Grado en Ing. Informática — Grado en Matemáticas
Examen Final de Cálculo Infinitesimal
Convocatoria extraordinaria
Curso 2020–2021

Nombre y apellidos: _____

Titulación: _____

1. (*0.75 puntos*) Determinad la región en el plano que determina la relación
$$z - \bar{z} = i.$$

2. (*1.5 puntos*) Razonad si son verdaderas o falsas las afirmaciones siguientes:
   (a) La sucesión recurrente $a_1 = 1$, $a_{n+1} = a_n + \dfrac{1}{n}$ es monótona creciente.
   (b) Todas sucesión monótona creciente es convergente.
   (c) $\lim\limits_{n\to\infty} \left(1 + \dfrac{1}{n^2}\right)^{n^2} = e.$

3. (*0.75 puntos*) Estudiad la convergencia de la serie
$$\sum_{n=1}^{\infty} (\sqrt{2n+1} - \sqrt{2n}).$$

4. (*1.5 puntos*) Demostrad sin usar relaciones trigonométricas
$$\operatorname{arctg} \frac{\operatorname{sen} x}{1 + \cos x} = \frac{1}{2}x, \quad x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right).$$
   Ayuda: derivad ambos lados de la igualdad.

5. (*1.5 puntos*) Desarrollad en serie de potencias de $x - \pi/6$ la función
$$f(x) = \operatorname{sen} x,$$
   indicando el radio de convergencia. Calculad $f^{(2021)}(\pi/6)$.

6. (*2 puntos*) Calculad las primitivas siguientes
   (a) $\displaystyle\int \frac{x^4}{x^4 + 5x^2 + 4}\, dx,$ (b) $\displaystyle\int \operatorname{tg}^3 x\, dx.$

7. (*2 puntos*) Calculad el valor de $a \geq 0$ para que el volumen generado al girar alrededor del eje OX la función $f(x) = \log x + \dfrac{a}{x}$ entre $x = 1$ y $x = e$, sea $V = (e-2)\pi$.

Tiempo para realizar el examen: **2,5 horas** .

El examen debe realizarse a bolígrafo azul o negro, nunca a lápiz.

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Nombre y apellidos:DNI:Fecha://Titulación:Asignatura:

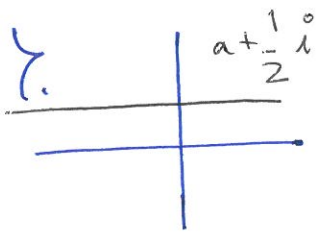
1) $z - \bar{z} = i$

$a+bi - (a-bi) = i$

$a+bi - a + bi = i$

$2bi = i$, $2b = 1$, $b = 1/2$, $a \in \mathbb{R}$

$R = \{ a + \frac{1}{2} i \mid a \in \mathbb{R} \}$.



2) a) Cierto. inducción $a_{n+1} \geq a_n$

$a_2 = a_1 + \frac{1}{1} = 1+1 = 2 \geq a_1 = 1$

$a_{n+2} = a_{n+1} + \frac{1}{n+1} \geq a_{n+1}$

b) Falso $a_n = n$, es monotona creciente y no convergente

c) $\lim_{n \to +\infty} \left(1 + \frac{1}{n^2}\right)^{n^2} = e$ cierto.

$\lim_{n \to +\infty} a_n^{b_n} = \lim_{n \to +\infty} e^{(a_n - 1) b_n} = e^1$

$\lim_{n \to +\infty} \left(1 + \frac{1}{a_n}\right)^{a_n} = e$ $a_n \to +\infty$

3) $\sum_{n=1}^{+\infty} (\sqrt{2n+1} - \sqrt{2n})$

$\sqrt{2n+1} - \sqrt{2n} = \frac{\sqrt{2n+1} + \sqrt{2n}}{\sqrt{2n+1} + \sqrt{2n}} \cdot \frac{\sqrt{2n+1} - \sqrt{2n}}{\sqrt{2n+1} + \sqrt{2n}} = \frac{2n+1-2n}{\sqrt{2n+1} + \sqrt{2n}} =$

$= \frac{1}{\sqrt{2n+1} + \sqrt{2n}} = a_n$, $b_n = \frac{1}{\sqrt{n}} = \frac{1}{n^{1/2}}$

$\lim_{n \to +\infty} \frac{a_n}{b_n} = \lim \frac{\frac{1}{\sqrt{2n+1} + \sqrt{2n}}}{\frac{1}{\sqrt{n}}} = \lim \frac{\sqrt{n}}{\sqrt{2n+1} + \sqrt{2n}} = \frac{1}{\sqrt{2} + \sqrt{2}} = \frac{1}{2\sqrt{2}}$

$\sum a_n = +\infty \iff \sum b_n = +\infty$, por tanto diverge

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4) $\operatorname{arctg} \frac{\operatorname{sen} x}{1+\cos x} = \frac{1}{2} x$, $x \in (-\pi/2, \pi/2)$.

$$\frac{\left(\frac{\operatorname{sen} x}{1+\cos x}\right)'}{1+\frac{\operatorname{sen}^2 x}{(1+\cos x)^2}} = \frac{\frac{\cos x(1+\cos x)+\operatorname{sen}^2 x}{(1+\cos x)^2}}{\frac{(1+\cos x)^2+\operatorname{sen}^2 x}{(1+\cos x)^2}} = \frac{\cos x+\cos^2 x+\operatorname{sen}^2 x}{(1+\cos x)^2+\operatorname{sen}^2 x}$$

$$= \frac{\cos x+1}{1+2\cos x+\cos^2 x+\operatorname{sen}^2 x} = \frac{\cos x+1}{2(1+\cos x)} = \frac{1}{2}$$

$f'(x) = g'(x)$, $x \in (-\pi/2, \pi/2)$

$f(x) = g(x) + C$

$f(0) = \operatorname{arctg} \frac{0}{1+1} = 0 + C$, $C = 0$.

5) $f(x) = \operatorname{sen} x$

$f'(x) = \cos x$

$f''(x) = -\operatorname{sen} x$

$f'''(x) = -\cos x$.

$$f(x) = \sum_{n=0}^{+\infty} \frac{f^{(n)}(c)\,(x-c)^n}{n!}$$

$$f(x) = \sum_{n=0}^{+\infty} \frac{\operatorname{sen}^{(n)}(\pi/6)}{n!}\,(x-\pi/6)^n$$

$f(\pi/6) = \operatorname{sen} \pi/6 = 1/2$

$f'(+\pi/6) = \cos \pi/6 = \frac{\sqrt{3}}{2}$

$f''(\pi/6) = -1/2$

$f'''(\pi/6) = -\sqrt{3}/2$

$2021 = 1+2020 = 1+4\cdot 505$

$f^{(2021)}(\pi/6) = f'(\pi/6) = \frac{\sqrt{3}}{2}$

$$= \sum_{j=0}^{+\infty} \frac{1/2}{(4j)!}\,(x-\pi/6)^{4j} + \sum_{j=0}^{+\infty} \frac{\sqrt{3}/2}{(4j+1)!}\,(x-\pi/6)^{4j+1} + \sum_{j=0}^{+\infty} \frac{(-1/2)}{(4j+2)!}\,(x-\pi/6)^{4j+2} + \sum_{j=0}^{+\infty} \frac{(-\sqrt{3}/2)}{(4j+3)!}\,(x-\pi/6)^{4j+3}$$

6) $\int \frac{x^4}{x^4+5x^2+4}\,dx = \int \frac{x^4+5x^2+4-5x^2-4}{x^4+5x^2+4}\,dx =$

$= \int 1\,dx - \int \frac{5x^2+4}{x^4+5x^2+4}\,dx = x - \int \frac{5x^2+4}{(x^2+4)(x^2+1)}\,dx$

$x^4+5x^2+4 = (t^2+5$

$(t^2+5t+4) = (t^2+4)(t^2+1).$

$\frac{5x^2+4}{(x^2+4)(x^2+1)} = \frac{Ax+B}{x^2+4} + \frac{Cx+D}{x^2+1} = \frac{(Ax+B)(x^2+1)+(Cx+D)(x^2+4)}{(x^2+4)(x^2+1)}.$

$(Ax+B)(x^2+1) + (Cx+D)(x^2+4) = 5x^2+4$

$Ax^3+Bx^2+Ax+B + Cx^3+Dx^2+4Cx+4D = 5x^2+4$

$A+C=0 \quad | \quad A=0, \quad C=0.$

$B+D=5$

$B+4D=4 \quad , \quad -3D=1, \quad D=-1/3$

$B = 5+D = 5+1/3 = \frac{16}{3}$

$\int \frac{16/3}{x^2+4}\,dx - \int \frac{1/3}{x^2+1}\,dx = \frac{16}{3\cdot 4}\int \frac{dx}{\left(\frac{x}{2}\right)^2+1} - \frac{1}{3}\int \frac{dx}{x^2+1} =$

$= \frac{8}{3}\operatorname{arctg}\frac{x}{2} - \frac{1}{3}\operatorname{arctg} x + C$

$= x - \frac{8}{3}\operatorname{arctg}\frac{x}{2} + \frac{1}{3}\operatorname{arctg} x + C$

6b) $\int \frac{\operatorname{sen}^3 x}{\cos^3 x}\,dx = \int \frac{\operatorname{sen}^2 x \operatorname{sen} x\,dx}{\cos^3 x} = \int \frac{(1-\cos^2 x)(\operatorname{sen} x\,dx)}{\cos^3 x}$

$\cos x = t, \quad dt = -\operatorname{sen} x\,dx$

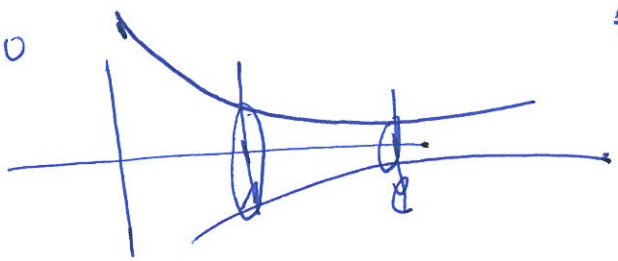
$= -\int \frac{(1-t^2)}{t^3}\,dt = +\int \frac{t^2}{t^3}\,dt - \int \frac{dt}{t^3} = \int \frac{dt}{t} - \int \frac{dt}{t^3} =$

$= \log t + \frac{1}{2t^2} + C = \log \cos x + \frac{1}{2}\frac{1}{\cos^2 x} + C$

$-\frac{\operatorname{sen} x}{\cos x} + \frac{1}{2}\frac{-2\cos x(-\operatorname{sen} x)}{\cos^4 x} = -\frac{\operatorname{sen} x}{\cos x} + \frac{\operatorname{sen} x}{\cos^3 x} = \frac{\operatorname{sen} x - \operatorname{sen} x(\cos^2 x)}{\cos^3 x}$

$= \frac{\operatorname{sen} x(1-\cos^2 x)}{\cos^3 x} = \frac{\operatorname{sen}^3 x}{\cos^3 x} = \operatorname{tg}^3 x$

7) $a > 0$



Eje OX $\quad V = \pi \int_1^e \left(\log x + \frac{a}{x}\right)^2 dx$

$\pi \int_1^e \log^2 x\,dx + \pi \int_1^e 2\log x \frac{a}{x}\,dx + \pi \int_1^e \frac{a^2}{x^2}\,dx = (e-2)\pi$

$\int_1^e \log^2 x\,dx + \int_1^e 2\log x \frac{a}{x}\,dx + \int_1^e a^2 \frac{dx}{x^2} = e-2$

$\int_1^e 2\log x \frac{a}{x}\,dx = \frac{2a}{2}\log^2 x\Big|_1^e = a\left((\log e)^2 - \log^2 1\right) = a$

$a^2 \int_1^e \frac{dx}{x^2} = a^2 - \frac{1}{x}\Big|_1^e = a^2\left(1 - \frac{1}{e}\right)$

$$\int \underbrace{\log^2 x}_{u}\,\underbrace{dx}_{dv} = \cancel{x}\,2\log x\,\frac{dx}{\cancel{x}} - \int 2\log x$$

$dv = dx$, $v = x$

$u = \log^2 x$ $\quad du = 2\log x \frac{1}{x}\,dx$

$$\int_1^e \underbrace{\log^2 x}_{u}\,\underbrace{dx}_{dv} = \log^2 x \cdot x\Big|_1^e - 2\int_1^e \log x \cdot \frac{1}{\cancel{x}} \cdot \cancel{x}\,dx =$$

$$= (1\cdot e - 0\cdot 1) - 2\left[\log x \cdot x\Big|_1^e - \int_1^e \frac{1}{x}\cdot x\,dx\right] =$$

$$= e - 0 - 2\left(e - (e-1)\right) = e - \cancel{2}e + \cancel{2}e - 2$$

$$e - 2 + a + a^2\left(1 - \frac{1}{e}\right) = e - 2, \quad \boxed{a = 0}$$