First-Year Electromagnetism: Problem Set 1

Hilary Term 2020, Prof Neville Harnew[1]

A. Electric Fields, Potentials and the Principle of Superposition

A.0 Background. State the definition of the electric field and potential and derive their relationship. Explain how the principle of superposition applies to charge distributions in electrostatics.

A.1 Assembly of point charges in the corners of a square. Charges $+q$, $+2q$, $-5q$ and $+2q$ are placed at the four corners ABCD of a square of side a, taken in cyclic order.

(a) Find the electric field $\mathbf{E}$ and the potential V at the centre of the square and verify that they are related by $\mathbf{E} = -\nabla V$.

(b) What is the potential energy of the charge configuration, i.e. the work done in assembling the configuration, starting with all the charges at infinity?

[Answers: $12q/(4\pi\epsilon_0 a^2)$ towards C; 0; $-q^2(32+\sqrt{2})/8\pi\epsilon_0 a$]

A.2 Electric dipole. Two point charges $\pm q$ are placed at points $(0,0,\pm d/2)$, defining an electric dipole moment $\mathbf{p} = q\mathbf{d}$.

(a) Using spherical polar coordinates, show that the potential V a large distance $r = |\mathbf{r}|$ from the dipole is given by

$$V = \frac{\mathbf{p}\cdot\mathbf{r}}{4\pi\epsilon_0 r^3}$$

(b) Derive expressions for the electric field vector $\mathbf{E} = (E_r, E_\theta, E_\phi)$ for large r.

(c) Determine the energy W of a dipole placed with its moment $\mathbf{p}$ at an angle α to the direction of an external electric field $\mathbf{E_{ext}}$.

A.3 Assembly of point charges on a line; multipoles. A system of charges consists of one charge $+q_2$ at the origin and two charges $-q_1$ at points (0, 0, $\pm a$).

(a) Using spherical polar coordinates, find the potential $V(r,\theta,\phi)$ created by these charges, taking θ to be the angle between $\mathbf{r}$ and the z-axis.

(b) Expand the potential as a power series in a/r, retaining only terms up to the second order. State which parts of your expression have monopole, dipole and quadrupole character.

(c) For the case of $q_2 = 2q_1$, state the potential and derive expressions for the radial and angular components of the associated electric field.

[1]With thanks to Prof Laura Herz

A.4 Uniformly charged rod. A thin rod of length $2l$ is uniformly charged with charge λ per unit length. By integrating the electric field components originating from small elements of the rod, calculate the total electric field outside the rod for:

(a) any point on the line of the rod (but beyond its ends) as a function of distance z from its mid point.

(b) any point a perpendicular distance x away from the midpoint of the rod.

A.5 Uniformly charged disk. A thin, circular disk has radius b and carries a surface charge density σ. Consider the disk to lie in the x-y-plane with its centre at the origin.

(a) Find the electric field $\mathbf{E}$ for any point P on the z-axis.

(b) What are the values of $\mathbf{E}$ for the limiting cases of $z \ll b$ and $z \gg b$?

(You can solve this problem by calculating the field at P arising from a ring of charge of radius r and width dr, and then integrating from $r = 0$ to $r = b$.)

A.6 Uniformly charged ring. The disk in the previous question is replaced by a thin, uniformly charged ring of radius a carrying charge q.

(a) Determine the points on the axis of the ring for which the magnitude of the electric field $|E_z|$ reaches its maximum value.

(b) Show that an electron placed on the z-axis at a small distance ($z \ll a$) from the centre of the ring will oscillate with frequency

$$\nu = \sqrt{\frac{eq}{16\pi^3\epsilon_0 a^3 m}}.$$

A.7 Uniformly charged hollow sphere. A charge is distributed uniformly with density σ over the surface of a hollow conducting sphere of radius a. Show by direct integration over contributions arising from infinitesimal surface elements of the sphere that the potential at any point P inside it is given by $a\sigma/\epsilon_0$.

(Hint: orienting the z-axis to contain P and using polar coordinates will make your life easier.)

First-Year Electromagnetism: Problem Set 2

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B. The Method of Image Charges

B.0 Background. Explain the factors that determine the parallel and normal components of the electric field near the surface of a flat metal plate.

B.1 Charge monopole near a flat metal surface. A point charge q is placed at a perpendicular distance d from a point O on a flat, infinite plate that is conducting and earthed.

(a) Use the method of images to show that the magnitude E of the electric field at the point P, a distance r along the plane from O, is

$$E(r) = \frac{qd}{2\pi\epsilon_0(r^2 + d^2)^{3/2}}.$$

(b) Sketch the resulting lines of the electric field and calculate the force F between the charge and the plate.

(c) Show that the charge on the plane is $-q$.

(d) Find the work done in moving the charge to an infinite distance from the plane. Hence find the minimum energy an electron must have in order to escape from a metal surface (assume that it starts at a distance 0.1 nm, which is about one atomic diameter above the surface). Express your answer in electron volts.

B.2 Two charges near a flat metal surface. Two charges $+Q$ and $-Q$ are a horizontal distance a apart and a vertical distance b above a large conducting sheet. Find the components of the forces acting on each charge.

B.3 Charge monopole near two orthogonal metal surfaces. Two semi-infinite plane conducting plates are joined together at right angle. A charge Q is situated near the join at a distance a from each plate.

(a) Show that the electric field is zero at any point along the join.

(b) Find the field just above the surface of one of the plates at the point closest to the charge.

(c) Sketch the equipotentials near the charge and near the plate.

(d) Calculate the surface charge density on the metal plates at the points closest to the charge Q.

(Hint: you need to consider three image charges.)

B.4 Uniformly charged rod near a metal surface. An infinite, thin, uniformly charged rod (line charge density λ) is placed parallel to a metal plate a distance d above it. Use the result obtained in Problem A.4 to calculate the electric field at a point P closely along the surface of the plate as a function of the distance $\overline{PM}$, where M is the closest midpoint between the charge and the image charge.

[1]With thanks to Prof Laura Herz

C. Electric Fields derived from Gauss' Law

C.0 Background. State Gauss' Law and explain how it may be used to determine the electric field arising from a spherically symmetric charge density distribution $\rho(r)$.

C.1 Uniformly charged sphere.

(a) Charge $+q$ is distributed uniformly throughout the volume of a sphere of radius a. Show that the electric field $\mathbf{E}$ and potential V at a distance r from the centre of the sphere are given by:

$$\mathbf{E} = \begin{cases} \dfrac{qr}{4\pi\epsilon_0 a^3}\,\hat{\mathbf{r}} \\ \\ \dfrac{q}{4\pi\epsilon_0 r^2}\,\hat{\mathbf{r}} \end{cases} \quad \text{and} \quad V = \begin{cases} \dfrac{q}{4\pi\epsilon_0 a^3}\left(\dfrac{3a^2}{2} - \dfrac{r^2}{2}\right) & \text{for } 0 \le r \le a \\ \\ \dfrac{q}{4\pi\epsilon_0 r} & \text{for } a \le r \end{cases}$$

(b) Repeat the calculation of fields and potentials for the charge now being uniformly distributed over the surface of a sphere of radius a.

(c) Draw graphs of the electric fields (magnitude) and the potentials for both cases (solid sphere and shell). Take care to illustrate the relation $E = -(\partial V/\partial r)$ everywhere and account for any discontinuities that occur at $r = a$.

C.2 Coulomb energy of the nucleus. The nucleus of an atom can be considered to be a charge $+Ze$ uniformly distributed throughout a sphere of radius a.

(a) Show that the potential energy W of a nucleus arising from the assembly of its charge is given by $W = 3\,(Ze)^2(20\,\pi\epsilon_0 a)^{-1}$.

(b) What would the potential energy be if the charge was instead spread uniformly over the surface of the nucleus?

C.3 Electron in a hydrogen atom. From a quantum mechanical treatment, the potential at a distance r from the nucleus that is generated by an electron in a hydrogen atom is given by:

$$V = \frac{q}{4\pi\epsilon_0}\left(\frac{\exp(-2r/a) - 1}{r} + \frac{\exp(-2r/a)}{a}\right)$$

where a is a constant and is a measure of the "size" of the atom.

(a) Sketch $V(r)$ for $0 \le r \le \infty$ and comment on the shape of the curve.

(b) Find the magnitude of the electric field at a distance $r \ll a$ from the nucleus.

(c) Show that, when an external electric field E_{ext} is applied, the atom develops a dipole moment of magnitude p (you may assume that the electron cloud remains spherical and merely moves relative to the nucleus). By considering the force on the nucleus, calculate p and show that the polarisability p/E_{ext} is equal to $3\pi\epsilon_0 a^3$.

(d) For the hydrogen atom, $a = 0.5 \times 10^{-10}$ m. Show that even for the largest accessible fields of $\sim 10^6$ Vm^{-1} the electron charge cloud moves relative to the nucleus by only about 10^{-17} m (which justifies the assumption $r \ll a$).

(e) Use Gauss law to calculate the total charge in the cloud.

First-Year Electromagnetism: Problem Set 3

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D. Capacitance and Electric-Field Energy

D.0 Background. State the definition of capacitance, and derive the electric field energy stored inside a capacitor.

D.1 Spherical and cylindrical capacitors. Use Gauss' law to derive expressions for the capacitance C of:

(a) two concentric spherical conducting shells with radii R_1 and R_2.

(b) two very long, coaxial cylindrical conducting shells with radii R_1 and R_2 and axial length L.

Show that as $|R_2 - R_1|$ becomes small, the systems described in (a) and (b) become equivalent to a parallel-plate capacitor.

D.2 Charge distribution inside a parallel-plate capacitor. A capacitor consists of two parallel large conducting planes separated by a distance d. The space between the plates is filled with a uniform, immobile space charge of density ρ. Find the magnitude of the electric field at a distance x from the positive plate when a potential difference V is applied to the capacitor. Discuss whether the capacitance of the capacitor is affected by the presence of the space charge.

[Answer: $(V/d) - \rho(d - 2x)/2\epsilon_0$]

D.3 Air breakdown thresholds inside a cylindrical capacitor. A capacitor consists of two air-spaced concentric cylinders, similar to that described in Problem D.1(b). The outer radius is fixed at b=10 mm, while the inner radius a is variable. Electric-field induced breakdown of air will occur for field strengths exceeding E_b=3 MVm^{-1}. Show that

(a) $a = b/e$ is required for maximixed potential difference across the capacitor

(b) $a = b/\sqrt{e}$ is needed for maximized energy per unit length stored in the capacitor

in the absence of air breakdown. [Here e is Euler's number.]

D.4 Forces between capacitor plates. Two parallel plates of a capacitor are spaced 10 mm apart, have an effective area of 0.01 m^2 each, and a potential difference of 600 V maintained between them through a connected battery.

(a) Determine the force between the two plates by considering the energy stored or supplied to the system. [Answer: 1.59×10^{-4} N]

(b) Find the work done when the plates are pulled apart slowly to a separation of 20 mm

i. while the potential difference is maintained at 600 V, and

ii. while the plates are charged to 600 V and isolated before separation.

[1]With thanks to Prof Laura Herz

(c) For case (i), show that the energy stored in the field between the plates is halved when they are slowly pulled apart. Yet work is done in pulling them apart against the attractive force between them. Explain.

E. Magnetostatics

E.0 Background. Describe how the laws of Biot-Savart and Ampère may be used to calculate the magnetic field generated by an electrical current. Explain how you would decide which of the two was the most appropriate to use for a given situation.

E.1 Magnetic fields from straight current segments and polygons.

(a) A straight wire of length $2b$ carries a current I. Find the magnitude of the magnetic field H at a distance a from the wire along its perpendicular bisector.

(b) N equal straight wires, carrying current I, form a closed regular polygon circumscribed about a circle of radius a. Calculate the magnetic field at the centre of the polygon and show that it takes the value at the centre of a circular loop of radius a when $N \to \infty$. [Answer: $NI\sin(\pi/N)/2\pi a$]

E.2 On-axis magnetic field of a coil and of a pair of Helmholtz coils

(a) A circular coil of N turns has radius a and negligible cross-section. Show that when a current I passes through it, the magnitude of the magnetic field H at a point on its axis a distance x from the centre of the coil is given by:

$$H = \frac{NIa^2}{2(a^2+x^2)^{3/2}}$$

(b) Hence show that when the current passes in the same direction in two identical coaxial coils separated by a distance equal to their radius, there is a small region on the axis mid-way between the coils in which the first, second and third differential coefficients of the field with respect to x are all zero. What is the practical importance of such a Helmholtz coil system?

E.3 In-plane magnetic field of a coil. A plane circular coil of N turns (of negligible cross-section) carries current I and has radius a. Calculate the magnetic flux density at a point lying in the plane of the coil, a distance $r \gg a$ from the centre of the coil. Show that the same result is obtained by substituting a magnetic dipole for the coil. [Answer: $\mu_0 NIa^2/4r^3$]

E.4 Magnetic field inside a solenoid A solenoid of radius a and length l is uniformly wound with n closely spaced turns of wire per unit axial length. A current I flows through the wire. The magnetic flux density on the axis of the solenoid is measured to be $\sqrt{2}$ times as large at the centre as it is at the ends of the solenoid. What is the ratio of l to a?

E.5 Magnetic field of a long cylindrical wire. A uniform current density J exists along the z-direction between cylinders $x^2+y^2=a^2$ and $x^2+y^2=b^2$.

(a) Determine the magnitude of the magnetic field in the regions (i) $r<a$, (ii) $a<r<b$, and (iii) $r>b$.

(b) Sketch the dependence of the magnetic field magnitude on r.

First-Year Electromagnetism: Problem Set 4

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F. Electromagnetic Induction and Self-Inductance

F.0 Background. State the laws of Faraday and Lenz. What is meant by the terms "self-inductance" and "mutual inductance"?

F.1 Rectangular coil moving away from current-carrying wire. A very long straight wire carries a current I. A plane rectangular coil of high resistance, with sides of length a and b, is coplanar with the wire. One of the sides of length a is parallel to the wire and a distance D from it; the opposite side is further from the wire. The coil is moving at a speed v in its own plane and away from the wire.

(a) Find the value of the e.m.f. induced in the coil by two methods:

i. by considering the e.m.f. induced in each of the sides separately;

ii. by considering the rate of change of magnetic flux through the loop.

(b) In this problem the resistance of the coil is stated as being "high". Why is this restriction necessary in order to find the answer given?

(c) The resistance of the coil is R. Calculate the force needed to move the coil with speed v as described, and show that the mechanical power used to move it is equal to the rate at which heat is generated in the coil.

[Answers: $\mu_0 Ivab/(2\pi D(D+b))$, $(\mu_0 Ivab)^2/(4\pi^2 RD^2(D+b)^2)$]

F.2 Sliding metal rod defining the edge of a circuit. Two horizontal metal rails, separated by a distance L, run parallel to the x-axis. At $x = 0$ a resistor R is connected between the rails. A closed circuit is formed by a metal rod which slides along the rails with a constant velocity v such that its position at time t is given by $x = vt$. There is a constant magnetic flux density B perpendicular to the plane of the rails. Neglecting the resistance of the rails and the rod, and the self-inductance of the circuit:

(a) calculate the current induced in the circuit;

(b) calculate the external force required to maintain steady motion of the rod;

(c) calculate the power P_1 supplied to maintain the steady motion of the rod;

(d) compare P_1 with the power P_2 dissipated in the resistor and comment on the result.

F.3 Homopolar generator. A conducting circular disc of radius a=3 m and mass m=10^4 kg rotates about its axis with angular frequency ω=3000 min^{-1} in a uniform field of magnetic flux density B=0.5 T parallel to its axis.

(a) Show that the potential difference V between the axis and the rim of the disc is $\omega a^2 B/2$.

[1] With thanks to Prof Laura Herz

(b) A load resistor of R=$10^{-3}\,\Omega$ is connected suddenly between the rim and the axis of the disc. What is the initial value of the current in the load (neglecting any other resistance in the circuit)? How long would it take for the flywheel to slow to half its initial speed in the absence of mechanical friction?

F.4 Self-inductance of a coax-cable. A co-axial cable is made from concentric cylindrical conductors. The radius of the inner conductor is a and the outer conductor has an inner radius b and an outer radius d. Calculate the self inductance per unit length of the cable. You may assume that $(b-a) \gg a$ and that $(b-a) \gg (d-b)$. Why is this assumption necessary?

F.5 Mutual induction between a small and a large coil. A small coil of N turns and area A, carrying a constant current I, and a circular ring with radius R have a common axis. The small coil moves along the axis so that its distance from the centre of the ring is given by $d = d_0 + a\cos(\omega t)$. Show that the voltage V induced in the ring is given by:

$$V = \frac{3}{2}\,\mu_0 N A I \omega\,\frac{aR^2 d}{(R^2+d^2)^{5/2}}\,\sin(\omega t).$$

F.6 Mutual inductance of two coaxial solenoids Two coaxial, completely overlapping solenoids, each having n turns per unit length and total length l, have radii a and $2a$.

(a) For each individual solenoid carrying a current I, find the magnetic flux density B generated within the solenoid at any point far removed from the ends.

(b) Neglecting end effects, calculate the self-inductance of each coil and the mutual inductance of the coils.

(c) The outer coil has a self-inductance of 40 mH. Calculate the e.m.f. induced in the inner coil when a current in the outer coil collapses at a constant rate of $2\,\mathrm{As}^{-1}$.

F.7 Energy of the magnetic field. Show that the energy stored in an inductor of self-inductance L carrying a current I can be written as $\frac{1}{2}LI^2$. Show that hence the magnetic energy per unit volume associated with the magnetic flux density B inside the coil can be expressed as $\frac{1}{2}B^2/\mu_0$.

First-Year Electromagnetism: Problem Set 5

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G. Motion of Charged Particles

G.0 Background. State an expression for the force experienced by a charged particle in the presence of electric and magnetic fields.

G.1 Bainbridge mass spectrometer. In a Bainbridge mass spectrometer, ions of charge q, mass m and velocity v enter an initial velocity filter, in which they experience both a uniform E and B-field. They subsequently pass through an aperture into a chamber in which only the B-field is present. Depending on their mass, ions are potentially recorderd by a detector situated behind an exit slit.

(a) Make a sketch of the spectrometer, indicating the orientation of E- and B-fields required for operation.

(b) If the spectrometer were used with E=100 V cm^{-1} and B=0.2 T, what would be the velocity of an ion that can pass through the velocity filter?

(c) If the ion beam leaving the velocity filter has a width of 1 mm, could this machine be used to separate two isotopes of helium, He^3 and He^4?

G.2 Charged particles moving in a constant magnetic field.

(a) Show that the path of a charged particle, moving in a constant and uniform magnetic field, is, in general, a helix.

(b) Particles with a charge e and mass m are emitted with velocity v from a point source. Their directions of emission make a small angle with the direction of a uniform constant flux density B. Show that the particles are focussed to a point at a distance $2\pi mv/Be$ from their source and at integral multiples of this distance.

(c) An electron in interstellar space has a component of velocity $0.01c$ in the direction of a magnetic flux density of 10^{-9} T. How many revolutions does it make in its helical path in travelling between two points of space one light-year apart, measured along a line of force? For an electron $e/m = 1.76\times10^{11}$ C kg^{-1}. [Answer: 8.8×10^{10} treated non-relativistically]

G.3 Magnetic quadrupole lens. A particle of mass m and charge q is projected along the z-direction with speed v, in a path close to the z-axis. It enters a long magnetic quadrupole lens, within which the magnetic flux density components are given by $B_x = Ay$, $B_y = Ax$ and $B_z = 0$, where A is a constant.

(a) Write down the equations of motion for the $x-$ and $y-$components of the particle's velocity v. Assume that the magnet is free from end effects, and that the particle's path always makes a small angle with the z-direction.

(b) Show that the lens has a focusing property in one plane, but is defocusing in the other.

[1]With thanks to Prof Laura Herz

(c) Show that for the focusing plane, the particle first meets the z-axis after travelling a distance $\dfrac{\pi}{2}\left(\dfrac{mv}{|qA|}\right)^{1/2}$

H. Electro-Magnetic Fields and Maxwell's Equations

H.0 Background. State Maxwell's equations in the presence of free charge and electric current and comment on the information each one contains.

H.1 Displacement current.

(a) Calculate the magnetic field around a straight, long current-carrying wire, using Ampere's law. Discuss whether an identical method can be used for calculating the magnetic field generated by a short segment of such a wire.

(b) Use a modified version of Ampere's law that includes a suitable displacement current, in order to determine the magnetic field H at a distance a from a straight current-carrying wire (length $2b$) along its perpendicular bisector. Compare your result with the one you previsouly obtained from Biot-Savart's law (Question E.1a).

H.2 Electro-magnetic waves in vacuo.

(a) Show that Maxwell's equations, in a vacuum devoid of charges and currents, lead to wave equations for the electric and magnetic fields.

(b) Show that plane wave solutions may be obtained and deduce the speed of propagation of these waves.

(c) Obtain expressions for the magnetic flux density vector $\mathbf{B}$ if the electric field vector is described by $\mathbf{E} = (E_0 \sin[kz - \omega t], 0, 0)$.

(d) Determine the characteristic impedance $|E|/|H|$ of free space.

H.3 Poynting vector of an electro-magnetic wave.

(a) Calculate the Poynting vector for plane electromagnetic waves propagating in free space.

(b) The sun has a total radiative power of 3.83×10^{26} W. Electromagnetic waves emanating from the sun take 8.3 min to reach the Earth's atmosphere. Calculate the magnitude of the pointing vector on the Earth's surface when 30% of the incident sunlight is absorbed by the atmosphere.

H.4 Poynting vector for a long resistive rod. Consider a long resistive rod of length l, radius r and resistance R, which carries a current I flowing uniformly through its circular cross section. Calculate the magnitude and direction of the Poynting vector (neglecting edge effects) and relate the rate of energy transfer between the rod and its exterior to the total power dissipated in the rod.