Theory

The frequency response of a system can be obtained by exciting the system with a sinusoidal signal of amplitude A and frequency ω_0, andobserving the response of the output. This output signal will comprise a transient component (which will die, eventually) and a steady-state component. If the input signal to a stable transfer function P(s) is

$$u(t) = \mathrm{A}\cos(\omega_0 t + \theta)\,, \tag{1}$$

then, the steady state component of the output signal is given by

$$y_{ss}(t) = A|P(j\omega_0)|\cos\big(\omega_0 t + \theta + arg(P\,(j\omega_0)\big)\,, \tag{2}$$

Thus, the steady state component of the output signal has a magnitude ratio $|P(j\omega_0)|$ and phase shift $arg\big(\,P(j\omega_0)\big)$ relative to the input signal.

For the Maglev plant transfer function, the expressions for magnitude and phase at a frequency ω are

$$|P(j\omega)| = \frac{(DC)\omega_n^2}{\sqrt{(\omega_n^2-\omega^2)^2+(2\xi\omega_n\omega)^2}}\,, \tag{3a}$$

$$\arg\big(P(j\omega)\big) = -\tan^{-1}\frac{2\xi\omega_n\omega}{(\omega_n^2-\omega^2)}\,,\ \text{(with proper consideration of the quadrant)} \tag{3b}$$

This information can be plotted at each frequency of interest, in the form of Bode plots. The Bode plot comprises two subplots: a subplot of the magnitude ratio vs $\log_{10}\omega$, and a subplot of the phase shift (in degrees) versus $\log_{10}\omega$. Other plots such as the Nyquist and Nichols plot also display the same information, but in somewhat different ways.

In the presence of a proportional controller with proportional gain K_c, the magnitude and phase of the series combination (or open loop transfer function) $K_c P(s)$ at a frequency ω are given by $K_c |P(j\omega)|$ and $\angle P(j\omega)$ respectively. (The proportional controller does not contribute any phase shift; hence the phase shift is only that of the plant).

Stability Margins

Important insight into the stability of the closed-loop system can be achieved by analyzing the gain and phase margins of the system before closing the loop. Gain and phase margins are determined by finding the crossover points of the frequency response, which can be done either analytically or from the interpolation of the response curve computed on a frequency grid.

Phase margin: At the gain crossover frequency (also called as the phase margin frequency, or P. M. frequency) ω_{gc}, the magnitude of the transfer function is unity. Using the magnitude expression given in (3a), we can find the expression for ω_{gc} as

$$x = \omega_n^2 \left\{(2 - 4\xi^2) \pm \sqrt{(2 - 4\xi^2)^2 + 4(K_c^2 (DC)^2 - 1)}\right\}/2 \ , \qquad (4)$$

$$\omega_{gc} = \sqrt{x} \ .$$

The phase margin can be obtained by substituting the value of ω_{gc} in (3b) to get the PM in radians as

$$PM = \pi + \tan^{-1} \frac{2\xi \omega_n \omega_{gc}}{\left(\omega_n^2 - {\omega_{gc}}^2\right)} \ ,$$

To get the PM in degrees, the expression is

$$PM = 180 + \left(\frac{180}{\pi}\right)\tan^{-1}\frac{2\xi\omega_n\omega_{gc}}{(\omega_n^2 - \omega_{gc}{}^2)} \ . \tag{5}$$

Gain Margin:

The gain margin is computed at the phase crossover frequency (also called as the gain margin frequency, or G. M. frequency) $\omega_{pc,}$ which is the frequency at which the phase angle equals -180 degrees.

Remark 1:

For the second order Maglev plant function under P control, the phase plot never crosses the -180 degrees phase value, so the phase crossover frequency (or G. M. frequency) ω_{pc} is infinity. This means that the gain margin GM turns out to be infinity for the maglev plant under P control.

We can confirm this analytically as follows:

The phase cross-over frequency is defined as:

$$\arg(P)\,|\omega_{pc} = -180° \ ,$$

Substituting in equation (3b),

$$-\pi = -\tan^{-1}\frac{2\xi\omega_n\omega_{pc}}{(\omega_n^2 - \omega_{pc}{}^2)} \ ,$$

Taking tan of both sides,

$$\tan(\pi) = 0 = \frac{2\xi\omega_n\omega_{pc}}{\omega_n^2 - \omega_{pc}^2} \ ,$$

which implies that, $\frac{2\xi\omega_n/\omega_{pc}}{\omega_n^2/\omega_{pc}^2} - 1 = 0$, which boils down to

$$\frac{2\xi\omega_n}{\omega_{pc}} = 0 \ . \qquad (6)$$

As the numerator is non-zero, we must have $\omega_{pc} \rightarrow \infty$ for equation (6) to hold. Thus, $\omega_{pc} = \infty$ => Gain Margin GM $=\infty$.