Due 23:59 March 24 (Sunday). There are 100 points in this assignment.
Submit your answers (**must be typed**) in pdf file to CourSys
`https://coursys.sfu.ca/2024sp-cmpt-307-d1/`.
Submissions received after 23:59 will get penalty of reducing points: 20 and 50 points deductions for submissions received at $[00:00, 00:10]$ and $(00:10, 00:30]$ of March 25, respectively; no points will be given to submissions after $00:30$ of March 25.

1. 10 points (Ex 19.1-2 and 19.1-3 of text book)

   (a) Prove that after all edges are processed by procedure Connected-Components discussed in class, two vertices are in the same connected component if and only if they are in the same set.

   (b) During the execution of Connected-Components (discussed in class) on an undirected graph G with k components, how many times is Find-Set called? How many times is Union called? Express your answers in terms of $|V|$, $|E|$ and k.

2. 10 points (Ex 19.2-3 of text book)

   Adapt the aggregate proof of slide 26 (Theorem 19.1 in the text book) to obtain amortized time bounds of $O(1)$ for Make-Set and Find-Set and $O(\log n)$ for Union using linked list representation and weighted-union heuristic.

3. 20 points (Ex 20.1-3 of text book)

   The **transpose** of a directed graph $G(V, E)$ is the graph $G^T(V, E^T)$, where $E^T = \{(u, v)|(v, u) \in E\}$. That is, G^T is obtained from G by reversing the direction of every edge of G. Give algorithms in pseudo code to compute G^T from G, one algorithm for G represented in adjacent-list and one for G represented in adjacent-matrix. Analyze the running times of your algorithms.

4. 10 points (Ex 20.2-4 of text book)

   Modify the BFS discussed in class (slide 14 or section 20.2 of text book) to a breadth first search algorithm in pseudo code for graphs represented by adjacent matrix, and give the running time of the modified breadth first search algorithm.

5. 10 points (P 20-2 of text book)

   An articulation point of an undirected graph G is a node whose removal disconnects G. Let G_π be a depth-first tree of G. Prove that the root of G_π is an articulation point of G if and only if it has at least two children in G_π.

6. 20 points (Ex 21.1-6 of text book)

   Show that a graph G has a unique minimum spanning tree if, for every cut of the graph, there is a unique light edge crossing the cut. Show that the converse is not true by giving a counterexample.

7. 20 points

   Write two programs for Prim's minimum spanning tree algorithm discussed in class, one implements the algorithm with the input graph G represented by adjacent list and the other implements the algorithm with G represented by adjacent matrix. Run your programs on the graph in Figure 1 starting from node a and show the minimum spanning tree found. (10 points) Run your programs on connected edge-weighted G with $n = 100, 200, 400, 800$ nodes, for each n, with $m \approx 3n, n^{1.5}, n(n-1)/2$ edges, and each edge is assigned a random number from $[1, a]$ for some $a > 1$. Report the running times of your programs on a computer. Write a procedure to create input graphs for both implementations and exclude the running time of the procedure from the times of your implmentations for Prim's algorithm. Please do not submit the codes. (Hint: For each n, first create a connected base graph of nodes $v_1, .., v_n$, e.g., a cycle, then add $m - n$ edges $\{u, v\}$ with u, v randomly selected from $\{v_1, .., v_n\}$ to the base graph.)

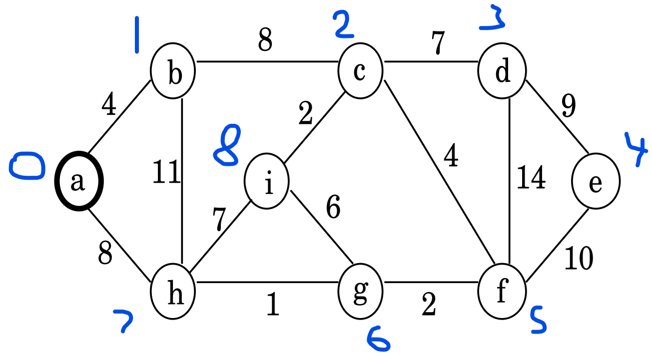

Figure 1: An input instance graph G.