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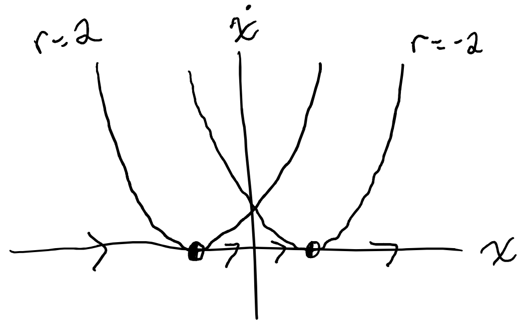
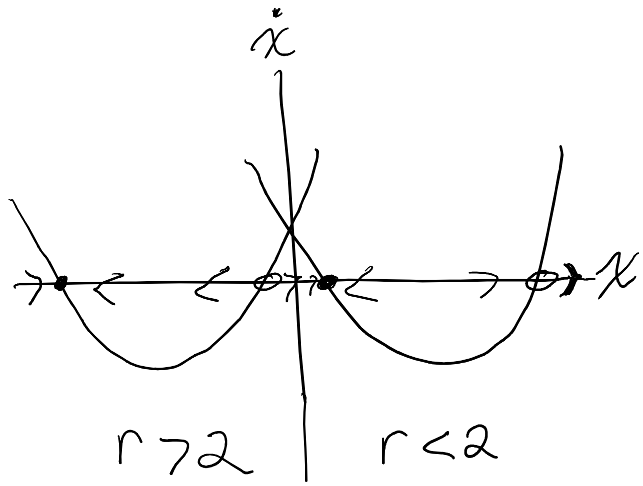
Homework #2

1. **Strogatz 3.1.1: Sketch the qualitatively different vector fields that occur as r is varied. Show that a saddle-node bifurcation occurs at a critical value of r, to be determined. Finally, sketch the bifurcation diagram of fixed points x* versus r:**

$$\dot{\boldsymbol{x}} = \mathbf{1} + \boldsymbol{r}\boldsymbol{x} + \boldsymbol{x}^2$$

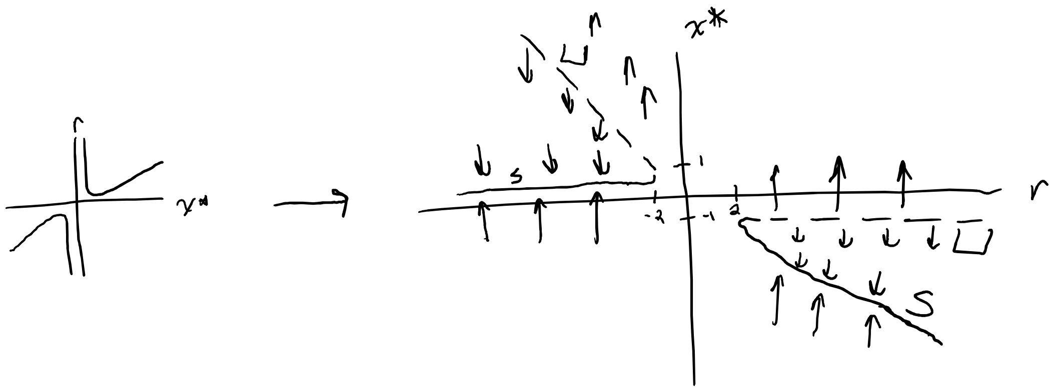
$\dot{x} = f(x,r) = 1 + rx + x^2$





$$1 + rx + x^2 = 0 \;\rightarrow\; r = \frac{x^2+1}{x}$$

$|r| < 2 \quad \rightarrow \quad$ No Fixed Points
$r = \pm 2 \quad \rightarrow \quad x^* = 1$ is half stable
$|r| > 2 \quad \rightarrow \quad x^* = \frac{-r-\sqrt{r^2+4}}{2}$ is stable, $x^* = \frac{-r+\sqrt{r^2+4}}{2}$ is unstable

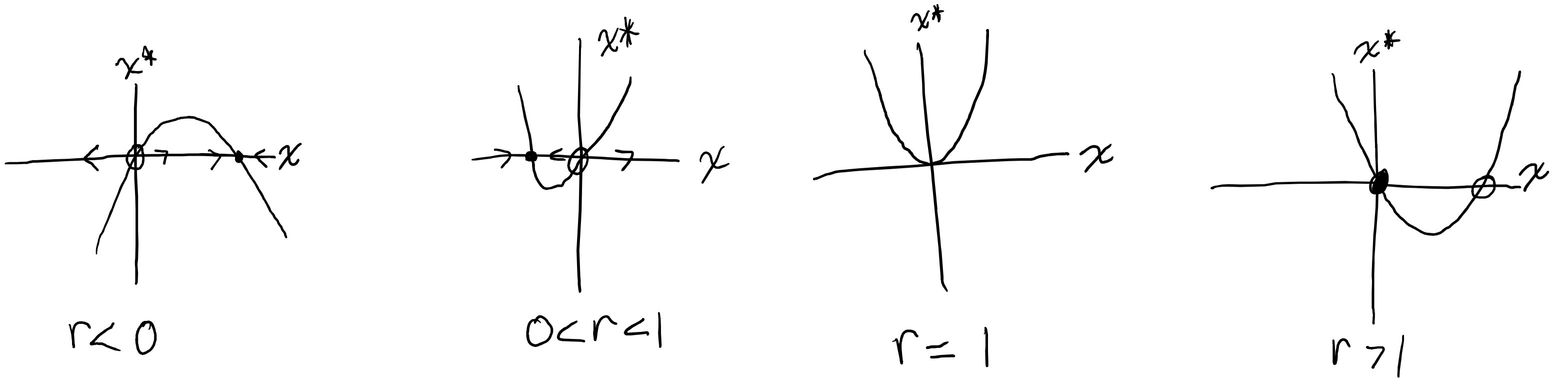


2. **Strogatz 3.2.3: Sketch the qualitatively different vector fields that occur as r is varied. Show that a transcritical bifurcation occurs at a critical value of r, to be determined. Finally, sketch the bifurcation diagram of fixed points x* versus r:**

$$\dot{\boldsymbol{x}} = \boldsymbol{x} - \boldsymbol{r}\boldsymbol{x}(\boldsymbol{1} - \boldsymbol{x})$$

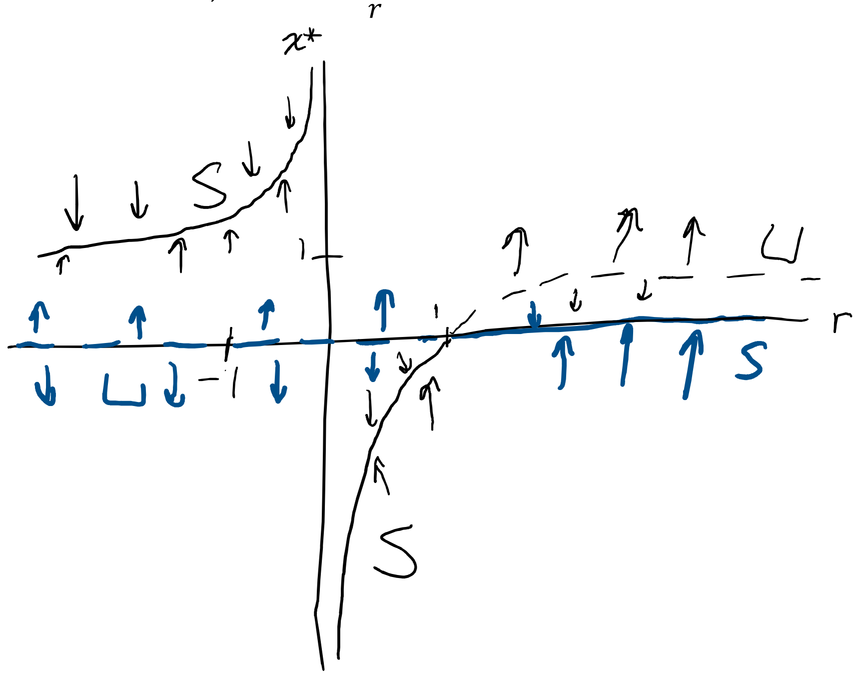
$\dot{x} = f(x, r) = x - rx(1 - x)$

$$x^* \in \{1, \frac{r-1}{r}\}$$



$r < 1 \quad \rightarrow \quad x^* = 0$ is unstable, $\quad x^* = \dfrac{r-1}{r}$ is stable

$r > 1 \quad \rightarrow \quad x^* = 0$ is stable, $\quad x^* = \dfrac{r-1}{r}$ is unstable



3. **Strogatz 3.3.2**

a.

$$\dot{P} = \gamma_1(ED - P) = 0 \therefore D = \frac{P}{E}$$
$$\dot{D} = \gamma_2(\lambda + 1 - D - \lambda EP) = 0 \therefore \lambda + 1 = D + \lambda EP$$
$$\lambda + 1 = \frac{P}{E} + \lambda EP \rightarrow E(\lambda + 1) = P(1 + \lambda E^2)$$
$$P = \frac{E(\lambda + 1)}{1 + \lambda E^2}$$
$$\dot{E} = k(P - E) = k(\frac{E(\lambda + 1)}{1 + \lambda E^2} - E) \ = kE\left(\frac{\lambda + 1 - 1 - \lambda E^2}{1 + \lambda E^2}\right)$$

$$\dot{E} = kE\lambda\left(\frac{1 - E^2}{1 + \lambda E^2}\right)$$

b.

The fixed points occur when E = 0 or $1 - E^2 = 0$

$$E^* \in \{0, \pm 1\}$$

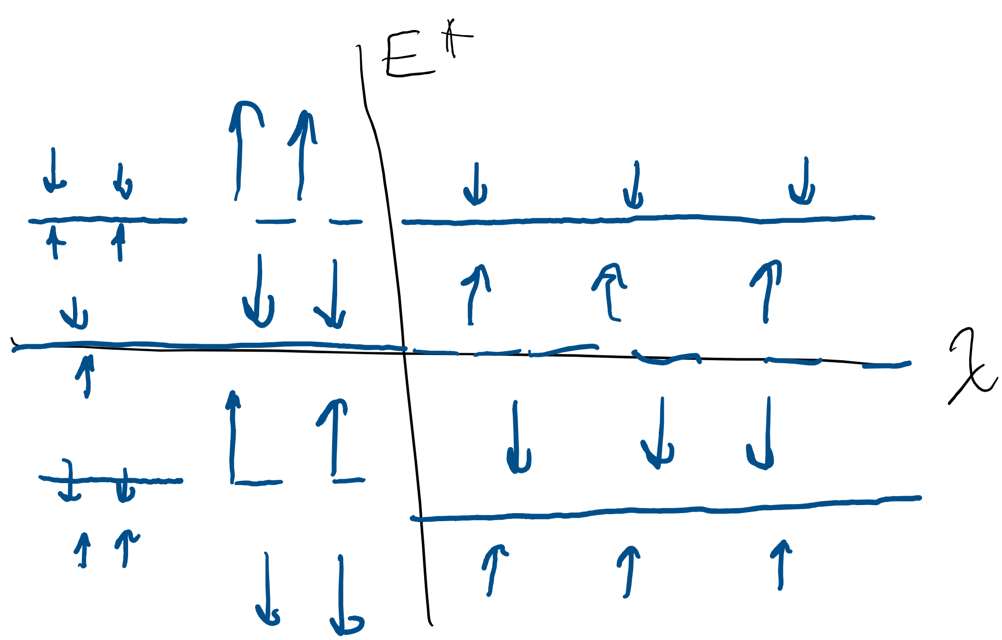
c.

$$f'(E) = -k\lambda * \frac{\lambda E^4 + (\lambda + 3)E^2 - 1}{(1 + \lambda E^2)^2}$$

When $E^* = 0, f'(E) = \ k\lambda$

When $E^* = \pm 1, f'(E) = \ -k\lambda * \frac{2(\lambda+1)}{(1+\lambda)(1+\lambda)} = \ -\frac{2k\lambda}{1+\lambda}$

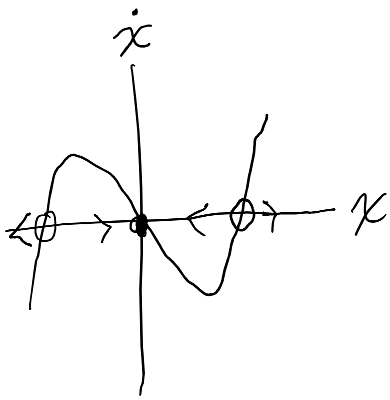
$\lambda < \ -1 \ \rightarrow E^* = 0$ is stable, $E^* = \pm 1$ is stable
$-1 < \lambda < \ 0 \ \rightarrow E^* = 0$ is stable, $E^* = \pm 1$ is unstable
$0 < \lambda < \ 1 \ \rightarrow E^* = 0$ is unstable, $E^* = \pm 1$ is stable
$\lambda > \ 1 \ \rightarrow E^* = 0$ is unstable, $E^* = \pm 1$ is stable

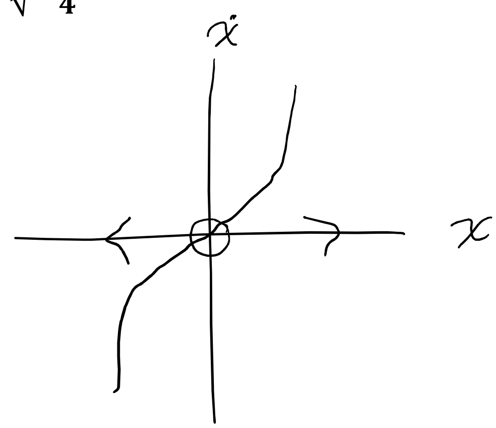


4. **Strogatz 3.4.1: Sketch the qualitatively different vector fields that occur as r is varied. Show that a pitchfork bifurcation occurs at a critical value of r, to be determined, and classify the bifurcation as supercritical or subcritical. Finally, sketch the bifurcation diagram of fixed points x* versus r:**

$$\dot{\boldsymbol{x}} = \boldsymbol{rx} + \boldsymbol{4x^3}$$

$$\boldsymbol{x^*} \in \{\boldsymbol{0}, \sqrt{-\frac{\boldsymbol{r}}{\boldsymbol{4}}}$$



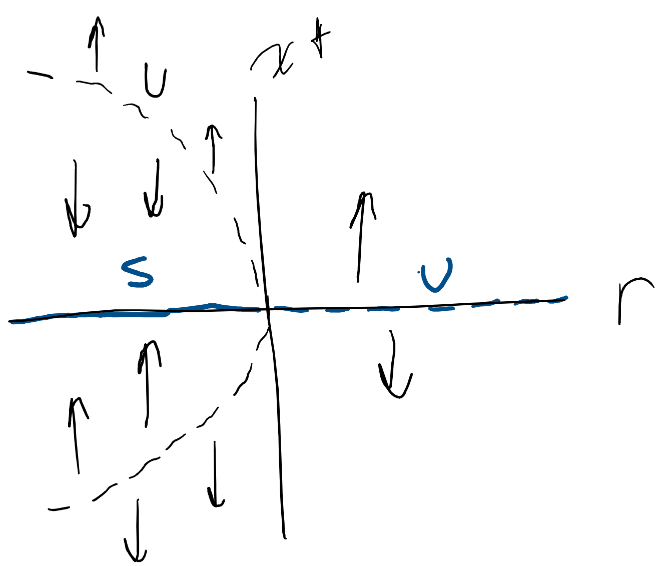


$r < 0$ $\qquad$ $r > 0$

$r < 0 \;\rightarrow x^* = 0$ is stable, $x^* = \pm\sqrt{-r}$ is unstable
$r > 0 \;\rightarrow x^* = 0$ is unstable

$$\dot{x} = f(x^*) = 0 \rightarrow x^* = 0 \text{ or } r = -(x^*)^2$$

This is a subcritical pitchfork bifurcation



5. **Strogatz 3.5.8**

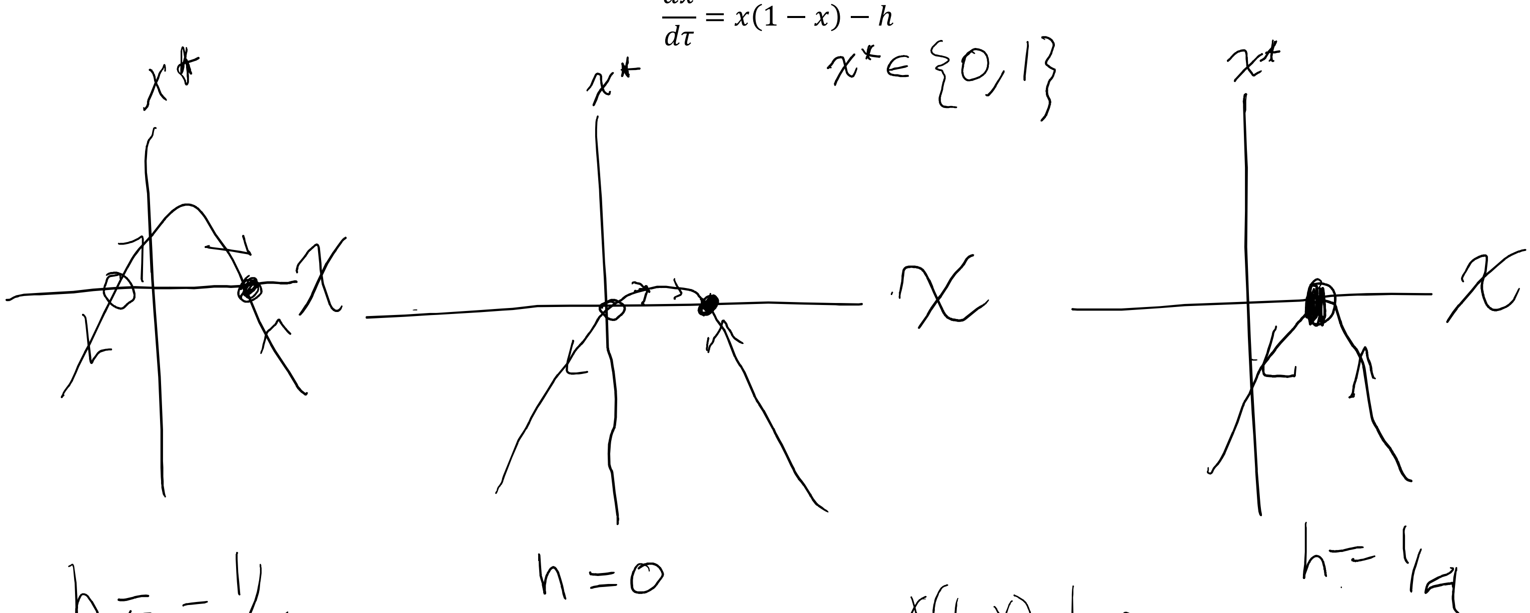
$$\frac{du}{dt} = au + bu^3 + cu^5$$
$$x = \frac{u}{U} \text{ and } \tau = \frac{\mathrm{t}}{T} \therefore u = xU \text{ and } t = \tau T$$
$$\frac{du}{dt} = \frac{dx}{d\tau}\frac{U}{T} = axU + bx^3U^3 - cx^5U^5$$
$$\frac{dx}{d\tau} = axT + bx^3U^2T - cx^5U^4T$$

Now, need to find a, b, c such that $r = aT, bU^2T = 1$ and $cU^4T = 1$

$$bU^2T = cU^4T \to bU^2 = cU^4 \to U^2 = \frac{b}{c} \to$$
$$U = \sqrt{\frac{b}{c}}$$
$$cU^4 = \frac{1}{T} \to c\left(\frac{b}{c}\right)^2 = \frac{1}{T} \to T = \frac{1}{c * \left(\frac{b}{c}\right)^2} = \frac{1}{\frac{b^2}{c}} \to$$
$$T = \frac{c}{b^2}$$
$$r = aT \to r = \frac{ac}{b^2}$$

6. **Strogatz 3.7.3**

$$\dot{N} = rN\left(1 - \frac{N}{k}\right) - H$$
$$x = \frac{N}{k} \to \frac{kdx}{dt} = rkx(1-x) - H$$
$$\left(\frac{1}{rk}\right)\frac{kdx}{dt} = \frac{rkx(1-x) - H}{rk} = \frac{dx}{rdt} = x(1-x) - \frac{H}{rk}$$
$$\tau = rt$$
$$\frac{dx}{d\tau} = x(1-x) - \frac{H}{\tau k}, h = \frac{H}{\tau k} \to$$
$$\frac{dx}{d\tau} = x(1-x) - h$$



$x(1-x) - h = 0 \Rightarrow$
$h = x(1-x)$

Any value of h > ¼ does not have a fixed point. When h is < ¼ there are two fixed points as seen in the graphs above. This is similar to the first problem in this homework set which was **saddle-node bifurcation**