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Homework #3

1. **Consider the *two-dimensional first-order differential equation***

$$\dot{\boldsymbol{x}} = \boldsymbol{A}\boldsymbol{x}$$

Where $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ **and** $\dot{x} = \begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix}$. **Let** $\tau = \mathrm{Tr}(A)$ **and** $\delta = \det(A)$.

Show $x(t)$ **is a solution to the *one-dimensional second-order equation***

$$\ddot{x} - \tau\dot{x} + \delta x = 0.$$

$$\dot{x} = ax + by$$
$$\dot{y} = cx + dy$$

$$\ddot{x} = a\dot{x} + b\dot{y} = a\dot{x} + b(cx + dy) = a\dot{x} + bcx + \frac{bd(\dot{x} - ax)}{b}$$
$$\ddot{x} = a\dot{x} + bcx + d\dot{x} - adx \rightarrow \ddot{x} - a\dot{x} - d\dot{x} + adx - bcx = 0$$
$$\ddot{x} - (a + d)\dot{x} + (ad - bc)x = 0; (a + d) = \tau, (ad - bc) = \delta$$
$$\ddot{x} - \tau\dot{x} + \delta x = 0$$

2. **Consider the linear system** $\dot{x} = -x$, $\dot{y} = -4y$. **Show that any nontrivial solution in the phase-plane lies on a curve of the form** $y(x) = cx^4$.

$$\frac{dy}{dx} = \frac{\dot{y}}{\dot{x}} = -\frac{4y}{-x} \rightarrow \frac{1}{4y}dy = \frac{1}{x}dx$$
$$\int \frac{1}{4y}dy = \int \frac{1}{x}dx = \frac{1}{4}\ln y = \ln x = 4\ln x \rightarrow$$
$$\boldsymbol{y(x) = cx^4, \text{where } c \text{ is a constant}}$$

3. **The motion of a damped harmonic oscillator is described by**

$$m\ddot{x} + \gamma\dot{x} + m\omega^2 x = 0$$

Where $m, \gamma, \text{and } \omega$ **are all positive constants.**

a. **Rewrite the equation as a two-dimensional linear system.**

$$\dot{x} = v$$
$$\dot{v} = \frac{-\gamma\dot{x} - m\omega^2 x}{m} = -\frac{\gamma v}{m} - \omega^2 x$$
$$\begin{pmatrix} \dot{x} \\ \dot{v} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -\omega^2 & -\frac{\gamma}{m} \end{pmatrix} \begin{pmatrix} x \\ v \end{pmatrix}$$

$$\dot{x} = v, \qquad \dot{v} = -\omega^2 x - \frac{\gamma v}{m}$$

b. **Classify the fixed point at the origin. Indicate all the different cases that can occur, depending on the relative sizes of the parameters.**

$$\tau = a + d = -\frac{\gamma}{m}$$
$$\delta = ad - bc = \omega^2$$
$$\lambda_{1,2} = \frac{\tau \pm \sqrt{\tau^2 - 4\delta}}{2}$$

If $\delta < 0$ then the fixed point is a saddle point. If $\delta = 0$ then there is not an isolated fix point→ There is either a line or a plane of fixed points. If $\tau^2 = 4\delta$ then the fixed point is a star node or a degenerate node.

If $\lambda_1 = \lambda_2 = \lambda$ then there is either one eigenvector or two eigenvectors that = λ. If there are two eigenvectors and $\lambda \neq 0$ the fixed point is a star node, but if $\lambda = 0$ there is a whole plane of fixed points. If there is only one eigenvector, then the fixed point is a degenerate node.

If $\tau^2 > 4\delta$, then the fixed point is a node. If $\lambda_1$and λ_2 > 0, then the fixed point is an unstable node. If λ_1 and λ_2 < 0 then the fixed point is a stable node.

If $\tau^2 < 4\delta$, then the fixed point is a spiral. If $\frac{\tau}{2} < 0$ then it is a stable spiral, and if $\frac{\tau}{2} > 0$ it is an unstable spiral. If $\frac{\tau}{2} = 0$ then the stable point is a center.

4. **Strogatz 5.2.1: Consider the system $\dot{x} = 4x - y, \dot{y} = 2x + y$.**
   a. **Write the system as $\dot{x} = Ax$. Show that the characteristic polynomial is $\lambda^2 - 5\lambda + 6$, and find the eigen values and eigenvectors of A.**

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} 4 & -1 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

$\boldsymbol{\tau = a + d = 4 + 1 = 5}$
$\boldsymbol{\delta = ad - bc = (4 * 1) - (-1 * 2) = 6}$

$$x(t) = e^{\lambda t} v \rightarrow Av = \lambda v$$
$$\mathrm{p}(\lambda) = 0 = \det(A - I\lambda) \rightarrow \det\begin{pmatrix} 4-\lambda & -1 \\ 2 & 1-\lambda \end{pmatrix} = (4-\lambda)(1-\lambda) - (-2)$$
$$\lambda^2 - 4\lambda - 1\lambda + 4 + 2 = \boldsymbol{\lambda^2 - 5\lambda + 6 = 0}$$

The eigenvalues are $(\lambda - 2)(\lambda - 3) \rightarrow$ $\boldsymbol{\lambda_1 = 2, \quad \lambda_2 = 3}$

When $\lambda = 2$:

$$\begin{pmatrix} 4-2 & -1 \\ 2 & 1-2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \rightarrow$$
$$2x = y$$
$$2x = y$$
$$\overrightarrow{\boldsymbol{v_1}} = \begin{pmatrix} \boldsymbol{1} \\ \boldsymbol{2} \end{pmatrix}$$

When $\lambda = 3$:

$$\begin{pmatrix} 4-3 & -1 \\ 2 & 1-3 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \rightarrow$$

$$x = y$$
$$2x = 2y$$
$$\overrightarrow{\boldsymbol{v_2}} = \begin{pmatrix} \mathbf{1} \\ \mathbf{1} \end{pmatrix}$$

The eigenvectors are: $\overrightarrow{\boldsymbol{v_1}} = \begin{pmatrix} \mathbf{1} \\ \mathbf{2} \end{pmatrix}, \overrightarrow{\boldsymbol{v_2}} = \begin{pmatrix} \mathbf{1} \\ \mathbf{1} \end{pmatrix}$

b. Find the general solution of the system

$$x(t) = c_1 e^{\lambda_1 t}\overrightarrow{v_1} + c_2 e^{\lambda_2 t}\overrightarrow{v_2}$$
$$\boldsymbol{x(t) = c_1 e^{2t}\begin{pmatrix} 1 \\ 2 \end{pmatrix} + c_2 e^{3t}\begin{pmatrix} 1 \\ 1 \end{pmatrix}}$$

c. Classify the fixed point at the origin

$$\tau = 5, \delta = 6$$
$$\tau^2 - 4\delta > 0, 25 - 24 = 1 > 0$$

$\tau^2 - 4\delta > 0$, $\lambda_1 > 0$, and $\lambda_2 > 0$ $\therefore$ The fixed point is an unstable node

d. Solve the system subject to the initial condition $(\boldsymbol{x_0}, \boldsymbol{y_o}) = (\mathbf{3}, \mathbf{4})$

$$x(0) = c_1 e^{2*0}\begin{pmatrix} 1 \\ 2 \end{pmatrix} + c_2 e^{3*0}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = c_1\begin{pmatrix} 1 \\ 2 \end{pmatrix} + c_2\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 3 \\ 4 \end{pmatrix}$$
$$c_1 + c_2 = 3$$
$$2c_1 + c_2 = 4$$
$$c_1 = 3 - (4 - 2c_1) \rightarrow \boldsymbol{c_1 = 1},$$
$$1 + c_2 = 3 \rightarrow \boldsymbol{c_2 = 2}$$

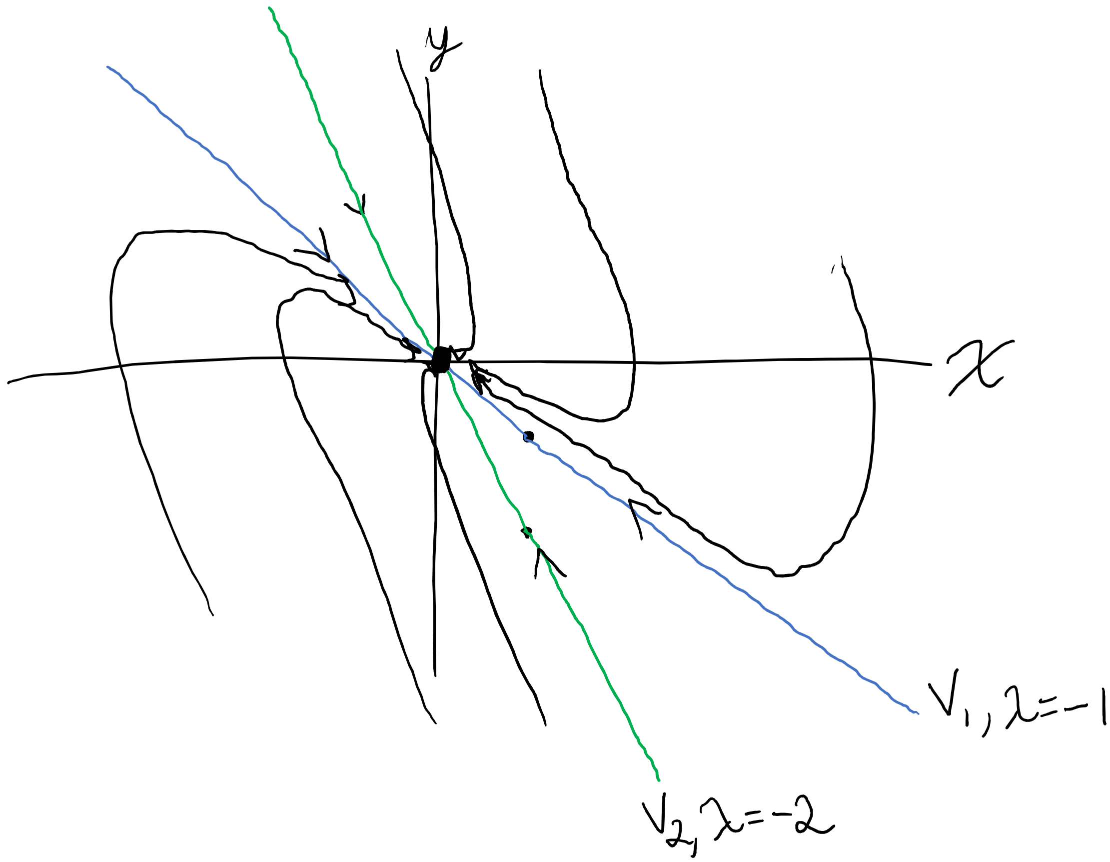
$$\boldsymbol{x(t) = e^{2t} + 2e^{3t}}$$
$$\boldsymbol{y(t) = 2e^{2t} + 2e^{3t}}$$

Plot the phase portrait and classify the fixed point of the following linear systems. If the eigenvectors are real, indicate them in your sketch

5. Strogatz 5.2.3:

$$\boldsymbol{\dot{x} = y}, \qquad \boldsymbol{\dot{y} = -2x - 3y}$$
$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -2 & -3 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$$
$$\tau = -3, \qquad \delta = 2, \qquad \lambda_{1,2} = -1, -2$$
$$\lambda_1 = -1 \rightarrow \begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} -1 & 1 \\ -2 & -2 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = 0 \rightarrow v_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, v_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$
$$\lambda_2 = -2 \rightarrow \begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} -2 & 1 \\ -2 & -1 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = 0 \rightarrow v_1 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}, v_2 = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$$

$$\lambda_1 < 0 \text{ and } \lambda_2 < 0 \rightarrow \text{ both eigensolutions decay exponentially}$$
$$\tau^2 - 4\delta > 0 \rightarrow 9 - 8 = 1$$
$$\therefore \text{The fixed point is stable node, eiganvectors are real}$$



6. Strogatz 5.2.4:

$$\dot{\boldsymbol{x}} = \mathbf{5}\boldsymbol{x} + \mathbf{10}\boldsymbol{y}, \qquad \dot{\boldsymbol{y}} = -\boldsymbol{x} - \boldsymbol{y}$$

$$\begin{pmatrix}\dot{x}\\ \dot{y}\end{pmatrix} = \begin{pmatrix}5 & 10\\ -1 & -1\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix}$$

$$\tau = 4, \qquad \delta = 5, \qquad \lambda_{1,2} = 2 \pm i$$

$$\tau^2 - 4\delta < 0 \to 16 - 20 = -4, \qquad \alpha = \frac{\tau}{2} = 2 > 0 \therefore \text{Unstable spiral}$$

$$\begin{pmatrix}\dot{x}\\ \dot{y}\end{pmatrix} = \begin{pmatrix}5 & 10\\ -1 & -1\end{pmatrix}\begin{pmatrix}1\\ 1\end{pmatrix} = (15, -1)$$

$$\begin{pmatrix}\dot{x}\\ \dot{y}\end{pmatrix} = \begin{pmatrix}5 & 10\\ -1 & -1\end{pmatrix}\begin{pmatrix}0\\ -1\end{pmatrix} = (-10, 1)$$

$\therefore$ the spiral is counterclockwise