1}{2L}$$

$$\Rightarrow c_2 = \frac{Rc_1}{2Lp}$$

$$\Rightarrow c_2 = \frac{R\left[\frac{E}{Rp}\right]}{2Lp}$$

$$\Rightarrow c_2 = \frac{E}{2Lp^2}$$

$$\Rightarrow c_2 = \frac{E}{2L \cdot \frac{1}{LC}}$$

$$\Rightarrow c_2 = \frac{EC}{2}$$

$$\therefore (5) \Rightarrow i(t) = \left[1 - \frac{Rt}{2L}\right]\left[\frac{-E}{Rp} p \sin pt + \frac{EC}{2} p \cos pt\right] - \frac{R}{2L}\left[\frac{E}{Rp} \cos pt + \frac{EC}{2} \sin pt\right] + \frac{E}{R} \sin pt$$

$$\Rightarrow i(t) = \cancel{\frac{-E}{R} \sin pt} + \frac{EC}{2} p \cos pt + \frac{Et}{2L} \sin pt - \frac{ECRt}{4L} p \cos pt - \cancel{\frac{E}{2Lp} \cos pt} - \frac{ERL}{4L} \sin pt + \cancel{\frac{E}{R} \sin pt}$$

$$\Rightarrow i(t) = \frac{Et}{2L} \sin pt \quad \left(\frac{R}{L} \text{ is small}\right)$$