Proper holomorphic mappings of balanced domains

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Talk at CEMS, Almora, November 16th 2016

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4. The map (z, w^2) is an example of a proper holomorphic self-map of $\mathbb{D} \times \mathbb{D}$.
5. Let D_1 and D_2 be domains in $\mathbb{C}^n$ and $\mathbb{C}^m$, respectively, and let $f : D_1 \to D_2$ be continuous. Then f is proper iff for all sequences $\{x_n\} \subset D_1$ that has no limit point in D_1, the sequence $\{f(x_n)\}$ has no limit point in D_2.

Some major results

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Slightly before the work of Alexander, Pinchuk, in 1973-74, had established the following:

Result (Pinchuk, 1973)

Let $D \subset \mathbb{C}^n$, $n > 1$, *be a bounded strictly pseudoconvex domain, and let* $f : D \to D$ *be a proper holomorphic mapping. If* f *extends to a* C^1 *mapping on* $\overline{D}$, *then* f *is an automorphism of* D.

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Let $D \subset \mathbb{C}^n, \; n > 1$, *be a bounded domain with smooth real-analytic boundary. Then any proper self-map of* D *that extends smoothly to* ∂D *must be an automorphism.*

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Note that in the above result, pseudoconvexity of D is not assumed.

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Result (Coupet, Pan and Sukhov, 1999)

Let $D \subset \mathbb{C}^2$ be a smoothly bounded balanced pseudoconvex domain of finite type. Then every proper holomorphic self mapping of D is an automorphism.

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- it is a bounded symmetric domain and hence homogeneous,
- it is balanced and convex, etc.

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Key question

What role, if any, did the various attributes of $\mathbb{B}^n$ listed above play in the phenomenon exhibited in Alexander's theorem?

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The above conjecture is nowhere close to being settled even with the additional hypotheses of pseudoconvexity and finiteness of type. However, significant progress has been made when the domain in question admits some symmetries. A case in point is the result of Coupet, Pan and Sukhov.

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Theorem (J., Math. Z., 2015)

Let $\Omega \subset \mathbb{C}^n,\ n > 1$, be a smoothly bounded balanced domain of finite type. Assume that Ω has a smooth defining function that is plurisubharmonic in Ω. Then every proper holomorphic self-map $F : \Omega \to \Omega$ is an automorphism.

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The special feature of the above theorem is that it is the first result in the literature for domains in $\mathbb{C}^n,\ n > 2$, with boundary *not necessarily* real-analytic or strictly pseudoconvex where the automorphism group need not be “large”.

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A C^2-smooth function $h : \Omega \to \mathbb{R},\ \Omega \subset \mathbb{C}^n$, is said to be plurisubharmonic if

$$\sum_{j,k=1}^{n} \frac{\partial^2 \rho}{\partial z_j \bar{z}_k}(p) v_j \bar{v}_k \geq 0 \quad \forall p \in \Omega, \forall (v_1, \ldots, v_n) \in \mathbb{C}^n$$

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2. Complex ellipsoids also satisfy all the hypotheses of our theorem.
3. The domain given by the defining function $|z_1|^2 + |z_2|^2|z_2 z_3|^2 - 1$ is an example of a balanced but *not* Reinhardt domain that satisfies all our hypotheses.

A structure theorem for the branch locus

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Theorem (Structure Theorem)

Let $\Omega \subset \mathbb{C}^n$, $n > 1$, be a smoothly bounded balanced pseudoconvex domain of finite type. Let $F : \Omega \to \Omega$ be a proper holomorphic mapping, and assume that the branch locus

$$V_F := \{z \in \Omega : \mathrm{Jac}_{\mathbb{C}} F(z) = 0\} \neq \emptyset.$$

Let X be an irreducible component of V_F. Then for each $z \in X$, the set $(\mathbb{C} \cdot z) \cap \Omega$ is contained in X.

Bell's result

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Lemma (Bell)

Let $f : D_1 \to D_2$ be a proper holomorphic map between bounded balanced domains. Assume that the intersection of every complex line passing through 0 with ∂D_1 is a circle. Then f extends holomorphically to a neighbourhood of $\overline{D}_1$.

Some dynamics

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Result

Let X be a taut manifold, and $f \in \mathrm{Hol}(X, X)$. Then either the sequence $\{f^k\}$ of iterates of f is compactly divergent, or there exists a complex submanifold M of X and a holomorphic retraction $\rho : X \to M$ (i.e., $\rho^2 = \rho$) such that every limit point $h \in \mathrm{Hol}(X, X)$ of $\{f^k\}$ is of the form $h = \gamma \circ \rho$, where γ is an automorphism of M. Moreover,

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1. even ρ is a limit point of the sequence $\{f^k\}$,
2. $f(M) \subset M$, and $f|_M$ is an automorphism of M;
3. The set of limit points of the iterates of f is a compact abelian group; in fact it is isomorphic to the closed subgroup of $\mathrm{Aut}M$ generated by $f|_M$.

Some dynamics

Definition

With the notation as in the above theorem, we say that f is *non-recurrent* if the sequence $\{f^k\}$ of iterates of f is compactly divergent. Otherwise, we say that f is *recurrent*, and we call the map ρ the *limit retraction*, and the manifold M the *limit manifold*.

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Result (Opshtein)

Let $D \subset \mathbb{C}^n$, $n > 1$, be a smoothly bounded pseudoconvex domain which admits a defining function that is plurisubharmonic. Let $f : D \to D$ be a proper holomorphic self-map that is recurrent. Then the limit manifold of f is necessarily of dimension higher than 1.

A classical result of Cartan

The behaviour of the iterates of a holomorphic self-map of a taut manifold X depends on whether f has a fixed point or not. The following theorem known as the Cartan-Carathéodory theorem gives a quantitative description of the behaviour of the differential f' at a fixed point of f.

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The role of finite type in our proof

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3. Another crucial property used is the analytical-disk lemma that was stated previously.

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- We may, after a change of co-ordinates, assume that

$$M = \{(z_1, z_2, \ldots, z_m, 0, \ldots, 0)\} \cap \Omega.$$

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Using the Cartan–Carathéodory theorem, it follows that $F|_M$ is of the form:

$$(z_1, z_2, \ldots, z_m, 0, \ldots, 0) \mapsto (e^{i\theta_1} z_1, e^{i\theta_2} z_2, \ldots, e^{i\theta_m} z_m, 0, \ldots, 0).$$

Finishing the proof

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- An analysis of the behaviour of the τ function on the iterates $F^k(p)$ reveals that τ cannot be upper-semicontinuous at p. This step uses conclusion (3) of the result from dynamics. The compact abelian group in question is a subgroup of the m-torus. This contradiction proves that $\dim M \leq 1$.

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- But $\dim M \leq 1$ contradicts Opshtein's theorem; so the assumption that F is branched must be false.
- The fact that F must be injective now follows from the monodromy theorem.

$\square$

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Result (Bell)

Let Ω_1 and Ω_2 be bounded domains in $\mathbb{C}^n$ and let $\{f_i : \Omega_1 \to \Omega_2\}$ be a sequence of proper holomorphic mappings all of which have multiplicity $\leq m$. Suppose f_i converge uniformly on compacts to a map $f : \Omega \to \mathbb{C}^n$. Then the following conditions are equivalent:

1. *f is a proper holomorphic mapping.*
2. $f(\Omega_1) \not\subset \partial OM_2$.
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2. What is the relationship between the above conjecture and Opshtein's result? Well, Opshtein's result follows if the words "strictly pseudoconvex" are replaced by the words "weakly pseudoconvex with plurisubharmonic defining function".

Proving the key conjecture for a general class of domains

We outline a general recipe for proving the key conjecture for a class of domains $\mathcal{D}$.

1. Establish an analogue of Wong's conjecture for the class $\mathcal{D}$.
2. Use the properties enjoyed by a domain in $\Omega \in \mathcal{D}$ to show that the iterate of any branched proper holomorphic self-map of Ω must converge to a non-proper holomorphic map of Ω to itself.
3. The conclusion of Step 2. is in conflict with the conclusion of Step 1. and this proves that any proper holomorphic self-map of Ω must be unbranched.
4. A result of Pinchuk shows that any unbrached proper holomorphic self-map of a bounded domain with smooth boundary is automatically an automorphism and we are done.

THANK YOU