Module - 1

OSCILLATIONS & WAVES

Free oscillations

Oscillations and vibrations play a significant role in our lives than we realize. When we strike a bell, the metal vibrates creating a sound wave. All musical instruments are based on some method to force air around the instrument to oscillate. Oscillations from the swing of pendulum in a clock to the vibrations of quartz crystal are used as timing devices. When we heat a substance, some of the energy supplied forces atoms to oscillate. The harmonic oscillators have close analogy in many other fields; mechanical example of weight on a spring, oscillations of charge flowing back and forth in an electrical circuit, vibrations of tuning fork, vibrations of electrons in an atom generating light waves, oscillation of electrons in an antenna etc. The oscillation of a body or system with its own natural frequency under no external influence other than the impulse that initiated the motion is called free oscillation. E.g.: when simple pendulum is slightly displaced from its mean position vibrates freely with its natural frequency that depends only upon length of pendulum.

Definitions

Displacement 'x': At a particular instant't', the distance of the location of the body from its mean position in linear oscillatory motion, or the angle at which the body is located from its mean position in angular motion.

Amplitude 'a': The maximum value of displacement of the body from equilibrium position.

Frequency 'γ': Number of oscillations executed by an oscillating body in unit time. SI unit is Hertz.

Angular frequency or angular velocity 'ω': The angle covered in unit time by a representative point moving on a circle whose motion is correlated to the motion of vibrating body. SI unit is radian per second.

$$\omega = 2\pi\gamma$$

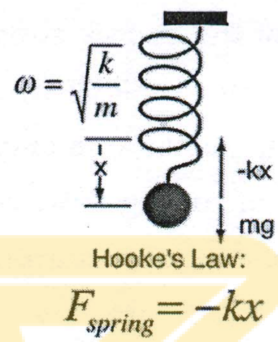
Period '*T*': Time taken by the body to complete one oscillation.

$$\gamma = \frac{1}{T}$$

Simple Harmonic Motion

Simple harmonic motion is the periodic or oscillatory motion executed by a body where the restoring force is directly proportional to the displacement and acts in the direction towards the mean position.

Example: A mass on a spring. More the mass stretches the spring, more it feels



a tug back towards the middle.

Restoring force α – displacement

$F = -kx$ where k is known as spring constant.

Characteristics of Simple Harmonic Motion:

1. Simple harmonic motion is a periodic motion.
2. The oscillating system must have inertia which in turn means mass.
3. There is a constant restoring force continuously acting on the body.
4. The acceleration developed in the motion due to the restoring force is directly proportional to displacement from its mean position.
5. The acceleration is always directed towards the mean position.
6. It can be represented by a sine or cosine function such as $x = a \sin \omega t$

Examples of SHM

1. A mass suspended to a spring when pulled down and left free executes SHM.
2. A pendulum set for oscillation.
3. Excited tuning fork.
4. A shock absorber after being bumped.
5. In an LC circuit the electrical charges move back & forth from an inductor to capacitor & vice versa causing electrical oscillations which are simple harmonic in nature.

Differential Equation of motion for SHM

Let a body of mass m be initiated to an oscillatory motion. Then the restoring force acting on the body

$$F_{restoring} = -kx$$

As per Newton's second law of motion, $F = m\frac{d^2x}{dt^2}$

$$\therefore \quad m\frac{d^2x}{dt^2} = -kx$$

$$\frac{d^2x}{dt^2} = \frac{-k}{m}x$$

$$\frac{d^2x}{dt^2} + \frac{k}{m}x = 0$$

Let $\omega^2 = \frac{k}{m}$ then $\omega = \sqrt{\frac{k}{m}}$

$$\frac{d^2x}{dt^2} + \omega^2 x = 0$$

The above equation represents the equation of motion for a body executing free vibrations.

The solution for above equation is $x = a \sin \omega t$, where a is the amplitude, ω is the angular frequency and t is the time elapsed.

Mechanical Simple Harmonic Oscillator

Force constant for a mass suspended to a spring

Consider an idealized spring in which

1. The spring is assumed to be light
2. There are no dissipative forces tending to decrease motion of the spring.
3. Restoring force exerted by the spring is directly proportional to its extension.

Consider a spring fixed to a rigid support at upper end be attached to a mass m at lower end due to which a load mg acts on the spring vertically downwards. Suppose the mass m is pulled down within the elastic limit by a distance x then the upward restoring force generated within the spring will be F which acts opposite to displacement.

By Hooke's law, $F\ a - x$

$$F = -k\,x$$

where k is known as force constant or stiffness factor. SI unit is newton per meter.

For $x = 1$, $k = F$.

Thus force constant is defined as the magnitude of the applied force that produces unit extension (or compression) in the spring while it is loaded within the elastic limit.

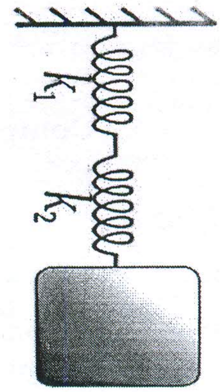
Time period of oscillation $T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{m}{k}}$ second

Frequency of oscillation $\gamma = \frac{1}{T} = \frac{1}{2\pi}\sqrt{\frac{k}{m}}$ Hertz

Physical significance of force constant

Force constant or stiffness factor k is a measure of stiffness of spring. It represents how much force it takes to stretch a material. Materials with larger spring constants are stiffer.

Expression for spring constant for series combination



Consider two springs S_1 and S_2 with spring constants k_1 and k_2 respectively. Let x_1 and x_2 be the respective extensions in S_1 and S_2 individually under the pulling action of a suspended mass m.

Then from Hooke's law, for S_1, $F = -k_1x_1$

Similarly for S_2, $F = -k_2x_2$

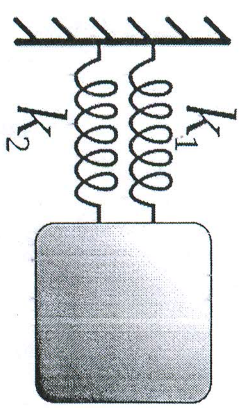
Now, let S_1 and S_2 suspended in series, are attached to a mass m at the bottom. Then S_1 and S_2 experience the same pull F by mass m. Thus the total elongation is given by

$$x = x_1 + x_2$$

Let the force constant for series combination be k_s.

$$\therefore -\frac{F}{k_s} = -\frac{F}{k_1} - \frac{F}{k_2} \quad \text{or} \quad \frac{1}{k_s} = \frac{1}{k_1} + \frac{1}{k_2}$$

Expression for spring constant for parallel combination



Consider two springs S_1 and S_2 with spring constants k_1 and k_2 respectively. Let x_1 and x_2 be the respective extensions in S_1 and S_2 individually under the pulling action of a suspended mass m.

Then from Hooke's law, for S_1, $F_1 = -k_1x_1$

Similarly for S_2, $F_2 = -k_2x_2$

Now, let S_1 and S_2 suspended in parallel from a rigid support, are attached to a mass m at the bottom. The two springs individually elongate by same distance x but experience the load non uniformly.

Thus the total load across the two springs is given by

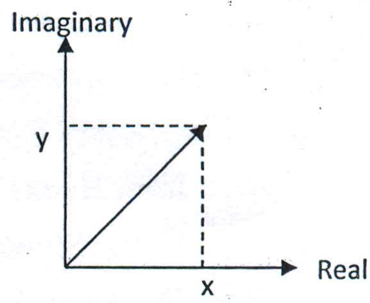
$$F = F_1 + F_2$$

Let the force constant for parallel combination be k_p.

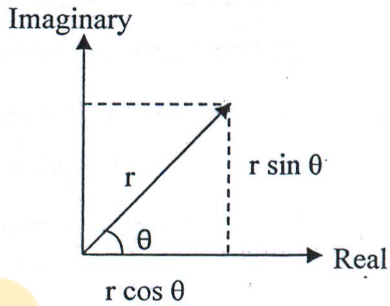
Then $$-k_p x = -k_1 x_1 - k_2 x_2$$

Since $x_1 = x_2 = x$, $$k_p = k_1 + k_2$$

Complex notation of SHM



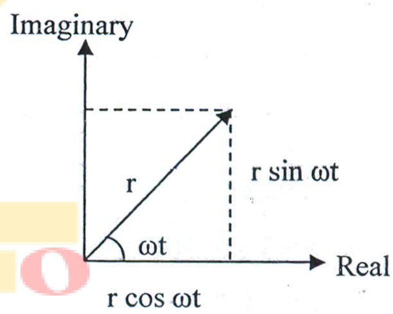
A complex number z has the form $z = x + iy$, where $i = \sqrt{-1}$ which is termed imaginary. x is the real part and is the projection of z on x-axis and y the imaginary part and is the projection of z on y-axis. This method of representation of a complex number in a coordinate form is called Argand diagram.



In polar coordinates, the Argand diagram representation is done in terms of r and θ. Here $z = r\cos\theta + ir\sin\theta$ where r is the magnitude of z.

$\therefore\ z = r(\cos\theta + i\sin\theta) = r\,e^{i\theta}$.

In case θ changes with time, then $\theta = \omega t$. The arrow r rotates about the origin with an angular velocity ω.



$\therefore\ z = r\,e^{i\omega t}$

At $t = 0$, if z is already making an angle ϕ, then $z = r\,e^{i(\omega t+\phi)}$. This is the complex notation used in SHM.

Phasor representation of SHM

Phasor is a complex representation of the magnitude and phase of a sinusoidal vibration. In the Argand diagram representation, we have for the rotation of z in complex notation as

$$z = re^{i(\omega t+\varphi)}$$

The length r of the arrow in the Argand diagram corresponds to amplitude and $\theta = \omega t$ gives the phase angle. The rotating arrow z is the phasor represented as

$$r\angle\phi = r\cos\phi + ir\sin\phi$$

Free Oscillations

If a body oscillates with its own natural frequency with undiminished amplitude, under the action of a restoring force as long as no external force intervenes in its motion then such oscillations are called as free oscillations.

E.g.: Free oscillations are ideal and don't exist in reality. For small displacement and negligible damping, the oscillation of a mass suspended by a spring, oscillation of simple pendulum, electrical oscillations can be considered under free oscillations.

If m is the mass of the oscillating body, k is the force constant and x is the displacement at the instant t, the equation of motion for free oscillations is given by

$$\frac{d^2x}{dt^2} + \frac{k}{m}x = 0$$

The frequency with which a body oscillates freely at its own under the influence of restoring force is called natural frequency.

Damped Oscillations

It is the type of motion executed by a body subjected to the combine action of both the restoring and resistive forces, and the motion always gets terminated with the body coming to rest at the equilibrium position in a finite interval of time.

E.g.: 1. Mechanical oscillations of a simple pendulum

2. Electrical oscillations in an LC circuit

3. A swing left free to oscillate after being pushed once.

Theory of damped vibrations

Consider a body of mass m executing vibrations in a resistive medium. The resistive or damping forces are proportional to the velocity $\frac{dx}{dt}$ of the body but opposite to its movement.

i.e. resistive force = $-r\frac{dx}{dt}$, where r is damping constant

Also, the restoring force = $-kx$

Hence the net force acting on the body = $-r\frac{dx}{dt} - kx$ --------------(1)

From Newton's second law of motion, resulting force = $m\frac{d^2x}{dt^2}$ ----------------(2)

$\therefore$ From (1) & (2), $\quad m\frac{d^2x}{dt^2} = -r\frac{dx}{dt} - kx$

Or $\quad m\frac{d^2x}{dt^2} + r\frac{dx}{dt} + kx = 0$ is the equation of motion for damped vibrations.

Dividing throughout by m, $\quad \frac{d^2x}{dt^2} + \frac{r}{m}\frac{dx}{dt} + \frac{k}{m}x = 0$

$$\frac{d^2x}{dt^2} + 2b\frac{dx}{dt} + \omega^2x = 0 \text{ ----------------(3)}$$

where $2b = \frac{r}{m}$ and $\omega^2 = \frac{k}{m}$ or $\omega = \sqrt{\frac{k}{m}}$ is the natural angular frequency of vibration in absence of damping forces.

Let the solution of the above equation be

$x = A\,e^{at}$ --------------------------(4) where A and α are constants.

Differentiating with respect to t, we get

$$\frac{dx}{dt} = Aa\,e^{at}$$

Differentiating again, $\frac{d^2x}{dt^2} = Aa^2\,e^{at}$

Substituting the values of $x, \frac{dx}{dt}, \frac{d^2x}{dt^2}$ in equation (3) we have,

$$Aa^2\,e^{at} + 2b\,Aa\,e^{at} + \omega^2A\,e^{at} = 0$$

$$A\,e^{at}\,[a^2 + 2ba + \omega^2] = 0$$

$$x\,[a^2 + 2ba + \omega^2] = 0$$

For above equation to be satisfied, either $x = 0$ or $a^2 + 2ba + \omega^2 = 0$

Since $x = 0$ corresponds to a trivial solution, we consider $a^2 + 2ba + \omega^2 = 0$

The solution for above quadratic equation is

$$\alpha = -b \pm\sqrt{b^2 - \omega^2}$$

Substituting for α in equation (4), the general solution can be written as,

$$x = A_1e^{(-b+\sqrt{b^2-\omega^2})t} + A_2e^{(-b-\sqrt{b^2-\omega^2})t} \text{ ----------- (5)}$$

where A_1 and A_2 are constants.

The constants A_1 and A_2 can be determined from the initial conditions and are given by

$$A_1 = \frac{x_0}{2}\left[1 + \frac{b}{\sqrt{b^2-\omega^2}}\right] \quad \text{and} \quad A_2 = \frac{x_0}{2}\left[1 - \frac{b}{\sqrt{b^2-\omega^2}}\right]$$

The quantity $\sqrt{b^2 - \omega^2}$ is imaginary, real or zero depends on relative values of b and ω. If $b^2 < \omega^2$ then $\sqrt{b^2 - \omega^2}$ is imaginary and is called under damped case. If $b^2 > \omega^2$, the situation is known as over damped and if $b^2 = \omega^2$, it is called critical damping.

Case 1: Over damping or dead beat, $b^2 > \omega^2$:

When $b^2 > \omega^2$, $(b^2 - \omega^2)$ is positive. But $\sqrt{b^2 - \omega^2} <$ b. Hence co-efficients of t in both the terms in equation $x = A_1e^{(-b+\sqrt{b^2-\omega^2})t} + A_2e^{(-b-\sqrt{b^2-\omega^2})t}$ are negative. This indicates exponential decay of the displacement with respect to time. The body after passing through its maximum displacement, simply comes to equilibrium position and motion is non oscillatory. Such motion is called dead beat or aperiodic or over damped motion. Its main application is in dead beat galvanometers. Example for this case is motion of a pendulum in a highly viscous liquid.

Case 2: Critical damping, $b^2 = \omega^2$:

For $b^2 = \omega^2$ cannot be analyzed as this doesn't satisfy the solution of equation of motion of damped vibrations. So let us assume $\sqrt{b^2 - \omega^2} = h$ to be a very small quantity tending to zero.

Therefore the equation $x = A_1e^{(-b+\sqrt{b^2-\omega^2})t} + A_2e^{(-b-\sqrt{b^2-\omega^2})t}$ becomes

$$x = A_1e^{(-b+h)t} + A_2e^{(-b-h)t}$$

$$x = e^{-bt}[A_1e^{ht} + A_2e^{-ht}]$$

$$x = e^{-bt}[A_1(1 + ht + \ldots\ldots\ldots) + A_2(1 - ht + \ldots\ldots\ldots)]$$

$$x = e^{-bt}[(A_1 + A_2) + ht\,(A_1 - A_2)]$$

$$x = e^{-bt}[\mathrm{p} + qt] \qquad \text{where } p = A_1 + A_2 \text{ and } q = h\,(A_1 - A_2)$$

The above equation is a product of two terms e^{-bt} and $[\mathrm{p} + qt]$, both contribute to variation of x with respect to time. For small values of t, $[\mathrm{p} + qt]$ has values comparable to e^{-bt}. Thus x decreases slowly initially. But as time elapses, the exponential term becomes relatively more predominant and displacement rapidly approaches to zero and the oscillatory motion does not occur. Such damping of motion is called critical damping.

Case 3: Underdamping, $b^2 < \omega^2$:

When $b^2 < \omega^2$, $(b^2- \omega^2)$ is negative.

$$\sqrt{b^2-\omega^2} = \sqrt{-1(\omega^2-b^2)} = i\sqrt{(\omega^2-b^2)}\text{ , where } i = \sqrt{-1}$$

Let $\sqrt{(\omega^2-b^2)} = n$ which is positive and real quantity, then $\sqrt{b^2-\omega^2} = in$

Therefore in equation $x = A_1 e^{(-b+\sqrt{b^2-\omega^2})t} + A_2 e^{(-b-\sqrt{b^2-\omega^2})t}$

$$x = A_1 e^{(-b+in)t} + A_2 e^{(-b-in)t}$$

$$x = e^{-bt}[A_1 e^{int} + A_2 e^{-int}]$$

$$x = e^{-bt}[A_1 (\cos nt + i\sin nt) + A_2(\cos nt - i\sin nt)]$$

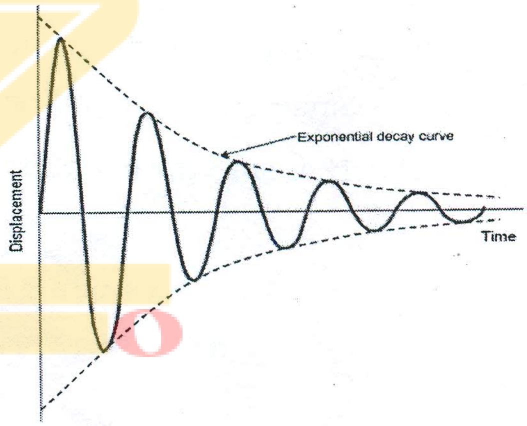
$$x = e^{-bt}[(A_1 + A_2)\cos nt + i\,((A_1 - A_2)\sin nt)]$$

As x is a real quantity, (A_1+A_2) and $i(A_1-A_2)$ must be real quantities. Clearly, A_1 and A_2 are complex quantities.

If $(A_1+A_2) = A_0 \sin\phi$ and $i(A_1-A_2) = A_0 \cos\phi$

then $x = e^{-bt}\,[A_0 \sin\phi \cos nt + A_0 \cos\phi \sin nt]$

$$x = A_0 e^{-bt}\sin(nt+\phi)$$



This equation represents damped harmonic motion. Here $A_0 e^{-bt}$ represents amplitude of vibration of body. As t increases e^{-bt} decreases. The amplitude $A_0 e^{-bt}$ therefore decreases progressively resulting in damping of vibrations. This motion is oscillatory or ballistic whose periodic time is given by

$$T = \frac{2\pi}{\omega} = \frac{2\pi}{\sqrt{\omega^2-b^2}}.$$

Thus damping increases the periodic time.

Quality factor "Q"

It is defined as the ratio of the energy of the oscillator to the energy lost radian.

It is given by

$$Q = \frac{2\pi \times Energy\ of\ the\ oscillator}{Energy\ lost\ per\ period\ of\ oscillation}$$

$$Q = \frac{2\pi E}{PT}$$

where E = energy stored in system, P = power dissipated, T = period of oscillation.

It is also given by

$$Q = \frac{\omega}{2b}; \ \omega \text{ – angular frequency, } 2b \text{ – damping frequency.}$$

Physical Significance:

Q is a measure of the extent to which oscillator is free from damping. High value of Q means the damping of oscillating system is low. For an undamped oscillator $r = 0$, so that Q is infinite. It specifies the degree damping.

Forced Oscillations

It is a steady state of sustained vibrations of a body vibrating in a resistive medium under the action of an external periodic force which acts independently of restoring force.

E.g.: 1. Oscillations of a swing which is pushed periodically by a person

2. The periodic variation of current in an LCR circuit driven by an AC source

3. The motion of diaphragm in a loudspeaker.

Theory of forced vibrations:

Consider a body of mass m executing vibrations in a damping medium acted upon by an external periodic force $F \sin(pt)$, where p is the angular frequency of external force. The oscillations experience different kinds of forces viz,

1. A restoring force proportional to the displacement but oppositely directed given by $F_{restoring} = -kx$, where k is the force constant.
2. A frictional or damping force proportional to velocity but oppositely directed given by $F_{damping} = -r\frac{dx}{dt}$, where r is the damping constant or frictional force/unit velocity.
3. The applied external periodic force $F \sin(pt)$, where F is the maximum value of the force and p is the angular frequency or the driving frequency.

The total force acting on the particle is given by,

$$F_{net} = F_{external} + F_{restoring} + F_{damping}$$

$$= F \sin (pt) - r\frac{dx}{dt} - kx$$

By Newton's second law of motion, $F = m\dfrac{d^2x}{dt^2}$

Hence, $m\dfrac{d^2x}{dt^2} = F \sin (pt) - r\dfrac{dx}{dt} - kx$

$$\frac{d^2x}{dt^2} + \frac{r}{m}\frac{dx}{dt} + \frac{k}{m}x = \frac{F \sin(pt)}{m} \text{ .... (1)}$$

Substitute $\frac{r}{m} = 2b$, $\frac{k}{m} = \omega^2$ and $\frac{F}{m} = f$

$\therefore$ eqn (1) becomes, $\dfrac{d^2x}{dt^2} + 2b\dfrac{dx}{dt} + \omega^2 x = f \sin (pt)$........... (2) where b is the damping coefficient and f is the amplitude of the external driving force.

The solution of the above equation is given by,

$x = A \sin (pt - \theta)$.............(3) where A is the amplitude of the forced vibrations.

Differentiating above equation twice

$\dfrac{dx}{dt} = pA \cos (pt - \theta)$ and $\dfrac{d^2x}{dt^2} = -p^2 A \sin (pt - \theta)$

Substituting x, $\dfrac{dx}{dt}$ and $\dfrac{d^2x}{dt^2}$ in equation (2) we get,

$$-p^2A \sin (pt - \theta) + 2bpA \cos (pt - \theta) + \omega^2 A \sin (pt - \theta) = f \sin [(pt - \theta) + \theta]$$

$$-p^2A \sin (pt - \theta) + 2bpA \cos (pt - \theta) + \omega^2 A \sin (pt - \theta) = f \sin (pt - \theta) \cos \theta + f \cos (pt - \theta) \sin \theta$$

...............(4)

Q. If equation (4) holds good for all values of t then, the coefficient of $\sin (pt - \theta)$ and $\cos (pt - \theta)$ must be equal on both sides.

$\therefore -p^2A + \omega^2 A = f \cos \theta$

$A(\omega^2 - p^2) = f \cos \theta$ ....(5)

and $2bpA = f \sin \theta$..............(6)

By squaring and adding eqn (5) and (6)

$$A^2(\omega^2 - p^2)^2 + 4b^2p^2A^2 = f^2[\cos^2 \theta + \sin^2 \theta]$$

$A^2 (\omega^2 - p^2)^2 + 4b^2p^2A^2 = f^2$

$A = \frac{f}{\sqrt{(\omega^2 - p^2)^2 + 4b^2p^2}}$ ....(7) is the eqn for amplitude of the forced vibrations.

Substituting this in eqn (3) we get,

$$x = \frac{f}{\sqrt{(\omega^2 - p^2)^2 + 4b^2p^2}} \sin (pt - \theta) \text{.....................(8)}$$

Dividing eqn (6) by eqn (5) we get,

$$\tan \theta = \frac{2bap}{a(\omega^2 - p^2)} = \frac{2bp}{(\omega^2 - p^2)}$$

$$\text{or} \quad \theta = \tan^{-1} \frac{2bp}{(\omega^2 - p^2)}$$

The above eqn gives the phase θ of the forced vibrations.

As per eqn (8), the frequency of the vibrating body is p which means that, after the application of external periodic force, the body adopts the frequency of the external force as its own in the steady state.

Dependence of amplitude and phase on the frequency of the applied force:

Case 1: ω >> p

For $\omega >> p$, p^2 will be very small, $(\omega^2 - p^2) \approx \omega^2$ & $2bp \approx 0$.

The amplitude will be $A = \frac{f}{\omega^2} = \frac{F/m}{k/m} = \frac{F}{k}$

Thus in this case amplitude does not depend on mass of the oscillating system or damping, but it depends only on the force constant k.

Phase is given by $\theta = \tan^{-1} (0) = 0$.

Thus the displacement and force will be in same phase.

Case 2: ω = p

For $\omega = p$, $(\omega^2 - p^2) = 0$

$$\therefore A = \frac{f}{2bp} = \frac{F/m}{2\frac{r}{2m}\omega} = \frac{F}{r\omega}$$

When frequency of the applied force is same as natural frequency of the oscillator, resonance occurs. Maximum transfer of energy takes place.

Phase is given by $\theta = \tan^{-1} (\frac{2bp}{0}) = \tan^{-1} (\infty) = \frac{\pi}{2}$

Thus the displacement has a phase lag of $\frac{\pi}{2}$ with respect to phase of applied force.

Case 3: ω << p

For ω << p, $(\omega^2 - p^2)^2 = (p^2)^2$

$\therefore A = \frac{f}{\sqrt{p^4+4b^2p^2}}$

∴ As p keeps increasing, A becomes smaller.

Since b is very small in this case, $4b^2\, p^2 << p^4$

$\therefore A = \frac{F/m}{\sqrt{p^4}} = \frac{F/m}{p2}$

Phase is given by $\theta = \tan^{-1}\left(\frac{-2b}{p}\right) = \tan^{-1}(-0) = \pi$.

Thus as p becomes large, the displacement develops a phase lag that approaches π with respect to phase of applied force.

Resonance:

The phenomenon of making a body vibrate with its natural frequency under the influence of another vibrating body with the same frequency is called **resonance**. The frequency at which resonance occurs is known as **resonant frequency**.

When the frequency of a periodic force acting on a vibrating body is equal to the natural frequency of vibrations of the body, the energy transfer from the periodic force to the body becomes maximum and the body vibrates with maximum amplitude.

E.g.: 1. Helmholtz resonator

2. A radio receiver set tuned to the broadcast frequency of a transmitting station

3. Vibrations caused by an excited tuning fork in another nearby identical tuning fork.

Condition for resonance:

Consider a body of mass m vibrating under the influence of an external force $F \sin pt$. If ω is the natural frequency of vibration for the body, then

$$A = \frac{f}{\sqrt{(\omega^2 - p^2)^2 + 4b^2p^2}}$$

For A to become maximum, the denominator in the above equation must be minimum. This can be achieved by

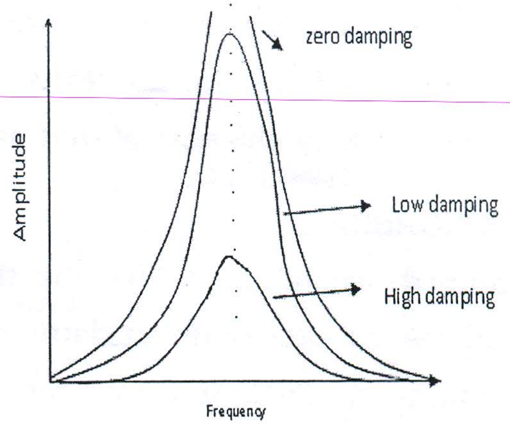
1. making b minimum i.e. when the damping caused by the medium is made minimum.
2. tuning the frequency p of the applied force to become equal to the natural frequency of vibration ω of the body, i.e., by making $p = \omega$

$\therefore$ The equation for maximum amplitude becomes $A = \frac{f}{\sqrt{4b^2\omega^2}} = \frac{F/m}{2b\omega}$.

Sharpness of resonance:

It is the rate of fall in amplitude with the change of forcing frequency on each side of the resonant frequency.

At resonance, the amplitude of the oscillating system becomes maximum. It decreases from this maximum value with the change of frequency of the applied force. For different values of damping, the graph is drawn as shown in the graph. When damping is low, amplitude falls rapidly on both sides of resonance frequency and we say that resonance is sharp. For high damping this resonance curve becomes flat. For zero damping, the amplitude becomes infinity at resonance, which case does not exist in reality.



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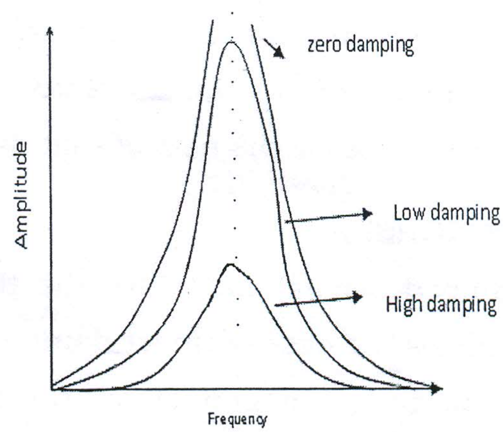
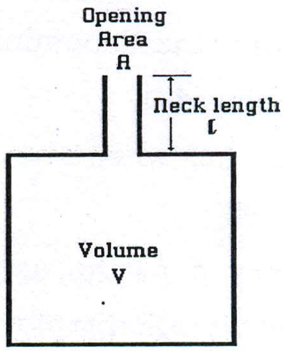
Sharpness of resonance:

It is the rate of fall in amplitude with the change of forcing frequency on each side of the resonant frequency.

At resonance, the amplitude of the oscillating system becomes maximum. It decreases from this maximum value with the change of frequency of the applied force. For different values of damping, the graph is drawn as shown in the graph. When damping is low, amplitude falls rapidly on both sides of resonance frequency and we say that resonance is sharp. For high damping this resonance curve becomes flat. For zero damping, the amplitude becomes infinity at resonance, which case does not exist in reality.



Helmholtz resonator



It is used to analyze a complex note or the quality of musical notes. It consists of a hollow sphere of thin glass or brass with an opening through a narrow neck. It is filled with air. The opening receives exciting sound waves and the ears are kept close to the neck. When air is pushed into the sphere and released, the pressure will drive it out. The volume of air in the container behaves as a mass on a spring which is pulled down and released. Compressed air tends to move out and creates low pressure inside. The air will oscillate into and out of the container at its natural frequency given by the expression

$$f = \frac{v}{2\pi}\sqrt{\frac{A}{l\,V}}$$

where v is the velocity of sound, l the length of the neck, A the area of the opening, V the volume of the resonator.

When one of component frequencies of the musical note carried by the incoming air is same as f, the air inside the sphere is set for vibrations with large amplitude. A small part of the air leaks out and is heard. Thus, the resonator is said to resonate for the note of that particular frequency.

IMPORTANT QUESTIONS:

1. What are damped oscillations? Give the theory of damped oscillations and hence discuss the case of under damping.
2. Define simple harmonic motion. Derive the equation for simple harmonic motion using Hooke's law
3. Discuss the theory of forced vibrations and hence obtain the expression for amplitude

4. Starting form Hookes' law derive the differential equation for SHM. Explain the Characteristics of SHM
5. What are damped oscillations? Derive the expression for decaying amplitude and hence discuss the case of critical damping.
6. Write a note on Helmholtz resonator.
7. What are forced oscillations? Derive the expression for steady state amplitude and phase angle in case of forced oscillations.
8. Define SHM. Mention the characteristics of SHM. Give one example of SHM.

Module – I

Shock waves

Sound wave is a disturbance that propagates in a given medium with time. The speed of the sound depends upon the bulk modulus of that medium. Sound waves cause gradual, small and continuous changes in pressure, density and temperature.

Acoustic waves

An acoustic wave is simply a sound wave. These are longitudinal waves that travel in a medium with the speed of sound (333 m/s in air at STP). These waves can propagate in solids, liquids and gases. These are classified into 3 types depending upon frequency.

1. **Infrasonic:** These acoustic waves have frequency less than 20 Hz. The human ears cannot detect these waves.
2. **Audible:** These acoustic waves have frequency between 20Hz and 20kHz. The human ear is sensitive to these waves.
3. **Ultrasonic:** These acoustic waves have frequency more than 20 kHz. The human ear is not sensitive to these waves. But they travel with the same speed as that of sound.

When an object moves through a medium with a speed less than the speed of sound in that medium, it is said to move with **subsonic** speed.

When an object moves through a medium with a speed greater than the speed of sound in that medium, it is said to move with **supersonic** speed.

Mach number:

It is defined as ratio of the speed of the object to the speed of the sound in a given medium. i.e.,

Mach number = object speed / speed of the sound in a given medium

It is denoted as M. If v is the speed of object and a is the speed of sound in the medium, then

$$M = v/a$$

where

$$a = \sqrt{\gamma RT}$$

γ – ratio of specific heats; R –gas constant; T – local temperature in Kelvin

Distinction between Acoustic, ultrasonic (based on frequency), Subsonic, Supersonic Waves

1) **Acoustic wave**

   Acoustic wave is simply a sound wave which moves with the speed of 330m/s, in air at STP. They have frequencies between 20 -20,000 Hz. Amplitude of acoustic wave is very small.

2) **Infrasonic waves**

   These are pressure waves having frequencies less 20 Hz. But they travel with the same speed as that of sound.

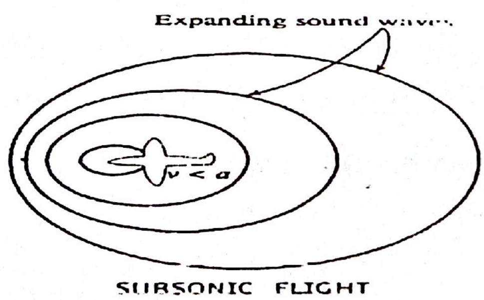
3) **Ultrasonic waves**

   These are pressure waves having frequencies beyond 20,000 Hz. But they travel with the same speed as that of sound. Amplitude of ultrasonic wave is very small.

Classification of waves based on Mach number:

1) **Subsonic waves**

   If the speed of mechanical wave or body moving in a fluid is lesser than that of sound then such a speed is referred to as subsonic and the wave is subsonic wave. All the subsonic waves have Mach number less than 1. For a body moving with subsonic speed sound emitted by it manages move ahead and away from the body since it is faster than the body.



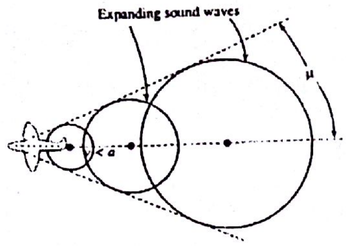
   Eg : speed of car and train.

2) **Supersonic waves**

   Supersonic waves are mechanical waves which travel with speeds greater than that of sound i.e., with speeds for which Mach no. >1.

A body with supersonic speed moves ahead leaving behind series of expanding sound waves. Amplitude of supersonic wave will be very high and it effects medium in which it is travelling.

Eg: fighter planes.



Mach angle:

A number of common tangents drawn to the expanding sound waves emitted from a body at supersonic speed forms a cone called the Mach cone .The angle made by the tangent with axes of the cone is called **Mach angle μ**.

μ is related to Mach number **M** through the equation $\mu = \sin^{-1}\left(\frac{1}{M}\right)$

Hypersonic waves: In supersonic waves we have a special class of waves which travel with speeds for which **Mach no $\geq$ 5** called as **hypersonic waves.**

Transonic wave: These are waves which overlap on the subsonic and supersonic ranges. In this domain there is a change of phase from subsonic to supersonic. It becomes very difficult to categorize certain parameters at speeds near or at $M = 1$, the speeds for which $0.8 < M < 1.2$ is called transonic speeds. This region is called grey area where there is a overlapping of some of the characteristics of both the subsonic and the supersonic speeds.

Shock waves:

A shock wave is also a pressure wave, with a very steep rise in pressure. Any fluid that propagates at supersonic speeds gives rise to a shock wave.

E.g.: Earth quakes (seismic waves), lightning strikes.

Though a shock wave cannot be heard we hear a booming sound as the velocity of the body increases from subsonic to supersonic. It is called "**sonic boom**".

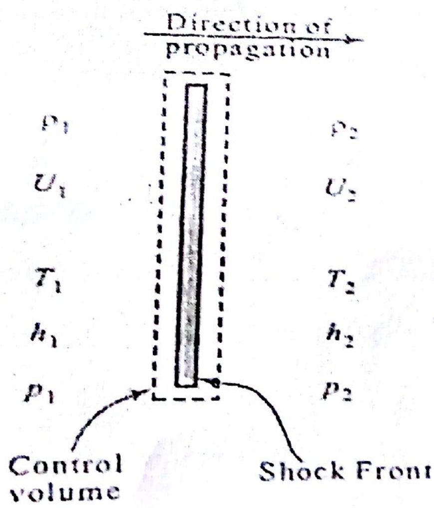
Shock waves can be produced by a sudden dissipation of mechanical energy in a medium enclosed in a small space. They are characterized by sudden increase in pressure, temperature and density of the gas through which it propagates. Shock waves are identified as strong or weak depending on the magnitude of the instantaneous changes in pressure and temperature in the medium that is held pressed in the space bound within the thickness of the shock front.

A Shock wave is a surface that manifests as a discontinuity in a fluid medium in which it is propagating with supersonic speed.

Properties of Shock Waves:

1. They always travel in the medium with Mach number exceeding 1
2. They obey the laws of fluid dynamics
3. The effects caused by shock waves results in increase in entropy
4. General wave properties are not applicable to shock waves
5. When they turn around a convex corner, they break-up into very large number of expanding supersonic waves diverging from a central spot. This process is called supersonic expansion fan.
6. Shock waves when produced in a medium lead to an enormous increase in pressure, density and temperature of the medium.
7. Across the shock wave, supersonic flow is decelerated into sub sonic flow. This process occurs adiabatically but with a change in the internal energy.

Control Volume:



CONTROL VOLUME

Control volume is an imaginary thin envelope that surrounds the shock wave within which there is a sharp increase in the pressure, density and temperature of the medium.

The characteristics of shock waves are analyzed with the help of this control volume. The boundary of this volume is the physical boundary of the region through which the fluid flows. It is a one dimensional confinement in the medium with two surfaces. One surface is on the re shock side and the other on the post shock side with a very small separation between them. However, the entire shock is bound between these two surfaces. On the pre shock side the density, flow velocity, temperature, enthalpy and pressure respectively are $\rho_1, u_1, T_1, h_1 and\ P_1$ and on the post shock side their values are $\rho_2, u_2, T_2, h_2\ and\ P_2$ respectively.

Basics of conservation of mass, momentum and energy

Conservation means the maintenance of certain quantities unchanged during a physical process. Conservation laws apply to closed system.

Law of conservation of mass

The total mass of any isolated system remains unchanged and is independent of any chemical and physical changes that could occur within the system

$$\rho_1 u_1 = \rho_2 u_2$$

where u_1, ρ_1 is the velocity and density of the fluid ahead of the shock and u_2, ρ_2 is the velocity and density of fluid following the shock.

Law of conservation of momentum

In a closed system, the total momentum remains a constant. It can also be stated as "when two objects collide in an isolated system the total momentum of the object before collision = the total momentum of the two objects after the collision.

$$P_1 + \rho_1 u_1^2 = P_2 + \rho_2 u_2^2$$

where P_1 and P_2 are the pressures ahead and following of the shock waves

Law if conservation of energy

The total energy of closed system remains constant and is independent of any changes occurring within the system.

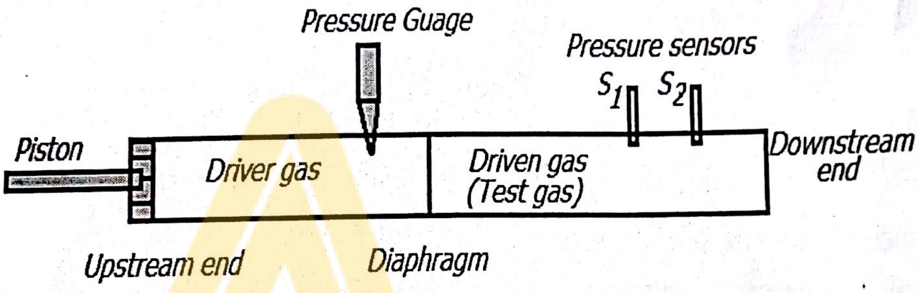
$$h_1 + \frac{u_1^2}{2} = h_2 + \frac{u_2^2}{2}$$

where h_1 and h_2 are the enthalpies ahead and following of the shock waves.

Reddy tube:

Reddy tube is hand operated shock tube capable of producing shock waves by using human energy. It is a long cylindrical tube with two sections separated by a diaphragm. It's one end is fitted with the piston and the other end is closed or open to the surroundings.

Description/Construction:



- Reddy tube consists of a long cylindrical stainless steel tube of about 30mm diameter.
- It is divided into 2 sections each of length about 50 cm. One is the **driver section** other is **driven section** separated by a 0.1 mm thick aluminum or paper diaphragm.
- It has a piston fitted at the far end of the driver section. Whereas the end of the driven section is closed.
- A digital pressure gauge is mounted in the driver section next to the diaphragm.
- Two piezoelectric sensors S_1 and S_2 are mounted on the driven section.
- The driver section is filled with gas termed as driver gas (high pressure) and the gas in the driven section is termed **driven gas.**

Working:

- The driver gas is compressed by pushing the piston hard into the driver tube until diaphragm ruptures. Following the rupture, the driver gas rushes into the driven section generating a moving shock wave that traverses the length of the driven section. The shock wave instantaneously raises the temperature and pressure of the driven (test) gas as the shock moves over it.

- The propagating primary shock wave is reflected from the downstream end. After the reflection, the test gas undergoes further compression which boosts its temperature and pressure further
- The state of high values of pressure and temperature is sustained at the downstream end until an expansion wave reflected from the upstream end of the driver tube arrives there and neutralizes the compression partially.
- The pressure rise caused by the primary shock waves and also the reflected shock wave are sensed as signals by the sensors S_1 and S_2 respectively and they are recorded in a digital cathode ray oscilloscope (CRO).
- Two signals of pressure are seen on the CRO display. The first one appears when the primary shock passes the first sensor (S_1) and the second one when it passes the second sensor (S_2).
- From the CRO display the time **'t'** taken to travel between the two pressure sensors can be measured.
- The distance **'x'** between the two pressure sensors can be measured with a graduated scale
- Using the data so obtained, Mach number, pressure and temperatures can be calculated as follows

Evaluation of Mach No. M:

Shock speed of primary shock wave; $U_s = \frac{x}{t}$

Mach number; $M = \frac{U_s}{a}$

Here 'a' is the speed of sound.

Characteristics of a Reddy tube.

- The Reddy tube operates on the principle of free piston driven shock tube (FPST)
- It is a hand operated shock producing device.
- It is capable of producing Mach no exceeding 1.5.
- The rupture pressure is a function of the thickness of the diaphragm.
- Temperatures exceeding 900K can be easily obtained by the Reddy tube by using helium as the driver gas and argon as the driven gas.

Applications of shock waves

- Cell information - by passing shock wave of appropriate strength DNA can be pushed inside a cell
- Gas dynamics studies
- Aerodynamics – hypersonic shock tunnels, scramjet engines.
- High temperature chemical kinetics
- Restoring of depleted bore wells
- Material studies – effect of sudden impact of pressure - blast protection materials
- Investigation of traumatic brain injuries
- Needle-less drug delivery
- Treatment of Kidney stones
- Industrial applications (E.g.- pencil)
- Preservation of wood & bamboo by injecting chemical preservatives using shock waves

Important Questions:

1. Describe the construction and working of a Reddy Shock Tube.
2. Discuss the laws of conservation of mass, energy and momentum in relation to shock waves.
3. What is a shock wave? Mention its properties.
4. Mention the applications of shock waves.
5. What is Mach number? Distinguish between subsonic and supersonic waves.