

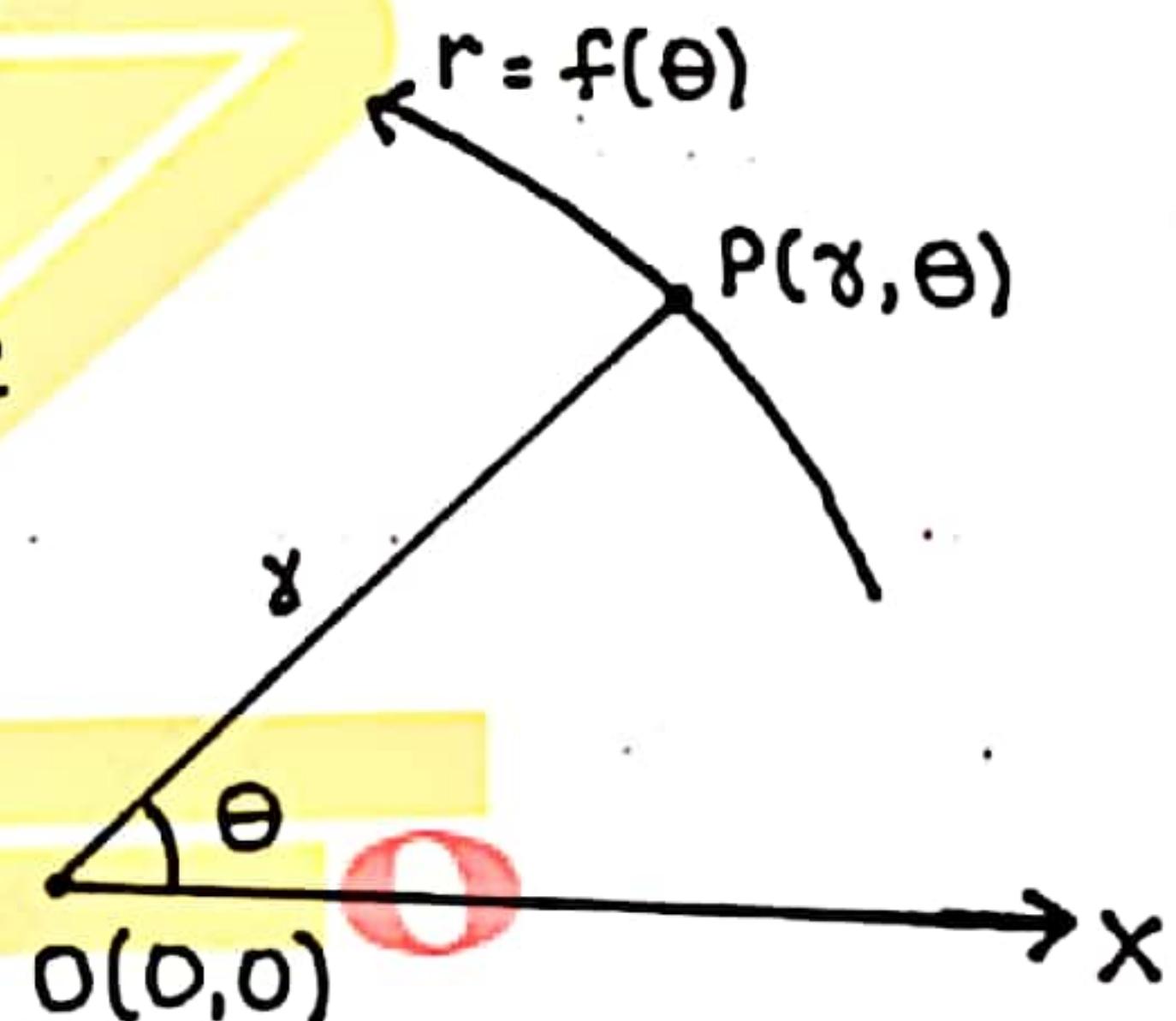
CALCULUS AND LINEAR ALGEBRA

MODULE - 01

POLAR CURVES AND EVALUATES

Introduction :-

Let \overrightarrow{Ox} be the initial line or initial ray, there is any point on the plane $P(r, \theta)$ with the radius of vector $OP = r$ and $\angle xOP = \theta$ and $r = f(\theta)$ be a polar curve at the point

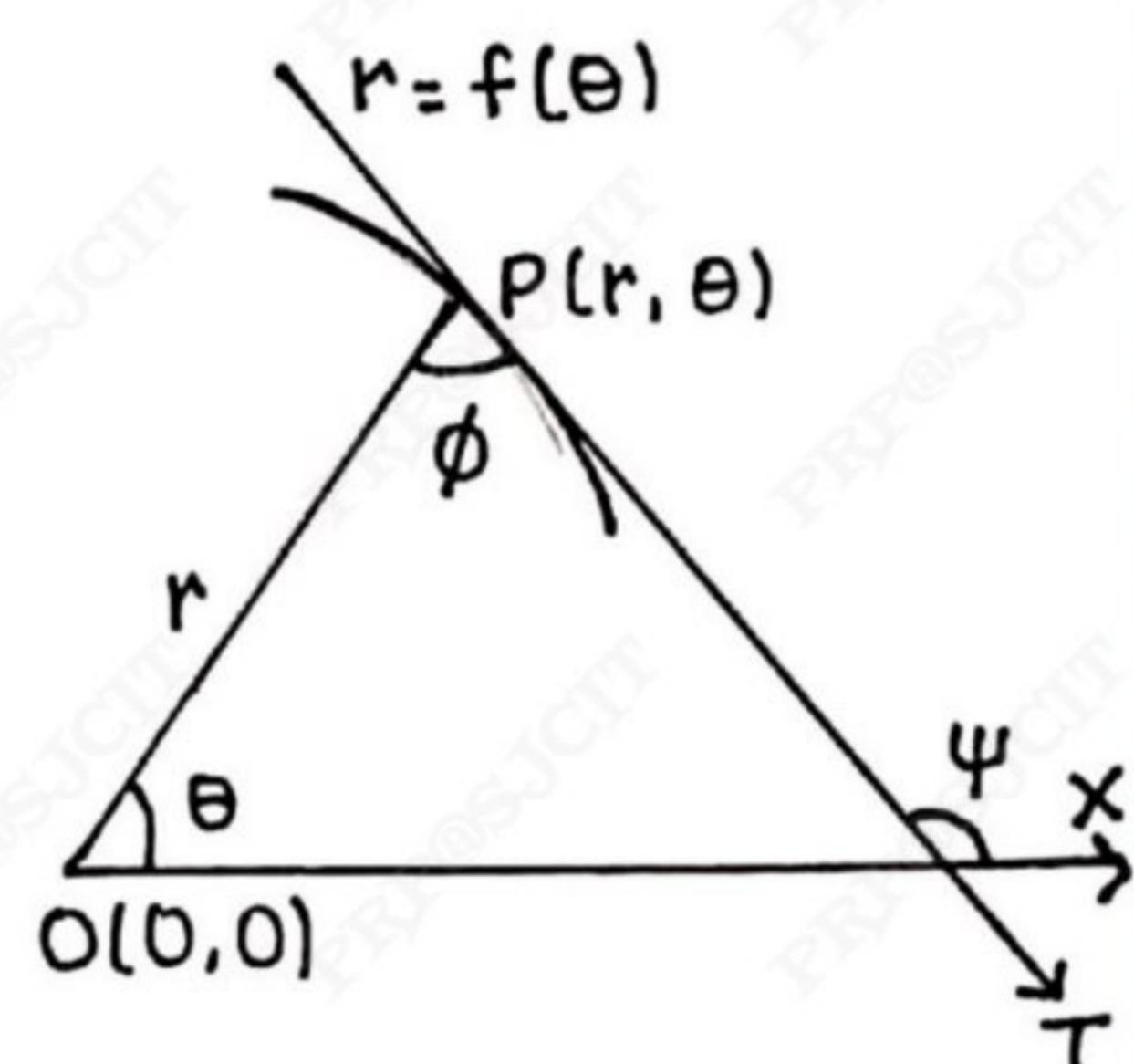


$P(r, \theta)$ Here, the coordinates of P is called as the Polar Coordinates.

A. ANGLE BETWEEN RADIUS VECTOR AND TANGENT TO THE POLAR CURVE

Let \overrightarrow{Ox} be the initial line $P(r, \theta)$ be a point in the plane and $\overrightarrow{OP} = r$ be the

radius of vector and let $r = f(\theta)$ be a polar curve at the point 'P', 'T' be the tangent to the curve $r = f(\theta)$ at P and making angle with the initial line ψ



let ϕ be the angle between radius of vector and tangent to the curve $r = f(\theta)$ at P

W.K.T

$$\psi = \phi + \theta$$

$$\Rightarrow \tan \psi = \tan(\phi + \theta)$$

$$\tan \psi = \frac{\tan \phi + \tan \theta}{1 - \tan \phi \tan \theta} \rightarrow ①$$

also W.K.T The slope of the tangent T

$$\text{is } m = \tan \psi = \frac{dy}{dx} \rightarrow ②$$

$$\text{let } x = r \cos \theta \text{ and } y = r \sin \theta$$

differentiate x and y w.r.t θ

$$\therefore \frac{dx}{d\theta} = \frac{dr}{d\theta} \cos \theta - r \sin \theta$$

$$\frac{dy}{d\theta} = \frac{dr}{d\theta} \sin \theta + r \cos \theta$$

$$② \Rightarrow \tan \psi = \frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta}$$

$$\Rightarrow \tan \psi = \frac{\frac{dr}{d\theta} \sin \theta + r \cos \theta}{\frac{dr}{d\theta} \cos \theta - r \sin \theta}$$

$$= \frac{\frac{\frac{dr}{d\theta} \sin \theta}{\frac{dr}{d\theta} \cos \theta} + \frac{r \cos \theta}{\frac{dr}{d\theta} \cos \theta}}{1 - \frac{r \sin \theta}{\frac{dr}{d\theta} \cos \theta}}$$

$$\Rightarrow \tan \psi = \tan \theta + r \frac{d\theta}{dr} / 1 - r \frac{d\theta}{dr} \tan \theta$$

$$\frac{\tan \theta + \tan \phi}{1 - \tan \phi \tan \theta} = \frac{\tan \theta + r \frac{d\theta}{dr}}{1 - r \frac{d\theta}{dr} \tan \theta}$$

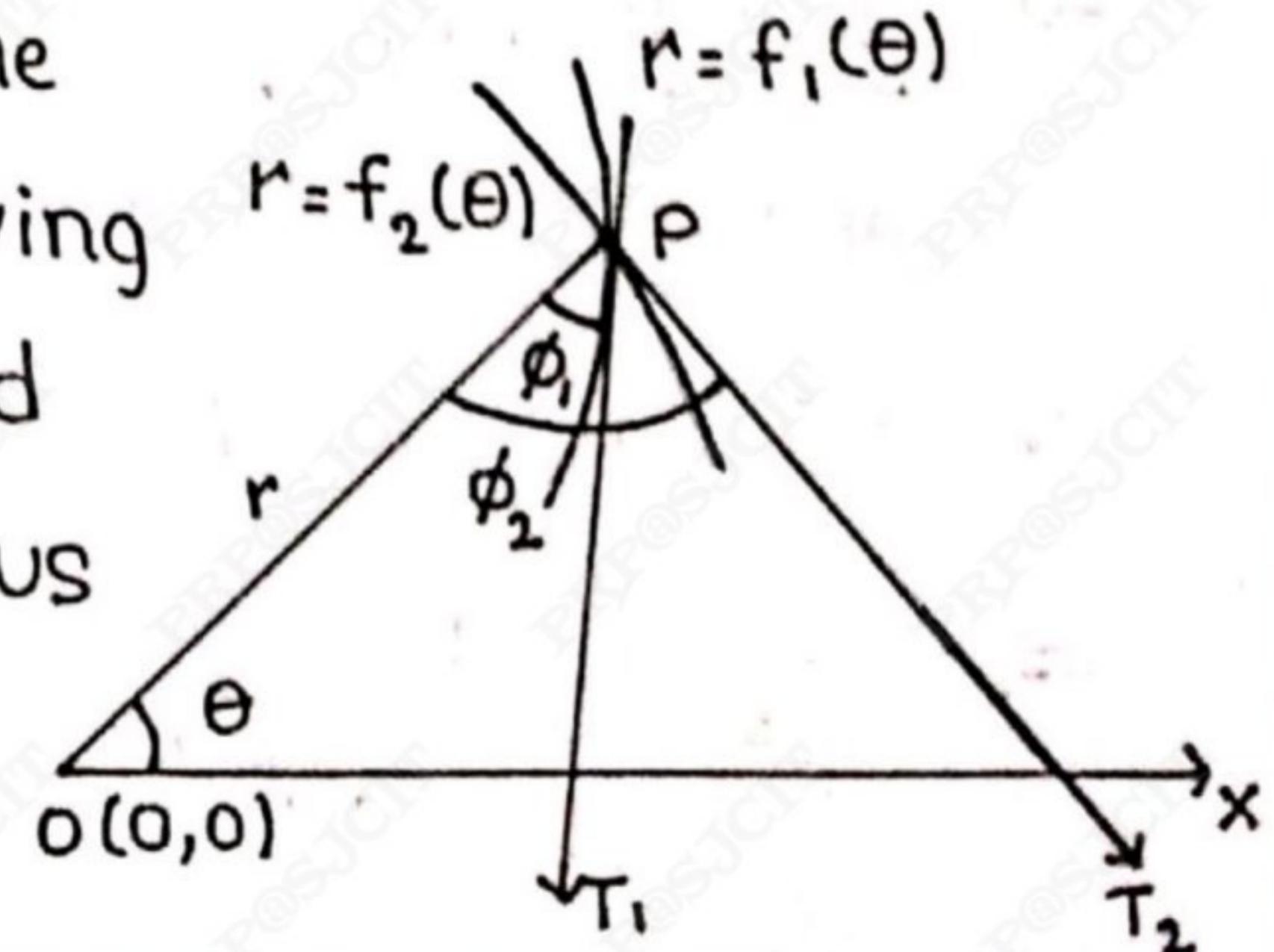
$$\boxed{\tan \phi = r \frac{d\theta}{dr}}$$

$$\phi = \tan^{-1} r \frac{d\theta}{dr}$$

In terms of, $\cot \phi = \frac{1}{r} \frac{dr}{d\theta}$ O

B. ANGLE BETWEEN TWO POLAR CURVES

Let $r = f_1(\theta)$, $r = f_2(\theta)$ be the given two polar curves having the tangents T_1 and T_2 and making an angles of radius of vector ϕ_1 and ϕ_2 then the angle between the



given two polar curves is the angle
between there two tangent = $|\phi_2 - \phi_1|$

WITH USUAL NOTATION, PROVE THAT

$$\frac{1}{P^2} = \frac{1}{r^2} + \frac{1}{r^4} \left(\frac{dr}{d\theta} \right)^2$$

Let \overrightarrow{Ox} be the intial line $r = f(\theta)$ be a polar curve at $P(r, \theta)$ and $OP = r$ be the radius of vector.

Let T be the tangent to the curve $r = f(\theta)$. At P and M is M the foot of the \perp^{lur} of the pole $O(0,0)$ having the perpendicular distance $OM = p$ from the ΔOPM we have $\angle OMP = 90^\circ$.

$$\therefore \sin \phi = \frac{OM}{OP}$$

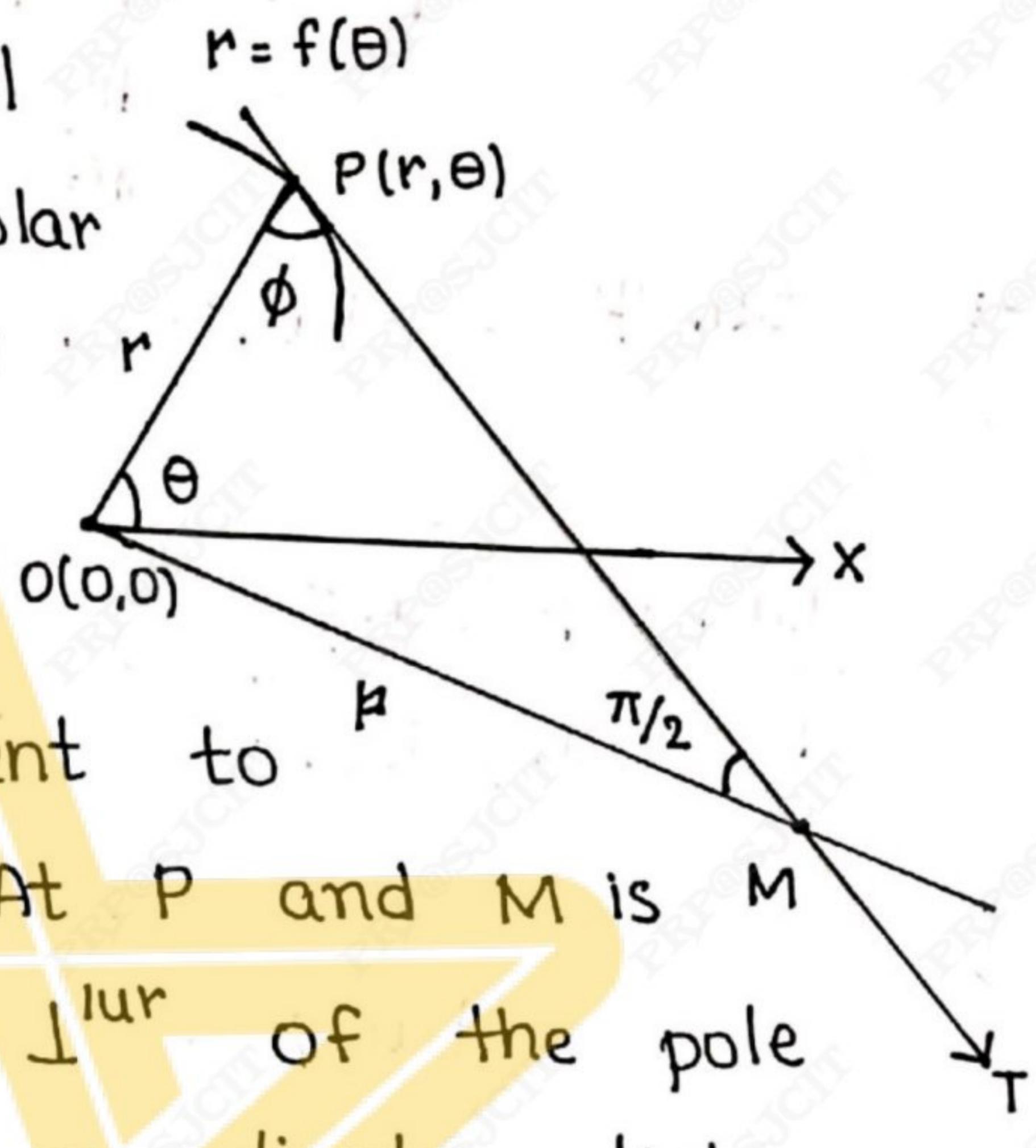
$$\Rightarrow \sin \phi = \frac{p}{r}$$

$$\Rightarrow P = r \sin \phi \quad \longrightarrow \textcircled{1}$$

Square on both Sides

$$\Rightarrow P^2 = r^2 \sin^2 \phi$$

$$\Rightarrow \frac{1}{P^2} = \frac{1}{r^2 \sin^2 \phi}$$



$$\Rightarrow \frac{1}{P^2} = \frac{1}{r^2} \csc^2 \phi$$

$$\Rightarrow \frac{1}{P^2} = \frac{1}{r^2} (1 + \cot^2 \phi)$$

$$\Rightarrow \frac{1}{P^2} = \frac{1}{r^2} \left[1 + \left(\frac{1}{r} \left(\frac{dr}{d\theta} \right)^2 \right) \right]$$

$$\Rightarrow \frac{1}{P^2} = \frac{1}{r^2} \left[1 + \frac{1}{r^2} \left(\frac{dr}{d\theta} \right)^2 \right]$$

$$\Rightarrow \frac{1}{P^2} = \frac{1}{r^2} + \frac{1}{r^4} \left(\frac{dr}{d\theta} \right)^2$$

I. Find the angle between the radius vector and tangent for the following polar curves.

① $r = a(1 - \cos\theta)$

Given,

$$r = a(1 - \cos\theta) \rightarrow ①$$

differentiate ① w.r.t. θ

$$\frac{dr}{d\theta} = a(\theta + \sin\theta)$$

$$\frac{dr}{d\theta} = a \sin\theta$$

$$\frac{1}{r} \frac{dr}{d\theta} = \frac{a \sin\theta}{r}$$

$$\frac{1}{r} \frac{dr}{d\theta} = \frac{a \sin\theta}{a(1 - \cos\theta)}$$

$$\Rightarrow \cot\phi = \frac{\sin\theta}{(1 - \cos\theta)}$$

$$\Rightarrow \cot \phi = \frac{2 \sin \theta/2 \cos \theta/2}{2 \sin^2 \theta/2}$$

$$\Rightarrow \cot \phi = \cot \theta/2$$

$$\boxed{\phi = \theta/2}$$

when, $\theta = \frac{\pi}{3}$

$$\phi = \frac{\pi/3}{2}$$

$$\phi = \frac{\pi}{6} = 30^\circ //$$

② $r = a(1 + \cos \theta)$

Given

$$r = a(1 + \cos \theta) \rightarrow ①$$

Differentiate ① w.r.t. θ

$$\frac{dr}{d\theta} = a(-\sin \theta)$$

$$\frac{dr}{d\theta} = a(-\sin \theta)$$

$$\frac{1}{r} \frac{dr}{d\theta} = \frac{-a \sin \theta}{r}$$

$$\frac{1}{r} \frac{dr}{d\theta} = \frac{-a \sin \theta}{a(1 + \cos \theta)}$$

$$\frac{1}{r} \frac{dr}{d\theta} = \frac{-\sin \theta}{(1 + \cos \theta)}$$

$$\frac{1}{r} \frac{dr}{d\theta} = -\frac{2 \sin \theta/2 \cos \theta/2}{2 \cos^2 \theta/2}$$

$$\Rightarrow \cot \phi = -\tan \theta/2$$

$$\Rightarrow \cot \phi = \cot (\pi/2 + \theta/2)$$

$$\phi = \frac{\pi}{2} + \frac{\theta}{2} //$$

③ $r = a(1 + \sin \theta)$

Given

$$r = a(1 + \sin \theta) \rightarrow ①$$

Differentiate ① w.r.t. θ

$$① \Rightarrow \frac{dr}{d\theta} = a(0 + \cos \theta)$$

$$\Rightarrow \frac{dr}{d\theta} = a \cos \theta$$

$$\Rightarrow \frac{1}{r} \frac{dr}{d\theta} = \frac{a \cos \theta}{r}$$

$$\Rightarrow \frac{1}{r} \frac{dr}{d\theta} = \frac{a \cos \theta}{a(1 + \sin \theta)}$$

$$\Rightarrow \cot \phi = \frac{\cos \theta}{1 + \sin \theta}$$

$$\Rightarrow \cot \phi = \frac{\cos^2 \theta/2 - \sin^2 \theta/2}{\cos^2 \theta/2 + \sin^2 \theta/2 + 2 \sin \theta/2 \cos \theta/2}$$

$$= \frac{(\cos \theta/2 + \sin \theta/2)(\cos \theta/2 - \sin \theta/2)}{(\cos \theta/2 + \sin \theta/2)^2}$$

$$= \frac{\cos \theta/2 - \sin \theta/2}{\cos \theta/2 + \sin \theta/2}$$

$$= \frac{1 - \tan \theta/2}{1 + \sin \theta/2 / \cos \theta/2}$$

$$= \frac{1 - \tan \theta/2}{1 + \tan \theta/2}$$

$$= \frac{\tan(\pi/4) - \tan(\theta/2)}{1 + \tan(\pi/4) \tan(\theta/2)}$$

$$\Rightarrow \cot \phi = \tan\left(\frac{\pi}{4} - \frac{\theta}{2}\right)$$

$$\Rightarrow \cot \phi = \cot\left[\frac{\pi}{2} - \left(\frac{\pi}{4} - \frac{\theta}{2}\right)\right]$$

$$\Rightarrow \phi = \frac{\pi}{2} - \frac{\pi}{4} + \frac{\theta}{2}$$

$$\Rightarrow \phi = \frac{\pi}{4} + \frac{\theta}{2} //$$

II. Show that the following pairs of curves intersect each other orthogonally.

$$①. r = a(1 + \cos \theta), r = b(1 - \cos \theta)$$

Given

$$r = a(1 + \cos \theta) \rightarrow ①$$

differentiate ① w.r.t. θ

$$\frac{dr}{d\theta} = -a \sin \theta$$

$$\frac{1}{r} \frac{dr}{d\theta} = \frac{-a \sin \theta}{r}$$

$$\Rightarrow \cot \phi_1 = \frac{-a \sin \theta}{a(1 + \cos \theta)}$$

$$\Rightarrow \cot \phi_1 = \frac{-\sin \theta}{1 + \cos \theta}$$

$$\Rightarrow \cot \phi_1 = \frac{-2 \sin \theta/2 \cos \theta/2}{2 \cos^2 \theta/2}$$

$$\Rightarrow \cot \phi_1 = -\tan \theta/2$$

$$\cot \phi = \cot \left(\frac{\pi}{2} + \frac{\theta}{2} \right)$$

$$\therefore \phi_1 = \frac{\pi}{2} + \frac{\theta}{2}$$

Similarly differentiate ② w.r.t. θ

$$\frac{dr}{d\theta} = b \sin \theta$$

$$\Rightarrow \frac{1}{r} \frac{dr}{d\theta} = \frac{b \sin \theta}{r}$$

$$\Rightarrow \cot \phi_2 = \frac{b \sin \theta}{b(1 - \cos \theta)}$$

$$\Rightarrow \cot \phi_2 = \frac{\sin \theta}{1 - \cos \theta}$$

$$\Rightarrow \cot \phi_2 = \frac{2 \sin \theta/2 \cos \theta/2}{2 \sin^2 \theta/2}$$

$$\therefore \cot \phi_2 = \theta/2$$

$$\therefore |\phi_1 - \phi_2| = \left| \frac{\pi}{2} + \frac{\theta}{2} - \frac{\theta}{2} \right| = \frac{\pi}{2} //$$

∴ the Given Curves are intersecting orthogonally.

$$\textcircled{2} \quad r = a(1 + \sin \theta), r = a(1 - \sin \theta)$$

Given

$$r = a(1 + \sin \theta) \longrightarrow \textcircled{1}$$

Differentiate \textcircled{1} w.r.t. \theta

$$\frac{dr}{d\theta} = a \cos \theta$$

$$\frac{1}{r} \frac{dr}{d\theta} = \frac{a \cos \theta}{r}$$

$$\Rightarrow \cot \phi_1 = \frac{a \cos \theta}{a(1 + \sin \theta)}$$

$$\Rightarrow \cot \phi_1 = \frac{\cos \theta}{(1 + \sin \theta)}$$

$$\Rightarrow \cot \phi_1 = \frac{\cos^2 \theta/2 - \sin^2 \theta/2}{\cos^2 \theta/2 + \sin^2 \theta/2 + 2 \sin \theta/2 \cos \theta/2}$$
$$= \frac{\cos \theta/2 - \sin \theta/2}{\cos \theta/2 + \sin \theta/2}$$

$$= \frac{1 - \tan \theta/2}{1 + \tan \theta/2} \quad \text{O}$$

$$\therefore \cot \phi_1 = \tan \left(\frac{\pi}{4} - \frac{\theta}{2} \right)$$

$$\Rightarrow \cot \phi_1 = \cot \left[\frac{\pi}{2} - \left(\frac{\pi}{4} - \frac{\theta}{2} \right) \right]$$

$$\phi_1 = \frac{\pi}{4} + \frac{\theta}{2}$$

Similarly differentiate \textcircled{2} w.r.t. \theta

$$\textcircled{2} \Rightarrow \frac{dr}{d\theta} = -a \cos \theta$$

$$\Rightarrow \frac{1}{r} \frac{dr}{d\theta} = -\frac{a \cos \theta}{a(1 - \sin \theta)}$$

$$\Rightarrow \cot \phi_2 = - \left[\frac{\cos^2 \theta/2 - \sin^2 \theta/2}{(\cos \theta/2 - \sin \theta/2)^2} \right]$$

$$\Rightarrow \cot \phi_2 = - \left[\frac{\cos \theta/2 + \sin \theta/2}{\cos \theta/2 - \sin \theta/2} \right]$$

$$= - \left[\frac{1 + \tan \theta/2}{1 - \tan \theta/2} \right]$$

$$= - \tan [\pi/4 + \theta/2]$$

$$\cot \phi_2 = \cot \left[\frac{\pi}{2} + \left(\frac{\pi}{4} + \frac{\theta}{2} \right) \right]$$

$$\therefore \phi_2 = \frac{\pi}{2} + \frac{\pi}{4} + \frac{\theta}{2}$$

$$\therefore |\phi_1 - \phi_2| = \left| \frac{\pi}{4} + \frac{\theta}{2} - \frac{\pi}{2} - \frac{\pi}{4} - \frac{\theta}{2} \right| = \frac{\pi}{2} //$$

\therefore The Given curves are intersecting orthogonally

$$③ r^n = a^n (\cos n\theta), r^n = b^n (\sin n\theta)$$

Given

$$r^n = a^n (\cos n\theta) \rightarrow ①$$

differentiate ① w.r.t. θ

$$\Rightarrow n r^{n-1} \frac{dr}{d\theta} = -a^n (\sin n\theta)$$

$$\Rightarrow \frac{r^n}{r} \frac{dr}{d\theta} = -a^n \sin n\theta$$

$$\Rightarrow \frac{1}{r} \frac{dr}{d\theta} = -\frac{a^n \sin n\theta}{r^n}$$

$$\cot \phi_1 = -\frac{a^n \sin n\theta}{a^n \cos n\theta}$$

$$\cot \phi_1 = -\tan \theta$$

$$\cot \phi_1 = \cot \left(\frac{\pi}{2} + n\theta \right)$$

$$\phi_1 = \frac{\pi}{2} + n\theta$$

Similarly differentiate ② w.r.t θ

$$② \Rightarrow n r^{n-1} \frac{dr}{d\theta} = b^n n \cos \theta$$

$$\Rightarrow \frac{r^n}{r} \frac{dr}{d\theta} = b^n \cos n\theta$$

$$\frac{1}{r} \frac{dr}{d\theta} = \frac{b^n \cos n\theta}{r^n}$$

$$\Rightarrow \cot \phi_2 = \frac{b^n \cos n\theta}{r^n}$$

$$\cot \phi_2 = \cot n\theta$$

$$\phi_2 = n\theta$$

$$\therefore |\phi_1 - \phi_2| = \left| \frac{\pi}{2} + n\theta - n\theta \right| = \pi/2 //$$

\therefore the Given curves are intersecting orthogonally

III. Find the angle between the Given pairs of Curves.

$$① r = \sin \theta + \cos \theta \text{ and } \gamma = 2 \sin \theta$$

Given :-

$$r = \sin \theta + \cos \theta \longrightarrow ①$$

$$r = 2 \sin \theta \rightarrow ②$$

differentiate equation ① w.r.t. θ

$$\text{①} \Rightarrow \frac{dr}{d\theta} = \cos \theta - \sin \theta$$

$$\Rightarrow \frac{1}{r} \frac{dr}{d\theta} = \frac{\cos \theta - \sin \theta}{r}$$

$$\Rightarrow \frac{1}{r} \frac{dr}{d\theta} = \frac{\cos \theta - \sin \theta}{\sin \theta + \cos \theta}$$

$$= \frac{1 - \tan \theta}{1 + \tan \theta}$$

$$= \frac{\tan(\pi/4) - \tan \theta}{\tan(\pi/4) + \tan \theta}$$

$$\Rightarrow \cot \phi_1 = \tan(\pi/4 - \theta)$$

$$\Rightarrow \cot \phi_1 = \cot[\pi/2 - (\pi/4 - \theta)]$$

$$\Rightarrow \phi_1 = \frac{\pi}{2} - \frac{\pi}{4} + \theta$$

$$\text{III}^{\text{q}} ② \Rightarrow \frac{dr}{d\theta} = 2 \cos \theta$$

$$\Rightarrow \frac{1}{r} \frac{dr}{d\theta} = \frac{2 \cos \theta}{r}$$

$$\Rightarrow \frac{1}{r} \frac{dr}{d\theta} = \frac{2 \cos \theta}{2 \sin \theta}$$

$$\Rightarrow \cot \phi_2 = \cot \theta$$

$$\Rightarrow \phi_2 = \theta$$

$$\therefore |\phi_1 - \phi_2| = \left| \frac{\pi}{2} - \frac{\pi}{4} + \theta - \theta \right|$$

$$\Rightarrow |\phi_1 - \phi_2| = \pi/4 //$$

\therefore The Given pairs of Curves are intersecting at
 $\frac{\pi}{4}$ [45°]

② $r = a \log \theta$ and $r = \frac{a}{\log \theta}$

Given

$$r = a \log \theta \rightarrow ①$$

$$r = \frac{a}{\log \theta} \rightarrow ②$$

differentiate equation ① and ② w.r.t. θ

$$① \Rightarrow \frac{dr}{d\theta} = a \cdot \left(\frac{1}{\theta} \right)$$

$$\Rightarrow \frac{dr}{d\theta} = \frac{a}{\theta}$$

$$\Rightarrow \frac{d\theta}{dr} = \frac{\theta}{a}$$

$$\Rightarrow r \frac{d\theta}{dr} = \frac{\theta}{a}$$

$$\Rightarrow \tan \phi_1 = \frac{(a \log \theta) \theta}{a}$$

$$\Rightarrow \tan \phi_1 = \theta \log \theta \rightarrow ③$$

III^{ly} differentiate equation ② w.r.t θ

$$② \Rightarrow \frac{dr}{d\theta} \log \theta + \frac{r}{\theta} = 0$$

$$\Rightarrow \log \theta \frac{dr}{d\theta} = -\frac{r}{\theta} = 0$$

$$\Rightarrow \frac{1}{r} \frac{dr}{d\theta} = -\frac{1}{\theta \log \theta}$$

$$\Rightarrow r \frac{d\theta}{dr} = -\theta \log \theta$$

$$\Rightarrow \tan \phi_2 = -\theta \log \theta \longrightarrow ④$$

W. K. T

$$\tan(\phi_1 - \phi_2) = \frac{\tan \phi_1 - \tan \phi_2}{1 + \tan \phi_1 \tan \phi_2}$$

$$\Rightarrow \tan(\phi_1 - \phi_2) = \frac{\theta \log \theta + \theta \log \theta}{1 + (\theta \log \theta)(-\theta \log \theta)}$$

$$\Rightarrow \tan(\phi_1 - \phi_2) = \frac{2 \theta \log \theta}{1 - \theta^2 (\log \theta)^2} \longrightarrow ⑤$$

∴ from equation ① and ②

$$a \log \theta = \frac{a}{\log \theta}$$

$$\Rightarrow (\log \theta)^2 = 1$$

$$\Rightarrow \log e^\theta = 1$$

$$\Rightarrow \theta = e^1 = e$$

$$\therefore ⑤ \Rightarrow \tan(\phi_1 - \phi_2) = \frac{2e}{1-e^2}$$

$$\therefore |\phi_1 - \phi_2| = \tan^{-1} \left(\frac{2e}{1-e^2} \right) = 2 \tan^{-1} e //$$

∴ The given two curves or pair of curves are intersecting at a tangent.

③ $r^2 \sin 2\theta = 4$ and $r^2 = 16 \sin 2\theta$

Given

$$r^2 \sin 2\theta = 4 \quad \rightarrow \textcircled{1} \quad \leftarrow \quad r^2 = \frac{4}{\sin 2\theta}$$
$$r^2 = 16 \sin 2\theta \quad \rightarrow \textcircled{2}$$

∴ from equation ① and ②

$$\frac{4}{\sin 2\theta} = 16 \sin 2\theta$$

$$\Rightarrow 4 \sin^2 2\theta = 1$$

$$\Rightarrow \sin^2 2\theta = \frac{1}{4}$$

$$\Rightarrow \sin 2\theta = \frac{1}{2}$$

$$\Rightarrow \sin 2\theta = \frac{1}{2}$$

$$\Rightarrow 2\theta = \sin^{-1}(1/2)$$

$$\Rightarrow 2\theta = \frac{\pi}{6}$$

$$\Rightarrow \theta = \frac{\pi}{12}$$

Differentiate equation ① and ② w.r.t. θ

$$\textcircled{1} \Rightarrow 2r \frac{dr}{d\theta} \sin 2\theta + r^2 \cos(2\theta) \cdot 2 = 0$$

$$\Rightarrow r \frac{dr}{d\theta} \cdot \sin 2\theta + r^2 \cos 2\theta = 0$$

$$\Rightarrow r \frac{dr}{d\theta} \cdot \sin 2\theta = -r^2 \cos 2\theta$$

$$\Rightarrow r \frac{dr}{d\theta} = - \frac{r^2 \cos 2\theta}{\sin \theta}$$

$$\Rightarrow \frac{1}{r} \frac{dr}{d\theta} = - \cot 2\theta$$

$$\Rightarrow \cot \phi_1 = \cot (-2\theta)$$

$$\Rightarrow \phi_1 = -2\theta$$

② $\Rightarrow 2r \cdot \frac{dr}{d\theta} = 16 (\cos 2\theta) (2)$

$$\Rightarrow r \frac{dr}{d\theta} = 16 \cos 2\theta$$

$$\Rightarrow \frac{1}{r} \frac{dr}{d\theta} = \frac{16 \cos 2\theta}{r^2}$$

$$\Rightarrow \cot \phi_2 = \frac{16 \cos 2\theta}{16 \sin 2\theta}$$

$$\Rightarrow \cot \phi_2 = \cot 2\theta$$

$$\Rightarrow \phi_2 = 2\theta$$

$$\therefore |\phi_1 - \phi_2| = |-2\theta - 2\theta| = 4\theta$$

$$\therefore |\phi_1 - \phi_2| = 4\left(\frac{\pi}{12}\right) = \frac{\pi}{3} //$$

\therefore The given pair of curves are intersecting at $\frac{\pi}{3}$

④ $r = a(1 - \cos \theta)$ and $r = 2a \cos \theta$

Given,

$$r = a(1 - \cos \theta) \rightarrow ①$$

$$r = 2a \cos \theta \rightarrow ②$$

differentiate equation ① and ② w.r.t. θ

$$① \Rightarrow \frac{dr}{d\theta} = a(-\sin \theta)$$

$$\Rightarrow \frac{dr}{d\theta} = a \sin \theta$$

$$\Rightarrow \frac{1}{r} \frac{dr}{d\theta} = \frac{a \sin \theta}{r}$$

$$\Rightarrow \frac{1}{r} \frac{dr}{d\theta} = \frac{a \sin \theta}{a(1 - \cos \theta)}$$

$$\Rightarrow \frac{1}{r} \frac{dr}{d\theta} = \frac{\sin \theta}{(1 - \cos \theta)}$$

$$\Rightarrow \cot \phi_1 = \frac{2 \sin \theta/2 \cos \theta/2}{2 \sin^2 \theta/2}$$

$$\Rightarrow \cot \phi_1 = \frac{\cos \theta/2}{\sin \theta/2}$$

$$\Rightarrow \cot \phi_1 = \cot \theta/2 \quad \therefore \phi_1 = \theta/2$$

III^{ly} ② d. w. r. t. θ

$$\frac{dr}{d\theta} = -2a \sin \theta$$

$$\Rightarrow \frac{1}{r} \frac{dr}{d\theta} = -\frac{2a \sin \theta}{2a \cos \theta}$$

$$\Rightarrow \frac{1}{r} \frac{dr}{d\theta} = -\tan \theta$$

$$\Rightarrow \cot \phi_2 = -\tan \theta$$

$$\Rightarrow \cot \phi_2 = \cot(\pi/2 + \theta)$$

$$\phi_2 = \frac{\pi}{2} + \theta$$

$$\therefore |\phi_1 - \phi_2| = \left| \frac{\theta}{2} - \frac{\pi}{2} - \theta \right|$$

$$|\phi_1 - \phi_2| = \left| \frac{\pi}{2} + \frac{\theta}{2} \right|$$

From ① and ②

$$a(1 - \cos \theta) = 2a \cos \theta$$

$$\Rightarrow (1 - \cos \theta) = 2 \cos \theta$$

$$\Rightarrow (1 - \cos \theta) = 2 \cos \theta$$

$$\Rightarrow 3 \cos \theta = 1$$

$$\Rightarrow \cos \theta = 1/3$$

$$\Rightarrow \theta = \cos^{-1}(1/3)$$

$$\therefore |\phi_1 - \phi_2| = \left| \pi/2 + 1/2 \cos^{-1}(1/3) \right|$$

∴ The Given two curves or pair of curves
are intersecting at $\left| \pi/2 + 1/2 \cos^{-1}(1/3) \right|$

⑤ $r = \frac{a}{1 + \cos \theta}$ and $r = \frac{b}{1 - \cos \theta}$

Given,

$$r = \frac{a}{1 + \cos \theta} \quad \text{or} \quad r(1 + \cos \theta) = a \rightarrow ①$$

① d ω. r t θ

$$(1 + \cos\theta) \frac{dr}{d\theta} - r \sin\theta = 0$$

$$\Rightarrow \frac{1}{r} \frac{dr}{d\theta} = \frac{\sin\theta}{1 + \cos\theta}$$

$$\Rightarrow \cot\phi_1 = \frac{2\sin\theta/2\cos\theta/2}{2\cos^2\theta/2}$$

$$\Rightarrow \cot\phi_1 = \tan\theta/2$$

$$\Rightarrow \cot\phi_1 = \cot(\pi/2 - \theta/2)$$

$$\therefore \phi_1 = (\pi/2 - \theta/2)$$

Similarly differentiate ② ω. r t. θ

$$r = \frac{b}{1 - \cos\theta} \quad \text{or} \quad r(1 - \cos\theta) = b \rightarrow ②$$

$$\Rightarrow \frac{1}{r} \frac{dr}{d\theta} = \frac{-\sin\theta}{1 - \cos\theta} = -\cot\theta/2$$

$$\Rightarrow \cot\phi_2 = -\frac{\theta}{2}$$

$$\therefore |\phi_2 - \phi_1| = \left| -\frac{\theta}{2} - \frac{\pi}{2} + \frac{\theta}{2} \right| = \frac{\pi}{2}$$

∴ The Given pair of curves are intersecting at the angle of $\frac{\pi}{2}$ or 90° .

IV. Find the pedal equation for the
Following polar curves.

$$\textcircled{1} \quad r = ae^{\theta \cot \alpha}$$

Let,

$$r = ae^{\theta \cot \alpha} \longrightarrow \textcircled{1}$$

\textcircled{1} equation differentiate w.r.t. \theta

$$\textcircled{1} \Rightarrow \frac{dr}{d\theta} = a \cdot e^{\theta \cot \alpha} \cdot \cot \alpha$$

$$\Rightarrow \frac{dr}{d\theta} = (a \cot \alpha) e^{\theta \cot \alpha}$$

$$\Rightarrow \frac{1}{r} \frac{dr}{d\theta} = \frac{(a \cot \alpha) e^{\theta \cot \alpha}}{r}$$

$$\Rightarrow \cot \phi = \frac{(a \cot \alpha) e^{\theta \cot \alpha}}{ae^{\theta \cot \alpha}}$$

$$\Rightarrow \cot \phi = \cot \alpha$$

$$\Rightarrow \phi = \alpha$$

w. K.T the pedal equation

$$P = r \sin \phi$$

$$\Rightarrow P = r \sin \alpha$$

$$\textcircled{2} \quad r^n = a^n \cos n\theta$$

Given,

$$r^n = a^n \cos n\theta \longrightarrow \textcircled{1}$$

\textcircled{1} equation differentiate w.r.t. \theta

$$\textcircled{1} \Rightarrow n r^{n-1} \frac{dr}{d\theta} = a^n (-\sin n\theta) n$$

$$\Rightarrow \frac{r^n}{r} \frac{dr}{d\theta} = -a^n \sin n\theta$$

$$\Rightarrow \frac{1}{r} \frac{dr}{d\theta} = -\frac{a^n \sin n\theta}{r^n}$$

$$\Rightarrow \cot \phi = \frac{-a^n \sin n\theta}{a^n \cos n\theta}$$

$$\Rightarrow \cot \phi = -\tan n\theta$$

W.K.T The pedal equation

$$\frac{1}{P^2} = \frac{1}{r^2} (1 + \cot^2 \phi)$$

$$\Rightarrow \frac{1}{P^2} = \frac{1}{r^2} [1 + \tan^2 n\theta]$$

$$\Rightarrow \frac{1}{P^2} = \frac{1}{r^2} \sec^2 n\theta$$

$$\Rightarrow \frac{1}{P^2} = \frac{1}{r^2 \cos^2 n\theta}$$

$$\Rightarrow P^2 = r^2 \cos^2 n\theta$$

$$\Rightarrow P = r \cos n\theta$$

$$\Rightarrow P = r \left(\frac{r^n}{a^n} \right) (\because ①)$$

$$\Rightarrow P = \frac{r^n + 1}{a^n}$$

$$\boxed{\Rightarrow a^n P = r^{n+1}}$$

$$③ \frac{2a}{r} = 1 + \cos \theta$$

Given.

$$\frac{\alpha}{r} = 1 + \cos\theta \quad \text{or} \quad \alpha = r(1 + \cos\theta) \rightarrow ①$$

② equation differentiate w.r.t. θ

$$① \Rightarrow (1 + \cos\theta) \frac{dr}{d\theta} + r(-\sin\theta) = 0$$

$$\Rightarrow (1 + \cos\theta) \frac{dr}{d\theta} - r \sin\theta = 0$$

$$\Rightarrow \frac{1}{r} \frac{dr}{d\theta} = \frac{\sin\theta}{1 + \cos\theta}$$

$$\Rightarrow \cot\phi = \frac{2 \sin\theta/2 \cos\theta/2}{2 \cos^2\theta/2}$$

$$\Rightarrow \cot\phi = \tan\theta/2$$

∴ W.K.T The pedal equation

$$\frac{1}{P^2} = \frac{1}{r^2} (1 + \cot^2\phi)$$

$$\Rightarrow \frac{1}{P^2} = \frac{1}{r^2} (1 + \tan^2\theta/2)$$

$$\Rightarrow \frac{1}{P^2} = \frac{1}{r^2} (\sec^2\theta/2)$$

$$\Rightarrow \frac{1}{P^2} = \frac{1}{r^2 \cos^2\theta/2}$$

$$\Rightarrow P^2 = r^2 \cos^2\theta/2$$

$$\Rightarrow P^2 = r^2 \left(\frac{1 + \cos\theta}{2} \right)$$

$$\Rightarrow 2P^2 = r^2 (1 + \cos\theta)$$

$$\Rightarrow \alpha p^2 = r^2 \left(\frac{\alpha}{r} \right)$$

$$\Rightarrow p^2 = ar$$

$$④ r^m = a^m (\cos m\theta + \sin m\theta)$$

Given

$$r^m = a^m [\cos m\theta + \sin m\theta] \rightarrow ①$$

① differentiate w.r.t. θ

$$① \Rightarrow m r^{m-1} \frac{dr}{d\theta} = a^m [-\sin m\theta(m) + \cos m\theta(m)]$$

$$\Rightarrow \frac{r^m}{r} \frac{dr}{d\theta} = \frac{a^m (\cos m\theta - \sin m\theta)}{r^m}$$

$$\Rightarrow \cot \phi = \frac{a^m (\cos m\theta - \sin m\theta)}{a^m (\cos m\theta + \sin m\theta)}$$

$$\Rightarrow \cot \phi = \frac{\cos m\theta - \sin m\theta}{\cos m\theta + \sin m\theta}$$

$$\therefore 1 + \cot^2 \phi = 1 + \frac{(\cos m\theta - \sin m\theta)^2}{(\cos m\theta + \sin m\theta)^2}$$

$$\Rightarrow 1 + \cot^2 \phi = \frac{(\cos m\theta + \sin m\theta)^2 + (\cos m\theta - \sin m\theta)^2}{(\cos m\theta + \sin m\theta)^2}$$

$$\Rightarrow 1 + \cot^2 \phi = \frac{2(\cos^2 m\theta + \sin^2 m\theta)}{\left(\frac{r^m}{a^m}\right)^2}$$

$$\Rightarrow 1 + \cot^2 \phi = \frac{2}{\frac{r^{2m}}{a^{2m}}}$$

$$\Rightarrow 1 + \cot^2 \phi = \frac{2a^2 m}{r^{2m}}$$

∴ the pedal equation

$$\frac{1}{p^2} = \frac{1}{r^2} (1 + \cot^2 \phi)$$

$$\Rightarrow \frac{1}{p^2} = \frac{1}{r^2} \cdot \frac{2a^2 m}{r^{2m}}$$

$$\Rightarrow \frac{1}{p^2} = \frac{2a^2 m}{r^{2m+2}}$$

$$\Rightarrow 2a^2 m p^2 = r^{2m+2}$$

$$⑤ \frac{l}{r} = 1 + e \cos \theta$$

Given,

$$\frac{l}{r} = 1 + e \cos \theta \rightarrow ①$$

differentiate w.r.t. θ

$$① \Rightarrow (1 + e \cos \theta) \frac{dr}{d\theta} + r(0 - e \sin \theta) = 0$$

$$\Rightarrow 1 + e \cos \theta \frac{dr}{d\theta} = re \sin \theta$$

$$\Rightarrow \frac{1}{r} \frac{dr}{d\theta} = \frac{e \sin \theta}{1 + e \cos \theta} = \cot \phi$$

$$\therefore 1 + \cot^2 \phi = \frac{1 + e^2 \sin^2 \theta}{(1 + e \cos \theta)^2}$$

$$= \frac{(1 + e \cos \theta)^2 + e^2 \sin^2 \theta}{(1 + e \cos \theta)^2}$$

$$= \frac{1 + 2e \cos \theta + e^2 \cos^2 \theta + e^2 \sin^2 \theta}{(1 + e \cos \theta)^2}$$

$$= \frac{1 + 2e \cos \theta + e^2(1)}{(1 + e \cos \theta)^2}$$

$$= \frac{1 + 2e \cos \theta + e^2}{(1 + e \cos \theta)^2}$$

$$\Rightarrow 1 + \cot^2 \phi = \frac{1 + e^2 + 2\left(\frac{l}{r} - 1\right)}{\left(\frac{l}{r}\right)^2}$$

$$= \frac{1 + e^2 + \frac{2l}{r} - 2}{\frac{l^2}{r^2}}$$

$$= \frac{e^2 + \frac{2l}{r} - 1}{\frac{l^2}{r^2}}$$

$$1 + \cot^2 \phi = \frac{r^2(e^2 + \frac{2l}{r} - 1)}{l^2}$$

∴ the pedal equation

$$\frac{1}{P^2} = \frac{1}{r^2} (1 + \cot^2 \phi)$$

$$\Rightarrow \frac{1}{P^2} = \frac{1}{r^2} \cdot r^2 \left(\frac{e^2 + \frac{2l}{r} - 1}{l^2} \right) \quad O$$

$$\Rightarrow \boxed{\frac{1}{P^2} = \frac{e^2 + \frac{2l}{r} - 1}{l^2}} \quad //$$

Radius of Curvature

Generally the Curvature of any curve can be denoted by 'k' and the reciprocal of the curvature will be called as the radius of curvature and it will be denoted

as,

$$P = \frac{1}{K}$$

Note:-

1. the radius of curvature can be evaluated in the Cartesian form by the following formula.

$$P = \frac{(1 + y_1^2)^{3/2}}{y_2}, y_2 \neq 0$$

where $y_1 = \frac{dy}{dx}$, $y_2 = \frac{d^2y}{dx^2}$ at any point P

In other way $P = \frac{(1 + x_1^2)^{3/2}}{x_2}, x_2 \neq 0$

where $x_1 = \frac{dx}{dy}$, $x_2 = \frac{d^2x}{dy^2}$ at any point P

2. The radius of curvature for the polar curve can be evaluated as,

$P = r \frac{dr}{dp}$, where $f(p, r, c)$ is the pedal equation.

V

1. Find the radius of curvature of the curve $y = a \log(\sec(x/a))$ at any point

Given,

$$y = a \log(\sec(x/a))$$

differentiate ① w.r.t.x

$$\text{①} \Rightarrow \frac{dy}{dx} = a \cdot \frac{1}{\sec(x/a)} \cdot \sec(x/a) \tan(x/a) \cdot 1/a$$

$$\Rightarrow \frac{dy}{dx} = \tan(x/a) = y_1 \rightarrow ②$$

differentiate ② w.r.t. x

$$② \Rightarrow \frac{d^2y}{dx^2} = \sec^2(x/a) \cdot 1/a$$

$$\Rightarrow y_2 = \frac{d^2y}{dx^2} = \frac{1}{a} \sec^2(x/a)$$

∴ W.K.T

$$\rho = \frac{(1 + y_1^2)^{3/2}}{y^2}$$

$$\Rightarrow \rho = \frac{(1 + \tan^2(x/a))^{3/2}}{1/a \sec^2(x/a)}$$

$$\Rightarrow \rho = \frac{a (\sec^2(x/a))^{3/2}}{\sec^2(x/a)}$$

$$\Rightarrow \rho = \frac{a \sec^3(x/a)}{\sec^2(x/a)}$$

$$\Rightarrow \boxed{\rho = a \sec(x/a)}$$

2. Find the radius of the curvature of the curve $x^3 + y^3 = 3axy$ at the point $P(3a/2, 3a/2)$.

Given,

$$x^3 + y^3 = 3axy \rightarrow ①$$

differentiate w.r.t. x

$$① \Rightarrow 3x^2 + 3y^2 \frac{dy}{dx} = 3a [1 \cdot y + x \frac{dy}{dx}]$$

$$\Rightarrow x^2 + y^2 y_1 = a[y + xy_1]$$

$$\Rightarrow x^2 + y^2 y_1 - ay - axy_1 = 0$$

$$\Rightarrow (y^2 - ax)y_1 + (x^2 - ay) = 0$$

$$\Rightarrow (y^2 - ax)y_1 + (x^2 - ay) = 0$$

$$\Rightarrow (y^2 - ax)y_1 = ay - x^2$$

$$\Rightarrow y_1 = \frac{ay - x^2}{y^2 - ax} \longrightarrow ②$$

$$\therefore (y_1)_P = \frac{a\left(\frac{3a}{2}\right) - \left(\frac{3a}{2}\right)^2}{\left(\frac{3a}{2}\right)^2 - a\left(\frac{3a}{2}\right)} = -\frac{\left(\frac{3a}{2}\right)^2 - a\left(\frac{3a}{2}\right)}{\left(\frac{3a}{2}\right)^2 - a\left(\frac{3a}{2}\right)} = -1$$

differentiate ② w.r.t x

$$② \Rightarrow y_2 = \frac{d^2 y}{dx^2} = \frac{(y^2 - ax)(ay_1 - 2x) - (ay - x^2)(2yy_1 - a)}{(y^2 - ax)^2}$$

$$(y_2)_P = \frac{\left[\left(\frac{3a}{2}\right)^2 - 4\left(\frac{3a}{2}\right)\right]a(-1) - 2\left(\frac{3a}{2}\right) - 4\left(\frac{3a}{2}\right) - \left(\frac{3a}{2}\right)^2}{\left[2\left(\frac{3a}{2}\right)(-1) - a\right] \quad 0}$$

$$\Rightarrow y_2 = \frac{\left(\frac{3a^2}{4}\right)(-4a) - \left(-\frac{3a^2}{4}\right)(-4a)}{\left(\frac{3a^2}{4}\right)^2}$$

$$\Rightarrow y_2 = \frac{-3a^3 - 3a^3}{9a^4}$$

$$\Rightarrow y_2 = -\frac{6a^3 \times 16}{9a^4} = -\frac{32}{3a}$$

$$\therefore \rho = \left| \frac{(1+y_1^2)^{3/2}}{y_2} \right| = \left| \frac{(1+(-1)^2)^{3/2}}{-32/3a} \right|$$

$$\Rightarrow \rho = \frac{2^{3/2}}{32/3a}$$

$$\Rightarrow \rho = \frac{2 \cdot 2^{1/2} \times 3a}{32}$$

$$\Rightarrow \rho = \frac{3\sqrt{2}a}{16}$$

$$\Rightarrow \rho = \frac{3\sqrt{2} \cdot a}{8 \times \sqrt{2} \times \sqrt{2}}$$

$$\boxed{\Rightarrow \rho = \frac{3a}{8\sqrt{2}}}$$

③ Find the radius of curvature of the curve $a^2y = x^2 - a^2$ at the point where the curve meets at x -axis

Given curve $a^2y = x^2 - a^2$ and the curve meet the x axis at the point 'P' and its y coordinate becomes zero.

\therefore the required point $P(a, 0)$

differentiate equation w.r.t x

$$a^2y = x^2 - a^2 \longrightarrow ①$$

$$\textcircled{1} \Rightarrow a^2 y_1' = 2x - 0$$

$$\Rightarrow y_1' = \frac{2x}{a^2} \rightarrow \textcircled{2}$$

$$\therefore (y_1)_P = \frac{2(a)}{a^2} = \frac{2}{a} \rightarrow \textcircled{3}$$

differentiate $\textcircled{3}$ w.r.t x

$$\textcircled{3} \Rightarrow y_2 = \frac{d^2y}{dx^2} = \frac{2}{a^2}$$

$$(y_2)_P = \frac{2}{a^2} \neq 0$$

$$\rho = \left| \frac{(1-y_1^2)^{3/2}}{y_2} \right|$$

$$\rho = \frac{(1+(2/a)^2)^{3/2}}{2/a^2}$$

$$\Rightarrow \rho = \frac{(1+4/a^2)^{3/2}}{2/a^3}$$

$$\Rightarrow \rho = \frac{a^2 \left(\frac{a^2+4}{a^2} \right)^{3/2}}{2}$$

$$\Rightarrow \rho = \frac{a^2 (a^2 + 4)^{3/2}}{a^3}$$

$$\Rightarrow \boxed{\rho = \frac{(a^2 + 4)^{3/2}}{2a}}$$

④ Find the radius of curvature of the curve $a^2y = x^3 - a^3$ and meets at x axis.

Given,

$$a^2y = x^3 - a^3 \rightarrow ①$$

differentiate w.r.t. x

$$① \Rightarrow a^2y_1 = 3x^2 \rightarrow ②$$

or

$$y_1 = \frac{3x^2}{a^2} \rightarrow ②$$

$$\therefore (y_1)_P = \frac{3(a^2)}{a^2} = 3$$

differentiate ② w.r.t x

$$② \Rightarrow y_2 = \frac{d^2y}{dx^2} = \frac{6x}{a^3}$$

$$(y_2)_P = \frac{6(a^2)}{a^3} = \frac{6}{a}$$

$$\Rightarrow \rho = \left| \frac{(1 + y_1^2)^{3/2}}{y_2} \right|$$

$$\Rightarrow \rho = \frac{(1 + 3^2)^{3/2}}{6/a}$$

$$\Rightarrow \rho = \frac{(1+9)^{3/2}}{6/a}$$

$$\Rightarrow \rho = \frac{a \cdot 10^{3/2}}{6}$$

$$\Rightarrow \rho = \frac{a \cdot 10 \cdot 10^{1/2}}{6}$$

$$\Rightarrow \boxed{\rho = \frac{5\sqrt{10} \cdot a}{3}}$$

⑤ Find the radius of the curvature of the curve $x^2y = a(x^2 + y^2)$ at point P(-2a, 2a)

Given,

$$x^2y = a(x^2 + y^2) \longrightarrow ①$$

differentiate ① w.r.t y

$$① \Rightarrow x^2(1) + y \frac{d}{dy}(x^2) = a \left(2x \frac{dx}{dy} + 2y \right)$$

$$\Rightarrow x^2 + 2xy \cdot x_1 = 2ax_1 + 2ay$$

$$\Rightarrow 2xy \cdot x_1 - 2ax \cdot x_1 = 2ay - x^2$$

$$\Rightarrow 2x x_1 (y-a) = 2ay - x^2$$

$$\Rightarrow x_1 = \frac{2ay - x^2}{2x(y-a)} = \frac{2ay - x^2}{2xy - 2ax} \quad ②$$

$$\therefore (x_1)_P = \frac{2a(2a) - (-2a)^2}{2(-2a)(2a-a)}$$

$$\Rightarrow (x_1)_P = 0$$

differentiate ② w.r.t y

$$② \Rightarrow x_2 = \frac{d^2x}{dy^2} = \frac{(2xy - 2ax)(2a - 2x \cdot x_1) - (2a - 2x \cdot x_1)}{[2x(y-a)]^2}$$

$$\quad \quad \quad - (2ay - x^2) [2x + 2x_1 y - 2ax_1]$$

$$\therefore (x_2)_p = \frac{[-8a^2 + 4a^2] [2a - 0]}{[-4a(a)]^2} - 0$$

$$\Rightarrow x_2 = -\frac{8a^3}{16a^4}$$

$$\Rightarrow x_2 = -\frac{1}{2a} \neq 0$$

$$\therefore P = \left| \frac{(1+x_1^2)^{3/2}}{x_2} \right| = \left| \frac{(1+0)^{3/2}}{-1/2a} \right| = 2a //$$

⑥ Find the radius of curvature of the polar curve $r = a(1 + \cos \theta)$

Given,

$$r = a(1 + \cos \theta) \rightarrow ①$$

differentiate ① w.r.t. θ

$$① \Rightarrow \frac{dr}{d\theta} = a(0 - \sin \theta) = -a \sin \theta$$

$$\Rightarrow \frac{1}{r} \frac{dr}{d\theta} = \frac{-a \sin \theta}{a(1 + \cos \theta)}$$

$$\Rightarrow \cot \phi = -\frac{2 \sin \theta / 2 \cos \theta / 2}{2 \cos^2 \theta / 2}$$

$$\Rightarrow \cot \phi = -\tan(\theta / 2)$$

w.k.t the pedal equation

$$\frac{1}{P^2} = \frac{1}{r^2} (1 + \cot^2 \phi)$$

$$\Rightarrow \frac{1}{P^2} = \frac{1}{r^2} (1 + \tan^2 \theta/2)$$

$$\Rightarrow \frac{1}{P^2} = \frac{1}{r^2} \sec^2 \theta/2$$

$$\Rightarrow \frac{1}{P^2} = \frac{1}{r^2 \cos^2 \theta/2}$$

$$\Rightarrow P^2 = r^2 \cos^2 \theta/2$$

$$\Rightarrow P^2 = r^2 \left(\frac{1 + \cos \theta}{2} \right)$$

$$\Rightarrow P^2 = \frac{r^2}{2} \left(\frac{\gamma}{\alpha} \right)$$

$$\Rightarrow 2\alpha P^2 = \gamma^3 \rightarrow ②$$

differentiate ②

$$② \Rightarrow 2\alpha \cdot 2P = 3\gamma^2 \frac{dP}{dP}$$

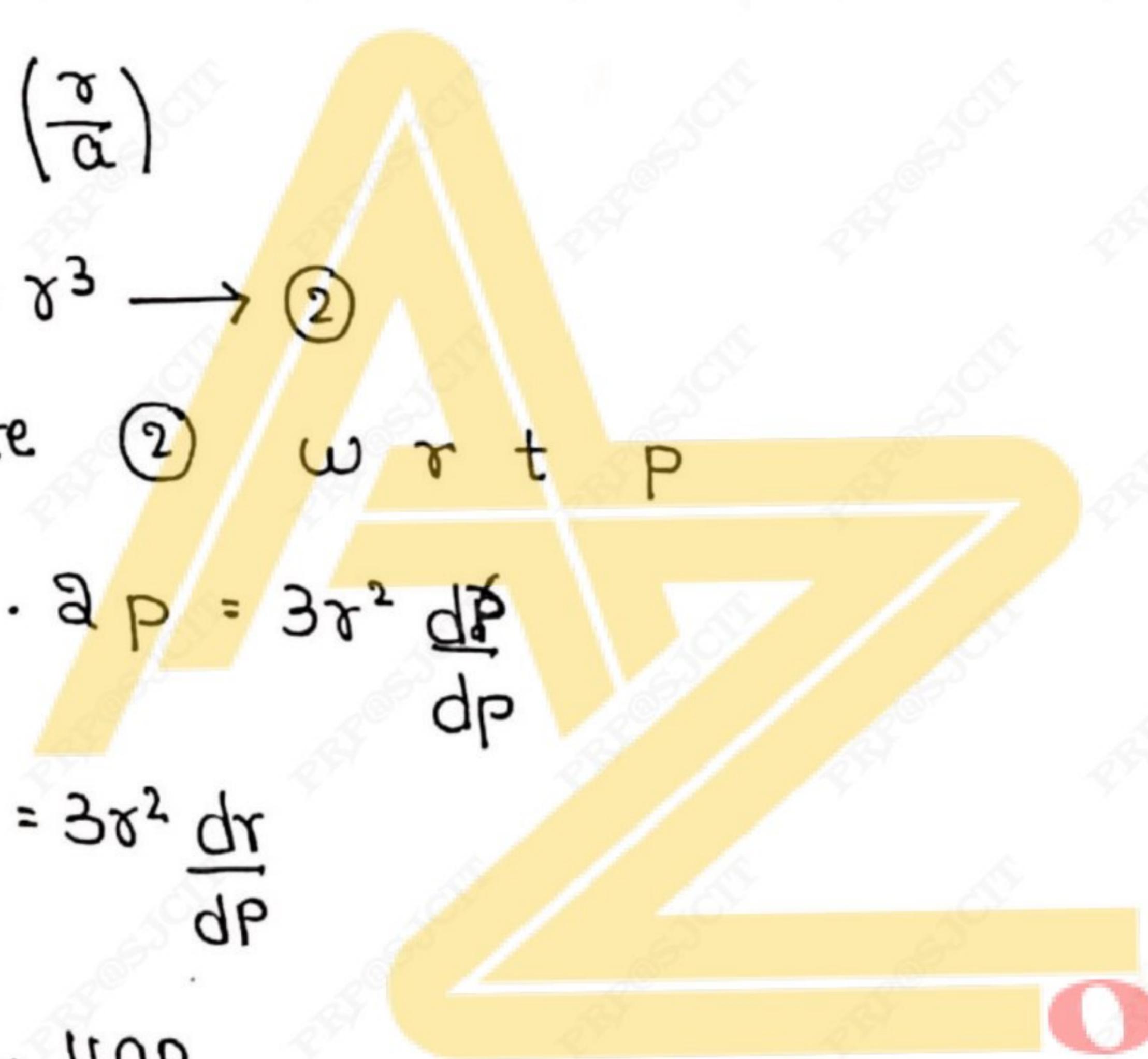
$$\Rightarrow 4\alpha P = 3\gamma^2 \frac{dr}{dp}$$

$$\Rightarrow \frac{dr}{dp} = \frac{4\alpha P}{3\gamma^2}$$

$$\Rightarrow \gamma \frac{dr}{dp} = \frac{4\alpha P}{3\gamma^2}$$

$$\Rightarrow \gamma \frac{dr}{dp} = \frac{4\alpha P}{3\gamma}$$

$$\Rightarrow f = \frac{4\alpha P}{3\gamma}$$



$$\Rightarrow \rho^2 = \frac{16a^2 p^2}{q r^2}$$

$$\Rightarrow \rho^2 = \frac{16a^2}{q} \cdot \frac{1}{r^2} \left(\frac{r^3}{2a} \right)$$

$$\Rightarrow \rho^2 = \left(\frac{8a}{q} \right) r$$

$$\boxed{\Rightarrow \rho^2 \propto r}$$

⑦ Show that for the curve $r(1 - \cos\theta) = 2a$

or $\frac{2a}{r} = (1 - \cos\theta)$, and ρ^2 varies as r^3

$$[\rho^2 \propto r^3]$$

Given,

$$r(1 - \cos\theta) = 2a \rightarrow ①$$

differentiate ① w.r.t. θ

$$① \Rightarrow (1 - \cos\theta) \frac{dr}{d\theta} + r(0 + \sin\theta) = 0$$

$$\Rightarrow (1 - \cos\theta) \frac{dr}{d\theta} = -r \sin\theta$$

$$\Rightarrow \frac{1}{r} \frac{dr}{d\theta} = \frac{-\sin\theta}{1 - \cos\theta}$$

$$\Rightarrow \omega t \phi = -\frac{2\sin\theta/2 \cos\theta/2}{2\sin^2\theta/2}$$

$$\Rightarrow \omega t \phi = -\cot(\theta/2)$$

$\therefore \omega \cdot K.T$ the pedal eqn.

$$\frac{1}{P^2} = \frac{1}{r^2} (1 + \omega t^2 \phi)$$

$$\Rightarrow \frac{1}{P^2} = \frac{1}{r^2} (1 + \omega t^2 \theta/2)$$

$$\Rightarrow \frac{1}{P^2} = \frac{1}{r^2} (\omega \sec^2 \theta/2)$$

$$\Rightarrow \frac{1}{P^2} = \frac{1}{r^2 \sin^2 \theta/2}$$

$$\Rightarrow P^2 = r^2 \sin^2 \theta/2$$

$$\Rightarrow P^2 = r^2 \left(1 - \frac{\cos \theta}{2} \right)$$

$$\Rightarrow P^2 = \frac{r^2}{2} \left(\frac{2a}{\gamma} \right)$$

$$\Rightarrow P^2 = ar \quad \text{--- (2)}$$

differentiate (2) w.r.t. P

$$(2) \Rightarrow \frac{\partial P}{\partial r} = a \frac{dr}{dp}$$

$$\Rightarrow \frac{dr}{dp} = \frac{\partial P}{a}$$

$$\Rightarrow r \frac{dr}{dp} = \frac{\partial P}{a}$$

$$\Rightarrow \rho = \frac{\partial P}{a}$$

$$\Rightarrow \rho^2 = \frac{4r^2 P^2}{a^2}$$

$$\Rightarrow \rho^2 = \frac{4r^2}{a^2} (ar)$$



$$\Rightarrow \rho^2 = \left(\frac{4}{a}\right) r^3$$

$$\Rightarrow \rho^2 \propto r^3$$

⑧ For the Cardioid, $r = a(1 - \cos\theta)$, show that $\frac{\rho^2}{r}$ is constant

Given,

$$r = a(1 - \cos\theta) \rightarrow ①$$

Differentiate ① w.r.t θ

$$\frac{dr}{d\theta} = \sin\theta \cdot a$$

$$\frac{1}{r} \frac{dr}{d\theta} = \frac{a \cdot 2 \sin\theta/2 \cos\theta/2}{a \cdot 2 \sin^2\theta/2}$$

$$\frac{1}{r} \frac{dr}{d\theta} = \cot\theta/2$$

$$\cot\phi = \cot\theta/2$$

$$\phi = \theta/2$$

∴ W.R.T the pedal equation

$$P = r \sin(\theta/2)$$

$$P^2 = r^2 \sin^2(\theta/2)$$

$$P^2 = r^2 \left(\frac{1 - \cos\theta}{2} \right)$$

$$\Rightarrow \rho^2 = \frac{3}{2} \left(\frac{x}{a} \right)$$

$$\Rightarrow \rho^2 = \frac{x^3}{2a}$$

then differentiate w.r.t P

$$2\rho = \frac{3x^2}{2a} \frac{dx}{dp}$$

$$x \frac{dx}{dp} = \frac{2\rho}{\frac{3x}{2a}}$$

$$\Rightarrow \rho = \frac{2\rho \times 2a}{3x}$$

$$\Rightarrow \rho^2 = \frac{16a}{9x^2} \left(\frac{x^3}{2a} \right)$$

$$\Rightarrow \boxed{\frac{\rho^2}{x} = \frac{8a}{9}} //$$



Q) Find the radius of curvature for the curve

$$\theta = \frac{\sqrt{r^2 - a^2}}{a} - \cos^{-1}\left(\frac{a}{r}\right) \text{ at any point on it}$$

Given,

$$\theta = \frac{\sqrt{r^2 - a^2}}{a} - \cos^{-1}\left(\frac{a}{r}\right) \rightarrow ①$$

Differentiate ① w.r.t. r

$$① \Rightarrow \frac{d\theta}{dr} = \frac{1}{a} \cdot \frac{1}{2\sqrt{r^2 - a^2}} (2r) - \left(\frac{-1}{\sqrt{1 - (a/r)^2}} \right) \frac{d}{dr} \left(\frac{a}{r} \right)$$

$$\begin{aligned} \Rightarrow \frac{d\theta}{dr} &= \frac{r}{a\sqrt{r^2 - a^2}} + \frac{1}{\sqrt{1 - a^2/r^2}} a \left(-\frac{1}{r^2} \right) \\ &= \frac{r}{a\sqrt{r^2 - a^2}} - \frac{ar}{\sqrt{r^2 - a^2}} \left(\frac{1}{r^2} \right) \\ &= \frac{r}{a\sqrt{r^2 - a^2}} - \frac{a}{\sqrt{r^2 - a^2}} \end{aligned}$$

$$\begin{aligned} \Rightarrow \frac{d\theta}{dr} &= \frac{1}{\sqrt{r^2 - a^2}} \left[\frac{r}{a} - \frac{a}{r} \right] \\ &= \frac{1}{\sqrt{r^2 - a^2}} \left[\frac{r^2 - a^2}{ar} \right] \\ &= \frac{1}{\sqrt{r^2 - a^2}} \frac{\sqrt{r^2 - a^2} \sqrt{r^2 - a^2}}{ar} \end{aligned}$$

$$\frac{d\theta}{dr} = \frac{\sqrt{r^2 - a^2}}{ar}$$

$$\Rightarrow \frac{dr}{d\theta} = \frac{ar}{\sqrt{r^2 - a^2}}$$

$$\Rightarrow \frac{1}{\gamma} \frac{dr}{d\theta} = \frac{a}{\sqrt{\gamma^2 - a^2}}$$

$$\Rightarrow \cot \phi = \frac{a}{\sqrt{\gamma^2 - a^2}}$$

$$\Rightarrow \cot \phi = \frac{a^2}{\gamma^2 - a^2}$$

$$\Rightarrow 1 + \cot^2 \phi = \frac{1 + a^2}{\gamma^2 - a^2} = \frac{\gamma^2 - a^2 + a^2}{\gamma^2 - a^2} = \frac{\gamma^2}{\gamma^2 - a^2}$$

∴ the pedal equation

$$\frac{1}{p^2} = \frac{1}{\gamma^2} (1 + \cot^2 \phi) = \frac{1}{\gamma^2} \frac{\gamma^2}{(\gamma^2 - a^2)}$$

$$\Rightarrow \frac{1}{p^2} = \frac{1}{\gamma^2 - a^2}$$

$$\Rightarrow p^2 = \gamma^2 - a^2 \rightarrow ①$$

differentiate ② w.r.t. p

$$② \Rightarrow 2p = 2r \frac{dr}{dp}$$

$$\Rightarrow r \frac{dr}{dp} = p$$

$$\Rightarrow p = \sqrt{\gamma^2 - a^2}$$

Q) Find the radius of curvature $\gamma^n = a^n (\sin n\theta)$

Given,

$$\gamma^n = a^n (\sin n\theta) \rightarrow ①$$

differentiate w.r.t. θ

$$n r^{n-1} \frac{dr}{d\theta} = a^n (\omega s n \theta)$$

$$\frac{\gamma^n}{r} \frac{dr}{d\theta} = \frac{a^n \cos n \theta}{r}$$

$$\frac{1}{r} \frac{dr}{d\theta} = \frac{a^n \cos n \theta}{a^n \sin n \theta}$$

$$\cot \phi = \cot n \theta$$

∴ The pedal equation

$$\frac{1}{P^2} = \frac{1}{r^2} (1 + \cot^2 \phi)$$

$$\frac{1}{P^2} = \frac{\operatorname{cosec}^2 n \theta}{r^2}$$

$$\Rightarrow \frac{1}{P^2} = \frac{1}{r^2 \sin^2 n \theta}$$

$$\Rightarrow P^2 = r^2 \sin^2 n \theta$$

$$\Rightarrow P^2 = r \sin n \theta$$

$$\Rightarrow P^2 = r \frac{r^n}{a^n} = \frac{r^{n+1}}{a^n}$$

$$\Rightarrow a^n P = r^{n+1} \longrightarrow ②$$

differentiate ② w.r.t p

$$② \Rightarrow a^n \cdot 1 = (n+1) r^n \frac{dr}{dp}$$

$$\Rightarrow r^n \frac{dr}{dp} = \frac{a^n}{n+1}$$

$$\Rightarrow r \cdot r^{n-1} \frac{dr}{dp} = \frac{a^n}{n+1}$$

$$\Rightarrow r \frac{dr}{d\theta} = \left(\frac{a^n}{n+1} \right) \cdot \left(\frac{1}{r^{n-1}} \right)$$

$$\Rightarrow r = \left(\frac{a^n}{n+1} \right) \left(\frac{1}{r^{n-1}} \right)$$

$$\Rightarrow r \propto \frac{1}{r^{n-1}}$$

(ii) Find the radius of curvature $r^n = a^n (\cos n\theta)$
Given,

$$r^n = a^n (\cos n\theta) \rightarrow ①$$

Differentiate ① w.r.t θ

$$① \Rightarrow n r^{n-1} \frac{dr}{d\theta} = -a^n \cdot n \sin^n \theta$$

$$\frac{1}{r} \frac{dr}{d\theta} = -\frac{a^n \sin^n \theta}{a^n \cos^n \theta}$$

$$\frac{1}{r} \frac{dr}{d\theta} = -\frac{a^n \sin^n \theta}{a^n \cos^n \theta}$$

$$\cot \phi = -\tan n\theta$$

$$\cot \phi = \cot(\pi/2 + n\theta)$$

$$\phi = \frac{\pi}{2} + n\theta$$

\therefore w.r.t the pedal equation

$$\Rightarrow P = r \sin \phi$$

$$\Rightarrow P = r \sin (\pi/2 + n\theta)$$

$$\Rightarrow P = r \cos n\theta$$

$$\Rightarrow P = r \left(\frac{r^n}{a^n} \right)$$

$$\Rightarrow P = \frac{r^{n+1}}{a^n}$$

$$\Rightarrow a^n P = r^{n+1}$$

→ ②

Then differentiate ω r t P

$$a^n = (n+1) r^n \frac{dr}{dP}$$

$$\frac{a^n}{(n+1)} \left(\frac{1}{r^{n-1}} \right) = \rho$$

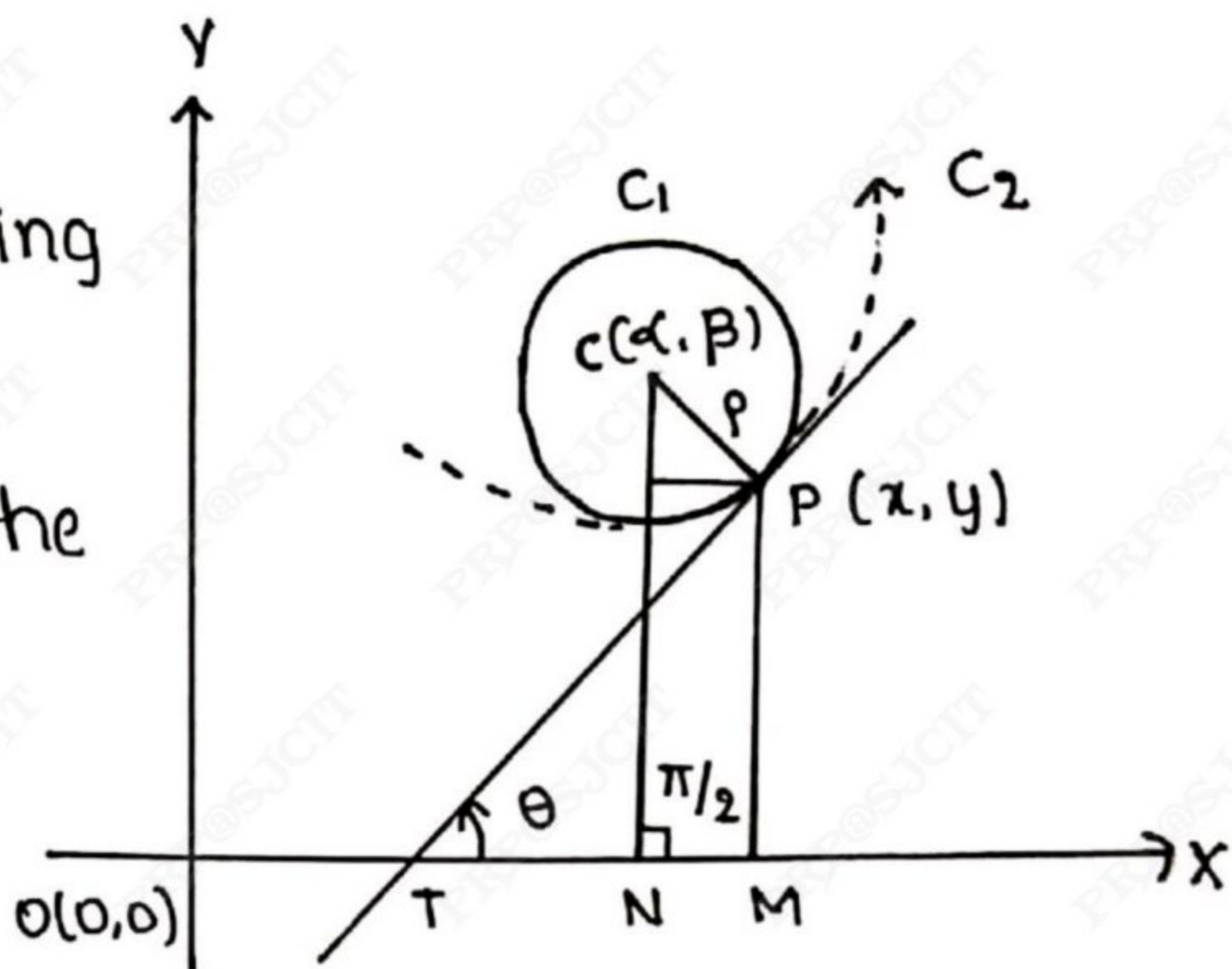
$$\Rightarrow \rho \propto \frac{1}{r^{n-1}}$$

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O

The Centre of Curvature, Invaluates and Evaluates

Let C_1 and C_2 be the two smooth curves passing through the same point $P(x, y)$. The centre of the curve C_1 is $C(\alpha, \beta)$ called as the centre of curvature, and its radius ρ is called circle curvature. The curve C_2 is called as the evaluate of the invaluate C_1 . The evaluate of an invaluate can be derived by the locus of the centre of the curvature $C(\alpha, \beta)$.



The centre of curvature $C(\alpha, \beta)$ can be evaluated by using the following formulas

$$\alpha = x - \rho \sin \psi$$

$$\beta = y + \rho \cos \psi$$

$$\Rightarrow \alpha = \frac{x - y_1(1 + y_1^2)}{y_2}, \quad \beta = \frac{y + (1 + y_1^2)}{y_2}$$

$$\text{where, } y_1 = \frac{dy}{dx}, \quad y_2 = \frac{dy^2}{dx^2}$$

Q1 Show that the evaluate of parabola,

$$y^2 = 4ax \text{ is } \frac{\partial}{\partial x} ay^2 = 4(x - 2a)^3$$

Given,

$$y^2 = 4ax \rightarrow ①$$

$$\Rightarrow y = 2a^{1/2} x^{1/2}$$

Differentiate ① w.r.t. x

$$① \Rightarrow \frac{\partial y}{\partial x} \frac{dy}{dx} = 4a$$

$$\Rightarrow y \frac{dy}{dx} = 2a$$

$$\Rightarrow \frac{dy}{dx} = y_1 = \frac{2a}{y} \rightarrow ②$$

Differentiate ② w.r.t. x

$$② \Rightarrow \frac{d^2y}{dx^2} = y_2 = \frac{y(a) 2ay_1}{y^2}$$

$$\Rightarrow y_2 = -\frac{2ay_1}{y^2}$$

$$\Rightarrow y_2 = -\frac{2a(2a/y)}{y^2}$$

$$\Rightarrow y_2 = -\frac{4a^2}{y^3}$$

$$\Rightarrow y_2 = -\frac{4a^2}{(2a^{1/2} \cdot x^{1/2})^3}$$

$$\Rightarrow y_2 = \frac{-4a^3}{8a^{3/2} x^{3/2}}$$

$$\Rightarrow y_2 = \frac{-a^{1/2}}{2x^{3/2}}$$

$$\therefore y_1 = \frac{2a}{2a^{1/3} \chi^{1/2}}$$

$$\Rightarrow y_1 = \frac{a^{1/2}}{\chi^{1/2}}$$

$\therefore w.k.t$

$$\alpha = \chi - y_1 (1 + y_1^2)$$

$$\Rightarrow \alpha = \chi - \frac{\left(\frac{a^{1/2}}{\chi^{1/2}} \right) \left(1 + a/\chi \right)}{-a^{1/2}}$$

$$\Rightarrow \alpha = \frac{\chi - 2a^{1/2} \chi^{3/2} \left(\frac{\chi+a}{\chi} \right)}{-a^{1/2} \chi^{1/2}}$$

$$\Rightarrow \alpha = \frac{\chi + 2\chi \cdot \chi^{1/2} \left(\frac{\chi+a}{\chi} \right)}{\chi^{1/2}}$$

$$\Rightarrow \alpha = \chi + 2(\chi+a)$$

$$\Rightarrow \alpha = 3\chi + 2a$$

$$\Rightarrow 3\chi = \alpha - 2a$$

$$\Rightarrow \chi = \frac{(\alpha - 2a)}{3}$$

$$\Rightarrow \chi^3 = \frac{(\alpha - 2a)^3}{27} \longrightarrow ③$$

$$\Rightarrow B = \frac{y + (1 + y_1)^2}{y_2} = \frac{2a^{1/2} \chi^{1/2} + \left(1 + \frac{a}{\chi} \right)}{-\frac{a^{1/2}}{2\chi^{3/2}}}$$

$$\Rightarrow \beta = \frac{2a^{1/2} \chi^{1/2} - 2\chi^{3/2} \left(\chi + \frac{a}{\chi}\right)}{a^{1/2}}$$

$$\Rightarrow \beta = \frac{2a^{1/2} \chi^{1/2} - 2\chi \chi^{1/2} \left(\frac{\chi + a}{\chi}\right)}{a^{1/2}}$$

$$\Rightarrow \beta = \frac{2a^{1/2} \chi^{1/2} - 2\chi^{1/2} (\chi + a)}{a^{1/2}}$$

$$\Rightarrow \beta = \frac{1}{a^{1/2}} \left| 2a\chi^{1/2} - 2\chi^{3/2} - 2a\chi^{1/2} \right|$$

$$\Rightarrow \beta = \frac{-2\chi^{3/2}}{a^{1/2}}$$

$$\Rightarrow -2\chi^{3/2} = \beta a^{1/2}$$

$$\Rightarrow \chi^{3/2} = \frac{-\beta a^{1/2}}{2}$$

$$\Rightarrow (\chi^{3/2})^2 = \left[-\frac{\beta a^{1/2}}{2} \right]^2$$

$$\Rightarrow \chi^3 = \frac{\beta^2 a}{4} \rightarrow \textcircled{4}$$

From $\textcircled{3}$ and $\textcircled{4}$

$$\frac{(\alpha - 2a)^3}{27} = \frac{\beta^2 a}{4}$$

$$\Rightarrow 4(\alpha - 2a)^3 = 27a\beta^2 \rightarrow \textcircled{5}$$

\therefore the locus of equation $\textcircled{5}$ is

$$4(\alpha - 2a)^3 = 27ay^2$$

② Show that evaluate of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$
 is $(ax)^{2/3} + (by)^{2/3} = (a^2 - b^2)^{2/3}$

Given.

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \rightarrow ①$$

$$\text{Let } x = a\cos\theta, y = b\sin\theta$$

$$\therefore \frac{dx}{d\theta} = -a\sin\theta, \frac{dy}{d\theta} = b\cos\theta$$

$$y_1 = \frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{b\cos\theta}{-a\sin\theta} = -\frac{b}{a}\cot\theta \rightarrow ②$$

differentiate

$$② \quad \omega \quad r + x$$

$$\Rightarrow y_2 = \frac{d^2y}{dx^2} = \frac{d}{dx} \left(\frac{dy}{dx} \right)$$

$$\Rightarrow y_2 = \frac{d}{d\theta} \left(\frac{dy}{dx} \right) \frac{d\theta}{dx}$$

$$\Rightarrow y_2 = \frac{d}{d\theta} \left(-\frac{b}{a}\cot\theta \right) \cdot \frac{1}{dx/d\theta} \quad ③$$

$$\Rightarrow y_2 = -\frac{b}{a} (-\operatorname{cosec}^2\theta) \cdot \frac{1}{-\sin\theta}$$

$$\Rightarrow y_2 = -\frac{b}{a^2} \frac{1}{\sin^2\theta} \cdot \frac{1}{\sin\theta}$$

$$\Rightarrow y_2 = -\frac{b}{a^2 \sin^3\theta}$$

∴ W.K.T

$$\ell = \frac{x - y_1(1 + y_1^2)}{y_2}$$

$$\alpha = \frac{a \cos \theta - (-b/a \cot \theta) \left(1 + \frac{b^2}{a^2} \cot^2 \theta \right)}{-b/a^2 \sin^3 \theta}$$

$$\Rightarrow \alpha = a \cos \theta - \frac{b}{a} \frac{\cos \theta}{\sin \theta} \left[1 + \frac{b^2}{a^2} \frac{\cos^2 \theta}{\sin^2 \theta} \right] \frac{b}{a^2 \sin^3 \theta}$$

$$\Rightarrow \alpha = a \cos \theta - (a \sin^3 \theta) \left(\frac{\cos \theta}{\sin \theta} \right) \left(\frac{a^2 \sin^2 \theta + b^2 \cos^2 \theta}{a^2 \sin^2 \theta} \right)$$

$$\Rightarrow \alpha = a \cos \theta - \frac{\cos \theta}{a} [a^2(1 - \cos^2 \theta) + b^2 \cos^2 \theta]$$

$$\Rightarrow \alpha = a \cos \theta - \frac{\cos \theta}{a} [a^2 - a^2 \cos^2 \theta + b^2 \cos^2 \theta]$$

$$\Rightarrow \alpha = a \cos \theta - a \cos \theta + \left[a \cos^3 \theta - \frac{b^2}{a} \cos^3 \theta \right]$$

$$\Rightarrow \alpha = \left(a - \frac{b^2}{a} \right) \cos^3 \theta$$

$$\Rightarrow \alpha = \left(\frac{a^2 - b^2}{a} \right) \cos^3 \theta$$

$$\Rightarrow \cos^3 \theta = \frac{a \alpha}{a^2 - b^2}$$

$$\Rightarrow (\cos^3 \theta)^{2/3} = (\alpha \ell)^{2/3} / (a^2 - b^2)^{2/3}$$

$$\Rightarrow \cos^2 \theta = \frac{(a\alpha)^{2/3}}{(a^2 - b^2)^{2/3}} \rightarrow ③$$

$$\beta = \frac{y + (1 + y_1^2)}{y_2}$$

$$\Rightarrow \beta = \frac{b \sin \theta + \left(1 + \frac{b^2}{a^2} \cot^2 \theta \right)}{-b/a^2 \sin^3 \theta}$$

$$\Rightarrow \beta = b \sin \theta - \frac{a^2 \sin^3 \theta}{b} - b \sin \theta + b \sin^3 \theta$$

$$\Rightarrow \beta = \left(b - \frac{a^2}{b} \right) \sin^3 \theta$$

$$\Rightarrow \beta = \left(\frac{b^2 - a^2}{b} \right) \sin^3 \theta$$

$$\Rightarrow \sin^3 \theta = \frac{b\beta}{b^2 - a^2}$$

$$\Rightarrow \sin^3 \theta = \frac{b\beta}{-(a^2 - b^2)}$$

$$\Rightarrow \sin^6 \theta = \frac{(b\beta)^2}{(a^2 - b^2)^{2/3}}$$

$$\Rightarrow \sin^2 \theta = \frac{(b\beta)^{2/3}}{(a^2 - b^2)^{2/3}} \rightarrow ④$$

$$③ + ④ = \cos^2 \theta + \sin^2 \theta = \frac{(a\alpha)^{2/3}}{(a^2 - b^2)^{2/3}} + \frac{(b\beta)^{2/3}}{(a^2 - b^2)^{2/3}}$$

$$\Rightarrow (ax)^{2/3} + (by)^{2/3} = (a^2 - b^2)^{2/3}$$

∴ the locus of (α, β)

$$(ax)^{2/3} + (by)^{2/3} = (a^2 - b^2)^{2/3} //$$

③ Show that the evaluate of the hyperbula

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1, \quad (ax)^{2/3} - (by)^{2/3} = (a^2 + b^2)^{2/3}$$

Given,

$$x = a \sec \theta, \quad y = b \tan \theta$$

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \rightarrow ①$$

$$\frac{dx}{d\theta} = a \sec \theta \tan \theta, \quad \frac{dy}{d\theta} = b \sec^2 \theta$$

$$\therefore y_1 = \frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{b \sec^2 \theta}{a \sec \theta \tan \theta} = \frac{b \sec \theta}{a \tan \theta}$$

$$y_1 = \frac{b}{a} \operatorname{cosec} \theta$$

$$y_2 = \frac{d}{dx} \left(\frac{dy}{dx} \right) = \frac{d}{d\theta} \left(\frac{dy}{dx} \right) \frac{d\theta}{dx}$$

$$= \frac{d}{d\theta} \left(\frac{b}{a} \operatorname{cosec} \theta \right) \cdot \frac{1}{dx/d\theta}$$

$$= \frac{b}{a} (-\operatorname{cosec} \theta \cdot \cot \theta) \cdot \frac{1}{a \sec \theta \tan \theta}$$

$$= -\frac{b}{a^2} \frac{1}{\sin \theta} \frac{\cos \theta}{\sin \theta} \cdot \cos \theta \cdot \frac{\cos \theta}{\sin \theta}$$

$$y_2 = -\frac{b}{a^2} \frac{\cos^3 \theta}{\sin^3 \theta}$$

$$\therefore d = \frac{x - y_1(1 + y_1^2)}{y_2}$$

$$d = a \sec \theta - \left(\frac{b}{a} \cosec \theta \right) \left[1 + \frac{b^2}{a^2} \cosec^2 \theta \right]$$

$$d = a \sec \theta + \frac{b}{a} \frac{1}{\sin \theta} \frac{\frac{a^2 \tan^3 \theta}{b \cos^3 \theta}}{\left[1 + \frac{b^2}{a^2} \cdot \frac{1}{\sin^2 \theta} \right]}$$

$$d = a \sec \theta + \frac{a \sin^2 \theta}{\cos^3 \theta} \left[\frac{a^2 \sin^2 \theta + b^2}{a^2 \sin^2 \theta} \right]$$

$$d = a \sec \theta + \frac{1}{a \cos^3 \theta} [a^2 \sin^2 \theta + b^2]$$

$$d = a \sec \theta + \frac{a \sin^2 \theta}{\cos^3 \theta} + \frac{b^2}{a \cos^3 \theta}$$

$$d = a \sec \theta + a \tan^2 \theta \sec \theta + \frac{b^2}{a} \sec^3 \theta$$

$$d = a \sec \theta + a \sec \theta (\sec^2 \theta - 1) + \frac{b^2}{a} \sec^3 \theta$$

$$\alpha = a \sec \theta + a \sec^3 \theta - a \sec \theta + \frac{b^2}{a} \sec^3 \theta$$

$$\alpha = \left(a + \frac{b^2}{a} \right) \sec^3 \theta$$

$$\alpha = \left(\frac{a^2 + b^2}{a} \right) \sec^3 \theta$$

$$\sec^3 \theta = \frac{a\alpha}{a^2 + b^2}$$

$$\sec^2 \theta = \frac{(a\alpha)^{2/3}}{(a^2 + b^2)^{2/3}} \rightarrow ④$$

Similarly $\beta = y + \frac{1 + y_1^2}{y_2}$

$$\tan^2 \theta = \frac{(b\beta)^{3/2}}{(a^2 + b^2)^{2/3}} \rightarrow ⑤$$

$$④ + ⑤ = 1 = \frac{(a\alpha)^{2/3} - (b\beta)^{2/3}}{(a^2 + b^2)^{2/3}}$$

$$(a^2 + b^2)^{2/3} = (a\alpha)^{2/3} - (by)^{2/3} //$$

A. Find the pedal equation.

① $r^n = a^n \sin n\theta$

Given,

$$r^n = a^n \sin n\theta \quad \rightarrow ①$$

① equation differentiate w.r.t. θ

$$\Rightarrow n r^{n-1} \frac{dr}{d\theta} = a^n \cos n\theta \cdot n$$

$$\Rightarrow \frac{r^n}{r} \frac{dr}{d\theta} = a^n \cos n\theta$$

$$\Rightarrow \frac{1}{r} \frac{dr}{d\theta} = \frac{a^n \cos n\theta}{r^n}$$

$$\Rightarrow \cot \phi = \frac{a^n \cos n\theta}{a^n \sin n\theta}$$

$$\Rightarrow \cot \phi = \cot n\theta$$

$$\Rightarrow \phi = n\theta$$

We know that, the pedal equation

$$\frac{1}{P^2} = \frac{1}{r^2} (1 + \cot^2 \phi)$$

$$\Rightarrow \frac{1}{P^2} = \frac{1}{r^2} (1 + \cot^2 n\theta)$$

$$\Rightarrow \frac{1}{P^2} = \frac{1}{r^2} \operatorname{cosec}^2 n\theta$$

$$\Rightarrow P^2 = r^2 \sin^2 n\theta$$

$$\Rightarrow P = r \sin n\theta$$

$$\Rightarrow P = r \left(\frac{r^n}{a^n} \right)$$

$$\Rightarrow P = \frac{r^{n+1}}{a^n}$$

B. Find the pedal equation for the following polar curves.

$$①. \gamma^m = a^m \cos m\theta + b^m \sin m\theta$$

Given.

$$\gamma^m = a^m \cos m\theta + b^m \sin m\theta \rightarrow ①$$

Differentiate ① w.r.t θ

$$m \gamma^{m-1} \frac{d\gamma}{d\theta} = -\sin m\theta \cdot m a^m + \cos m\theta \cdot m b^m$$

$$\frac{\gamma^m}{\gamma} \frac{d\gamma}{d\theta} = b^m \cos m\theta - a^m \sin m\theta$$

$$\frac{1}{\gamma} \frac{d\gamma}{d\theta} = \frac{b^m \cos m\theta - a^m \sin m\theta}{a^m \cos m\theta + b^m \sin m\theta}$$

$$\cot \phi = \frac{b^m \cos m\theta - a^m \sin m\theta}{a^m \cos m\theta + b^m \sin m\theta}$$

$$1 + \cot^2 \phi = \frac{1 + (b^m \cos m\theta - a^m \sin m\theta)^2}{(a^m \cos m\theta + b^m \sin m\theta)^2}$$

$$= \frac{(a^m \cos m\theta + b^m \sin m\theta)^2 + (b^m \cos m\theta - a^m \sin m\theta)^2}{(a^m \cos m\theta + b^m \sin m\theta)^2}$$

$$1 + \cot^2 \phi = \frac{2(a^{2m} + b^{2m})}{\gamma^{2m}}$$

∴ W.K.T the pedal equation

$$\frac{1}{P^2} = \frac{1}{\gamma^2} (1 + \cot^2 \phi)$$

$$\frac{1}{P^2} = \frac{1}{\gamma^2} \left(\frac{a^{2m} + b^{2m}}{\gamma^{2m}} \right)$$

$$\Rightarrow \frac{1}{P^2} = \frac{a^{2m} + b^{2m}}{\gamma^2 + 2am}$$

$$\Rightarrow \gamma^{2m+2} = P^2(a^{2m} + b^{2m})$$

