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Homework #5

1. **The system $\dot{x} = y - 16x^3, \dot{y} = -4x + 2y - 2y^3$ has a single fixed-point at the origin. Prove the existence of a limit cycle by considering the flow across the boundary of the rectangle $\{|x| \leq 1, |y| \leq 2\}$ and applying the *Poincare-Bendixson* theorem.**

$$J(x,y) = \begin{pmatrix} -32x^2 & 1 \\ -4 & 2-6y^2 \end{pmatrix}$$
$$J(0,0) = \begin{pmatrix} 0 & 1 \\ -4 & 2 \end{pmatrix}, \qquad \tau = 2, \qquad \delta = 4 \therefore \text{Spiral Source}$$

$$\dot{\boldsymbol{y}} \to \boldsymbol{y}(+) = \mathbf{2} \to -4x + 2(2) - 2(2)^3 = -4x - 16 \to$$
$$\boldsymbol{x} = -\mathbf{1}: -4(-1) - 16 = -12,$$
$$\boldsymbol{x} = \mathbf{1}: -4(1) - 16 = -8$$
$$\therefore \mathbf{-12 < \dot{y} < -8}$$

$$\dot{\boldsymbol{y}} \to \boldsymbol{y}(-) = -\mathbf{2} \to -4x + 2(-2) - 2(-2)^3 = -4x + 12 \to$$
$$\boldsymbol{x} = -\mathbf{1}: -4(-1) + 12 = 16,$$
$$\boldsymbol{x} = \mathbf{1}: -4(1) + 12 = 8$$
$$\therefore \mathbf{8 < \dot{y} < 16}$$

$$\dot{\boldsymbol{x}} \to \boldsymbol{x}(+) = \mathbf{1} \to y - 16(1)^3 = y - 16 \to$$
$$\boldsymbol{y} = -\mathbf{2}: -2 - 16 = -18,$$
$$\boldsymbol{y} = \mathbf{2}: 2 - 16 = -14$$
$$\therefore \mathbf{-16 < \dot{x} < -14}$$

$$\dot{\boldsymbol{x}} \to \boldsymbol{x}(-) = -\mathbf{1} \to y - 16(-1)^3 = y + 16 \to$$
$$\boldsymbol{y} = -\mathbf{2}: -2 + 16 = 14,$$
$$\boldsymbol{y} = \mathbf{2}: 2 + 16 = 18$$
$$\therefore \mathbf{14 < \dot{x} < 18}$$

$\dot{y}$ is negative along y=2 and it is positive along y=-2
x is negative along x=1 and it is positive along x=-1
This means outside of the boundaries flows towards the inside, and since there is a fixed point at the origin that is a spiral source there is a limit cycle.

2. **This exercise uses a *Liapunov* function to prove the existence of a stable limit cycle. Consider the 2D system**

$$\boldsymbol{\dot{x} = x(r^4 - a^4x^4 - y^4) - by^3}$$
$$\boldsymbol{\dot{y} = y(r^4 - a^4x^4 - y^4) + a^4bx^3}$$

Where r, a, and b are positive constants.

a. **Classify the fixed-point at the origin**

$$J(0,0) = \begin{pmatrix} r^4 & 0 \\ 0 & r^4 \end{pmatrix}, \qquad \tau = 2r^4, \qquad \delta = r^8 \ \therefore \textbf{Source (star node)}$$

b. **Let $V(x,y) = \left(r^4 - a^4x^4 - y^4\right)^2$ and show $\dot{V} = c(x,y) * V \le 0$**

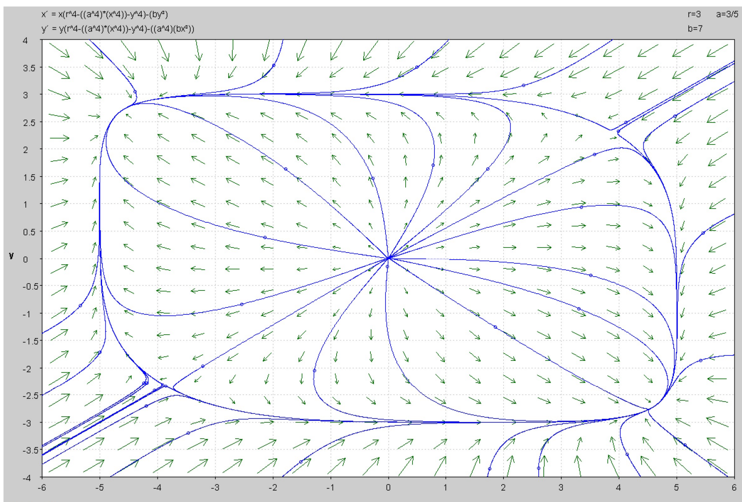
$$\frac{dV}{dt} = 2(r^4 - a^4x^4 - y^4)(-4a^4x^3\dot{x} - 4y^3\dot{y}) \rightarrow$$
$$= 2(r^4 - a^4x^4 - y^4) * -4(a^4x^3(x(r^4 - a^4x^4 - y^4) - by^3) - y^3(y(r^4 - a^4x^4 - y^4) + a^4bx^3))$$
$$= 2(r^4 - a^4x^4 - y^4) * (-4a^4x^4(r^4 - a^4x^4 - y^4) + 4a^4x^3by^3 - 4a^4x^3by^3 - 4y^4(r^4 - a^4x^4 - y^4)$$
$$= -4(a^4x^4 + y^4) * 2(r^4 - a^4x^4 - y^4)^2$$
$$= \mathbf{-8\left(a^4x^4 + y^4\right)V \le 0}$$

c. **Find where $\dot{V} = 0$ and give an argument for the existence of a stable limit cycle**

$$\dot{V} = -8(a^4x^4 + y^4) * V$$

Since there is a -8 in front of $\dot{V}$, V is always decreasing until it stops. Where $\dot{V} = 0$ and $V(x,y) = 0$, but the origin is a source.

d. **Use a computer to generate a phase portrait with $r = 3, a = \frac{3}{5}$, and $b = 7$**



3. **Balthasar van der Pol was a Dutch physicist and electrical engineer who was a pioneer in the experimental study of nonlinear phenomena related to radio communications. The triode vacuum tube, which was able to produce stable self-excited oscillations, is described by the *van der Pol Oscillator* differential equations:**

$$\ddot{x} + \mu\left(x^2 - 1\right)\dot{x} + x = 0$$

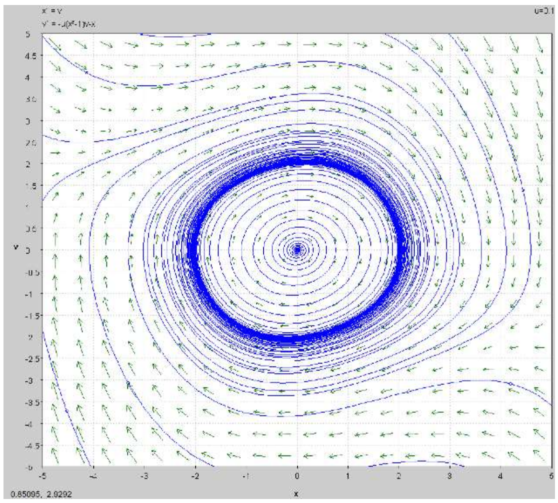
a. **Rewrite the second-order system as two first-order differential equations**

$$\dot{x} = v$$
$$\dot{v} = -\mu(x^2 - 1)v - x$$

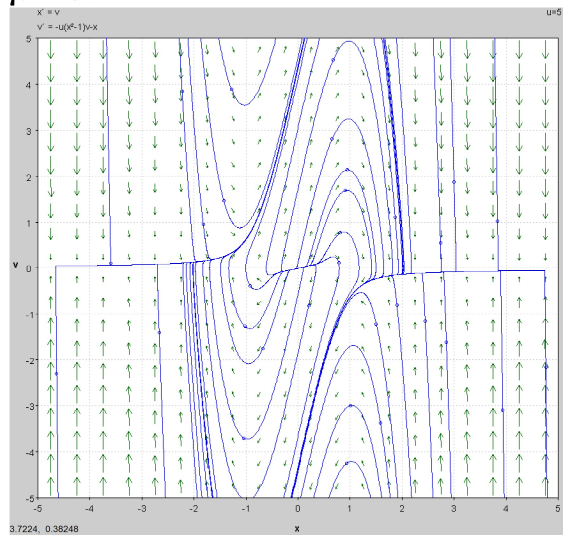
$$\begin{pmatrix}\dot{x}\\ \dot{v}\end{pmatrix} = \begin{pmatrix}0 & 1\\ -1 & -\mu(x^2 - 1)\end{pmatrix}\begin{pmatrix}x\\ v\end{pmatrix}$$

b. **Use a computer to generate a phase portrait with $\mu = 0.1$ and $\mu = 5$**

$\mu = 0.1$:



$\mu = 5$:

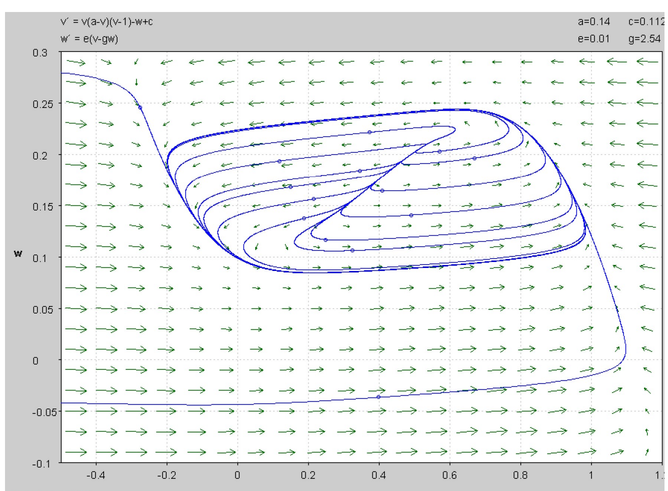


4. **The *FitzHugh-Nagumo* oscillator is used to model the action-potential of a neuron. It is a two-variable simplification of the four-dimensional *Hodgkin-Huxley* model (1952) for which Hodgkin and Huxley received the 1963 Nobel Prize in Physiology and Medicine. The *FitzHugh-Nagumo* oscillator equations are**

$$\dot{v} = v(a - v)(v - 1) - w + v_c$$
$$\dot{w} = \epsilon(v - \gamma w)$$

where v is a voltage, w is the recovery of voltage, a is a threshold, γ is a shunting variable, and v_c is a constant voltage. The parameter ϵ influences the rate of approach to equilibrium but does not affect the equilibrium value. For certain parameter values the solution exhibits *relaxation oscillations*.

Use a computer to generate a phase portrait which shows a limit cycle for $a = 0.14, v_c = 0.112, \epsilon = 0.01$ and $\gamma = 2.54$.



5. **Strogatz 7.1.8: Consider $\ddot{x} + a\dot{x}(x^2 + \dot{x}^2 - 1) + x = 0$, where $a > 0$**

   a. **Find and classify all the fixed points**

$$\dot{x} = y$$
$$\dot{y} = -ay(x^2 + y^2 - 1) - x$$

$$(x^*, y^*) = (0,0)$$
$$J(0,0) = \begin{pmatrix} 0 & 1 \\ -1 & a \end{pmatrix}, \qquad \tau = a, \qquad \delta = 1$$
$$\therefore \mathrm{a} > 2 \rightarrow \text{unstable node}, \qquad 0 < \mathrm{a} < 2 \rightarrow \text{unstable spiral}$$

   b. **Show the system has a circular limit cycle, and find its amplitude and period**

$$r\dot{r} = x\dot{x} + y\dot{y} = xy - ay^2(x^2 + y^2 - 1) - xy = -ay^2(r^2 - 1) \rightarrow$$
$$\boldsymbol{\dot{r} = -ar\sin^2\theta\,(r^2 - 1)}$$
$$r^2\theta = x\dot{y} - y\dot{x} = -axy(x^2 + y^2 - 1) + x^2 - y^2 = -axy(r^2 - 1) - r^2 \rightarrow$$
$$\boldsymbol{\dot{\theta} = -a\cos\theta\sin\theta\left(r^2 - 1\right) - 1}$$

$$\dot{r} = 0 \rightarrow r = 1 \; or \; r = 0$$
$$\dot{\theta}: r = 0 \rightarrow \; \dot{\theta} = -\,\mathrm{a}\cos\theta\sin\theta - 1, \boldsymbol{r = 1} \rightarrow \; \boldsymbol{\dot{\theta} = -1}$$
When $\dot{r} = 0, r = 1$ and $\theta = -1$ so this is a clockwise cycle, $\mathbf{amplitude = 1}$, $\mathbf{period = 2\pi}$

   c. **Determine the stability of the limit cycle**

   When $r = 1$, $\theta = -1$ so the cycle goes clockwise
   When $r > 1$ $\dot{r} < 0$, toward the cycle
   When $r < 1$ $\dot{r} > 0$, toward the cycle
   Therefore, the limit cycle is **stable**

   d. **Give an argument which shows that the limit cycle is unique, i.e., there are no other periodic trajectories**

   When r is 0 it is the fixed point, but the only other place $\dot{r} = 0$ is when $r = 1$
   For every value of $r > 1, \dot{r}$ is positive, and for every value of $r < 1, \dot{r}$ is negative.
   Therefore, every value of r goes towards the limit cycle except for the fixed point when $r = 0$.

6. **Show that the system $\dot{x} = x - y - x^3$, $\dot{y} = x + y - y^3$ has a periodic solution. It may be helpful to apply the *Poincare-Bendixson* theorem to the region $1 \le r \le \sqrt{2}$ and make use of the trigonometric identities**

$$\mathbf{4\cos^4\theta + 4\sin^4\theta = 3 + \cos 4\theta}$$
$$\mathbf{4\cos^3\theta\sin\theta - 4\cos\theta\sin^3\theta = \sin 4\theta}$$

Fixed points: $(x^*, y^*) = (0,0)$

$$J(x,y) = \begin{pmatrix} 1 - 3x^2 & -1 \\ 1 & 1 - 3y^2 \end{pmatrix}$$
$$J(0,0) = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}, \qquad \tau = 2, \qquad \delta = 2, \qquad \therefore \text{spiral source}$$

$$r\dot{r} = x\dot{x} + y\dot{y} \rightarrow \; r\dot{r} = x(x - y - x^3) + y(x + y - y^3)$$
$$r\dot{r} = x^2 - xy - x^4 + xy + y^2 - y^4 = r^2 - x^4 - y^4$$
$$r\dot{r} = r^2 - r^4\cos^4\theta - r^4\sin^4\theta = r - r^3(\cos^4\theta - \sin^4\theta)$$

$$4\dot{r} = 4r - r^3(4\cos^4\theta - 4\sin^4\theta) = 4r - r^3(3 + \cos 4\theta)$$
$$\boldsymbol{\dot{r} = r - \frac{r^3(3 + \cos 4\theta)}{4}}$$

$$r^2\dot{\theta} = x\dot{y} - y\dot{x} = x(x + y - y^3) - y(x - y - x^3) = x^2 + xy - xy^3 - xy + y^2 + yx^3$$
$$r^2\dot{\theta} = r^2 - xy^3 + yx^3 = r^2 - r^4\cos\theta\sin^3\theta + r^4\sin\theta\cos^3\theta$$
$$\dot{\theta} = 1 - r^2(\cos\theta\sin^3\theta - \cos^3\theta\sin\theta) \rightarrow\ 4\dot{\theta} = 4 + r^2(-4\cos\theta\sin^3\theta + 4\cos^3\theta\sin\theta)$$
$$\boldsymbol{\dot{\theta} = 1 - \frac{r^2\sin 4\theta}{4}}$$

$$\text{When } r < 1 \rightarrow\ \dot{r} = 1 - \frac{3 + \cos 4\theta}{4} \rightarrow$$
$$\dot{r} > 0 \rightarrow 1 < \frac{3 + \cos 4\theta}{4} \rightarrow 1 < \cos 4\theta$$

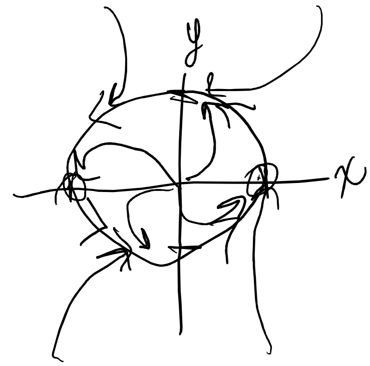
Since $\cos 4\theta$ is always less than 1, **$\dot{r}$ is always greater than 0 when $r = 1$**

$$\text{When } r > \sqrt{2} \rightarrow\ \dot{r} = \sqrt{2} - 2\sqrt{2}\left(\frac{3 + \cos 4\theta}{4}\right) \rightarrow \dot{r} = \sqrt{2}\left(1 - \frac{3}{2} - \frac{\cos 4\theta}{2}\right)$$
$$\dot{r} = -\frac{\sqrt{2}}{2} * (1 + \cos 4\theta)$$
$$\dot{r} < 0 \rightarrow 1 + \cos 4\theta > 0 \rightarrow \cos 4\theta > -1$$

Since $\cos 4\theta$ is always greater than -1, **$\dot{r}$ is always less than 0 when $r = \sqrt{2}$**

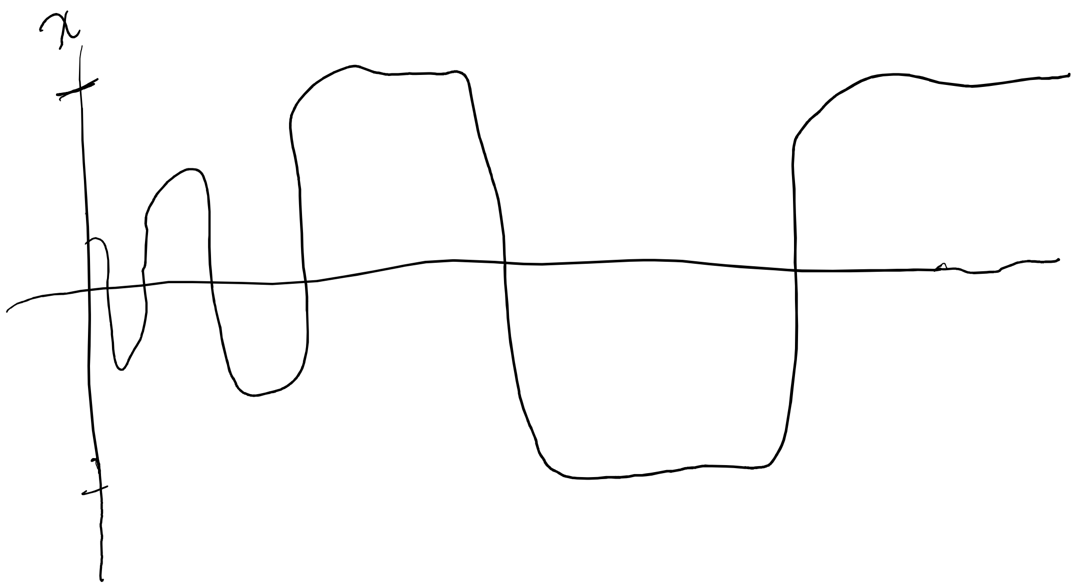
$$\boldsymbol{\therefore \dot{r} > 0 \text{ when } r < 1, \dot{r} < 0 \text{ when } r > \sqrt{2}}$$

7. **Strogatz 7.3.11: (Cycle graphs) Suppose $\dot{x} = f(x)$ is a smooth vector on field R^2. An improved version of the Poincare-Bendixson theorem states that if a trajectory is trapped in a compact region, then it must approach a fixed point, a closed orbit, or something exotic called a *cycle graph* (an invariant set containing a finite number of fixed points connected by a finite number of trajectories, all oriented either clockwise or counterclockwise). Cycle graphs are rare in practice; here's a contrived by simple example.**
   a. **Plot the phase portrait for the system**
   $$\boldsymbol{\dot{r} = r(1 - r^2)\left[r^2 sin^2\theta + (r^2 cos^2\theta - 1)^2\right]}$$
   $$\boldsymbol{\dot{\theta} = r^2 sin^2\theta + (r^2 cos^2\theta - 1)^2}$$
   Where r, θ are polar coordinates. (Hint: Note the common factor in the two equations; examine where it vanishes.
   $\dot{r} = r(1 - r^2)\dot{\theta}$, and $\dot{\theta}$ is always positive or 0 which means it is either stopped or it is moving counter clockwise
   $$\dot{r} = 0 \rightarrow r = 1, \dot{\theta} = 0 \rightarrow r = 1, \theta = c * \pi$$
   $$\dot{r} = r(1 - r^2)\dot{\theta} > 0 \rightarrow 1 - r^2 > 0 \rightarrow r < 1$$
   $$\dot{r} = r(1 - r^2) < 0 \rightarrow 1 - r^2 < 0 \rightarrow r > 1$$
   $$\therefore\ \dot{r} < 0 \text{ when } r > 1, \qquad \dot{r} > 0 \text{ when } r < 1$$



b. Sketch x vs. t for a trajectory starting away from the unit circle. What happens as $t \to \infty$?

When x is at the fixed points the slope is 0. The fixed point occurs at $r = 1$, and $x = r * \cos\theta$, so as r go towards 1 the fixed point is reached.



8. Strogatz 8.2.1: Consider the biased van der Pol oscillator $\ddot{x} + \mu(x^2 - 1)\dot{x} + x = a$. Find the curves in (μ, a) space at which Hopf bifurcations occur.

$$\dot{x} = y$$
$$\dot{y} = -\mu(x^2 - 1)y - x + a$$
$$J(x,y) = \begin{pmatrix} 0 & 1 \\ -2\mu xy - 1 & -\mu(x^2 - 1) \end{pmatrix}$$
$$J(a,0) = \begin{pmatrix} 0 & 1 \\ -1 & -\mu(a^2 - 1) \end{pmatrix}, \qquad \tau = -\mu(a^2 - 1), \qquad \delta = 1$$
$$\tau > 0 \to \mu > 0, |a| < 1 \ or \ \mu < 0, |a| > 1$$
$$\tau < 0 \to \mu > 0, |a| > 1 \ or \ \mu < 0, |a| < 1$$
$$\therefore \mu = 0, a = 1, a = -1$$