# Hyperbolicity in Complex Analysis

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6. **The de Franchis Theorem (1913):** Let $R$ and $S$ be compact Riemann surfaces of genus higher than 1. Then there are only finitely many non-constant holomorphic mappings from $R$ into $S$.

# Conformal Metrics

## Conformal Metric

Let $D \subset \mathbb{C}$ be a domain. A *conformal metric* on $D$ is a $C^2$-smooth function $\rho : D \longrightarrow \mathbb{R}^+ \cup \{0\}$. We also allow $\rho$ to have only a discrete set of zeroes at which points $\rho$ is not required to be differentiable. If $z \in D$ and $v \in \mathbb{C}$ is a vector, we define the *length of $v$ at $z$* to be

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Angles can now defined using the inner-product and they agree with the Euclidean angles at those points where the metric is non-singular.

# Length of curves

If $\gamma : [a, b] \longrightarrow D$ is a piecewise smooth curve then we define the length of $\gamma$ with respect to $\rho$ as

$$\ell(\gamma) := \int_a^b \rho(\gamma(t))|\gamma'(t)|dt.$$

# The distance associated to a conformal metric

If $D \subset \mathbb{C}$ is a domain and $\rho$ is a conformal metric, we define

$$d_\rho(z, w) := \inf\{\ell(\gamma) : \gamma : [0, 1] \to D \text{ is a curve with } \gamma(0) = a \text{ and } \gamma(1) = b\}.$$

It is straightforward to show that $d_\rho$ defines a distance (i.e., a metric in the usual sense) on $D$.

# Pullback of a conformal metric

**Definition**

Let $D_1, D_2 \subset \mathbb{C}$ be domains and let $f : D_1 \longrightarrow D_2$ be a thrice continuously differentiable function with isolated zeroes. We define the *pullback* of the conformal metric $\rho$ of $D_2$ under $f$ to be the conformal metric on $D_1$ given by

$$f^*\rho(z) = \rho(f(z)) \cdot |\partial_z f|.$$

If $\rho_1$ is another conformal metric such that for some holomorphic mapping $g : D_1 \longrightarrow D_2$, $g^*\rho \equiv \rho_1$ then we say $g$ is an *isometry* of the pair $(D_1, \rho_1)$ with $(D_2, \rho)$.

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Straightforward computations reveal that holomorphic isometries preserve lengths of curves and therefore distances as well.

# The Schwarz–Pick Lemma

A nice exercise in a first course in complex analysis is to deduce the following result known as the Schwarz–Lemma using the standard Schwarz Lemma.

## Schwarz–Pick Lemma, 1916

Let $f : \mathbb{D} \longrightarrow \mathbb{D}$ be holomorphic. Then

1. $$\frac{|f'(z)|}{1 - |f(z)|^2} \leq \frac{1}{1 - |z|^2} \; \forall z \in \mathbb{D}.$$

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1. $$\frac{|f'(z)|}{1-|f(z)|^2} \leq \frac{1}{1-|z|^2} \;\forall z \in \mathbb{D}.$$
2. $$\frac{|f(z_1)-f(z_2)|}{|1-f(z_1)\overline{f(z_2)}|} \leq \frac{|z_1-z_2|}{|1-z_1\overline{z}_2|} \;\forall z_1, z_2 \in \mathbb{D}.$$

## The Poincaré metric, 1881-84

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2. Consider the unit disk. It is easy to construct a conformal metric $\rho$ on $\mathbb{D}$ under which each automorphism is an isometry. In fact, it is immediate from the Schwarz–Pick lemma that

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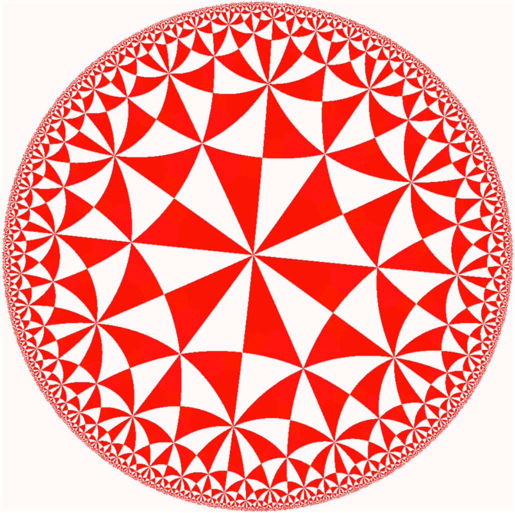
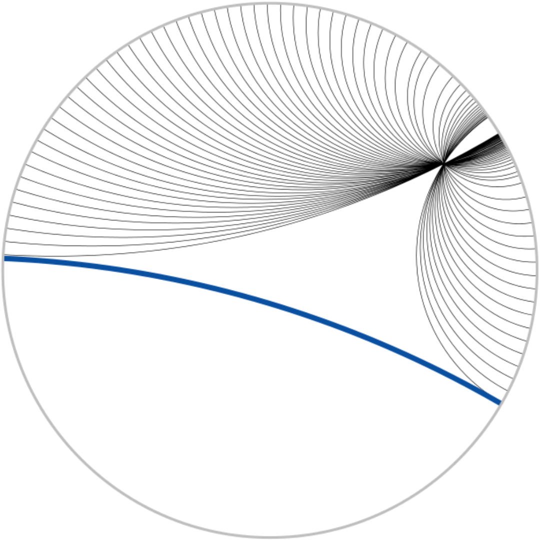
   is such a metric.
3. The above metric is known as the *Poincaré metric*. It is straightforward to show that the only conformal metrics on the disk that preserve the length of curves under automorphisms are constant multiples of the Poincaré metric.

# Artistic visualization



# Geodesics



# The distance decreasing property

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2. Observe that the Schwarz–Pick lemma implies the more general fact that holomorphic self-maps are contracting under the Poincaré distance.
3. This geometric interpretation of the Schwarz lemma led Ahlfors to formulate a far-reaching generalization of the Schwarz lemma. This work kick-started the study of invariant metrics which is now a prominent field of research in complex analysis.

# Curvature

## Definition

Let $D \subset \mathbb{C}$ be a domain equipped with a conformal metric $\rho$. The *curvature of* $\rho$ is defined to be

$$\kappa_{D,\rho}(z) := -\frac{1}{\rho^2(z)} \nabla^2 \log \rho(z).$$

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3. An easy computation shows that if $D_1$ is a domain and $f : D_1 \longrightarrow D$ is a holomorphic map with $f'(z) \neq 0$ for some $z \in D_1$ then

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4. This means that the notion of curvature is invariant under biholomorphism.

# The Ahlfors–Schwarz lemma

## Theorem (Ahlfors, 1938)

*Let $\rho$ be the Poincaré metric on $\mathbb{D}$ and let $U$ be a domain equipped with a conformal metric $\sigma$ such that $\kappa_{U,\sigma}(w) \leq -4 \; \forall w \in U$. Then for any holomorphic map $f : \mathbb{D} \longrightarrow U$, we have*

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The proof of the lemma is not hard. What is more interesting is that several classical results including Montel's theorem and Picard's theorems can be given easy proofs using this theorem.
The only non-trivial step in the proofs is the construction of a conformal metric on $\mathbb{C} \setminus \{0, 1\}$ whose curvature is bounded above by $-4$. There are several such constructions available.

# First definition of hyperbolicity

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2. Using the famous uniformization theorem for Riemann surfaces, one can prove that the hyperbolic domains are precisely those whose universal cover is the unit disk.
3. It also turns out that the distance generated on a hyperbolic domain makes it into a complete metric space.

# On to higher dimensions: The Carathéodory pseudodistance

Let $M$ be a complex manifold. The Carathéodory pseudodistance $C_M(z, w)$ is defined as follows:

$$C_M(z, w) := \sup\{\rho_{\mathbb{D}}(f(z), f(w)) : f \in \mathcal{O}(M, \mathbb{D})\}.$$

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It is a triviality that holomorphic mappings are contractions when both the domain and codomain are equipped with the respective Carathéodory pseudodistances.
Observe also that the Carathéodory pseudodistance of a compact complex manifold is trivial.

# The second definition of hyperbolicity

**Definition (Carathéodory, 1926)**

We say a complex manifold $M$ is *C-hyperbolic* if the Carathéodory pseudodistance of $M$ is a distance. If the distance is complete, we say that $M$ is *complete C-hyperbolic*

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The unit disk is (complete) $C$-hyperbolic whereas $\mathbb{C}$ and $\mathbb{C} \setminus \{0, 1\}$ and compact complex manifolds are not.

# Invariant metrics

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2. This prompted the study of the more general idea of a *contractible family of invariant metrics*. A system of invariant metrics is an association of a pseudodistance to each complex manifold such that holomorphic mappings become distance decreasing. We also stipulate that the pseudodistance associated to $\mathbb{D}$ is the Poincaré distance.

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3. In other words, it is a way of putting a pseudodistance on complex manifolds so that the analogue of the Schwarz lemma is built-in.
4. The Carathéodory pseudodistance provides one way of creating such a system.

# The Kobayashi pseudodistance

1. If $\rho_M$ is a pseudodistance on the complex manifold $M$ that is distance decreasing for holomorphic maps $f : M \to \mathbb{D}$ then

$$\rho_{\mathbb{D}}(f(z), f(w)) \leq \rho_M(z, w) \ \forall z, w \in M.$$

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2. Taking supremum on the LHS reveals that Carathéodory system of assigning pseudodistances gives the "smallest" possible pseudodistance on each complex manifold.
3. The Kobayashi pseudodistance is the dual of the Carathéodory distances and gives the "largest" possible distance on each complex manifold.

# The Kobayashi pseudodistance, 1967

If $z, w \in M$ define the *Lempert function* $\delta_M(z, w)$ to be

$$\delta_M(z, w) := \inf \rho(x, y)$$

where the infimum ranges over all holomorphic mappings $f : \mathbb{D} \to M$ with $f(x) = z$ and $f(y) = w$.

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where the infimum ranges over all holomorphic mappings $f : \mathbb{D} \to M$ with $f(x) = z$ and $f(y) = w$.
The Lempert function need not satisfy the triangle inequality so we use the Kobayashi trick to create a pseudodistance as follows: we define the *Kobayashi pseudodistance* $K(z, w)$ to be

$$K(z, w) := \inf \left\{ \sum_{i=1}^{n} \delta_M(p_i, p_{i+1}) \right\},$$

where the sum runs through all finite chains of points $z = p_1, \ldots, p_n = w$.

## Basic properties of the Kobayashi pseudodistance

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2. Again, it is trivial the holomorphic functions are contractions under the Kobayashi pseudodistance.
3. Analogous definitions of $K$-hyperbolicity and completeness can be made. If a complex manifold is $K$-hyperbolic then the topology generated by the Kobayashi distance coincides with the manifold topology.
4. An extremely useful property of the Kobayashi pseudodistance is its good behaviour under coverings: If a complex manifold $M'$ is a covering manifold of the complex manifold $M$ then $M$ is (complete) $K$-hyperbolic iff $M'$ is.

# A central fact

The following fact is one of the central facts that make invariant metrics so powerful in geometric function theory:

**Equicontinuity of the space of holomorphic mappings**

If $M$ and $N$ are complex manifolds with $N$ being $K$-hyperbolic then the space of holomorphic mappings $\mathcal{O}(M, N)$ is an equicontinuous family.

# Wu's version of Montel's theorem

H. Wu initiated the study of normal families in a seminal paper in 1967. Using the classical Arzela–Ascoli theorem and the central fact from the previous slide the following theorem becomes straightforward.

**Theorem (Wu)**

*Let $M$ and $N$ be complex manifolds and let $N$ be complete $K$-hyperbolic. Then the family $\mathcal{O}(M, N)$ is a normal family.*

# A version of Montel's theorem for Riemann surfaces

## Theorem (J., 2014)

*Let $X$ be a connected complex manifold and let $R$ be a hyperbolic open connected subset of a compact Riemann surface $S$. Then, given any sequence $\{f_\nu\} \subset \mathcal{O}(X, R)$, there exists a subsequence $\{f_{\nu_k}\}$ and a holomorphic map $f_0 : X \longrightarrow \overline{R}$ (the closure taken in $S$ whenever $R$ is non-compact) such that $f_{\nu_k} \to f_0$ uniformly on compact subsets of $X$.*

# But where is the geometry?

So far, all the geometry in this talk has been differential and the familiar classical geometry has disappeared. However, there is one deep and shocking result that is poorly understood where Euclidean geometry crops up in an unexpected manner.

**Theorem (Lempert, 1984)**

*Let $D \subset \mathbb{C}^n, n > 1$ be a bounded and convex domain. Then $D$ is $K$-hyperbolic and the Lempert function, the Kobayashi distance and the Carathéodory distance all coincide.*

# Structure theorem for products of Riemann surfaces

**Theorem (J., 2014)**

*Let $R_j$ and $S_j$, $j = 1, \ldots, n$, be compact Riemann surfaces, and let $X_j$ (resp. $Y_j$) be a connected, hyperbolic open subset of $R_j$ (resp. $S_j$) for each $j = 1, \ldots, n$. Let $F = (F_1, \ldots, F_n) : X_1 \times \cdots \times X_n \longrightarrow Y_1 \times \cdots \times Y_n$ be a finite proper holomorphic map. Then, each $F_i$ is of the form $F_i(z_{\pi(i)})$, where $\pi$ is a permutation of $\{1, \ldots, n\}$.*

# Finiteness of the space of holomorphic mappings

**Theorem (Divakaran and J., 2017)**

*Let $X := X_1 \times \cdots \times X_n$ be a product of hyperbolic Riemann surfaces of finite type and let $Y = \Omega/\Gamma$ be an m-dimensional complex manifold where $\Omega \subset \mathbb{C}^m$ is a bounded domain and $\Gamma$ is fixed-point-free discrete subgroup of $Aut(\Omega)$.*

1. *If $N$ is a tautly embedded complex submanifold of $Y$ then $\mathcal{O}_{dom}(X, N)$ is a finite set.*
2. *If $Y$ is geometrically finite and $\Omega$ is complete hyperbolic then $\mathcal{O}_{dom}(X, Y)$ is a finite set.*
3. *If in addition to the conditions in (2), the essential boundary dimension of $\Omega$ is zero, then $\mathcal{O}_{*}(X, Y)$ is a finite set.*

# Structure theorem for the symmetric product of a Riemann surface

**Theorem (Biswas, Bharali, Divakaran and J., 2018)**

*Let $X = X_1 \times \cdots \times X_n$ be a complex manifold where each $X_j$ is a connected non-compact Riemann surface obtained by excising a non-empty indiscrete set from a compact Riemann surface $R_j$. Let $Y$ be a connected bordered Riemann surface with $C^2$-smooth boundary. Let $F : X \longrightarrow Sym^n(Y)$ be a proper holomorphic map. Then, there exist proper holomorphic maps $F_j : X_j \longrightarrow Y$, $j = 1, \ldots, n$, such that*

$$F(x_1, \ldots, x_n) = \pi_{Sym} \circ \big(F_1(x_1), \ldots, F_n(x_n)\big) \quad \forall (x_1, \ldots, x_n) \in X.$$

THANK YOU