Finiteness theorems for holomorphic mappings from products of hyperbolic Riemann surfaces

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- All function spaces will given the compact-open topology.

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A *Riemann surface of finite type* (g, n) is a Riemann surface that is biholomorphic to a Riemann surface obtained by removing n points from a compact Riemann surface of genus g.

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2. Recall that the Poincaré metric on the unit disk is given by the formula

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3. The integrated version of the above metric, also denoted ρ, is called the Poincaré distance.
4. The Schwarz–Pick lemma can now be reinterpreted to say that holomorphic self-maps of the unit disk are distance decreasing under the Poincaré metric and distance.

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3. The crucial observation is that one *cannot* equip such a distance on non-hyperbolic Riemann surfaces. One way to see this is to observe that one can embedd as large a analytic disk as one desires inside a non-hyperbolic Riemann surface.
4. Using the seminal work of Ahlfors on the Schwarz lemma, one can give illuminating and unified proofs of Liouville’s theorem and the theorems of Picard and several other related results like Schotkky’s theorem and Montel’s theorem.

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3. If the Kobayashi pseudodistance on X is a distance then we say that X is *Kobayashi hyperbolic.*
4. It follows from the very definition of the Kobayashi pseudodistance that holomorphic mappings are distance decreasing under the Kobayashi pseudodistance.
5. It also follows trivially that there are no complex lines sitting inside a hyperbolic manifold.

Shiga's theorem

The following result of Shiga is a higher-dimensional analogue of Imayoshi's theorem.

Result (Shiga, 2004)

Let $X = \mathbb{B}^n/G$ be a complex hyperbolic manifold of divergence type and let $Y = \Omega/\Gamma$ be a geometrically finite n-dimensional complex manifold where $\Omega \subset \mathbb{C}^m$ is a bounded domain and Γ is fixed-point-free discrete subgroup of $Aut(\Omega)$. Suppose G is finitely generated and that Ω is complete with respect to the Kobayashi distance. Then $\mathcal{O}_{dom}(X, Y)$ is a finite set. Furthermore, if the essential boundary dimension of Ω is zero, then $\mathcal{O}_*(X, Y)$ is a finite set

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We will explain the meaning of the terms *geometrically finite* and *essential boundary dimension* 0 in a later slide.

Our main result

Theorem (Divakaran and Jaikrishnan, 2017, IJM)

Let $X := X_1 \times \cdots \times X_n$ be a product of hyperbolic Riemann surfaces of finite type and let $Y = \Omega/\Gamma$ be an m-dimensional complex manifold where $\Omega \subset \mathbb{C}^m$ is a bounded domain and Γ is fixed-point-free discrete subgroup of $Aut(\Omega)$.

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1. *If N is a tautly embedded complex submanifold of Y then $\mathcal{O}_{dom}(X, N)$ is a finite set.*
2. *If Y is geometrically finite and Ω is complete hyperbolic then $\mathcal{O}_{dom}(X, Y)$ is a finite set.*

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1. *If N is a tautly embedded complex submanifold of Y then $\mathcal{O}_{dom}(X, N)$ is a finite set.*
2. *If Y is geometrically finite and Ω is complete hyperbolic then $\mathcal{O}_{dom}(X, Y)$ is a finite set.*
3. *If in addition to the conditions in (2), the essential boundary dimension of Ω is zero, then $\mathcal{O}_*(X, Y)$ is a finite set.*

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1. Our result extends the result of Imayoshi to product manifolds.
2. The proof also works *mutatis mutandis* to give a proof of Shiga's result.
3. We borrow a number of ideas from the proofs given by Imayoshi and Shiga.
4. However, the crux of our argument involves a (to our knowledge) new normal families argument that neatly clarifies the underlying role of (Kobayashi) hyperbolicity in the above results.

Tautness and normal families

- If M and N are two hyperbolic complex manifolds, then by the distance decreasing property the space $\mathcal{O}(M, N)$ is an equicontinuous family. Is $\mathcal{O}(M, N)$ relatively compact as a subspace of $\mathcal{C}(M, N)$?

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Definition

A subset $\mathcal{F} \subset C(M, N)$ is said to be a *normal family* if every sequence $\{f_n\} \subset \mathcal{F}$ has either a subsequence that converges uniformly on compacts to a function in $C(M, N)$ or has a compactly divergent subsequence.

A complex manifold N is said to be *taut* if for every complex manifold M the set $\mathcal{O}(M, N)$ is a normal family.

Let N be a complex manifold and let Y be a complex submanifold. We say that Y is *tautly embedded* in N if every sequence of holomorphic mappings $\{f_n : M \to Y\}$, where M is any complex manifold, admits a subsequence that converges uniformly on compacts to a holomorphic map $f : M \to N$.

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- Complete hyperbolic manifolds are taut.

Meaning of technical terms

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4. The essential boundary dimension of a bounded domain in $\mathbb{C}^n$ is roughly the maximal dimension of analytic sets sitting in ∂D. The unit ball in $\mathbb{C}^n$ and more generally strictly pseudoconvex domains have essential boundary dimension 0 whereas the polydisk has essential boundary dimension $n - 1$.

A rigidity result

Theorem

Let $X := X_1 \times \cdots \times X_n$ be a product of hyperbolic Riemann surfaces of finite type and let $Y = \Omega/\Gamma$ be an m-dimensional complex manifold where $\Omega \subset \mathbb{C}^m$ is a bounded domain and Γ is fixed-point-free discrete subgroup of $Aut(\Omega)$. Write X as $\mathbb{D}^n/G$, where $G := \bigoplus_{i=1}^n G_i$ and G_i is the Fuchsian group of divergence type such that $X_i = \mathbb{D}/G_i$. Suppose $\phi, \psi : X \to Y$ are holomorphic mappings such that we can find lifts $\widetilde{\phi}, \widetilde{\psi} : \mathbb{D}^n \to \Omega$ that induce the same homomorphism on G.

1. *If ϕ (or ψ) is dominant, then $\phi = \psi$.*
2. *If ϕ (or ψ) is non-constant and Ω has essential boundary dimension zero, then $\phi = \psi$.*

Some remarks about the rigidity result

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4. A somewhat different rigidity result has been established by Shiga for his purposes.
5. Our proof can be easily adapted to the situation where X is a complex hyperbolic manifold of divergence type.

Proof sketch of our main result

We will now give a sketch of the proof of the first assertion our main result.

- Let $\{f_k\} \subseteq \mathcal{O}_{\mathrm{dom}}(X, N)$ be a sequence of distinct dominant holomorphic mappings. As N is tautly embedded in Y, we may assume that the sequence $\{f_k\}$ converges in the compact-open topology to a map $f : X \to Y$.

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- We can find a connected and compact set K such that finitely many closed loops contained in K based at a point $x \in K$ generate $\pi_1(X, x)$.
- We show that for suitably large k, we can find lifts $\widetilde{f_k}$ and $\widetilde{f}$ that induce the same homomorphism on G.

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- We first choose k suitably large so that $z_k := f_k(x)$ and $y := f(x)$ belong to an evenly covered coordinate ball in Y, say U.

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- Let χ and χ_k be the homomorphism induced by $\widetilde{f}$ and $\widetilde{f_k}$, respectively.
- Each $g \in G$ can be represented by a closed loop based at x, say γ. Then $f \circ \gamma$ and $f_k \circ \gamma$ are loops in Y based at y and z_k, respectively.

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- Let $\sigma := f \circ \gamma$ and $\sigma_k := \overline{\delta}_k * (f_k \circ \gamma) * \delta_k$ be two loops based at the point y, where δ_k is a curve lying in U that connects y to z_k.

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- Now $\sigma_k \to \sigma$ uniformly. A folklore result now shows that σ and σ_k are equivalent in $\pi_1(Y, y)$ for suitably large k.
- Let $\widetilde{\sigma}$ and $\widetilde{\sigma}_k$ be the lifts of σ and σ_k, respectively, that start at $\widetilde{f}(\widetilde{x})$. As σ and σ_k represent the same element in $\pi_1(Y, y)$, the endpoints of $\widetilde{\sigma}$ and $\widetilde{\sigma}_k$ must be the same and equal to $\chi(g)\left(\widetilde{f}\,(\widetilde{x})\right)$.

Conclusion of proof sketch

Since the quotient map is a homeomorphism from $\widetilde{U}$ to U and δ_k lies entirely in U, a lift of δ_k starting at $\widetilde{f}(\widetilde{x})$ ends in $\widetilde{U}$. Similarly, a lift of $\overline{\delta}_k$ that ends in $\chi(g)\left(\widetilde{U}\right)$ has to begin in $\chi(g)\left(\widetilde{U}\right)$. Thus a lift of $f_k \circ \gamma$ starting in $\widetilde{U}$ (at $\delta_k(1)$) has to end in $\chi(g)\left(\widetilde{U}\right)$. Thus, $\chi_k(g)\left(\widetilde{U}\right) \cap \chi(g)\left(\widetilde{U}\right) \neq \emptyset$. Since U is an evenly covered neighborhood, it follows that $\chi_k(g) = \chi(g)$.

THANK YOU