CALCULUS AND LINEAR ALGEBRA

MODULE - 02

SERIES OF EXPANSION

Introduction :-

Suppose $y = f(x)$ be a real valued funtion defined on $[a, b]$ with the equal length $h = b - a$, then the Series of expansion can be defined as,

$$f(b) = f(a) + \frac{(b-a)}{1!} f'(a) + \frac{(b-a)^2}{2!} f''(a) + \frac{(b-a)^3}{3!} f'''(a) + \ldots$$

[equation 01]

when $(b-a) = h \Rightarrow b = a + h$, then

$$① \Rightarrow f(a+h)$$

$$= f(a) + \frac{h}{1!} f'(a) + \frac{h^2}{2!} f''(a) + \frac{h^3}{3!} f'''(a) + \ldots\ldots$$

[equation 02]

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when $(a+h) = x \Rightarrow h = x - a$

$$f(x) = f(a) + \frac{(x-a)}{1!} f'(a) + \frac{(x-a)^2}{2!} f''(a) + \frac{(x-a)^3}{3!} f'''(a) \ldots$$

[equation 03]

when $a = 0$, then

$$③ \Rightarrow f(x) = f(0) + \frac{x}{1!} f'(0) + \frac{x^2}{2!} f''(0) + \frac{x^3}{3!} f'''(0) + \ldots$$

is called the Maclaurin Series of expansion.

A.

1. Expand $\sin x$ in the powers of $(x - \pi/2)$ upto IVth degree term.

$\Rightarrow$ Let $f(x) = \sin x$ and $a = \frac{\pi}{2}$

$\Rightarrow f(\pi/2) = \sin(\pi/2) = 1$

$f'(x) = \cos x \Rightarrow f'\left(\frac{\pi}{2}\right) = \cos\left(\frac{\pi}{2}\right) = 0$

$f''(x) = -\sin x \Rightarrow f''\left(\frac{\pi}{2}\right) = -\sin\left(\frac{\pi}{2}\right) = -1$

$f'''(x) = -\cos x \Rightarrow f'''\left(\frac{\pi}{2}\right) = -\cos\left(\frac{\pi}{2}\right) = 0$

$f^{IV}(x) = \sin x \Rightarrow f^{IV}\left(\frac{\pi}{2}\right) = \sin\left(\frac{\pi}{2}\right) = 1$

$\therefore$ W K T

The expansion of tailor's Series in the powers of $x - a$

$$f(x) = f(a) + \frac{(x-a)}{1!} f'(a) + \frac{(x-a)^2}{2!} f''(a) + \ldots$$

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$$\Rightarrow \sin x = 1 + \frac{\left(x-\frac{\pi}{2}\right)}{1!}(0) + \frac{\left(x-\frac{\pi}{2}\right)^2}{2!}(-1) + \frac{\left(x-\frac{\pi}{2}\right)^3}{3!}(0)$$

$$+ \frac{\left(x-\frac{\pi}{2}\right)^4}{4!}(1) \ldots\ldots$$

$$\Rightarrow \sin x = 1 - \frac{\left(x-\frac{\pi}{2}\right)^2}{2!} + \frac{\left(x-\frac{\pi}{2}\right)^4}{4!} + \ldots\ldots$$

2. Expand $\log_e x$ in the powers of $(x-1)$ and hence find $\log_e(1.1)$

Given

$f(x) = \log_e x$; $a = 1$

$\Rightarrow f(a) = f(1) = \log_e(1) = 0$

$\Rightarrow f'(x) = \frac{1}{x} \Rightarrow f'(a) = f'(1) = \frac{1}{1} = 1$

$\Rightarrow f''(x) = -\frac{1}{x^2} \Rightarrow f''(a) = f''(1) = \frac{-1}{1^2} = -1$

$\Rightarrow f'''(x) = \frac{2}{x^3} \Rightarrow f'''(a) = f'''(1) = \frac{2}{1^3} = 2$

$\therefore$ W.K.T

$$f(x) = f(a) + \frac{(x-a)}{1!}f'(a) + \frac{(x-a)^2}{2!}f''(a) + \frac{(x-a)^3}{3!}f'''(a) + \ldots$$

$$\Rightarrow \log_e x = 0 + \frac{(x-1)}{1!}(1) + \frac{(x-1)^2}{2!}(-1) + \frac{(x-1)^3}{3!}(2) + \ldots$$

$$\Rightarrow \log_e x = (x-1) - \frac{(x-1)^2}{2!} + \frac{(x-1)^3}{3!} + \ldots\ldots$$

$$\therefore \log_e(1.1) = (1.1-1) - \frac{(1.1-1)^2}{2!} + \frac{(1.1-1)^3}{3!}$$

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$\Rightarrow \log_e(1.1) = 0.1 - \frac{0.01}{2} + \frac{0.001}{3}$

$\Rightarrow \log_e(1.1) \cong$

B.

1. using Maclaurin Series, Expand $\log(\sec x)$

Let $f(x) = \log(\sec x)$

$\Rightarrow f(0) = \log(\sec 0) = \log 1 = 0$

$f'(x) = \frac{1}{\sec x} \sec x . \tan x$

$\Rightarrow f'(x) = \tan x \Rightarrow f'(0) = \tan 0 = 0$

$\Rightarrow f''(x) = \sec^2 x \Rightarrow f''(0) = \sec^2(0) = 1$

$\Rightarrow f'''(x) = 2\sec^2 x . \tan x$

$= 2(1 + \tan^2 x)\tan x$

$= 2(\tan x + \tan^3 x)$

$\Rightarrow f'''(0) = 2(\tan 0 + \tan^3 0) = 0$

$\Rightarrow f^{iv}(x) = 2[\sec^2 x + 3\tan^2 x . \sec^2 x]$

$\Rightarrow f^{iv}(0) = 2[\sec^2 0 + 3\tan^2(0)\sec^2(0)] = 2[1+0] = 2$

$\therefore$ W.K.T

$f(x) = f(0) + \frac{x}{1!} f'(0) + \frac{x^2}{2!} f''(0) + \ldots$

$\Rightarrow \log(\sec x) = 0 + \frac{x}{1!}(0) + \frac{x^2}{2!}(1) + \frac{x^3}{3!}(0) + \frac{x^4}{4!}(2) \ldots$

$\Rightarrow \log(\sec x) = \frac{x^2}{2} + \frac{x^4}{12} + \ldots\ldots$

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2. Using Maclaurin Series, Prove that $\sqrt{1+\sin 2x}$

$= 1 + x - \frac{x^2}{2} - \frac{x^3}{6} + \frac{x^4}{24} \ldots\ldots$

Let $f(x) = \sqrt{1+\sin 2x}$

$\Rightarrow f(x) = \sqrt{\sin^2 x + \cos^2 x + 2\sin x \cos x}$

$\Rightarrow f(x) = \sqrt{(\sin x + \cos x)^2}$

$\Rightarrow f(x) = \sin x + \cos x \Rightarrow f(0) = \sin 0 + \cos 0 = 0+1 = 1$

$\Rightarrow f'(x) = \cos x - \sin x \Rightarrow f'(0) = \cos 0 - \sin 0 = 1-0 = 1$

$\Rightarrow f''(x) = \sin x - \cos x \Rightarrow f''(0) = -\sin 0 - \cos 0 = 0-1 = -1$

$\Rightarrow f'''(x) = -\cos x + \sin x \Rightarrow f'''(0) = -\cos 0 + \sin 0 = -1+0 = -1$

$\Rightarrow f^{IV}(x) = \sin x + \cos x \Rightarrow f^{(IV)}(0) = \sin 0 + \cos 0 = 0+1 = 1$

$\therefore$ W.K.T

$f(x) = f(0) + \frac{x}{1!} f'(0) + \frac{x^2}{2!} f''(0) + \ldots\ldots$

$\Rightarrow \sqrt{1+\sin 2x} = 1 + \frac{x}{1!}(1) + \frac{x^2}{2!}(-1) + \frac{x^3}{3!}(-1) + \frac{x^4}{4!}(1) \ldots$

$\Rightarrow \sqrt{1+\sin 2x} = 1 + x - \frac{x^2}{2} - \frac{x^3}{6} + \frac{x^4}{24} + \ldots\ldots$

3. Expand $\log(1+\cos x)$ by using Maclurin Series upto the term x^4

$\Rightarrow$ Let $f(x) = \log(1+\cos x) \Rightarrow f(0) = \log(1+\cos 0)$

$= \log(1+1) = \log 2$

$\Rightarrow f(x) = \log[2\cos^2(x/2)]$

$\Rightarrow f(x) = \log 2 + \log[\cos^2 \frac{x}{2}]$

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$\Rightarrow f(x) = \log 2 + 2\log[\cos x/2]$

$f'(x) = 0 + 2\dfrac{1}{\cos(x/2)}\left[-\sin(x/2)\cdot\dfrac{1}{2}\right]$

$\Rightarrow f'(x) = -\tan(x/2) \Rightarrow f'(0) = \tan(0) = 0$

$\Rightarrow f''(x) = -\sec^2(x/2)\dfrac{1}{2} \Rightarrow f''(0) = \dfrac{1}{2}\sec^2(0) = -\dfrac{1}{2}(1) = \dfrac{-1}{2}$

$\Rightarrow f'''(x) = -\dfrac{1}{2}\, 2\sec^2(x/2)\tan(x/2)\cdot\dfrac{1}{2} \Rightarrow f'''(0) = 0$

$= -\dfrac{1}{2}[1+\tan^2(x/2)]\tan(x/2)$

$= -\dfrac{1}{2}[\tan(x/2) + \tan^3(x/2)]$

$\Rightarrow f^{IV}(x) = -\dfrac{1}{2}\left[\dfrac{1}{2}\sec^2(x/2) + 3\tan^2(x/2)\sec^2(x/2)\cdot\dfrac{1}{2}\right]$

$\Rightarrow f^{IV}(0) = -\dfrac{1}{2}\left[\dfrac{1}{2}(1) + 0\right] = -\dfrac{1}{4}$

$\therefore$ W.K.T

$f(x) = f(0) + \dfrac{x}{1!}f'(0) + \dfrac{x^2}{2!}(f''(0)) + \ldots\ldots$

$\therefore \log(1+\cos x) = \log 2 + 0 + \dfrac{x^2}{2!}\left(-\dfrac{1}{2}\right) + 0 + \dfrac{x^4}{4!}\left(-\dfrac{1}{4}\right)\ldots$

$\Rightarrow \log(1+\cos x) = \log 2 - \dfrac{x^2}{4} - \dfrac{x^4}{96} + \ldots\ldots$

4. Expand $f(x) = \log|\sec x + \tan x|$ using Maclurin Series.

$\Rightarrow f(0) = \log|\sec 0 + \tan 0|$

$= \log(0|+0) = 0$

$\Rightarrow f'(x) = \dfrac{1}{\sec x + \tan x}[\sec x\tan x + \sec^2 x]$

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$\Rightarrow f'(x) = \dfrac{\sec x(\sec x + \tan x)}{(\sec x + \tan x)}$

$\Rightarrow f'(x) = \sec x \Rightarrow f'(0) = \sec 0 = 1$

$\Rightarrow f''(x) = \sec x \tan x \Rightarrow f''(0) = 0$

$\Rightarrow f'''(x) = \sec x . \sec^2 x + \sec x \tan x . \tan x$

$= \sec^3 x + \sec x \tan x$

$\Rightarrow f'''(0) = 1 + 0 = 1$

$\therefore$ W.K.T

$f(x) = f(0) + \dfrac{x}{1!} f'(0) + \dfrac{x^2}{2!} f''(0) + \ldots$

$\Rightarrow \log[\sec x + \tan x] = 0 + \dfrac{x}{1!}(1) + \dfrac{x^2}{2!}(0) + \dfrac{x^3}{3!}(1) + \ldots$

$\Rightarrow \log[\sec x + \tan x] = \dfrac{x}{1!} + \dfrac{x^3}{3!} + \dfrac{x^5}{5!} + \ldots$

5. obtain Maclurin Series $\log(1+x)$ upto iv^{th} term.

let, $f(x) = \log(1+x)$

$f(0) = \log(1+0) = \log 1 = 0$

$f'(x) = \dfrac{1}{1+x}(0+1) \Rightarrow \dfrac{1}{1+x} \Rightarrow f'(0) = \dfrac{1}{1+0} = 1$

$f''(x) = \dfrac{-1}{(1+x)^2}(1) = \dfrac{-1}{(1+x)^2} \Rightarrow f''(0) = \dfrac{-1}{(1+0)^2} = -1$

$f'''(x) = \dfrac{-2}{(1+x)^3} = \dfrac{-2}{(1+x)^3} \Rightarrow f'''(0) = \dfrac{-2}{(1+0)^3} = -2$

$f^{iv}(x) = \dfrac{2.(-3)}{(1+x)^4} = \dfrac{-6}{(1+0)^4} = -6$

$\therefore$ W.K.T

Maclurin Series.

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$$f(x) = f(0) + \frac{x}{1!} f'(0) + \frac{x^2}{2!} f''(0) + \frac{x^3}{3!} f'''(0) + \ldots$$

$$= 0 + \frac{x}{1!}(1) + \frac{x^2}{2!}(-1) + \frac{x^3}{3!}(-2) + \frac{x^4}{4!}(-6) + \ldots$$

$$\Rightarrow f(x) = 0 + \frac{x}{1} - \frac{x^2}{2} - \frac{x^3}{2} - \frac{x^4}{4} + \ldots$$

⇒ <u>Indeterminate forms</u> :-

Suppose $f(x)$ and $g(x)$ be the two functions and $\underset{x \to a}{Lt} f(x) = 0$, $\underset{x \to a}{Lt} g(x) = 0$ then we get,

$\underset{x \to a}{Lt} \frac{f(x)}{g(x)} = \frac{0}{0}$ is called Indeterminate form. and which will be evaluated by using 'L' Hospital rule and follows as

a. If $\underset{x \to a}{Lt} \frac{f(x)}{g(x)} = \frac{0}{0}$, then differentiate both the Numerator and denomenator with respect to Same variable then we get,

$$l = \underset{x \to a}{Lt} \frac{f'(x)}{g'(x)} = \frac{0}{0}$$

b. follow the Same procedure again as in the above Step

$$l = \underset{x \to a}{Lt} \frac{f''(x)}{g''(x)} = \frac{0}{0}$$ then Contiune the Same procedure untill to get the denomenator as to be not zero.

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C.

1. Find $\lim_{x \to 0} \frac{\sin x}{x}$

Let $l = \lim_{x \to 0} \frac{\sin x}{x} = \frac{\sin 0}{0} = \frac{0}{0}$

Apply 'L' Hospital Rule [LHR]

$l = \lim_{x \to 0} \frac{\cos x}{1}$

$\Rightarrow l = \frac{\cos 0}{1}$

$\Rightarrow l = \frac{1}{1}$

$\Rightarrow l = 1$

2. Find, $\lim_{x \to 0} \frac{\tan x}{x^2}$

Let, $l = \lim_{x \to 0} \frac{\tan x}{x^2} = \frac{\tan 0}{0^2} = \frac{0}{0}$

Apply 'L' Hospital [Rule]

$\therefore l = \lim_{x \to 0} \frac{\sec^2 x}{2x}$

$\Rightarrow l = \lim_{x \to 0} \frac{2\sec x \cdot \sec x \tan x}{2}$

$\Rightarrow l = \lim_{x \to 0} \sec^2 x \cdot \tan x = 1(0) = 0$

3. Find, $\lim_{x \to 0} \frac{a^x - b^x}{x}$

Let $l = \lim_{x \to 0} \frac{a^x - b^x}{x} = \frac{a^0 - b^0}{0} = \frac{1-1}{0} = \frac{0}{0}$

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Apply 'L' Hospital Rule [LHR]

$$\therefore\ l = \lim_{x \to 0} \frac{a^x \log a - b^x \log b}{1}$$

$$= a^0 \log a - b^0 \log b$$

$$= \log a - \log b$$

$$\Rightarrow l = \log(a/b)$$

4. Find, $\lim_{x \to \frac{\pi}{2}} \frac{\log(x - \frac{\pi}{2})}{\tan x}$

let, $l = \lim_{x \to \frac{\pi}{2}} \frac{\log(x - \pi/2)}{\tan x} = \frac{\infty}{0}$

Apply 'L' Hospital Rule [LHR]

$$\therefore\ l = \lim_{x \to \pi/2} \frac{\frac{1}{x - \pi/2}(1)}{\sec^2 x}$$

$$\Rightarrow l = \lim_{x \to \frac{\pi}{2}} \frac{\cos^2 x}{x - \pi/2} = \frac{0}{0}$$

$$\Rightarrow l = \lim_{x \to \pi/2} \frac{2\cos x(-\sin x)}{1 - 0}$$

$$\Rightarrow l = \lim_{x \to \pi/2} -2\sin x . \cos x$$

$$\Rightarrow l = 2\sin(\pi/2)\cos(\pi/2)$$

$$\Rightarrow l = -2(1)(0)$$

$$\Rightarrow l = 0$$

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Indeterminate forms, are of the types 1^∞, 0^0, ∞^0

D. Evaluate the following Indeterminate forms

1. $\lim_{x \to 0} \left(\frac{a^x + b^x}{2} \right)^{1/x}$

let $l = \lim_{x \to 0} \left(\frac{a^x + b^x}{2} \right)^{1/x} = 1^\infty$

$\Rightarrow \log l = \lim_{x \to 0} \log \left(\frac{a^x + b^x}{2} \right)^{1/x}$

$\Rightarrow \log l = \lim_{x \to 0} \frac{1}{x} \log \left[\frac{a^x + b^x}{2} \right]$

$\Rightarrow \log l = \lim_{x \to 0} \frac{\log(a^x + b^x) - \log 2}{x} = \frac{0}{0}$

Apply 'L' Hospital Rule

$\therefore \log l = \lim_{x \to 0} \frac{\frac{1}{a^x + b^x} [a^x \log a + b^x \log b] - 0}{1}$

$\Rightarrow \log l = \lim_{x \to 0} \frac{a^x \log a + b^x \log b}{a^x + b^x}$

$\Rightarrow \log l = \frac{\log a + \log b}{2}$

$\Rightarrow \log l = \frac{1}{2} \log(ab)$

$\Rightarrow \log l = \log (ab)^{1/2}$

$\Rightarrow l = (ab)^{1/2}$

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2. $\lim_{x\to 0} \left(\frac{a^x+b^x+c^x+d^x}{4}\right)^{1/x}$

let $l = \lim_{x\to 0} \frac{(a^x+b^x+c^x+d^x)}{4} = 1^\infty$

$\Rightarrow \log l = \lim_{x\to 0} \log\left|\left(\frac{(a^x+b^x+c^x+d^x)}{4}\right)^{1/x}\right|$

$\Rightarrow \log l = \lim_{x\to 0} \frac{1}{x}\log\left|\frac{a^x+b^x+c^x+d^x}{4}\right|$

$\Rightarrow \log l = \lim_{x\to 0} \frac{\log(a^x+b^x+c^x+d^x)-\log 4}{x} = \frac{0}{0}$

Apply 'L' Hospital Rule

$\Rightarrow \log l = \lim_{x\to 0} \frac{\frac{1}{a^x+b^x+c^x+d^x}\left[a^x\log a+b^x\log b+c^x\log c+d^x\log d\right]}{1}$

$\Rightarrow \log l = \lim_{x\to 0} \frac{a^x\log a+b^x\log b+c^x\log c+d^x\log d}{a^x+b^x+c^x+d^x}$

$\Rightarrow \log l = \frac{\log a+\log b+\log c+\log d}{4}$

$\Rightarrow \log l = \frac{1}{4}\log(abcd)$

$\Rightarrow \log l = \log(abcd)^{1/4}$

$\Rightarrow \cancel{\log} l = \cancel{\log}(abcd)^{1/4}$

$\Rightarrow l = (abcd)^{1/4}$

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3. $\lim_{x\to 0} \left(\frac{a^x+b^x+c^x}{3}\right)^{1/x}$

let $l = \lim_{x\to 0} \frac{(a^x+b^x+c^x)^{1/x}}{3} = 1^\infty$

$\Rightarrow \log l = \lim_{x\to 0} \log\left(\frac{a^x+b^x+c^x}{3}\right)^{1/x}$

$\Rightarrow \log l = \lim_{x\to 0} \frac{1}{x}\log\left(\frac{a^x+b^x+c^x}{3}\right)$

$\Rightarrow \log l = \lim_{x\to 0} \frac{\log(a^x+b^x+c^x)-\log 3}{x} = \frac{0}{0}$

Apply 'L' Hospital Rule

$\Rightarrow \log l = \lim_{x\to 0} \frac{\frac{1}{a^x+b^x+c^x}\left[a^x\log a + b^x\log b + c^x\log c\right]}{1}$

$\Rightarrow \log l = \lim_{x\to 0} \frac{a^x\log a + b^x\log b + c^x\log c}{a^x+b^x+c^x}$

$\Rightarrow \log l = \frac{\log a + \log b + \log c}{3}$

$\Rightarrow \log l = \frac{1}{3}\log(abc)$

$\Rightarrow \log l = \log(abc)^{1/3}$

$\Rightarrow l = (abc)^{1/3}$

4. $\lim_{x\to 0} \left(\frac{\tan x}{x}\right)^{1/x}$

let $l = \lim_{x\to 0} \left(\frac{\tan x}{x}\right)^{1/x} = \infty^\infty$

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$$\Rightarrow \log l = \underset{x\to 0}{Lt} \log\left(\left(\frac{\tan x}{x}\right)^{1/x}\right)$$

$$\Rightarrow \log l = \underset{x\to 0}{Lt} \frac{1}{x} \log\left(\frac{\tan x}{x}\right)$$

$$\Rightarrow \log l = \underset{x\to 0}{Lt} \frac{\log(\tan x) - \log x}{x} = \frac{\infty}{0}$$

$$\Rightarrow \log l = \underset{x\to 0}{Lt} \frac{\frac{1}{\tan x} \sec^2 x - \frac{1}{x}}{1}$$

$$\Rightarrow \log l = \underset{x\to 0}{Lt} \frac{\sec^2 x}{\tan x} - \frac{1}{x}$$

$$\Rightarrow \log l = \underset{x\to 0}{Lt} \frac{\sec^2 x}{x(\tan x/x)} - \frac{1}{x}$$

$$\Rightarrow \log l = \underset{x\to 0}{Lt} \frac{\sec^2 x}{x} - \frac{1}{x} \quad \left[\because \underset{x\to 0}{Lt} \frac{\tan x}{x} = 1\right]$$

$$\Rightarrow \log l = \underset{x\to 0}{Lt} \frac{\sec^2 x - 1}{x} = \frac{0}{0}$$

$$\Rightarrow \log l = \underset{x\to 0}{Lt} \frac{2\sec^2 x \tan x - 0}{1}$$

$$\Rightarrow \log l = \underset{x\to 0}{Lt} 2\sec^2 x \cdot \tan x$$

$$\Rightarrow \log_e l = 0$$

$$\Rightarrow l = e^0$$

$$\Rightarrow l = 1$$

E. <u>Find the following indeterminate forms.</u>

1. $\underset{x\to\infty}{Lt} \left(1 + \frac{a}{x}\right)^x$

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Let $l = \underset{x\to\infty}{Lt} \left(1+\frac{a}{x}\right)^x = 1^\infty$

$\Rightarrow \log l = \underset{x\to\infty}{Lt} \log\left(1+\frac{a}{x}\right)^x$

$\Rightarrow \log l = \underset{x\to\infty}{Lt}\, x \log\left(1+\frac{a}{x}\right)$

$\Rightarrow \log l = \underset{x\to\infty}{Lt}\, x \log\left(\frac{x+a}{x}\right)$

$\Rightarrow \log l = \underset{x\to\infty}{Lt} \dfrac{\log(x+a) - \log x}{\frac{1}{x}} = \dfrac{\infty}{0}$

$\Rightarrow \log l = \underset{x\to\infty}{Lt} \dfrac{\frac{1}{x+a} - \frac{1}{x}}{(-1/x^2)}$

$\Rightarrow \log l = \underset{x\to\infty}{Lt} \dfrac{x-x-a}{(-1/x^2)(x(x+a))}$

$\Rightarrow \log l = \underset{x\to\infty}{Lt} \dfrac{x}{x+a}$

$\Rightarrow \log l = a \underset{x\to\infty}{Lt} \dfrac{x/x}{\frac{x+a}{x}}$

$\Rightarrow \log l = a \underset{x\to\infty}{Lt} \dfrac{1}{1+a/x}$

$\Rightarrow \log l = a\left(\frac{1}{1+0}\right)$

$\Rightarrow \log l = a(1)$

$\Rightarrow \log_e l = a$

$\Rightarrow l = e^a$

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2. $\lim_{x \to \frac{\pi}{2}} (\sin x)^{\tan x}$

let $l = \lim_{x \to \frac{\pi}{2}} (\sin x)^{\tan x}$

$\Rightarrow \log l = \lim_{x \to \frac{\pi}{2}} \tan x \log(\sin x)$

$\Rightarrow \log l = \lim_{x \to \frac{\pi}{2}} \frac{\log(\sin x)}{\cot x}$

Apply 'L' Hospital Rule

$\Rightarrow \log l = \lim_{x \to \pi/2} \frac{\cot x}{-\operatorname{cosec}^2 x}$

$\Rightarrow \log l = \lim_{x \to \frac{\pi}{2}} \frac{-0}{1}$

$\Rightarrow \log l = 0$

$\Rightarrow l = e^0$

$\Rightarrow l = 1$

3. $\lim_{x \to 0} \left(\frac{\sin x}{x}\right)^{1/x^2}$

Let $l = \lim_{x \to 0} \left(\frac{\sin x}{x}\right)^{1/x^2} = 1^\infty$

$\Rightarrow \log l = \lim_{x \to 0} \log\left[\left(\frac{\sin x}{x}\right)^{1/x^2}\right]$

$\Rightarrow \log l = \lim_{x \to 0} \frac{\log\left(\frac{\sin x}{x}\right)}{x^2}$

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$$\Rightarrow \log l = \lim_{x \to 0} \frac{\log(\sin x) - \log x}{x^2}$$

$$\Rightarrow \log l = \frac{\cot x - 1/x}{2x}$$

$$\Rightarrow \log l = \lim_{x \to 0} \frac{x \cot x - 1}{2x^2} = \frac{-1}{0}$$

$$\Rightarrow \log l = \lim_{x \to 0} \frac{x \cos x - \sin x}{2x^2 \cdot \sin x}$$

$$= \lim_{x \to 0} \frac{x \cos x \cdot \sin x}{2x^3}$$

$$= \lim_{x \to 0} \frac{\cos x - x \sin x - \cos x}{6x^2}$$

$$= \lim_{x \to 0} \frac{-x \sin x}{6x^2}$$

$$= \frac{1}{6} \lim_{x \to 0} \left(\frac{\sin x}{x} \right)$$

$$\Rightarrow \log l = \frac{-1}{6}$$

$$\Rightarrow l = e^{-1/6}$$

⇒ <u>Partial</u> <u>differentiation.</u>

Suppose $z = f(x, y)$ be a function in the variables of x and y and the first and Second order partial derivatives of $z = f(x, y)$ are defined as,

a. $\frac{\partial z}{\partial x} = \frac{\partial t}{\partial x} = z_x = f_x$

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b. $\frac{\partial z}{\partial y} = \frac{\partial t}{\partial y} = z_y = f_y$

c. $\frac{\partial^2 z}{\partial x^2} = \frac{\partial^2 t}{\partial x^2} = z_{xx} = f_{xx}$

d. $\frac{\partial^2 z}{\partial y^2} = \frac{\partial^2 z}{\partial y^2} = z_{yy} = f_{yy}$

e. $\frac{\partial}{\partial x}\left(\frac{\partial^2 z}{\partial y}\right) = \frac{\partial^2 z}{\partial x \partial y} = \frac{\partial^2 f}{\partial x \partial y} = z_{xy} = f_{xy}$

f. $\frac{\partial}{\partial y}\left(\frac{\partial z}{\partial z}\right) = \frac{\partial^2 z}{\partial y \partial z} = \frac{\partial^2 f}{\partial y \partial x} = z_{yx} = f_{yz}$

where,

$$\boxed{\frac{\partial^2 z}{\partial x \partial y} = \frac{\partial^2 z}{\partial y \partial x}}$$

Note :-

1. Differentiating a function with respect to the partially one independed variable, others have to be consider as a constants.

F.1. Find the first and Second order partial derivatives of the function $z = x^3 + y^3 - 3xy$

$\Rightarrow$ Let $z = x^3 + y^3 - 3xy \longrightarrow (1)$

d eq^n (1) w r t partial derivative.

$(1) \Rightarrow \frac{\partial z}{\partial x} = \frac{\partial}{\partial x}(x^3) + \frac{\partial}{\partial x}(y^3) - \frac{\partial}{\partial x}(3xy)$

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$$\Rightarrow \frac{\partial z}{\partial x} = \frac{\partial}{\partial x}(x^3) + y^3\frac{\partial}{\partial x}(1) - 3y\frac{\partial}{\partial x}(x)$$

$$\Rightarrow \frac{\partial z}{\partial x} = 3x^2 + y^3(0) - 3y(1)$$

$$\Rightarrow \frac{\partial z}{\partial x} = 3x^2 - 3y$$

Similarlly, $\frac{\partial z}{\partial y} = \frac{\partial}{\partial y}(x^3) + \frac{\partial}{\partial y}(y^3) - \frac{\partial}{\partial y}(3xy)$

$$\Rightarrow \frac{\partial z}{\partial y} = x^3\frac{\partial}{\partial y}(1) + \frac{\partial}{\partial y}(y^3) - 3x\frac{\partial}{\partial y}(y)$$

$$\Rightarrow \frac{\partial z}{\partial y} = 0 + 3y^2 - 3x(1)$$

$$\Rightarrow \frac{\partial z}{\partial y} = 3y^2 - 3x$$

$$\therefore \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial^2 z}{\partial x^2} = \frac{\partial}{\partial x}(3x^2) - \frac{\partial}{\partial x}(3y)$$

$$\Rightarrow \frac{\partial^2 z}{\partial x^2} = 3(2x) - 0 = 6x$$

Then,

$$\frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial^2 z}{\partial y^2} - \frac{\partial}{\partial y}(3y^2) - \frac{\partial}{\partial x}(3x) = 3(2y) - 0 = 6y$$

$$\therefore \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial^2 z}{\partial x \partial y} = \frac{\partial}{\partial x}[3y^2 - 3x]$$

$$= 3y^2\frac{\partial}{\partial x}(1) - 3\frac{\partial}{\partial x}(x) = 3y^2(0) - 3(1) = -3$$

$$\frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial^2 z}{\partial y \partial x} = \frac{\partial}{\partial y}(3x^2 - 3y)$$

$$= 3x^2\frac{\partial}{\partial y}(y) = 3x^2(0) - 3(1) = -3$$

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$$\therefore \frac{\partial^2 z}{\partial x \partial y} = \frac{\partial^2 z}{\partial y \partial z}$$

2. Find the first and Second order partial derivative of the function $z = x^2 + 2x\cos y + y^2$

Let, $z = x^3 + 2x\cos y + y^2 \longrightarrow (1)$

$$\frac{\partial z}{\partial x} = \frac{\partial}{\partial x}(x^2) + 2\cos y \frac{\partial}{\partial x}(x) + y^2 \frac{\partial}{\partial x}(1)$$

$$\Rightarrow \frac{\partial z}{\partial x} = 2x + 2\cos y + 0$$

$$\Rightarrow \frac{\partial z}{\partial x} = 2x + 2\cos y$$

$$\Rightarrow \frac{\partial z}{\partial y} = x^2 \frac{\partial}{\partial y}(1) + 2x \frac{\partial}{\partial y}(\cos y) + \frac{\partial}{\partial y}(y^2)$$

$$\Rightarrow \frac{\partial z}{\partial y} = 2y - 2x\sin y$$

$$\Rightarrow \frac{\partial^2 z}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right) = 2\frac{\partial}{\partial x}(x) + 2\cos y \frac{\partial}{\partial x}(1)$$

$$\Rightarrow \frac{\partial^2 z}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right) = 2\frac{\partial}{\partial y}(y) - 2x\frac{\partial}{\partial y}(\sin y)$$

$$\Rightarrow \frac{\partial^2 z}{\partial y^2} = 2 - 2x\cos y$$

$$\Rightarrow \frac{\partial^2 z}{\partial x \partial y} = \frac{\partial}{\partial x}\left[\frac{\partial z}{\partial y}\right] = 2y\frac{\partial}{\partial x}(1) - 2\sin y \frac{\partial}{\partial x}(x)$$

$$\frac{\partial^2 z}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right) = 2x\frac{\partial}{\partial y}(1) + 2\frac{\partial}{\partial y}[\cos y]$$

$$= 2x(0) + 2[-\sin y] = -2\sin y$$

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$$\therefore \frac{\partial^2 z}{\partial x \partial y} = \frac{\partial^2 z}{\partial y \partial x}$$

3. If $u = x^3 - 3xy^2 + x + e^x \cos y + 1$, show that

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

Given,

$$y = x^3 - 3xy^2 + x + e^x \cos y + 1$$

$$\frac{\partial u}{\partial x} = \frac{\partial}{\partial x}(x^3) - 3y^2 \frac{\partial}{\partial x}(x) + \cos y \frac{\partial}{\partial x}(e^x) + \frac{\partial}{\partial x}(0)$$

$$\Rightarrow \frac{\partial u}{\partial x} = 3x^2 - 3y^2(1) + 1 + \cos y\, e^x + 0$$

$$\Rightarrow \frac{\partial u}{\partial x} = 3x^2 - 3y^2 + e^x \cos y + 1 \longrightarrow (1)$$

$$\Rightarrow \frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial y}{\partial x}\right) = 3\frac{\partial}{\partial x}(x^2) - 3y^2 \frac{\partial}{\partial x}(1) + \cos y \frac{\partial}{\partial x}(e^x) + \frac{\partial}{\partial x}(1)$$

$$\Rightarrow \frac{\partial^2 y}{\partial x^2} = 6x - 0 + e^x \cos y + 0 = 6x + e^x \cos y \longrightarrow (2)$$

Similary,

$$\frac{\partial u}{\partial y} = x^3 \frac{\partial}{\partial y}(1) - 3x \frac{\partial}{\partial y}(y^2) + x \frac{\partial}{\partial y}(1) + e^x \frac{\partial}{\partial y} \cos y + \frac{\partial}{\partial y}(1)$$

$$\Rightarrow \frac{\partial u}{\partial y} = 0 - 6xy + 0 - e^x \sin y + 0$$

$$\therefore \frac{\partial^2 u}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial y}\right) = -6x \frac{\partial}{\partial y}(y) - \cos y\, e^x \longrightarrow (2)$$

$$\therefore \text{LHS} = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 6x + e^x \cos y - 6x - e^x \cos y = 0$$

$$\therefore \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

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4. If $r^2 = (x-a)^2 + (y-b)^2 + (z-c)^2$, than prove that

$$\frac{\partial^2 r}{\partial x^2} + \frac{\partial^2 r}{\partial y^2} + \frac{\partial^2 r}{\partial z^2} = \frac{2}{r}$$

let $r^2 = (x-a)^2 + (y-b)^2 + (z-c)^2 \longrightarrow (1)$

differentiate (1) w.r.t. x partially

$$(1) \Rightarrow \frac{\partial}{\partial r}(r^2) = 2(x-a)(1) + 0 + 0$$

$$\Rightarrow \frac{\partial}{\partial r}(r^2)\frac{\partial r}{\partial x} = 2(x-a)$$

$$\Rightarrow 2r\frac{\partial r}{\partial x} = 2(x-a)$$

$$\Rightarrow r\frac{\partial r}{\partial x} = x-a$$

$$\Rightarrow \frac{\partial r}{\partial x} = \frac{x-a}{r} \longrightarrow (2)$$

differentiate (2) w r t x partially

$$(2) \Rightarrow \frac{\partial}{\partial x}\left(\frac{\partial r}{\partial x}\right) = \frac{\partial}{\partial x}\left[\frac{x-a}{r}\right]$$

$$\Rightarrow \frac{\partial^2 r}{\partial x^2} = \frac{r\frac{\partial}{\partial x}(x-a) - (x-a)\frac{\partial r}{\partial x}}{r^2}$$

$$\Rightarrow \frac{\partial^2 r}{\partial x^2} = \frac{r(1) - (x-a)(x-a/r)}{r^2}$$

$$\Rightarrow \frac{\partial^2 r}{\partial x^2} = \frac{r(1) - \frac{(x-a)^2}{r}}{r^2}$$

$$\Rightarrow \frac{\partial^2 r}{\partial x^2} = \frac{1}{r}\frac{(x-a)^2}{r^3} \longrightarrow (3)$$

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Similarlly

$$\frac{\partial^2 r}{\partial y^2} = \frac{1}{r} - \frac{(y-b)^2}{r^3} \longrightarrow (4)$$

$$\frac{\partial^2 r}{\partial z^2} = \frac{1}{r} - \frac{(z-c)^2}{r^3} \longrightarrow (5)$$

$$\therefore \text{LHS} = \frac{\partial^2 r}{\partial x^2} + \frac{\partial^2 r}{\partial y^2} + \frac{\partial^2 r}{\partial z^2} = \frac{1}{r} - \frac{(x-a)^2}{r^3} + \frac{1}{r} \frac{(y-b)^2}{r^3} + \frac{1}{r} - \frac{(z-c)^2}{r^3}$$

$$= \frac{3}{r} - \frac{1}{r^3}\left((x-a)^2 + (y-b)^2 + (z-c)^2\right)$$

$$= \frac{3}{r} - \frac{r^2}{r^3}$$

$$= \frac{3}{r} - \frac{1}{r}$$

$$= \frac{2}{r} \;//$$

⇒ <u>Total derivative of a function</u>

Suppose $u = f(x, y, z)$ be a function having the Independent variables (x, y, z), then the total derivative of $u = f(x, y, z)$ can be defined as

$$du = \frac{\partial u}{\partial x} \cdot dx + \frac{\partial u}{\partial y} dy + \frac{\partial u}{\partial z} \cdot dz$$

Suppose $u = f(x, y, z)$ be a function having 3 Independent Variables and (x, y, z) are in the function of the independent variable 't' i.e

$x = x(t)$, $y = y(t)$, $z = z(t)$

Then,

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$$\frac{du}{dt} = \frac{\partial u}{\partial x}\frac{dx}{dt} + \frac{\partial u}{\partial y}\frac{dy}{dt} + \frac{\partial u}{\partial z}\frac{dz}{dt}$$

If $u = f(p, q, r)$ and $p = p(x, y, z)$, $q = q(x, y, z)$ $r = r(x, y, z)$.

Then,

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial p}\frac{\partial p}{\partial x} + \frac{\partial u}{\partial q}\frac{\partial q}{\partial x} + \frac{\partial u}{\partial r}\frac{\partial r}{\partial x}$$

$$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial p}\frac{\partial p}{\partial y} + \frac{\partial u}{\partial q}\frac{\partial q}{\partial y} + \frac{\partial u}{\partial r}\frac{\partial r}{\partial y}$$

$$\frac{\partial u}{\partial z} = \frac{\partial u}{\partial p}\cdot\frac{\partial p}{\partial z} + \frac{\partial u}{\partial q}\frac{\partial q}{\partial z} + \frac{\partial u}{\partial r}\frac{\partial r}{\partial z}$$

G.

a. Find the total derivative of the following function and also verify the result by direct substitution.

1. $z = xy^2 + x^2y$ where $x = at$, $y = 2at$

Given,

$z = xy^2 + x^2y \longrightarrow (1)$

$x = at$, $y = 2at$

$$\frac{\partial z}{\partial x} = y^2 + 2xy$$

$$\frac{\partial z}{\partial y} = 2xy + x^2$$

$$\frac{\partial x}{\partial t} = u \cdot \frac{\partial v}{\partial t} = 2a$$

$\therefore$ W.K.T

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$$\frac{\partial z}{\partial t} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}$$

$$\Rightarrow \frac{\partial z}{\partial t} = (y^2 + 2xy)\,a + (2xy + x^2)\,2a$$

$$\Rightarrow \frac{\partial z}{\partial t} = a\left[(2at)^2 + 2(at)(2at)\right] + \left[2(at)(2at) + (at)^2\right]$$

$$\Rightarrow \frac{\partial z}{\partial t} = a\left[4a^2t^2 + 4a^2t^2\right] + 2a\left[4a^2t^2 + a^2t^2\right]$$

$$= 8a^3t^2 + 10a^3t^2$$

$$\Rightarrow \frac{\partial z}{\partial t} = 18a^3t^2$$

Verification,

① $\Rightarrow z = (at)(2at)^2 + (at)^2(2at)$

$$z = 4a^3t^3 + 2a^3t^3$$

$$z = 6a^3t^3$$

$$\therefore \frac{\partial z}{\partial t} = 6a^3(3t^2)$$

$$\Rightarrow \frac{\partial z}{\partial t} = 18a^3t^2$$

2. $u = xy + yz + zx$ where $x = t\cos t$, $y = t\sin t$, $z = t$, at $t = \frac{\pi}{4}$

Given,

$x = t\cos t$

$y = t\sin t$

$z = t$

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$$\therefore \frac{\partial u}{\partial x} = y + 0 + z = y + z$$

$$\frac{\partial u}{\partial y} = x + z + 0 = x + z$$

$$\frac{\partial u}{\partial z} = 0 + y + x = y + x$$

$$\therefore \frac{dx}{dt} = \cos t - t\sin t$$

$$\frac{dy}{dt} = \sin t + t\cos t$$

$$\frac{dz}{dt} = 1$$

W.K.T

$$\frac{du}{dt} = \frac{\partial u}{\partial x}\frac{dx}{dt} + \frac{\partial u}{\partial y}\frac{dy}{dt} + \frac{\partial u}{\partial z}\frac{dz}{dt}$$

$$\Rightarrow \frac{du}{dt} = (y+z)(\cos t - t\sin t) + (x+z)(\sin t + t\cos t) + (y+x)(1)$$

$$\Rightarrow \frac{du}{dt} = (t\sin t + t)(\cos t - t\sin t) + (t\cos t + t)(\sin t + t\cos t) + (t\sin t + t\cos t)$$

$$\Rightarrow \frac{du}{dt} = t\sin t \cdot \cos t - t^2\sin^2 t + t\cos t - t^2\sin t + t\sin t\cos t + t^2\cos^2 t + t\sin t + t^2\cos t + t\sin t + t\cos t$$

$$\Rightarrow \frac{du}{dt} = 2t\sin t\cos t + t^2(\cos^2 t - \sin^2 t) + (\cos t - \sin t)t^2 + 2t(\sin t + \cos t)$$

when,

$$t = \frac{\pi}{4}$$

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$$\Rightarrow \frac{du}{dt} = 2\left(\frac{\pi}{4}\right)\left(\frac{1}{\sqrt{2}}\right)\left(\frac{1}{\sqrt{2}}\right) + \frac{\pi^2}{16}\left(\frac{1}{2} - \frac{1}{2}\right) + \left(\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}\right)\frac{\pi^2}{16}$$

$$+ \frac{\pi}{2}\left(\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}\right)$$

$$\frac{du}{dt} = \frac{\pi}{4} + \frac{\pi}{2} \cdot \frac{2}{\sqrt{2}}$$

$$\frac{du}{dt} = \frac{\pi}{4} + \frac{\pi}{\sqrt{2}}$$

Verification :-

① $\Rightarrow u = (t\cos t)(t\sin t) + (t\sin t)t + (t\cos t)t$

$\Rightarrow u = \frac{1}{2} t^2 (2\sin t \cdot \cos t) + t^2\cos t + t^2\sin t$

$\Rightarrow u = \frac{1}{2} t^2 (\sin 2t) + \cos t + t^2 \sin t \longrightarrow$ ②

differentiate ② w.r.t

$$\therefore \frac{du}{dt} = \frac{1}{2}\left[2t\sin 2t + 2t^2\cos 2t\right] + 2t\cos t - t^2\sin t$$

$$+ 2t\sin t + t^2\cos t$$

$$\Rightarrow \frac{du}{dt} = t\sin 2t + t^2\cos 2t + 2t\cos t - t^2\sin t + 2t\sin t$$

$$+ t^2\cos t$$

$\therefore$ WKT

$t = \frac{\pi}{4}$

$$\frac{du}{dt} = \frac{\pi}{4}(1) + \frac{\pi^2}{16}(0) + \frac{\pi}{2}\left(\frac{1}{\sqrt{2}}\right) - \frac{\pi^2}{16} \cdot \left(\frac{1}{\sqrt{2}}\right) + \frac{\pi}{2}\left(\frac{1}{\sqrt{2}}\right) + \frac{\pi^2}{16}\left(\frac{1}{\sqrt{2}}\right)$$

$$\frac{du}{dt} = \frac{\pi}{4} + \frac{2\pi}{2\sqrt{2}}$$

$$\Rightarrow \frac{du}{dt} = \frac{\pi}{4} + \frac{\pi}{\sqrt{2}}$$

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3. Find $\frac{du}{dt}$ at $t=0$, from $u = e^{x^2+y^2+z^2}$ where $x = t^2+1$, $y = t\cos t$, $z = \sin t$

Given,

$$u = e^{x^2+y^2+z^2} \longrightarrow (1)$$

$$x = t^2+1$$

$$y = t\cos t$$

$$z = \sin t$$

$$\frac{\partial u}{\partial x} = 2x e^{x^2+y^2+z^2}$$

$$\frac{\partial u}{\partial y} = 2y e^{x^2+y^2+z^2}$$

$$\frac{\partial u}{\partial z} = 2z e^{x^2+y^2+z^2}$$

$$\therefore \frac{dx}{dt} = 2t$$

$$\frac{dy}{dt} = \cos t - t\sin t$$

$$\frac{dz}{dt} = \cos t$$

W.K.T

$$\frac{du}{dt} = \frac{\partial u}{\partial x}\frac{dx}{dt} + \frac{\partial u}{\partial y}\frac{dy}{dt} + \frac{\partial u}{\partial z}\frac{dz}{dt}$$

$$\Rightarrow \frac{du}{dt} = \left[2x e^{x^2+y^2+z^2}\right](2t) + \left[2y e^{x^2+y^2+z^2}\right]\left[\cos t - t\sin t\right]$$

$$+ \left[2z e^{x^2+y^2+z^2}\right](\cos t)$$

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$$\Rightarrow \frac{du}{dt} = 2[2xt + y(\cos t - t\sin t) + z\cos t]\, e^{x^2+y^2+z^2}$$

$$\Rightarrow \frac{du}{dt} = 2[2t(t^2+1) + t\cos t(\cos t - t\sin t) + \sin t.\cos t]\, e^{(t^2+1)^2 + t^2\cos^2 t + \sin^2 t}$$

$\therefore$ At $t = 0$

$$\Rightarrow \frac{du}{dt} = 0$$

4. If $u = f[2x-3y, 3y-4z, 4z-2x]$ Show that $\frac{1}{2}u_x + \frac{1}{3}u_y + \frac{1}{4}u_z = 0$

Given,

$u = f(2x-3y, 3y-4z, 4z-2x]$

Let,

$p = 2x - 3y$

$q = 3y - 4z$

$r = 4z - 2x$

$\therefore u = f(p, q, r)$

then,

$$\frac{\partial p}{\partial x} = 2, \quad \frac{\partial p}{\partial y} = -3, \quad \frac{\partial p}{\partial z} = 0$$

$$\frac{\partial q}{\partial x} = 0, \quad \frac{\partial q}{\partial y} = -3, \quad \frac{\partial q}{\partial z} = -4$$

$$\frac{\partial r}{\partial x} = -2, \quad \frac{\partial r}{\partial y} = 0, \quad \frac{\partial r}{\partial z} = 4$$

$$\therefore \frac{\partial u}{\partial x} = \frac{\partial u}{\partial p}\frac{\partial p}{\partial x} + \frac{\partial u}{\partial q}\cdot\frac{\partial q}{\partial x} + \frac{\partial u}{\partial r}\frac{\partial r}{\partial x}$$

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$$\Rightarrow \frac{\partial u}{\partial x} = 2\frac{\partial u}{\partial p} + 0 - 2\frac{\partial u}{\partial r}$$

$$\Rightarrow \frac{1}{2}\frac{\partial u}{\partial x} = \frac{\partial u}{\partial p} - \frac{\partial u}{\partial r} \longrightarrow (1)$$

$$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial p}\frac{\partial p}{\partial y} + \frac{\partial u}{\partial q}\frac{\partial q}{\partial y} + \frac{\partial u}{\partial r}\frac{\partial r}{\partial y}$$

$$\Rightarrow \frac{\partial u}{\partial y} = -3\frac{\partial u}{\partial p} + 3\frac{\partial u}{\partial q}$$

$$\Rightarrow \frac{1}{3}\frac{\partial u}{\partial y} = \frac{\partial u}{\partial p} + \frac{\partial u}{\partial q} \longrightarrow (2)$$

$$\therefore \frac{\partial u}{\partial z} = \frac{\partial u}{\partial p}\frac{\partial p}{\partial z} + \frac{\partial u}{\partial q}\cdot\frac{\partial q}{\partial z} + \frac{\partial u}{\partial r}\frac{\partial r}{\partial z}$$

$$\Rightarrow \frac{\partial u}{\partial z} = -4\frac{\partial u}{\partial q} + 4\frac{\partial u}{\partial r}$$

$$\Rightarrow \frac{1}{4}\frac{\partial u}{\partial z} = -\frac{\partial u}{\partial q} + \frac{\partial u}{\partial r}$$

$$\Rightarrow \frac{1}{4}\frac{\partial u}{\partial z} = -\frac{\partial u}{\partial q} + \frac{\partial u}{\partial r} \longrightarrow (3)$$

$\therefore$ From (1) + (2) + (3)

$$\Rightarrow \frac{1}{2}\frac{\partial u}{\partial x} + \frac{1}{3}\frac{\partial u}{\partial y} + \frac{1}{4}\frac{\partial u}{\partial z} = \frac{\partial u}{\partial p} - \frac{\partial u}{\partial r} - \frac{\partial u}{\partial p} + \frac{\partial u}{\partial q} - \frac{\partial u}{\partial q} + \frac{\partial u}{\partial r}$$

$$\Rightarrow \frac{1}{2}u_x + \frac{1}{3}u_y + \frac{1}{4}u_z = 0$$

5. If $u = f((x-y), (y-z), (z-x))$ then Show that $\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = 0$

Given,

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$u = f(x-y, y-z, z-x)$

Let, $p = x-y$

$q = y-z$

$r = z-x$

$\therefore u = f(p, q, r)$

$\therefore \frac{\partial p}{\partial x} = 1, \frac{\partial p}{\partial y} = -1, \frac{\partial p}{\partial z} = 0$

$\frac{\partial p}{\partial y} = 0, \frac{\partial q}{\partial y} = 1, \frac{\partial r}{\partial z} = -1$

$\frac{\partial p}{\partial z} = 0, \frac{\partial q}{\partial z} = -1, \frac{\partial r}{\partial z} = 01$

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial p}\frac{\partial p}{\partial x} + \frac{\partial u}{\partial q}\frac{\partial q}{\partial x} + \frac{\partial u}{\partial r}\frac{\partial r}{\partial x}$$

$$= \frac{\partial u}{\partial p}(1) + \frac{\partial u}{\partial q}(0) + \frac{\partial u}{\partial r}(-1)$$

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial p} - \frac{\partial u}{\partial r} \longrightarrow (1)$$

$$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial p}\frac{\partial p}{\partial y} + \frac{\partial u}{\partial q}\frac{\partial q}{\partial y} + \frac{\partial u}{\partial r}\frac{\partial r}{\partial y}$$

$$= \frac{\partial u}{\partial p}(-1) + \frac{\partial u}{\partial q}(1) + \frac{\partial u}{\partial r}(0)$$

$$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial q} - \frac{\partial u}{\partial p} \longrightarrow (2)$$

$$\frac{\partial u}{\partial z} = \frac{\partial u}{\partial p}\frac{\partial p}{\partial z} + \frac{\partial u}{\partial q}\frac{\partial q}{\partial z} + \frac{\partial u}{\partial r}\frac{\partial r}{\partial z}$$

$$= \frac{\partial u}{\partial p}(0) + \frac{\partial u}{\partial q}(-1) + \frac{\partial u}{\partial p} \longrightarrow (3)$$

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Now ① + ② + ③

$$\Rightarrow \frac{\partial u}{\partial p} - \frac{\partial u}{\partial r} + \frac{\partial u}{\partial q} - \frac{\partial u}{\partial p} + \frac{\partial u}{\partial r} - \frac{\partial u}{\partial q}$$

$$\Rightarrow \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = 0$$

6. If $u = f(x/y, y/z, z/x)$, $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} = 0$ then prove the equation.

Given,

$u = f(x/y, y/z, z/x)$

Let,

$p = \frac{x}{y}$, $q = \frac{y}{z}$, $r = \frac{z}{x}$

$\frac{\partial p}{\partial x} = \frac{1}{y}$, $\frac{\partial p}{\partial y} = \frac{-x}{y^2}$, $\frac{\partial p}{\partial z} = 0$

$\frac{\partial q}{\partial x} = 0$, $\frac{\partial q}{\partial y} = \frac{1}{z}$, $\frac{\partial q}{\partial z} = \frac{-y}{z^2}$

$\frac{\partial r}{\partial x} = \frac{z}{x^2}$, $\frac{\partial r}{\partial y} = 0$, $\frac{\partial r}{\partial z} = \frac{1}{x}$

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial p}\frac{\partial p}{\partial x} + \frac{\partial u}{\partial q}\frac{\partial q}{\partial x} + \frac{\partial u}{\partial r}\frac{\partial r}{\partial x}$$

$$\frac{\partial u}{\partial x} = \frac{1}{y}\frac{\partial u}{\partial p} + z\frac{\partial u}{\partial r} \longrightarrow ①$$

$$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial p}\frac{\partial p}{\partial y} + \frac{\partial u}{\partial q}\frac{\partial q}{\partial y} + \frac{\partial u}{\partial r}\frac{\partial r}{\partial y}$$

$$\frac{\partial u}{\partial y} = -\frac{1}{y^2}\frac{\partial u}{\partial p} + \frac{1}{z}\frac{\partial u}{\partial q}$$

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$$\frac{\partial u}{\partial z} = \frac{\partial u}{\partial p}\frac{\partial p}{\partial z} + \frac{\partial u}{\partial q}\frac{\partial q}{\partial z} + \frac{\partial u}{\partial r}\frac{\partial r}{\partial z}$$

$$\frac{\partial u}{\partial z} = \frac{-1}{z^2}\frac{\partial u}{\partial q} + \frac{1}{x}\frac{\partial u}{\partial r} \longrightarrow (3)$$

From (1) + (2) + (3)

$$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = \frac{1}{y}\frac{\partial u}{\partial p} + 2\frac{\partial u}{\partial r} + \left(\frac{-1}{y^2}\right)\frac{\partial u}{\partial p} + \frac{1}{z}\frac{\partial u}{\partial q} - \frac{1}{z^2}\frac{\partial u}{\partial q} + \frac{\partial u}{\partial r}\frac{1}{x}$$

$$\therefore \frac{\partial u}{\partial x} + y\frac{du}{\partial y} + z\frac{\partial u}{\partial z} = 0$$

⇒ JACOBIANS

Suppose $u = u(x,y)$, $v = v(x,y)$ are the two functions in x and y then the Jacobian of u, v with respect to (x,y) can be defined as

$$\frac{\partial(u,v)}{\partial(x,y)} = \frac{J(u,v)}{(x,y)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix}$$

Similarlly if $u = u(x,y,z)$, $v = v(x,y,z)$ and $w = w(x,y,z)$ then Jacobian of u, v, w with respect to x, y, z can be defined as.

$$\frac{\partial(u,v,w)}{\partial(x,y,z)} = \frac{J(u,v,w)}{(x,y,z)}$$

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$$\frac{\partial(u,v,w)}{\partial(x,y,z)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{vmatrix}$$

H.

1. Find the Jacobian of u, v, w with respect to x, y, z for $u = x+y+z$, $v = y+z$, $w = z$

Given,

$u = x+y+z$

$v = y+z$

$w = z$

$\therefore u_x = 1 \quad v_x = 0 \quad w_x = 0$

$u_y = 1 \quad v_y = 1 \quad w_y = 0$

$u_z = 1 \quad v_z = 1 \quad w_z = 1$

$$J\frac{(u,v,w)}{(x,y,z)} = \begin{vmatrix} u_x & u_y & u_z \\ v_x & v_y & v_z \\ w_x & w_y & w_z \end{vmatrix} = \begin{vmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{vmatrix}$$

$J = 1(1-0) - 1(0-0) + 1(0-0)$

$J = 1$

2. If $u = x^2+y^2+z^2$, $v = xy+yz+zx$, $w = x+y+z$ then find $\frac{\partial(u,v,w)}{\partial(x,y,z)}$

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Given,

$u = x^2 + y^2 + z^2$, $v = xy + yz + zx$, $w = x + y + z$

$u_x = 2x$ $\quad v_x = y + z$ $\quad w_x = 1$

$u_y = 2y$ $\quad v_y = x + z$ $\quad w_y = 1$

$u_z = 2z$ $\quad v_z = x + y$ $\quad w_z = 1$

$$\therefore J\left(\frac{u,v,w}{x,y,z}\right) = \frac{\partial(u,v,w)}{\partial(x,y,z)} = \begin{vmatrix} u_x & u_y & u_z \\ v_x & v_y & v_z \\ w_x & w_y & w_z \end{vmatrix}$$

$$= \begin{vmatrix} 2x & 2y & 2z \\ y+z & z+x & x+y \\ 1 & 1 & 1 \end{vmatrix}$$

$$= 2\begin{vmatrix} x & y & z \\ y+z & z+x & x+y \\ 1 & 1 & 1 \end{vmatrix}$$

$$= 2\begin{vmatrix} x & y & z \\ x+y+z & x+y+z & x+y+z \\ 1 & 1 & 1 \end{vmatrix}$$

$[R_2 : R_2 + R_1]$

$$= 2(x+y+z)\begin{vmatrix} x & y & z \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{vmatrix}$$

$$J\left(\frac{u,v,w}{x,y,z}\right) = 0$$

③ If $u = x + 3y^2 - z^3$, $v = 4x^2yz$, $w = 2z^2 - xy$ find $\frac{\partial(u,v,w)}{\partial(x,y,z)}$ at the point $(1,-1,0)$

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Given

$u = x + 3y^2 - z^3$

$v = 4x^2yz$

$w = 2z^2 - xy$

$P = (1, -1, 0)$

$u_x = 1 \Rightarrow (u_x)_p = 1$

$u_y = 6y \Rightarrow (u_y)_p = 6(-1) = -6$

$u_z = -3z^2 \Rightarrow (u_z)_p = 0$

$v_x = 8xyz \Rightarrow (v_x)_p = 0$

$v_y = 4x^2z \Rightarrow (v_y)_p = 0$

$v_z = 4x^2y \Rightarrow (v_z)_p = 4(-1)(1^2) = -4$

$w_x = -y \Rightarrow (w_x)_p = -(-1) = 1$

$w_y = -x \Rightarrow (w_y)_p = -1$

$w_z = 4z \Rightarrow (w_z)_p = 0$

$$\frac{\partial(u,v,w)}{\partial(x,y,z)} = \begin{vmatrix} u_x & u_y & u_z \\ v_x & v_y & v_z \\ w_x & w_y & w_z \end{vmatrix} = \begin{vmatrix} 1 & -6 & 0 \\ 0 & 0 & -4 \\ 1 & -1 & 0 \end{vmatrix}$$

$\Rightarrow J = 1(0-4) + 6(0+4)$

$\Rightarrow J = -4 + 24$

$\Rightarrow J = 20$

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4. If $u = \frac{yz}{x}$, $v = \frac{xz}{y}$, $w = \frac{xy}{z}$ then find $\frac{\partial(u,v,w)}{\partial(x,y,z)}$

Given,

$u = yz/x$

$v = xz/y$

$w = xy/z$

$\therefore u_x = -\frac{yz}{x^2}$ $\quad v_x = \frac{z}{y}$ $\quad w_x = \frac{y}{z}$

$u_y = \frac{z}{x}$ $\quad v_y = -\frac{zx}{y^2}$ $\quad w_y = \frac{x}{z}$

$u_z = \frac{y}{x}$ $\quad v_z = \frac{x}{y}$ $\quad w_z = -\frac{xy}{z^2}$

$$\frac{\partial(u,v,w)}{\partial(x,y,z)} = \begin{vmatrix} u_x & u_y & u_z \\ v_x & v_y & v_z \\ w_x & w_y & w_z \end{vmatrix}$$

$$J = \begin{vmatrix} -\frac{yz}{x^2} & \frac{z}{x} & \frac{y}{x} \\ \frac{z}{y} & -\frac{zx}{y^2} & \frac{x}{y} \\ \frac{y}{z} & \frac{x}{z} & -\frac{xy}{z^2} \end{vmatrix} = \begin{vmatrix} -yz/x^2 & zx/x^2 & \frac{xy}{x^2} \\ yz/y^2 & -zx/y^2 & \frac{xy}{y^2} \\ yz/z^2 & \frac{xz}{z^2} & -xy/z^2 \end{vmatrix}$$

$$= \frac{1}{x^2y^2z^2} \begin{vmatrix} -zy & zx & xy \\ yz & -zx & xy \\ yz & zx & -xy \end{vmatrix}$$

$$= \frac{(yz)(zx)(xy)}{x^2y^2z^2} \begin{vmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{vmatrix}$$

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$= (-1)(1-1) - 1(-1-1) + 1(1+1)$

$= 2 + 2$

$J = 4$

⑤ If $x = r\cos\theta(\cos\phi)$, $y = r\sin\theta(\sin\phi)$, $z = r\cos\theta$, then Show that $\dfrac{\partial(x,y,z)}{\partial(r,\theta,\phi)} = r^2(\sin^2\phi\cos\theta)$

Given,

$x = r\cos\theta(\cos\phi)$

$y = r\sin\theta(\sin\phi)$

$z = r\cos\theta$

$\dfrac{\partial x}{\partial r} = \cos\theta\cos\phi$; $\dfrac{\partial y}{\partial r} = \sin\theta\sin\phi$; $\dfrac{\partial z}{\partial r} = \cos\theta$

$\dfrac{\partial x}{\partial \theta} = -r\sin\theta\cos\phi$; $\dfrac{\partial y}{\partial \theta} = r\cos\theta\sin\phi$; $\dfrac{\partial z}{\partial \theta} = -r\sin\theta$

$\dfrac{\partial x}{\partial \phi} = -r\sin\theta\cos\theta$; $\dfrac{\partial y}{\partial \phi} = r\sin\theta\cos\phi$; $\dfrac{\partial z}{\partial \phi} = 0$

$$\therefore \frac{\partial(x,y,z)}{\partial(r,\theta,\phi)} = \begin{vmatrix} \cos\theta\cos\phi & -r\sin\theta\cos\phi & -r\cos\theta\sin\phi \\ \sin\theta\sin\phi & r\cos\theta\sin\phi & r\sin\theta\cos\phi \\ \cos\theta & -r\sin\theta & 0 \end{vmatrix}$$

$$= r^2 \begin{vmatrix} \cos\theta\cos\phi & -\sin\theta\cos\phi & -\cos\theta\sin\phi \\ \sin\theta\sin\phi & \cos\theta\sin\phi & \sin\theta\cos\phi \\ \cos\theta & -\sin\theta & 0 \end{vmatrix}$$

$\Rightarrow J = r^2\{\cos\theta\cos\phi[0-[\sin^2\theta\cos\phi] + [\sin\theta\cos\phi][0 - \sin\theta\cos\theta\cos\phi] - \cos\theta\sin\phi[-\sin^2\theta\sin\phi - \cos^2\theta\sin\phi]\}$

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$J = r^2 \{ \sin^2\phi \cos\theta [\sin^2\theta + \cos^2\theta]$

$J = r^2 (\sin^2\phi \cos\theta)$

6. If $x = r\sin\theta(\cos\phi)$, $y = r\sin\theta(\sin\phi)$, $z = r\cos\theta$ then show that $\dfrac{\partial(x,y,z)}{\partial(r,\theta,\phi)} = r^2\sin\theta$

Given,

$x = r\sin\theta\cos\phi$

$y = r\sin\theta\sin\phi$

$z = r\cos\theta$

$\dfrac{\partial x}{\partial r} = \sin\theta\cos\phi$; $\dfrac{\partial y}{\partial r} = \sin\theta\sin\phi$; $\dfrac{\partial z}{\partial r} = \cos\theta$

$\dfrac{\partial x}{\partial \theta} = r\cos\theta\cos\phi$; $\dfrac{\partial y}{\partial \theta} = r\cos\theta\sin\phi$; $\dfrac{\partial z}{\partial \theta} = -r\sin\theta$

$\dfrac{\partial x}{\partial \phi} = r\sin\theta\sin\phi$; $\dfrac{\partial y}{\partial \phi} = r\sin\theta\cos\phi$; $\dfrac{\partial z}{\partial \phi} = 0$

$$J = \frac{\partial(x,y,z)}{\partial(r,\theta,\phi)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial \phi} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial \phi} \\ \frac{\partial z}{\partial r} & \frac{\partial z}{\partial \theta} & \frac{\partial z}{\partial \phi} \end{vmatrix}$$

$$J = \begin{vmatrix} \sin\theta\cos\phi & r\cos\theta\cos\phi & -r\sin\theta\sin\phi \\ \sin\theta\sin\phi & r\cos\theta\sin\phi & r\sin\theta\cos\phi \\ \cos\theta & -r\sin\theta & 0 \end{vmatrix}$$

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$$\Rightarrow J = \begin{vmatrix} \sin\theta\cos\phi & r\cos\theta\cos\phi & -r\sin\theta\sin\phi \\ \sin\theta\sin\phi & r\cos\theta\sin\phi & r\sin\theta\cos\phi \\ \cos\theta & -r\sin\theta & 0 \end{vmatrix}$$

$$= r^2\sin\theta \begin{vmatrix} \sin\theta\cos\phi & \cos\theta\cos\phi & -\sin\phi \\ \sin\theta\sin\phi & \cos\theta\sin\phi & \cos\phi \\ \cos\theta & -\sin\theta & 0 \end{vmatrix}$$

$$= r^2\sin\theta \{ \sin\theta\cos\phi[0+\sin\theta\cos\phi] - \cos\theta\cos\phi[0-\cos\theta\cos\phi] - \sin\phi[-\sin^2\theta\sin\phi - \cos^2\theta\sin\phi] \}$$

$$= r^2\sin\theta \{ \sin^2\theta\cos^2\phi + \cos^2\theta\cos^2\phi + \sin^2\phi[\sin^2\theta + \cos^2\theta] \}$$

$$= r^2\sin\theta \{ \cos^2\phi[\sin^2\theta + \cos^2\theta] + \sin^2\phi(1) \}$$

$$= r^2\sin\theta[\cos^2\phi + \sin^2\phi]$$

$$= r^2\sin\theta$$

Maxima and Minima having two Independent Variables

Step 1 :- Write the given function as $f(x,y) = 0$

Step 2 :- find $\frac{\partial f}{\partial x} = f_x$, $\frac{\partial f}{\partial y} = f_y$

Step 3 :- For maxima or minima take $f_x = 0$ and $f_y = 0$ and find the points of intersection, say (x_0, y_0), (x_1, y_1), (x_2, y_2) .....

Step 4 :- Find $A = f_{xx}$, $B = f_{xy}$, $C = f_{yy}$ and check the status by the following ways.

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Case 1 :- If $AC - B^2 > 0$ and $A < 0$, then we can say that $f(x, y)$ may have the maximum value at the particular point.

Case 2 :- If $AC - B^2 > 0$ and $A > 0$, then we can say that $f(x, y)$ may have the minimum value at the particular point.

Case 3 :- If $AC - B^2 < 0$ or $AC - B^2 = 0$ or $A = 0$ then we can say that, it is neither maximum nor minimum and the point is called saddel point.

I.

1. Find the extream values of the function,

$f(x, y) = x^3 + y^3 - 3x - 12y + 20$

Given,

$f(x, y) = x^3 + y^3 - 3x - 12y + 20 \longrightarrow (1)$

$\frac{\partial f}{\partial x} = 3x^2 - 3 \longrightarrow (2)$

$\frac{\partial f}{\partial y} = 3y^2 - 12 \longrightarrow (3)$

For maxima and minima

$\frac{\partial f}{\partial x} = 0$, $\frac{\partial f}{\partial y} = 0$

$\Rightarrow 3x^2 - 3 = 0$; $3y^2 - 12 = 0$

$\Rightarrow x^2 - 1 = 0$; $y^2 - 4 = 0$

$\Rightarrow x^2 = 1$; $y^2 = 4$

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$\Rightarrow x = \pm 1 \; ; \; y = \pm 2$

The points are,

$A = \frac{\partial^2 f}{\partial x^2} = 6x$

$B = \frac{\partial^2 f}{\partial x \partial y} = 0$

$C = \frac{\partial^2 f}{\partial y^2} = 6y$

Points	A	B	C	$AC - B^2$	Conculsion
$(-1,-2)$	$-6<0$	0	-12	$72>0$	Maximum
$(-1,2)$	$-6<0$	0	12	$-72<0$	Saddel
$(1,-2)$	$6>0$	0	-12	$-72<0$	Saddel
$(1,2)$	$6>0$	0	12	$72>0$	Minimum

2, Examine the function $f(x,y) = 2(x^2-y^2) - x^4 + y^4$ For it's extream values.

Given

$f(x,y) = 2(x^2-y^2) - x^4 + y^4$.

$\Rightarrow f(x,y) = 2x^2 - 2y^2 - x^4 + y^4 \longrightarrow (1)$

$\frac{\partial f}{\partial x} = 4x - 4x^3 \longrightarrow (2)$

$\frac{\partial f}{\partial y} = -4y + 4y^3 \longrightarrow (3)$

For maxima and minima

$\frac{\partial f}{\partial x} = 0 \; ; \; \frac{\partial f}{\partial y} = 0$

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$\Rightarrow 4x - 4x^3 = 0 \quad , \quad -4y + 4y^3 = 0$

$\Rightarrow 4x(1-x^2) = 0 \quad , \quad 4y(y^2-1) = 0$

$\Rightarrow x = 0 \; , \; 1-x^2 = 0 \; , \quad y = 0 \; , \; y^2 - 1 = 0$

$\Rightarrow x = 0, -1, +1 \quad , \quad y = 0, -1, +1$

$\therefore$ The points of x, y are

$(0,0), (0,-1), (0,1), (-1,0), (-1,-1), (-1,1), (1,0) (1,1) (1,-1)$

$\therefore A = \dfrac{\partial^2 f}{\partial x^2} = 4 - 12x^2$

$B = \dfrac{\partial^2 f}{\partial x \partial y} = 0$

$C = \dfrac{\partial^2 f}{\partial y^2} = -4 + 12y^2$

Point	A	B	C	$AC - B^2$	Conculsion
(0,0)	$4 > 0$	0	-4	$-16 < 0$	Saddle
(0,-1)	$4 > 0$	0	8	$32 > 0$	Maximum
(0,1)	$4 > 0$	0	8	$32 > 0$	minimum
(-1,0)	$-8 < 0$	0	-4	$32 > 0$	maximum
(-1,-1)	$-8 < 0$	0	8	$-64 < 0$	Saddle
(-1,1)	$-8 < 0$	0	8	$-64 < 0$	Saddle
(1,0)	$-8 < 0$	0	-4	$32 > 0$	Maximum
(1,-1)	$-8 < 0$	0	8	$-64 < 0$	Saddle
(1,1)	$-8 < 0$	0	8	$-64 < 0$	Saddle

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3. Examine the extream values of the function $f(x,y) = x^3 + 3xy^2 - 15x^2 - 15y^2 + 72x$.

Given,

$f(x,y) = x^3 + 3xy^2 - 15x^2 - 15y^2 \longrightarrow (1)$

$\frac{\partial f}{\partial x} = 3x^2 + 3y^2 - 30x - 72 \longrightarrow (2)$

$\frac{\partial f}{\partial y} = 6xy - 30y \longrightarrow (3)$

$\therefore$ For maxima and minima

$\frac{\partial f}{\partial x} = 0, \quad \frac{\partial f}{\partial y} = 0$

$\Rightarrow 3x^2 + 3y^2 - 30x + 72 = 0$

$\Rightarrow x^2 + y^2 - 10x + 24 \longrightarrow (4)$

$6xy - 30y = 0$

$\Rightarrow xy - 5y = 0$

$\Rightarrow y(x-5) = 0$

When $x = 5$

(1) $\Rightarrow 25 + y^2 - 50 + 24 = 0$

$\Rightarrow y^2 - 1 = 0$

$\Rightarrow y^2 - 1 = 0$

$\Rightarrow y^2 = 1$

$\Rightarrow y = \pm 1$

$\Rightarrow y = -1$

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the points are for $x = 5$

$(5,-1)$ $(5,1)$

$\therefore A = \frac{\partial^2 f}{\partial x^2} = 6x - 30$

$B = \frac{\partial^2 f}{\partial x \partial y} = 6y$

$C = \frac{\partial^2 f}{\partial y^2} = 6x - 30$

Point	A	B	C	$AC - B^2$	Concu
(6,0)	$6>0$	0	6	$36>0$	Minimum
(4,0)	$-6<0$	0	-6	$36>0$	Maximum
(5,-1)	0	-6	0	$-36<0$	Saddle
(5,1)	0	6	0	$-36<0$	Saddle

4. Examine the function in the example has four Stationary lets Consider $f(x,y) = 2x^3 + 6xy^2 - 3y^3 - 150$

$\frac{\partial f}{\partial x} = 6x^2 + 6y^2 - 150$

$\frac{\partial f}{\partial y} = 12yx - 9y^2$

$\therefore$ For maxima and minima

$\frac{\partial f}{\partial y} = 0$

$12xy - 9y^2 = 0$

$4xy - 3y^2 = 0$

$y = (4x - 3y) = 0$

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$y=0$, $4x-3y=0$, $\Rightarrow 4x=0 \Rightarrow x=0$

$$\frac{\partial f}{\partial x}=0$$

$$6x^2-6y^2-150=0$$

$$3x^2+3y^2-75=0$$

$$x^2+y^2-25=0$$

The pair of points for $y=0$

$$x^2+y^2-25=0$$

$$x^2+0=25$$

$$x=\sqrt{25}$$

$$x=\pm 5$$

$$(5,0),(-5,0)$$

The pair of points for $x=0$

$$x^2+y^2-25=0$$

$$y^2=25$$

$$y=\pm 5$$

$$(0,-5)\ (0,5)$$

$\therefore$ The points are $(5,0)\ (0,5),(-5,0),(0,-5)$

$$\therefore A=\frac{\partial^2 f}{\partial x^2}=12x$$

$$\therefore B=\frac{\partial^2 f}{\partial x \partial y}=12y$$

$$\therefore C=\frac{\partial^2 f}{\partial y^2}=12x-18y$$

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Point	A	B	C	$AC - B^2$	Conculsion
(5,0)	60	0	60	$3600 > 0$	Minimum
(-5,0)	-60	0	-60	$3600 > 0$	Maximum
(0,5)	0	60	-90	$-3600 < 0$	Saddle
(0,-5)	0	-60	+90	$-3600 < 0$	Saddle

5. Examine the function $f(x,y) = x^4 + y^4 - a[x-y]^2$

Given,

$$f(x,y) = x^4 + y^4 - a(x-y)^2 \longrightarrow (1)$$

$$\therefore \frac{\partial f}{\partial x} = 4x^3 + 0 - 4(x-4)(01)$$

$$\Rightarrow \frac{\partial f}{\partial x} = 4x^3 - 4(x-4) \longrightarrow (2)$$

$$\frac{\partial f}{\partial y} = 0 + 4y^3 - 4(x-y)(-1)$$

$$\Rightarrow \frac{\partial f}{\partial y} = 4y^3 + 4(x-y) \longrightarrow (3)$$

For maximum or minima

$$\frac{\partial f}{\partial x} = 0 \quad , \quad \frac{\partial f}{\partial y} = 0$$

$$\Rightarrow 4x^2 - 4(x-4) = 0 \quad , \quad 4y^3 + 4(x-4) = 0$$

$$\Rightarrow x^2 - (x-4) = 0 \rightarrow (4) \quad , \quad y^3 + (x-4) = 0 \longrightarrow (5)$$

$$\therefore (4) + (5) = x^3 - (x-4) + y^3 + (x-4) = 0$$

$$\Rightarrow x^3 + y^3 = 0$$

$$\Rightarrow (x+y)(x^2 - xy + y^2) = 0$$

$$\Rightarrow x = -y \longrightarrow (6)$$

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⑤ $= y^3 + |-y-y| = 0$

$\Rightarrow y^2 - 2y = 0$

$\Rightarrow y(y^2 - 2) = 0$

$\Rightarrow y = 0 \;,\; y^2 = 2 = 0$

$\Rightarrow y = 0 \;, \Rightarrow y^2 = 2 \Rightarrow y = \pm\sqrt{2}$

$\Rightarrow$ The point are

$(0,0)\;,\; (\sqrt{2}, -\sqrt{2})\;,\; (-\sqrt{2}, \sqrt{2})$

$\therefore\; A = \dfrac{\partial^2 f}{\partial x^2} = 12x^2 - 4$

$B = \dfrac{\partial^2 f}{\partial x \partial y} = 4$

$C = \dfrac{\partial^2 f}{\partial y^2} = 12y^2 - 4$

Point	A	B	C	$AC - B^2$	conculsion
(0,0)	$-4 < 0$	4	-4	0	Saddle
$(\sqrt{2}, -\sqrt{2})$	$20 > 0$	4	20	$384 > 0$	Minimum
$(-\sqrt{2}, \sqrt{2})$	$20 > 0$	4	20	$384 > 0$	minimum

$\therefore\; \text{Min } f = (-\sqrt{2})^4 = (\sqrt{2})^4 - 2(-\sqrt{2} - \sqrt{2})^2$

$\Rightarrow \text{Min } f = 4 + 4 - 2(8)$

$\Rightarrow \text{Min } f = 8 - 16 = -8$

Applications of partial differentiation

Step ① : Consider the function $u(x, y, z)$ to be maximized or minimized, Subject to be the Condition $v(x, y, z) = c$

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Step ② :- Consider $F = u(x,y,z) + \lambda v(x,y,z)$

Step ③ :- Take the total derivative on both the Side

$$dF = \left[\frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy + \frac{\partial u}{\partial z}dz\right] + \lambda\left[\frac{\partial v}{\partial x}dx + \frac{\partial v}{\partial x}dy + \frac{\partial v}{\partial x}dz\right]$$

Step ④ :- For maxima and minima take $dF = 0$

$\Leftrightarrow$ $u_x + v_y\lambda = 0$, $v_y + \lambda v_y = 0$, $u_z + \lambda v_z = 0$

Step ⑤ :- Find λ and x, y, z, by the help of $v(x,y,z) = c$ and x, y, z, by the help of Same in the $u(x,y,z)$, which is to be maximized or minimized.

① Find the maximum of $x^2 + y^2 + z^2$ when $x + y + z = 3a$

$u = x^2 + y^2 + z^2 \rightarrow ①$

$v = x + y + z = 3a$

$v = x + y + z - 3a = 0 \rightarrow ②$

Consider

$F = u + \lambda v$

$\Rightarrow F = x^2 + y^2 + z^2 + \lambda(x + y + z - 3a) \rightarrow ③$

$\Rightarrow dF = [2x\,dx + 2y\,dy + 2z\,dz] + \lambda[(1)\,dx + 1\,dy + 1\,dz)$

$\Rightarrow dF = (2x + \lambda)\,dx + (2y + \lambda)\,dy + (2z - \lambda)\,dz \rightarrow ④$

$\therefore$ For maxima and minima

$dF = 0$

$\Rightarrow 2x + \lambda = 0 \Rightarrow x = -\lambda/2$

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$\Rightarrow 2y + \lambda = 0 \Rightarrow y = -\frac{\lambda}{2}$

$\Rightarrow 2z + \lambda = 0 \Rightarrow z = -\frac{\lambda}{2}$

$\therefore$ (2) $\Rightarrow -\frac{\lambda}{2} - \frac{\lambda}{2} - \frac{\lambda}{2} = 3a$

$\Rightarrow -\frac{3\lambda}{2} = 3a$

$\Rightarrow -\frac{\lambda}{2} = a$

$\Rightarrow \lambda = -2a$

$\therefore x = -\frac{\lambda}{2} = \frac{-(-2a)}{2} = a$

$y = -\frac{\lambda}{2} = \frac{-(-2a)}{2} = a$

$z = -\frac{\lambda}{2} = \frac{-(-2a)}{2} = a$

$\therefore p(x, y, z) = (a, a, a)$

$\therefore \max u = a^2 + a^2 + a^2 = 3a^2$

2. Find the minimum value of $x^2 + y^2 + z^2$, subject to the condition $xy + yz + zx = 3a$.

Given,

$u = x^2 + y^2 + z^2 \longrightarrow$ (1)

$\Rightarrow v = xy + yz + zx = 3a$

Consider

$F = u + \lambda v$

$\Rightarrow F = (x^2 + y^2 + z^2) + \lambda(xy + yz + zx - 3a) \longrightarrow$ (3)

$\Rightarrow dF = [2x\,dx + 2y\,dy + 2z\,dz] + \lambda[(y+z)dx + (z+x)dy + (x+y)dz$

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$\Rightarrow dF = [2x + \lambda(y+z)]dx + (2y + \lambda(z+x))dy + (2z + \lambda)(x+y)dz$

$\therefore$ For maxima and minima

$dF = 0$

$\Rightarrow 2x + \lambda(y+z) = 0 \longrightarrow (4)$

$\Rightarrow 2y + \lambda(z+x) = 0 \longrightarrow (5)$

$\Rightarrow 2z + \lambda(x+y) = 0 \longrightarrow (6)$

$\therefore (4) + (5) + (6)$

$\Rightarrow 2(x+y+z) + 2\lambda(x+y+z) = 0$

$\Rightarrow 1 + \lambda = 0$

$\Rightarrow \lambda = -1$

$\therefore (4) \Rightarrow 2x - y - z = 0 \longrightarrow (7)$

$(5) \Rightarrow 2y - z - x = 0 \longrightarrow (8)$

$(6) \Rightarrow 2z - x - y = 0 \longrightarrow (9)$

$(7) - (8) \Rightarrow 2x - 2y - y + x = 0$

$\Rightarrow 2(x-y) - y + x = 0$

$\Rightarrow 3(x-y) = 0$

$\Rightarrow x - y = 0$

$\Rightarrow x = y \longrightarrow (10)$

$(8) - (9) \Rightarrow 2y - 2z - z + y = 0$

$\Rightarrow 2(y-z) + 1(y-z) = 0$

$\Rightarrow 3(y-z) = 0$

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$\Rightarrow y - z = 0$

$\Rightarrow y = z \longrightarrow (11)$

$\therefore x = y = z = K$

$\therefore$ (2) $\Rightarrow K.K + K.K + K.K = 3a$

$\Rightarrow K^2 + K^2 + K^2 = 3a$

$\Rightarrow 3K^2 = 3a$

$\Rightarrow K^2 = a$

$\Rightarrow K = \sqrt{a}$

$\therefore P(x, y, z) = (\sqrt{a}, \sqrt{a}, \sqrt{a})$

$\therefore \text{Min } u = (\sqrt{a})^2 + (\sqrt{a})^2 + (\sqrt{a})^2$

$\text{Min } u = a + a + 3a$

3. Find the Stationary value of $x^2y^3z^4$ Subject to the condition $x + y + z = 5$ Given.

$v = x + y + z = 5 \longrightarrow (1)$

Consider

$F = u + \lambda v$

$\Rightarrow F = (x^2y^3z^4) + \lambda(x + y + z = 5)$

$\Rightarrow dF = [2xy^2z^4 . dx + 3x^2y^2 dy + 4x^2y^3z^3 . dz] + \lambda[1.dx + 1.dy + 1dz]$

$\Rightarrow dF = [2xy^3z^4 + \lambda] dx + [3x^2y^2z^4 + \lambda] dy + [4x^2y^3z^3 + \lambda] z \longrightarrow (2)$

$\therefore$ For maximum or minimum

$dF = 0$

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$\Rightarrow 2xy^3z^4 + \lambda = 0 \Rightarrow \lambda = -2xy^3z^4 \longrightarrow (3)$

$\Rightarrow 3x^2y^2z^4 + \lambda = 0 \Rightarrow \lambda = -3x^2y^2z^4 \longrightarrow (4)$

$\Rightarrow 4x^2y^3z^3 + \lambda = 0 \Rightarrow \lambda = -4x^2y^3z^3 \longrightarrow (5)$

$\therefore$ from (3), (4), (5)

$$-2xy^3z^4 = -3x^2y^2z^4 = -4x^2y^3z^3$$

$$\Rightarrow \frac{2xy^3z^4}{x^2y^3z^4} = \frac{3x^2y^2z^4}{x^2y^3z^4} = \frac{4x^2y^3z^3}{x^2y^3z^4}$$

$$\Rightarrow \frac{2}{x} = \frac{3}{y} = \frac{4}{z}$$

$$\Rightarrow \frac{x}{2} = \frac{y}{3} = \frac{z}{4} = K$$

$$\Rightarrow \frac{x}{2} = K \Rightarrow x = 2K$$

$$\frac{y}{3} = K \Rightarrow y = 3K$$

$$\frac{z}{4} = K \Rightarrow z = 4K$$

$\therefore$ (1) $\Rightarrow 2K + 3K + 4K = 5$

$$\Rightarrow 9K = 5$$

$$\Rightarrow K = \frac{5}{9}$$

$$\therefore x = 2\left(\frac{5}{9}\right) = \frac{10}{9}$$

$$y = 3\left(\frac{5}{9}\right) = \frac{5}{3}$$

$$z = 4\left(\frac{5}{9}\right) = \frac{20}{9}$$

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$$P(x,y,z) = \left(\frac{10}{9}, \frac{5}{3}, \frac{20}{9}\right)$$

$$\therefore u = \left(\frac{10}{9}\right)^2 \left(\frac{5}{3}\right)^2 \left(\frac{20}{9}\right)^2$$

(4) A rectangular box is opened in the top is having volume $32\,cm^3$. Find it's dimension if the total surface Area is minima.

→ let x, y, z be the dimension of a open Rectangular box length, bredth, height respectively

Given that the value of the box

$$V = xyz = 32 \rightarrow (1)$$

let u be the total surface area of a open Rectangular box

$$u = xy + 2yz + 2zx \longrightarrow (2)$$

Consider

$$F = u + \lambda v$$

$$\Rightarrow F = (xy + 2yz + 2zx) + (xyz)$$

$$\Rightarrow dF = [(y+2z)\,dx + (x+2z)\,dy + 2(y+x)\,dz] + \lambda [yz\,dx + 2z\,dy + xy\,dz]$$

$$\Rightarrow dF = [(y+2z) + \lambda yz]\,dx + [x+2z) + \lambda zx]\,dy + [2(y+x) + \lambda xy]\,dx$$

$\therefore$ For maximum and minimum

$$dF = 0$$

$$(y+2z) + \lambda yz = 0 \rightarrow (3) \Rightarrow \lambda = \frac{-(y+2z)}{yz}$$

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$(x + az) + \lambda zx = 0 \rightarrow (4) \Rightarrow \lambda = -\dfrac{(x+az)}{zx}$

$a(x+y) + \lambda xy = 0 \rightarrow (5) \Rightarrow \lambda = -\dfrac{a(x+y)}{xy}$

$\therefore -\dfrac{(y+az)}{yz} = -\dfrac{(x+az)}{zx} = -\dfrac{a(x+y)}{xy}$

$\Rightarrow \dfrac{y+az}{yz} = \dfrac{x+az}{zx} = \dfrac{ax+ay}{xy}$

let $\dfrac{y+az}{yz} = \dfrac{x+az}{zx}$

$\Rightarrow \dfrac{y+az}{y} = \dfrac{x+az}{x}$

$\Rightarrow xy + azx = xy + azy$

$\Rightarrow azx = azy$

$\Rightarrow x = y$

and $\dfrac{x+az}{zx} = \dfrac{ax+ay}{xy}$

$\Rightarrow xy = azx$

$\Rightarrow y = az$

$\therefore x = y = az = K$

$\therefore x = K, \; y = K, \; az = K$

$\Rightarrow z = \dfrac{K}{a}$

$(1) \Rightarrow (K)(K)(K/2) = 32$

$\Rightarrow K^3 = 64$

$\Rightarrow K = 4$

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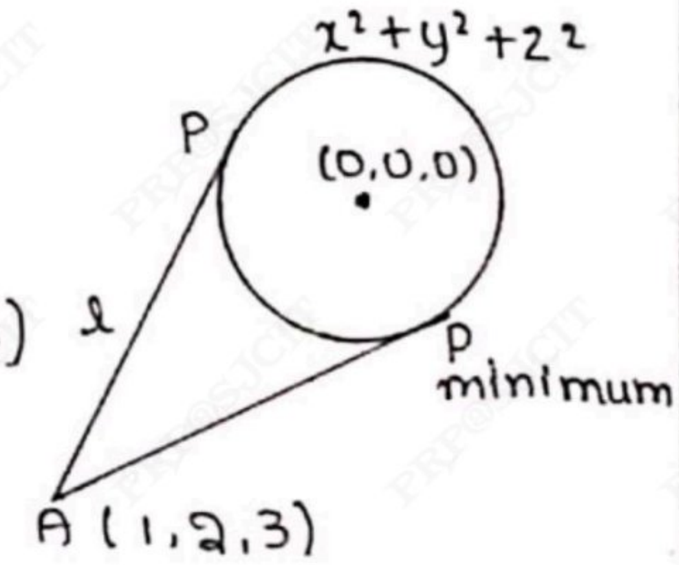
$\therefore P(x,y,z) = (4,4,2)$

$\therefore \min u = (4)(4) + 2(4)(2) + 2(2)(4)$

$= 16 + 16 + 16$

$\min u = 48$ sq units

⑤ Find the maximum and minimum distance From the point $(1,2,3)$ to the Sphere $x^2+y^2+z^2 = 56$



$\Rightarrow$ let $P(x,y,z)$ be a point on the Sphere $x^2+y^2+z^2 = 56$, And $A = (1,2,3)$

$\therefore$ let $V = x^2+y^2+z^2 = 56$

$AP = \sqrt{(x-1)^2 + (y-2)^2 + (z-3)^2}$

$\Rightarrow AP^2 = (x-1)^2 + (y-2)^2 + (z-3)^2 = u$

Consider

$F = u + \lambda v$

$\Rightarrow dF = du + \lambda dv$

$\Rightarrow dF = [2(x-1)dx + 2(y-2)dy + 2(z-3)dz]$
$+ \lambda[2x\,dx + 2y\,dy + 2z\,dz]$

$\Rightarrow dF = 2[(x-1) + \lambda x]\,dx + 2[(y-2) + \lambda y] + 2[(z-3)$
$+ \lambda z]\,dz$

For maximum or minimum

$dF = 0$

$\therefore (x-1) + \lambda x = 0$

$\Rightarrow (1+\lambda)x - 1 = 0 \rightarrow ② \Rightarrow 1 + \lambda = \frac{1}{x}$

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$(y-2) + \lambda y = 0$

$\Rightarrow (1+\lambda) y - 2 = 0 \longrightarrow (3)$

$(z-3) + \lambda z = 0$

$\Rightarrow (1+\lambda z - 3) = 0 \longrightarrow (4) \Rightarrow 1+\lambda = \frac{3}{2}$

$\therefore \frac{1}{x} = \frac{2}{y} = \frac{3}{z}$

$\Rightarrow \frac{x}{1} = \frac{y}{2} = \frac{z}{3} = K$

$\therefore x = K, \; y = 2K, \; z = 3K$

$\therefore (1) \Rightarrow K^2 + 4K^2 + 9K^2 = 56$

$\Rightarrow 14K^2 = 56$

$\Rightarrow K^2 = \frac{56}{14}$

$\Rightarrow K^2 = 4$

$\Rightarrow K = \pm 2$

if $K = -2$

$\therefore P(-2, -4, -6)$

$AP = \sqrt{4 + 36 + 81} = 11.225$

if $K = 2$

$\therefore P(2, 4, 6)$

$AP = \sqrt{1+4+9} = \sqrt{14} = 3.74$

$\therefore$ max at $P(-2, -4, -6)$

and min at $P(2, 4, 6)$

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6. Find the dimensions of the rectangular box open at the top of maximum capacity whose Surface is 432 Sq. cm

let x, y, z, be the dimension of a open rectangular box

$\therefore$ length, width, height respectively

Given that the total Surface area of a open rectangular area $= 432$ Sq cm

$V = xy + 2yz + 2zx = 432$ Sq cm

Let u be the volume of the box $u = xyz$

Consider,

$F = u + \lambda V$

$F = (xyz) + \lambda(xy + 2yz + 2zx)$

$dF = [yz\,dx + xz\,dy + xy\,dz) + \lambda[(y+2z)dx + (x+2y]$
$dy + 2(y+x)\lambda dz]$

$dF = [(y+2z)\lambda + yz]dx + [xz + (x+2y)dy] +$
$[xy + 2(y+x)\lambda dz]$

For maximum

$dF = 0$

$yz + \lambda(y+2z) \Rightarrow \lambda = \dfrac{-yz}{y+2z}$

$xz + \lambda(x+2z) \Rightarrow \lambda = \dfrac{-xz}{x+2z}$

$xy + \lambda(2x+2y) \Rightarrow \lambda = -xy/2(x+y)$

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$$\therefore \frac{-yz}{y+2z} = \frac{-xz}{x+2z} = \frac{-xy}{2(x+y)}$$

$$\frac{-yz}{y+2z} = \frac{-xz}{x+2z}$$

$$\frac{y}{2z(y+1)} = \frac{x}{2z(x+1)}$$

$$yx + y = xy + x$$

$$x = y$$

$$\frac{-xz}{x-2z} = \frac{-xy}{2x+2y}$$

$$(2x+2y)z = y(x+2z)$$

$$2zx + 2zy = yx + 2zy$$

$$2z(x+y) = 2z(yx+1)$$

$$x + y = xy + 1$$

Now, $x = y = 2z = K$

$$x = K, \quad y = K, \quad z = \frac{K}{2}$$

W.K.T

$$V = xy + 2yz + 2zx = 432$$

$$= K(K) + 2(K)\left(\frac{K}{2}\right) + 2\left(\frac{K}{2}\right)(K) = 432$$

$$= 3K^2 = 432$$

$$= K^2 = \frac{432}{3}$$

$$= K^2 = 144$$

$$\boxed{K = 12}$$

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$x = 12, \quad y = 12, \quad z = 6$

$P(x, y, z) = (12, 12, 6)$

$\text{Max } u = xyz = 12 \times 12 \times 6 = 864 \text{ cm}^3$

$\text{Max } u = 864 \text{ cm}^3$

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