RESEARCH STATEMENT

JAIME SANTOS-RODRÍGUEZ

1. INTRODUCTION

My mathematical research is in the area of geometry, more specifically, curvature bounds on metric spaces. During my PhD I have studied isometric actions on non-smooth spaces that admit a synthetic notion of lower Ricci curvature bound. My results so far include a proof that the isometry group of an $RCD^*(K,N)-$space is a Lie group (joint work wih Prof. Guijarro [7], and independently proved by Sosa in [14]) and that given a compact subgroup G of isometries acting on an $RCD^*(K,N)-$space one can find a $G-$invariant reference measure such that the curvature bound is preserved [12].

I have used techniques coming from metric geometry as well as structural results available for calculus in metric measure spaces. I would like to continue examining problems that involve the presence of symmetries in metric measures spaces. In what follows I will talk about some of the ongoing work and also about some open problems that I would like to address.

2. FUTURE WORK AND ONGOING PROJECTS

2.1. Wasserstein isometries.

This is ongoing work with Prof. Guijarro. Consider a metric measure space $(X, d, \mathfrak{m})$ and its associated Wasserstein space $(\mathbb{P}_2(X), \mathbb{W}_2)$ of probability measures. The geometry of $\mathbb{P}_2(X)$ encloses some of the geometry of X, since there is an isometric embedding of X into $\mathbb{P}_2(X)$ via the map that sends each point $x \in X$ to its point measure δ_x. But it is still unclear to what point the geometry of X restricts the geometry of $\mathbb{P}_2(X)$.

A first approach to this question consists on studying isometry groups. We can use the embedding $X \hookrightarrow \mathbb{P}_2(X)$ to make the isometry group of X, $\mathrm{Iso}(X)$, act isometrically on $\mathbb{P}_2(X)$ via pushforwards. An interesting problem is to see when $\mathrm{Iso}(X) = \mathrm{Iso}(\mathbb{P}_2(X))$; if this happen, we will say that $\mathbb{P}_2(X)$ is *isometrically rigid.*

For Euclidean and Hadamard spaces, this has already been studied by Kloeckner [8] and by Bertrand and Kloeckner [2]. Surprisingly, the real line $\mathbb{R}$ results to be non isometrically rigid, with the presence of a one parameter group of exotic isometries that pull mass away to infinity.

It is then natural to ask what happens when X is compact and has a lower curvature bound. In Alexandrov spaces it is known that if X is non-negatively curved then $\mathbb{P}_2(X)$ is also non-negatively curved (see for example Proposition $2.10, iv)$ in [15]). In general for a non-zero curvature bound only a weaker notion holds as is proved by Ohta in [11]. I propose then the following problem:

Open problem 1. *Let (X, d) be a closed non-negatively curved Alexandrov space, then $\mathbb{P}_2(X)$ is isometrically rigid.*

A first approach addressed in [13] is to what extent isometries of the Wasserstein space can move the embedded image of X:

Date: December 22, 2019.

Theorem 2. *Let (M, g) be a closed Riemannian manifold. Denote by*

$$\Delta_1 := \{\, \mu \in \mathbb{P}_2(M) \,|\, \mu = \delta_x \text{ for some } x \in M\}.$$

Then for all $\Phi \in \operatorname{Iso}\mathbb{P}_2(M)$, we have that $\Phi(\Delta_1) = \Delta_1$.

The proof relies on a Theorem of Gigli [4] which characterizes regular measures as those that have Tangent spaces isometric to Hilbert spaces. In order to extend the previous result to Alexandrov spaces it would be sufficient to prove the following:

Open problem 3. *Let (X, d) be a non-negatively curved Alexandrov space and $\mu \in \mathbb{P}_2(X)$. Then the following are equivalent:*

(1) The tangent space $Tan_\mu \mathbb{P}_2(X)$ at μ is a Hilbert space.
(2) The functional $\mathbb{W}_2^2(\mu, \cdot) : \mathbb{P}_2(X) \to [0, \infty)$ is strictly convex.

Studying how isometries behave when restricted to measures supported on geodesics has proved useful in describing $\operatorname{Iso}\mathbb{P}(M)$. (see [8] and [2]). I propose then:

Open problem 4. *Let $\Phi \in \operatorname{Iso}\mathbb{P}_2(M)$ such that $\Phi(\delta_x) = \delta_x$ for all $x \in M$. Let $\gamma \in \operatorname{Geo}(M)$ and denote by $\mathbb{P}(\gamma)$ be the set of measures supported at γ. Then $\Phi(\mathbb{P}(\gamma)) = \mathbb{P}(\gamma)$.*

This is seen in [13] to hold under the assumption that $Sec(M) > 0$, however it should still be true for $Sec(M) \geq -1$.

2.2. **Isometric quotients by non-compact groups.**

The study isometric Lie group actions on metric measure spaces arises from the papers [7] and [14] where it is proven that the isometry group of an $RCD^*(K, N)$ space is a Lie group.

Later on, in [3] Galaz-García, Kell, Mondino and Sosa studied isometric actions in the case when the group is compact and acts by measure preserving isometries. Therefore, it is natural to study the case when $G \leq \operatorname{Iso}(X)$ is a non-compact subgroup acting on an $RCD^*(K, N)$ space $(X, d, \mathfrak{m})$.

However, even if one assumes that G acts by measure preserving isometries, it is not clear that $(X/G, d^*)$ has bounded lower Ricci curvature. The first issue we encounter is that if $p : X \to X/G$ is the quotient map, then $p_\#\mathfrak{m}$ is not necessarily a Radon measure. One possible way to solve this may be by answering the following conjecture:

Open problem 5. *There exists a density $\rho \in \mathfrak{L}^1_{loc}(\mathfrak{m})$ such that $(X/G, d^*, p_\#(\rho\mathfrak{m}))$ is an $RCD^*(K, N)$ space.*

For Riemannian manifolds it is known that sectional curvature is non-decreasing under a Riemannian submersion. The same is true for Alexandrov spaces and submetries. In [10], Lott addressed this same problem in the context of manifolds with lower bounds on their Bakry-Émery tensor. More concretely, he considered Riemannian submersions such that the transport map between the fibers preserves the induced volume measure up to constants.

It would be interesting to find something similar for metric measure spaces that admit a lower Ricci curvature bound. In [3] this was studied under the assumption that the fibers are compact. Following [5], we consider the following

Definition 6. Let $(X, d_X, \mathfrak{m})$ and $(Y, d_Y, \mathfrak{n})$ be metric measure spaces. A map $F : X \to Y$ will be a called a *map of bounded deformation* if

- F is a submetry.
- There exists $C > 0$ such that $F_\#\mathfrak{m} \leq C\mathfrak{n}$.

Open problem 7. *Find appropriate conditions such that the synthetic Ricci curvature bound is non-decreasing under maps of bounded deformation.*

2.3. **Upper Ricci curvature bounds.**
In [16], Sturm discusses various characterizations of upper Ricci curvature bounds on metric measure spaces. His definition involves requiring semiconcavity of the entropy functional along Wasserstein geodesics with endpoints supported in arbitrarily small neighbourhoods of given points.

Although it is known that there are no topological obstructions for a manifold to admit a metric with Ricci bounded from above there are some geometric restrictions. I believe that the same can be said in the context of metric measure spaces.

In relation to isometric Lie group actions, we may try to extend the classic theorem of Bochner:

Open problem 8. *Let $(X, d, \mathfrak{m})$ be a compact metric measure space with Ricci $\leq K < 0$. Then its isometry group is finite.*

Another problem that I find interesting is defining metric measure spaces that that are Einstein, that is, spaces with constant Ricci curvature. A first naive attempt at this would be to ask for equality instead of convexity of entropy functionals in the definition of an $RCD^*(K, N)$ space. However, by doing this, certain spaces appear that intuitively should not be considered "Einstein." For example consider the Euclidean cone over $M = (\frac{1}{\sqrt{3}}\mathbb{S}^2) \times (\frac{1}{\sqrt{3}}\mathbb{S}^2)$ equipped with the measure $dm_5(x, s) = dVol_M(x) \otimes s^5 ds$. Then the punctured cone has vanishing Ricci curvature. (See Example 1 in [1] for details).

The advantage of Sturm's definition of upper Ricci curvature bound is that it detects the curvature concentrated at the vertex of the cone. (see Theorem 2.5 and Corollary 2.6 [16]). Therefore the natural definition to give is:

Definition 9. A metric measure space $(X, d, \mathfrak{m})$ which is both $RCD^*(K, N)$ and has a synthetic upper Ricci curvature bound by K will be called an *Einstein metric measure space*.

We consider the general problem to be far from reach. Therefore, we simplify it by looking at small dimensions; we think that it would be interesting to prove the following conjecture.

Open problem 10. *Let $(X, d, \mathfrak{m})$ be an Einstein metric measure space. Assume that $N \leq 2, 3$. Then X is locally isometric to domains in space forms with constant curvature.*

2.4. **Super-Ricci flow and isometries.**
A time dependent metric measure space $(X, d_t, \mathfrak{m}_t)_{t\in[0,T)}$ consists of a Polish space X equipped with

- A $1-$parameter family of intrinsic metrics d_t which generate the topology of X.
- A $1-$parameter family of Borel measures $\mathfrak{m}_t$, absolutely continuous with respect to each other.

In [17], Sturm defined the notion of Super-Ricci flow of $(X, d_t, \mathfrak{m}_t)_{t\in[0,T)}$. Briefly, a time dependent metric measure space will be a Super-Ricci flow if the time dependent Boltzmann entropy

$$S : (t, \mu) \to \mathrm{Ent}(\mu \,|\, \mathfrak{m}_t)$$

is strongly dynamical convex. (Refer to [17] for details).

It is known that if $(M, g_t)_{t\in[0,T)}$ is a Ricci flow of complete metrics with bounded curvature, then the isometry group of the starting manifold remains unchanged under the flow, i.e. $\mathrm{Iso}(M, g_0) = \mathrm{Iso}(M, g_t)$ for all $t \in [0, T)$. It is of interest then to see if this is still true in a non-smooth context.

Open problem 11. *Let* $(X, d_t, \mathfrak{m}_t)_{t\in[0,T)}$ *be a Super-Ricci flow, then*

$$\mathrm{Iso}(X, d_0, \mathfrak{m}_0) = \mathrm{Iso}(X, d_t, \mathfrak{m}_t) \quad \textit{for all } t \in [0, T).$$

In order to tackle this problem it will be useful for to use the equivalent formulations of a Super-Ricci Flow developed in [9] by Kopfer and Sturm. More precisely we will use the characterization of the heat flow on functions as the unique forward EVI-flow for the (time-dependent) energy in L^2-Hilbert space and the dual heat flow on probability measures as the unique backward EVI-flow for the (time-dependent) Boltzmann entropy in L^2-Wasserstein space. Therefore we will study the effect of $\mathrm{Iso}(X, d_0, \mathfrak{m}_0)$ on the EVI-flows.

References

[1] Bacher, K.; Sturm, K.-T., *Ricci bounds for Euclidean and spherical cones.* Singular phenomena and scaling in mathematical models, 3-23, Springer, Cham, 2014.

[2] Bertrand, J.; Kloeckner, B., *A geometric study of Wasserstein spaces: Hadamard spaces.* J. Topol. Anal. 4 (2012), no. 4, 515-542.

[3] Galaz-García, F.;Kell, M.; Mondino, A.; Sosa, G.,*On quotients of spaces with Ricci curvature bounded below,* Journal of Functional Analysis (2018) https://doi.org/10.1016/j.jfa.2018.06.002

[4] Gigli, N., *On the inverse implication of Brenier-McCann theorems and the structure of* $(\mathbb{P}_2(M), W_2)$.

[5] Gigli, N., *Nonsmooth differential geometryan approach tailored for spaces with Ricci curvature bounded from below.* Mem. Amer. Math. Soc. 251 (2018), no. 1196, v+161 pp. ISBN: 978–1–4704–2765–8; 978–1–4704–4266–8

[6] Gigli, N.; Mondino, A.; Rajala, T., *Euclidean spaces as weak tangents of infinitesimally Hilbertian metric measure spaces with Ricci curvature bounded below* J. Reine Angew. Math. 705 (2015), 233-244.

[7] Guijarro, L; Santos-Rodríguez J., *On the isometry group of* $RCD^*(K, N)$*-spaces,* Manuscripta Math. (2018). https://doi.org/10.1007/s00229-018-1010-7

[8] Kloeckner, B., *A geometric study of Wasserstein spaces: Euclidean spaces.* Ann. Sc. Norm. Super. Pisa Cl. Sci. (5) 9 (2010), no. 2, 297-323.

[9] Kopfer, E., Sturm, K.-T. *Heat flows on Time-dependent Metric Measure Spaces and Super-Ricci Flows.* Comm. Pure Appl. Math. 71 (2018),no. 12 ,2500–2608

[10] Lott, J., *Some geometric properties of the Bakry-Émery-Ricci tensor.* Comment. Math. Helv. 78 (2003), no. 4, 865-883.

[11] Ohta, S., *Gradient flows on Wasserstein spaces over compact Alexandrov spaces.* Amer. J. Math. 131 (2009), no. 2, 475-516.

[12] Santos-Rodríguez, J., *Invariant measures and lower Ricci curvature bounds.* Potential Analysis (2019)https://doi.org/10.1007/s11118-019-09790-y

[13] Santos-Rodríguez, J., *Wasserstein isometries on positively curved Riemannian manifolds.* (In preparation).

[14] Sosa, G., *The Isometry Group of an* RCD^* *space is Lie.* Potential Anal (2018) 49:267. https://doi.org/10.1007/s11118-017-9656-4

[15] Sturm, K.-T., *On the geometry of metric measure spaces I*, Acta Math. 196 (2006), no. 1, 65-131.

[16] Sturm, K.-T., *Remarks about synthetic upper Ricci bounds for metric measure spaces,* arXiv:/1711.01707

[17] Sturm, K.-T., *Super-Ricci flows for metric measure spaces.* J. Funct. Anal. 275 (2018), no. 12, 3504-3569.

Department of Mathematics, Universidad Autónoma de Madrid and ICMAT CSIC-UAM-UCM-UC3M, Spain

E-mail address: jaime.santos@icmat.es