13. Residues and Its Applications

- isolated singular points
- residues
- Cauchy's residue theorem
- applications of residues

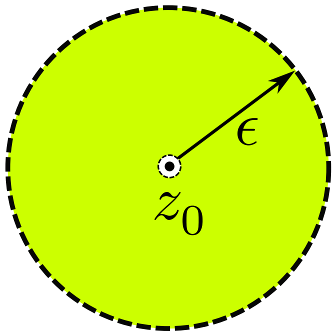
Isolated singular points

z_0 is called a **singular point** of f if

- f fails to be analytic at z_0
- but f is analytic at *some* point in *every* neighborhood of z_0

a singular point z_0 is said to be **isolated** if f is analytic in *some* punctured disk

$$0 < |z - z_0| < \epsilon$$



centered at z_0 (also called a *deleted neighborhood* of z_0)

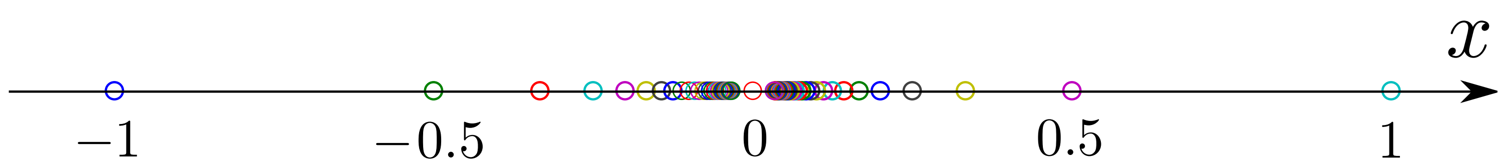
example: $f(z) = 1/(z^2(z^2 + 1))$ has the three isolated singular points at

$$z = 0, \quad z = \pm j$$

Non-isolated singular points

example: the function $\dfrac{1}{\sin(\pi/z)}$ has the singular points

$$z = 0, \quad z = \frac{1}{n}, \quad (n = \pm 1, \pm 2, \ldots)$$



- each singular point except $z = 0$ is isolated
- 0 is nonisolated since *every* punctured disk of 0 contains other singularities
- for any $\varepsilon > 0$, we can find a positive integer n such that $n > 1/\varepsilon$
- this means $z = 1/n$ always lies in the punctured disk $0 < |z| < \varepsilon$

Residues

assumption: z_0 is an isolated singular point of f, *e.g.*,

there exists a punctured disk $0 < |z - z_0| < r_0$ throughout which f is analytic

consequently, f has a Laurent series representation

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n + \frac{b_1}{z - z_0} + \cdots + \frac{b_n}{(z - z_0)^n} + \cdots, \quad (0 < |z - z_0| < r_0)$$

let C be any positively oriented simple closed contour lying in the disk

$$0 < |z - z_0| < r_0$$

the coefficient b_n of the Laurent series is given by

$$b_n = \frac{1}{j2\pi} \int_C \frac{f(z)}{(z - z_0)^{-n+1}}\, dz, \qquad (n = 1, 2, \ldots)$$

the coefficient of $1/(z - z_0)$ in the Laurent expansion is obtained by

$$\int_C f(z)dz = j2\pi b_1$$

b_1 is called the **residue** of f at the **isolated singular point** z_0, denoted by

$$b_1 = \operatorname*{Res}_{z=z_0} f(z)$$

this allows us to write

$$\int_C f(z)dz = j2\pi \operatorname*{Res}_{z=z_0} f(z)$$

which provides a powerful method for evaluating integrals around a contour

example: find $\int_C e^{1/z^2} dz$ when C is the positive oriented circle $|z| = 1$

$1/z^2$ is analytic everywhere except $z = 0$; 0 is an isolated singular point

the Laurent series expansion of f is

$$f(z) = e^{1/z^2} = 1 + \frac{1}{z^2} + \frac{1}{2!z^4} + \frac{1}{3!z^6} + \cdots \quad (0 < |z| < \infty)$$

the residue of f at $z = 0$ is zero ($b_1 = 0$), so the integral is zero

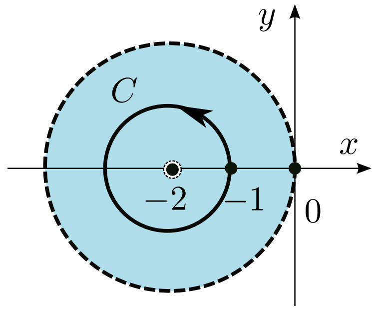
remark: the analyticity of f within and on C is a *sufficient condition* for $\int_C f(z)dz$ to be zero; however, it is not a *necessary condition*

example: compute $\int_C \frac{1}{z(z+2)^3}\,dz$ where C is circle $|z+2|=1$

f has the isolated singular points at 0 and -2

choose an annulus domain: $0 < |z+2| < 2$

on which f is analytic and contains C



f has a Laurent series on this domain and is given by

$$
\begin{aligned}
f(z) &= \frac{1}{(z+2-2)(z+2)^3} = -\frac{1}{2}\cdot\frac{1}{1-(z+2)/2}\cdot\frac{1}{(z+2)^3} \\
&= -\frac{1}{2(z+2)^3}\sum_{n=0}^{\infty}\frac{(z+2)^n}{2^n} \;=\; -\sum_{n=0}^{\infty}\frac{(z+2)^{n-3}}{2^{n+1}}, \qquad (0 < |z+2| < 2)
\end{aligned}
$$

the residue of f at $z=-2$ is $-1/2^3$ which is obtained when $n=2$

therefore, the integral is $j2\pi(-1/2^3) = -j\pi/4$ (check with the Cauchy formula)

Cauchy's residue theorem

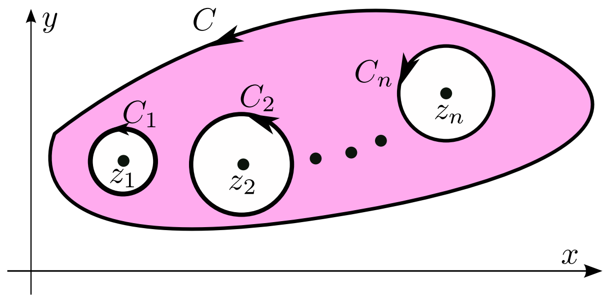
let C be a positively oriented simple closed contour

Theorem: if f is analytic inside and on C except for a finite number of singular points $z_1, z_2, \ldots, z_n$ inside C, then

$$\int_C f(z)dz = j2\pi \sum_{k=1}^{n} \operatorname*{Res}_{z=z_k} f(z)$$

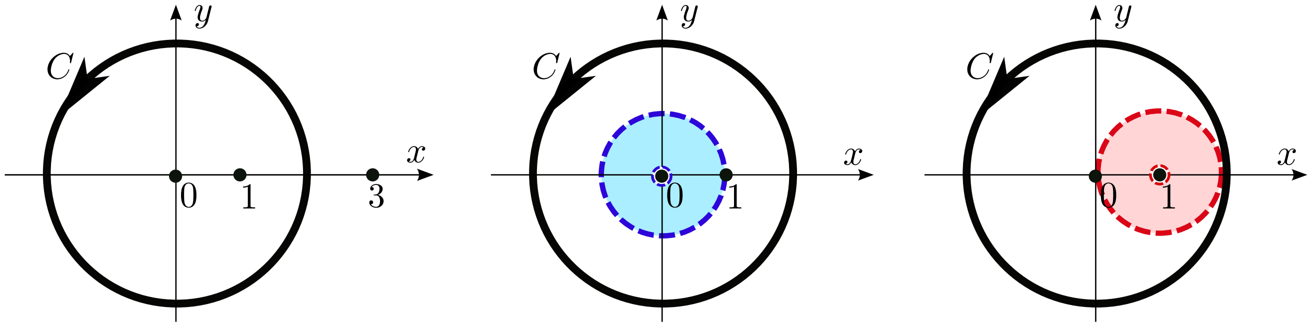
Proof.

- since z_k's are isolated points, we can find small circles C_k's that are mutually disjoint
- f is analytic on a multiply connected domain
- from the Cauchy-Goursat theorem:
  $\int_C f(z)dz = \sum_{k=1}^{n} \int_{C_k} f(z)dz$



example: use the Cauchy residue theorem to evaluate the integral

$$\int_C \frac{3(z+1)}{z(z-1)(z-3)}dz, \quad C \text{ is the circle } |z| = 2, \text{ in counterclockwise}$$



C encloses the two singular points of the integrand, so

$$I = \int_C f(z)dz = \int_C \frac{3(z+1)}{z(z-1)(z-3)}dz = j2\pi \left[\operatorname*{Res}_{z=0} f(z) + \operatorname*{Res}_{z=1} f(z) \right]$$

- calculate $\operatorname{Res}_{z=0} f(z)$ via the Laurent series of f in $0 < |z| < 1$
- calculate $\operatorname{Res}_{z=1} f(z)$ via the Laurent series of f in $0 < |z-1| < 1$

rewrite $f(z) = \frac{1}{z} - \frac{3}{z-1} + \frac{2}{z-3}$

- the Laurent series of f in $0 < |z| < 1$

$$f(z) = \frac{1}{z} + \frac{3}{1-z} - \frac{2}{3(1-z/3)} = \frac{1}{z} + 3(1+z+z^2+\ldots) - \frac{2}{3}(1+(z/3)+(z/3)^2+\ldots)$$

the residue of f at 0 is the coefficient of $1/z$, so $\mathrm{Res}_{z=0}\, f(z) = 1$

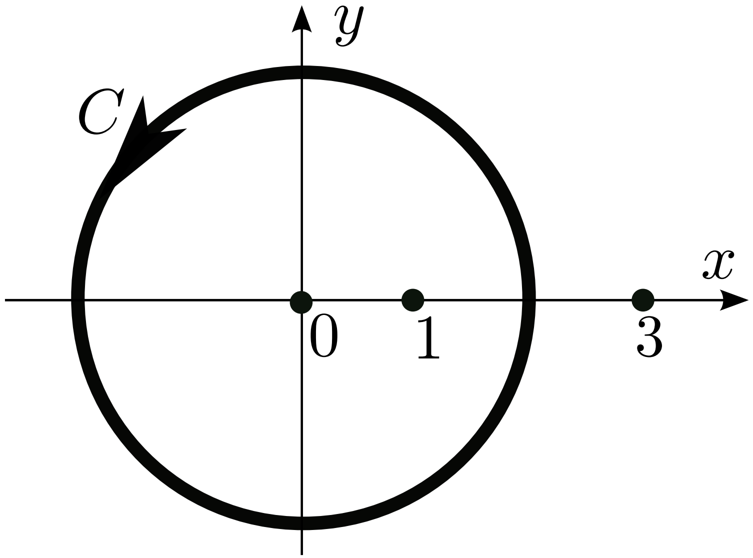
- the Laurent series of f in $0 < |z-1| < 1$

$$\begin{aligned} f(z) &= \frac{1}{1+z-1} - \frac{3}{z-1} - \frac{1}{1-(z-1)/2} \\ &= 1 - (z-1) + (z-1)^2 + \ldots - \frac{3}{z-1} - \left(1 + \frac{z-1}{2} + \left(\frac{z-1}{2}\right)^2 + \ldots\right) \end{aligned}$$

the residue of f at 1 is the coefficient of $1/(z-1)$, so $\mathrm{Res}_{z=0}\, f(z) = -3$

therefore, $I = j2\pi(1-3) = -j4\pi$

alternatively, we can compute the integral from the Cauchy integral formula



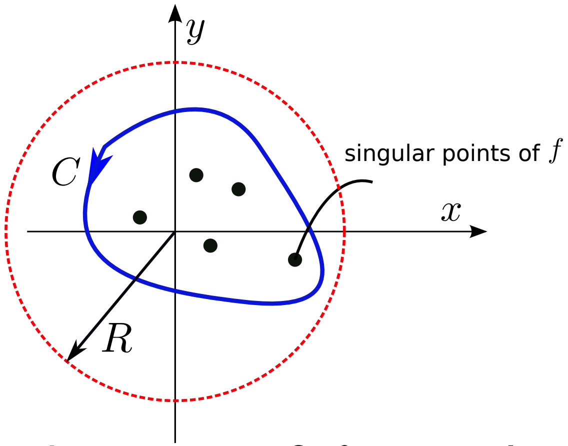
$$I = \int_C \left(\frac{1}{z} - \frac{3}{z-1} + \frac{2}{z-3} \right) dz$$

$$= j2\pi(1 - 3 + 0) = -j4\pi$$

Residue at infinity

f is said to have an **isolated point at** $z_0 = \infty$ if

$$\text{there exists } R > 0 \text{ such that } f \text{ is analytic for } R < |z| < \infty$$



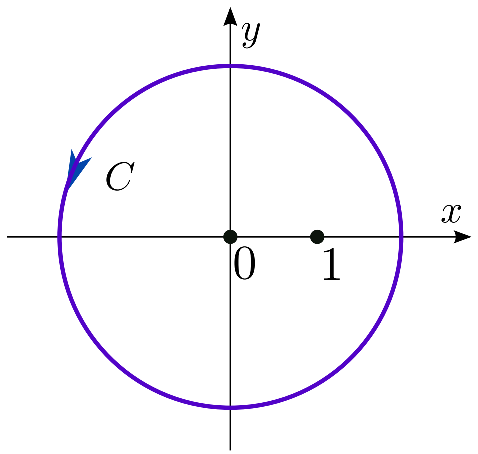
C is a positive oriented simple closed contour

Theorem: if f is analytic everywhere *except* for a finite number of singular points interior to C, then

$$\int_C f(z)dz = j2\pi \operatorname*{Res}_{z=0} \left[\frac{1}{z^2} f\left(\frac{1}{z}\right)\right]$$

(see a proof on section 71, Churchill)

example: find $I = \int_C \frac{z-3}{z(z-1)} dz$, C is the circle $|z| = 2$ (counterclockwise)



$$I = j2\pi \operatorname*{Res}_{z=0} \left[(1/z^2) f(1/z)\right]$$
$$= j2\pi \operatorname*{Res}_{z=0} \left[\frac{1-3z}{z(1-z)}\right] \triangleq j2\pi \operatorname*{Res}_{z=0} g(z)$$

find the residue via the Laurent series of g in $0 < |z| < 1$

write $g(z) = \left(\frac{1}{z} - 3\right)(1 + z + z^2 + \cdots) \quad \Longrightarrow \quad \operatorname*{Res}_{z=0} g(z) = 1$

compare the integral with other methods ✎

- Cauchy integral formula (write the partial fraction of f)
- Cauchy residue theorem (have to find two residues; hence two Laurent series)

Principal part

f has an isolated singular point at z_0, so f has a Laurent seires

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n + \frac{b_1}{(z - z_0)} + \frac{b_2}{(z - z_0)^2} + \cdots + \frac{b_n}{(z - z_0)^n} + \cdots$$

in a punctured disk $0 < |z - z_0| < R$

the portion of the series that involves **negative powers** of $z - z_0$

$$\frac{b_1}{(z - z_0)} + \frac{b_2}{(z - z_0)^2} + \cdots + \frac{b_n}{(z - z_0)^n} + \cdots$$

is called the **principal part of** f

Types of isolated singular points

three possible types of the principal part of f

- no principal part

$$f(z) = \cos z = 1 - \frac{z^2}{2!} + \frac{z^4}{4!} + \cdots, \quad (0 < |z| < \infty)$$

- finite number of terms in the principal part

$$f(z) = \frac{1}{z^2(1+z)} = \frac{1}{z^2} - \frac{1}{z} + 1 - z + z^2 + \cdots, \quad (0 < |z| < 1)$$

- infinite number of terms in the principal part

$$f(z) = e^{1/z} = 1 + \frac{1}{z} + \frac{1}{2!z^2} + \frac{1}{3!z^3} + \cdots, \quad (0 < |z| < \infty)$$

classify the number of terms in the principal part in a general form

- none: z_0 is called a **removable singular point**

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n$$

- finite (m terms): z_0 is called a **pole of order** m

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n + \frac{b_1}{z - z_0} + \frac{b_2}{(z - z_0)^2} + \cdots + \frac{b_m}{(z - z_0)^m}$$

- infinite: z_0 is said to be an **essential singular point of** f

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n + \frac{b_1}{z - z_0} + \frac{b_2}{(z - z_0)^2} + \cdots + \frac{b_n}{(z - z_0)^n} + \cdots$$

examples:

$$
\begin{aligned}
f_1(z) &= \cos z = 1 - \frac{z^2}{2!} + \frac{z^4}{4!} + \cdots \\
f_2(z) &= \frac{3}{(z-1)(z-2)} = -\left(\frac{1}{z-2} + 1 + (z-2) + (z-2)^3 + \cdots\right) \\
f_3(z) &= \frac{1}{z^2(1+z)} = \frac{1}{z^2} - \frac{1}{z} + 1 - z + z^2 + \cdots \\
f_4(z) &= e^{1/z} = 1 + \frac{1}{z} + \frac{1}{2!z^2} + \frac{1}{3!z^3} + \cdots
\end{aligned}
$$

- 0 is a removeable singular point of f_1
- 2 is a pole of order 1 (or **simple pole**) of f_2
- 0 is a pole of order 2 (or **double pole**) of f_3
- 0 is an essential singular point of f_4

note: for f_2, f_3 we can determine the pole/order from the denominator of f

Residue formula

if f has a pole of order m at z_0 then

$$\operatorname*{Res}_{z=z_0} f(z) = \frac{1}{(m-1)!} \lim_{z \to z_0} \frac{d^{m-1}}{dz^{m-1}} (z-z_0)^m f(z)$$

Proof. if f has a pole of order m, its Laurent series can be expressed as

$$f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n + \frac{b_1}{(z-z_0)} + \frac{b_2}{(z-z_0)^2} + \cdots + \frac{b_m}{(z-z_0)^m}$$

$$(z-z_0)^m f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^{m+n} + b_1 (z-z_0)^{m-1} + b_2 (z-z_0)^{m-2} + \cdots + b_m$$

to obtain b_1, we take the $(m-1)$th derivative and take the limit $z \to z_0$

example 1: find $\mathrm{Res}_{z=0} f(z)$ and $\mathrm{Res}_{z=2} f(z)$ where $f(z) = \dfrac{(z+1)}{z^2(z-2)}$

$$\begin{aligned}
\operatorname*{Res}_{z=0} f(z) &= \lim_{z\to 0} \frac{d}{dz}\left(\frac{z+1}{z-2}\right) = -3/4 \qquad \text{(0 is a double pole of } f\text{)} \\
\operatorname*{Res}_{z=2} f(z) &= \lim_{z\to 2} \frac{z+1}{z^2} = 3/4
\end{aligned}$$

example 2: find $\mathrm{Res}_{z=0}\, g(z)$ where $g(z) = \dfrac{z+1}{1-2z}$

g is analytic at 0 (0 is a removable singular point of g), so $\mathrm{Res}_{z=0}\, g(z) = 0$

check ✎ apply the results from the above two examples to compute

$$\int_C \frac{(z+1)}{z^2(z-2)} dz, \quad C \text{ is the circle } |z| = 3 \text{ (counterclockwise)}$$

by using the Cauchy residue theorem and the formula on page 13-12 sometimes the pole order cannot be readily determined

example 3: find $\operatorname{Res}_{z=0} f(z)$ where $f(z) = \dfrac{\sinh z}{z^4}$

use the Maclaurin series of $\sinh z$

$$f(z) = \frac{1}{z^4} \cdot \left(z + \frac{z^3}{3!} + \frac{z^5}{5!} + \cdots \right) = \left(\frac{1}{z^3} + \frac{1}{3! z} + \frac{z}{5!} + \cdots \right)$$

0 is the **third**-order pole with residue $1/3!$

here we determine the residue at $z = 0$ from its definition (the coeff. of $1/z$)

no need to use the residue formula on page 13-18

when the pole order (m) is unknown, we can

- assume $m = 1, 2, 3, \ldots$
- find the corresponding residues until we find the first finite value

example 4: find $\mathrm{Res}_{z=0} f(z)$ where $f(z) = \dfrac{1+z}{1-\cos z}$

- assume $m = 1$

$$\operatorname*{Res}_{z=0} f(z) = \lim_{z\to 0} \frac{z(1+z)}{1-\cos z} = 0/0 = \lim_{z\to 0} \frac{1+2z}{\sin z} = 1/0 = \infty \implies \text{(not 1st order)}$$

- assume $m = 2$

$$\operatorname*{Res}_{z=0} f(z) = \lim_{z\to 0} \frac{d}{dz}\left(\frac{z^2(1+z)}{1-\cos z}\right) = 2 \;\; \text{(finite)} \quad \implies \quad 0 \text{ is a double pole}$$

note: use L'Hôpital's rule to compute the limit

Summary

many ways to compute a contour integral ($\int_C f(z)dz$)

- parametrize the path (feasible when C is easily described)
- use the principle of deformation of paths (if f is analytic in the region between the two contours)
- use the Cauchy integral formula (typically requires the partial fraction of f)
- use the Cauchy's residue theorem on page 13-8 (requires the residues at singular points enclosed by C)
- use the theorem of the residue at infinity on page 13-12 (find one residue at 0)

to find the residue of f at z_0

- read from the coeff of $1/(z - z_0)$ in the Laurent series of f
- apply the residue formula on page 13-18

Application of the residue theorem

- calculating real definite integrals
  - integrals involving sines and cosines
  - improper integrals
  - improper integrals from Fourier series
- inversion of Laplace transforms

Definite integrals involving sines and cosines

we consider a problem of evaluating definite integrals of the form

$$\int_0^{2\pi} F(\sin\theta, \cos\theta)d\theta$$

since θ varies from 0 to 2π, we can let θ be an argument of a point z

$$z = e^{j\theta} \quad (0 \leq \theta \leq 2\pi)$$

this describe a positively oriented circle C centered at the origin

make the substitutions

$$\sin\theta = \frac{z - z^{-1}}{j2}, \quad \cos\theta = \frac{z + z^{-1}}{2}, \quad d\theta = \frac{dz}{jz}$$

this will transform the integral into the *contour* integral

$$\int_C F\left(\frac{z - z^{-1}}{j2}, \frac{z + z^{-1}}{2}\right)\frac{dz}{jz}$$

- the integrand becomes a function of z
- if the integrand reduces to a rational function of z, we can apply the Cauchy's residue theorem

example:

$$\begin{aligned}\int_0^{2\pi}\frac{d\theta}{5+4\sin\theta} &= \int_C \frac{1}{5+4\frac{(z-z^{-1})}{2j}}\frac{dz}{jz} = \int_C \frac{dz}{2z^2 + j5z - 2} \triangleq \int_C g(z)dz \\ &= \int_C \frac{dz}{2(z+2j)(z+j/2)} = j2\pi\left(\operatorname*{Res}_{z=-j/2} g(z)\right) = 2\pi/3\end{aligned}$$

where C is the positively oriented circle $|z| = 1$

the above idea can be summarized in the following theorem

Theorem: if $F(\cos\theta, \sin\theta)$ is a rational function of $\cos\theta$ and $\sin\theta$ which is finite on the closed interval $0 \leq \theta \leq 2\pi$, and if f is the function obtained from $F(\cdot, \cdot)$ by the substitutions

$$\cos\theta = \frac{z + z^{-1}}{2}, \quad \sin\theta = \frac{z - z^{-1}}{j2}$$

then

$$\int_C^{2\pi} F(\cos\theta, \sin\theta)\, d\theta = j2\pi \left(\sum_k \operatorname*{Res}_{z=z_k} \frac{f(z)}{jz} \right)$$

where the summation takes over all z_k's that lie within the circle $|z| = 1$

example: compute $I = \displaystyle\int_0^{2\pi} \frac{\cos 2\theta}{1 - 2a\cos\theta + a^2}\, d\theta, \quad -1 < a < 1$

make change of variables

- $\cos 2\theta = \dfrac{e^{j2\theta} + e^{-j2\theta}}{2} = \dfrac{z^2 + z^{-2}}{2} = \dfrac{z^4 + 1}{2z^2}$
- $1 - 2a\cos\theta + a^2 = 1 - 2a(z + z^{-1})/2 + a^2 = -\dfrac{az^2 - (a^2+1)z + a}{z}$

we have $\int_0^{2\pi} F(\theta) d\theta = \int_C \frac{f(z)}{jz} dz \triangleq \int_C g(z) dz$ where

$$g(z) = -\frac{(z^4+1)z}{jz \cdot 2z^2(az^2 - (a^2+1)z + a)} = \frac{(z^4+1)}{j2z^2(1 - az)(z - a)}$$

we see that only the poles $z = 0$ and $z = a$ lie inside the unit circle C

therefore, the integral becomes

$$I = \int_C g(z)dz = j2\pi \left(\operatorname*{Res}_{z=0} g(z) + \operatorname*{Res}_{z=a} g(z) \right)$$

- note that $z = 0$ is a double pole of $g(z)$, so

$$\operatorname*{Res}_{z=0} g(z) = \lim_{z=0} \frac{d}{dz}(z^2 g(z)) = -\frac{1}{j2} \cdot \frac{a^2+1}{a^2}$$

- $\operatorname*{Res}_{z=a} g(z) = \lim_{z=a} (z-a)g(z) = \dfrac{1}{j2} \cdot \dfrac{a^4+1}{a^2(1-a^2)}$

hence, $I = \dfrac{2\pi a^2}{1-a^2}$

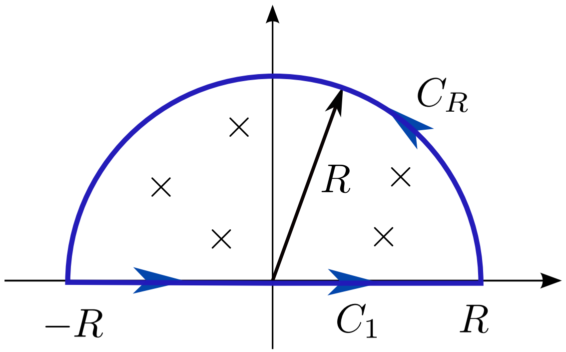
Improper integrals

let's first consider a well-known improper integral

$$I = \int_{-\infty}^{\infty} \frac{dx}{1+x^2} = \pi$$

of course, this can be evaluated using the inverse tangent function

we will derive this kind of integral by means of **contour integration**



some poles of the integrand lie in the upper half plane

let C_R be a semicircular contour with radius $R \to \infty$

$$\int_{-R}^{R} f(x)dx + \int_{C_R} f(z)dz = j2\pi \sum_k \operatorname*{Res}_{z=z_k} f(z)$$

and show that $\int_{C_R} f(z)dz \to 0$ as $R \to \infty$

Theorem: if all of the following assumptions hold

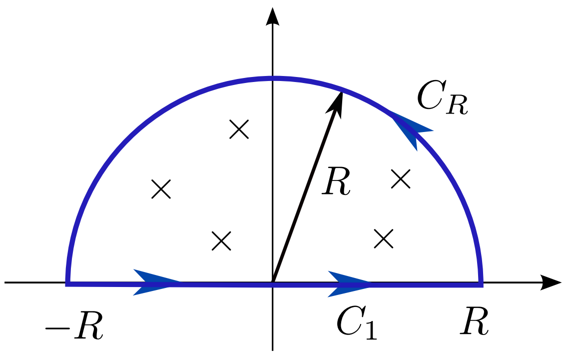
1. $f(z)$ is analytic in the upper half plane except at a finite number of poles

2. none of the poles of $f(z)$ lies on the real axis

3. $|f(z)| \leq \dfrac{M}{R^k}$ when $z = Re^{j\theta}$; M is a constant and $k > 1$

then the real improper integral can be evaluated by a contour integration, and

$$\int_{-\infty}^{\infty} f(x)dx = j2\pi \left[\begin{array}{l} \text{sum of the residues of } f(z) \text{ at the poles} \\ \text{which lie in the } \textbf{upper half plane} \end{array} \right]$$



- assumption 2: f is analytic on C_1
- assumption 3: $\int_{C_R} f(z)dz \to 0$ as $R \to \infty$

Proof. consider a semicircular contour with radius R large enough to include all the poles of $f(z)$ that lie in the upper half plane

- from the Cauchy's residue theorem

$$\int_{C_1 \cup C_R} f(z)dz = j2\pi \left[\sum \mathrm{Res}\, f(z) \ \text{ at all poles within } C_1 \cup C_R\right]$$

(to apply this, $f(z)$ cannot have singular points on C_1, $i.e.$, the real axis)

- the integral along the real axis is our desired integral

$$\lim_{R\to\infty} \int_{-R}^{R} f(x)dx + \lim_{R\to\infty} \int_{C_R} f(z)dz = \lim_{R\to\infty} \int_{C_1 \cup C_R} f(z)dz$$

- hence, it suffices to show that

$$\lim_{R\to\infty} \int_{C_R} f(z)dz = 0 \quad \text{by using } |f(z)| \leq M/R^k, \text{ where } k > 1$$

- apply the modulus of the integral and use $|f(z)| \leq M/R^k$

$$\left| \int_{C_R} f(z)dz \right| \leq \frac{M}{R^k} \cdot \text{length of } C_R = \frac{M\pi R}{R^k}$$

hence, $\lim_{R\to\infty} \int_{C_R} f(z)dz = 0$ if $k > 1$

remark: an example of $f(z)$ that satisfies all the conditions in page 13-30

$$f(x) = \frac{p(x)}{q(x)}, \quad p \text{ and } q \text{ are polynomials}$$

$q(x)$ has **no real roots** and $\deg q(x) \geq \deg p(x) + 2$

(relative degree of f is greater than or equal to 2)

example: show that

$$\int_{C_R} f(z)dz = 0$$

as $R \to \infty$ where C_R is the arc $z = Re^{j\theta},\ 0 \leq \theta \leq \pi$

- $f(z) = (z+2)/(z^3+1)$ (relative degree of f is 2)

$$|z+2| \leq |z| + 2 = R + 2, \quad |z^3+1| \geq ||z^3| - 1| = |R^3 - 1|$$

hence, $|f(z)| \leq \frac{R+2}{R^3-1}$ and apply the modulus of the integral

$$\left|\int_C f(z)dz\right| \leq \int_C |f(z)|dz \leq \frac{R+2}{R^3-1} \cdot \pi R = \pi \cdot \frac{1+\frac{2}{R^2}}{R-\frac{1}{R^2}}$$

the upper bound tends to zero as $R \to \infty$

- $f(z) = 1/(z^2 + 2z + 2)$

$$z^2 + 2z + 2 = (z - (1 + j))(z - (1 - j)) \triangleq (z - z_0)(z - \bar{z_0})$$

hence, $|z - z_0| \geq ||z| - |z_0|| = R - |1 + j| = R - \sqrt{2}$ and similarly,

$|z - z_0| \geq ||z| - |\bar{z_0}|| = R - \sqrt{2}$

then it follows that

$$|z^2 + 2z + 2| \geq (R - \sqrt{2})^2 \quad \Rightarrow \quad |f(z)| \leq \frac{1}{(R - \sqrt{2})^2}$$

$$\left| \int_C f(z) dz \right| \leq \int_C |f(z)| dz \leq \frac{1}{(R - \sqrt{2})^2} \cdot \pi R = \frac{\pi}{(1 - \frac{\sqrt{2}}{R})^2}$$

the upper bound tends to zero as $R \to \infty$

example: compute $I = \int_{-\infty}^{\infty} \frac{dx}{1+x^2}$

- define $f(z) = \frac{1}{1+z^2}$ and create a contour $C = C_1 \cup C_R$ as on page 13-29
- relative degree of f is 2, so $\int_{C_R} f(z)dz = 0$ as $R \to \infty$
- $f(z)$ has poles at $z = j$ and $z = -j$ (no poles on the real axis)
- only the pole $z = j$ lies in the upper half plane
- by the residue's theorem

$$j2\pi \cdot \sum \operatorname*{Res}_{z=z_k} f(z) = \oint_C f(z)dz = \underbrace{\int_{-R}^{R} f(x)dx}_{=I \text{ as } R\to\infty} + \underbrace{\int_{C_R} f(z)dz}_{=0 \text{ as } R\to\infty}$$

$$I \quad = \quad j2\pi \operatorname*{Res}_{z=j} f(z) \quad = \quad j2\pi \lim_{z\to j} (z-j)f(z) \quad = \quad \pi$$

example: compute

$$I = \int_{-\infty}^{\infty} \frac{x^2}{(x^2+a^2)(x^2+b^2)}\, dx$$

- define $f(z) = \dfrac{z^2}{(z^2+a^2)(z^2+b^2)}$ and create $C = C_1 \cup C_R$ as on page 13-29
- relative degree of f is 2, so $\int_{C_R} f(z)dz = 0$ as $R \to \infty$
- $f(z)$ has poles at $z = \pm ja$ and $z = \pm jb$ (no poles on the real axis)
- only the poles $z = ja$ and $z = jb$ lie in the upper half plane
- by the residue's theorem

$$j2\pi \cdot \sum \operatorname*{Res}_{z=z_k} f(z) = \oint_C f(z)dz = \underbrace{\int_{-R}^{R} f(x)dx}_{=I \text{ as } R\to\infty} + \underbrace{\int_{C_R} f(z)dz}_{=0 \text{ as } R\to\infty}$$

$$I = j2\pi \left[\operatorname*{Res}_{z=ja} f(z) + \operatorname*{Res}_{z=jb} f(z) \right] = j2\pi \left[\frac{a}{j2(a^2-b^2)} + \frac{b}{j2(b^2-a^2)} \right] = \frac{\pi}{a+b}$$

Improper integrals from Fourier analysis

we can use residue theory to evaluate improper integrals of the form

$$\int_{-\infty}^{\infty} f(x)\sin mx\, dx, \quad \int_{-\infty}^{\infty} f(x)\cos mx\, dx, \quad \text{or} \int_{-\infty}^{\infty} e^{jmx} f(x)\, dx$$

we note that e^{jmz} is analytic everywhere, moreover

$$|e^{jmz}| = e^{jm(x+jy)} = e^{-my} < 1 \quad \text{for all } y \text{ in the upper half plane}$$

therefore, if $|f(z)| \leq M/R^k$ with $k > 1$, then so is $|e^{jmz}f(z)|$

hence, if $f(z)$ satisfies the conditions in page 13-30 then

$$\int_{-\infty}^{\infty} e^{jmx} f(x) dx = j2\pi \left[\begin{array}{l} \text{sum of the residues of } e^{jmz}f(z) \text{ at the poles} \\ \text{which lie in the } \textbf{upper half plane} \end{array} \right]$$

denote

$$S = \left[\begin{array}{l} \text{sum of the residues of } e^{jmz}f(z) \text{ at the poles} \\ \text{which lie in the } \textbf{upper half plane} \end{array} \right]$$

and note that S can be *complex*

by comparing the real and imaginary part of the integral

$$\int_{-\infty}^{\infty} e^{jmx} f(x) dx = \int_{-\infty}^{\infty} (\cos mx + j \sin mx) f(x) dx = j2\pi S$$

we have

$$\begin{aligned} \int_{-\infty}^{\infty} \cos mx f(x) \, dx &= \mathrm{Re}(j2\pi S) = -2\pi \cdot \mathrm{Im}\, S \\ \int_{-\infty}^{\infty} \sin mx f(x) \, dx &= \mathrm{Im}(j2\pi S) = 2\pi \cdot \mathrm{Re}\, S \end{aligned}$$

example: compute $I = \displaystyle\int_{-\infty}^{\infty} \frac{\cos mx \; dx}{1+x^2}$

- define $f(z) = \frac{e^{jmz}}{1+z^2}$ and create $C = C_1 \cup C_R$ as on page 13-29
- relative degree of f is 2, so $\int_{C_R} f(z)dz = 0$ as $R \to \infty$
- f has poles at $z = j$ and $z = -j$ (no poles on the real axis)
- the pole $z = j$ lies in the upper half plane
- by residue's theorem

$$j2\pi \cdot \sum \operatorname*{Res}_{z=z_k} f(z) = \oint_C f(z)dz = \underbrace{\int_{-R}^{R} f(x)dx}_{=I \text{ as } R\to\infty} + \underbrace{\int_{C_R} f(z)dz}_{=0 \text{ as } R\to\infty}$$

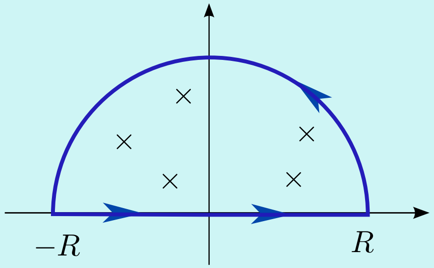
- therefore,

$$\int_{-\infty}^{\infty} \frac{e^{jmx}}{1+x^2} dx = j2\pi \operatorname*{Res}_{z=j} \frac{e^{jmz}}{1+z^2}$$
$$= j2\pi \lim_{z\to j} \frac{(z-j)e^{jmz}}{1+z^2} = \pi e^{-m}$$

- our desired integral can be obtained by

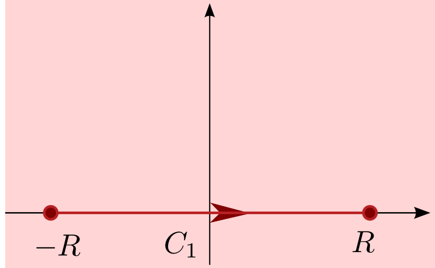
$$\int_{-\infty}^{\infty} \frac{\cos mx\ dx}{1+x^2} = \mathrm{Re}(\pi e^{-m}) = \pi e^{-m},$$
$$\int_{-\infty}^{\infty} \frac{\sin mx\ dx}{1+x^2} = \mathrm{Im}(\pi e^{-m}) = 0$$

Summary of improper integrals



$$\oint_C f(z)dz = j2\pi \sum_{z=z_k} \mathrm{Res} f(z)$$

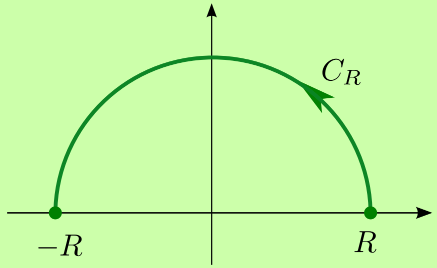
$$=$$



$$\int_{-R}^{R} f(x)dx$$

our desired integral as $R \to \infty$

$$+$$



$$\int_{C_R} f(z)dz$$

show that this term is zero as $R \to \infty$

the examples of f we have seen so far are in the form of

$$f(x) = \frac{p(x)}{q(x)}$$

where p, q are polynomials and $\deg p(x) \geq \deg q(x) + 2$

the assumption on the degrees of p, q is *sufficient* to guarantee that

$$\int_{C_R} f(z)e^{jaz}dz = 0 \quad (a > 0)$$

as $R \to \infty$ where C_R is the arc $z = Re^{j\theta}, \;\; 0 \leq \theta \leq \pi$

we can relax this assumption to consider function f such as

$$\frac{z}{z^2+2z+2}, \quad \frac{1}{z+1} \qquad \text{(relative degree is 1)}$$

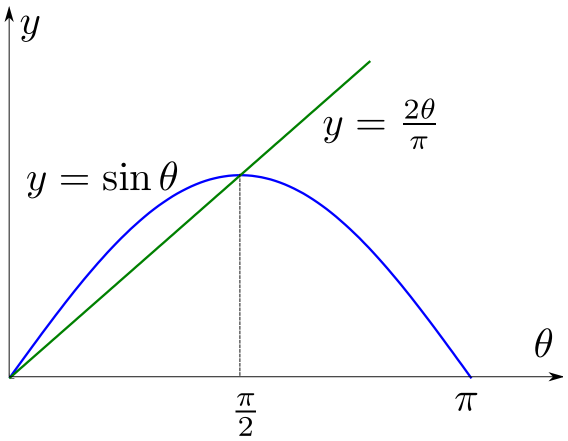
and obtain the same result by making use of **Jordan's inequality**

Jordan inequality

for $R > 0$,

$$\int_0^{\pi} e^{-R\sin\theta} d\theta \; < \; \frac{\pi}{R}$$

Proof.



$$\begin{aligned} \sin\theta &\geq 2\theta/\pi, \quad 0 \leq \theta \leq \frac{\pi}{2} \\ e^{-R\sin\theta} &\leq e^{-2R\theta/\pi}, \;\; R > 0, \;\; 0 \leq \theta \leq \frac{\pi}{2} \\ \int_0^{\pi/2} e^{-R\sin\theta} d\theta &\leq \frac{\pi}{2R} \end{aligned}$$

the last line is another form of the Jordan inequality

because the graph of $y = \sin\theta$ is symmetric about the line $\theta = \pi/2$

example: let $f(z) = \dfrac{z}{z^2+2z+2}$ show that $\int_{C_R} f(z)e^{jaz}dz = 0$ for $a > 0$ as $R \to \infty$

- first note that $|e^{jaz}| = |e^{ja(x+jy)}| = |e^{jax} \cdot e^{-ay}| = e^{-ay} < 1 \quad$ (since $a > 0$)

- similar to page 13-34, we see that $|f(z)| \leq R/(R-\sqrt{2})^2 \triangleq M_R$ and

$$\left| \int_{C_R} f(z)e^{jaz}dz \right| \leq \int_{C_R} \frac{R}{(R-\sqrt{2})^2} \cdot \pi R = \frac{\pi}{(1-\frac{\sqrt{2}}{R})^2}$$

  which **does not** tend to zero as $R \to \infty$

- however, for z that lies on C_R, *i.e.*, $z = Re^{j\theta}$

$$f(z)e^{jaz} = f(z)e^{jaRe^{j\theta}} = f(z)e^{jaR(\cos\theta + j\sin\theta)} = f(z)e^{-aR\sin\theta} \cdot e^{jaR\cos\theta}$$

- if we find an upper bound of the integral, **and** use **Jordan's inequality**:

$$
\begin{aligned}
\left|\int_{C_R} f(z)e^{jaz}dz\right| &= \left|\int_0^\pi f(z)e^{-aR\sin\theta}\cdot e^{jaR\cos\theta}jRe^{j\theta}d\theta\right| \\
&\leq \int_0^\pi \left|f(z)e^{-aR\sin\theta}\cdot e^{jaR\cos\theta}jRe^{j\theta}\right|d\theta \\
&= RM_R\int_0^\pi e^{-aR\sin\theta}d\theta \\
&< \frac{\pi M_R}{a}
\end{aligned}
$$

the final term approach 0 as $R\to\infty$ because $M_R\to 0$

conclusion: then we can apply the residue's theorem to integrals like

$$\int_{-\infty}^{\infty}\frac{x\cos(ax)}{x^2+2x+2}dx$$

Inversion of Laplace transforms

recall the definitions:

$$\begin{aligned} F(s) &\triangleq \mathcal{L}[f(t)] &\triangleq& \int_0^\infty f(t)e^{-st}dt \\ f(t) &= \mathcal{L}^{-1}[F(s)] &=& \frac{1}{j2\pi}\int_{a-j\infty}^{a+j\infty} F(s)e^{st}ds \end{aligned}$$

Theorem: suppose $F(s)$ is analytic everywhere except at the poles

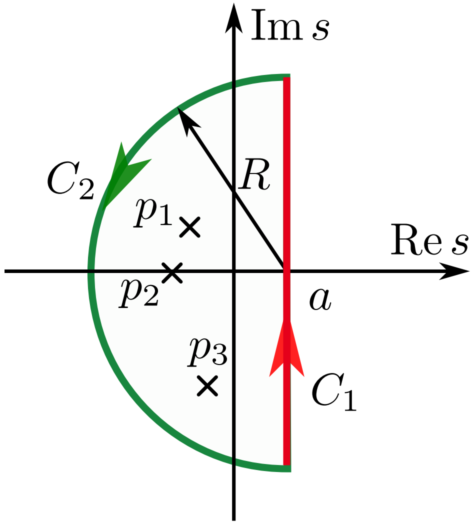
$$p_1, p_2, \ldots, p_n,$$

all of which lie to the **left** of the vertical line $\mathrm{Re}(s) = a$ (a convergence factor)

if $|F(s)| \leq M_R$ **and** $M_R \to 0$ as $s \to \infty$ through the half plane $\mathrm{Re}(s) \leq a$ then

$$\mathcal{L}^{-1}[F(s)] = \sum_{i=1}^{n} \operatorname*{Res}_{s=p_i} F(s)e^{st}$$

Proof sketch.



parametrize C_1 and C_2 by

$$C_1 = \{z \mid z = a + jy, \quad -R \leq y \leq R\}$$

$$C_2 = \left\{z \mid z = a + Re^{j\theta}, \quad \frac{\pi}{2} \leq \theta \leq \frac{3\pi}{2}\right\}$$

1. create a huge semicircle that is large enough to contain all the poles of $F(s)$
2. apply the Cauchy's residue theorem to conclude that

$$\int_{C_1} e^{st}F(s)ds = j2\pi \sum_{k=1}^{n} \operatorname*{Res}_{s=p_k}[e^{st}F(s)] - \int_{C_2} e^{st}F(s)ds$$

3. prove that the integral along C_2 is zero when the circle radius goes to ∞

choose a and R: choose the center and radius of the circle

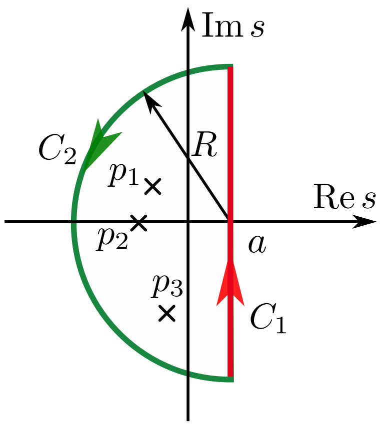
- $a > 0$ is so large that all the poles of $F(s)$ lie to the left of C_1

$$a > \max_{k=1,2,\ldots,n} \mathrm{Re}(p_k)$$

- $R > 0$ is large enough so that all poles of $F(s)$ are enclosed by the semicircle

  if the maximum modulus of $p_1, p_2, \ldots, p_n$ is R_0 then

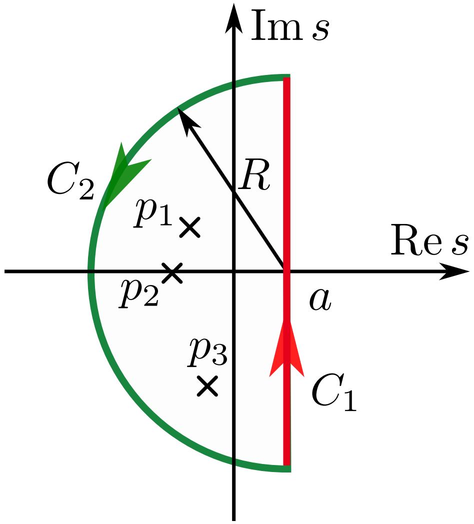
$$\forall k, \quad |p_k - a| \leq |p_k| + a \leq R_0 + a \quad \Longrightarrow \quad \text{pick } R > R_0 + a$$



$$C_1 = \{z \mid z = a + jy, \quad -R \leq y \leq R\}$$

$$C_2 = \left\{z \mid z = a + Re^{j\theta}, \quad \frac{\pi}{2} \leq \theta \leq \frac{3\pi}{2}\right\}$$

integral along C_2 is zero



$$C_1 = \{z \mid z = a + jy, \quad -R \leq y \leq R\}$$

$$C_2 = \left\{z \mid z = a + Re^{j\theta}, \quad \frac{\pi}{2} \leq \theta \leq \frac{3\pi}{2}\right\}$$

- for $s = a + Re^{j\theta}$ and $ds = jRe^{j\theta}d\theta$, the integral becomes

$$\left|\int_{C_2} e^{st}F(s)ds\right| = \left|\int_{\pi/2}^{3\pi/2} e^{at} \cdot e^{Rt\cos\theta + jRt\sin\theta}F(a+Re^{j\theta})Rje^{j\theta}d\theta\right|$$

- apply the modolus of the integral

$$\left|\int_{C_2} e^{st}F(s)ds\right| \leq \int_{\pi/2}^{3\pi/2} \left|e^{at}e^{Rt\cos\theta} \cdot e^{jRt\sin\theta}F(a+Re^{j\theta})Rje^{j\theta}\right| \, d\theta$$

- since $|F(s)| \leq M_R$ for s that lies on C_2

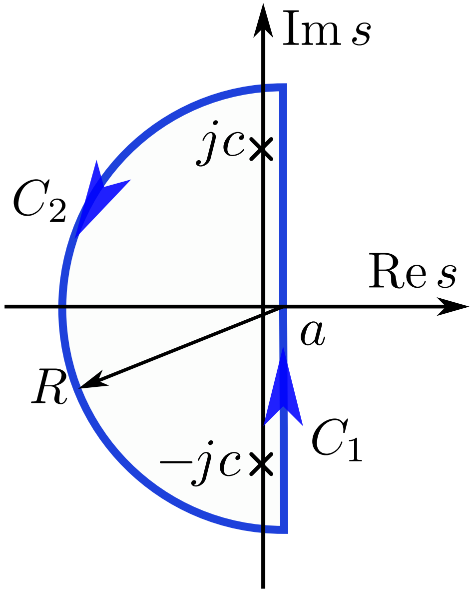
$$\left|\int_{C_2} e^{st}F(s)ds\right| \leq M_R R e^{at} \int_{\pi/2}^{3\pi/2} e^{Rt\cos\theta} \, d\theta$$

- make change of variable $\phi = \theta - \pi/2$ and apply the **Jordan inequality**

$$\left|\int_{C_2} e^{st}F(s)ds\right| \leq M_R R e^{at} \underbrace{\int_{0}^{\pi} e^{-Rt\sin\phi} \, d\phi}_{<\pi/Rt} \; < \; \frac{\pi M_R e^{at}}{t}$$

the last term approaches zero as $R \to \infty$ because $M_R \to 0$ (by assumption)

example: find $\mathcal{L}^{-1}[F(s)]$ where $F(s) = \dfrac{s}{(s^2+c^2)^2}$ and $c > 0$



$$C_2 = \left\{z \mid z = a + Re^{j\theta}, \quad \tfrac{\pi}{2} \leq \theta \leq \tfrac{3\pi}{2}\right\}$$

poles of $F(s)$ are $s = \pm jc$ so we choose $a > 0$

the semicircle must enclose all the pole

so we have $R > a + c$

first we verifty that $|F(s)| \leq M_R$ and $M_R \to 0$ as $s \to \infty$ for s on C_2

we note that $|s| = |a + Re^{j\theta}| \leq a + R$ and $|s| \geq |a - R| = R - a$

since $|s^2 + c^2| \geq ||s|^2 - c^2| \geq (R-a)^2 - c^2 > 0$, then

$$|F(s)| = \frac{|s|}{|s^2+c^2|^2} \leq \frac{(R+a)}{[(R-a)^2 - c^2]^2} \triangleq M_R \to 0 \quad \text{as } R \to \infty$$

therefore, we can apply the theorem on page 13-46

$$\mathcal{L}^{-1}[F(s)] = \sum \operatorname*{Res}_{s=s_k}[e^{st}F(s)] = \operatorname*{Res}_{s=jc} \frac{se^{st}}{(s^2+c^2)^2} + \operatorname*{Res}_{s=-jc} \frac{se^{st}}{(s^2+c^2)^2}$$

poles of $F(s)$ are $s = \pm jc$ (double poles)

$$\begin{aligned}
\operatorname*{Res}_{s=jc} e^{st}F(s) &= \lim_{s\to jc} \frac{d}{ds}\left[\frac{se^{st}}{(s+jc)^2}\right] = \left[\frac{e^{st}(1+ts)}{(s+jc)^2} - \frac{2se^{st}}{(s+jc)^3}\right]_{s=jc} \\
&= \frac{te^{jct}}{j4c} \\
\operatorname*{Res}_{s=-jc} e^{st}F(s) &= \lim_{s\to -jc} \frac{d}{ds}\left[\frac{se^{st}}{(s-jc)^2}\right] = \left[\frac{e^{st}(1+ts)}{(s-jc)^2} - \frac{2se^{st}}{(s-jc)^3}\right]_{s=-jc} \\
&= -\frac{te^{-jct}}{j4c}
\end{aligned}$$

hence $\mathcal{L}^{-1}[F(s)] = \frac{t}{4jc}(e^{jct} - e^{-jct}) = \dfrac{t\sin ct}{2c}$

example: find $\mathcal{L}^{-1}[F(s)]$ where $F(s) = \dfrac{1}{(s+a)^2+b^2}$

$F(s)$ has poles at $s = -a \pm jb$ (simple poles)

$$\mathcal{L}^{-1}[F(s)] = \operatorname*{Res}_{s=-a+jb} e^{st}F(s) + \operatorname*{Res}_{s=-a-jb} e^{st}F(s)$$

(provided that $|F(s)| \leq M_R$ and $M_R \to 0$ as $s \to \infty$ on C_2 ... please check ✎)

$$\operatorname*{Res}_{s=-a+jb} = \lim_{s=-a+jb} \frac{e^{st}}{s+a+jb} = \frac{e^{(-a+jb)t}}{j2b}$$

$$\operatorname*{Res}_{s=-a-jb} = \lim_{s=-a-jb} \frac{e^{st}}{s+a-jb} = \frac{e^{(-a-jb)t}}{-j2b}$$

hence, $\mathcal{L}^{-1}[F(s)] = \dfrac{e^{-at}(e^{jbt} - e^{-jbt})}{2jb} = \dfrac{e^{-at}\sin(bt)}{b}$

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