Chapter One

Euclidean Three-Space

1.1 Introduction.

Let us briefly review the way in which we established a correspondence between the real numbers and the points on a line, and between ordered pairs of real numbers and the points in a plane. First, the line. We choose a point on a line and call it the ***origin***. We choose one direction from the origin and call it the ***positive*** direction. The opposite direction, not surprisingly, is called the ***negative*** direction. In a picture, we generally indicate the positive direction with an arrow or a plus sign:
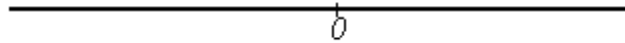
Now we associate with each real number r a point on the line. First choose some unit of measurement on the line. For $r > 0$, associate with r the point on the line that is a distance r units from the origin in the positive direction. For $r < 0$, associate with r the point on the line that is a distance r units from the origin in the negative direction. The number 0 is associated with the origin. A moments reflection should convince you that this procedure establishes a so-called ***one-to-one correspondence*** between the real numbers and the points on a line. In other words, a real number determines exactly one point on a line, and, conversely, a point on the line determines exactly one real number. This line is called a ***real line***.

Next we establish a one-to-one correspondence between ordered pairs of real numbers and points in a plane. Take a real line, called the ***first axis***, and construct another real line, called the ***second axis***, perpendicular to it and passing through the origin of the first axis. Choose this point as the origin for the second axis. Now suppose we have an ordered pair (x_1, x_2) of reals. The point in the plane associated with this ordered pair is

found by constructing a line parallel to the second axis through the point on the first axis corresponding to the real number x_1, and constructing a line parallel to the first axis through the point on the second axis corresponding to the real number x_2. The point at which these two lines intersect is the point associated with the ordered pair (x_1, x_2). A moments reflection here will convince you that there is exactly one point in the plane thus associated with an ordered pair (a, b), and each point in the plane is the point associated with some ordered pair (a, b):
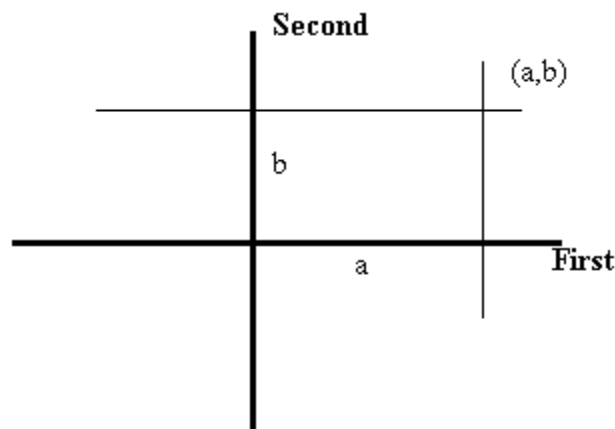

It is traditional to assume the point of view we have taken in this picture, in which the first axis is horizontal, the second axis is vertical, the positive direction on the first axis is to the right, and the positive direction on the second axis is up. We thus usually speak of the **horizontal** axis and the **vertical** axis, rather than the first axis and the second axis. We also frequently abuse the language by speaking of a *point* (x_1, x_2) when, of course, we actually mean the point associated with the ordered pair (x_1, x_2). The numbers x_1 and x_2 are called the **coordinates** of the point- x_1 is the first coordinate and x_2 is the second coordinate.

Given any collection of ordered pairs(A collection of ordered pairs is called a *relation*.), a picture of the collection is obtained by simply looking at the set of points in the plane corresponding to the pairs in the given collection. Suppose we have an equation

involving two variables, say x and y. Then this equation defines a collection of ordered pairs of numbers, namely all (x, y) that satisfy the equation. The corresponding picture in the plane is called the **graph** of the equation. For example, consider the equation $y^2 = x^4$. Let's take a look at the graph of this equation. A little algebra (very little, actually), convinces us that

$$\{(x, y): y^2 = x^4\} = \{(x, y): y = x^2\} \quad \{(x, y): y = -x^2\},$$

and we remember from the sixth grade that each of the sets on the right hand side of this equation is a parabola:
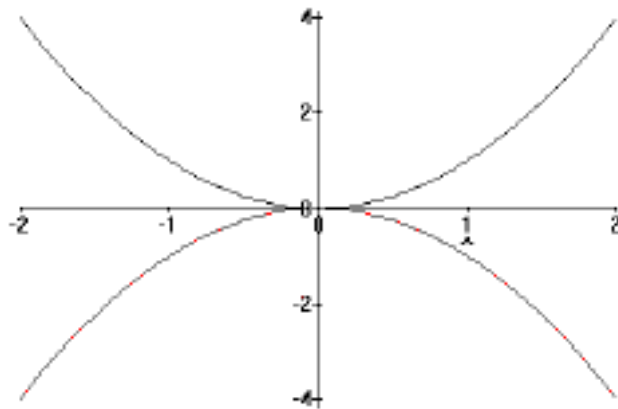
What do we do with all this? These constructions are, of course, the bases of analytic geometry, in which we join the subjects of algebra and geometry, to the benefit of both. A geometric figure (a subset of the plane) corresponds to a collection of ordered pairs of real numbers. Algebraic facts about the collection of ordered pairs of real are reflected by geometric facts about the subset of the plane, and, conversely, geometric

facts about the plane subset are reflected by algebraic facts about the collection of pairs of reals.

Exercises

Draw a picture of the given relation:

1. $R = \{(x,y): 0 \leq x \leq 1, \text{ and } 1 \leq y \leq 4\}$

2. $R = \{(x,y): -4 \leq x \leq 4, \text{ and } -x^2 \leq y \leq x^2\}$

3. $R = \{(x,y): 1 \leq y \leq 2\} \cap \{(x,y): y \leq x^2\}$

4. $S = \{(x,y): x^2 + y^2 = 1, \text{ and } x \geq 0\}$

5. $S = \{(x,y): x^2 + y^2 \leq 1\} \cap \{(x,y): y \geq x^2\}$

6. $E = \{(r,s): |r| = |s|\}$

7. $T = \{(u,v): |u| + |v| = 1\}$

8. $R = \{(u,v): |u| + |v| \leq 1\}$

9. $T = \{(x,y): x^2 = y^2\}$

10. $A = \{(x,y): x^2 \leq y^2\}$

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11. $G = \{(s,t): \max\{|s|,|t|\} = 1\}$

**12.** $B = \{(s,t): \max\{|s|,|t|\} \quad 1\}$

1.2 Coordinates in Three-Space

Now let's see what's doing in three dimensions. We shall associate with each ordered triple of real numbers a point in three space. We continue from where we left off in the previous section. Start with the plane constructed in the previous section, and construct a line perpendicular to both the first and second axes, and passing through the origin. This is the ***third***, axis. Now we must be careful about which direction on this third axis is chosen as the positive direction; it makes a difference. The positive direction is chosen to be the direction in which a right-hand threaded bolt would advance if the positive first axis is rotated to the positive second axis:
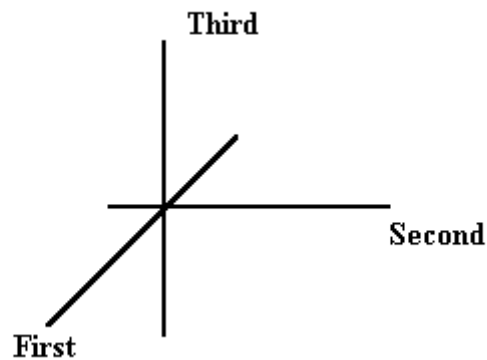
We now see how to define a one-to-one correspondence between ordered triples of real numbers (x_1, x_2, x_3) and the points in space. The association is a simple extension of the way in which we established a correspondence between ordered pair and points in

a plane. Here's what we do. Construct a plane perpendicular to the first axis through the point x_1, a plane perpendicular to the second axis through x_2, and a plane perpendicular to the third axis through x_3. The point at which these three planes intersect is the point associated with the ordered triple (x_1, x_2, x_3). Some meditation on this construction should convince you that this procedure establishes a one-to-one correspondence between ordered triples of reals and points in space. As in the two dimensional, or plane, case, x_1 is called the first coordinate of the point, x_2 is called the second coordinate of the point, and x_3 is called the third coordinate of the point. Again, the point corresponding to $(0,0,0)$ is called the *origin*, and we speak of the *point (x_1, x_2, x_3)*, when we actually mean the point which corresponds to this ordered triple.
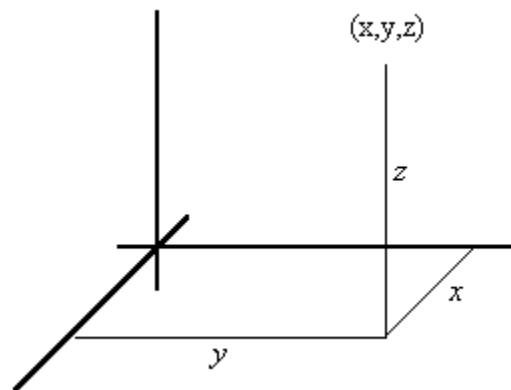
The three axes so defined is called a *coordinate system* for three space, and the three numbers x, y, and z, where (x, y, z) is the triple corresponding to the point P, are called the *coordinates* of P. The coordinate axes are sometimes given labels-most commonly, perhaps, the first axis is called the *x* axis, the second axis is called the *y* axis, and the third axis is called the *z* axis.

1.3 Some Geometry

Suppose P and Q are two points, and suppose space is endowed with a coordinate system such that $P = (x, y, z)$ and $Q = (u, v, w)$. How do we find the distance between P and Q ? This simple enough; look at the picture:
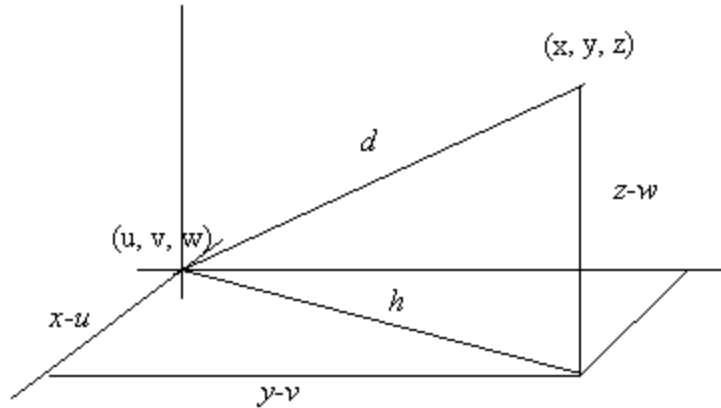
We can see that $d^2 = h^2 + (z - w)^2$ and $h^2 = (x - u)^2 + (y - v)^2$. Thus we have

$$d^2 = (x - u)^2 + (y - v)^2 + (z - w)^2 \text{, or}$$

$$d = \sqrt{(x - u)^2 + (y - v)^2 + (z - w)^2} \text{ .}$$

We saw that in the plane an equation in two variables defines in a natural way a collection of ordered pairs of numbers. The analogous situation obtains in three-space: an equation in three variables defines a collection of ordered triples. We thus speak of the collection of triples (x, y, z) which satisfy the equation

$$x^2 + y^2 + z^2 = 1$$

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The collection of all such points is the **graph** of the equation. In this example, it is easy to see that the graph is precisely the set of all points at a distance of 1 from the origin-a sphere of radius 1 and center at the origin.

The graph of the equation $x = 0$ is simply the set of all points with first coordinate 0, and this is clearly the plane determined by the second axis and the third axis, or the y axis and the z axis. When the axes are labeled x, y, and z, this is known as **the yz plane.** . Similarly, the plane $y = 0$ is the **xz plane**, and $z = 0$ is the **xy plane**. These special planes are also called the **coordinate planes.**

More often than not, it is difficult to see exactly what a graph of equation looks like, and even more difficult for most of us to draw it. Computers can help, but they usually draw rather poor pictures whose main application is in stimulating your own imagination sufficiently to allow you to see the graph in your mind's eye. An example:
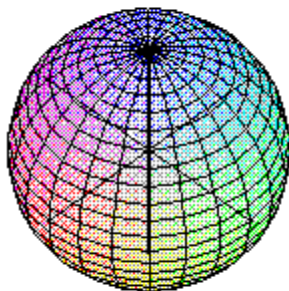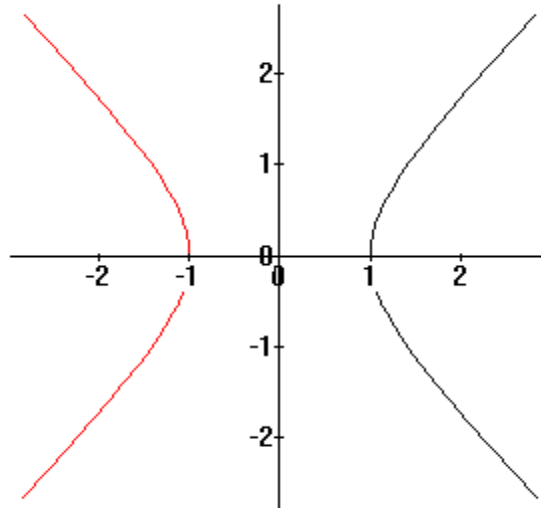
This picture was drawn using *Maple.*

Let's look at a more complicated example. What does the graph of

$$x^2 + y^2 - z^2 = 1$$

look like? We'll go after a picture of this one by slicing the graph with the coordinate planes. First, let's slice through it with the plane $z = 0$; then we see

$$x^2 + y^2 = 1,$$

a circle of radius 1 centered at the origin. Next, let's slice with the plane $y = 0$. Here we see $x^2 - z^2 = 1$, a hyperbola:

We, of course, see the same hyperbola when we slice the graph with the plane $x = 0$. What the graph looks like should be fairly clear by now:
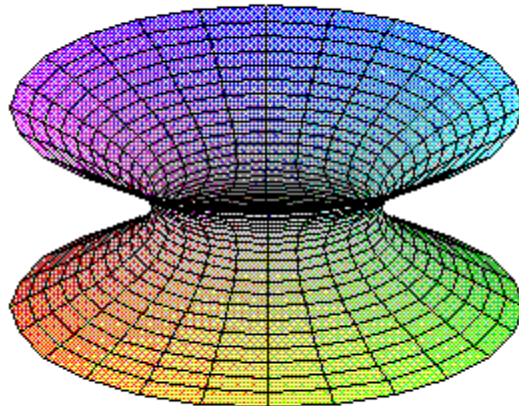
This graph has a name; it is called a **hyperboloid**.

Exercises

13. Describe the set of points $S = \{(x, y, z): x \geq 0, y \geq 0, \text{ and } z \geq 0\}$.

14. Describe the following sets

    a) $S = \{(x, y, z): z \geq 0\}$ b) $S = \{(x, y, z): x \geq 5\}$

    c) $R = \{(x, y, z): x^2 + y^2 \geq 1\}$ d) $T = \{(r, s, t): r^2 + s^2 + t^2 \leq 4\}$

15. Let G be the graph of the equation $x^2 + 4y^2 + 9z^2 = 36$.

    a) Sketch the graphs of the curves sliced from G by the coordinate planes $x = 0$, $y = 0$, and $z = 0$.

    b) Sketch G. (This graph is called an **ellipsoid.**)

16. Let G be the graph of the equation $x^2 - 3y^2 + 4z^2 = 12$.

    a) Sketch the graphs of the curves sliced from G by the coordinate planes $x = 0$, $y = 0$, and $z = 0$.

    b) Sketch G. (Does this set look at all familiar to you?.)

1.4 Some More Geometry-Level Sets

The curves that result from slicing the graphs with the coordinate planes are special cases of what are called level sets of a set. Specifically, if S is a set, the intersection of S with a plane $z = \text{constant}$ is called a **level set**. In case the level set is a

curve, it is frequently called a ***level curve***. (The slices by planes $x =$ constant, or $y =$ constant are also level sets.) A family of level sets can provide a nice stimulant to your powers of visualization. Everyday examples of the use of level sets to describe a set are contour maps, in which the contours are, of course, just level curves ; and weather maps, in which, for instance, the isoclines on a 500mb chart are simply level curves for the 500mb surface. Let's illustrate with an example.

Let S be the graph of

$$z^2 - y^2 - x^2 = 1$$

Now we look at the level set $z = c$:

$$c^2 - y^2 - x^2 = 1 \text{, or}$$

$$x^2 + y^2 = c^2 - 1.$$

Notice first that we have the same curve for $z = c$ and $z = -c$. The graph is symmetric about the plane $z = 0$. We shall thus look at just that part of the graph that is above the xy plane.
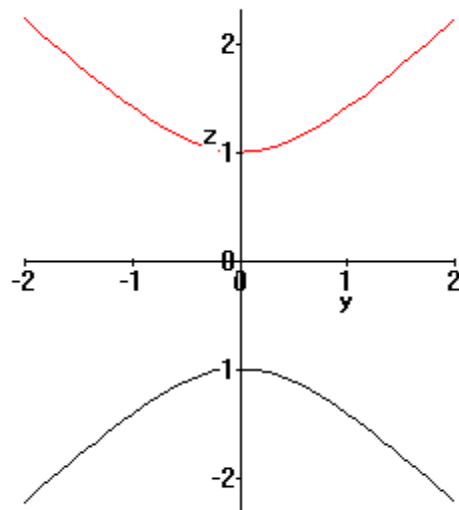
It is clear that these curves are concentric circles of radius $\sqrt{c^2 - 1}$ centered at the origin. There are no level sets for $|c| < 1$, and for $c = 1$ or -1, the level set is a single point, the origin.
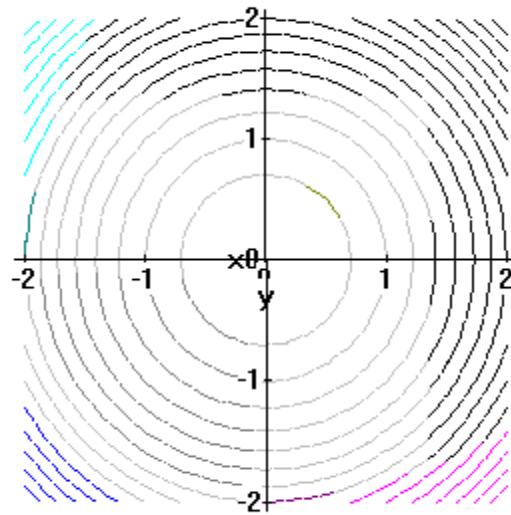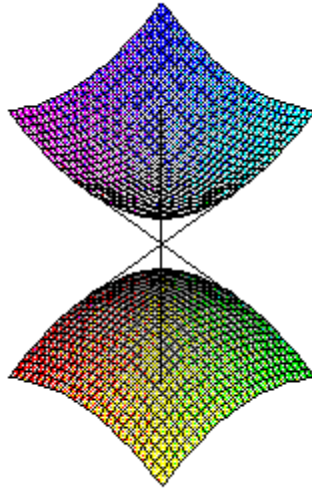
Next, slice with the planes $x = 0$ and $y = 0$ to get a better idea of what this thing looks like. For $x = 0$, we see

$$z^2 - y^2 = 1,$$

a hyperbola:

The slice by $y = 0$, of course, is the same. It is rather easy to visualize this graph. Here is a *Maple* drawn picture:

This also is called a hyperboloid. This is a hyperboloid of *two sheets*, while the previously described hyperboloid is a hyperboloid of *one sheet*.

Exercises

17. Let $S = \{(x_1, x_2, x_3): |x_1| + |x_2| + |x_3| = 1\}$.

a)Sketch the coordinate plane slices of S.

b)Sketch the set S.

18. Let C be the graph of the equation $z^2 = 4(x^2 + y^2)$.

a)Sketch some level sets $z = c$.

b)Sketch the slices by the planes $x = 0$ and $y = 0$.

c)Sketch C. What does the man on the street call this set?

19. Using level sets, coordinate plane slices, and whatever, describe the graph of the equation $z = x^2 + y^2$. (This one has a name also; it is a *paraboloid*.).

20. Using level sets, coordinate plane slices, and whatever, describe the graph of the equation $z = x^2 - y^2$.