IEOR 4102, HMWK 1, Professor Sigman

1. An asset price starts off initially at price \$7.00 at the end of a day (day 0), and at the end of each consecutive day, independent of the past, the price goes up by one dollar (with probability $p = 0.6$) or down by one dollar (with probability $q = 0.4$).

   (a) What is the probability that the stock will reach \$10.00 before going down to 0?

   (b) What is the probability that the stock will reach \$10.00 before going down to a low of \$2.00?

   (c) What is the probability that the stock will (as time goes on) become infinitely valuable without ever hitting 0?

   (d) (*Continuation:*) Answer (a)– (c) in the case when the two probabilities 0.6 and 0.4 are reversed.

   (e) (*Continuation:*) Answer (a)– (c) in the case when $p = q = 0.5$.

2. An insurance risk business has a reserve of money (in units of millions of dollars). Initially, it has 1 unit. Every day, it earns 1 unit (interest), but also (each day) there is a chance that a claim will come in, independent of past days, of size 2 units with probability 0.45 (with probability 0.55 no such claim comes in).

   (a) What is the probability that the risk business will get ruined (run out of money)?

   (b) What is the smallest value i (units) the business would need to have started with to ensure that the probability of ruin is *less* than 1/2?

3. A jumping bean moves on the integers according to a simple random walk taking one step per unit time. $R_n =$ the position at time $n \geq 0$. Assume that $p = 0.35$; the probability that a step takes the bean forward (to the right), and $q = 1 - p = 0.65$ is the probability that a step takes the bean backward (to the left). It starts off initially at position $R_0 = 4$.

   (a) Does this random walk have positive drift or negative drift?

   (b) What is the probability that the bean will go down to 0 before ever reaching \$5?

   (c) What is the probability that the bean will go below 0 before ever reaching \$5?

   (d) What is the probability that the bean will *never* reach as high as 6.00?

4. As a more realistic model for asset pricing, suppose that the price of an asset moves (day by day) as

   $$X_n = 8 \times 2^{R_n}, \ n \geq 0,$$

   where $R_0 = 0$, and $R_n = \sum_{k=1}^{n} \Delta_k, \ k \geq 1$, is a simple symmetric random walk; $P(\Delta = 1) = 1/2 = P(\Delta = -1)$.

   (a) What is the probability that the asset price reaches a high of 32 before a low of 1/2?

   (b) What is the probability that the asset price will ever reach as high as 2^{500}?

5. *Simulating simple random walks:* For each of $p = .4$ and $p = .55$ and $p = 0.5$: Simulate (using MATLAB or PYTHON) the simple random walks starting from $R_0 = 0$ out to $n = 1000$ steps to:

   (a) Compute R_n/n to see if it is close to $E(\Delta) = 2p - 1$, as it should be by the Strong Law Of Large Numbers.

   (b) Estimate the probability that the random walk will ever reach as high as $a = 100$ by time $n = 1000$. If we define

   $$M_n = \max_{0 \le k \le n} R_k,$$

   the maximum of the random walk during the first n time units (steps), then we want to estimate $P(M_n \ge 100)$, for $n = 1000$.

   Here is the pseudo-code:

   1 With $R_0 = 0$, start simulating the random walk sequentially via $R_{k+1} = R_k + \Delta_{k+1}$.

   2 If (before or at time $n = 1000$) $R_k = 100$, then stop and output $I = 1$.

   3 If $R_k < 100,\ 0 \le k \le n = 1000$, then stop and output $I = 0$

   4 Repeat (1–3) above (independently) $m = 5,000$ times to obtain 5000 independent copies of I, denoted by $I_1, \ldots, I_{5000}$.

   5 Use estimate

   $$P(M_n \ge 100) \approx \frac{1}{m} \sum_{i=1}^{m} I_i.$$