YET ANOTHER TABLE OF INTEGRALS

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ABSTRACT. This collection of sums and integrals has been harvested from the mathematical and physical literature in unstructured ways. Its main use is backtracking the original sources whenever an integral of the reader's application resembles one of the items in the collection.

INTRODUCTION

Dealing with the analysis of real numbers in the physical sciences shows a strange attraction towards integrals. Closed-form integration beats numerical integration, and often adaptive series expansion helps to crumble cumbersome integral kernels to digestable pieces.

The current table started as a incoherent list of bookmarks pointing to "interesting" formulas that complement or correct the Gradstein-Rhyshik tables [91], see http://www.mathtable.com/gr/. As such it does not replicate the original sources in full but is to be merely regarded as an aid to find places at which certain forms and classes of integrals or sums have been targeted.

The notation is generally not harmonized. Stirling numbers appear in bracketed and indexed notations, and at least two different meanings of harmonic numbers H with lower and upper indices are met.

There is only one hint of use: The list of references appears *prior* to each formula.

0.1. **Finite Series.** [181, 176]

$$\sum_{j=1}^{n} j^k = \frac{1}{n}\left(\rho(n,i) + \sum_{i=1}^{n}\sigma_{k+1}(i)\right) = \frac{B_{k+1}(n+1) - B_{k+1}}{k+1} \tag{0.1}$$

where $\rho(n,k) \equiv \sum_{d=1}^{n} d^k (n \mod d)$ is a sum over $n \mod d$ multiplied, then summed, over d^k, and $\sigma_k(n) = \sum_{d|n} d^k$.

[55, 67]

$$\sum_{k=1}^{n-1} k^r = \sum_{k=0}^{r} \frac{B_k}{k!}\frac{r!}{(r-k+1)!} n^{r-k+1}. \tag{0.2}$$

[182]

$$\sum_{k=0}^{n} k^m = \sum_{j=0}^{m} \left\{ \begin{matrix} m \\ j \end{matrix} \right\} \binom{n+1}{j+1} j!. \tag{0.3}$$

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By telescoping the last term in this formula becomes

$$n^m = \sum_{j=0}^{m} \left\{ \begin{array}{c} m \\ j \end{array} \right\} \binom{n}{j} j!. \tag{0.4}$$

[186]

$$\sum_{0\le j\le n/m} \binom{n-jm}{k} = P(n+m,k,m) - P(r,k,m) \tag{0.5}$$

for $n \ge 0$, $k \ge 1$, $m \ge 1$ where

$$P(x,k,m) \equiv \frac{1}{m} \sum_{j=1}^{k+1} \binom{x}{j} A(m,k+1-j) \tag{0.6}$$

with g.f.

$$\frac{mx}{(1+x)^m - 1} = \sum_{j=0}^{\infty} A(m,j)x^j. \tag{0.7}$$

[166]

$$\sum_{k=0}^{n} \binom{r+k}{k} = \binom{r+n+1}{n}, \quad n = 0,1,2,\ldots \tag{0.8}$$

[182]

$$\sum_{k} \binom{n}{2k} k = n2^{n-3}, \quad n \ge 2. \tag{0.9}$$

[182]

$$\sum_{k} \binom{n}{2k+1} k = (n-2)2^{n-3}, \quad n \ge 2. \tag{0.10}$$

[182]

$$\sum_{k} \binom{n}{2k} k^{\underline{m}} = n(n-m-1)^{\underline{m-1}} 2^{n-2m-1}, \quad n \ge m+1, \tag{0.11}$$

[182]

$$\sum_{k} \binom{n}{2k+1} k^{\underline{m}} = n(n-m-1)^{\underline{m}} 2^{n-2m-1}, \quad n \ge m+1, \tag{0.12}$$

where $k^{\underline{m}}$ is the falling factorial $k(k-1)(k-2)\cdots(k-m+1)$.
[182]

$$\sum_{k=0}^{n} \binom{n}{k} k^m = \sum_{j=0}^{m} \left\{ \begin{array}{c} m \\ j \end{array} \right\} \binom{n}{j} j! 2^{n-j}. \tag{0.13}$$

[182]

$$\sum_{k=0}^{n} \binom{n}{k} (-1)^k k^m = \left\{ \begin{array}{c} m \\ n \end{array} \right\} (-1)^n n!. \tag{0.14}$$

[182]

$$\sum_{k=0}^{n} \binom{n}{2k} k^m = n \sum_{j=1}^{\min(m,n-1)} \left\{ \begin{matrix} m \\ j \end{matrix} \right\} \binom{n-j-1}{j-1} (j-1)! 2^{n-2j-1}. \tag{0.15}$$

[182]

$$\sum_{k=0}^{n} \binom{n}{2k+1} k^m = \sum_{j=1}^{\min(m,n-1)} \left\{ \begin{matrix} m \\ j \end{matrix} \right\} \binom{n-j-1}{j-1} j! 2^{n-2j-1}. \tag{0.16}$$

[107]

(0.17)

$$\sum_{0\le k\le N, k\ne K} \binom{N}{k} (-1)^k \frac{1}{(k-K)^m} = \binom{N}{K} (-1)^{K+1} \frac{1}{m!} Y_m(\ldots, (i-1)! [H_{N-K}^{(i)} + (-1)^i H_K^{(i)}], \ldots)$$

where $0 \le K \le N$, $Y_m(\ldots, x_i, \ldots)$ are the Bell polynomials and $H_r^{(i)} = \sum_{j=1}^{r} j^{-i}$ Harmonic numbers of the i-th order.

[107]

$$\sum_{0\le k\le N, k\ne K} \binom{N}{k} (-1)^k \frac{1}{k-K} = \binom{N}{K} (-1)^{K+1} (H_{N-K} - H_K) \tag{0.18}$$

where $0 \le K \le N$, $H_r^{(i)} = \sum_{j=1}^{r} j^{-i}$ Harmonic numbers of the i-th order.

[107]

$$\sum_{0\le k\le N} \binom{N}{k} (-1)^k \frac{1}{k^m} = -\frac{1}{m!} Y_m(\ldots, (i-1)! H_N^{(i)}, \ldots) \tag{0.19}$$

where $H_r^{(i)} = \sum_{j=1}^{r} j^{-i}$ Harmonic numbers of the i-th order.

[107]

(0.20)

$$\sum_{0\le k\le N} \binom{N}{k} (-1)^k \frac{1}{(k-\xi)^m} = \Gamma(-\xi) \frac{\Gamma(N+1)}{\Gamma(N+1-\xi)} \frac{1}{(m-1)!} Y_{m-1}(\ldots, (i-1)! \zeta_N(i, -\xi), \ldots),$$

where $\zeta_N(i, -\zeta) = \sum_{j=0}^{N} (j-\xi)^{-j}$.

[161, §4.3]

$$a_n = \sum_{k=0}^{n} \binom{p+k}{k} b_{n-k} \Leftrightarrow b_n = \sum_{k=0}^{n} (-1)^k \binom{p+k}{k} a_{n-k}. \tag{0.21}$$

[161, §4.3]

$$a_n = \sum_{k=0}^{n} \binom{p+k}{k} b_{n-qk} \Leftrightarrow b_n = \sum_{k=0}^{n} (-1)^k \binom{p+k}{k} a_{n-qk}. \tag{0.22}$$

[161, §4.3]

$$a_n = \sum_{k=0}^{n} \binom{2k}{k} b_{n-k} \Leftrightarrow b_n = \sum_{k=0}^{n} \frac{1}{1-2k} \binom{2k}{k} a_{n-k}. \tag{0.23}$$

[161, §4.3]

$$a_n = \sum_{k=0}^{n} \frac{1}{k+1}\binom{2k}{k} b_{n-k} \Leftrightarrow b_n = a_n - \sum_{k=1}^{n} \frac{1}{k}\binom{2k}{k} a_{n-k}. \tag{0.24}$$

[156]

$$x_n = a_n + 2^{1-n}\sum_{k=0}^{n}\binom{n}{k} x_k \Leftrightarrow x_n = x_0 + \sum_{k=0}^{n}(-1)^k\binom{n}{k}\frac{k\bar{x}_1 + \hat{a}_k - \bar{x}_0}{1-2^{1-k}} \tag{0.25}$$

where $\bar{x}_0 \equiv a_0 + x_0$, $\bar{x}_1 = a_1 + x_0$ and

$$\hat{a}_n = \sum_{k=0}^{n}\binom{n}{k}(-1)^k a_k. \tag{0.26}$$

[156]

$$x_{n+1} = a_{n+1} + 2^{1-n}\sum_{k=0}^{n}\binom{n}{k} x_k \Leftrightarrow x_n = -\sum_{k=0}^{n}(-1)^k\binom{n}{k}\hat{x}_{k-2} \tag{0.27}$$

where $\bar{x}_0 = 0$,

$$\hat{x}_n = Q_n \sum_{i=1}^{n+1}(\hat{a}_i - \hat{a}_{i+1} - a_1)/Q_{i-1}, \quad Q_n \equiv \prod_{k=1}^{n}(1-2^{-k}). \tag{0.28}$$

[106]

$$u_n = Au_{n-1} - Bu_{n-2},\ v_n = Av_{n-1} - Bv_{n-2}, \tag{0.29}$$

$$\rightsquigarrow u_{n+1} = \prod_{j=1}^{n}\left(A - 2i\sqrt{-B}\cos\frac{\pi j}{n+1}\right) = (i\sqrt{-B})^n \frac{\sin\left((n+1)\cos^{-1}\frac{-iA}{2\sqrt{-B}}\right)}{\sin\left(\cos^{-1}\frac{-iA}{2\sqrt{-B}}\right)}, \tag{0.30}$$

$$\rightsquigarrow v_n = \prod_{k=1}^{n}\left(A - 2i\sqrt{-B}\cos\frac{\pi(k-1/2)}{n}\right) = 2(i\sqrt{-B})^n\cos\left(n\cos^{-1}\frac{-iA}{2\sqrt{-B}}\right), \tag{0.31}$$

where $u_0 = 0$, $u_1 = 1$, $v_0 = 2$, $v_1 = A$

[201]

$$\sum_{x=1}^{p-1}F(x) + \sum_{x=1}^{p-1}\left(\frac{x}{p}\right)F(x) = \sum_{x=1}^{p-1}F(x^2), \tag{0.32}$$

if p is an odd prime and $F(x) = F(x+p)$, where $(\frac{x}{p})$ is the Legendre Symbol.

[201]

$$\sum_{x=1}^{p-1}\left(\frac{x}{p}\right)e(k(x+\bar{x})) = \left(\frac{k}{p}\right)i^{(p-1)^2/4}p^{1/2}(e(2k)+e(-2k)). \tag{0.33}$$

where $\bar{x}$ is the unique solution to $x\bar{x} \equiv 1 \pmod{p}$, and $e(t) = e^{2\pi i t/p}$.

[48]

$$\sum_{k=0}^{m}\binom{m}{k}\frac{(n+k)!}{(n+k+s)!}a_{n+k+s}=\sum_{k=0}^{n}\binom{n}{k}\frac{(-1)^{n-k}(m+k)!}{(m+k+s)!}b_{m+k+s}$$
$$+\sum_{j=0}^{s-1}\sum_{i=0}^{s-1-j}\binom{s-1-j}{i}\binom{s-1}{j}\frac{(-1)^{n+1+i}a_j}{(s-1)!(m+n+1+i)\binom{m+n+i}{n}}, \tag{0.34}$$

where $b_n \equiv \sum_{k=0}^{n}\binom{n}{k}a_k$.

[48]

$$\sum_{k=0}^{m}\binom{m}{k}\binom{n+k}{s}a_{n+k+s}=\sum_{k=0}^{n}\binom{n}{k}\binom{m+k}{s}(-1)^{n-k}b_{m+k+s}, \tag{0.35}$$

where $b_n \equiv \sum_{k=0}^{n}\binom{n}{k}a_k$.

[48]

$$\sum_{k=0}^{m}\frac{\binom{m}{k}}{\binom{n+k+s}{s}}x^{m-k}A_{n+k+s}(y)=\sum_{k=0}^{n}\frac{\binom{n}{k}}{\binom{m+k+s}{s}}(-1)^{n+m+s}x^{n-k}A^*_{m+k+s}(z)$$
$$+\sum_{j=0}^{s-1}\sum_{i=0}^{s-1-j}\binom{s-1-j}{i}\binom{s-1}{j}\frac{(-1)^{n+1+i}x^{m+n+s-j}sA_j(y)}{(m+n+1+i)\binom{m+n+i}{n}}, \tag{0.36}$$

and

(0.37)
$$\sum_{k=0}^{m}\binom{m}{k}\binom{n+k}{s}x^{m-k}A_{n+k+s}(y)=\sum_{k=0}^{n}\binom{n}{k}\binom{m+k}{s}(-1)^{n+m+s}x^{n-k}A^*_{m+k+s}(z)$$

where $A_n(x) \equiv \sum_{k=0}^{n}\binom{n}{k}(-1)^k a_k x^{n-k}$ and $A^*_n(x) \equiv \sum_{k=0}^{n}\binom{n}{k}(-1)^k a^*_k x^{n-k}$ and $a^*_n \equiv \sum_{k=0}^{n}\binom{n}{k}(-1)^k a_k$ and $x+y+z=1$.

[48]

$$a_{n,m}=\sum_{k=0}^{n}\binom{n}{k}\alpha^{n-k}\beta^k a_{0,m+k}, \tag{0.38}$$

and

$$\sum_{k=0}^{n}\binom{n}{k}\alpha^{n-k}\beta^k a_{0,m+k}=\sum_{k=0}^{m}\binom{m}{k}(-\alpha)^{m-k}\beta^{-m}a_{n+k,0}, \tag{0.39}$$

where $a_{n,m} \equiv \alpha a_{n-1,m}+\beta a_{n-1,m+1}$ for $n \geq 1, m \geq 0$.

[92]

(0.40)
$$x_{nm}=\sum_{k=0}^{m}x_{mk}\binom{n+k}{2m} \rightsquigarrow x_{nm}=\sum_{k=0}^{m}\frac{2k+1}{m+k+1}\binom{n+k}{m+k}\binom{n-1-k}{m-k}x_{kk},$$

for $m<n$, $n \geq 0$, $0 \leq m \leq n$.

[33]

(0.41)
$$F(x) \equiv \sum_{k\geq 1}f(x/k); \quad G(x) \equiv \sum_{k\geq 1}(-)^k f(x/k) \rightsquigarrow F(x)=2^{-n}F(2^n x)+\sum_{k\geq 1}2^{-k}G(2^k x).$$

[33]
(0.42)
$$F(x) = r_1F(m_1x) + r_2G(m_2x) \rightsquigarrow r_2\sum_{k=1}^{n} r_1^{k-1}G(m_1^{k-1}m_2x) = F(x) - r_1^nF(m_1^nx).$$

[182]

$$\sum_{k=0}^{n}\binom{n}{k}H_k = 2^n\left(H_n - \sum_{k=1}^{n}\frac{1}{k2^k}\right). \tag{0.43}$$

where H_k are the harmonic numbers.
[182]

$$\sum_k \binom{n}{2k}\frac{1}{k+1} = \frac{n2^{n+1}+2}{(n+1)(n+2)}. \tag{0.44}$$

[182]

$$\sum_k \binom{n}{2k+1}\frac{1}{k+1} = \frac{2^{n+1}-2}{n+1}. \tag{0.45}$$

[164]

$$\sum_{j=0}^{n}\binom{m-a+b}{j}\binom{n+a-b}{n-j}\binom{a+j}{m+n} = \binom{a}{m}\binom{b}{n}, \tag{0.46}$$

where n, m are integer and a, b real.
[164]

$$\sum_{m=0}^{P}(-)^m\binom{P}{m}\binom{a-m}{M} = \binom{a-P}{M-P}, \tag{0.47}$$

where P, M are integer and a, b real.
[166]
(0.48)
$$\sum_k \binom{r}{k}\binom{s}{n+k} = \binom{r+s}{r+n}, \quad r = 0, 1, 2, \ldots \quad , n = \ldots - 2, -1, 0, 1, 2, 3, \ldots$$

[164, 166]

$$\sum_{m=0}^{n}(-)^m\binom{n}{m}\binom{a+m}{p} = (-)^n\binom{a}{p-n}, \tag{0.49}$$

where n, p are integer and a is real.
[166]

$$\sum_{k\ge 0}\binom{n+k}{m+2k}\binom{2k}{k}\frac{(-1)^k}{k+1} = \binom{n-1}{m-1}. \tag{0.50}$$

[166]

$$\sum_{\nu=0}^{s}(-)^\nu\binom{\beta}{\nu}\binom{\beta+s-\nu}{\beta}\frac{\alpha}{\alpha+s-\nu} = \frac{(\alpha-\beta)_s}{(\alpha+1)_s}. \tag{0.51}$$

[166]
(0.52)
$$\sum_{k=-l}^{l} (-)^k \binom{2l}{l+k}\binom{2m}{m+k}\binom{2n}{n+k} = \frac{(l+m+n)!(2l)!(2m)!(2n)!}{(l+m)!(m+n)!(n+l)!l!m!n!}, \quad l = \min(l,m,n).$$

[199]
$$\sum_k \sum_j \binom{n}{k}\binom{n+k}{k}\binom{k}{j}^3 = \sum_k \binom{n}{k}^2\binom{n+k}{k}^2. \tag{0.53}$$

[199]
$$\sum_{i=0}^{n}\sum_{j=0}^{n} \binom{i+j}{i}^2 \binom{4n-2i-2j}{2n-2i} = (2n+1)\binom{2n}{n}. \tag{0.54}$$

[199]
(0.55)
$$\sum_{k_1}\sum_{k_2\le k_1}\sum_{k_3\le k_2} (k_1-k_2)(k_1-k_3)(k_2-k_3)\binom{n}{k_1}\binom{n}{k_2}\binom{n}{k_3} = n^2(n-1)8^{n-2}\frac{(3/2)_{n-2}}{(3)_{n-2}}.$$

[199]
$$\sum_i\sum_j \binom{i+j}{i}\binom{n-i}{j}\binom{n-j}{n-i-j} = \sum_{k=0}^{n}\binom{2k}{k}. \tag{0.56}$$

[199]
$$\sum_{i=0}^{n}\sum_{j=0}^{m} \binom{i+j}{j}^2\binom{m+m-i-j}{n-j}^2 = \frac{1}{2}\binom{2m+2n+2}{2n+1}. \tag{0.57}$$

[199]
$$\sum_{s=0}^{k}\sum_{b\ge 0} (-)^b \binom{s}{b}\binom{k-s}{2v-b}\binom{k-2v}{s-b} = \binom{k-v}{k-2v}2^{k-2v}, \quad k \ge 2v. \tag{0.58}$$

[199]
(0.59)
$$\sum_{j,k} (-)^{j+k}\binom{j+k}{k+l}\binom{r}{j}\binom{n}{k}\binom{s+n-j-k}{m-j} = (-)^l\binom{n+r}{n+l}\binom{s-r}{m-n-l}.$$

[49]
(0.60)
$$\sum_{i=0}^{n}\sum_{j=0}^{n}\binom{i+j}{i}\binom{m-i+j}{j}\binom{n-j+i}{i}\binom{m+n-i-j}{m-i} = \frac{(m+n+1)!}{m!n!}\sum_k \frac{1}{2k+1}\binom{m}{k}\binom{n}{k}.$$

[49]
$$\sum_{i=0}^{n}\sum_{j=0}^{n} \binom{i+j}{i}^2\binom{4n-2i-2j}{2n-2i} = (2n+1)\binom{2n}{n}^2. \tag{0.61}$$

[49]

(0.62)
$$\sum_{i=0}^{\lfloor m/2\rfloor}\sum_{j=0}^{\lfloor n/2\rfloor}\binom{i+j}{i}^2\binom{m+n-2i-2j}{n-2i}=\frac{\lfloor (m+n+1)/2\rfloor!\lfloor (m+n+2)/2\rfloor!}{\lfloor m/2\rfloor!\lfloor (m+1)/2\rfloor!\lfloor (n+1)/2\rfloor!}.$$

[49]

$$\sum_i\sum_j\binom{n}{j}\binom{n+j}{j}\binom{j}{i}^2\binom{2i}{i}^2\binom{2i}{j-i}=\sum_k\binom{n}{k}^3\binom{n+k}{k}^3. \quad (0.63)$$

[49]

(0.64)
$$\sum_j\sum_k(-1)^{j+k}\binom{j+k}{k+l}\binom{r}{j}\binom{n}{k}\binom{s+n-j-k}{m-j}=(-1)^l\binom{n+r}{n+l}\binom{s-r}{m-n-l}.$$

[49, 199]

$$\sum_r\sum_s(-1)^{n+r+s}\binom{n}{r}\binom{n}{s}\binom{n+s}{s}\binom{n+r}{r}\binom{2n-r-s}{n}=\sum_k\binom{n}{k}^4. \quad (0.65)$$

[172]

$$\binom{n}{i}^k=\binom{n+ik-i}{ik}+A_2\binom{n+ik-i-1}{ik}+\cdots A_{ik-i}\binom{n+1}{ik}+\binom{n}{ik}, \quad (0.66)$$

$$\begin{aligned}S_i^k(n,0) &\equiv \binom{n}{i}^k;\\ S_i^k(n,p) &\equiv S_i^k(1,p-1)+S_i^k(2,p-1)+\cdots S_i^k(n,k-1)\\ &= \binom{n+ik-i+p}{ik+p}+A_2\binom{n+ik-i+p-1}{ik+p}+\cdots+A_{ik-i}\binom{n+p+1}{ik+p}+\binom{n+p}{ik+p}\end{aligned}$$

where $A_j=A_{ik-i-j+2}$ for $j=2,3,\ldots,ik-i$,

$$A_j=\begin{vmatrix}1 & 0 & 0 & \cdots & 0 & \binom{i}{i}^k\\ \binom{ik+1}{ik} & 1 & 0 & \cdots & 0 & \binom{i+1}{i}^k\\ \binom{ik+2}{ik} & \binom{ik+1}{ik} & 1 & \cdots & 0 & \binom{i+2}{i}^k\\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots\\ \binom{ik+j-1}{ik} & \binom{ik+j-2}{ik} & \binom{ik+j-3}{ik} & \cdots & \binom{ik+1}{ik} & \binom{i+j-1}{i}^k\end{vmatrix} \quad (0.67)$$

[189]

$$\sum_{k=0}^{2n}(-)^k\frac{\binom{2n}{k}}{\binom{4n}{2k}}=\frac{4n+1}{2n+1}. \quad (0.68)$$

[189]

$$\sum_{k=0}^{2n}(-)^k\frac{\binom{4n}{2k}}{\binom{2n}{k}}=-\frac{1}{2n-1}. \quad (0.69)$$

[189]

$$\sum_{k=0}^{m} \frac{\binom{m}{k}}{\binom{n+m}{p+k}} = \frac{n+m+1}{n+1} \binom{n}{p}^{-1}, \tag{0.70}$$

with m, n, p nonnegative integers and $p \leq n$.

[189]

$$\sum_{k=0}^{n} (-)^k \frac{1}{\binom{n+m}{m+k}} = \frac{n+m+1}{m+n+2} \left(\binom{m+n+1}{m}^{-1} + (-)^n \right), \tag{0.71}$$

with m and n nonnegative integers.

[182]

$$\sum_{k=0}^{n} \left[\begin{array}{c} n \\ k \end{array} \right] H_k = n! \sum_{k=1}^{n} \frac{c_{k-1}}{k!}, \tag{0.72}$$

where H_k are the harmonic numbers and $[\ldots]$ unsigned Stirling numbers of the first kind.

[182]

$$\sum_{k=0}^{n} s(n,k) = s(n-1,0) + s(n-1,-1), \tag{0.73}$$

where $s(.,.)$ are the Stirling numbers of the first kind.

[182]

$$\sum_{k=0}^{n} s(n,k)k = s(n-1,1) + s(n-1,0). \tag{0.74}$$

[182]

$$n! \sum_{k=0}^{n} s(k,m)(-1)^{n-k}/k! = s(n+1,m+1). \tag{0.75}$$

[182]

$$\sum_{k=0}^{n} s(n,k) k^{\underline{m}} = m![s(n-1,m) + s(n-1,m-1)], \tag{0.76}$$

where $k^{\underline{m}} = k(k-1)(k-2)\cdots(k-m+1)$.

[182]

$$\sum_{k=0}^{n} s(n,k) k^m = \sum_{j=0}^{m} \left\{ \begin{array}{c} m \\ j \end{array} \right\} (s(n-1,j) + s(n-1,j-1))\, j!. \tag{0.77}$$

[182]

$$\sum_{k=0}^{n} \frac{s(n,k)}{k+1} = b_n, \tag{0.78}$$

where $b_n = \int_0^1 x^{\underline{n}} dx$ are the Cauchy numbers of the first type [175, A006232].

[182]

$$n! \sum_{k=0}^{n} \left[\begin{array}{c} k \\ m \end{array} \right] \frac{1}{k!} = \left[\begin{array}{c} n+1 \\ m+1 \end{array} \right]. \tag{0.79}$$

where $[\cdots]$ are the unsigned Stirling numbers of the first kind.

[182]

$$\sum_{k=0}^{n} \left[\begin{array}{c} n \\ k \end{array}\right] k = \left[\begin{array}{c} n+1 \\ 2 \end{array}\right]. \tag{0.80}$$

[182]

$$\sum_{k=0}^{n} \left[\begin{array}{c} n \\ k \end{array}\right] k^{\underline{m}} = \left[\begin{array}{c} n+1 \\ m+1 \end{array}\right] m!, \tag{0.81}$$

where $k^{\underline{m}} = k(k-1)(k-2)\cdots(k-m+1)$.

[182]

$$\sum_{k=0}^{n} \left[\begin{array}{c} n \\ k \end{array}\right] k^{m} = \sum_{j=0}^{m} \left[\begin{array}{c} n+1 \\ j+1 \end{array}\right] \left\{\begin{array}{c} m \\ j \end{array}\right\} j!. \tag{0.82}$$

[182]

$$\sum_{k=0}^{n} \left[\begin{array}{c} n \\ k \end{array}\right] \frac{1}{k+1} = c_n, \tag{0.83}$$

where $c_n = \int_0^1 (x)_n dx$ are the Cauchy numbers of the second type [175, A002657], using Pochhammer's symbol.

[48]

$$\begin{aligned} \sum_{k=0}^{m} \binom{m}{k} (n+k)! \left\{\begin{array}{c} n+k+s \\ q \end{array}\right\} &= \sum_{k=0}^{n} \binom{n}{k} (m+k)!(-1)^{n-k} \frac{(m+k+s)^q}{(m+k+s)!} \\ &+ \sum_{j=0}^{s-1} \sum_{i=0}^{s-1-j} \binom{s-1-j}{i} \frac{(-1)^{n+1+i} \left\{\begin{array}{c} j \\ q \end{array}\right\}}{(s-1-j)!(m+n+1+i)\binom{m+n+i}{n}} \end{aligned} \tag{0.84}$$

and

$$\sum_{k=0}^{m} (n+k)! \left\{\begin{array}{c} n+k-s \\ q \end{array}\right\} = \sum_{k=0}^{n} \binom{n}{k} (m+k)!(-1)^{n-k} \frac{(m+k-s)^q}{(m+k-s)!}. \tag{0.85}$$

[138]

$$\left\{\begin{array}{c} n \\ m \end{array}\right\}_r = \sum_{k=2}^{n} \binom{n}{k} \sum_{l=1}^{k-1} (-1)^{l-1} \binom{l+r-2}{l-1} \left\{\begin{array}{c} k-l \\ m-1 \end{array}\right\}_{r-1} \tag{0.86}$$

where $\{\}_r$ are r-Stirling numbers of the second kind, namely

$$\left\{\begin{array}{c} n \\ m \end{array}\right\}_r = \begin{cases} 0, & n < r \\ \delta_{mr} & n = r \\ m \left\{\begin{array}{c} n-1 \\ m \end{array}\right\}_r + \left\{\begin{array}{c} n-1 \\ m-1 \end{array}\right\}_r, & n > r. \end{cases} \tag{0.87}$$

0.2. **Numerical Series.** [56, 52][175, A152649]

$$\sum_{n=1}^{\infty} \frac{H_n^{(1)}}{n^3} = \frac{1}{2}\zeta^2(2), \tag{0.88}$$

where $H_n^{(r)} \equiv \sum_{k=1}^{n} \frac{1}{k^r}$.
[52]

$$\sum_{n=1}^{\infty} \frac{{H_n^{(1)}}^2}{n^4} = \frac{97}{24}\zeta(6) - 2\zeta^2(3) \approx 1.22187994531988. \tag{0.89}$$

[52]

$$\sum_{n=1}^{\infty} \frac{{H_n^{(1)}}^3}{n^3} = \zeta^2(3) - \frac{1}{3}\zeta(6) \approx 1.1058264444388. \tag{0.90}$$

[56]

$$\sum_{n=1}^{\infty} \frac{H_n^{(1)}}{n^{2p+1}} = \frac{1}{2}\sum_{j=2}^{2p}(-1)^j\zeta(j)\zeta(2p-j+2), \tag{0.91}$$

where $H_n^{(r)} \equiv \sum_{k=1}^{n} \frac{1}{k^r}$.
[35, 137, 179]

$$\sum_{n=1}^{\infty} \frac{H_n^{(1)}}{n^m} = \frac{m+2}{2}\zeta(m+1) - \frac{1}{2}\sum_{k=1}^{m-2}\zeta(m-k)\zeta(k+1), \tag{0.92}$$

where $H_n^{(r)} \equiv \sum_{k=1}^{n} \frac{1}{k^r}$.
[179]

$$\sum_{n=1}^{\infty} \frac{H_n}{n(n+\alpha)} = \frac{1}{2\alpha}[3\zeta(2) + \psi^2(\alpha) + 2\gamma\psi(\alpha) + \gamma^2 - \psi'(x)], \tag{0.93}$$

and

$$\sum_{n=1}^{\infty} \frac{H_n}{(n+\alpha)^2} = \gamma\psi'(\alpha) + \psi(\alpha)\psi'(\alpha) - \frac{1}{2}\psi''(\alpha), \tag{0.94}$$

and
(0.95)

$$\sum_{n=1}^{\infty} \frac{H_n}{(n+x)^q} = \frac{(-)^q}{(q-1)!}\left[\left(\psi(x)+\gamma\right)\psi^{(q-1)}(x) - \frac{1}{2}\psi^{(q)}(x) + \sum_{m=1}^{q-2}\binom{q-2}{m}\psi^{(m)}\psi^{(q-m-1)}(x)\right].$$

The paper also demonstrates a finite expansion of $\sum_{n>=1} H_n/[n^q\binom{an+k}{k}^t]$ in terms of ζ and ψ functions for $t=1$ and 2.
[137]

$$S(r,m) = S(1,m) + \sum_{k=1}^{r-1}\frac{S(k,m-1)-B(k,m)}{k}. \tag{0.96}$$

where $H_n^{(r)} \equiv \sum_{k=1}^{n} \frac{1}{k^r}$, where $S(r,m) \equiv \sum_{n=1}^{\infty} \frac{H_n^{(r)}}{n^m}$, where $B(k,m) \equiv {}_{m+1}F_m(1,1,\ldots,1,k+1;2,2,\ldots,2;1)$.

[137]

$$S(2,3)=\frac{\pi^4}{72}-\frac{\pi^2}{6}+2\zeta(3)\approx 2.1120837816098848. \tag{0.97}$$

[137]

$$S(2,4)=\frac{\pi^4}{72}+3\zeta(5)-\zeta(3)\left(1+\frac{\pi^2}{6}\right). \tag{0.98}$$

[21, 52]

$$\sum_{k=1}^{\infty}\frac{H_k^{(1)2}}{k^2}=\frac{17}{4}\zeta(4). \tag{0.99}$$

[21]

$$\sum_{k=1}^{\infty}\frac{H_k^{(1)2}}{(k+1)^n}=\frac{1}{3}n(n+1)\zeta(n+2)+\zeta(2)\zeta(n)-\frac{1}{n}\sum_{k=0}^{n-2}\zeta(n-k)\zeta(k+2)\\ +\frac{1}{3}\sum_{k=2}^{n-2}\zeta(n-k)\sum_{j=1}^{k-1}\zeta(j+1)\zeta(k+1-j)+\sum_{k=1}^{\infty}\frac{H_k^{(2)}}{(k+1)^n}. \tag{0.100}$$

[21]

$$\sum_{k=1}^{\infty}\frac{H_k^{(2)}}{(k+1)^2}=\frac{1}{2}\zeta^2(2)-\frac{1}{2}\zeta(4)=\frac{1}{120}\pi^4. \tag{0.101}$$

[167, 21][175, A214508]

$$A_4\equiv\sum_{k=1}^{\infty}(-)^{k+1}\frac{1}{(k+1)^2}H_k^{(2)}=-4\operatorname{Li}_4(1/2)+\frac{13\pi^4}{288}+\log(2)[-\frac{7}{2}\zeta(3)+\frac{\pi^2}{6}\log 2-\frac{\log^3 2}{6}]. \tag{0.102}$$

[21]

$$\sum_{k=1}^{\infty}\frac{H_k^{(2)}}{(k+1)^4}=-6\zeta(6)+\frac{8}{3}\zeta(2)\zeta(4)+\zeta^2(3)=\zeta^2(3)-\frac{4}{2835}\pi^6. \tag{0.103}$$

[21]

$$\sum_{k=1}^{\infty}\frac{H_k^{(1)}}{(k+1)^n}=\frac{1}{2}n\zeta(n+1)-\frac{1}{2}\sum_{k=1}^{n-2}\zeta(n-k)\zeta(k+1). \tag{0.104}$$

[21]

$$\sum_{k=1}^{\infty}\frac{H_k^{(2)}}{(k+1)^{2n-1}}=-\frac{1}{2}(2n^2+n+1)\zeta(2n+1)+\zeta(2)\zeta(2n-1)+\sum_{k=1}^{n-1}2k\zeta(2k+1)\zeta(2n-2k). \tag{0.105}$$

[21]

(0.106)
$$\sum_{k=1}^{\infty}\frac{H_k^{(1)2}}{(k+1)^{2n-1}}=\frac{1}{6}(2n^2-7n-3)\zeta(2n+1)+\zeta(2)\zeta(2n-1)-\frac{1}{2}\sum_{k=1}^{n-2}(2k-1)\zeta(2n-1-2k)\zeta(2k+2)$$
$$+\frac{1}{3}\sum_{k=1}^{n-2}\zeta(2k+1)\sum_{j=1}^{n-2-k}\zeta(2j+1)\zeta(2n-1-2k-2j).$$

[138]

(0.107)
$$\sum_{n=1}^{\infty}\frac{H_n^{(2)}}{n!}=e\left[1+\frac{1}{4}\,{}_2F_2\left(\begin{matrix}1 & 1\\ 3 & 3\end{matrix}\;|\;-1\right)\right],$$

(0.108)
$$\sum_{n=1}^{\infty}\frac{H_n^{(4)}}{2^n n!}=\sqrt{e}\left[\frac{H_1^{(3)}}{2^1 1!}+\frac{H_2^{(2)}}{2^2 2!}+\frac{H_3^{(1)}}{2^3 3!}+\frac{3!}{2^4(4!)^2}\,{}_2F_2\left(\begin{matrix}1 & 1\\ 5 & 5\end{matrix}\;|\;-\frac{1}{2}\right)\right],$$

where $H_n^{(r)}$ are hyperharmonic numbers (1.77).

[74]

(0.109)
$$\sum_{-\infty}^{\infty}\frac{1}{n^2+q^2}=\frac{\pi\coth\pi q}{q}.$$

[74]

(0.110)
$$\sum_{-\infty}^{\infty}\frac{1}{(n^2+q_1^2)((N-n)^2+q_2^2}=\frac{2\pi}{2q_1 2q_2}\Big[\frac{1+n_b(q_1)+n_b(q_2)}{Ni+q_1+q_2}+\frac{n_b(q_1)-n_b(q_2)}{Ni-q_1+q_2}$$
$$-\frac{n_b(q_1)-n_b(q_2)}{Ni+q_1-q_2}-\frac{1+n_b(q_1)+n_b(q_2)}{Ni-q_1-q_2}\Big]$$

where $n_b(z)\equiv 1/(e^{2\pi z}-1)$.

[74] Define the Matsubara sum associated to the Graph G (loopless multigraph such that the degree of each vertex is at least 2, with I lines) by

(0.111)
$$S_G\equiv\sum_{n_1=-\infty}^{\infty}\sum_{n_2=-\infty}^{\infty}\cdots\sum_{n_I=-\infty}^{\infty}\frac{\delta_g(n_1,n_2,\cdots n_I;\{N_v\})}{(n_1^2+q_1^2)(n_2^2+q_2^2)\cdots(n_I^2+q_I^2)},$$

where

(0.112)
$$\delta_g(n_1,\ldots;\{N_v\})=\prod_{v=1}^{V}\delta_{T_v,N_v}$$

imposes a series of constraints over the vertices v, and $T_v\equiv\sum_i s_i^v n_i$ is an algebraic sum at vertex v, with s_i^v having values ±1 or 0 depending on the orientation of the line i with respect to the vertex v. The q_i are weights associated with the lines i. Then the integral

(0.113)
$$I(N,q_1,q_2,\ldots)\int_{-\infty}^{\infty}dx_1\int_{-\infty}^{\infty}dx_2\cdots\frac{1}{(x_1^2+q_1^2)(x_2^2+q_2^2)\cdots((N-x_1-x_2\cdots)^2+q_I^2)}$$

is related to the sum via

$$S_G = \hat{O}_G I_G \tag{0.114}$$

where the operator $\hat{O}_G = \prod_{i=1}^{I}[1 + n_{b_i}(1 - \hat{R}_i)]$ is composed of the functions n_b of the previous formula and the reflection operator $\hat{R}_i$ (which switches the sign of the variable q_i).

[175, A152416][126]

$$\sum_{n=2}^{\infty} \frac{1}{n^s(n-1)} = s - \sum_{l=2}^{s} \zeta(l). \tag{0.115}$$

[36]

$$\sum_{n\ge 1} \frac{1}{n(n^2+1)} = \gamma + \Re\psi(1+i) \approx 0.67186598552400983. \tag{0.116}$$

[36]

$$\sum_{n\ge 1} \frac{1}{n^2(n^2+1)} = \frac{\pi^2}{6} - \frac{\pi\coth\pi - 1}{2} \approx 0.56826001937964526. \tag{0.117}$$

[132]

$$\sum_{k=1}^{\infty} \frac{1}{(2k)^{2s}(2k+1)^{2s}} = \sum_{t=1}^{2s} \binom{4s-t-1}{2s-1} \left\{ [1 - \frac{1-(-)^t}{2^t}]\zeta(t) - 1 \right\}. \tag{0.118}$$

[132]

$$\sum_{k=1}^{\infty} \frac{1}{(2k)^2(2k+1)^2} = -3 + \frac{\pi^2}{6} + 2\log 2. \tag{0.119}$$

[132]

$$\sum_{k=1}^{\infty} \frac{1}{(2k)^4(2k+1)^4} = -35 + \frac{\pi^4}{90} + 3\zeta(3) + \frac{5\pi^2}{3} + 20\log 2. \tag{0.120}$$

[132]

$$\sum_{n=1}^{\infty} \frac{1}{n^{2s}(n+1)^{2s}} = \sum_{t=1}^{2s} \binom{4s-t-1}{2s-1} \left\{ [1 + (-)^t]\zeta(t) - 1 \right\}. \tag{0.121}$$

[132]

$$\sum_{n=1}^{\infty} \frac{1}{n^2(n+1)^2} = \frac{\pi^2}{3} - 3. \tag{0.122}$$

[132]

$$\sum_{n=1}^{\infty} \frac{1}{n^4(n+1)^4} = -35 + \frac{10\pi^2}{3} + \frac{\pi^4}{45}. \tag{0.123}$$

[119, (1)]

$$\sum_{n=0}^{\infty} \frac{\binom{2n}{n}}{8^n} = \sqrt{2}. \tag{0.124}$$

[119, (1)]

$$\sum_{n=0}^{\infty} \frac{\binom{2n}{n}}{10^n} = \sqrt{5/3}. \tag{0.125}$$

[119, (1)]

$$\sum_{n=0}^{\infty} \frac{(-1)^n \binom{2n}{n}}{8^n} = \sqrt{2/3}. \tag{0.126}$$

[119][175, A145439]

$$\sum_{n=0}^{\infty} \frac{\binom{4n}{2n}}{64^n} = \frac{3\sqrt{2}+\sqrt{6}}{6} \tag{0.127}$$

[119]

$$4\sum_{n=0}^{\infty} \frac{\binom{8n}{4n}}{8^{4n}} = \frac{3\sqrt{2}+\sqrt{6}}{3} + \frac{2\sqrt{2+\sqrt{5}}}{\sqrt{5}}. \tag{0.128}$$

[119, (6)]

$$\sum_{n=1}^{\infty} \frac{\binom{2n}{n}}{n4^n} = \log 4. \tag{0.129}$$

[119, (6)][175, A157699]

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}\binom{2n}{n}}{n4^n} = 2\log\frac{1+\sqrt{2}}{2}, \tag{0.130}$$

The formula above corrects a factor 2 in [119], see [127].
[175, A091648]

$$\sum_{n=1,3,5,7,\ldots}^{\infty} \frac{\binom{2n}{n}}{n4^n} = \log(1+\sqrt{2}). \tag{0.131}$$

[119, (7)]

$$\sum_{n=1}^{\infty} \frac{\binom{2n}{n}}{n(n+1)4^n} = \log 4 - 1. \tag{0.132}$$

[119, (8)]

$$\sum_{n=1}^{\infty} \frac{n\binom{2n}{n}}{8^n} = 1/\sqrt{2}. \tag{0.133}$$

[119]

$$\sum_{n=1}^{\infty} \frac{n^2\binom{2n}{n}}{8^n} = \frac{5\sqrt{2}}{4}. \tag{0.134}$$

[119][175, A019670]

$$\sum_{n=0}^{\infty} \frac{\binom{2n}{n}}{(2n+1)16^n} = \frac{\pi}{3}. \tag{0.135}$$

[175, A086466]

$$\sum_{m=1}^{\infty} \frac{(-1)^{m-1}}{m\binom{2m}{m}} = \frac{2}{\sqrt{5}} \log \frac{1+\sqrt{5}}{2}. \tag{0.136}$$

This corrects a factor 2 in [119] and two typos in [17, 4.1.42], see [127].

[119, (15)] [66, 189][175, A073016]

$$\sum_{m=1}^{\infty} \frac{1}{\binom{2m}{m}} = \frac{9+2\pi\sqrt{3}}{27}. \tag{0.137}$$

[189]

$$\sum_{k=0}^{\infty} \frac{1}{\binom{mk}{nk}} = \int_0^1 \frac{1+(m-1)t^n(1-t)^{m-n}}{(1-t^n(1-t)^{m-n})^2} dt, \tag{0.138}$$

where m and n are positive integers with $m > n$.

[189]

$$\sum_{k=0}^{\infty} \frac{1}{\binom{4k}{2k}} = \frac{16}{15} + \frac{\pi\sqrt{3}}{27} - \frac{2\sqrt{5}}{25} \ln \frac{1+\sqrt{5}}{2}. \tag{0.139}$$

[119][175, A086465]

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{\binom{2n}{n}} = \frac{1}{5} + \frac{4\sqrt{5}}{25} \log \frac{1+\sqrt{5}}{2}. \tag{0.140}$$

[119, 66][175, A145429]

$$\sum_{m=1}^{\infty} \frac{m}{\binom{2m}{m}} = \frac{2}{27}(\pi\sqrt{3}+9). \tag{0.141}$$

[39] And by linear combination with (0.137):

$$\sum_{n\ge 1} \frac{18-9n}{\binom{2n}{n}} = 2\frac{\pi}{\sqrt{3}}. \tag{0.142}$$

[119, 66]

$$\sum_{m=1}^{\infty} \frac{m^2}{\binom{2m}{m}} = \frac{2}{81}(5\pi\sqrt{3}+54). \tag{0.143}$$

[119, 66]

$$\sum_{m=1}^{\infty} \frac{m^3}{\binom{2m}{m}} = \frac{2}{243}(37\pi\sqrt{3}+405). \tag{0.144}$$

[119][175, A145433]

$$\sum_{m=1}^{\infty} \frac{(-1)^{m-1}m}{\binom{2m}{m}} = \frac{2}{125}(2\sigma+15), \quad \sigma \equiv \sqrt{5} \log \frac{1+\sqrt{5}}{2}. \tag{0.145}$$

[119]

$$\sum_{m=1}^{\infty} \frac{(-1)^{m-1}m^2}{\binom{2m}{m}} = \frac{2}{125}(5-\sigma). \tag{0.146}$$

Erratum to [17, 4.1.40]:

$$\sum_{n=0}^{\infty} \frac{(-1)^{n-1} n^2}{\binom{2n}{n}} = \frac{4}{125}\left[5 - \sqrt{5}\ln\left(\frac{1+\sqrt{5}}{2}\right)\right]. \tag{0.147}$$

$$\sum_{m=1}^{\infty} \frac{(-1)^{m-1} m^3}{\binom{2m}{m}} = \frac{2}{625}(14\sigma - 5), \tag{0.148}$$

which corrects a factor 2 in [119], see [127].

[119, 66]

$$\sum_{m=1}^{\infty} \frac{2^m}{m^2\binom{2m}{m}} = \frac{\pi^2}{8}, \tag{0.149}$$

a special case of (1.17).

[119]

$$\sum_{m=1}^{\infty} \frac{2^m}{m\binom{2m}{m}} = \frac{\pi}{2}. \tag{0.150}$$

[119]

$$\sum_{m=1}^{\infty} \frac{2^m}{\binom{2m}{m}} = \frac{\pi}{2} + 1. \tag{0.151}$$

[119]

$$\sum_{m=1}^{\infty} \frac{m 2^m}{\binom{2m}{m}} = \pi + 3. \tag{0.152}$$

[119, 66]

$$\sum_{m=1}^{\infty} \frac{m^2 2^m}{\binom{2m}{m}} = \frac{1}{5}\sum_{m=1}^{\infty} \frac{m^3 2^m}{\binom{2m}{m}} = \frac{7\pi}{2} + 11. \tag{0.153}$$

[119, 66]

$$\sum_{m=1}^{\infty} \frac{m^4 2^m}{\binom{2m}{m}} = 113\pi + 355. \tag{0.154}$$

[119, 66]

$$\sum_{m=1}^{\infty} \frac{m^{10} 2^m}{\binom{2m}{m}} = 229093376\pi + 719718067. \tag{0.155}$$

[119]

$$\sum_{m=1}^{\infty} \frac{3^m}{m^2\binom{2m}{m}} = \frac{2\pi^2}{9}, \tag{0.156}$$

a special case of (1.17).

[119][175, A186706]

$$\sum_{m=1}^{\infty} \frac{3^m}{m\binom{2m}{m}} = \frac{2\pi}{\sqrt{3}} \equiv \nu. \tag{0.157}$$

[119]

$$\sum_{m=1}^{\infty}\frac{3^m}{\binom{2m}{m}} = 2\nu+3. \tag{0.158}$$

[119]

$$\sum_{m=1}^{\infty}\frac{m3^m}{\binom{2m}{m}} = 10\nu+18. \tag{0.159}$$

[119]

$$\sum_{m=1}^{\infty}\frac{m^2 3^m}{\binom{2m}{m}} = 2(43\nu+78). \tag{0.160}$$

[119]

$$\sum_{m=1}^{\infty}\frac{(-1)^{m-1}2^m}{m\binom{2m}{m}} = \rho/3, \quad \rho\equiv\sqrt{3}\log(2+\sqrt{3}). \tag{0.161}$$

[12]

$$\sum_{m=1}^{\infty}\frac{(2t)^{2m+2k}}{m(2m+2k)\binom{2m}{m}} = \arcsin^2(t)\binom{2k}{k}+\sum_{j=1}^{k}\binom{2k}{k-j}\frac{(-)^{j+1}}{j^2} + \sum_{j=1}^{k}(-)^j\binom{2k}{k-j}\left(\frac{2\arcsin t\sin(2j\arcsin t)}{j}+\frac{\cos(2j\arcsin t)}{j^2}\right), \tag{0.162}$$

and a similar logarithmic result for an alternating sign sum on the left hand side.
[12]

$$\sum_{m=1}^{\infty}\frac{(2t)^{2m+2k}}{m^2(2m+2k)\binom{2m}{m}} = -\frac{1}{k}\arcsin^2(t)+\frac{(2t)^{2k}}{k}\arcsin^2 t-\sum_{j=1}^{k}\binom{2k}{k-j}\frac{(-)^{j+1}}{kj^2} - \sum_{j=1}^{k}(-)^j\binom{2k}{k-j}\left(\frac{2\arcsin t\sin(2j\arcsin t)}{kj}+\frac{\cos(2j\arcsin t)}{kj^2}\right), \tag{0.163}$$

and a similar logarithmic result for an alternating sign sum on the left hand side.
[119]

$$\sum_{m=1}^{\infty}\frac{(-1)^{m-1}2^m}{\binom{2m}{m}} = \frac{\rho+3}{9}. \tag{0.164}$$

[119]

$$\sum_{m=1}^{\infty}\frac{(-1)^{m-1}m2^m}{\binom{2m}{m}} = \frac{1}{3}. \tag{0.165}$$

[119]

$$\sum_{m=1}^{\infty}\frac{(-1)^{m-1}m^2 2^m}{\binom{2m}{m}} = \frac{1}{27}(3-\rho). \tag{0.166}$$

[119]

$$\sum_{m=1}^{\infty}\frac{(-1)^m m^3 2^m}{\binom{2m}{m}}=\frac{1}{81}(\rho+15). \tag{0.167}$$

[119]

$$\sum_{m=1}^{\infty}\frac{(2-\sqrt{2})^m}{\binom{2m}{m}}=\frac{3-2\sqrt{2}}{4}(\pi\sqrt{2}+4). \tag{0.168}$$

[119]

$$\sum_{m=1}^{\infty}\frac{(-1)^{m-1}3^{2m}}{4^m\binom{2m}{m}}=\frac{48}{125}(\log 2+\frac{15}{16}). \tag{0.169}$$

[119][175, A152422]
(0.170)

$$\sum_{m=1}^{\infty}\frac{2^m(2-\sqrt{3})^m}{m^2\binom{2m}{m}}=2\left(\arcsin\tau\right)^2, \quad \tau=\frac{\sqrt{3}-1}{2}=\sqrt{2}\sin\frac{\pi}{12}=\sin\frac{\pi}{3}-\sin\frac{\pi}{6},$$

where both right hand sides in [119] are erroneous, see [127].

[39]

$$\sum_{n\geq 0}\frac{50n-6}{\binom{3n}{n}2^n}=\pi. \tag{0.171}$$

[39]

$$\sum_{n\geq 1}\frac{1}{\binom{3n}{n}2^n}=\frac{2}{25}-\frac{6}{125}\ln 2+\frac{11}{250}\pi. \tag{0.172}$$

[39]

$$\sum_{n\geq 1}\frac{n}{\binom{3n}{n}2^n}=\frac{81}{625}-\frac{18}{3125}\ln 2+\frac{79}{3125}\pi. \tag{0.173}$$

[39]

$$\sum_{n\geq 1}\frac{n^2}{\binom{3n}{n}2^n}=\frac{561}{3125}+\frac{42}{15625}\ln 2+\frac{673}{31250}\pi. \tag{0.174}$$

[39]

$$\sum_{n\geq 1}\frac{-150n^2+230n-36}{\binom{3n}{n}2^n}=\pi. \tag{0.175}$$

[39]

$$\sum_{n\geq 1}\frac{575n^2-965n+273}{\binom{3n}{n}2^n}=6\log 2. \tag{0.176}$$

[39]

$$\sum_{n\geq 1}\frac{(-1)^n}{\binom{3n}{n}4^n}=-\frac{1}{28}-\frac{3}{32}\ln 2+\frac{13}{112}\frac{\arctan(\sqrt{7}/5)}{\sqrt{7}}. \tag{0.177}$$

[39]

$$\sum_{n\ge 1}\frac{(-1)^n n}{\binom{3n}{n}4^n} = -\frac{81}{1568}-\frac{9}{256}\ln 2+\frac{17}{6272}\frac{\arctan(\sqrt{7}/5)}{\sqrt{7}}. \tag{0.178}$$

[39]

$$\sum_{n\ge 1}\frac{1}{n\binom{3n}{n}2^n} = \frac{1}{10}\pi-\frac{1}{5}\ln 2. \tag{0.179}$$

[39]

$$\sum_{n\ge 1}\frac{1}{n^2\binom{3n}{n}2^n} = \frac{1}{24}\pi^2-\frac{1}{2}\ln^2 2. \tag{0.180}$$

[34]

$$\sum_{j=1}^{\infty}\frac{(-1)^j}{j!}(j/2)! = 1-\frac{1}{2}e^{1/4}\sqrt{\pi}(1-\operatorname{erf}(1/2)). \tag{0.181}$$

Erratum to [17, 4.1.47][175, A145438]: Using twice 9.14.+13, then [91, 9.121.6] and [91, 9.121.26], then substituting $tt'/4=z^2$, then $y=\sqrt{t'/4}$, then $\arcsin y = v$

$$\sum_{n=1}^{\infty}\frac{1}{n^3\binom{2n}{n}} = \frac{1}{2}\,{}_4F_3(1,1,1,1;2,2,3/2;1/4) = \frac{1}{2}\int_0^1 dt\,{}_3F_2(1,1,1;2,3/2;t/4)$$

$$= \frac{1}{4}\int_0^1 dt\int_0^1 dt'\,{}_2F_1(1,1;3/2;tt'/4) = \frac{1}{4}\int_0^1 dt\int_0^1 dt'\frac{1}{\sqrt{1-tt'/4}}\,{}_2F_1(1/2,1/2;3/2;tt'/4)$$

$$= \frac{1}{4}\int_0^1 dt\int_0^1 dt'\frac{1}{\sqrt{1-tt'/4}}\frac{\arcsin(\sqrt{tt'/4})}{\sqrt{tt'/4}} = \int_0^1\frac{dt'}{t'}\int_0^{\sqrt{t'/4}}dz\frac{\arcsin z}{\sqrt{1-z^2}} = \frac{1}{2}\int_0^1\frac{dt'}{t'}\arcsin^2(\sqrt{t'/4})$$

$$= \int_0^{\sqrt{1/2}}\frac{dy}{y}\arcsin^2 y = \int_0^{\sin\sqrt{1/2}} v^2\cot v\,dv.$$

This becomes [39, (35)][119][38, Theorem 3.3]

$$\frac{2\pi}{3}\Im\operatorname{Li}_2(e^{i\pi/4})-\frac{4}{3}\zeta(3) = \frac{2\pi}{3}\operatorname{Cl}_2(\pi/3)-\frac{4}{3}\zeta(3) = \frac{\pi\sqrt{3}}{18}\{\psi'(1/3)-\psi'(2/3)\}-\frac{4\zeta(3)}{3}. \tag{0.182}$$

So the r.h.s. of [119] is missing a factor 4, and [15, 4.1.47] is in addition misleading to imply that the digamma (instead of the trigamma) functions are in effect [127].

[178]

$$\sum_{n=1}^{\infty}\frac{t^n}{n^{k+1}\binom{an+j+1}{j+1}} = \begin{cases} \frac{(j+1)t(-)^k}{k!}\int_0^1\int_0^1\frac{(1-x)^j x^a(\ln y)^k}{1-tx^a y}dxdy, & k\ge 1\\ at\int_0^1\frac{(1-x)^{j+1}x^{a-1}}{1-tx^a}dx, & k=1\end{cases} \tag{0.183}$$

$$= T_0\,{}_{a+k+1}F_{a+k}\left(\begin{matrix}1,1,\ldots,1;(a+1)/a,\ldots,(2a-1)/a\\ 2,2,\ldots,2;(a+j+2)/a,\ldots,(a+j+a+1)/a\end{matrix}\mid t\right)$$

where $T_0 = t(j+1)B(j+1,a+1)$.

[67]

$$\frac{1}{2^m}\sum_{i=1}^{\infty}\binom{i+1}{2}^{-m} = (-)^{m-1}\binom{2m-1}{m}+(-)^m 2\sum_{i=1}^{\lfloor m/2\rfloor}\binom{2m-2i-1}{m-1}\zeta(2i). \tag{0.184}$$

[39, 119][175, A073010]

$$\sum_{n\ge 1}\frac{1}{n\binom{2n}{n}}=\frac{\pi}{3\sqrt{3}}. \tag{0.185}$$

$$\sum_{n\ge 1}\frac{1}{n^2\binom{2n}{n}}=\frac{1}{3}\zeta(2). \tag{0.186}$$

a special case of (1.17).

[94]

$$\sum_{n\ge 1}\frac{1}{n^4\binom{2n}{n}}=\frac{17}{36}\zeta(4). \tag{0.187}$$

[94]

$$\sum_{n\ge 1}\frac{1}{n^5\binom{2n}{n}}=2\pi\operatorname{Cl}_4(\pi/3)-\frac{19}{3}\zeta(5)+\frac{2}{3}\zeta(3)\zeta(2). \tag{0.188}$$

[94]

$$\sum_{n\ge 1}\frac{1}{n^6\binom{2n}{n}}=-\frac{4\pi}{3}\Im L_{4,1}(e^{i\pi/3})+\frac{3341}{1296}\zeta(6)-\frac{4}{3}\zeta^2(3). \tag{0.189}$$

[94]

$$\sum_{n\ge 1}\frac{(-1)^n}{n\binom{2n}{n}}=-2\frac{\operatorname{arctanh}(1/\sqrt{5})}{\sqrt{5}}. \tag{0.190}$$

Correction of a sign error in [119], see [127]:

$$\sum_{n\ge 1}\frac{(-1)^n}{n^2\binom{2n}{n}}=-2\left(\ln\frac{\sqrt{5}-1}{2}\right)^2=-2\left(\ln\frac{\sqrt{5}+1}{2}\right)^2. \tag{0.191}$$

$$\sum_{n\ge 1}\frac{(-1)^n}{n^3\binom{2n}{n}}=-\frac{2}{5}\zeta(3). \tag{0.192}$$

(0.193)
$$\sum_{n\ge 1}\frac{(-1)^n}{n^4\binom{2n}{n}}=-4K_4(\rho)+4K_4(-\rho)+\frac{1}{2}\ln^4\rho+7\zeta(4),\quad K_k(x)\equiv\sum_{r=0}^{k-1}\frac{(-\ln|x|)^r}{r!}L_{k-r}(x).$$

$$\sum_{n\ge 1}\frac{1}{n^3\binom{3n}{n}2^n}=-\frac{33}{16}\zeta(3)+\frac{1}{6}\ln^3 2-\frac{1}{24}\pi^2\ln 2+\pi\Im L_2(i). \tag{0.194}$$

(0.195)
$$\sum_{n\ge 1}\frac{1}{n^4\binom{3n}{n}2^n}=-\frac{143}{16}\zeta(3)\ln 2+\frac{91}{640}\pi^4-\frac{3}{8}\ln^4 2+\frac{3}{8}\pi^2\ln^2 2-8L_4(1/2)-8\Re L_{3,1}(\frac{1+i}{2})-8\Re L_4(\frac{1+i}{2}).$$

$$\sum_{n\ge 1}\frac{(-1)^n}{8^n n\binom{6n}{2n}}=\left(-\frac{1}{3}+\frac{2}{57}\sqrt{114\sqrt{57}-342}\right)\ln 2-\frac{1}{114}\sqrt{114\sqrt{57}-342}\ln\left(13+\sqrt{57}+\sqrt{-30+26\sqrt{57}}\right)$$
$$+\frac{1}{57}\sqrt{114\sqrt{57}+342}\arctan\frac{\sqrt{2\sqrt{57}+9}}{7}.$$

$$\sum_{n\ge 1}\frac{(-1)^n}{2^n n\binom{4n}{n}}=\sum_{13+12r+2r^3=0}\frac{\ln(r+2)}{r+3}-\frac{6}{7}\ln 3+\frac{1}{7}\ln 2. \tag{0.196}$$

$$\sum_{n\ge 1}\frac{(-1)^n}{n\binom{3n}{n}}=\sum_{8+4r+r^3=0}\frac{\ln(r+2)}{r+3}-\ln 2. \tag{0.197}$$

[206] Let $A_k\equiv\sum_{n=0}^{\infty}\binom{n+k}{n}/\binom{2n}{n}$, then

$$A_{k+1}=-\frac{2}{3(k+1)}+\frac{2(2k+3)}{3(k+1)}A_k. \tag{0.198}$$

[206] Let $B_k\equiv\sum_{n=0}^{\infty}(-1)^n\binom{n+k}{n}/\binom{2n}{n}$, then

$$B_{k+1}=-\frac{2}{5(k+1)}+\frac{2(2k+3)}{5(k+1)}B_k. \tag{0.199}$$

and similar recurrences with an additional factor n or n^2 in the denominator of the sums.

[33]

$$\sum_{k=1}^{\infty}\arctan\frac{2}{k^2}=\frac{3\pi}{4}. \tag{0.200}$$

[33][175, A091007]

$$\sum_{k=1}^{\infty}\arctan\frac{1}{k^2}=\arctan\frac{\tan(\pi/\sqrt{2})-\tanh(\pi/\sqrt{2})}{\tan(\pi/\sqrt{2})+\tanh(\pi/\sqrt{2})}. \tag{0.201}$$

[33]

$$\sum_{k=1}^{\infty}\arctan\frac{a}{a^2k^2+a(a+2b)k+1+ab+b^2}=\frac{\pi}{2}-\arctan(a+b). \tag{0.202}$$

[33]

(0.203)
$$\sum_{k=1}^{\infty}\arctan\frac{a^2k^2+a^2k-1-b^2}{a^4k^4+2a^3(a+2b)k^3+a^2(2+a^2+6ab+6b^2)k^2+2a(a+2b)(1+ab+b^2)+(1+b^2)(1+[a+b]^2)}$$
$$=\frac{1}{1+(a+b)^2}.$$

[33]

$$\sum_{k=1}^{\infty}\arctan\frac{2ak+a+b}{a_4k^4+a_3k^3+a_2k^2+a_1k+a_0}=\frac{\pi}{2}-\arctan(a+b+c). \tag{0.204}$$

[33]

(0.205)
$$\sum_{k=1}^{n}\arctan\frac{f(k+1)-f(k-1)}{1+f(k+1)f(k-1)}=\arctan f(n+1)-\arctan f(1)+\arctan f(n)-\arctan f(0).$$

[33]

$$\sum_{k=1}^{\infty}\arctan\frac{8k}{k^4-2k^2+5}=\pi-\arctan\frac{1}{2}. \tag{0.206}$$

[33]

$$\sum_{k=1}^{\infty} \arctan \frac{4ak}{k^4+a^2+4} = \arctan \frac{a}{2} + \arctan a. \tag{0.207}$$

[33]

$$\sum_{k=1}^{\infty} \arctan \frac{2xy}{k^2-x^2+y^2} = \arctan \frac{y}{x} - \arctan \frac{\tanh \pi y}{\tan \pi x}. \tag{0.208}$$

[33]

$$\sum_{k=1}^{\infty} \arctan \frac{1}{2k^2} = \pi/4. \tag{0.209}$$

[33]

$$\sum_{k=1}^{\infty} \frac{k^2}{k^4+4x^4} = \frac{\pi}{4x} \frac{\sin 2\pi x - \sinh 2\pi x}{\cos 2\pi x - \cosh 2\pi x}. \tag{0.210}$$

[33]

$$\sum_{k=1}^{\infty} \frac{k^2}{k^4+4} = \frac{\pi}{4} \coth \pi. \tag{0.211}$$

[33]

$$\sum_{k=1}^{\infty} \frac{k^2}{k^4+x^2k^2+x^4} = \frac{\pi}{2x\sqrt{3}} \frac{\sinh \pi x\sqrt{3} - \sqrt{3}\sin \pi x}{\cosh \pi x\sqrt{3} - \cos \pi x}. \tag{0.212}$$

[153]

$$\sum_{k=1}^{\infty} \frac{A_0+B_0k+C_0k^2}{(k^2-a^2)(k^2-b^2)} = \sum_{n=1}^{\infty} \frac{d_n}{\prod_{m=1}^{n}(m^2-a^2)(m^2-b^2)} \tag{0.213}$$

for $|a|<1$, $|b|<1$, with d_n defined via a recurrence in the reference.

$$\sum_{n=0}^{\infty} nx^n = \frac{x}{(1-x)^2}. \tag{0.214}$$

$$\sum_{n=0}^{\infty} n^2x^n = \frac{x(1+x)}{(1-x)^3}. \tag{0.215}$$

[175, A000578]

$$\sum_{n=0}^{\infty} n^3x^n = \frac{x(1+4x+x^2)}{(1-x)^4}. \tag{0.216}$$

[175, A000583]

$$\sum_{n=0}^{\infty} n^4x^n = \frac{x(1+x)(1+10x+x^2)}{(1-x)^5}. \tag{0.217}$$

[175, A000584]

$$\sum_{n=0}^{\infty} n^5 x^n = \frac{x(1+26x+66x^2+26x^3+x^4)}{(1-x)^6}. \tag{0.218}$$

[175, A001014]

$$\sum_{n=0}^{\infty} n^6 x^n = \frac{x(x+1)(1+56x+246x^2+56x^3+x^4)}{(1-x)^7}. \tag{0.219}$$

[175, A001015]

$$\sum_{n=0}^{\infty} n^7 x^n = \frac{x(1+120x+1191x^2+2416x^3+1191x^4+120x^5+x^6)}{(1-x)^8}. \tag{0.220}$$

See [175, A008292] for coefficients in the numerator polynomial if exponents of n are higher than 7. The pole at $x=1$ in the generating functions obviously delimits the radius of convergence.

[98] Let

$$S_{a,p}(n) \equiv \sum_{k=0}^{n} a^k k^p; \tag{0.221}$$

then

$$S_{a,p}(n) = \frac{a^{n+1}}{a-1}\sum_{r=0}^{p-1}\binom{p}{r} f_r(a)(n+1)^{p-r} + f_p(a)\frac{a^{n+1}-1}{a-1} \tag{0.222}$$

with

$$f_0(a) = 1, \quad a\sum_{j=0}^{r}\binom{r}{j} f_j(a) - f_r(a) = 0, \quad r = 1,2,\ldots \tag{0.223}$$

[6]

$$\sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)^2} = G. \tag{0.224}$$

[6, 42]

$$2\sum_{k=0}^{\infty} \frac{1}{(4k+1)^2} = G + \frac{\pi^2}{8}. \tag{0.225}$$

[6, 42]

$$-2\sum_{k=0}^{\infty} \frac{1}{(4k+3)^2} = G - \frac{\pi^2}{8}. \tag{0.226}$$

[42]

$$\sum_{n=0}^{\infty} \frac{1}{2^{n+1}} \sum_{k=0}^{\infty} \binom{n}{k} \frac{(-1)^k}{(2k+1)^2} = G. \tag{0.227}$$

[42]

$$1 - \sum_{n=1}^{\infty} \frac{n\zeta(2n+1)}{16^n} = G. \tag{0.228}$$

[42]

$$\frac{1}{4}\sum_{n=1}^{\infty} n16^{-n}(3^{2n}-1)\zeta(2n+1) = G - \frac{1}{6} \tag{0.229}$$

(and similar ζ-sums).

[42]

$$\sum_{n=0}^{\infty} \frac{1}{n+1}\binom{n+1/2}{n}^{-2} = {}_4F_3(1,1,1,1;2,\frac{3}{2},\frac{3}{2};1) = 2\pi G - \frac{7}{2}\zeta(3). \tag{0.230}$$

[42]

$$2\sum_{n=0}^{\infty} \frac{(-1)^{n+1}}{n^2}\sum_{k=0}^{n-1}\frac{1}{2k+1} = 2\pi G - \frac{7}{2}\zeta(3). \tag{0.231}$$

[42]

$$\sum_{n=0}^{\infty} \frac{2^n}{(2n+1)\binom{2n}{n}}\sum_{k=0}^{n}\frac{1}{2k+1} = 2G. \tag{0.232}$$

[42]

$$\sum_{n=0}^{\infty} \frac{4^n}{(2n+1)^2\binom{2n}{n}} = 2G. \tag{0.233}$$

[42]

$$\frac{3}{8}\sum_{n=0}^{\infty} \frac{1}{(2n+1)^2\binom{2n}{n}} = G - \frac{1}{8}\pi\log(2+\sqrt{3}). \tag{0.234}$$

[42]

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1}\sum_{k=1}^{n}\frac{1}{k} = G - \frac{1}{2}\pi\log 2. \tag{0.235}$$

[42]

$$-2\sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1}\sum_{k=0}^{n-1}\frac{1}{2k+1} = G - \frac{1}{4}\pi\log 2. \tag{0.236}$$

[42]

$$-\frac{1}{32}\pi\sum_{n=0}^{\infty} \frac{(2n+1)^2}{(n+1)^3 16^n}\binom{2n}{n}^2 = G - \frac{1}{2}\pi\log 2. \tag{0.237}$$

[42]

$$\sum_{n=0}^{\infty} \frac{\sqrt{2}}{(2n+1)^2 8^n}\binom{2n}{n} = G + \frac{1}{4}\pi\log 2. \tag{0.238}$$

[42]

$$\sum_{n=1}^{\infty} \frac{\sin(n\pi/4)}{n^2 2^{n/2}} = G - \frac{1}{8}\pi\log 2. \tag{0.239}$$

[42]

$$\sum_{n=0}^{\infty} \frac{2^{n+1}(n!)^2}{(2n+1)!(n+1)^2} = 2\pi G - \frac{35}{8}\zeta(3) + \frac{1}{4}\pi^2 \log 2. \tag{0.240}$$

[42]

$$\frac{1}{4}\pi\, {}_3F_2(1/2, 1/2, n+1/2; 1, n+3/2; 1) - \frac{1}{2}\sum_{k=0}^{n-1} \frac{(k!)^2}{(\frac{3}{2})_k^2} = G. \tag{0.241}$$

[76, 77] Let T denote Tornheim double sums

$$T(a,b,c) \equiv \sum_{r=1}^{\infty}\sum_{s=1}^{\infty} \frac{1}{r^a s^b (r+s)^c} \tag{0.242}$$

with $a+c>1$, $b+c>1$ and $a+b+c>2$. Then

$$T(m,k,n) = \sum_{i=1}^{m} \binom{m+k-i-1}{m-i} T(i,0,N-i) + \sum_{i=1}^{k} \binom{m+k-i-1}{k-i} T(i,0,N-i). \tag{0.243}$$

$$\begin{aligned} T(m,0,n) = (-1)^m &\sum_{j=0}^{\lfloor (n-1)/2 \rfloor} \binom{m+n-2j-1}{m-1} \zeta(2j)\zeta(m+n-2j) \\ &+ (-1)^m \sum_{j=0}^{\lfloor m/2 \rfloor} \binom{m+n-2j-1}{n-1} \zeta(2j)\zeta(m+n-2j) - \frac{1}{2}\zeta(m+n), \end{aligned}$$

if $N \equiv m+k+n$ is odd.

$$T(1,0,5) = -\frac{1}{2}\zeta(3)^2 + \frac{3}{4}\zeta(6). \tag{0.244}$$

$$T(2,0,4) = \zeta^2(3) - \frac{4}{3}\zeta(6). \tag{0.245}$$

$$T(3,0,3) = \frac{1}{2}\zeta(3)^2 - \frac{1}{2}\zeta(6). \tag{0.246}$$

$$T(4,0,2) = -\zeta^2(3) + \frac{25}{12}\zeta(6). \tag{0.247}$$

$$T(0,0,N) = \zeta(N-1) - \zeta(N), \quad N \ge 3; \quad T(0,0,4) = \zeta(3) - \frac{\pi^4}{90}. \tag{0.248}$$

$$T(n,0,n) = \frac{1}{2}\zeta^2(n) - \frac{1}{2}\zeta(2n). \quad T(2,0,2) = \frac{\pi^4}{120}. \tag{0.249}$$

$$T(0,0,8) = \zeta(7) - \zeta(8). \tag{0.250}$$

$$T(1,0,7) = \frac{5}{4}\zeta(8) - \zeta(3)\zeta(5). \tag{0.251}$$

$$T(4,0,4) = \frac{1}{12}\zeta(8). \tag{0.252}$$

$$T(0,0,14) = \zeta(13) - \zeta(14). \tag{0.253}$$

$$T(1,0,13)=\frac{11}{4}\zeta(14)-\zeta(3)\zeta(11)-\zeta(5)\zeta(9)-\frac{1}{2}\zeta(7)^2. \tag{0.254}$$

$$T(1,0,2)=\zeta(3). \tag{0.255}$$

$$T(1,1,1)=2\zeta(3). \tag{0.256}$$

$$T(1,0,3)=\frac{\pi^4}{360}. \tag{0.257}$$

$$T(1,1,2)=\frac{\pi^4}{180}. \tag{0.258}$$

$$T(2,1,1)=\frac{\pi^4}{72}. \tag{0.259}$$

$$T(2,2,0)=\frac{\pi^4}{36}. \tag{0.260}$$

$$T(1,0,5)=\frac{3}{4}\zeta(6)-\frac{1}{2}\zeta^2(3). \tag{0.261}$$

$$T(1,1,4)=\frac{3}{2}\zeta(6)-\zeta^2(3). \tag{0.262}$$

$$T(3,3,0)=\zeta^2(3). \tag{0.263}$$

$$T(4,1,1)=\frac{7}{6}\zeta(6)-\frac{1}{2}\zeta^2(3). \tag{0.264}$$

$$T(4,2,0)=\frac{7}{4}\zeta(6). \tag{0.265}$$

$$T(1,1,4)=\frac{3}{2}\zeta(6)-\zeta^2(3). \tag{0.266}$$

$$\sum_{n=0}^{\infty}\frac{1}{(n+a)(n+b)}=\frac{\psi(a)-\psi(b)}{a-b}. \tag{0.267}$$

(0.268)
$$\sum_{n=0}^{\infty}\frac{(-)^n}{(n+a)(n+b)}=-\frac{1}{2}\,\frac{\psi(1/2+a/2)-\psi(a/2)-\psi(1/2+b/2)+\psi(b/2)}{a-b}.$$

[89]

$$\log 2=\frac{1}{2}-\sum_{k\geq 1}\frac{(-1)^k}{k(4k^4+1)}. \tag{0.269}$$

After inserting $x = 1$ in (1.65), [17, 4.1.13]

$$2\ln 2 = 1 + \sum_{k=1}^{\infty}(-1)^{k+1}\frac{1}{k(k+1)} = 1 + 2\sum_{k=1,3,5,\ldots}^{\infty}\frac{1}{k(k+1)(k+2)} \tag{0.270}$$
$$= 1 + 4\sum_{k=1,5,9,13\ldots}^{\infty}\frac{6+4k+k^2}{k(k+1)(k+2)(k+3)(k+4)}.$$

[17, 4.1.20]

$$2\ln 2 = \frac{5}{4} + \sum_{k=1}^{\infty}(-1)^{k+1}\frac{1}{k(k+1)(k+2)} = \frac{5}{4} + 3\sum_{k=1,3,5,\ldots}^{\infty}\frac{1}{k(k+1)(k+2)(k+3)}$$
$$= \frac{17}{12} - 3\sum_{k=2,4,6,\ldots}^{\infty}\frac{1}{k(k+1)(k+2)(k+3)}.$$

$$\frac{4}{3}\ln 2 = \frac{8}{9} + \sum_{k=1}^{\infty}(-1)^{k+1}\frac{1}{k(k+1)(k+2)(k+3)} = \frac{8}{9} + 4\sum_{k=1,3,5,\ldots}^{\infty}\frac{1}{k(k+1)(k+2)(k+3)(k+4)}. \tag{0.271}$$

More irregular denominators follow from hybridization. For example we can multiply the penultimate formula by 4, the previous formula by 3, and subtract

$$4\ln 2 = \frac{7}{3} + 12\sum_{k=1,3,5,\ldots}^{\infty}\frac{1}{k(k+1)(k+2)(k+4)}. \tag{0.272}$$

$$\frac{2}{3}\ln 2 = \frac{131}{288} + \sum_{k=1}^{\infty}(-1)^{k+1}\frac{1}{k(k+1)(k+2)(k+3)(k+4)}$$
$$= \frac{131}{288} + 5\sum_{k=1,3,5,\ldots}^{\infty}\frac{1}{k(k+1)(k+2)(k+3)(k+4)(k+5)}.$$

Inserting $x = -1/2$ in (1.65) [89, 22] yields

$$\ln 2 = 1 - \sum_{k=1}^{\infty}\frac{1}{2^k k(k+1)}. \tag{0.273}$$

$$\ln 2 = \frac{1}{2} + \sum_{k=1}^{\infty}\frac{1}{2^{k-1}k(k+1)(k+2)}. \tag{0.274}$$

$$\ln 2 = \frac{5}{6} - 3\sum_{k=1}^{\infty}\frac{1}{2^{k-1}k(k+1)(k+2)(k+3)}. \tag{0.275}$$

More irregular denominators follow from hybridization. For example we can multiply the penultimate formula by 3 and add to the previous formula,

$$4\ln 2 = \frac{7}{3} + 3\sum_{k=1}^{\infty}\frac{1}{2^{k-1}k(k+1)(k+3)}. \tag{0.276}$$

$$\ln 2 = \frac{7}{12} + 3\sum_{k=1}^{\infty} \frac{1}{2^{k-3}k(k+1)(k+2)(k+3)(k+4)}. \tag{0.277}$$

[89]

$$\log 2 = \frac{1327}{1920} + \frac{45}{4}\sum_{k\geq 4} \frac{(-1)^k}{k(k^2-1)(k^2-4)(k^2-9)}. \tag{0.278}$$

[89]

$$\log 2 = \sum_{k\geq 1} \left(\frac{1}{3^k} + \frac{1}{4^k}\right)\frac{1}{k}. \tag{0.279}$$

$$\log 2 = \sum_{k\geq 0} \left(\frac{1}{8k+8} + \frac{1}{4k+2}\right)\frac{1}{4^k}. \tag{0.280}$$

[89]

$$\log 2 = \frac{2}{3} + \sum_{k\geq 1} \left(\frac{1}{2k} + \frac{1}{4k+1} + \frac{1}{8k+4} + \frac{1}{16k+12}\right)\frac{1}{16^k}. \tag{0.281}$$

[89]

$$\log 2 = \frac{2}{3}\sum_{k\geq 0} \frac{1}{(2k+1)9^k}. \tag{0.282}$$

$$\log 2 = \frac{3}{4}\sum_{k\geq 0} \frac{(-1)^k k!^2}{2^k(2k+1)!}. \tag{0.283}$$

[89]

$$\log 2 = \frac{3}{4} + \frac{1}{4}\sum_{k\geq 1} \frac{(-1)^k(5k+1)}{k(2k+1)16^k}\binom{2k}{k}. \tag{0.284}$$

[175, A154920][23]

$$\log 3 = \sum_{k\geq 1} \left[\frac{9}{2k-1} + \frac{1}{2k}\right]\frac{1}{9^k}. \tag{0.285}$$

[175, A164985][23]

$$27\log 5 = 4\sum_{k\geq 0} \left[\frac{9}{4k+1} + \frac{3}{4k+2} + \frac{1}{4k+3}\right]\frac{1}{81^k}. \tag{0.286}$$

$$\log 5 = 2\log 3 - \log 2 + \sum_{k=1}^{\infty} \frac{1}{k10^k}. \tag{0.287}$$

$$4\log 7 = 5\log 2 + \log 3 + 2\log 5 + \sum_{k=1}^{\infty} \frac{(-1)^k}{k2400^k}. \tag{0.288}$$

TABLE 1. Formulas of the type $s\log p = t\log q + \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k} r^k$. [128]

s	p	t	q	r
1	3	1	2	$1/2 \approx 0.50000000$
1	3	2	2	$-1/4 \approx -0.25000000$
2	3	3	2	$1/8 \approx 0.12500000$
5	3	8	2	$-13/256 \approx -0.05078125$
12	3	19	2	$7153/524288 \approx 0.01364326$
1	5	2	2	$1/4 \approx 0.25000000$
3	5	7	2	$-3/128 \approx -0.02343750$
1	7	2	2	$3/4 \approx 0.75000000$
1	7	3	2	$-1/8 \approx -0.12500000$
5	7	14	2	$423/16384 \approx 0.02581787$
1	11	3	2	$3/8 \approx 0.37500000$
2	11	7	2	$-7/128 \approx -0.05468750$
11	11	38	2	$10433763667/274877906944 \approx 0.03795781$
1	13	3	2	$5/8 \approx 0.62500000$
1	13	4	2	$-3/16 \approx -0.18750000$
3	13	11	2	$149/2048 \approx 0.07275391$
7	13	26	2	$-4360347/67108864 \approx -0.06497423$
10	13	37	2	$419538377/137438953472 \approx 0.00305254$
1	17	4	2	$1/16 \approx 0.06250000$
1	19	4	2	$3/16 \approx 0.18750000$
4	19	17	2	$-751/131072 \approx -0.00572968$
1	23	4	2	$7/16 \approx 0.43750000$
1	23	5	2	$-9/32 \approx -0.28125000$
2	23	9	2	$17/512 \approx 0.03320312$
1	29	4	2	$13/16 \approx 0.81250000$
1	29	5	2	$-3/32 \approx -0.09375000$
7	29	34	2	$70007125/17179869184 \approx 0.00407495$
1	31	4	2	$15/16 \approx 0.93750000$
1	31	5	2	$-1/32 \approx -0.03125000$
1	37	5	2	$5/32 \approx 0.15625000$
4	37	21	2	$-222991/2097152 \approx -0.10633039$
5	37	26	2	$2235093/67108864 \approx 0.03330548$
1	41	5	2	$9/32 \approx 0.28125000$
2	41	11	2	$-367/2048 \approx -0.17919922$
3	41	16	2	$3385/65536 \approx 0.05165100$
1	43	5	2	$11/32 \approx 0.34375000$
2	43	11	2	$-199/2048 \approx -0.09716797$
5	43	27	2	$12790715/134217728 \approx 0.09529825$
7	43	38	2	$-3059295837/274877906944 \approx -0.01112965$
1	47	5	2	$15/32 \approx 0.46875000$
1	47	6	2	$-17/64 \approx -0.26562500$
2	47	11	2	$161/2048 \approx 0.07861328$
1	53	5	2	$21/32 \approx 0.65625000$
1	53	6	2	$-11/64 \approx -0.17187500$
3	53	17	2	$17805/131072 \approx 0.13584137$
4	53	23	2	$-498127/8388608 \approx -0.05938137$
1	59	5	2	$27/32 \approx 0.84375000$
1	59	6	2	$-5/64 \approx -0.07812500$
1	61	5	2	$29/32 \approx 0.90625000$
1	61	6	2	$-3/64 \approx -0.04687500$
1	67	6	2	$3/64 \approx 0.04687500$

[23]

$$3^5\log 7=\sum_{k\geq 0}\left[\frac{405}{6k+1}+\frac{81}{6k+2}+\frac{72}{6k+3}+\frac{9}{6k+4}+\frac{5}{6k+5}\right]\frac{1}{3^{6k}}. \tag{0.289}$$

[23]

(0.290)
$$2\times 3^9\log 11=\sum_{k\geq 0}\Big[\frac{85293}{10k+1}+\frac{10935}{10k+2}+\frac{9477}{10k+3}+\frac{1215}{10k+4}+\frac{648}{10k+5}+\frac{135}{10k+6}+\frac{117}{10k+7}+\frac{15}{10k+8}+\frac{13}{10k+9}\Big]\frac{1}{3^{10k}}.$$

$$\log 11=\log 2+\log 5+\sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{k10^k}. \tag{0.291}$$

$$\log 11=2\log 2+\log 3-\sum_{k=1}^{\infty}\frac{1}{k12^k}. \tag{0.292}$$

$$\log 11=2(\log 2+\log 5-\log 3)-\sum_{k=1}^{\infty}\frac{1}{k100^k}. \tag{0.293}$$

$$\log 13=2\log 2+\log 3+\sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{k12^k}. \tag{0.294}$$

$$3\log 17=3\log 2+4\log 5-\sum_{k=1}^{\infty}\frac{1}{k}\left(\frac{87}{5000}\right)^k. \tag{0.295}$$

$$\log 19=2\log 2+\log 5-\sum_{k=1}^{\infty}\frac{1}{k20^k}. \tag{0.296}$$

[115]

$$\log 11=\sum_{k\geq 1}\frac{1}{k}(\frac{3}{2^k}+\frac{1}{2^{2k}}+\frac{1}{2^{5k}}-\frac{1}{2^{10k}})=\frac{1}{2}\sum_{k\geq 1}\frac{1}{k}(\frac{13}{3^k}-\frac{4}{3^{2k}}-\frac{1}{3^{5k}}). \tag{0.297}$$

[115]

$$\log 13=\sum_{k\geq 1}\frac{1}{k}(\frac{3}{2^k}+\frac{1}{2^{2k}}+\frac{1}{2^{3k}}+\frac{1}{2^{4k}}-\frac{1}{2^{12k}})=\sum_{k\geq 1}\frac{1}{k}(\frac{7}{3^k}-\frac{2}{3^{2k}}-\frac{1}{3^{3k}}). \tag{0.298}$$

[115]

$$\log 17=\sum_{k\geq 1}\frac{1}{k}(\frac{4}{2^k}+\frac{1}{2^{4k}}-\frac{1}{2^{8k}}). \tag{0.299}$$

[115]

$$\log 19=\sum_{k\geq 1}\frac{1}{k}(\frac{3}{2^k}+\frac{3}{2^{2k}}+\frac{1}{2^{9k}}-\frac{1}{2^{18k}}). \tag{0.300}$$

Similar formulas as the four above are obtained by inserting
(0.301)
$$\log[(2^s-1)^\tau] = s\tau\log 2 - \sum_{k\ge 1}\frac{\tau}{k2^{ks}} = -s\tau\log\frac{1}{2} - \sum_{k\ge 1}\frac{\tau}{k2^{ks}} = \sum_{k\ge 1}\frac{\tau}{k}\left(\frac{s}{2^k}-\frac{1}{2^{ks}}\right),$$
—immediate consequence of putting $x = 1/2$ in (1.64)—into the right hand sides of [46]:

$$\log 41 = \log(2^{20}-1)-\log(2^{10}-1)+\log[(2^2-1)^2]-\log[(2^4-1)^2]. \tag{0.302}$$

$$\log 43 = \log(2^{14}-1)-\log(2^2-1)-\log(2^7-1). \tag{0.303}$$

$$\log 73 = \log(2^9-1)-\log(2^3-1). \tag{0.304}$$

$$\log 151 = \log(2^{15}-1)-\log(2^3-1)-\log(2^5-1). \tag{0.305}$$

$$\log 241 = \log(2^{24}-1)-\log(2^{12}-1)-\log(2^8-1)+\log(2^4-1). \tag{0.306}$$

$$\log 257 = \log(2^{16}-1)-\log(2^8-1). \tag{0.307}$$

$$\log 331 = \log(2^{30}-1)-\log(2^{15}-1)-\log(2^{10}-1)+\log(2^5-1)-\log(2^2-1). \tag{0.308}$$

$$\log 337 = \log(2^{21}-1)-\log(2^7-1)-\log[(2^6-1)^2]+\log[(2^2-1)^4]. \tag{0.309}$$

$$\log 683 = \log(2^{22}-1)-\log(2^{11}-1)-\log(2^2-1). \tag{0.310}$$

$$\log 2731 = \log(2^{26}-1)-\log(2^{13}-1)-\log(2^2-1). \tag{0.311}$$

$$\log 5419 = \log(2^{42}-1)-\log(2^{21}-1)-\log(2^{14}-1)+\log(2^7-1)-\log(2^2-1). \tag{0.312}$$

$$\log 43691 = \log(2^{34}-1)-\log(2^{17}-1)-\log(2^2-1). \tag{0.313}$$

$$\log 61681 = \log(2^{40}-1)-\log(2^{20}-1)-\log(2^8-1)+\log(2^4-1). \tag{0.314}$$

$$\log 174763 = \log(2^{38}-1)-\log(2^{18}-1)-\log(2^2-1). \tag{0.315}$$

$$\log 262657 = \log(2^{27}-1)-\log(2^9-1). \tag{0.316}$$

$$\log 599479 = \log(2^{33}-1)-\log(2^{11}-1)-\log(2^3-1). \tag{0.317}$$

$$\sum_{k\ge 0}\frac{\log(2k+1)}{(2k+1)^l} = -\frac{\log 2}{2^l}\zeta(l)-(1-2^{-l})\zeta'(l). \tag{0.318}$$

$$\sum_{k\ge 0}\frac{\log(ak+b)}{(ak+b)^l} = \frac{\log a}{a^l}\zeta(l,b/a)-\frac{1}{a^l}\zeta'(l,b/a). \tag{0.319}$$

[34]

$$\sum_{k=1}^{\infty}\frac{kk!}{(2k)!} = \frac{1}{8}\left(2+3e^{1/4}\sqrt{\pi}\,\mathrm{erf}(1/2)\right). \tag{0.320}$$

$$\sum_{k=1}^{\infty}\frac{(-1)^k\alpha^k x^k}{kk!} = -\gamma-\Gamma(0,\alpha x)-\ln\alpha-\ln x. \tag{0.321}$$

[6]

$$\frac{3}{2}\sum_{k=0}^{\infty}\frac{1}{16^k}\Big(\frac{1}{(8k+1)^2}-\frac{1}{(8k+2)^2}+\frac{1}{2(8k+3)^2}-\frac{1}{4(8k+5)^2}+\frac{1}{4(8k+6)^2}-\frac{1}{8(8k+7)^2}\Big)$$
$$-\frac{1}{4}\sum_{k=0}^{\infty}\frac{1}{4096^k}\Big(\frac{1}{(8k+1)^2}+\frac{1}{2(8k+2)^2}+\frac{1}{8(8k+3)^2}-\frac{1}{64(8k+5)^2}-\frac{1}{128(8k+6)^2}-\frac{1}{512(8k+7)^2}\Big)=G,$$

which is Catalan's constant [175, A006752].

[44]

(0.322) $$\sum_{k\geq 0}\frac{1}{16^k}\left(\frac{4}{8k+1}-\frac{2}{8k+4}-\frac{1}{8k+5}-\frac{1}{8k+6}\right)=\pi.$$

[90]

(0.323)
$$\frac{1}{16807}\sum_{n=0}^{\infty}\frac{1}{2^n\binom{7n}{2n}}\left(\frac{59296}{7n+1}-\frac{10326}{7n+2}-\frac{3200}{7n+3}-\frac{1352}{7n+4}-\frac{792}{7n+5}+\frac{552}{7n+6}\right)=\pi.$$

[6]

(0.324) $$\frac{1}{2}\sum_{k=0}^{\infty}\frac{4^k k!^2}{(2k)!(2k+1)^2}=G.$$

[6]

(0.325) $$-\frac{\pi}{32}\sum_{k=0}^{\infty}\frac{(2k+1)!^2}{16^k k!^4(k+1)^3}=G-\frac{\pi}{2}\log 2.$$

[6]

(0.326) $$\sqrt{2}\sum_{k=0}^{\infty}\frac{(2k)!}{8^k k!^2(2k+1)^2}=G+\frac{\pi}{4}\log 2.$$

[6, 41]

(0.327) $$\frac{3}{8}\sum_{k=0}^{\infty}\frac{k!^2}{(2k)!(2k+1)^2}=G+\frac{\pi}{8}\log(2+\sqrt{3}).$$

[41]

(0.328) $$\frac{5}{8}\sum_{k=0}^{\infty}\frac{L(2k+1)}{(2k+1)^2\binom{2k}{k}}=G-\frac{\pi}{8}\log(\frac{10+\sqrt{50-22\sqrt{5}}}{10-\sqrt{50-22\sqrt{5}}}).$$

where L are the Lucas numbers.

[155]

(0.329) $$\sqrt{e}=\frac{16}{31}\left[1+\sum_{n=1}^{\infty}\frac{1+n^3/2+n/2}{2^n n!}\right],$$

which is a special case of $\frac{1}{2}P_3(1/2)+\frac{1}{2}P_1(1/2)+P_0(1/2)$ of [100]

(0.330) $$\sum_{n=0}^{\infty}n^j\frac{t^n}{n!}=P_j(t)\exp(t)$$

where

$$P_0(t) \equiv 1; \quad P_j(t) \equiv t\left[P'_{j-1}(t) + P_{j-1}(t)\right], j > 0. \tag{0.331}$$

[4]

$$\sum_{k\ge 1} \left[\begin{array}{c} k \\ p \end{array}\right] \frac{1}{k!k} = \zeta(p+1). \tag{0.332}$$

[4]

$$\sum_{k\ge 2} \left[\begin{array}{c} k \\ 2 \end{array}\right] \frac{1}{k!k^q} = \frac{q+1}{2}\zeta(q+2) - \frac{1}{q}\sum_{k=1}^{q-1} k\zeta(k+1)\zeta(q+1-k). \tag{0.333}$$

[4]

$$\sum_{k\ge 1} \left[\begin{array}{c} k \\ p \end{array}\right] \frac{z^k}{k!k} = \zeta(p+1) + \sum_{k=1}^{p} (\frac{(-)^{k-1}}{k!} \operatorname{Li}_{p+1-k}(1-z)\log^k(1-z). \tag{0.334}$$

[4]

$$\sum_{k\ge p} \left[\begin{array}{c} k \\ p \end{array}\right] \frac{1}{k!k^q} = \sum_{k\ge q} \left[\begin{array}{c} k \\ q \end{array}\right] \frac{1}{k!k^p} = \frac{(-)^{q-1}}{(q-1)!p!} \lim_{\beta\to 0}\lim_{\alpha\to 0} \frac{d^{q+p-1}}{d\alpha^q d\beta^{q-1}} \frac{\Gamma(1-\alpha)\Gamma(1+\beta)}{\Gamma(\beta)\Gamma(1-\alpha+\beta)}, \tag{0.335}$$

which can always be represented in finite terms of zeta functions.

0.3. **Formulae from Differential Calculus.** [112]

$$\left(\frac{d}{dx}\right)^m e^{g(x)} = e^{g(x)} Y_m\left(g'(x), g''(x), \ldots, g^{(m)}(x)\right) \tag{0.336}$$

where Y_m is the mth exponential complete Bell polynomial,

$$Y_n(x_1, \ldots, x_n) = \sum_{\pi(n)} \frac{n!}{k_1! \cdots k_n!} \left(\frac{x_1}{1!}\right)^{k_1} \cdots \left(\frac{x_n}{n!}\right)^{k_n}. \tag{0.337}$$

$$\frac{d}{dx}\left[f(x)^{g(x)}\right] = f(x)^{g(x)}\left[(\log f)g' + g\frac{f'}{f}\right] \tag{0.338}$$

[62]

$$\sum_{n\ge 0} \frac{g^{(n)}(0)}{n!} f(n)x^n = \sum_{n\ge 0} \frac{f^{(n)}(0)}{n!} \sum_{k=0}^{n} \left\{\begin{array}{c} n \\ k \end{array}\right\} x^k g^{(k)}(x). \tag{0.339}$$

where the braces are Stirling numbers of the second kind.

1. Elementary Functions

1.1. **Powers of Binomials.** [30]

$$\left(\frac{1+\sqrt{1+4x}}{2}\right)^n = (1+x)^{n/2} \sum_{k\ge 0} \frac{(-)^{k+1}\Gamma(\frac{3k-2}{2})}{\Gamma(\frac{3k-n}{2} - k + 1)k!} \left(\frac{x}{(1+x)^{3/2}}\right)^k. \tag{1.1}$$

[119]

$$\sum_{n=0}^{\infty} \binom{2n}{n} x^n = \frac{1}{\sqrt{1-4x}}. \tag{1.2}$$

[119][175, A000108]

$$\sum_{n=0}^{\infty}\binom{2n}{n}\frac{x^n}{n+1}=\frac{1-\sqrt{1-4x}}{2x}. \tag{1.3}$$

[119]

$$\sum_{n=1}^{\infty}\frac{1}{n}\binom{2n}{n}x^n=2\log\frac{1-\sqrt{1-4x}}{2x}. \tag{1.4}$$

[119]

$$x\sum_{n=1}^{\infty}\frac{1}{n(n+1)}\binom{2n}{n}x^n=2x\log\frac{1-\sqrt{1-4x}}{x}+\frac{\sqrt{1-4x}}{2}-x(\log 4-1)-\frac{1}{2}, \tag{1.5}$$

which corrects a sign error in [119], see [127].

[40]

$$\sum_{n=0}^{\infty}\frac{1}{n+1}\binom{2n}{n}H_n x^{n+1}=\sqrt{1-4x}\log(2\sqrt{1-4x})-(1+\sqrt{1-4x})\log(1+\sqrt{1-4x})+\log 2, \tag{1.6}$$

$$\sum_{n=0}^{\infty}\binom{2n}{n}H_n x^n=\frac{2}{\sqrt{1-4x}}\log(\frac{1+\sqrt{1-4x}}{2\sqrt{1-4x}}), \tag{1.7}$$

$$\sum_{n=0}^{\infty}\binom{2n}{n}H_n(-1)^{n-1}x^n=\frac{2}{\sqrt{1+4x}}\log(\frac{2\sqrt{1+4x}}{1+\sqrt{1+4x}}), \tag{1.8}$$

where $H_n=\sum_{k=1}^{n}1/k$.

[40]

$$\sum_{n=0}^{\infty}\binom{2n}{n}P_q(n)x^{n+1}=\frac{1}{\sqrt{1-4x}}\sum_{k=0}^{q}\binom{2k}{k}k!\left(\frac{x}{1-4x}\right)^k\sum_{m=k}^{q}a_m S(m,k) \tag{1.9}$$

where $P_q(z)=a_q z^q+a_{q-1}z^{q-1}+\ldots+q_0$ is a polynomial and S are the Stirling Numbers of the Second Kind.

[175, A005430]

$$\sum_{n=1}^{\infty}n\binom{2n}{n}x^n=2x(1-4x)^{-3/2}. \tag{1.10}$$

[119][175, A002736]

$$\sum_{n=1}^{\infty}n^2\binom{2n}{n}x^n=\frac{2x(2x+1)}{(1-4x)^{5/2}}. \tag{1.11}$$

[119]

$$\sum_{n=0}^{\infty}\frac{1}{2n+1}\binom{2n}{n}x^n=\frac{1}{2x}\arcsin(2x). \tag{1.12}$$

[40]

$$\sum_{n=0}^{\infty}\frac{1}{n+m+1}\binom{2n}{n}x^n=\frac{1}{2^{2m+1}x^{m+1}}\sum_{k=0}^{m}\binom{m}{k}\frac{(-)^k}{2k+1}[1-(1-4x)^{k+1/2}]. \tag{1.13}$$

[40]

$$\sum_{n=0}^{\infty}\frac{1}{n^2}\binom{2n}{n}x^n = 2\,\mathrm{Li}_2\left(\frac{1-\sqrt{1-4x}}{2}\right) - \log^2(1+\sqrt{1-4x}) - 2\log 2\log\frac{1-\sqrt{1-4x}}{x} + 3\log^2 2. \tag{1.14}$$

[119]

$$\sum_{m=1}^{\infty}\frac{(-1)^{m-1}}{\binom{2m}{m}}(2x/y)^{2m} = \frac{xy^2}{h^3}\left[\log\frac{x+h}{y}+\frac{xh}{y^2}\right] \tag{1.15}$$

where $h \equiv \sqrt{x^2+y^2}$.

[119]

$$\sum_{m=1}^{\infty}\frac{(2x)^{2m}}{m\binom{2m}{m}} = \frac{2x\arcsin x}{\sqrt{1-x^2}}. \tag{1.16}$$

[119]

$$\sum_{m=1}^{\infty}\frac{(2x)^{2m}}{m^2\binom{2m}{m}} = 2(\arcsin x)^2. \tag{1.17}$$

[119]

$$\sum_{m=1}^{\infty}\frac{(2x)^{2m}}{\binom{2m}{m}} = \frac{x^2}{1-x^2}+\frac{x\arcsin x}{(1-x^2)^{3/2}}. \tag{1.18}$$

$$\frac{1}{(\sigma^2+\sigma_L^2)^{\gamma/2}} = \frac{1}{\sigma_L^\gamma}\exp\left(-\frac{\gamma}{2}\frac{\sigma^2}{\sigma_L^2}\right)\left(1+\frac{\gamma}{4}\left(\frac{\sigma}{\sigma_L}\right)^4-\frac{\gamma}{6}\left(\frac{\sigma}{\sigma_L}\right)^6+\frac{\gamma(\gamma+4)}{32}\left(\frac{\sigma}{\sigma_L}\right)^8 -\frac{\gamma(5\gamma+12)}{120}\left(\frac{\sigma}{\sigma_L}\right)^{10}+\frac{\gamma(3\gamma^2+52\gamma+96)}{1152}\left(\frac{\sigma}{\sigma_L}\right)^{12}-\frac{\gamma(35\gamma^2+308\gamma+480)}{6720}\left(\frac{\sigma}{\sigma_L}\right)^{14}+\cdots\right) \tag{1.19}$$

[162, 3.41(a)]

$$\frac{1}{a^2-x^2} = \frac{2}{a\sqrt{a^2-1}}{\sum_{j=0}^{\infty}}'(a-\sqrt{a^2-1})^{2j}T_{2j}(x) \tag{1.20}$$

where the prime at the sum symbols means taking only half of the value at index zero.

[31]

$$(1+x+x^{-1})^n = \sum_{j=-n}^{n}\binom{n}{j}_2 x^j,\quad \binom{n}{m}_2 \equiv \sum_{j\ge 0}\frac{n!}{j!(m+j)!(n-2j-m)!}. \tag{1.21}$$

[153]

$$\sum_{s=0}^{\infty}\zeta(4s+3)x^{4s} = \sum_{k=1}^{\infty}\frac{k}{k^4-x^4} = \frac{5}{2}\sum_{k=1}^{\infty}\frac{(-)^{k-1}}{\binom{2k}{k}}\frac{k}{k^4-x^4}\prod_{m=1}^{k-1}\frac{m^4+4x^4}{m^4-x^4}. \tag{1.22}$$

[28] Let $B_0(x)$ be a period function of period 1. Assume $B_0(x)$ has a continuous derivative in the open interval (0,1). Let $a_0 \equiv \int_0^1 B_0(x)dx$ and define

$$B_n(x) = \int_0^2 B_{n-1}(y)dy + \int_0^1 (y-1)B_{n-1}(y)dy. \tag{1.23}$$

Then

$$\sum_{k=0}^{\infty} B_k(x)t^k = \frac{te^{xt}}{e^t-1}\left[a_0 - \int_0^1 B_0'(1-y)\frac{e^{ty}-1}{t}dy\right] + \int_0^x e^{t(x-y)}B_0'(y)dy, \tag{1.24}$$

and

$$B_n(x) = \Re\left[2\sum_{j=1}^{\infty}\frac{e^{2\pi ijx}}{(2\pi ij)^n}(a_j - a_0 - ib_j)\right] \tag{1.25}$$

where a_j and b_j are the Fourier coefficients of $B_0(x)$. For example $B_0(x) = \cos(\pi x)$ yields $B_2(x) = (1-2x-\cos(\pi x))/\pi^2$ and

$$\frac{\pi^3}{2}B_2(x) = -\sum_{j=1}^{\infty}\frac{\sin(2\pi jx)}{j(4j^2-1)}. \tag{1.26}$$

Consider also a periodic sequence $\lambda_{j+T} = \lambda_j$, the Dirichlet series

$$f(s) = \sum_{j=1}^{\infty}\frac{a_j - a_0}{j^s}\lambda_j, \quad g(s) = \sum_{j=1}^{\infty}\frac{b_j}{j^s}\lambda_j, \tag{1.27}$$

and the Fourier coefficients

$$\alpha_j \equiv \sum_{k=1}^{T}\lambda_k\cos(2\pi jk/T), \quad \beta_j \equiv \sum_{k=1}^{T}\lambda_k\sin(2\pi jk/T). \tag{1.28}$$

If $\lambda_{T-j} = \lambda_j$ for $1 \le j < T$, then

$$f(2n) = \frac{(-)^n}{2T}(2\pi)^{2n}\sum_{j=1}^{T}\alpha_j B_{2n}(j/T), \tag{1.29}$$

$$g(2n+1) = \frac{(-)^{n+1}}{2T}(2\pi)^{2n+1}\sum_{j=1}^{T}\alpha_j B_{2n+1}(j/T), \tag{1.30}$$

and similar expressions for other even-odd symmetries of the λ.

[28]

$$8\log 2 - 4 = \zeta(3) + \sum_{j=1}^{\infty}\frac{1}{j^2(4j^2-1)} = \sum_{k=0}^{\infty}\frac{\zeta(3+2k)}{4^k}. \tag{1.31}$$

[28]

$$\sum_{j=1}^{\infty}\frac{\cos(2\pi j/3)}{j^{2k}(4\pi^2j^2+1)} = \frac{(-)^k}{4}(2\pi)^{2k}\alpha_{2k}, \tag{1.32}$$

with g.f.

(1.33)

$$\sum_{k=0}^{\infty}\alpha_k t^k = \frac{e + e^{4/3} - e^{1+t} - e^{t+4/3} + t^3[e^{(2t+5)/3} - e^{(2+2t)/3} - e^{(2+t)/3} + e^{(5+t)/3}]}{(e-1)e^{2/3}(e^t-1)(t^2-1)} - \frac{2te^{t/3}}{e^t-2}.$$

1.2. **The Exponential Function.** [84], [175, A001469]

$$\frac{2t}{e^t+1} = \sum_{N=0}^{\infty} \frac{G_N}{N!} t^N; \tag{1.34}$$

with $(G+1)^N + G_N = 1$ for $N > 1$, $G_1 = 1$, $G_2 = -1$, $G_4 = 1$, $G_6 = -3$, $G_8 = 17$, $G_{10} = -155$, $G_{12} = 2073$, Genocchi numbers.

[31]

$$\exp(xt+yt^2) = \sum_{n=0}^{\infty} \frac{t^n}{n!} H_n(x,y), \quad H_n(x,y) \equiv n! \sum_{k=0}^{[n/2]} \frac{x^{n-2k}y^k}{k!(n-2k)!}. \tag{1.35}$$

1.3. **Fourier Series.** [61, B2a]

$$\cos(m\theta) = \sum_{k=0}^{\lfloor m/2 \rfloor} (-1)^k \binom{m}{2k} \cos^{m-2k}\theta \sin^{2k}\theta. \tag{1.36}$$

$$\sin^{2k}\alpha + \cos^{2k}\alpha = 1 - \sum_{l=1}^{k-1} \binom{k}{l} \sin^{2l}\alpha \cos^{2k-2l}\alpha. \tag{1.37}$$

[60]

$$S_\nu(\alpha) \equiv \sum_{k=0}^{\infty} \frac{\sin(2k+1)\alpha}{(2k+1)^\nu}; \tag{1.38}$$

$$C_\nu(\alpha) \equiv \sum_{k=0}^{\infty} \frac{\cos(2k+1)\alpha}{(2k+1)^\nu}; \tag{1.39}$$

$$S_{2n+1}(\alpha) = \frac{(-1)^n}{4(2n)!} \pi^{2n+1} E_{2n}\left(\frac{\alpha}{\pi}\right); \tag{1.40}$$

$$C_{2n}(\alpha) = \frac{(-1)^n}{4(2n-1)!} \pi^{2n} E_{2n-1}\left(\frac{\alpha}{\pi}\right); \tag{1.41}$$

$$S_\nu(2\pi p/q) = \frac{1}{q^\nu} \sum_{s=1}^{q-1} \zeta(\nu, s/q) \left[\sin(s2\pi p/q) - \frac{\sin(s4\pi p/q)}{2^\nu}\right]; \tag{1.42}$$

$$C_\nu(2\pi p/q) = \frac{1}{q^\nu} \left\{\zeta(\nu)\left(1 - \frac{1}{2^\nu}\right) + \sum_{s=1}^{q-1} \zeta(\nu, s/q) \left[\cos(s2\pi p/q) - \frac{\cos(s4\pi p/q)}{2^\nu}\right]\right\}. \tag{1.43}$$

[91, 1.448.1]

$$\sum_{k=1}^{\infty} \frac{p^k \sin(kx)}{k} = \arctan\frac{p\sin x}{1-p\cos x}, \quad 0 < x < 2\pi, p^2 \le 1, \tag{1.44}$$

with special case

$$\sum_{k=1}^{\infty} \frac{\sin(kx)}{2^k k} = \arctan\frac{\sin x}{2-\cos x}, \quad 0 < x < 2\pi. \tag{1.45}$$

[91, 1.448.2]

$$\sum_{k=1}^{\infty} \frac{p^k \cos(kx)}{k} = \ln \frac{1}{\sqrt{1-2p\cos x + p^2}}, \quad 0 < x < 2\pi, \tag{1.46}$$

with special case

$$\sum_{k=1}^{\infty} \frac{\cos(kx)}{2^k k} = -\ln \sqrt{1-\cos x + 1/4}, \quad 0 < x < 2\pi. \tag{1.47}$$

[99, (1.10)]

$$-\sqrt{2-2\cos\theta} = -\frac{4}{\pi} + \frac{2}{\pi}\sum_{k=1}^{\infty} \frac{\cos k\theta}{(k-1/2)(k+1/2)} \tag{1.48}$$

[184]

$$\mathrm{Cl}_{2n+1}(\pi/2) = -2^{-2n-1}(1-2^{-2n})\zeta(2n+1), n \in \mathbb{N}. \tag{1.49}$$

[184]

$$\mathrm{Cl}_{2n+1}(\pi/3) = \frac{1}{2}(1-2^{-2n})(1-3^{-2n})\zeta(2n+1), n \in \mathbb{N}. \tag{1.50}$$

[184]

$$\mathrm{Cl}_{2n+1}(2\pi/3) = -\frac{1}{2}(1-3^{-2n})\zeta(2n+1), n \in \mathbb{N}. \tag{1.51}$$

[184]

$$\sum_{k\geq 1} \frac{\cos(k\pi/2)}{k^s} = -2^{-s}(1-2^{1-s})\zeta(s), \Re s > 1. \tag{1.52}$$

[119]

$$\sum_{m=1}^{\infty} \frac{m^{k-2}(2x)^{2m}}{\binom{2m}{m}} = \frac{x}{2^{k-2}(1-x^2)^{k-1/2}}\left[\arcsin x V_x(x^2) + x\sqrt{1-x^2}W_k(x^2)\right], \quad k \geq 0, \tag{1.53}$$

[119]

$$\sum_{m=1}^{\infty} \frac{m^{k-2}4^m(\sin\theta)^{2m}}{\binom{2m}{m}} = \frac{\sin 2\theta}{(2\cos^2\theta)^k}\left[2\theta V_k(\sin^2\theta) + \sin 2\theta W_k(\sin^2\theta)\right], \tag{1.54}$$

[119]

$$\sum_{m=1}^{\infty} \frac{(-1)^{m-1}m^{k-2}4^m(\sinh z)^{2m}}{\binom{2m}{m}} \tag{1.55}$$
$$= \frac{\sinh 2z}{(2\cosh^2 z)^k}\left[2\log\{\sinh z + \cosh z\}V_k(-\sinh^2 z) + \sinh 2z W_k(-\sinh^2 z)\right],$$

where $V_1(t) = 1$, (see [175, A156919]) $W_1(t) = 0$,

$$V_{k+1}(t) = \{(2k-2)t+1\}V_k(t) + 2(1-t)\delta V_k(t)$$
$$W_{k+1}(t) = \{(2k-4)t+2\}W_k(t) + 2(1-t)\delta W_k(t) + V_k(t)$$

and δ is the operator $x\frac{d}{dx}$.

1.4. **Expansions of Hyperbolic Functions.** [30]

$$\cos x \cosh x = \sum_{k\ge 0} \frac{(-)^k (2x^2)^{2k}}{(4k)!}. \tag{1.56}$$

[30]

$$\sin x \sinh x = \sum_{k\ge 0} \frac{(-)^k (2x^2)^{2k+1}}{(4k+2)!}. \tag{1.57}$$

[30]

$$\coth(2x)\tanh x = 1 - \sum_{k\ge 1} \frac{2^{2k-1}(2^{2k}-1)(2k-1)B_{2k}x^{2k-2}}{(2k)!}. \tag{1.58}$$

[33]

$$\sum_{j=0}^{\infty} \frac{x}{\sinh 2^{-j}x} - 2^j = 1 - \frac{x}{\tanh x}. \tag{1.59}$$

[33]

$$\sum_{j=0}^{\infty} \frac{2^j - \coth 2^{-j}}{2^j \sinh 2^{-j}} = \frac{1+4e^2-e^4}{1-2e^2+e^4}. \tag{1.60}$$

[86]

$$\sum_{m,n=-\infty}^{\infty} \frac{F(|2m+2n+1|)}{\cosh(2m+1)u\ \cosh 2nu} = 2\sum_{n=0}^{\infty} \frac{(2n+1)F(2n+1)}{\sinh(2n+1)u}, \tag{1.61}$$

for any summable function $F(x)$.
[86]

$$\sum_{m,n=-\infty}^{\infty} \frac{F(m+n+1)+F(m-n)}{\sinh(2m+1)u\ \cosh(2n+1)u} = 8\sum_{n=1}^{\infty} \frac{nF(n)}{\cosh(2nu)}, \tag{1.62}$$

and

$$\sum_{k,m,n=-\infty}^{\infty} \frac{F(k+m+n+1)+F(k-m+n)}{\cosh(2ku)\cosh(2m+1)u\ \sinh(2n+1)u} = 8\sum_{n=1}^{\infty} \frac{n^2F(n)}{\sinh(2nu)}, \tag{1.63}$$

where F is any sine transform (and hence odd).

1.5. **The Logarithmic Function.**

$$\ln(1-x) = -\sum_{k=1}^{\infty} \frac{x^k}{k}. \tag{1.64}$$

(1.65)

$$(1+x)\ln(1+x) = x + \frac{1}{1\cdot 2}x^2 - \frac{1}{2\cdot 3}x^3 + \frac{1}{3\cdot 4}x^4 - \frac{1}{4\cdot 5}x^5 + \cdots = x + \sum_{k=1}^{\infty} (-1)^{k+1} \frac{x^{k+1}}{k(k+1)}.$$

By integration of this w.r.t. x [91, 2.729], [135]

$$\frac{1}{2}(1+x)^2\ln(1+x) = \frac{x}{2} + \frac{3x^2}{4} + \sum_{k=1}^{\infty} (-1)^{k+1} \frac{x^{k+2}}{k(k+1)(k+2)}. \tag{1.66}$$

$$\frac{1}{6}(1+x)^3\ln(1+x)=\frac{x}{6}+\frac{5x^2}{12}+\frac{11x^3}{36}+\sum_{k=1}^{\infty}(-1)^{k+1}\frac{x^{k+3}}{k(k+1)(k+2)(k+3)}. \tag{1.67}$$

(1.68)

$$\frac{1}{24}(1+x)^4\ln(1+x)=\frac{x}{24}+\frac{7x^2}{48}+\frac{13x^3}{72}+\frac{25x^4}{288}+\sum_{k=1}^{\infty}(-1)^{k+1}\frac{x^{k+4}}{k(k+1)(k+2)(k+3)(k+4)}.$$

$$x+(1-x)^l\ln(1-x)=\sum_{i=2}^{l}\tau_{i,l}x^i-(-1)^l l!\sum_{k=1}^{\infty}\frac{x^{k+l}}{k(k+1)(k+2)\cdots(k+l)}, \tag{1.69}$$

where

$$\tau_{2,l}=l-1/2,\quad l\geq 2;\quad i\tau_{i,l}=(-1)^i\binom{l-1}{i-1}-l\tau_{i-1,l-1},\quad i\geq 3. \tag{1.70}$$

[116]

$$\log n=-\sum_{s\geq 1}\frac{\alpha(s,n)}{s} \tag{1.71}$$

with

$$\alpha(s,n)\equiv\begin{cases} n-1, & n\mid s;\\ -1 & n\nmid s.\end{cases} \tag{1.72}$$

Superposition of [91, 1.513.1] and [91, 1.511], see [175, A165998]:

$$\frac{1}{3x}\ln\frac{1+x}{(1-x)^2}=1+\frac{x}{6}+\frac{x^2}{3}+\frac{x^3}{12}+\cdots+\frac{x^{2j}}{2j+1}+\frac{x^{2j+1}}{3(2j+1)}+\cdots \tag{1.73}$$

[11]

$$\sum_{j=k}^{\infty}\frac{|S_j^{(k)}|b^j}{j!}=\frac{\ln^k(1+b)}{k!}. \tag{1.74}$$

[101]

(1.75)

$$\sum_{k=0}^{\infty}\frac{x^k}{\binom{k+L}{L}}={}_2F_1(1,1;L+1;x)=L\sum_{j=0}^{L-2}\frac{(x-1)^j}{(L-j-1)x^{j+1}}-L(x-1)^{L-1}\frac{\ln(1-x)}{x^L};\quad L\geq 1.$$

[43]

$$\frac{1}{2x}\left\{1-\ln(1+x)-\frac{1-x}{\sqrt{x}}\arctan\sqrt{x}\right\}=\sum_{k=1}^{\infty}\frac{(-1)^{k+1}x^{k-1}}{(2k-1)2k(2k+1)}; \tag{1.76}$$

$0<x\leq 1$.

[138]

$$-\frac{\ln(1-t)}{(1-t)^r}=\sum_{n=0}^{\infty}H_n^{(r)}t^n, \tag{1.77}$$

where the Hyperharmonic numbers are defined by $H_n=\sum_{k=1}^{n}(1/k)$ and $H_n^{(r)}=\sum_{k=1}^{n}H_k^{(r-1)}=\binom{n+r-1}{r-1}(H_{n+r-1}-H_{r-1})$.

1.6. **The Inverse Trigonometric and Hyperbolic Function.** [33]

$$\sum_{k=1}^{\infty} 2^{-k}\tan\frac{x}{2^k} = \frac{1}{2^n}\cot\frac{x}{2^n} - \cot x. \tag{1.78}$$

[33]

$$\sum_{k=1}^{\infty} \csc\frac{x}{2^{k-1}} = \cot\frac{x}{2^n} - \cot x. \tag{1.79}$$

[33]

$$\sum_{k=1}^{\infty} \arctan\frac{2x^2}{k^2} = \frac{\pi}{4} - \arctan\frac{\tanh\pi x}{\tan\pi x}. \tag{1.80}$$

[33]

$$\sum_{k=1}^{\infty} (-)^{k-1}\arctan\frac{2x^2}{k^2} = -\frac{\pi}{4} - \arctan\frac{\sinh\pi x}{\sin\pi x}. \tag{1.81}$$

2. Indefinite Integrals of Elementary Functions

2.1. **Rational Functions.** Aids to partial fraction decompositions: [205]

$$\frac{1}{s^n(s^2+as+b)} = \frac{-\alpha_{n-1}s+\alpha_n}{b^n(s^2+as+b)} + \sum_{k=0}^{n-1}\frac{\alpha_k}{b^{k+1}s^{n-k}}; \tag{2.1}$$

where $b>0$, $a^2-4b<0$, $\alpha_0\equiv 1$,

$$\alpha_m \equiv (-1)^m\sqrt{b^m}U_m\left(\frac{a}{2\sqrt{b}}\right) = \sum_{k=0}^{\lfloor m/2\rfloor}(-1)^{k+m}\binom{m-k}{k}a^{m-2k}b^k. \tag{2.2}$$

[205]

(2.3)
$$\frac{\alpha s+\beta}{(s^2+as+b)(s^2+kas+kb)} = \frac{1}{(k-1)b^2}\left(\frac{L(s+a)+\beta b}{s^2+as+b} - \frac{L(s+ka)+\beta b}{s^2+kas+kb}\right),$$

$k\neq 1$, $ab\neq 0$, $L\equiv\begin{vmatrix}\alpha & \beta\\ a & c\end{vmatrix}$.

[205]

$$\frac{1}{(pq+ap+b)(pq+cp+b)} = \begin{vmatrix}a & c\\ b & b\end{vmatrix}^{-1}\left(\frac{q+a}{pq+ap+b} - \frac{q+c}{pq+cp+b}\right), \tag{2.4}$$

$a\neq c$, $b\neq 0$.

[205]

$$\frac{\alpha p+\beta}{(pq+ap+b)(pq+cp+b)} = \begin{vmatrix}a & c\\ b & b\end{vmatrix}^{-1}\left(\frac{\begin{vmatrix}\alpha & \beta\\ q+c & b\end{vmatrix}}{pq+cp+b} - \frac{\begin{vmatrix}\alpha & \beta\\ q+a & b\end{vmatrix}}{pq+ap+b}\right), \tag{2.5}$$

$a\neq c$, $b\neq 0$.

[205]

(2.6)

$$\frac{as+b}{(s^2+\alpha s+\beta)(s^2+\gamma s+\delta)} = \frac{1}{k}\left(\frac{\left(b-a\frac{\delta-\beta}{\gamma-\alpha}\right)s+b\alpha-a\beta-b\frac{\delta-\beta}{\gamma-\alpha}}{s^2+\alpha s+\beta} - \frac{\left(b-a\frac{\delta-\beta}{\gamma-\alpha}\right)s+b\gamma-a\delta-b\frac{\delta-\beta}{\gamma-\alpha}}{s^2+\gamma s+\delta}\right),$$

$\alpha \neq \gamma$.

[132]

$$\frac{1}{n^{2s}(n+1)^{2s}} = \sum_{t=1}^{2s}\binom{4s-t-1}{2s-1}\left[\frac{(-1)^t}{n^t}+\frac{1}{(n+1)^t}\right]. \tag{2.7}$$

[65, 170.]

$$\int \frac{dx}{a^4+x^4} = \frac{1}{4a^3\sqrt{2}}\log\frac{x^2+ax\sqrt{2}+a^2}{x^2-ax\sqrt{2}+a^2} + \frac{1}{2a^3\sqrt{2}}\arctan\frac{ax\sqrt{2}}{a^2-x^2}. \tag{2.8}$$

[65, 171.]

$$\int \frac{dx}{a^4-x^4} = \frac{1}{4a^3}\log\left|\frac{a+x}{a-x}\right| + \frac{1}{2a^3}\arctan\frac{x}{a}. \tag{2.9}$$

[65, 170.1]

$$\int \frac{xdx}{a^4+x^4} = \frac{1}{2a^2}\arctan\frac{x^2}{a^2}. \tag{2.10}$$

[65, 170.2]

$$\int \frac{x^2dx}{a^4+x^4} = -\frac{1}{4a\sqrt{2}}\log\frac{x^2+ax\sqrt{2}+a^2}{x^2-ax\sqrt{2}+a^2} + \frac{1}{2a\sqrt{2}}\arctan\frac{ax\sqrt{2}}{a^2-x^2}. \tag{2.11}$$

[65, 170.3]

$$\int \frac{x^3dx}{a^4+x^4} = \frac{1}{4}\log(a^4+x^4). \tag{2.12}$$

[65, 171.1]

$$\int \frac{xdx}{a^4-x^4} = \frac{1}{4a^2}\log\left|\frac{a^2+x^2}{a^2-x^2}\right|. \tag{2.13}$$

[65, 171.2]

$$\int \frac{x^2dx}{a^4-x^4} = \frac{1}{4a}\log\left|\frac{a+x}{a-x}\right| - \frac{1}{2a}\arctan\frac{x}{a}. \tag{2.14}$$

[65, 171.3]

$$\int \frac{x^3dx}{a^4-x^4} = -\frac{1}{4}\log|a^4-x^4|. \tag{2.15}$$

[65, 173]

$$\int \frac{dx}{x(a+bx^m)} = \frac{1}{am}\log|\frac{x^m}{a+bx^m}|. \tag{2.16}$$

2.2. **Algebraic Functions.** [65, 186.11,188.11]

$$\int \frac{dx}{(a^2+b^2x)x^{1/2}} = \frac{2}{ab}\arctan\frac{bx^{1/2}}{a}. \tag{2.17}$$

$$\int \frac{dx}{(a^2-b^2x)x^{1/2}} = \frac{1}{2ab}\log\left|\frac{a+bx^{1/2}}{a-bx^{1/2}}\right|. \tag{2.18}$$

[65, 185.11,187.11]

$$\int \frac{x^{1/2}dx}{a^2+b^2x} = \frac{2x^{1/2}}{b^2} - \frac{2a}{b^3}\arctan\frac{bx^{1/2}}{a}. \tag{2.19}$$

$$\int \frac{x^{1/2}dx}{a^2-b^2x} = -\frac{2x^{1/2}}{b^2} + \frac{a}{b^3}\log\left|\frac{a+bx^{1/2}}{a-bx^{1/2}}\right|. \tag{2.20}$$

[65, 185.13,187.13]

$$\int \frac{x^{3/2}dx}{a^2+b^2x} = \frac{2}{3}\frac{x^{3/2}}{b^2} - \frac{2a^2x^{1/2}}{b^4} + \frac{2a^3}{b^5}\arctan\frac{bx^{1/2}}{a}. \tag{2.21}$$

$$\int \frac{x^{3/2}dx}{a^2-b^2x} = -\frac{2}{3}\frac{x^{3/2}}{b^2} - \frac{2a^2x^{1/2}}{b^4} + \frac{a^3}{b^5}\log\left|\frac{a+bx^{1/2}}{a-bx^{1/2}}\right|. \tag{2.22}$$

[65, 186.21,188.21]

$$\int \frac{dx}{(a^2+b^2x)^2x^{1/2}} = \frac{x^{1/2}}{a^2(a^2+b^2x)} + \frac{1}{a^3b}\arctan\frac{bx^{1/2}}{a}. \tag{2.23}$$

$$\int \frac{dx}{(a^2-b^2x)^2x^{1/2}} = \frac{x^{1/2}}{a^2(a^2-b^2x)} + \frac{1}{2a^3b}\log\left|\frac{a+bx^{1/2}}{a-bx^{1/2}}\right|. \tag{2.24}$$

[65, 185.21,187.21]

$$\int \frac{x^{1/2}dx}{(a^2+b^2x)^2} = -\frac{x^{1/2}}{b^2(a^2+b^2x)} + \frac{1}{ab^3}\arctan\frac{bx^{1/2}}{a}. \tag{2.25}$$

$$\int \frac{x^{1/2}dx}{(a^2-b^2x)^2} = \frac{x^{1/2}}{b^2(a^2-b^2x)} - \frac{1}{2ab^3}\log\left|\frac{a+bx^{1/2}}{a-bx^{1/2}}\right|. \tag{2.26}$$

[65, 185.23,187.23]

$$\int \frac{x^{3/2}dx}{(a^2+b^2x)^2} = \frac{2x^{3/2}}{b^2(a^2+b^2x)} + \frac{3a^2x^{1/2}}{b^4(a^2+b^2x)} - \frac{3a}{b^5}\arctan\frac{bx^{1/2}}{a}. \tag{2.27}$$

$$\int \frac{x^{3/2}dx}{(a^2-b^2x)^2} = \frac{3a^2x^{1/2}-2b^2x^{3/2}}{b^4(a^2-b^2x)} - \frac{3a}{2b^5}\log\left|\frac{a+bx^{1/2}}{a-bx^{1/2}}\right|. \tag{2.28}$$

[65, 186.13,188.13]

$$\int \frac{dx}{(a^2+b^2x)x^{3/2}} = -\frac{2}{a^2x^{1/2}} - \frac{2b}{a^3}\arctan\frac{bx^{1/2}}{a}. \tag{2.29}$$

$$\int \frac{dx}{(a^2-b^2x)x^{3/2}} = -\frac{2}{a^2x^{1/2}} + \frac{b}{a^3}\log\left|\frac{a+bx^{1/2}}{a-bx^{1/2}}\right|. \tag{2.30}$$

[65, 186.23,188.23]

$$\int \frac{dx}{(a^2+b^2x)^2x^{3/2}} = -\frac{2}{a^2(a^2+b^2x)x^{1/2}} - \frac{3b^2x^{1/2}}{a^4(a^2+b^2x)} - \frac{3b}{a^5}\arctan\frac{bx^{1/2}}{a}. \tag{2.31}$$

$$\int \frac{dx}{(a^2-b^2x)^2x^{3/2}} = -\frac{2}{a^2(a^2-b^2x)x^{1/2}} + \frac{3b^2x^{1/2}}{a^4(a^2-b^2x)} + \frac{3b}{2a^5}\log\left|\frac{a+bx^{1/2}}{a-bx^{1/2}}\right|. \tag{2.32}$$

[65, 189.2,189.4]

$$\int \frac{dx}{(a^4+x^2)x^{1/2}} = \frac{1}{2a^3\sqrt{2}}\log\frac{x+a\sqrt{2x}+a^2}{x-a\sqrt{2x}+a^2} + \frac{1}{a^3\sqrt{2}}\arctan\frac{a\sqrt{2x}}{a^2-x}. \tag{2.33}$$

$$\int \frac{dx}{(a^4-x^2)x^{1/2}} = \frac{1}{2a^3}\log\left|\frac{a+x^{1/2}}{a-x^{1/2}}\right| + \frac{1}{a^3}\arctan\frac{x^{1/2}}{a}. \tag{2.34}$$

[65, 188.23,189.3]

$$\int \frac{x^{1/2}dx}{a^4+x^2} = -\frac{1}{2a\sqrt{2}}\log\frac{x+a\sqrt{2x}+a^2}{x-a\sqrt{2x}+a^2} + \frac{1}{a\sqrt{2}}\arctan\frac{a\sqrt{2x}}{a^2-x}. \tag{2.35}$$

$$\int \frac{x^{1/2}dx}{a^4-x^2} = \frac{1}{2a}\log\left|\frac{a+x^{1/2}}{a-x^{1/2}}\right| - \frac{1}{a}\arctan\frac{x^{1/2}}{a}. \tag{2.36}$$

$$\int \frac{x^n}{\sqrt{a+bx}}dx = \frac{2\sqrt{a+bx}}{b^{n+1}}(-a)^n\sum_{k=0}^{n}\binom{n}{k}\frac{(-z/a)^k}{2k+1}. \tag{2.37}$$

This corrects two sign errors in [91] at the b:

$$\begin{aligned}\int \frac{z^m dx}{t^n\sqrt{z}} &= -z^m\sqrt{z}\Big\{\frac{1}{(n-1)\Delta}\frac{1}{t^{n-1}} \\ &+ \sum_{k=2}^{n-1}\frac{(2n-2m-3)(2n-2m-5)\cdots(2n-2m-2k+1)(-b)^{k-1}}{2^{k-1}(n-1)(n-2)\cdots(n-k)\Delta^k}\frac{1}{t^{n-k}}\Big\} \\ &- \frac{(2n-3m-3)(2n-3m-5)\cdots(-2m+3)(-2m+1)(-b)^{n-1}}{2^{n-1}(n-1)!\Delta^n}\int\frac{z^m dx}{t\sqrt{z}}\end{aligned} \tag{2.38}$$

where $z=a+bx$ and $t=\alpha+\beta x$ and $\Delta\equiv a\beta-\alpha b$.

[191]

$$\int \frac{f_m(z)}{\sqrt{D_m(z)}}dz = \log(x_m(z)+y_m(z)\sqrt{D_m(z)}), \tag{2.39}$$

if for example

$$f=4z+2;\quad D=z^4+8(z+1);\quad x=z^4-2z^3+2z^2+4z-4;\quad y=z^2-2z+2. \tag{2.40}$$

or

$$f=5z+1;\quad D=(z^2+1)^2+4z;\quad x=z^5-z^4+3z^3+z^2+2;\quad y=z^3-z^2+2z. \tag{2.41}$$

or

$$f=6z+2;\quad D=(z^2+2)^2+8z;\quad x=z^6-2z^5+8z^4-4z^3+8z^2+8z;\quad y=z^4-2z^3+6z^2-4z+4. \tag{2.42}$$

or
(2.43)
$f=3z-s;\quad D=(z^2-s^2)^2+t(z-s);\quad x=1+2(z+s)(z^2-s^2)/t;\quad y=2(z+s)/t.$

2.3. **The Exponential Function.**

$$\int x^{m+2}e^{-ax^2}dx=\frac{m+1}{2a}\left[-x^{m+1}e^{-ax^2}+\int x^m e^{-ax^2}dx\right]; \tag{2.44}$$

$m\neq -1$.

2.4. **Hyperbolic Functions.**

2.5. **Trigonometric Functions.**

$$\begin{aligned}\int \sin x\frac{(a+\sin x)^2}{(a+\sin x)^2+(b+\cos x)^2}dx &= \frac{1}{4}\left[\frac{2b^2}{a^2+b^2}+\frac{a^2-b^2}{(a^2+b^2)^2}-3\right]\cos(x)\\ &-\frac{ab}{2}\left[\frac{1}{a^2+b^2}-\frac{1}{(a^2+b^2)^2}\right]\sin(x)\\ +\frac{1}{8}\frac{b\cos(2x)-a\sin(2x)}{a^2+b^2}&+\frac{b}{4}\left[1+\frac{2(a^2-b^2)}{(a^2+b^2)^2}+\frac{-3a^2+b^2}{(a^2+b^2)^3}\right]\ln\sqrt{(\cos x+b)^2+(\sin x+a)^2}\\ +\frac{a}{4}\left[1-\frac{4b^2}{(a^2+b^2)^2}-\frac{a^2-3b^2}{(a^2+b^2)^3}\right]&\arctan\frac{\sin x+a}{\cos x+b}+\frac{a}{4}\left[\frac{4b^2}{(a^2+b^2)^2}+\frac{-3b^2+a^2}{(a^2+b^2)^3}\right]x.\end{aligned}$$

2.6. **Rational Functions of Trigonometric Functions.**

2.7. **The Logarithm.** [137]

$$\int\frac{\log z}{(1-z)z}dz=\mathrm{Li}_2(1-z)+\frac{1}{2}\log^2 z. \tag{2.45}$$

[17, 3.1.6.]

$$\int\frac{\ln x}{1+ax}dx=\frac{1}{a}[\ln x\ln(1+ax)+\mathrm{Li}_2(-ax)]. \tag{2.46}$$

Correcting a sign error in [17, 3.1.7]:

$$\int\frac{\ln(a+bx)}{c+hx}dx=\frac{1}{h}[\ln\left(\frac{ah-bc}{h}\right)\ln(c+hx)-\mathrm{Li}_2(\frac{bc+bhx}{bc-ah})]. \tag{2.47}$$

(2.48)
$$\begin{aligned}\int z\ln(c+z+1/z)dz=\frac{z^2-c^2/2+1}{2}\ln(z^2+cz+1)-\frac{c\sqrt{c^2/4-1}}{2}\ln\frac{z+c/2+\sqrt{c^2/4-1}}{z+c/2-\sqrt{c^2/4-1}}\\ -\frac{(z-c)^2}{4}-\frac{1}{2}z^2\ln z.\end{aligned}$$

3. Definite Integrals of Elementary Functions I.

3.1. General formulae. [8]

$$\int_0^\infty f([ax-b/x]^2)dx = \frac{1}{a}\int_0^\infty f(y^2)dy, \qquad a,b>0. \tag{3.1}$$

[34] Let

$$\varphi(x) = \sum_{k=0}^\infty A_k x^k,$$

then

$$\int_0^\infty x^{\beta-1}e^{-x}\varphi(x)dx = \sum_{k=0}^\infty A_k\Gamma(k+\beta), \quad \beta>0. \tag{3.2}$$

Let

$$\varphi(x) = \sum_{k=0}^\infty A_k x^{k/p},$$

then

$$\int_0^\infty x^{\beta-1}e^{-x}\varphi(x)dx = \sum_{k=0}^\infty A_k\Gamma(k/p+\beta), \quad \beta>0. \tag{3.3}$$

Let

$$\varphi(x) = \sum_{k=0}^\infty A_k x^k,$$

then

$$p\int_0^\infty x^{\beta-1}e^{-x^p}\varphi(x)dx = \sum_{k=0}^\infty A_k\Gamma(\frac{\beta+k}{p}), \quad \beta>0. \tag{3.4}$$

[117]

$$\int_a^u \frac{xdx}{\sqrt{(x-a)(x-b)(x-c)}} = \frac{2}{\sqrt{a-c}}[(a-b)\Pi(\mu,1,q)+bF(\mu,q)],\ u>a>b>c. \tag{3.5}$$

[117]

$$\int_u^c \frac{dx}{(r-x)\sqrt{(a-x)(b-x)(c-x)}} = \frac{2(c-b)}{(r-b)(r-c)\sqrt{a-c}} \times \Pi(\beta,\frac{r-b}{r-c},p) + \frac{2}{(r-b)\sqrt{a-c}}F(\beta,p),\ a>b>c>u, r\neq c. \tag{3.6}$$

[117]

$$\int_a^u \frac{dx}{(x-r)\sqrt{(x-a)(x-b)(x-c)}} = \frac{2}{(b-r)(a-r)\sqrt{a-c}} \times \left[(b-a)\Pi(\mu,\frac{b-r}{a-b},q)+(a-r)F(\mu,q)\right],\ u>a>b>c, r\neq a. \tag{3.7}$$

[117]

$$\int_u^b \sqrt{\frac{(x-c)(b-x)}{a-x}}dx = \frac{2}{3}\sqrt{a-c}[2(b-a)F(\delta,q) + (2a-b-c)E(\delta,q)] + \frac{2}{3}(b+c-a-u)\sqrt{\frac{(b-u)(u-c)}{a-u}},\ a>b>u\ge c. \tag{3.8}$$

[117]

$$\int_a^u \sqrt{\frac{(x-b)(x-c)}{x-a}}dx = \frac{2}{3}\sqrt{a-c}[2(a-b)F(\mu,q) + (b+c-2a)E(\mu,q)] + \frac{2}{3}(u+2a-2b-c)\sqrt{\frac{(u-a)(u-c)}{u-b}},\ u>a>b>c. \tag{3.9}$$

[117]

$$\int_a^u \sqrt{\frac{(x-a)(x-c)}{x-b}}dx = \frac{2}{3}\sqrt{a-c}[(a+c-2b)E(\mu,q) - (a-b)F(\mu,q)] + \frac{2}{3}(u+b-a-c)\sqrt{\frac{(u-a)(u-c)}{u-b}},\ u>a>b>c. \tag{3.10}$$

[58]

$$\int_0^\infty \frac{u^\alpha du}{(au+1)^\beta(bu+1)^\gamma} = A_0\log a + B_0\log b + \sum_{r=1}^{\beta-1}\frac{A_r}{r} + \sum_{s=1}^{\gamma-1}\frac{B_s}{s}; \tag{3.11}$$

where

$$A_r = (-1)^{\alpha+\beta+r+1}a^{\gamma-\alpha-1}\sum_{j=0}^{\beta-\gamma-1}\binom{\alpha}{j}\binom{\beta+\gamma-r-j-2}{\gamma-1}\frac{b^{\beta-r-j-1}}{(a-b)^{\beta+\gamma-r-j-1}}; \tag{3.12}$$

$$B_s = (-1)^\beta b^{\beta-\alpha-1}\sum_{j=0}^{\gamma-s-1}\binom{\alpha}{j}\binom{\beta+\gamma-s-j-2}{\beta-1}\frac{a^{\gamma-s-j-1}}{(a-b)^{\beta+\gamma-s-j-1}}. \tag{3.13}$$

$$r=0,1,\ldots,\beta-1;\quad s=0,1,\ldots,\gamma-1. \tag{3.14}$$

3.2. **Powers of x, of binomials of the form $\alpha+\beta x^p$, and of polynomials in x.** [64]

$$ {}_3F_2(-n,b/2,(b+1)/2;c/2,(c+1)/2;x) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)}\int_0^1 t^{b-1}(1-t)^{c-b-1}(1-xt^2)^n dt,\quad \Re c>\Re b>0. \tag{3.15}$$

[32]

$$\int_0^\infty \left[\frac{x^2}{x^4+2ax^2+1}\right]^r\cdot\frac{x^2+1}{x^b+1}\frac{dx}{x^2} = \int_0^\infty \left[\frac{x^2}{x^4+2ax^2+1}\right]^r\frac{dx}{x^2} = \int_0^\infty \left[\frac{x^2}{x^4+2ax^2+1}\right]^r dx = \int_0^\infty \left[\frac{x^2}{x^4+2ax^2+1}\right]^r\frac{x^2+1}{x^2}dx = 2^{-1/2-r}(1+a)^{1/2-r}B(r-1/2,1/2), \tag{3.16}$$

with B Euler's beta function, $a >= 1, r > 1/2$, any b.

From this by specialization [32]

$$\int_0^\infty \frac{x^4}{(x^4+x^2+1)^3}dx = \frac{\pi}{48\sqrt{3}}. \tag{3.17}$$

[32]

$$\int_0^\infty \frac{x^3}{(x^4+7x^2+1)^{5/2}}dx = \frac{2}{243}. \tag{3.18}$$

[32]

$$\int_0^\infty \frac{\sqrt{x}}{(x^4+14x^2+1)^{5/4}}dx = \frac{\Gamma^2(3/4)}{4\sqrt{2\pi}}. \tag{3.19}$$

[32]

$$\int_0^\infty \left[\frac{x^2}{bx^4+2ax^2+c}\right]^r dx = \frac{B(r-1/2,1/2)}{2^{r+1/2}\sqrt{b}[a+\sqrt{bc}]^{r-1/2}} \tag{3.20}$$

with $b > 0$, $c \geq 0$, $a > -\sqrt{bc}$ and $r > 1/2$.

[32]

$$\int_0^\infty \left[\frac{x^2}{x^4-x^2+1}\right]^r \frac{dx}{x^2} = \frac{\sqrt{\pi}}{2}\frac{\Gamma(r-1/2)}{\Gamma(r)} \tag{3.21}$$

[10, 143]

$$\int_0^\infty \frac{dx}{(x^4+2ax^2+1)^{m+1}} = \frac{\pi}{2}\frac{\sum_{l=0}^m d_l(m)a^l}{[2(a+1)]^{m+1/2}} \tag{3.22}$$

where

$$d_l(m) \equiv 2^{-2m}\sum_{k=l}^m 2^k \binom{2m-2k}{m-k}\binom{m+k}{m}\binom{k}{l}. \tag{3.23}$$

[143]

$$\int_0^\infty \frac{dx}{bx^4+2ax^2+1} = \frac{\pi}{2\sqrt{2}}\frac{1}{\sqrt{a+\sqrt{b}}}. \tag{3.24}$$

[121]

(3.25)

$$\frac{(-)^n}{4^{n-1}}\int_0^1 \frac{x^{4n}(1-x)^{4n}}{1+x^2}dx = \pi - \sum_{k=0}^{n-1}(-)^k \frac{2^{4-2k}(4k)!(4k+3)!}{(8k+7)!}(820k^3+1533k^2+902k+165).$$

[121]

(3.26)

$$\int_0^1 \frac{x^m(1-x)^n}{1+x^2}dx = \frac{\sqrt{\pi}\Gamma(m+1)\Gamma(n+1)}{2^{m+n+1}}\, {}_3F_2\left(\begin{matrix}1, (m+1)/2, (m+2)/2 \\ (m+n+2)/2, (m+n+3)/2\end{matrix} \mid -1\right).$$

[19]

$$\int_0^1 \frac{x^m(1-x)^n}{1+x^2}dx = R_{m,n} + A_{m,n}\pi + B_{m,n}\ln\sqrt{2} \tag{3.27}$$

induced by the partial fraction decomposition

$$\frac{x^m(1-x)^n}{1+x^2} = Q_{m,n}(x) + \frac{A_{m,n}}{1+x^2} + \frac{B_{m,n}x}{1+x^2}. \tag{3.28}$$

[170][175, A093341]

$$\int_0^\infty \frac{dx}{\sqrt{1+x^4}} = \mathbf{K}(\frac{1}{\sqrt{2}}). \tag{3.29}$$

[170]

$$\int_0^\infty \frac{dx}{\sqrt{x+x^4}} = \mathbf{K}(\frac{\sqrt{6}-\sqrt{2}}{4}). \tag{3.30}$$

[170]

$$\int_0^1 \frac{dx}{\sqrt[4]{1-x^2}} = \sqrt{3}\left[2\mathbf{E}(\frac{1}{\sqrt{2}} - \mathbf{K}(\frac{1}{\sqrt{2}})\right]. \tag{3.31}$$

[170][175, A062539]

$$\int_0^1 \frac{dx}{(1-x^2)^{3/4}} = \sqrt{2}\mathbf{K}(\frac{1}{\sqrt{2}}). \tag{3.32}$$

[170]

$$\int_0^1 \frac{x^2}{(1-x^2)^{3/4}}dx = \frac{2^{3/2}}{3}\mathbf{K}(\frac{1}{\sqrt{2}}). \tag{3.33}$$

[170]

$$\int_0^1 \frac{1}{\sqrt{1-x^6}}dx = \frac{1}{\sqrt[4]{3}}\mathbf{K}(\frac{\sqrt{6}-\sqrt{2}}{4}). \tag{3.34}$$

[170]

$$\int_0^\infty \frac{1}{\sqrt{1+x^6}}dx = \frac{2}{\sqrt[4]{27}}\mathbf{K}(\frac{\sqrt{6}-\sqrt{2}}{4}). \tag{3.35}$$

[170]

$$\int_0^1 \frac{1}{\sqrt{1-x^8}}dx = \frac{1}{\sqrt{2}}\mathbf{K}(\sqrt{2}-1). \tag{3.36}$$

[170]

$$\int_0^1 \frac{x^2}{\sqrt{1-x^8}}dx = (1-\frac{1}{\sqrt{2}})\mathbf{K}(\sqrt{2}-1). \tag{3.37}$$

[170]

$$\int_0^1 \frac{x^{a-1}}{\sqrt{1-x^n}}dx = \cos(a\pi/n)\int_0^\infty \frac{z^{a-1}}{\sqrt{1+z^n}}dz, \quad 2a<n. \tag{3.38}$$

[170]

$$\int_0^\infty \frac{z^{n-a-1}}{\sqrt{1+z^n}}dz \cdot \int_0^1 \frac{x^{a-1}}{\sqrt{1-x^n}}dx = \frac{2\pi}{n(2a-n)\sin(\pi a/n)}, \quad n/2<a<n. \tag{3.39}$$

[170]

$$\int_0^\infty \frac{1}{\sqrt{1+x^8}}dx = \int_0^\infty \frac{x^2}{\sqrt{1+x^8}}dx = \sqrt{2-\sqrt{2}}\mathbf{K}(\sqrt{2}-1). \tag{3.40}$$

[170]

$$\int_0^1 \frac{x^4}{\sqrt{1-x^8}}dx = \frac{\pi}{8}\frac{\sqrt{2}}{\mathbf{K}(\sqrt{2}-1)}. \tag{3.41}$$

[170]

$$\int_0^1 \frac{x^6}{\sqrt{1-x^8}}dx = \frac{\pi}{24}\frac{2+\sqrt{2}}{\mathbf{K}(\sqrt{2}-1)}. \tag{3.42}$$

[170]

$$\int_0^\infty \frac{1}{\sqrt[3]{1+x^6}}dx = \frac{\sqrt[3]{4}}{\sqrt[4]{3}}\mathbf{K}(\frac{\sqrt{6}-\sqrt{2}}{4}). \tag{3.43}$$

[170]

$$\int_0^1 \frac{1}{\sqrt[3]{1-x^6}}dx = \frac{\sqrt[3]{4}}{\sqrt[4]{27}}\mathbf{K}(\frac{\sqrt{6}-\sqrt{2}}{4}). \tag{3.44}$$

[170]

(3.45)
$$\int_1^\infty \frac{1}{\sqrt[m]{(x^n+a)(x^n+b)}}dx = \frac{m}{2n-m}F_1\left(\begin{matrix} \frac{2n-m}{mn}; 1/m, 1/n \\ \frac{2n-m+mn}{mn} \end{matrix} \mid -a, -b\right), \quad 2n-m>0, \quad a,b>0.$$

[170]

(3.46)
$$\int_0^1 \frac{1}{\sqrt[m]{(x^n+a)(x^n+b)}}dx = \frac{1}{\sqrt[m]{ab}}F_1\left(\begin{matrix} 1/n; 1/m, 1/m \\ 1+1/n \end{matrix} \mid -1/a, -1/b\right), \quad a,b>0.$$

3.3. **The Exponential function.** [11]

$$-\int_0^2 \frac{x^p-x^q}{1-x}dx = \psi(p+1)-\psi(q+1) \tag{3.47}$$

[110]

$$\int_u^\infty \exp(-\frac{x^2}{4\beta}-\gamma x)dx = \sqrt{\pi\beta}e^{\beta\gamma^2}\left[1-\Phi(\gamma\sqrt{\beta}+\frac{u}{2\sqrt{\beta}})\right], \Re\beta>0, u\geq 0. \tag{3.48}$$

[110]

$$\int_{-\infty}^\infty \exp(-p^2x^2 \pm qx)dx = \exp(\frac{q^2}{4p^2})\frac{\sqrt{\pi}}{|p|}. \tag{3.49}$$

[110]

(3.50)
$$\int_0^\infty \frac{x^n e^{-\mu x}}{x+\beta}dx = (-1)^{n-1}\beta^n e^{\beta\mu}\operatorname{Ei}(-\beta\mu) + \sum_{k=1}^n (k-1)!(-\beta)^{n-k}\mu^{-k}, \ |\arg\beta|<\pi, \ \Re\mu>0, \ n\geq 0.$$

3.4. **Rational functions of powers and exponentials.** [110]

$$\int_0^\infty \frac{x^{\nu-1}e^{-\mu x}}{1-\beta e^{-x}}dx = \Gamma(\nu)\sum_{n=0}^\infty (\mu+n)^{-\nu}\beta^\mu. \tag{3.51}$$

[110]

$$\int_0^\infty \frac{(1+ix)^{2n-1}-(1-ix)^{2n-1}}{i}\frac{dx}{e^{\pi x}+1} = \frac{1}{2n}[1-2^{2n}B_{2n}]. \tag{3.52}$$

[110]

$$\int_0^\infty \frac{x^q e^{-px}dx}{(1-ae^{-px})^2} = \frac{\Gamma(q+1)}{ap^{q+1}}\sum_{k=1}^\infty \frac{a^k}{k^q}, \ -1\leq a<1, \ q>-1, \ p>0. \tag{3.53}$$

[110]

$$\int_0^\infty \frac{(1+a)e^x+a}{(1+e^x)^2}e^{-ax}x^n dx = -n!\sum_{k=1}^\infty \frac{(-1)^k}{(a+k)^n},\ a>-1,\ n=1,2,\ldots \tag{3.54}$$

[34]

$$p\int_0^\infty x^{\beta-1}e^{ax-x^p}dx = \sum_{k=0}^\infty \frac{a^k}{k!}\Gamma(\frac{k+\beta}{p}). \tag{3.55}$$

$$p^2\int_0^\infty t^{p-1}e^{at-t^p}dt = \sum_{k=1}^\infty \frac{a^k}{(k-1)!}\Gamma(k/p). \tag{3.56}$$

$$\int_0^\infty x^{\beta-1}e^{-x-x^p}dx = \sum_{j=0}^\infty \frac{(-1)^j}{j!}\Gamma(\beta+jp). \tag{3.57}$$

[171][175, A002161]

$$\int_0^\infty (1-e^{-1/x^2})dx = \sqrt{\pi}. \tag{3.58}$$

[110][175, A155739]

$$\int_0^\infty \left\{e^{-x^2}-\frac{1}{1+x^{2^n}}\right\}\frac{dx}{x} = -\frac{1}{2}C \approx -0.288607. \tag{3.59}$$

[192]

$$\int_0^\infty \exp(nx-\beta\operatorname{sh}x)dx = \frac{1}{2}\left[S_n(\beta)-\pi\mathbf{E}_n(\beta)-\pi N_n(\beta)\right]. \tag{3.60}$$

[192]

$$\int_{-\infty}^\infty \frac{\exp(\nu\operatorname{Arsh}x-iax)}{\sqrt{1+x^2}} = \begin{cases} 2\exp\left(-\frac{i\nu\pi}{2}\right)K_\nu(a) & \text{if } a>0, \\ 2\exp\left(\frac{i\nu\pi}{2}\right)K_\nu(-a) & \text{if } a<0. \end{cases} \tag{3.61}$$

$[|\Re\nu|<1]$

3.5. **Hyperbolic Functions.** [6, 42]

$$\frac{1}{2}\int_0^\infty \frac{x}{\cosh x}dx = G. \tag{3.62}$$

[14]

$$\int_0^\infty \frac{du}{(b^2+u^2)\sinh(au)} = \frac{1}{2b}\left[\psi(\frac{ab}{2\pi}+\frac{3}{4})-\psi(\frac{ab}{2\pi}+\frac{1}{4})\right] \tag{3.63}$$

for $\Re a>0$, $\Re b>\max(-\Re a,-\Re 3a)$.

[14]

$$\int_0^\infty \frac{udu}{(b^2+u^2)\sinh(au)} = \frac{1}{2}\left[\psi(\frac{ab}{2\pi}+\frac{1}{2})-\psi(\frac{ab}{2\pi})\right]-\frac{\pi^2}{4a^2b} \tag{3.64}$$

for $\Re a>0$, $\Re b>0$.

[34]

$$\int_0^\infty x^{\beta-1}e^{-x}\frac{\sinh b\sqrt{x}}{b\sqrt{x}}dx = \sum_{j=0}^\infty \frac{\Gamma(j+\beta)}{(2j+1)!}b^{2j}. \tag{3.65}$$

(3.66) $$\int_0^\infty t^{2\beta-2}e^{-t^2/b^2}\sinh t\,dt = \frac{1}{2}\sum_{j=0}^\infty \frac{\Gamma(j+\beta)}{(2j+1)!}b^{2(j+\beta)}.$$

(3.67) $$\int_0^\infty x^{\beta-1}e^{-x}\sinh\sqrt{x}\,dx = \sum_{k=1}^\infty \frac{\Gamma(\beta+k)}{\Gamma(2k)}.$$

[42]

(3.68) $$\int_0^{\pi/2}\sinh^{-1}(\sin x)dx = \int_0^{\pi/2}\sinh^{-1}(\cos x)dx = G.$$

[42]

(3.69) $$\int_0^{\pi/2}\operatorname{csch}^{-1}(\csc x)dx = \int_0^{\pi/2}\operatorname{csch}^{-1}(\sec x)dx = G.$$

3.6. **Rational Functions of Sines and Cosines.** [110]

(3.70) $$\int_0^{\pi/2}\frac{\sin 2nx\cos^{2m+1}x}{\sin x}dx = \frac{\pi}{2},\ n > m \geq 0.$$

[61, B2b]

(3.71) $$\int_0^{2\pi}\sin^m\theta\cos^n\theta d\theta = 2\pi\epsilon_m\epsilon_n\frac{(m-1)!!(n-1)!!}{(m+n)!!}$$

where $\epsilon_j = 1$ if j is even and $\epsilon_j = 0$ otherwise.

[26]

(3.72)
$$\int_0^z \sin^\mu t\sin^\nu(z-t)dt = \frac{\sqrt{\pi}\Gamma(\mu+1)\Gamma(\nu+1)}{2^{(\mu+\nu+1)/2}\Gamma(\mu/2+\nu/2+1)}\sin^{(\mu+\nu+1)/2}zP^{-(\mu+\nu+1)/2}_{(\mu-\nu-1)/2}(\cos z), \Re\mu > -1, \Re\nu > -1.$$

[26]

(3.73)
$$\frac{2^m\Gamma(m+1/2)}{\sqrt{\pi}\Gamma(m+n+1)\Gamma(m-n)}\int_0^z \sin^{m+n}t\sin^{m-n-1}(z-t)dt = \sin^m zP_n^{-m}(\cos z).$$

[26]

(3.74) $$\int_0^z\left(\frac{\sin t}{\sin(z-t)}\right)^\mu dt = \frac{\pi\sin\mu z}{\sin\mu\pi}, \quad -1 < \Re\mu < 1.$$

[124]

(3.75) $$\int_0^{\arcsin q}\left(\cos\phi \pm \sqrt{q^2-\sin^2\phi}\right)^{n+2}T_m(\cos\phi)d\phi = \dots$$

[6]

(3.76) $$\int_0^{\pi/2}\sinh^{-1}(\sin x)dx = G.$$

[6]

(3.77) $$\int_0^{\pi/2}\sinh^{-1}(\cos x)dx = G.$$

[6]

$$\int_0^{\pi/2} \operatorname{csch}^{-1}(\csc x)dx = G. \tag{3.78}$$

[6]

$$\int_0^{\pi/2} \operatorname{csch}^{-1}(\sec x)dx = G. \tag{3.79}$$

[122, p40]

$$\int_0^{\infty} \cos(T_n(t,-x))dt = \frac{\pi\sqrt{x}}{2n\sin\frac{\pi}{2n}}\left[J_{1/n}(2x^{n/2}) - J_{-1/n}(2x^{n/2})\right], \tag{3.80}$$

$$\int_0^{\infty} \cos(T_{2m}(t,x))dt = \frac{\pi\sqrt{x}}{4m\sin\frac{\pi}{4m}}\left[J_{-1/(2m)}(2x^{m}) - J_{1/(2m)}(2x^{m})\right], \quad m = 1,2,3\ldots \tag{3.81}$$

$$\int_0^{\infty} \cos(T_{2m+1}(t,x))dt = \frac{2\sqrt{x}\cos\frac{\pi}{4m+2}}{2m+1}K_{1/(2m+1)}(2x^{m+1/2}), \quad m = 1,2,3\ldots \tag{3.82}$$

where x real positive, where

$$T_n(t,x) \equiv t^n\, {}_2F_1(-\frac{n}{2},\frac{1-n}{2};1-n;-\frac{4x}{t^2}), \quad n = 2,3,4,\ldots \tag{3.83}$$

for example $T_2 = t^2 + 2x$, $T_3 = t^3 + 3tx$.

3.7. Trigonometric and Rational Functions. [6, 42]

$$\frac{1}{2}\int_0^{\pi/2} \frac{x}{\sin x} = G. \tag{3.84}$$

[6]

$$\frac{3}{4}\int_0^{\pi/6} \frac{x}{\sin x} = G - \frac{\pi}{8}\log(1+\sqrt{3}). \tag{3.85}$$

[6, 42]

$$-\frac{\pi^2}{4}\int_0^{1} (x-\frac{1}{2})\sec(\pi x)dx = G. \tag{3.86}$$

[11]

$$\frac{2^{s+2}}{\pi}\int_0^{\pi/2} x\cos^s x\sin(sx)dx - \gamma = \psi(s+1). \tag{3.87}$$

[110]

$$\int_0^{\pi/4} x^m \tan x dx = 2\pi G - \frac{7}{2}\zeta(3),\ m = 2. \tag{3.88}$$

[57]

$$\int_0^{1/2} x^n \cot\pi x dx = \frac{n!}{2^n}\sum_{k=1,k\mathrm{odd}}^{n} \frac{(-)^{(k-1)/2}}{\pi^k}\frac{\eta(k)}{(n-k+1)!} + \frac{(-)^n+1}{2}\frac{4n!(1-2^{-n-1})}{(2\pi)^{n+1}}\zeta(n+1), \tag{3.89}$$

where $\eta(s) \equiv (1-2^{1-s})\zeta(s)$.

[149, p 6]

$$\int_0^\infty \frac{\cos(xy)}{(a^2+x^2)^{\nu+1/2}}dx = \sqrt{\pi}\left(\frac{y}{2a}\right)^\nu \frac{1}{\Gamma(\nu+1/2)}K_\nu(ay). \tag{3.90}$$

[149, p 10]

(3.91)
$$\int_0^\infty \frac{x^\nu}{(a^2+x^2)^{\mu+1}}\cos(xy)dx = \frac{a^{\nu-2\mu-1}}{2}B\left(\frac{1}{2}+\frac{1}{2}\nu, \mu-\frac{1}{2}\nu+\frac{1}{2}\right)\,{}_1F_2(\frac{\nu+1}{2};\frac{\nu+1}{2}-\mu,\frac{1}{2},\frac{a^2y^2}{4})$$

(3.92)
$$+\sqrt{\pi}\frac{2^{-2\mu+\nu-2}}{\Gamma(1+\mu-\frac{1}{2}\nu)}y^{2\mu-\nu+1}\Gamma(\frac{1}{2}\nu-\mu-\frac{1}{2})\,{}_1F_2(\mu+1-\frac{\nu}{2};\mu-\frac{\nu}{2}+\frac{3}{2};\frac{a^2y^2}{4}).$$

[110]

(3.93)
$$\int_0^\infty \frac{x^{2m}\cos(ax)dx}{(\beta^2+x^2)^{n+1/2}} = \frac{(-1)^m\sqrt{\pi}}{2^n\beta^n\Gamma(n+1/2)}\cdot\frac{d^{2m}}{da^{2m}}\{a^nK_n(a\beta)\},\ a>0,\ \Re\beta>0,\ 0\le m\le n.$$

[149, p 116]

$$\int_0^\infty \frac{\sin(xy)}{x(a^2+x^2)^{\nu+1/2}}dx = \frac{\pi y}{2a^{2\nu}}[K_\nu\mathbf{L}_{\nu-1}(ay)+\mathbf{L}_\nu(ay)K_{\nu-1}(ay)] \tag{3.94}$$

where $\mathbf{L}$ are Struve functions.

[6]

$$\frac{1}{2\pi}\int_0^{\pi/2}\frac{x^2}{\sin x} = G-\frac{7}{4\pi}\zeta(3). \tag{3.95}$$

[42]

$$\int_0^{\pi/4}\frac{x^2}{\sin^2 x}dx = G-\frac{1}{16}\pi^2+\frac{1}{4}\pi\log 2. \tag{3.96}$$

[6, 42]

$$\int_0^{\pi/2}\frac{x\csc x}{\cos x+\sin x} = G+\frac{\pi}{4}\log 2. \tag{3.97}$$

[6, 42]

$$-2\int_0^{\pi/2}\frac{x\cos x}{\cos x+\sin x} = G-\frac{\pi^2}{8}-\frac{\pi}{4}\log 2. \tag{3.98}$$

[6, 42]

$$2\int_0^{\pi/2}\frac{x\sin x}{\cos x+\sin x} = G+\frac{\pi^2}{8}-\frac{\pi}{4}\log 2. \tag{3.99}$$

[42]

$$\frac{3}{4}\int_0^{\pi/6}\frac{x}{\sin x}dx = G-\frac{1}{8}\pi\log(2+\sqrt{3}). \tag{3.100}$$

[9]

$$\int_0^\infty x^{-p}\cos^{2n+1}(x+b)dx = \frac{\Gamma(1-p)}{2^{2n}}\sum_{k=0}^n\binom{2n+1}{n-k}\frac{\sin[\pi p/2-(2k+1)b]}{(2k+1)^{1-p}}. \tag{3.101}$$

[9]

$$\int_0^\infty x^{-p}\sin^{2n+1}(x+b)dx = \frac{\Gamma(1-p)}{2^{2n}}\sum_{k=0}^{n}\binom{2n+1}{n-k}\frac{\cos[\pi p/2-(2k+1)b]}{(2k+1)^{1-p}} \quad (3.102)$$

[9]

$$\int_0^\infty x^{-p}\cos^{2n+1}xdx = \frac{\Gamma(1-p)}{2^{2n}}\sin\left(\frac{\pi p}{2}\right)\sum_{k=0}^{n}\frac{\binom{2n+1}{n-k}}{(2k+1)^{1-p}} \quad (3.103)$$

for $0<p<1$.

[9]

$$\int_0^\infty x^{-p}\sin^{2n+1}xdx = \frac{\Gamma(1-p)}{2^{2n}}\cos\left(\frac{\pi p}{2}\right)\sum_{k=0}^{n}(-1)^k\frac{\binom{2n+1}{n-k}}{(2k+1)^{1-p}} \quad (3.104)$$

for $0<p<1$.

[9]

$$\int_0^\infty \cos^{2n+1}x^p dx = \frac{1}{2^{2n}}\Gamma\left(\frac{p+1}{p}\right)\cos\left(\frac{\pi}{2p}\right)\sum_{k=0}^{n}\frac{\binom{2n+1}{n-k}}{(2k+1)^{1/p}} \quad (3.105)$$

for $p>1$.

[9]

$$\int_0^\infty \sin^{2n+1}x^p dx = \frac{1}{2^{2n}}\Gamma\left(\frac{p+1}{p}\right)\sin\left(\frac{\pi}{2p}\right)\sum_{k=0}^{n}(-1)^k\frac{\binom{2n+1}{n-k}}{(2k+1)^{1/p}} \quad (3.106)$$

for $p>1$.

[9]

$$\int_0^{\pi/2} x^p\cos^{2n}xdx = \sum_{j=0}^{\lfloor p/2\rfloor} a_{n,p,p+1-2j}\pi^{p+1-2j}+\delta_{odd,p}\cdot a^*_{n,p}, \quad (3.107)$$

where for $p\ge 2$ and $0\le j\le\lfloor p/2\rfloor$

$$a_{n,p,p+1-2j} = \frac{(-)^j\binom{2n}{n}p!}{2^{2n+p+1}(p+1-2j)!}\sum_{1\le k_1\le k_2\le\cdots\le k_j\le n}\frac{1}{k_1^2k_2^2\cdots k_j^2}. \quad (3.108)$$

and $a^*_{n,p}$ is a similar multinomial sum. A similar form exists for odd powers of the cosine.

[110]

$$\int_0^\infty \sin^{2m+1}x\frac{xdx}{a^2+x^2} = \frac{\pi}{2^{2m+1}}e^{-(2m+1)a}\sum_{k=0}^{m}(-1)^{m+k}\binom{2m+1}{k}e^{2ka}. \quad (3.109)$$

[110]

$$\int_0^\infty \cos^{2m}x\frac{xdx}{a^2+x^2} = \frac{\pi}{2^{2m+1}a}\binom{2m}{m}+\frac{\pi}{2^{2m}a}\sum_{k=1}^{m}\binom{2m}{m+k}e^{-2ka},\ a>0. \quad (3.110)$$

[69]

$$I(a,b)\equiv\int_0^\infty x^{-a}\left(1-\frac{\sin^b x}{x^b}\right)dx.$$

then

$$I(a,b) = \frac{\pi\sec(\pi a/2)}{2^b\Gamma(a+b)} \sum_{k=0}^{\lfloor (b-1)/2 \rfloor} (-1)^{k+1}\binom{b}{k}(b-2k)^{a+b-1}, \tag{3.111}$$

with special cases

$$I(2,b) = \frac{\pi}{2^b(b+1)!} \sum_{k=0}^{\lfloor (b-1)/2 \rfloor} (-1)^{k+1}\binom{b}{k}(b-2k)^{b+1}, \tag{3.112}$$

$$I(a,1) = -\frac{\pi\sec(\pi a/2)}{2\Gamma(1+a)} \qquad \int_0^\infty x^{-3/2}(1-\frac{\sin x}{x})dx = \frac{2\sqrt{2\pi}}{3}, \tag{3.113}$$

$$I(a,2) = -\frac{\pi 2^{a-1}\sec(\pi a/2)}{\Gamma(2+a)}, \quad \int_0^\infty x^{-3/2}(1-\frac{\sin^2 x}{x^2})dx = \frac{16\sqrt{\pi}}{15}, \tag{3.114}$$

(3.115)
$$I(a,3) = \frac{(3-3^{2+a})\pi\sec(\pi a/2)}{8\Gamma(3+a)} \qquad \int_0^\infty x^{-3/2}(1-\frac{\sin^3 x}{x^3})dx = \frac{2}{35}(9\sqrt{3}-1)\sqrt{2\pi}.$$

3.8. **Trigonometric Functions and Exponentials.** [120]

$$\int_0^\infty e^{-tx^2}\cos x^2 dx = \sqrt{\pi/8}\sqrt{\frac{\sqrt{1+t^2}+t}{1+t^2}}. \tag{3.116}$$

[120]

$$\int_0^\infty e^{-tx^2}\sin x^2 dx = \sqrt{\pi/8}\sqrt{\frac{\sqrt{1+t^2}-t}{1+t^2}}. \tag{3.117}$$

4. Definite Integrals of Elementary Functions II

[167] In terms of the constant (0.102) we have

$$\int_0^1 \frac{\log^2 u}{u}\log(1+u)du = \frac{7\pi^4}{360}, \tag{4.1}$$

$$\int_0^1 \frac{\log^2 u}{u}\log(1-u)du = -\frac{\pi^4}{45}, \tag{4.2}$$

$$\int_0^1 \frac{\log^2 u}{u}\log\frac{1+u}{1-u}du = \frac{\pi^4}{24}, \tag{4.3}$$

$$\int_0^1 \frac{\log u}{u}\log^2(1+u)du = A_4 - \frac{\pi^4}{288}, \tag{4.4}$$

$$\int_0^1 \frac{\log u}{u}\log^2(1-u)du = -\frac{\pi^4}{180}, \tag{4.5}$$

$$\int_0^1 \frac{\log u}{u}\log^2\frac{1+u}{1-u}du = 2A_4 - \frac{\pi^4}{60}, \tag{4.6}$$

$$\int_0^1 \frac{1}{u}\log^3(1+u)du = \frac{3}{2}A_4 - \frac{\pi^4}{960}, \tag{4.7}$$

$$\int_0^1 \frac{1}{u}\log^3(1-u)du = -\frac{\pi^4}{15}, \tag{4.8}$$

[167]

$$\int_0^1 \frac{\log u}{u}\log(1+u)du = -\frac{3}{4}\zeta(3), \tag{4.9}$$

$$\int_0^1 \frac{\log u}{u}\log(1-u)du = \zeta(3), \tag{4.10}$$

$$\int_0^1 \frac{\log u}{u}\log\frac{1+u}{1-u}du = -\frac{7}{4}\zeta(3), \tag{4.11}$$

$$\int_0^1 \frac{1}{u}\log^2(1+u)du = \frac{1}{4}\zeta(3), \tag{4.12}$$

$$\int_0^1 \frac{1}{u}\log^2(1-u)du = 2\zeta(3). \tag{4.13}$$

[110]

$$2\int_0^{\pi/2} \ln|1-\sin x|dx = -\pi\ln 2 - 4G. \tag{4.14}$$

[42]

$$-2\int_0^{\pi/4} \log(2\sin x)dx = G. \tag{4.15}$$

A factor 2 is missing in [6].
[42]

$$\frac{1}{4}\int_0^{\pi/2} \log\frac{1+\cos x}{1-\cos x}dx = G. \tag{4.16}$$

[42]

$$\frac{1}{4}\int_0^{\pi/2} \log\frac{1+\sin x}{1-\sin x}dx = G. \tag{4.17}$$

[110]

$$\int_0^{\pi/2} (\ln\tan x)^{2n}dx = (\pi/2)^{2n+1}|E_{2n}|. \tag{4.18}$$

[110]

$$\int_0^{\infty} \frac{\ln x dx}{(x+a)^2} = \frac{\ln a}{a},\ 0<a. \tag{4.19}$$

[6, 42]

$$2\int_0^{\pi/4} \log(2\cos x)dx = G. \tag{4.20}$$

[11, 144][175, A115252]

$$\int_{\pi/4}^{\pi/2} \ln\ln\tan x dx = \int_0^1 \ln\ln\left(\frac{1}{x}\right)\frac{dx}{1+x^2} = \frac{\pi}{2}\ln\left(\frac{\Gamma(3/4)\sqrt{2\pi}}{\Gamma(1/4)}\right) \tag{4.21}$$

[142]

$$\frac{1}{4}\int_0^1 \frac{x^4-6x^2+1}{(1+x^2)^3}\log\log(1/x)dx = -(\gamma+\log 4)[\zeta(-1,1/4)-\zeta(-1,3/4)]+\zeta'(-1,1/4)-\zeta'(-1,3/4). \tag{4.22}$$

[6]

$$-\int_0^1 \frac{\log(x)}{x^2+1}dx = G. \tag{4.23}$$

[56]

$$\int_0^1 \frac{\log t}{1-(1-t)(1-u)}dt = \frac{1}{1-u}\operatorname{Li}_2\left(-\frac{u-1}{u}\right) \tag{4.24}$$

[11]

$$\int_0^b \frac{\ln t}{(1+t)^{n+1}}dt = \frac{1}{n}[1-(1+b)^{-n}]\ln b - \frac{1}{n}\ln(1+b) - \frac{1}{n(1+b)^{n-1}}\sum_{j=1}^{n-1}\frac{1}{j!}\binom{n-1}{j}|S_{j+1}^{(2)}|b^j \tag{4.25}$$

[4]

$$\frac{z^p}{(p-1)!}\int_0^1 \frac{t^{z-1}}{(1-t)^z}\log^{p-1}\frac{1}{t}dt = {}_{p+1}F_p\left(\begin{matrix} z,z,\dots,z \\ z+1,\dots,z+1\end{matrix}\mid 1\right). \tag{4.26}$$

[4]

$$\frac{1}{\Gamma(1-z)\Gamma(p)}\int_0^1 \frac{t^{z-1}}{(1-t)^z}\log^{p-1}(t)dt = \left[\begin{matrix} z \\ p\end{matrix}\right]. \tag{4.27}$$

[11]

$$\int_0^x \frac{\ln t}{(1+t^2)^{n+1}}dt = \frac{\binom{2n}{n}}{2^{2n}}\left[g_0(x)+p_n(x)\ln x - \sum_{k=0}^{n-1}\frac{\tan^{-1}x+p_k(x)}{2k+1}\right] \tag{4.28}$$

where

$$g_0(x) \equiv \ln x\tan^{-1}x - \int_0^x \frac{\tan^{-1}t}{t}dt. \tag{4.29}$$

[144]

$$\begin{aligned}\int_0^\infty \frac{\ln^{n-1}x dx}{(x-1)(x+a)} = \frac{(-)^n(n-1)!}{1+a}\big\{&[1+(-)^n]\zeta(n) \\ &- \sum_{j=0}^{\lfloor n/2\rfloor}\binom{n}{2j}(2^{2j}-2)(-)^j B_{2j}\pi^{2j}\log^{n-2j}a\big\}\end{aligned} \tag{4.30}$$

for $n \geq 2$, $a > 0$.

[177]

$$-2t\int_0^1 \frac{(1-x)^{j+1}\log(1-x)}{(1-tx(1-x))^3}dx = \sum_{n\geq 1}\frac{t^n}{C_n(j)}\sum_{r=1}^n \frac{1}{r+j+n} \tag{4.31}$$

where $C_n(j) = \binom{2n+j}{n}/(n+1)$ are Catalan related numbers. For $t = 2$ the integral becomes a sum of G, $\zeta(2)$, $\pi \ln 2$, π and 1 with rational coefficients, and similar results are given for $t = 1/2$.

[20]

$$\int_0^\infty \frac{\log(1+x)}{1+x+x^2}dx = -\int_0^1 \frac{(1+t)\log t}{1+t^3}dt = -\frac{3}{2}L_{-3}(2), \tag{4.32}$$

where $L_{-3}(s)$ is a Dirichlet series [175, A086724].

[11]

$$\int_0^1 \frac{\ln t}{(1+t^2)^{n+1}}dt = -2^{-2n}\binom{2n}{n}\left(G + \sum_{k=0}^{n-1}\frac{\frac{\pi}{4}+p_k(1)}{2k+1}\right) \tag{4.33}$$

where

$$p_k(1) = \sum_{j=1} k\frac{2^j}{2j\binom{2j}{j}}. \tag{4.34}$$

[6]

$$\int_0^1 \left(\frac{2}{x^2-4x+8} - \frac{3}{x^2+2x+2}\right)\log x dx = C. \tag{4.35}$$

[56]

$$(-1)^{p+1}n\int_0^1 (1-t)^{n-1}\log^p t dt = p!\sum_{k=1}^n \binom{n}{k}\frac{(-1)^k}{k^p}. \tag{4.36}$$

[56][175, A152648]

(4.37)
$$\int_0^1 \frac{\log^2 t}{1-t}dt = \sum_{n=1}^\infty \frac{H_n^{(1)}}{n^2} = 2\zeta(3) = 2\operatorname{Li}_3(t) - 2\operatorname{Li}_2(t)\log t - \log(1-t)\log^2 t + c$$

where $H_n^{(r)} \equiv \sum_{k=1}^n \frac{1}{k^r}$.

[56]

$$\int_0^1 \frac{\log^3 u}{1-u}du = -6\zeta(4). \tag{4.38}$$

[196]

$$\int_0^{\pi/2} \frac{\ln(2\cos x)}{x^2+\ln^2(2\cos x)}dx = \pi/4. \tag{4.39}$$

[196]

$$\int_0^{\pi/2} \ln[x^2+\ln^2(2\cos x)]dx = 0. \tag{4.40}$$

[196]

$$\int_0^{\pi/2} \ln[x^2+\ln^2(2e^{-a}\cos x)]dx = x\ln\frac{a}{e^b-1}, \tag{4.41}$$

and

$$\int_0^{\pi/2} \ln[x^2+\ln^2(2e^{-a}\cos x)]\cos 2x dx = \frac{\pi}{2}\left(1-\frac{1}{a}-e^b+\frac{1}{e^b-1}\right) \tag{4.42}$$

where $b = \min(a, \ln 2)$.

[196]

$$\int_0^{\pi/2} \ln[x^2 + \ln^2(\cos x)]dx = \frac{\pi}{2} \ln\ln 2. \tag{4.43}$$

[196]

$$\int_0^{\pi/2} \ln[x^2 + \ln^2(\cos x)] \cos 2x dx = -\frac{\pi}{\ln 2}. \tag{4.44}$$

[196]

$$\int_0^{\pi/2} \frac{\ln\cos x}{x^2 + \ln^2(\cos x)} dx = \frac{\pi}{2}\left(1 - \frac{1}{\ln 2}\right). \tag{4.45}$$

[196]

$$\int_0^{\pi/2} \frac{x \sin 2x}{x^2 + \ln^2(\cos x)} dx = \frac{\pi}{4\ln^2 2}. \tag{4.46}$$

[196]

$$\int_0^{\pi/2} \frac{x \sin 2x}{x^2 + \ln^2(2\cos x)} dx = \frac{13\pi}{48}. \tag{4.47}$$

[196]

$$\int_{-\pi/2}^{\pi/2} \frac{(1+e^{-2ix})^\beta}{\ln(1+e^{-2ix}) - a} dx = -\frac{\pi}{a} + \pi\frac{e^{(\beta+1)a}}{e^a - 1} H(\ln 2 - a), \tag{4.48}$$

and

$$\int_0^{\pi/2} \frac{x \sin x}{x^2 + \ln^2(2e^{-a}\cos x)} dx = \frac{\pi}{4a^2} + \frac{\pi e^a}{4}\left(1 - \frac{1}{(e^a-1)^2}\right) H(\ln 2 - a), \tag{4.49}$$

where H is the unit step function.

[11, 196]

$$\frac{4}{\pi}\int_0^{\pi/2} \frac{x^2 dx}{x^2 + \ln^2(2\cos x)} = \frac{1}{2}(1 + \ln(2\pi) - \gamma). \tag{4.50}$$

[11]

(4.51)

$$\frac{4}{\pi}\int_0^{\pi/2} \frac{x^2 dx}{x^2 + \ln^2(2e^{-a}\cos x)} = \frac{\gamma}{a} + \frac{a + \ln(1-e^{-a}) - \gamma - \ln a}{1 - e^{-a}} + \frac{a}{1-e^{-a}}\int_0^1 e^{-at} \ln\Gamma(t) dt$$

$$= \frac{\gamma}{a} + \frac{a + \ln(1-e^{-a}) + \Gamma(0,a)}{1 - e^{-a}} + \frac{1}{1-e^{-a}}\int_0^1 e^{-at}\psi(t+1) dt \tag{4.52}$$

where $0 < a < \ln 2$.

[11]

$$\frac{4}{\pi}\int_0^{\pi/2} \frac{x^2 dx}{x^2 + \ln^2(2e^{-a}\cos x)} = \frac{\gamma}{a} + \int_0^\infty e^{-at}\psi(t+1) dt \tag{4.53}$$

where $a > \ln 2$.

[11]

$$\int_0^{\pi/2} \frac{x^2 \ln(2\cos x) dx}{(x^2+\ln^2(2\cos x))^2} = \frac{7\pi}{192} + \frac{\pi \ln 2\pi}{96} - \frac{\zeta'(2)}{16\pi}. \tag{4.54}$$

[196]

$$\frac{1}{2i}\int_{-\pi/2}^{\pi/2} \frac{x(1+\exp(-2ix))^\beta}{\ln(1+\exp(-2ix))} dx = \frac{\pi}{8}[1+\ln 2\pi - \gamma(2\beta+1) - 2\ln\Gamma(\beta+1)] \tag{4.55}$$

with $\Re\beta > -1$.
[196]

$$\int_{-\pi/2}^{\pi/2} \frac{(1+\exp(-2ix))^\beta}{\ln(1+\exp(-2ix))} dx = \frac{\pi}{2}(1+2\beta). \tag{4.56}$$

4.1. **Logarithmic functions of compound arguments and powers.** [6]

$$-\int_0^1 \frac{\log(\frac{1}{\sqrt{2}}(1-x))}{x^2+1} = G. \tag{4.57}$$

[6]

$$-\int_0^1 \frac{\log[\frac{1}{2}(1-x^2)]}{x^2+1} = G. \tag{4.58}$$

[32]

$$\int_0^\infty \left[\frac{1}{x^4-x^2+1}\right]^r \ln\frac{x^2}{x^4-x^2+1}\frac{dx}{x^2} = \frac{\sqrt{\pi}}{2}\frac{\Gamma(r)\Gamma'(r-1/2)-\Gamma(r-1/2)\Gamma'(r)}{\Gamma^2(r)} \tag{4.59}$$

[32]

$$\int_0^\infty \frac{1}{x^4-x^2+1}\ln^2\frac{x^2}{x^4-x^2+1}\frac{dx}{x^2} = \frac{\pi}{2}(\frac{\pi^2}{3}+4\ln^2 2). \tag{4.60}$$

[32]

$$\int_0^\infty \left[\frac{x}{x^2+1}\right]^{2r} \ln\frac{x}{x^2+1}\frac{dx}{x^2} = \frac{\sqrt{\pi}}{4}G(r)[\psi(r-1/2)-\psi(r)-2\ln 2], \tag{4.61}$$

where $G \equiv 2^{1-2r}B(r-1/2,1/2)/\sqrt{\pi}$.
[32]

$$\int_0^\infty \left[\frac{x}{x^2+1}\right]^{2r} \ln\frac{x}{x^2+1}\frac{x^2+1}{x^2(x^b+1)}dx = \frac{\sqrt{\pi}}{4}G'(r) \tag{4.62}$$

where $G \equiv 2^{1-2r}B(r-1/2,1/2)/\sqrt{\pi}$.
[32]

$$\int_0^\infty \left[\frac{x}{x^2+1}\right]^{2r} \ln^2\frac{x}{x^2+1}\frac{dx}{x^2} = \frac{\sqrt{\pi}}{8}G''(r) \tag{4.63}$$

where $G'' \equiv G(r)[\psi'(r-1/2)-\psi'(r)+(\psi(r-1/2)-\psi(r)-2\ln 2)^2]$
[11]

$$\int_0^\infty e^{-ax}\ln x\,dx = -\frac{\gamma+\ln a}{a}. \tag{4.64}$$

[11]

(4.65)
$$\int_0^\infty \frac{\ln x}{e^x+e^{-x}-1}dx = \int_0^1 \ln\ln\left(\frac{1}{x}\right)\frac{dx}{1-x+x^2} = \frac{2\pi}{\sqrt{3}}\left(\frac{5}{6}\ln 2\pi - \ln\Gamma\left(\frac{1}{6}\right)\right).$$

[112]

(4.66)
$$\frac{a^\nu}{\Gamma(\nu)}\int_0^\infty e^{-ax}x^{\nu-1}\ln^m x dx = \phi^m(a,\nu) + \sum_{j=1}^m \binom{m}{j}\eta_j\phi^{m-j}(a,\nu),$$

where

(4.67)
$$\phi(a,\nu) \equiv \psi(\nu) - \ln a,$$

(4.68)
$$\eta_j(\nu) \equiv (-1)^j \sum_{\pi_0(j)} (j;0,k2,\dots,k_j)^*\zeta^{k_2}(2,\nu)\cdots\zeta^{k_j}(j,\nu),$$

(4.69)
$$\zeta(k,\nu) \equiv \sum_{l=0}^\infty \frac{1}{(l+\nu)^k}, \quad k \geq 2.$$

[9]

$$\int_0^\infty \log x\cos^{2n+1}x^2dx = -\frac{\sqrt{\pi}}{2^{2n+3}}(\pi+2\gamma+4\log 2)\sum_{k=0}^n\binom{2n+1}{n-k}\frac{1}{\sqrt{4k+2}} - \frac{\sqrt{\pi}}{2^{2n+2}}\sum_{k=0}^n\binom{2n+1}{n-k}\frac{\log(2k+1)}{\sqrt{4k+2}}.$$

[11]

(4.70)
$$\int_0^{\pi/2} x\ln(2\cos x)dx = -\frac{7}{16}\zeta(3).$$

[11]

(4.71)
$$\int_0^{\pi/2} x^2\ln(2\cos x)dx = -\frac{\pi}{4}\zeta(3).$$

[11]

(4.72)
$$\int_0^{\pi/2} x^2\ln^2(2\cos x)dx = \frac{11\pi}{16}\zeta(4) = \frac{11\pi^5}{1440}.$$

[52]

(4.73)
$$\int_0^{\pi/2} x^4\ln^2(2\cos x)dx = \frac{5\pi^7}{8064} + \frac{3\pi}{4}\zeta^2(3).$$

[52]

(4.74)
$$\int_0^{\pi/2} x^2\ln^4(2\cos x)dx = \frac{33\pi^7}{4480} + \frac{3\pi}{2}\zeta^2(3).$$

[52]

(4.75)
$$\int_0^{\pi/2} x^3 \ln(\cos x)\sin[(p-1)x]\cos^{p-1} x dx = -\frac{\pi}{15}2^{-(p+6)}[60\gamma^4-60\gamma^2\pi^2+\pi^4+60\gamma(\pi^2-2\gamma^2)\ln 2$$
$$+60\psi'''(p)+60\{-(\pi^2+6\gamma(-\gamma+\ln 2))\psi^2(p)+(4\gamma-2\ln 2)\psi^3(p)+\psi^4(p)+6\gamma\ln 2\psi'(p)$$
$$-3[\psi'(p)]^2-2(\gamma+\ln 2)\psi''(p)+(8\gamma-4\ln 2)\zeta(3)+\psi(p)(4\gamma^3-2\gamma\pi^2+(\pi^2-6\gamma^2)\ln 2$$
$$+6\ln 2\psi'(p)-2\psi''(p)+8\zeta(3))\}].$$

[6, 42]

$$\int_0^{\pi/4}\log(\cot x)dx = -\int_0^{\pi/4}\log(\tan x)dx = G. \tag{4.76}$$

[75, 134]

$$\int_{\pi/4}^{\pi/2}\ln\ln\tan x\, dx = \frac{\pi}{2}\ln\frac{\Gamma(3/4)\sqrt{2\pi}}{\Gamma(1/4)}. \tag{4.77}$$

[134, 3]

$$\int_0^1 x^j\log\log\frac{1}{x}dx = -\frac{\gamma+\log(j+1)}{j+1}. \tag{4.78}$$

[3]
(4.79)
$$\int_0^1\frac{x^{p-1}}{1+x^n}\log\log\frac{1}{x}dx = \frac{1}{2n}[\log(2n)+\gamma]\left(\psi(\frac{p}{2n})-\psi(\frac{n+p}{2n})\right)+\frac{1}{2n}\left(\zeta'(1,\frac{p}{2n})-\zeta'(1,\frac{n+p}{2n})\right)$$
for $\Re p>0$ and $\Re n>0$, and a similar expression if $(1+x^n)^2$ or $(1+x^n)^3$ are in the denominator.

[3]
(4.80)
$$\int_0^1\frac{x^{nr-1}}{1+x^n}\log\log\frac{1}{x}dx = \frac{1}{2n}[\log(2n)+\gamma]\left(\psi(r/2)-\psi(\frac{r+1}{2})\right)+\frac{1}{2n}\left(\zeta'(1,r/2)-\zeta'(1,(r+1)/2)\right).$$

[3]
(4.81)
$$\int_0^1 x^{p-1}\frac{1-x}{1-x^n}\log\log\frac{1}{x}dx = \frac{1}{n}[\log(n)+\gamma]\left(\psi(p/n)-\psi(\frac{p+1}{n})\right)+\frac{1}{n}\left(\zeta'(1,p/n)-\zeta'(1,(p+1)/n)\right).$$

[134] Let

$$R_{m,j}(a)\equiv\int_0^1\frac{x^j\log\log 1/x}{(x+a)^{m+1}}dx, \tag{4.82}$$

and (Eulerian numbers)

$$A_{m,j} = \sum_{k=0}^{j}(-)^k\binom{m+1}{k}(j-k)^m \tag{4.83}$$

and

$$E_m\equiv\int_0^1\frac{T_{m-1}(x)\log\log 1/x}{(x+1)^{m+1}}dx \tag{4.84}$$

defined via polynomials

$$T_m(x) \equiv \sum_{j=0}^{m} (-)^j A_{m+1,j+1} x^j, \tag{4.85}$$

then

$$E_m = (1-2^m)\zeta'(1-m) - (-)^m[\gamma(2^m-1) + 2^m \log 2]\frac{B_m}{m} \tag{4.86}$$

where B_m are the Bermoulli numbers. The $R_{m,j}$ are then recursively

$$R_{0,0}(1) = -(\log^2 2)/2; \quad R_{m,0}(1) = \frac{E_m}{b_0(m)} - \sum_{k=1}^{m-1} \frac{b_k(m)}{b_0(m)} R_{m-k,0}(1), \tag{4.87}$$

$$R_{0,0}(a) = -\gamma \log(1+1/a) - \mathrm{Li}_1'(-1/a); \quad \mathrm{Li}_c'(x) = -\sum_{n>=1} \frac{\log n}{n^c} x^n, \tag{4.88}$$

and for $m > 0$

(4.89)

$$R_{m,0}(a) = -\frac{\gamma}{a^m(1+a)m} - \frac{\gamma}{a^{m+1}m!} \sum_{j=2}^{m} \frac{S_1(m,j)T_{j-2}(1/a)}{(1+1/a)^j} - \frac{1}{a^m m!} \sum_{j=1}^{m} S_1(m,j)\,\mathrm{Li}_{1-j}'(-1/a),$$

where

$$b_k(m) \equiv (-)^k \sum_{j=0}^{m-1} \binom{j}{k} A_{m,j+1}, \tag{4.90}$$

and the unsigned Stirling numbers of the first kind are $S_1(m,j)$ as in $(t)_m = \sum_{j=1}^{m} S_j(m,j)t^j$. For larger parameters m then

$$R_{m,0}(a) = \sum_{j=0}^{r} \alpha_{j,r}(a) R_{m-r+j,j}(a), \quad \alpha_{j,r}(a) \equiv (-)^j \binom{r}{j} a^{-r}. \tag{4.91}$$

[134] Let

$$D_{m,j}(r,\theta) =\equiv \int_0^1 \frac{x^j \log\log 1/x}{(x^2 - 2rx\cos\theta + r^2)^{m+1}} dx, \tag{4.92}$$

then

$$D_{0,0}(1,\theta) = \frac{\pi}{2\sin\theta}\left[(1-\theta/\pi)\log 2\pi + \log\frac{\Gamma(1-\theta/2\pi)}{\Gamma(\theta/2\pi)}\right], \tag{4.93}$$

$$D_{0,0}(r,\theta) = -\frac{\gamma}{r\sin\theta}\tan^{-1}\frac{\sin\theta}{r-\cos\theta} + \frac{1}{2ri\sin\theta}(\mathrm{Li}_1'(e^{i\theta}/r) - \mathrm{Li}_1'(e^{-i\theta}/r)), \tag{4.94}$$

(4.95)

$$D_{0,1}(r,\theta) = -\frac{\gamma}{2}\log\frac{r^2 - 2r\cos\theta + 1}{r^2} - \gamma\cot\theta\tan^{-1}\frac{\sin\theta}{r-\cos\theta} + \frac{1}{2ri\sin\theta}[\Phi'(e^{i\theta}/r,1,1) - \Phi'(e^{-i\theta}/r,1,1)],$$

$$D_{m,j}(r\theta) = -\frac{1}{2rm\sin\theta}\frac{\partial}{\partial\theta} D_{m-1,j-1}(r,\theta), \quad m,j > 0. \tag{4.96}$$

[134]

$$\int_0^1 \frac{\log(1-x)}{x}\log\log 1/x dx = \int_0^\infty \log t \log(1-e^{-t})dt = \frac{\gamma\pi^2}{6} - \zeta'(2). \tag{4.97}$$

[134]

$$\int_0^1 \frac{\log(1+x)}{x} \log\log 1/x dx = \frac{\pi^2}{12}(\log 2 - \gamma) + \zeta'(2)/2, \tag{4.98}$$

and other examples involving the kernel $\log\log 1/x$.

[75]

$$\int_0^1 q^n \ln(\sin \pi q) dq = -\frac{\ln 2}{n+1} + n! \sum_{k=1}^{\lfloor n/2 \rfloor} \frac{(-1)^k (\zeta(2k+1)}{(2\pi)^{2k}(n+1-2k)!}. \tag{4.99}$$

[80, 129] by differentiation of [91, 3.761.4] w.r.t. the parameter:

$$\int_0^\infty \frac{\sin x}{x^s} \ln x dx = \frac{\pi}{2} \frac{1}{[\Gamma(s)\sin\frac{\pi s}{2}]^2} \left[\Gamma'(s)\sin\frac{s\pi}{2} + \frac{\pi}{2}\Gamma(s)\cos\frac{\pi s}{2}\right]. \tag{4.100}$$

$$\begin{aligned} \int_0^\infty \frac{\sin x}{x^s} \ln^2 x dx = \pi \frac{1}{[\Gamma(s)\sin\frac{\pi s}{2}]^3} \left\{\Gamma'(s)\sin\frac{s\pi}{2} + \frac{\pi}{2}\Gamma(s)\cos\frac{\pi s}{2}\right\} \\ - \frac{\pi}{2} \frac{1}{[\Gamma(s)\sin\frac{\pi s}{2}]^3} \left\{\Gamma''(s)\sin\frac{s\pi}{2} + \pi\Gamma'(s)\cos\frac{\pi s}{2} - \frac{\pi^2}{4}\Gamma(s)\sin\frac{s\pi}{2}\right\}. \end{aligned} \tag{4.101}$$

[108] Define

$$s_{n,p} \equiv \frac{(-)^{n+p-1}}{(n-1)!p!} \int_0^1 t^{-1} \log^{n-1} t \log^p(1-t) dt \tag{4.102}$$

then

$$s_{n,p} = s_{p,n} = \sum_{k=1}^{p} \frac{(-)^{k+1}}{k!} \sum_{m_i} \frac{H_p(m_1, \ldots m_k)}{m_1 \cdots m_k} \zeta(m_1)\cdots\zeta(m_k), \tag{4.103}$$

where

$$H_p(m_1, \ldots, m_k) = \sum_{p_i} \binom{m_1}{p_1} \cdots \binom{m_k}{p_k}, \tag{4.104}$$

the sum over m_i over all sets of integers which satisfy $m_i \geq 2$, $\sum_{i=1}^k = n + p$, and the sum over p_i over all sets of integers which satisfy $1 \leq p_i \leq m_i - 1$, $\sum_{i=1}^k p_i = p$. Examples with $s_{n,p} = \sum_{k=1}^p (-)^{k+1} \alpha_k(n,p)/k!$ are $\alpha_1(n,p) = (n+p-1)!\zeta(n+p)/(n!p!)$ or $\alpha_2(n,2) = \sum_{\nu=2}^n \zeta(\nu)\zeta(n-\nu+2)$. The reference provides an explicit table for $n, p \leq 4$.

[109]

$$r_{np} \equiv \int_0^{\pi/2} \log^n \cos x \log^p \sin x dx; \tag{4.105}$$

$$r_{10} = -\frac{\pi}{2} \log 2, \tag{4.106}$$

$$r_{11} = \frac{\pi}{2} \left(-\frac{\pi^2}{24} + \log^2 2 \right), \tag{4.107}$$

$$r_{20} = \frac{\pi}{2} \left(\frac{\pi^2}{12} + \log^2 2 \right), \tag{4.108}$$

$$r_{21} = \frac{\pi}{2} \left(-\log^3 2 + \frac{1}{4}\zeta(3) \right), \tag{4.109}$$

$$r_{22} = \frac{\pi}{2} \left(\frac{\pi^4}{160} + \log^4 2 - \zeta(3) \log 2 \right). \tag{4.110}$$

[56]

$$\int_0^1 \frac{\log u}{1-u} \operatorname{Li}_2 \left(\frac{u-1}{u} \right) du = \frac{17}{4}\zeta(4). \tag{4.111}$$

[56]

$$-\int_0^1 \frac{\operatorname{Li}_{q-1}(1-t) \log t}{1-t} dt = \sum_{n=1}^{\infty} \frac{H_n^{(1)}}{n^q}, \tag{4.112}$$

where $H_n^{(r)} \equiv \sum_{k=1}^n \frac{1}{k^r}$.
[56]

$$\int \frac{\operatorname{Li}_2(1-t) \log t}{1-t} dt = \frac{1}{2} \left[\operatorname{Li}_2(1-t) \right]^2 + c. \tag{4.113}$$

[56]

$$\int_0^1 \frac{\operatorname{Li}_{2p}(1-t) \log t}{1-t} dt = \frac{1}{2} \sum_{j=2}^{2p} (-1)^j \zeta(j) \zeta(2p-j+2), \tag{4.114}$$

[42]

$$-\frac{1}{4} \int_0^1 \frac{\log x}{(x+1)\sqrt{x}} dx = \frac{1}{4} \int_1^{\infty} \frac{\log x}{(x+1)\sqrt{x}} dx = G. \tag{4.115}$$

[42]

$$\frac{1}{\sqrt{2}} \int_0^{\pi/2} \log\left(\frac{1+\frac{1}{\sqrt{2}}\sin x}{1-\frac{1}{\sqrt{2}}\sin x}\right) \frac{dx}{1+\cos^2 x} = G. \tag{4.116}$$

[42]

$$\frac{1}{2} \int_0^{\pi/4} \log\left(\frac{1+\sin x}{1-\sin x}\right) \frac{dx}{\cos x \sqrt{\cos 2x}} = G. \tag{4.117}$$

[42]

$$\int_0^{\frac{\sqrt{2}+1}{\sqrt{2}-1}} \frac{(x+1)\log x}{4x\sqrt{6x-x^2-1}} dx = G. \tag{4.118}$$

[42]

$$-\int_0^1 \frac{\log x}{1+x^2}dx = \int_1^\infty \frac{\log x}{1+x^2}dx = G. \tag{4.119}$$

[42]

$$-\int_0^1 \log(\frac{1-x}{\sqrt{2}})\frac{dx}{1+x^2} = G. \tag{4.120}$$

[42]

$$-\int_0^1 \log(\frac{1-x^2}{2})\frac{dx}{1+x^2} = G. \tag{4.121}$$

[42]

$$\int_1^\infty \log(\frac{x+1}{\sqrt{2}})\frac{dx}{1+x^2} = G. \tag{4.122}$$

[42]

$$\int_0^\infty \frac{\log(1+x)}{1+x^2}dx = G + \frac{1}{4}\pi\log 2. \tag{4.123}$$

[42]

$$-\int_0^1 \frac{\log(1+x^2)}{1+x^2}dx = G - \frac{1}{2}\pi\log 2. \tag{4.124}$$

[42]

$$-\int_1^{\sqrt{2}} \frac{2\log x}{x\sqrt{x^2-1}}dx = G - \frac{1}{2}\pi\log 2. \tag{4.125}$$

[42]

$$\int_0^{\pi/2} \log(\cos x + \sin x)dx = G - \frac{1}{4}\pi\log 2. \tag{4.126}$$

[42]

$$-\frac{3}{2}\int_0^{2-\sqrt{3}} \frac{\log x}{1+x^2}dx = G. \tag{4.127}$$

[42]

$$\frac{3}{2}\int_{2+\sqrt{3}}^\infty \frac{\log x}{1+x^2}dx = G. \tag{4.128}$$

4.2. **Inverse Trigonometric Functions.** [6, 42]

$$\frac{2}{\pi}\int_0^1 \frac{\tan^{-1}x}{x} = G - \frac{7}{4\pi}\zeta(3). \tag{4.129}$$

[42]

$$\frac{3}{2}\int_0^{2-\sqrt{3}} \frac{\tan^{-1}x}{x}dx = G - \frac{1}{8}\pi\log(2+\sqrt{3}). \tag{4.130}$$

[42]

$$-\int_0^1 (\tan^{-1} x)^2 dx = G - \frac{1}{16}\pi^2 - \frac{1}{4}\pi \log 2. \tag{4.131}$$

[6, 42]

$$\int_0^1 \frac{\tan^{-1} x}{x} dx = G. \tag{4.132}$$

[42]

$$2\int_0^1 (\frac{1}{4}\pi - \tan^{-1} x)\frac{dx}{1-x^2} = G. \tag{4.133}$$

[42]

$$-\int_0^1 \frac{\sin^{-1} x}{\sqrt{1+x^2}} dx = G - \frac{1}{2}\pi \log(1+\sqrt{2}). \tag{4.134}$$

[88]

$$\int_0^b \tan^{-1} \frac{a}{\sqrt{1+x^2}} \cdot \frac{dx}{\sqrt{1+x^2}} = \frac{1}{2}[g(a,b) + g(b,a)] \tag{4.135}$$

where

$$\begin{aligned} g(a,b) = \tan^{-1} \frac{b}{a} \ln[4(a^2+1) - \frac{4a\sqrt{a^2+1}}{2a^2+1}] - 2\eta \ln(\sqrt{a^2+1} - a) \\ - \mathrm{Cl}_2(2\tan^{-1}\frac{b}{a}) + \frac{1}{2}\mathrm{Cl}_2(4\tan^{-1}\frac{b}{a} - 2\eta) + \frac{1}{2}\mathrm{Cl}_2(2\eta) \end{aligned} \tag{4.136}$$

and

$$\eta = \tan^{-1} \frac{ab}{(\sqrt{a^2+1} - a + 1)a^2 + (\sqrt{a^2+1} - a - 1)b^2}. \tag{4.137}$$

4.3. **Multiple Integrals.** [20] Let

$$C_n = \frac{4}{n!}\int_0^\infty \cdots \int_0^\infty \frac{1}{[\sum_{j=1}^n (u_j + 1/u_j)]^2} \frac{du_1}{u_1} \cdots \frac{du_n}{u_n} \tag{4.138}$$

then

$$C_2 = 1; \quad C_3 = L_{-3}(s); \quad C_4 = 7\zeta(3)/12 \tag{4.139}$$

see (4.32).

[20] Let

(4.140)

$$D_n = \frac{4}{n!}\int_0^\infty \cdots \int_0^\infty \frac{\prod_{i<j}(\frac{u_i - u_j}{u_i + u_j})^2}{[\sum_{j=1}^n (u_j + 1/u_j)]^2} \frac{du_1}{u_1} \cdots \frac{du_n}{u_n} = \frac{1}{n!}\int d^n x \frac{\prod_{i<j} \tanh^2[(x_i - x_j)/2]}{(\cosh x_1 + \cdots + \cosh x_n)^2}$$

then

(4.141)

$$D_1 = 2; \quad D_2 = 1/3; \quad D_3 = 8 + 4\pi^2/3 - 27L_{-3}(2); \quad D_4 = 4\pi^2/9 - 1/6 - 7\zeta(3)/2,$$

see (4.32).

[42]

$$\frac{2}{\pi}\int_0^{\pi/2}\int_0^{\pi/2} \tan^{-1}(\sin x \sin y)\frac{dxdy}{\sin x} = G. \tag{4.142}$$

[42]

$$\frac{1}{2}\int_0^1\int_0^{\pi/2}\frac{d\theta dx}{\sqrt{1-x^2\sin^2\theta}}=G. \tag{4.143}$$

[42]

$$\int_0^1\int_0^{\pi/2}\sqrt{1-x^2\sin^2\theta}d\theta dx=G+\frac{1}{2}. \tag{4.144}$$

[42]

$$8\int_0^1\int_0^1\frac{\tan^{-1}(xy)dxdy}{1+x^2y^2}=2\pi G-\frac{7}{2}\zeta(3). \tag{4.145}$$

[42]

$$4\int_0^1\int_0^1\frac{\tan^{-1}x}{1+x^2y^2}dxdy=2\pi G-\frac{7}{2}\zeta(3). \tag{4.146}$$

[42]

$$-\int_0^1\int_0^1\frac{\log(1-x^2y^2)}{xy\sqrt{(1-x^2)(1-y^2)}}dxdy=2\pi G-\frac{7}{2}\zeta(3). \tag{4.147}$$

[30]

$$\int_0^{\pi/2}\int_0^{\pi/2}\frac{\tan(\phi/2)d\theta d\phi}{\sqrt{1-x\cos^2\theta\cos^2\phi}}=\frac{\pi}{4}\int_0^{\pi/2}\frac{d\phi}{\sqrt{1-(1-x)\sin^2\phi}}+\frac{1}{4}\log x\int_0^{\pi/2}\frac{d\phi}{\sqrt{1-x\sin^2\phi}}. \tag{4.148}$$

[6, 42]

$$\frac{1}{4}\int_0^1\int_0^1\frac{1}{\sqrt{(1-x)(1-y)}}\frac{dxdy}{x+y}=G. \tag{4.149}$$

[96]

$$\int_0^1\cdots\int_0^1\prod_{1\le i<j\le n}|t_i-t_j|^{2z}\prod_{j=1}^n t_j^{x-1}(1-t_j)^{y-1}dt_j=\prod_{j=1}^n\frac{\Gamma(x+(j-1)z)\Gamma(y+(j-1)z)\Gamma(jz+1)}{\Gamma(x+y+(n+j-2)z)\Gamma(z+1)}, \tag{4.150}$$

where n is a positive integer, x, y, z are in $\mathbb{C}$, and $\Re x, \Re y>0$, $\Re z>-\max\{1/n, \Re x/(n-1), \Re y/(n-1)\}$.

[56]

$$\int_0^1\int_0^1\frac{\mathrm{Li}_{q-2}[(1-t)(1-u)]\log t\log u\,dudt}{(1-t)(1-u)}=\sum_{n=1}^\infty\frac{\left[H_n^{(1)}\right]^2}{n^q} \tag{4.151}$$

where $H_n^{(r)}\equiv\sum_{k=1}^n\frac{1}{k^r}$.

[56]

$$\int_0^1\int_0^1\frac{\log t\log u\,dudt}{1-(1-t)(1-u)}=\sum_{n=1}^\infty\frac{\left[H_n^{(1)}\right]^2}{n^2} \tag{4.152}$$

where $H_n^{(r)}\equiv\sum_{k=1}^n\frac{1}{k^r}$.

[56]

$$-\int_0^1\int_0^1 \frac{\log[1-(1-t)(1-u)]\log t\log u\,dudt}{(1-t)(1-u)} = \sum_{n=1}^{\infty}\frac{\left[H_n^{(1)}\right]^2}{n^3} \tag{4.153}$$

where $H_n^{(r)} \equiv \sum_{k=1}^n \frac{1}{k^r}$.
[56]

$$n^2\int_0^1 (1-t)^{n-2}\log t\,dt\int_0^1(1-u)^{n-1}\log u\,du = \left[H_n^{(1)}\right]^2, \tag{4.154}$$

where $H_n^{(r)} \equiv \sum_{k=1}^n \frac{1}{k^r}$.
[9]
(4.155)

$$\int_0^\infty\int_0^\infty \frac{\cos^{2n+1}(x+y)}{x^py^q}dxds = -\Gamma(1-p)\Gamma(1-q)\cos\frac{\pi(p+q)}{2}\sum_{k=0}^n\binom{2n+1}{n-k}\frac{(2k+1)^{p+1-2}}{2^{2n}}$$

[9]

$$\int_0^\infty\int_0^\infty \frac{\cos(x+y)}{x^py^q}dxdy = -\Gamma(1-p)\Gamma(1-q)\cos\frac{\pi(p+q)}{2}. \tag{4.156}$$

[9]

$$\int_0^\infty\int_0^\infty \frac{\log x\log y}{\sqrt{xy}}\cos(x+y)dxdy = (\gamma+2\log 2)\pi^2. \tag{4.157}$$

[9]

$$\int_{R_+^n}(\cos||x||^2)\cdot\prod_{j=1}^n\log x_j dV = \frac{(-)^{\Delta_n}\pi^{n/2}}{2^{2n}}\times\begin{cases}\Re\psi_n & n\,\text{even}\\ \Im\psi_n & n\,\text{odd}\end{cases} \tag{4.158}$$

where $\Delta_n \equiv n(n+1)/2$ and $\psi_n \equiv (\gamma+2\log 2+\pi i/2)^n e^{\pi i n/4}$.
[95]

$$\int_0^1\int_0^1 \frac{x^{u-1}y^{v-1}}{1-xyz}(-\ln xy)^s dxdy = \Gamma(s+1)\frac{\Phi(z,s+1,v)-\Phi(z,s+1,u)}{u-v}, \tag{4.159}$$

[95]

$$\int_0^1\int_0^1 \frac{(xy)^{u-1}}{1-xyz}(-\ln xy)^s dxdy = \Gamma(s+1)\Phi(z,s+2,u), \tag{4.160}$$

where

$$\Phi(z,s,u) = \sum_{k=0}^\infty \frac{z^k}{(u+k)^s} = \frac{1}{\Gamma(s)}\int_0^\infty \frac{e^{-(u-1)t}}{e^t-z}t^{s-1}dt \tag{4.161}$$

is the Lerch transcendent. [95]

$$\int_0^1\int_0^1 \frac{1}{1+xyi}dxdy = G - \frac{\pi^2 i}{48}, \tag{4.162}$$

[95]

$$\int_0^1\int_0^1 \frac{-x\ln xy}{1+x^2y^2}dxdy = G - \frac{\pi^2}{48}, \tag{4.163}$$

[95]

$$\int_0^1\int_0^1 \frac{-x\ln xy}{1-x^2y^2}dxdy = \frac{\pi^2}{12}, \tag{4.164}$$

[95]

$$\int_0^1\int_0^1 \frac{(-\ln xy)^n}{1-xyz}dxdy = \frac{(n+1)!\,\mathrm{Li}_{n+2}(z)}{z}, \tag{4.165}$$

[95]

$$\int_0^1\int_0^1 \frac{-1}{(1-xyz)\ln xy}dxdy = -\frac{\ln(1-z)}{z}, \tag{4.166}$$

[95]

$$\int_0^1\int_0^1 \frac{-1}{(2-xyz)\ln xy}dxdy = \ln 2, \tag{4.167}$$

[95]

$$\int_0^1\int_0^1 \frac{1}{2-xy}dxdy = \frac{\pi^2}{12} - \frac{\ln^2 2}{2}. \tag{4.168}$$

[95]

$$\int_0^1\int_0^1 \frac{-\ln xy}{2-xy}dxdy = \frac{7\zeta(3)}{4} - \frac{\pi^2\ln 2}{6} + \frac{\ln^3 2}{3}. \tag{4.169}$$

[95]

$$\int_0^1\int_0^1 \frac{-1}{(\varphi-xy)\ln xy}dxdy = \ln\varphi, \tag{4.170}$$

[95]

$$\int_0^1\int_0^1 \frac{1}{\varphi-xy}dxdy = \frac{\pi^2}{10} - \ln^2\varphi, \tag{4.171}$$

[95]

$$\int_0^1\int_0^1 \frac{-1}{(\varphi^2-xy)\ln xy}dxdy = \ln\varphi, \tag{4.172}$$

etc where

$$\varphi \equiv (1+\sqrt{5})/2. \tag{4.173}$$

[95]

$$\int_0^1\int_0^1 \frac{1-2xy}{(8+xy)(9-xy)}dxdy = \frac{1}{2}\ln^2\frac{9}{8}. \tag{4.174}$$

[95]

$$\int_0^1\int_0^1 \frac{52-7xy}{(2+xy)(9-xy)}dxdy = \frac{\pi^3}{3} + 3\ln^2 2 + 2\ln^2 3 - 6\ln 2\ln 3. \tag{4.175}$$

etc [95]

$$\int_0^1\int_0^1 \frac{(-\ln xy)^s}{1-xy}dxdy = \Gamma(s+2)\zeta(s+2), \Re s > -1. \tag{4.176}$$

[95]

(4.177)
$$\int_0^1 \int_0^1 \frac{(-\ln xy)^s}{1+xy} dxdy = \Gamma(s+2)\zeta^*(s+2), \Re s > -2, \quad \zeta^*(s) \equiv (1-2^{1-s})\zeta(s).$$

[95]

$$\int_0^1 \int_0^1 \frac{(-\ln xy)^s}{1+x^2y^2} dxdy = \Gamma(s+2)\beta(s+2), \Re s > -2. \tag{4.178}$$

[95]

$$\int_0^1 \int_0^1 \frac{1}{1-xy} dxdy = \zeta(2). \tag{4.179}$$

[95]

$$\int_0^1 \int_0^1 \frac{-\ln xy}{1-xy} dxdy = 2\zeta(3). \tag{4.180}$$

[95]

$$\int_0^1 \int_0^1 \frac{-1}{(1+x^2y^2)\ln xy} dxdy = \pi/4. \tag{4.181}$$

[95, 42]

$$\int_0^1 \int_0^1 \frac{1}{1+x^2y^2} dxdy = G. \tag{4.182}$$

[95]

$$\int_0^1 \int_0^1 \frac{\ln xy}{1+x^2y^2} dxdy = \frac{\pi^3}{16}. \tag{4.183}$$

[95]

$$\int_0^1 \int_0^1 \frac{(-\ln xy)^s}{1+x^2y^2z^2} dxdy = \Gamma(s+1)\frac{\chi_{s+2}(z)}{z}, \quad \Re s > -2, \Re z \neq 0. \tag{4.184}$$

[95]

$$\int_0^1 \int_0^1 \frac{1}{1-x^2y^2\tan^2(\pi/8)} dxdy = \frac{\pi^2}{16\tan\frac{\pi}{8}} - \frac{\ln^2\tan\frac{\pi}{8}}{4\tan\frac{\pi}{8}}. \tag{4.185}$$

[95]

$$\int_0^1 \int_0^1 \frac{x^{u-1}y^{v-1}}{-\ln xy} dxdy = \frac{1}{u-v}\ln\frac{u}{v}. \tag{4.186}$$

[95]

$$\int_0^1 \int_0^1 \frac{(xy)^{u-1}}{-\ln xy} dxdy = \frac{1}{u}. \tag{4.187}$$

[95]

$$\int_0^1 \int_0^1 \frac{-x^{u-1}y^{v-1}}{(1+xy)\ln xy} dxdy = \frac{1}{u-v}\ln\frac{\Gamma(u/2)\Gamma(\frac{u+1}{2})}{\Gamma(v/2)\Gamma(\frac{v+1}{2})}, \quad u>0, v>0. \tag{4.188}$$

[95]

$$\int_0^1\int_0^1 \frac{-(xy)^{u-1}}{(1+xy)\ln xy}dxdy = \frac{1}{2}\left[\psi(\frac{u+1}{2} - \psi(\frac{u}{2})\right]. \tag{4.189}$$

[95][175, A053510]

$$\int_0^1\int_0^1 \frac{1+x}{-(1+xy)\ln xy}dxdy = \ln\pi. \tag{4.190}$$

[95][175, A094640]

$$\int_0^1\int_0^1 \frac{1-x}{-(1+xy)\ln xy}dxdy = \ln\frac{4}{\pi}. \tag{4.191}$$

[95]

$$\int_0^1\int_0^1 \frac{-x}{(1+x^2y^2)\ln xy}dxdy = \ln\frac{\sqrt{2\pi}}{\Gamma^2(3/4)}. \tag{4.192}$$

[95]

$$\int_0^1\int_0^1 \frac{x^{u-1}y^{v-1}}{1-xy}dxdy = \frac{\psi(u)-\psi(v)}{u-v};\quad \int_0^1\int_0^1 \frac{(xy)^{u-1}}{1-xy}dxdy = \psi'(u). \tag{4.193}$$

[95][175, A073010]

$$\int_0^1\int_0^1 \frac{y}{(1-x^3y^3)}dxdy = \frac{\pi}{3\sqrt{3}}. \tag{4.194}$$

[95]

$$\int_0^1\int_0^1 x^{u-1}y^{v-1}(-\ln xy)^s dxdy = \Gamma(s+1)\frac{v^{-s-1}-u^{-s-1}}{u-v} \tag{4.195}$$

and others of similar shape.

[37]

$$W_n(2k) = \sum_{a_1+a_2+\cdots+a_n=k}\left(\begin{array}{c} k \\ a_1,\ldots,a_n\end{array}\right)^2, \tag{4.196}$$

where the sum is over all compositions (unordered partitions) with n terms, and

$$W_n(s) \equiv \int_{[0,1]^n}\left|\sum_{k=1}^n e^{2\pi i x_k}\right|^s d^n x. \tag{4.197}$$

[37]

$$W_3(k) = \Re\, {}_3F_2\left(\begin{array}{c} 1/2,-k/2,-k/2 \\ 1,1\end{array} \mid 4\right). \tag{4.198}$$

[37]

(4.199)

$$W_n(s) = n^s\sum_{m\ge 0}(-1)^m\binom{s/2}{m}\sum_{k=0}^m\frac{(-)^k}{n^{2k}}\binom{m}{k}\sum_{a_1+a_2+\cdots+a_n=k}\left(\begin{array}{c} k \\ a_1,\ldots,a_n\end{array}\right)^2.$$

5. Indefinite Integrals of Special Functions

5.1. **Elliptic Integrals and Functions.**

5.2. **The Exponential Integral.**

5.3. **The Sine Integral and Cosine Integral.**

5.4. **The Error Funcation and Fresnel Integrals.**

5.5. **Cylinder Functions.**

$$\int x^2 Z_{\nu+1}(x)dx = -2x^2 Z_\nu(x) + 4\int xZ_\nu(x)dx + \int x^2 Z_{\nu-1}(x)dx, \tag{5.1}$$

by partial integration of $\int xZ_\nu dx$ with [91, 8.471.2], where Z is a Bessel Function.

$$\int x^{\mu+1} Z_{\nu-1}(x)dx = x^{\mu+1}Z_\nu(x) + (\nu-\mu-1)\int x^\mu Z_\nu(x)dx, \tag{5.2}$$

$\mu \neq -1$, by partial integration of $\int x^\mu Z_\nu dx$ with [91, 8.472.1], where Z is a Bessel Function. Equivalent formula for spherical Bessel functions $j_n(z) \equiv \sqrt{\pi/(2z)}J_{n+1/2}(z)$:

$$(n-m)\int x^m j_n(x)dx = \int x^{m+1} j_{n-1}(x)dx - x^{m+1}j_n(x). \tag{5.3}$$

$$\int x^{\mu+1} Z_{\nu+1}(x)dx = -x^{\mu+1}Z_\nu(x) + (\mu+1+\nu)\int x^\mu Z_\nu(x)dx, \tag{5.4}$$

$\mu \neq -1$, by partial integration of $\int x^\mu Z_\nu dx$ with [91, 8.472.2], where Z is a Bessel Function.

[154]

$$\int \frac{\sin(x)Z_\nu(x)}{x^{3/2}}dx = \frac{2[(2\nu+1)\sin(x) - 2x\cos(x)]}{x^{1/2}(2\nu-1)(2\nu+1)}Z_\nu(x) - \frac{4x^{1/2}\sin(x)}{(2\nu-1)(2\nu+1)}Z_{\nu+1}(x) \tag{5.5}$$

where Z is a Bessel function J or Y.

[154]

$$\int \frac{\cos(x)Z_\nu(x)}{x^{3/2}}dx = \frac{2[(2\nu+1)\cos(x) + 2x\sin(x)]}{x^{1/2}(2\nu-1)(2\nu+1)}Z_\nu(x) - \frac{4x^{1/2}\cos(x)}{(2\nu-1)(2\nu+1)}Z_{\nu+1}(x) \tag{5.6}$$

where Z is a Bessel function J or Y.

[154]

$$\begin{aligned} \int \frac{Z_\mu(x)\bar{Z}_\nu(x)}{x^2}dx = &-\frac{1+\mu+\nu+2\mu\nu+\mu\nu^2+\mu^2\nu-\mu^2-\mu^3-\nu^2-\nu^3+2x^2}{x(-1+\mu-\nu)(-1+\mu+\nu)(1+\mu-\nu)(1+\mu+\nu)}Z_\mu\bar{Z}_\nu \\ &+ \frac{1}{(1-\mu-\nu)(1-\mu+\nu)}Z_\mu\bar{Z}_{\nu+1} + \frac{1}{(1-\mu-\nu)(1+\mu-\nu)}Z_{\mu+1}\bar{Z}_\nu \\ &- \frac{2x}{(1-\mu-\nu)(1-\mu+\nu)(1+\mu-\nu)(1+\mu+\nu)}Z_{\mu+1}\bar{Z}_{\nu+1}. \end{aligned} \tag{5.7}$$

[154]

(5.8)
$$\int \frac{Z_\mu(x)\bar Z_\nu(x)}{x^3}dx = -\frac{(2+\mu+\nu)(4\mu+4\nu+\mu\nu^2+\mu^2\nu-\mu^3-\nu^3+4x^2}{x^2(\mu+\nu)[\nu^2-(2-\mu)^2][\nu^2-(2+\mu)^2]}Z_\mu\bar Z_\nu - \frac{4\mu\nu^2+2\mu^2\nu^2-4\mu^2-4\nu^3-\mu^4+4\nu^2-\nu^4+8x^2}{x(\mu^2-\nu^2)[\nu^2-(2-\mu)^2][\nu^2-(2+\mu)^2]}Z_\mu\bar Z_{\nu+1} + \frac{4\mu^2\nu+2\mu^2\nu^2+4\mu^2-\mu^4-4\nu^2-4\nu^3-\nu^4+8x^2}{x(\mu^2-\nu^2)[\nu^2-(2-\mu)^2][\nu^2-(2+\mu)^2]}Z_{\mu+1}\bar Z_\nu - \frac{4}{[-\nu^2+(2-\mu)^2][-\nu^2+(2+\mu)^2]}Z_{\mu+1}\bar Z_{\nu+1}.$$

[154]

(5.9)
$$\int \frac{Z_\nu^2(x)}{x^2}dx = \frac{1+2\nu+2x^2}{(4\nu^2-1)x}Z_\nu^2(x) - \frac{2}{-1+2\nu}Z_\nu(x)Z_{\nu+1}(x) - \frac{2x}{1-4\nu^2}Z_{\nu+1}^2(x).$$

[154]

$$\int \frac{Z_\nu^2(x)}{x^4}dx = \frac{-9-6\nu+x^2(6+16\nu+8\nu^2)+36\nu^2+24\nu^3+16x^4}{3x^3(1-4\nu^2)(9-4\nu^2)}Z_\nu^2(x) - \frac{2(-3+4\nu+4\nu^2+8x^2)}{3x^2(1-2\nu)(9-4\nu^2)}Z_\nu(x)Z_{\nu+1}(x) - \frac{2(1-4\nu^2-8x^2)}{3x(1-4\nu^2)(9-4\nu^2)}Z_{\nu+1}^2(x). \tag{5.10}$$

[154]

(5.11)
$$\int x^2 Z_{1/3}^3(x)dx = (-\frac{4}{9}x - \frac{16}{81x})Z_{1/3}^3(x) - \frac{4x}{3}Z_{1/3}(x)Z_{4/3}^2(x) + (\frac{8}{9}+x^2)Z_{1/3}^2(x)Z_{4/3}(x) + \frac{2}{3}x^2Z_{4/3}^3(x).$$

[154]

(5.12)
$$\int \frac{Z_1^4(x)}{x}dx = \frac{x^2}{4}Z_2^4(x) + (\frac{3}{4}+\frac{x^2}{4})Z_1^4(x) - \frac{3x}{2}Z_1(x)Z_2^3(x) + 6(\frac{1}{2}+\frac{x^2}{12})Z_1^2(x)Z_2^2(x) + 4(-\frac{3x}{8}-\frac{1}{2x})Z_1^3(x)Z_2(x).$$

[154]

$$\int \frac{Z_3^4(x)}{x^3}dx = (\frac{1}{24}+\frac{1}{2x^2}+\frac{2}{x^4}+\frac{x^2}{378})Z_3^4(x) + (\frac{5}{216}+\frac{2}{27x^2}+\frac{x^2}{378})Z_4^4(x) + 4(-\frac{x}{108}-\frac{5}{54x}-\frac{1}{3x^3})Z_3(x)Z_4^3(x) + 6(\frac{7}{216}+\frac{1}{3x^2}+\frac{4}{3x^4}+\frac{x^2}{1134})Z_3^2(x)Z_4^2(x) + 4(-\frac{x}{108}-\frac{1}{8x}-\frac{1}{x^3}-\frac{4}{x^5})Z_3^3(x)Z_4(x). \tag{5.13}$$

where Z and $\bar Z$ are Bessel functions J or Y.

[154]

(5.14)
$$\int x^l Z_\mu(x)\bar Z_\nu(x)dx = A_{00}(x)Z_\mu(x)\bar Z_\nu(x) + A_{01}(x)Z_\mu(x)\bar Z_{\nu+1}(x) + A_{10}(x)Z_{\mu+1}(x)\bar Z_\nu(x) + A_{11}(x)Z_{\mu+1}(x)\bar Z_{\nu+1}(x),$$

where Z and $\bar{Z}$ are Bessel functions J or Y, where

$$A_{00} = \frac{x}{2(\mu+\nu)}D^3A_{11} + \frac{3+\mu+\nu}{2(\mu+\nu)}D^2A_{11} + \frac{-7-3\mu-3\nu-2\mu\nu+\mu^2+\nu^2-4x^2}{2x(\mu+\nu)}DA_{11} + \frac{(-2-\mu-\nu)(-4-2\mu\nu+\mu^2+\nu^2-2x^2)}{2x^2(\mu+\nu)}A_{11} + \frac{x^{l+1}}{\mu+\nu},$$

$$A_{01} = \frac{-x^2}{2(\mu^2-\nu^2)}D^3A_{11} + \frac{3x}{2(\mu^2-\nu^2)}D^2A_{11} - \frac{7-3\mu^2-\nu^2+4x^2}{2(\mu^2-\nu^2)}DA_{11} + \frac{4+\mu\nu^2-3\mu^2-\mu^3-\nu^2+2x^2}{x(\mu^2-\nu^2)}A_{11} + \frac{x^{l+2}}{\mu^2-\nu^2},$$

$$A_{10} = \frac{x^2}{2(\mu^2-\nu^2)}D^3A_{11} - \frac{3x}{2(\mu^2-\nu^2)}D^2A_{11} + \frac{7-\mu^2-3\nu^2+4x^2}{2(\mu^2-\nu^2)}DA_{11} - \frac{4+\mu^2\nu-\mu^2-3\nu^2-\nu^3+2x^2}{x(\mu^2-\nu^2)}A_{11} - \frac{x^{l+2}}{\mu^2-\nu^2},$$

$$A_{11}(x) = x^{l+3}\sum_{n=0}^{n'} d_n x^{2n},$$

$$d_0 = \frac{2(l+1)}{(l+3)^4-8(l+3)^3+2(12-\mu^2-\nu^2)(l+3)^2-8(l+3)(4-\mu^2-\nu^2)+[(2-\mu)^2-\nu^2][(2+\mu)^2-\nu^2]},$$

$\{(3+2n+l)^4-8(3+2n+l)^3+2(12-\mu^2-\nu^2)(3+2n+l)^2-8(3+2n+l)(4-\mu^2-\nu^2)+[(2-\mu)^2-\nu^2][(2+\mu)^2-\nu^2]\}d_n = -4(1+2n+l)(2n+l)d_{n-1}, n' \geq n > 0,$
and $d_n = 0$ if $n > n'$.

$$n' = \begin{cases} 0, & l=-1 \\ \frac{|l|}{2}-1, & l<-1, even \\ \frac{|l|-3}{2}, & l<-1, odd \\ \infty, & l \geq 0 \end{cases}$$

[154]

(5.15) $$\int x^l Z_\nu^2(x)dx = A_{00}(x)Z_\nu^2(x) + 2A_{01}(x)Z_\nu(x)Z_{\nu+1}(x) + A_{11}(x)Z_{\nu+1}^2(x)$$

where

(5.16) $$A_{00}(x) = \frac{1}{2}D^2A_{11}(x) - \frac{3+2\nu}{2x}DA_{11}(x) + \frac{(2+2\nu+x^2)}{x^2}A_{11}(x),$$

(5.17) $$A_{01}(x) = \frac{1}{2}DA_{11}(x) - \frac{1+\nu}{x}A_{11}(x),$$

(5.18) $$A_{11}(x) = xy(x),$$

(5.19) $$y = \sum_{n=0}^{(l-1)/2} b_n x^{2n+1}, \quad \frac{b_{l-1}}{2} = 1/2l,$$

(5.20) $$\frac{b_{l-1}}{2} = 1/2l, \quad b_n = \frac{2(n+1)[\nu^2-(n+1)^2]}{2n+1}b_{n+1},$$

if $0 \le n \le (l-3)/2$, and if $l \ge 3$ a positive odd integer.

[198, 136,1]

(5.21)
$$\int z^{-\mu-\nu-1}\mathcal{C}_{\mu+1}(z)\mathcal{D}_{\nu+1}(z)dz = -\frac{z^{-\mu-\nu}}{2(\mu+\nu+1)}\left\{\mathcal{C}_\mu(z)\mathcal{D}_\nu(z)+\mathcal{C}_{\mu+1}(z)\mathcal{D}_{\nu+1}(z)\right\}$$

where $\mathcal{C}$ and $\mathcal{D}$ are arbitrary Bessel functions.

[198, 136,2]

$$\int z^{\mu+\nu+1}\mathcal{C}_\mu(z)\mathcal{D}_\nu(z)dz = \frac{z^{\mu+\nu+2}}{2(\mu+\nu+1)}\left\{\mathcal{C}_\mu(z)\mathcal{D}_\nu(z)+\mathcal{C}_{\mu+1}(z)\mathcal{D}_{\nu+1}(z)\right\} \tag{5.22}$$

where $\mathcal{C}$ and $\mathcal{D}$ are arbitrary Bessel functions.

[198]

$$(\rho+\mu+\nu)\int z^{\rho-1}\mathcal{C}_\mu(z)\mathcal{D}_\nu(z)dz + (\rho-\mu-\nu-2)\int z^{\rho-1}\mathcal{C}_{\mu+1}(z)\mathcal{D}_{\nu+1}(z)dz = \\ z^\rho\left\{\mathcal{C}_\mu(z)\mathcal{D}_\nu(z)+\mathcal{C}_{\mu+1}(z)\mathcal{D}_{\nu+1}(z)\right\}$$

where $\mathcal{C}$ and $\mathcal{D}$ are arbitrary Bessel functions.

[198, 136,5]

$$(\mu+\nu)\int \mathcal{C}_\mu(z)\mathcal{D}_\nu(z)\frac{dz}{z} - (\mu+\nu+2n)\int \mathcal{C}_{\mu+n}(z)\mathcal{D}_{\nu+n}(z)\frac{dz}{z} = \\ \mathcal{C}_\mu(z)\mathcal{D}_\nu(z) + 2\sum_{m=1}^{n-1}\mathcal{C}_{\mu+m}(z)\mathcal{D}_{\nu+m}(z) + \mathcal{C}_{\mu+n}(z)\mathcal{D}_{\nu+n}(z)$$

where $\mathcal{C}$ and $\mathcal{D}$ are arbitrary Bessel functions.

[198, 137,1]

(5.23)
$$\int \mathcal{C}_n(z)\mathcal{D}_n(z)\frac{dz}{z} = -\frac{1}{2n}\left[\mathcal{C}_0(z)\mathcal{D}_0(z) + 2\sum_{m=1}^{n-1}\mathcal{C}_m(z)\mathcal{D}_m(z) + \mathcal{C}_n(z)\mathcal{D}_n(z)\right]$$

where $\mathcal{C}$ and $\mathcal{D}$ are arbitrary Bessel functions.

[198, 138]

$$(\mu+2)\int z^{\mu+2}\mathcal{C}_\nu^2(z)dz = (\mu+1)\left\{\nu^2-\frac{1}{4}(\mu+1)^2\right\}\int z^\mu\mathcal{C}_\nu^2(z)dz \\ +\frac{1}{2}\left[z^{\mu+1}\left\{z\mathcal{C}_\nu'(z)-\frac{1}{2}(\mu+1)\mathcal{C}_\nu(z)\right\}^2 + z^{\mu+1}\left\{z^2-\nu^2+\frac{1}{4}(\mu+1)^2\right\}\mathcal{C}_\nu^2(z)\right] \tag{5.24}$$

where $\mathcal{C}$ and $\mathcal{D}$ are arbitrary Bessel functions.

With [1, 10.121], then partial integration for a product of three spherical Bessel functions

$$(n+m+l+2)\int j_n(x)j_m(x)j_l(kx)dx = -(2n-1)\,j_{n-1}(x)j_m(x)j_l(kx) \\ +(n-3-m-l)\int j_{n-2}(x)j_m(x)j_l(kx)dx + (2n-1)\int j_{n-1}(x)j_{m-1}(x)j_l(kx)dx \\ +(2n-1)\,k\int j_{n-1}(x)j_m(x)j_{l-1}(kx)dx. \tag{5.25}$$

$$\int xJ_1(x)dx = \frac{\pi}{2}x[J_1(x)\mathbf{H}_0(x) - J_0(x)\mathbf{H}_1(x)] \tag{5.26}$$

where $\mathbf{H}$ are Struve functions [1, §12].

5.6. **Orthogonal Polynomials.** [154]

(5.27)

$$\int \ln(1\pm x)P_\nu(x)dx = \left[-\frac{x}{\nu}\ln(1\pm x) \pm \frac{1}{\nu(\nu+1)} - \frac{x}{\nu^2}\right]P_\nu + \left[\frac{1}{\nu}\ln(1\pm x) + \frac{1}{\nu^2(\nu+1)}\right]P_{\nu+1}.$$

where P are Legendre functions.

[154]

(5.28)

$$\int P_\mu(x)\bar{P}_\nu(x)dx = -\frac{x}{1+\mu+\nu}P_\mu(x)\bar{P}_\nu(x) - \frac{1+\nu}{(\mu-\nu)(1+\mu+\nu)}P_\mu(x)\bar{P}_{\nu+1}(x) + \frac{1+\mu}{(\mu-\nu)(1+\mu+\nu)}P_{\mu+1}(x)\bar{P}_\nu(x).$$

[154]

(5.29)

$$\int x[P_\nu(x)]^2dx = -\frac{1+\nu}{2\nu}\left[\frac{x^2+\nu}{1+\nu}[P_\nu(x)]^2 - 2xP_\nu(x)P_{\nu+1}(x) + [P_{\nu+1}(x)]^2\right].$$

[154]

(5.30)

$$\int [P_{1/2}(x)]^2dx = \frac{9}{2}x[P_{3/2}(x)]^2 + \frac{x(-7+16x^2)}{2}[P_{1/2}(x)]^2 - 3(-1+2x)(1+2x)P_{1/2}(x)P_{3/2}(x).$$

[154]

(5.31)

$$\int [P_{3/2}(x)]^2dx = \frac{x(93-480x^2+512x^4)}{18}x[P_{3/2}(x)]^2 + \frac{25x(-3+8x^2)}{18}[P_{5/2}(x)]^2 - \frac{5}{9}(3-42x^2+64x^4)P_{3/2}(x)P_{5/2}(x).$$

[154]

$$\int x^5P_{1/3}(x)P_{2/3}(x)dx = \left(-\frac{335}{2352} - \frac{512x^2}{392} + \frac{235x^4}{336} + \frac{x^6}{3}\right)P_{1/3}(x)P_{2/3}(x) + \left(\frac{685x}{784} - \frac{573x^3}{1176} - \frac{5x^5}{48}\right)P_{1/3}(x)P_{5/3}(x) + \left(\frac{235x}{196} - \frac{295x^3}{588} - \frac{2x^5}{21}\right)P_{2/3}(x)P_{4/3}(x) + \left(-\frac{5}{6} + \frac{65x^2}{196} + \frac{25x^4}{588}\right)P_{4/3}(x)P_{5/3}(x). \tag{5.32}$$

[154]

(5.33)

$$\int x[P_{1/3}(x)]^3dx = \left(\frac{125x^4}{12} - \frac{14}{3}x^2 - \frac{5}{12}\right)[P_{1/3}(x)]^3 + (-4+20x^2)P_{1/3}(x)[P_{4/3}(x)]^2 + (9x-25x^3)[P_{1/3}(x)]^2P_{4/3}(x) - \frac{16}{3}x[P_{4/3}(x)]^3.$$

[154]

$$\int x[P_{1/2}(x)]^4 dx = (-\frac{5}{16} - \frac{19}{4}x^2)[P_{1/2}(x)]^4 + \frac{81}{4}xP_{1/2}(x)[P_{3/2}(x)]^3 \tag{5.34}$$
$$+6(-\frac{9}{16}-\frac{9}{2}x^2)[P_{1/2}(x)]^2[P_{3/2}(x)]^2+4(\frac{33x}{16}+3x^3)[P_{1/2}(x)]^3P_{3/2}(x)-\frac{81}{16}[P_{3/2}(x)]^4.$$

[154]

$$\int x^l P_\mu(x)P_\nu(x)dx = A_{00}(x)P_\mu(x)P_\nu(x) + A_{01}(x)P_\mu(x)P_{\nu+1}(x) + A_{10}(x)P_{\mu+1}(x)P_\nu(x) + A_{11}(x)P_{\mu+1}(x)P_{\nu+1}(x), \tag{5.35}$$

[154]

$$\int e^{-x^2}H_\nu(x)dx = -e^{-x^2}H_{\nu-1}(x). \tag{5.36}$$

[154]

$$\int H_\nu(x)x^{-(\nu+3)}dx = \left[\frac{2x^{-\nu}}{(\nu+1)(\nu+2)} - \frac{x^{-(\nu+2)}}{\nu+2}\right]H_\nu(x) - \frac{2\nu}{(\nu+1)(\nu+2)}x^{-(\nu+1)}H_{\nu-1}(x). \tag{5.37}$$

[154]

$$\int xH_\nu(x)dx = \frac{1+2x^2}{2(\nu+2)}H_\nu(x) - \frac{\nu x}{\nu+2}H_{\nu-1}(x). \tag{5.38}$$

[154]

$$\int e^{-x^2}H_\mu(x)\bar{H}_\nu dx = \frac{e^{-x^2}}{2(\mu-\nu)}[-H_\mu(x)\bar{H}_{\nu+1}(x) + H_{\mu+1}(x)\bar{H}_\nu(x)]. \tag{5.39}$$

[154]

$$\int xe^{-x^2}H_\mu(x)\bar{H}_\nu dx = \frac{e^{-x^2}}{2}[-\frac{1+\mu+\nu}{(1-\mu+\nu)(1+\mu-\nu)}H_\mu(x)\bar{H}_{\nu+1}(x)+\frac{x}{1+\mu-\nu}H_{\mu+1}(x)\bar{H}_\nu(x) + \frac{x}{1-\mu+\nu}H_\mu(x)\bar{H}_{\nu+1}(x) - \frac{1}{(1-\mu+\nu)(1+\mu-\nu)}H_{\mu+1}(x)\bar{H}_{\nu+1}(x)]. \tag{5.40}$$

[154]

$$\int x^2e^{-x^2}H_\mu(x)\bar{H}_\nu dx = -e^{-x^2}\frac{H_\mu(x)\bar{H}_\nu(x)x(\mu+\nu)}{(2-\mu+\nu)(2+\mu-\nu)} \tag{5.41}$$
$$+ H_{\mu+1}(x)\bar{H}_\nu(x)\frac{2+\mu+3\nu+2\mu x^2-2\nu x^2-\mu^2x^2-\nu^2x^2+2\mu\nu x^2}{2(\mu-\nu)(2-\mu+\nu)(2+\mu-\nu)}$$
$$+ H_\mu(x)\bar{H}_{\nu+1}(x)\frac{2+3\mu+\nu-2\mu x^2+2\nu x^2-\mu^2x^2-\nu^2x^2+2\mu\nu x^2}{2(\mu-\nu)(2-\mu+\nu)(2+\mu-\nu)}$$
$$- H_{\mu+1}(x)\bar{H}_{\nu+1}(x)\frac{x}{(2-\mu+\nu)(2+\mu-\nu)}.$$

[154]

(5.42)
$$\int e^{-3x^2}x^2H_{2/3}^3(x)dx = e^{-3x^2}[-\frac{1}{12}x(5+6x^2)H_{2/3}^3(x)+\frac{1}{8}(1+6x^2)H_{2/3}^2(x)H_{5/3}(x) - \frac{3}{8}xH_{2/3}(x)H_{5/3}^2(x) + \frac{1}{16}H_{5/3}^3(x)].$$

where H and $\bar{H}$ are Hermite functions.

[154]

$$\int xe^{-(\nu+1)x}L_\nu(x)dx = \frac{e^{-(\nu+1)x}}{\nu+1}[-(1+x)L_\nu(x) + L_{\nu-1}(x)]. \tag{5.43}$$

[154]

(5.44)
$$\int x(1+x)^{-(\nu+3)}L_\nu(x)dx = \frac{(1+x)^{-(\nu+1)}}{\nu+2}\left[\left(\frac{x-\nu}{\nu+1} - \frac{x}{1+x}\right)L_\nu(x) + \frac{\nu}{\nu+1}L_{\nu-1}(x)\right].$$

[154]

(5.45)
$$\int e^{-x}L_\mu(x)\bar{L}_\nu(x)dx = e^{-x}[L_\mu(x)\bar{L}_\nu(x)+\frac{1+\nu}{\mu-\nu}L_\mu(x)\bar{L}_{\nu+1}(x)-\frac{1+\mu}{\mu-\nu}L_{\mu+1}(x)\bar{L}_\nu(x)].$$

[154]

(5.46)
$$\int xe^{-x}L_\mu(x)\bar{L}_\nu(x)dx = e^{-x}[-\frac{1+\mu+\nu-x+2\mu\nu+\mu^2x+\nu^2x-2\mu\nu x}{(1-\mu+\nu)(1+\mu-\nu)}L_\mu(x)\bar{L}_\nu(x) + \frac{(1+\nu)(1+2\mu-\mu x+\nu x)}{(\mu-\nu)(1-\mu+\nu)}L_\mu(x)\bar{L}_{\nu+1}(x) - \frac{(1+\mu)(1+2\nu+\mu x-\nu x)}{(\mu-\nu)(1+\mu-\nu)}L_{\mu+1}(x)\bar{L}_\nu(x) - \frac{2(1+\mu)(1+\nu)}{(1-\mu+\nu)(1+\mu-\nu)}L_{\mu+1}(x)\bar{L}_{\nu+1}(x)].$$

[154]

(5.47)
$$\int e^{-3x}xL_{2/3}^3(x)dx = e^{-3x}\Big\{L_{2/3}^3(x)[\frac{125}{24}-\frac{625x}{24}+\frac{853x^2}{16}-\frac{675x^3}{16}+\frac{225x^4}{16}-\frac{27x^5}{16}] + L_{5/3}^3(x)[-\frac{125}{24}+\frac{125x}{12}-\frac{125x^2}{16}]+3[\frac{125}{24}-\frac{125x}{8}+\frac{275x^2}{16}-\frac{75x^3}{16}]L_{2/3}(x)L_{5/3}^2(x) + 3[-\frac{125}{24}+\frac{125x}{6}-\frac{515x^2}{16}+\frac{135x^3}{8}-\frac{45x^4}{16}]L_{2/3}^2(x)L_{5/3}(x)\Big\}.$$

where L and $\bar{L}$ are Laguerre functions.

6. Definite Integrals of Special Functions

6.1. **Elliptic Integrals and Functions.** [6]

$$\frac{1}{2}\int_0^1 K(x^2)dx = G. \tag{6.1}$$

[6]

$$\int_0^1 E(x^2)dx = G + \frac{1}{2}. \tag{6.2}$$

6.2. **The Exponential Integral and Related Functions.** [34]

$$\int_0^\infty t^{2\beta} e^{-3t^2} \operatorname{erf}(t)dt = \frac{\Gamma(1+\beta)}{\sqrt{\pi}3^{\beta+1}}\, {}_2F_1(\frac{1}{2}, 1+\beta; \frac{3}{2}; -\frac{1}{3}). \tag{6.3}$$

6.3. **The Gamma Function and Related Functions.** [11][175, A075700]

$$\int_0^1 \ln\Gamma(t)dt = \frac{1}{2}\ln 2\pi. \tag{6.4}$$

[11]

$$\int_0^1 t\ln\Gamma(t)dt = \frac{\zeta'(2)}{2\pi^2} + \frac{1}{6}\ln 2\pi - \frac{\gamma}{12}. \tag{6.5}$$

[11]

$$\int_0^\infty 2^{-t}\ln\Gamma(t)dt = 2\int_0^1 2^{-t}\ln\Gamma(t)dt - \frac{\gamma+\ln\ln 2}{\ln 2}. \tag{6.6}$$

[11]

$$\int_0^\infty 2^{-t}t\ln\Gamma(t)dt = 2\int_0^1 2^{-t}(t+1)\ln\Gamma(t)dt - \frac{(\gamma+\ln\ln 2)(1+2\ln 2)-1}{\ln^2 2}. \tag{6.7}$$

[75]

$$\int_0^1 \ln\Gamma(q)\cos((2n+1)\pi q)dq \frac{2}{\pi^2}\left(\frac{\gamma+2\ln\sqrt{2\pi}}{(2n+1)^2} + 2\sum_{k=1}^\infty \frac{\ln k}{4k^2-(2n+1)^2}\right). \tag{6.8}$$

[75]

$$\int_0^1 B_{2m}(q)\ln\Gamma(q)dq = (-)^{m+1}\frac{(2m)!\zeta(2m+1)}{2(2\pi)^{2m}}. \tag{6.9}$$

[75]

$$\int_0^1 B_{2m-1}(q)\ln\Gamma(q)dq = \frac{B_{2m}}{2m}\left[\frac{\zeta'(2m)}{\zeta(2m)} - A\right], \tag{6.10}$$

where $A = 2\ln\sqrt{2\pi}+\gamma$.
[75]

$$\begin{aligned}\int_0^1 q^n\ln\Gamma(q)dq = \frac{1}{n+1}\sum_{k=1}^{\lfloor (n+1)/2\rfloor}(-)^k\binom{n+1}{2k-1}\frac{(2k!)}{k(2\pi)^{2k}}[A\zeta(2k)-\zeta'(2k)] \\ -\frac{1}{n+1}\sum_{k=1}^{\lfloor n/2\rfloor}(-)^k\binom{n+1}{2k}\frac{(2k!)}{2(2\pi)^{2k}}\zeta(2k+1) + \frac{\ln\sqrt{2\pi}}{n+1},\end{aligned} \tag{6.11}$$

where $A = 2\ln\sqrt{2\pi}+\gamma$.
[75]

$$\int_0^1 (q-1/2)\ln\Gamma(q)dq = \frac{1}{12}\left(\frac{6\zeta'(2)}{\pi^2} - 2\ln\sqrt{2\pi} - \gamma\right). \tag{6.12}$$

[75]

$$\int_0^{1/2} \ln\Gamma(q+1)dq = \frac{\gamma}{8} + \frac{3\ln\sqrt{2\pi}}{4} - \frac{13\ln 2}{24} - \frac{3\zeta'(2)}{4\pi^2} - \frac{1}{2}. \tag{6.13}$$

[75]

(6.14)

$$\int_0^1 q^n \psi^{(m)}(q)dq = (-1)^m \frac{n!}{(n-m)!}\Big[\frac{\gamma}{n-m+1} + (n-m)!\sum_{k=0}^{m-2}\frac{\Gamma(m-k)\zeta(m-k)}{(n-k)!} + \sum_{k=0}^{n-m-1}(-)^k\binom{n-m}{k}[H_k\zeta(-k)+\zeta'(-k)]\Big],$$

where H_k are harmonic sums.

[75]

$$\int_0^1 q^n\psi(q)dq = \zeta'(0) + \sum_{k=1}^{n-1}(-)^k\binom{n}{k}[H_k\zeta(-k)+\zeta'(-k)], \tag{6.15}$$

where H_k are harmonic sums.

[16]

$$\int_0^\infty \frac{t^{\alpha+z}\psi(\alpha+z+1)}{\Gamma(\alpha+z+1)}dz = \frac{t^\alpha}{\Gamma(\alpha+1)} + \nu(t,\alpha)\ln t, \quad \Re\alpha > -1. \tag{6.16}$$

$$\int_0^\infty \frac{\psi(\alpha+z+1)}{\Gamma(\alpha+z+1)}dz = \frac{1}{\Gamma(\alpha+1)}, \quad \Re\alpha > -1. \tag{6.17}$$

$$\int_1^\infty \frac{\psi(z)}{\Gamma(z)}dz = 1. \tag{6.18}$$

$$\int_0^\infty \frac{t^z\psi(z+1)}{\Gamma(z+1)}dz = 1 + \nu(t)\ln t, t \neq 1. \tag{6.19}$$

[16]

(6.20)

$$\int_0^\infty \frac{t^{\alpha+z}}{\Gamma(\alpha+z+1)}\{\psi(\alpha+z+1)^2 - \psi^{(1)}(\alpha+z+1)\}dz = \frac{2t^\alpha\ln t}{\Gamma(\alpha+1)} + (\ln t)^2\nu(t,\alpha) + L^{-1}\left\{\frac{\ln s}{s^{\alpha+1}}\right\}, \quad \Re\alpha > -1$$

where L^{-1} is the inverse Laplace transform.

$$\int_0^\infty \frac{t^z}{\Gamma(z+1)}\{\psi(z+1)^2 - \psi^{(1)}(z+1)\}dz = -\gamma + \ln t(1+\nu(t)\ln t). \tag{6.21}$$

$$\int_1^\infty \frac{1}{\Gamma(z)}\{\psi^{(1)}(z) - \psi(z)^2\}dz = \gamma. \tag{6.22}$$

(6.23)

$$\int_0^\infty \frac{t^{\alpha+z}}{\Gamma(\alpha+z+1)}\{\psi(\alpha+z+1)^2 - \psi^{(1)}(\alpha+z+1)\}dz = \frac{t^\alpha}{\Gamma(\alpha+1)}[\psi(\alpha+1)+\ln t] + (\ln t)^2\nu(t,\alpha).$$

(6.24)

$$\int_0^\infty \frac{t^{\alpha+z}z^\beta}{\Gamma(\alpha+z+1)}\frac{\psi(\alpha+z+1)}{\Gamma(\beta+1)}dz = \mu(t,\beta-1,\alpha) + \ln t\,\mu(t,\beta,\alpha), \quad \Re\alpha > -1, \Re\beta > -1.$$

$$\int_0^\infty \frac{t^{\alpha+z}z^\beta}{\Gamma(\alpha+z+1)}\frac{\psi(\alpha+z+1)^2-\psi^{(1)}(\alpha+z+1)}{\Gamma(\beta+1)}dz$$
$$=\mu(t,\beta-2,\alpha)+2\ln t\mu(t,\beta-1,\alpha)+(\ln t)^2\mu(t,\beta,\alpha),\quad \Re\alpha>-1,\Re\beta>1. \tag{6.25}$$

[16] Let $L\{f(t)\}\equiv\int_0^\infty e^{-st}f(t)dt=F(s)$ be the Laplace transform and

$$\nu(z)\equiv\int_0^\infty \frac{z^t dt}{\Gamma(t+1)}, \tag{6.26}$$

$$\nu(z,\alpha)\equiv\int_0^\infty \frac{z^{\alpha+t} dt}{\Gamma(\alpha+t+1)}, \tag{6.27}$$

$$\mu(z,\beta)\equiv\int_0^\infty \frac{z^t t^\beta dt}{\Gamma(\beta+1)\Gamma(t+1)}, \tag{6.28}$$

$$\mu(z,\beta,\alpha)\equiv\int_0^\infty \frac{z^{\alpha+t} t^\beta dt}{\Gamma(\beta+1)\Gamma(\alpha+t+1)}, \tag{6.29}$$

then

$$L\{\nu(t)\}=\frac{1}{s\ln s}. \tag{6.30}$$

[16]

$$L\{\nu(t,\alpha)\}=\frac{1}{s^{1+\alpha}\ln s}. \tag{6.31}$$

[16]

$$L\{\mu(t,\beta)\}=\frac{1}{s(\ln s)^{\beta+1}}. \tag{6.32}$$

[16]

$$L\{\mu(t,\beta,\alpha)\}=\frac{1}{s^{\alpha+1}(\ln s)^{\beta+1}}. \tag{6.33}$$

Above $\Re\alpha>-1$, $\Re\beta>-1$, $\Re s>1$. [16]

$$\frac{F(\ln s)}{s\ln s}=L\{\int_0^\infty \nu(t,x)f(x)dx\}. \tag{6.34}$$

[16]

$$\frac{F(\ln s)}{s^{\alpha+1}\ln s}=L\{\int_0^\infty \nu(t,\alpha+x)f(x)dx\}. \tag{6.35}$$

[16]

$$\frac{F(\ln s)}{s(\ln s)^{\beta+1}}=L\{\int_0^\infty \mu(t,\beta,x)f(x)dx\}. \tag{6.36}$$

[16]

$$\frac{F(\ln s)}{s^{\alpha+1}(\ln s)^{\beta+1}}=L\{\int_0^\infty \mu(t,\beta,\alpha+x)f(x)dx\}. \tag{6.37}$$

[16]

$$\int_0^\infty x^{\beta-1}\nu(t,\alpha+x)dx=\Gamma(\beta)\mu(t,\beta,\alpha),\quad \Re\alpha>-1,\Re\beta>0. \tag{6.38}$$

[16]

$$\int_0^\infty x^{\beta-1}\nu(t,x)dx = \Gamma(\beta)\mu(t,\beta). \tag{6.39}$$

[16]

$$\int_0^\infty \nu(t,\alpha+x)dx = \mu(t,1,\alpha). \tag{6.40}$$

[16]

$$\int_0^\infty \nu(t,x)dx = \mu(t,1). \tag{6.41}$$

[16]

(6.42)

$$\int_0^\infty t^{\lambda-1}\mu(t,\beta,\alpha+x)dx = \Gamma(\lambda)\mu(t,\beta+\lambda,\alpha), \quad \Re\alpha > -1, \Re\beta > -1, \Re\lambda > 0.$$

[16]

$$\frac{1}{3\pi}\int_0^\infty (\frac{x}{t})^{1/2}K_{1/2}(\phi)\mu(x,\beta,\alpha)dx = 3^{\beta+1}\mu(t,\beta,\alpha/3), \phi \equiv 2(x^3/27t)^{1/2}, \tag{6.43}$$

and others,

6.4. **Cylinder Functions.** [147]

(6.44)

$$2\pi(-1)^{(n-m)/2}\int_0^\infty dkJ_{n+1}(2\pi k)J_m(2\pi kr) = R_n^m(r) = \sum_{s=0}^{(n-m)/2}(-1)^s\binom{n-s}{s}\binom{n-2s}{(n-m)/2-s}r^{n-2s},$$

for $n \geq 0$, $0 \leq m \leq n$, $n-m$ even.

[27]

$$\int_0^\infty J_\mu(ct\sin\phi)J_\nu(ct\sin\Phi)K_\rho(ct\cos\phi\cos\Phi)dt = ... \tag{6.45}$$

[27]

(6.46)

$$\int_0^\infty J_\mu(ct\sin\phi\sin\Phi)J_\nu(ct\cos\phi\sin\Phi)J_\rho(ct)dt = \frac{\Gamma(\frac{1+\mu+\nu+\rho}{2})\sin^\mu\phi\cos^\mu\Phi\cos^\nu\phi\sin^\nu\Phi}{c^{\mu+\nu+1}\Gamma(\mu+1)\Gamma(\nu+1)\Gamma(\frac{1-\mu-\nu+\rho}{2})}$$

(6.47)

$$\times\,{}_2F_1(\frac{1+\mu+\nu-\rho}{2},\frac{1+\mu+\nu+\rho}{2};\mu+1;\sin^2\phi)\,{}_2F_1(\frac{1+\mu+\nu-\rho}{2},\frac{1+\mu+\nu+\rho}{2};\nu+1;\sin^2\Phi)$$

where ϕ and Φ anre positive angles whose sum is acute.

6.4.1. *Cylinder Functions combined with x and x^2.* [203]

$$\int_0^1 xJ_\nu(\lambda_n x)J_\nu(\lambda_m x)dx = \frac{1}{2}\left\{(1-\nu^2/\lambda_n^2)J_\nu^2(\lambda_n) + J_\nu'^2(\lambda_n)\right\}\delta_{nm} \tag{6.48}$$

for $\{\lambda_n\}$ $(n = 1, 2, \cdots)$ a sequence of succesive positive roots of the equation $xJ_\nu'(x) + HJ_\nu(x) = 0$, where H is a real number and $\nu \geq -1$.

[125]

$$\int_0^1 x^{2+l+2n} j_l(2\pi\sigma x)dx = \frac{1}{2\pi\sigma}\sum_{k=0}^{n}\frac{(-n)_k}{(\pi\sigma)^k} j_{l+k+1}(2\pi\sigma); \quad l,n=0,1,2,3,\ldots \tag{6.49}$$

[152]

$$\int_0^\infty k^2 j_l(kr) j_l(kr')dk = \frac{\pi\delta(r-r')}{2r^2}. \tag{6.50}$$

[93, 136]

$$\begin{aligned}\frac{4p_1p_2p_3}{\pi}\int_0^\infty x^2 j_1(p_1x)j_2(p_2x)j_3(p_3x)dx &= \Delta(p_1,p_2,p_3)(-1)^{(l_1+l_2+l_3)/2}\\ &\times \frac{1}{2}\sum_{k_1=0}^{l_1}\sum_{k_2=0}^{l_2}\sum_{k_3=0}^{l_3}\frac{(-1)^{k_1+k_2+k_3}}{(k_1+k_2+k_3)!}\prod_{i=1}^{3}\frac{(l_i+k_i)!}{k_i!(l_i-k_i)!}(2p_i)^{-k_i}\\ &\times [(-1)^{l_1+k_1}(p_2+p_3-p_1)^{k_1+k_2+k_3}+(-1)^{l_2+k_2}(p_3+p_1-p_2)^{k_1+k_2+k_3}\\ &\quad +(-1)^{l_3+k_3}(p_1+p_2-p_3)^{k_1+k_2+k_3}-(p_1+p_2+p_3)^{k_1+k_2+k_3}]\end{aligned} \tag{6.51}$$

supposed that p_1, p_2 and p_3 can be the sides of a plane triangle, that is where $\Delta(.)=1$ if they form a non-denerate triangle, $\Delta(.)=1/2$ if they form a degenerate triangle, and $\Delta(.)=0$ otherwise.

$$\int_0^\infty x j_0(ax)j_0(bx)j_1(cx)dx = \frac{\pi}{8abc^2}(c^2-(a-b)^2) \tag{6.52}$$

for $c>0$, $a>0$, $|c-a|<b<c+a$.

[102]

$$\int_0^\infty J_{n_1}(k_1\rho)J_{n_2}(k_2\rho)J_{n_3}(k_3\rho)\rho d\rho = \frac{\Delta}{6\pi A}[\cos(n_1\alpha_2-n_2\alpha_1)+\cos(n_2\alpha_3-n_3\alpha_2)+\cos(n_3\alpha_1-n_1\alpha_3)] \tag{6.53}$$

if $n_1+n_2+n_3=0$, Δ as above, the area of the triangle of k_1, k_2 and k_3 given by $2A=k_1k_3\sin\theta_{13}$, and α three external angles in that triangle.

6.4.2. *Cylinder Functions and Rational Functions.* [82]

$$\int_0^\infty \frac{J_\nu(x)}{x^2+a^2}dx = \frac{i}{a}[S_{0,\nu}(ia - e^{-i\nu\pi/2}K_\nu(a)] = \frac{1}{a}[is_{0,\nu}(ia)+\frac{\pi}{2}\sec\frac{\nu\pi}{2}I_\nu(a)]. \tag{6.54}$$

[180]

$$\begin{aligned}\int_0^\infty x^{1-2n}J_\nu(ax)J_\nu(bx)\frac{dx}{x^2+c^2} &= (-1)^n c^{-2n}\Big\{I_\nu(bc)K_\nu(ac)\\ &-\frac{1}{2}\left(\frac{b}{a}\right)^\nu\frac{\pi}{\sin\pi\nu}\sum_{p=0}^{n-1}\frac{(ac/2)^{2p}}{p!\Gamma(1-\nu+p)}\sum_{k=0}^{n-1-p}\frac{(bc/2)^{2k}}{k!\Gamma(1+\nu+k)}\Big\}\end{aligned} \tag{6.55}$$

for $0<b<a$, $\Re c>0$, $\Re\nu>n-1$, $n=1,2,\ldots$. For $0<a<b$, the arguments a and b should be interchanged.

6.4.3. *Cylinder Functions and Powers.* [110, 114]

(6.56)

$$\int_0^1 x^\mu J_\nu(ax)dx = 2^\mu \frac{\Gamma(\frac{1}{2}+\frac{1}{2}\mu+\frac{1}{2}\nu)}{a^{\mu+1}\Gamma(\frac{\nu}{2}+\frac{1}{2}-\frac{1}{2}\mu)} + a^{-\mu}\left\{(\mu+\nu-1)J_\nu(a)S_{\mu-1,\nu-1}(a) - J_{\nu-1}(a)S_{\mu,\nu}(a)\right\}$$

$$[a>0, \Re(\mu+\nu)>-1].$$

$$(6.57) \quad \int_0^u z^\nu K_\nu(x)dx = \frac{2^\nu-1}{1+2\nu}[(1+2\nu)\Gamma(\nu)u\,{}_1F_2(1/2;3/2,1-\nu;u^2/4) + 2(u/2)^{1+2\nu}\Gamma(-\nu)\,{}_1F_2(\nu+1/2;1+\nu,3/2+\nu;u^2/4)].$$

[147]

$$(6.58) \quad \int_0^\infty x^{-P}[1-J_0(bx)]dx = \frac{\pi b^{P-1}}{2^P\Gamma^2(\frac{P+1}{2})\sin[\pi(P-1)/2]}.$$

[72, p 22]

(6.59)

$$\int_0^1 x^\mu J_\nu(xy)\sqrt{xy}dx = y^{-\mu-1}\big[(\nu+\mu-\frac{1}{2})yJ_\nu(y)S_{\mu-1/2,\nu-1}(y) - yJ_{\nu-1}(y)S_{\mu+1/2,\nu}(y) + 2^{\mu+1/2}\frac{\Gamma(\frac{\mu}{2}+\frac{\nu}{2}+\frac{3}{4})}{\Gamma(\frac{\nu}{2}-\frac{\mu}{2}+\frac{1}{4})}\big]$$

for $\Re(\mu+\nu)>-3/2$.

[82]

$$(6.60) \quad \int_0^\infty \frac{x^{\rho-1}J_\nu(ax)}{(x^2+k^2)^{\mu+1}}dx = \frac{a^\nu k^{\rho+\nu-2\mu-2}\Gamma(\rho/2+\nu/2)\Gamma(\mu+1-\rho/2-\nu/2)}{2^{\nu+1}\Gamma(\mu+1)\Gamma(\nu+1)} \times {}_1F_2\left(\frac{\rho+\nu}{2};\frac{\rho+\nu}{2}-\mu,\nu+1;\frac{a^2k^2}{4}\right) + \frac{a^{2\mu+2-\rho}\Gamma(\nu/2+\rho/2-\mu-1)}{2^{2\mu+3-\rho}\Gamma(\mu+2+\nu/2-\rho/2)} \times {}_1F_2\left(\mu+1;\mu+2+\frac{\nu-\rho}{2},\mu+2-\frac{\nu+\rho}{2};\frac{a^2k^2}{4}\right),\ a>0, -\Re\nu<\Re\rho<2\Re\mu+\frac{7}{2}, \Re k>0.$$

[204]

$$\int_0^\infty \frac{J_{n+1}(k)J_{n'+1}(k)}{k(k^2+k_0^2)^{1+\gamma 2}}dk = k_0^{n+n'-\gamma}\frac{\Gamma(\frac{\gamma-n-n'}{2})\Gamma(\frac{n+n'}{2}+1)}{2^{n+n'+3}\Gamma(n+2)\Gamma(n'+2)\Gamma(1+\gamma/2)} \times {}_3F_4\left(\begin{matrix}\frac{n+n'}{2}+1, \frac{n+n'}{2}+2, \frac{n+n'+3}{2} \\ n+2, n'+2, n+n'+3, 1+\frac{n+n'-\gamma}{2}\end{matrix} \mid k_0^2\right)$$

$$+\frac{\Gamma(\frac{n+n'-\gamma}{2})\Gamma(3+\gamma)}{2^{3+\gamma}\Gamma(3+\frac{\gamma+n+n'}{2})\Gamma(2+\frac{\gamma+n-n'}{2})\Gamma(2+\frac{\gamma+n'-n}{2})}\,{}_3F_4\left(\begin{matrix}2+\frac{\gamma}{2}, 1+\frac{\gamma}{2}, \frac{3+\gamma}{2} \\ 2+\frac{\gamma+n-n'}{2}, 2+\frac{\gamma+n'-n}{2}, 3+\frac{\gamma+n+n'}{2}, 1+\frac{\gamma-n-n'}{2}\end{matrix} \mid k_0^2\right)$$

[122, p50]

$$\int_0^\infty J_{\nu+n}(at)J_{\nu-n-1}(bt)dt = \begin{cases} \frac{b^{\nu-n-1}\Gamma(\nu)}{a^{\nu-n}n!\Gamma(\nu-n)}\,{}_2F_1(\nu,-n;\nu-n;\frac{b^2}{a^2}), & 0<b<a, \\ (-1)^n/(2a), & 0<b=a, \\ 0, & 0<a<b. \end{cases} \tag{6.61}$$

where $n = 0, 1, 2, \ldots$, $\Re\nu > 0$.

[122, p50]

$$\int_0^\infty J_\nu(at)J_{\nu+1}(bt)dt = \begin{cases} a^\nu b^{-\nu-1}, & 0<b<a \\ 1/(2a), & 0<b=a \\ 0, & 0<a<b \end{cases} \tag{6.62}$$

where a, b real positive, $\Re\nu > -1$.

[122, p50]

$$\int_0^\infty J_\mu(at)J_\nu(at)dt = \frac{2}{\pi}\frac{\sin(\frac{\nu-\mu}{2}\pi)}{\nu^2-\mu^2}, \tag{6.63}$$

$\Re(\nu+\mu) > 0$, $a > 0$.

[147]

$$\int_0^\infty x^{-P}\left(1-\frac{4J_1^2(x)}{x^2}\right)dx = \frac{\pi\Gamma(P+2)}{2^P\Gamma^2(\frac{P+3}{2})\Gamma(\frac{P+5}{2})\Gamma(\frac{P+1}{2})\sin[\pi(P-1)/2]}. \tag{6.64}$$

[79]

$$\int_0^\infty t^{\rho-\mu-\nu-3}J_\mu(at)J_\nu(bt)J_\rho(ct)dt = \frac{2^{\rho-\mu-\nu-3}a^\mu b^\nu\Gamma(\rho-1)}{c^{\rho-2}\Gamma(\mu+1)\Gamma(\nu+1)}\left(1-\frac{\rho-1}{\mu-1}\frac{a^2}{c^2}-\frac{\rho-1}{\nu-1}\frac{b^2}{c^2}\right). \tag{6.65}$$

[190] Let

$$G_{lmn} \equiv \int_0^\infty dx x^\alpha J_l(x)J_m(x)J_n(\beta x) \tag{6.66}$$

with $\Re\alpha < 3/2$ and $\Re(\alpha+l+m+n+1) > 0$, then for $\beta/2 < 1$

$$\begin{aligned} G_{lmn} &= \frac{\Gamma(\frac{\alpha+n}{2})(\beta/2)^{-\alpha}}{2\Gamma(\frac{-l+m+1}{2})\Gamma(\frac{l-m+1}{2})\Gamma(\frac{-\alpha+n+2}{2})}\,{}_4F_3\left(\begin{array}{c} \frac{l+m+1}{2},\frac{l-m+1}{2},\frac{-l-m+1}{2},\frac{-l+m+1}{2} \\ \frac{-\alpha+n+2}{2},\frac{1}{2},\frac{-\alpha-n+2}{2} \end{array} \mid \beta^2/4\right) \\ &- \frac{\Gamma(\frac{l+m+2}{2})\Gamma(\frac{\alpha+n-1}{2})(\beta/2)^{-\alpha+1}}{\Gamma(\frac{-l+m}{2})\Gamma(\frac{l+m}{2})\Gamma(\frac{l-m}{2})\Gamma(\frac{-\alpha+n+3}{2})}\,{}_4F_3\left(\begin{array}{c} \frac{l+m+2}{2},\frac{l-m+2}{2},\frac{-l-m+2}{2},\frac{-l+m+2}{2} \\ \frac{-\alpha+n+3}{2},\frac{3}{2},\frac{-\alpha-n+3}{2} \end{array} \mid \beta^2/4\right) \\ &+ \frac{2^{\alpha+n}\Gamma(-\alpha-n)\Gamma(\frac{\alpha+l+m+n+1}{2})(\beta/2)^n}{\Gamma(\frac{-\alpha-l+m-n+1}{2})\Gamma(\frac{-\alpha+l+m-n+1}{2})\Gamma(\frac{-\alpha+l-m-n+1}{2})\Gamma(n+1)} \\ &\times\,{}_4F_3\left(\begin{array}{c} \frac{\alpha+l+m+n+1}{2},\frac{\alpha+l-m+n+1}{2},\frac{\alpha-l-m+n+1}{2},\frac{\alpha-l+m+n+1}{2} \\ n+1,\frac{n+\alpha+2}{2},\frac{\alpha+n+1}{2} \end{array} \mid \beta^2/4\right), \end{aligned} \tag{6.67}$$

and for $\beta/2 > 1$

$$G_{lmn}(\alpha,\beta) = \frac{2^{-l-m-1}\Gamma(\frac{\alpha+l+m+n+1}{2})(\beta/2)^{-\alpha-l-m-1}}{\Gamma(\frac{-\alpha-l-m+n+1}{2})\Gamma(l+1)\Gamma(m+1)}\,{}_4F_3\left(\begin{array}{c} \frac{\alpha+l+m+n+1}{2},\frac{l+m+1}{2},\frac{l+m+2}{2},\frac{\alpha+l+m-n+1}{2} \\ m+1,l+1,m+l+1 \end{array} \mid 4/\beta^2\right) \tag{6.68}$$

[187]

(6.69)
$$\int_0^\infty t\left(\frac{t}{\sqrt{u^2+t^2}}-1\right)J_0(\gamma t)dt = \frac{u^2}{2}[I_1(u\gamma/2)K_1(u\gamma/2)-I_0(u\gamma/2)K_0(u\gamma/2)].$$

6.4.4. *Cylinder Functions and Exponentials.*

6.4.5. *Cylinder Functions, Exponentials and Powers.* With

$$l_1(a,b,c) \equiv \frac{1}{2}\left[\sqrt{(a+b)^2+c^2}-\sqrt{(a-b)^2+c^2}\right]$$

$$l_2(a,b,c) \equiv \frac{1}{2}\left[\sqrt{(a+b)^2+c^2}+\sqrt{(a-b)^2+c^2}\right]$$

at $a>0$, $b>0$, $c>0$ [79]

(6.70)
$$\int_0^\infty e^{-cx}J_1(ax)J_{1/2}(bx)\frac{dx}{x^{3/2}} = \frac{\sqrt{2}}{\sqrt{\pi b}a}\left[\frac{l_1}{2}\sqrt{a^2-l_1^2}+\frac{a^2}{2}\sin^{-1}(\frac{l_1}{a})+c(\sqrt{b^2-l_1^2}-b)\right].$$

(6.71)
$$\int_0^\infty e^{-cx}J_1(ax)J_{1/2}(bx)\frac{dx}{\sqrt{x}} = \frac{\sqrt{2}}{\sqrt{\pi b}a}(b-\sqrt{b^2-l_1^2}).$$

(6.72)
$$\int_0^\infty e^{-cx}J_1(ax)J_{1/2}(bx)\sqrt{x}dx = \frac{\sqrt{2}}{\sqrt{\pi b}a}\frac{l_1\sqrt{a^2-l_1^2}}{l_2^2-l_1^2}.$$

(6.73)
$$\int_0^\infty e^{-cx}J_1(ax)J_{3/2}(bx)\sqrt{x}dx = \frac{2l_1^2\sqrt{b^2-l_1^2}}{\sqrt{2\pi}b^{3/2}a(l_2^2-l_1^2)}.$$

(6.74)
$$\int_0^\infty e^{-cx}J_1(ax)J_{3/2}(bx)\frac{dx}{\sqrt{x}} = \frac{1}{\sqrt{2\pi}b^{3/2}a}\left(-l_1\sqrt{a^2-l_1^2}+a^2\sin^{-1}(l_1/a)\right).$$

(6.75)
$$\int_0^\infty e^{-cx}J_1(ax)J_{5/2}(bx)\frac{dx}{\sqrt{x}} = \frac{2^{-1/2}c}{\sqrt{\pi}b^{5/2}a}\left(l_1\sqrt{a^2-l_1^2}+\frac{2a^2l_1}{\sqrt{a^2-l_1^2}}-3a^2\sin^{-1}(l_1/a)\right).$$

and similar combinations of even and odd-indexed $J(ax)$ and $J(bx)$.

[7]

(6.76)
$$\int_0^\infty x^\nu e^{-x/2}J_\nu(\mu x)L_n^{2\nu}(x)dx = 2^\nu\Gamma(\nu+\frac{1}{2})\frac{1}{\sqrt{\pi\mu}}(\sin\theta)^{\nu+\frac{1}{2}}C_n^{\nu+\frac{1}{2}}(\cos\theta),$$

$\mu\ge 0$, $\nu>-\frac{1}{2}$, $\cos\theta\equiv\frac{\mu^2-1/4}{\mu^2+1/4}$, $C_n^\lambda(x)$ ultraspherical polynomial.

[150]

(6.77)
$$\int_0^\infty \frac{x^{\nu+1}}{a^2+x^2}J_\nu(xy)dx = \frac{a^\nu}{\sqrt{y}}K_\nu(ay), \quad 1<\Re\nu<3/2.$$

[150]

(6.78)
$$\int_0^\infty \frac{x^{\nu+1}}{(a^2+x^2)^\mu}J_\nu(xy)dx = \frac{2^{1-\mu}a^{\nu-\mu+1}y^{\mu-1}}{\Gamma(\mu)}K_{\nu-\mu+1}(ay), \quad \Re\nu>-1, \Re(2\mu-\nu)>1/2.$$

[150]

$$\int_0^\infty \frac{x^{1-\nu}}{(a^2+x^2)^\mu} J_\nu(xy)dx = a^{-\mu-\nu+1}y^{\mu-1}\big[2^{-\mu}\frac{\Gamma(1-mu)}{1-\nu} I_{\nu+\mu-1}(ay) \\ - 2\frac{1-\mu}{\Gamma(\nu)} e^{-i\frac{\pi}{2}(\nu-\mu+1)} s_{-\mu+\nu,-\mu-\nu+1}(iay)\big], \quad \Re(\nu+2\mu) > 1/2. \tag{6.79}$$

[34]

$$\int_0^\infty x^{\beta-1}e^{-x}I_n(x)dx = \sum_{j=0}^\infty \frac{\Gamma(\beta+2j+n)}{2^{2j+n}j!(j+n)!}. \tag{6.80}$$

[34]

$$\int_0^\infty x^{2n+1}e^{-x^2/4a}I_0(x)dx = 2^{2n+1}a^{n+1}n!e^aL_n(-a) = 2^{2n+1}a^{n+1}n!\,{}_1F_1(n+1;1;a). \tag{6.81}$$

$$\int_0^\infty x^3e^{-x^2/4a}I_0(x)dx = 8a^2(1+a)e^a. \tag{6.82}$$

[2]

$$\int_0^\infty xe^{-x^2/z}J_2(x)Y_2(x)dx = -\frac{2}{\pi} + \frac{4}{\pi z} - \frac{zK_2(z/2)}{2\pi\exp(z/2)}. \tag{6.83}$$

[2]

$$\int_0^\infty xe^{-x^2/z}I_3(x)K_3(x)dx = -\frac{32+16z+3z^2}{2z^2} + \frac{\exp(z/2)zK_3(z/2)}{4}. \tag{6.84}$$

[2]

$$\int_0^\infty x^3e^{-x^2/z}J_2(x)Y_2(x)dx = -\frac{4}{\pi} + \frac{z^2(2+z)K_0(z/2)}{4\pi\exp(z/2)} + \frac{z(8+4z+z^2)K_1(z/2)}{4\pi\exp(z/2)}. \tag{6.85}$$

[2]

$$\int_0^\infty x^5e^{-x^2/z}I_3(x)K_3(x)dx = -32 + \frac{1}{8}e^{z/2}z^2(32-16z+5z^2-z^3)K_0(z/2) \\ + \frac{1}{8}e^{z/2}z(128-64z+24z^2-6z^3+z^4)K_1(z/2). \tag{6.86}$$

[68] Let

$$I(\mu,\nu,\lambda) \equiv \int_0^\infty J_\mu(at)J_\nu(bt)e^{-ct}t^\lambda dt, \tag{6.87}$$

then

$$I(n,n;0) = \frac{(-)^nk}{\pi\sqrt{ab}} \int_0^{\pi/2} \frac{\cos(2n\psi)d\psi}{\sqrt{1-k^2\sin^2\psi}}, \tag{6.88}$$

$$I(n,n;1) = \frac{(-)^nck^3}{4\pi(ab)^{3/2}} \int_0^{\pi/2} \frac{\cos(2n\psi)d\psi}{(1-k^2\sin^2\psi)^{3/2}}, \tag{6.89}$$

in particular

$$I(0,0;0) = \frac{k}{2\sqrt{ab}} F_0(k), \tag{6.90}$$

$$I(1,1;0) = \frac{1}{k\sqrt{ab}}[(1-k^2/2)F_0(k) - E_0(k)], \tag{6.91}$$

$$I(0,0;1) = \frac{ck^3 E_0(k)}{8k'^2(ab)^{3/2}}, \tag{6.92}$$

and more results on $I(n+1,n;\pm 1)$ and $I(n+1,n;0)$ in terms of Elliptic Integrals. Associated recurrences:

$$a[I(\mu+1,\nu;\lambda) + I(\mu-1,\nu;\lambda)] = 2\mu I(\mu,\nu;\lambda-1); \tag{6.93}$$

$$b[I(\mu,\nu+1;\lambda) + I(\mu,\nu-1;\lambda)] = 2\nu I(\mu,\nu;\lambda-1); \tag{6.94}$$

$$aI(\mu+1,\nu;\lambda) - bI(\mu,\nu-1;\lambda) = C_{\mu,\nu} + (\mu-\nu+\lambda)I(\mu,\nu;\lambda-1) - cI(\mu,\nu;\lambda), \tag{6.95}$$

with

$$C_{\mu,\nu} \equiv \begin{cases} \frac{a^\mu b^\nu}{2^{\mu+\nu}\Gamma(\mu+1)\Gamma(\nu+1)}, & \text{if } \lambda+\mu+\nu = 0; \\ 0 & \text{if } \lambda+\mu+\nu > 0. \end{cases} \tag{6.96}$$

6.4.6. *Cylinder and Trigonometric Functions and Powers.* [79]

(6.97)
$$\int_0^\infty \sin(cx) x^{\nu-\mu-4} J_\mu(ax) J_\nu(bx)\, dx = \frac{\Gamma(\nu) a^\mu b^{-\nu} c}{2^{\mu-\nu+3}\Gamma(\mu+1)} \left(\frac{b^2}{\nu-1} - \frac{a^2}{\mu+1} - \frac{2c^2}{3} \right).$$

[79]

(6.98)
$$\int_0^\infty \cos(cx) x^{\nu-\mu-3} J_\mu(ax) J_\nu(bx)\, dx = \frac{\Gamma(\nu) a^\mu b^{-\nu}}{2^{\mu-\nu+3}\Gamma(\mu+1)} \left(\frac{b^2}{\nu-1} - \frac{a^2}{\mu+1} - 2c^2 \right).$$

[165]

$$\int_0^\infty J_{2n}(a\sqrt{t^2+2bt}) e^{-pt}\, dt = (-1)^n \Bigg\{ \frac{e^{b(p-\sqrt{p^2+a^2})}}{\sqrt{p^2+a^2}} + \frac{2e^{bp}}{a} \tag{6.99}$$
$$\times \sum_{k=1}^n (-)^k (2k-1) I_{k-1/2}[b(\sqrt{p^2+a^2}-a)/2)] K_{k-1/2}[b(\sqrt{p^2+a^2}+a)/2] \Bigg\}.$$

[165]

(6.100)
$$\int_0^\infty J_{2n}(a\sqrt{t^2+2bt}) e^{-pt}\, dt = (-1)^n \Bigg\{ \frac{e^{b(p-\sqrt{p^2+a^2})}}{\sqrt{p^2+a^2}} + 2n \sum_{\lambda=1}^n \frac{(n-1+\lambda)!}{\lambda!(n-\lambda)!} \frac{1}{b^{2\lambda-1}a^{2\lambda}}$$
$$\times \sum_{\mu=0}^{\lambda-1} \frac{(2\lambda-2-\mu)!}{\mu!(\lambda-1-\mu)!} (2b\sqrt{p^2+a^2})^\mu \left(e^{b(p-\sqrt{p^2+a^2})} - \sum_{\nu=0}^{2\lambda-2-\mu} \frac{[b(p-\sqrt{p^2+a^2})]^\nu}{\nu!} \right) \Bigg\}.$$

[34]

$$\int_0^\infty x^{\beta-1} e^{-x} I_0(2\sqrt{ax})\, dx = \sum_{k=0}^\infty \frac{a^k}{k!^2} \Gamma(\beta+k). \tag{6.101}$$

7. Definite Integrals of Special Functions II.

7.1. **Associated Legendre Functions and Powers.** [159]

$$\int_{-1}^{1} P_{l+\alpha}^{-\alpha}(x) P_{k+\beta}^{-\beta}(x)(1-x^2)^{-p-1}dx = 2^{-(\alpha+\beta)}$$
$$\times \frac{\Gamma[\frac{1}{2}(k+1)]\Gamma(-\frac{1}{2}k-\beta)\Gamma[\frac{1}{2}(\alpha+\beta)-p]\Gamma[\frac{1}{2}(\alpha-\beta-p)]\Gamma[\frac{1}{2}(l-k)]\Gamma[\frac{1}{2}(l+k+1)+\beta]}{\Gamma(\beta+1)\Gamma(-\beta)\Gamma(\alpha+1)\Gamma(-\frac{1}{2}k)\Gamma[\frac{1}{2}(k+1)+\beta]\Gamma[\frac{1}{2}(l-k+\alpha-\beta)-p]\Gamma[\frac{1}{2}(l+k+\alpha+\beta+1)-p]}$$
$$\times\ {}_4F_3\left(\begin{matrix} \frac{1}{2}(\alpha-\beta)+p+1, & \frac{1}{2}(\alpha-\beta)-p, & -\frac{1}{2}(l-1), & -\frac{1}{2}l \\ \frac{1}{2}(k-l)+1, & -\frac{1}{2}(l+k-1)-\beta, & \alpha+1 & \end{matrix} ;1\right)$$

for $\Re[\frac{1}{2}(\alpha+\beta)-p] > 0$, for k, l both even or both odd.

[159]

$$\int_{-1}^{1} P_l^m(x) P_k^n(x)(1-x^2)^{-p-1}dx = (-1)^{(l-k+m-n)/2}2^{-(m+n)} \tag{7.1}$$
$$\times \frac{(l+m)!\Gamma[\frac{1}{2}(m+n)-p]\Gamma[\frac{1}{2}(m-n)-p]\Gamma[\frac{1}{2}(l+k+1-m+n)]}{(l-m)!m![\frac{1}{2}(k-l+m-n)]!\Gamma[\frac{1}{2}(l-k)-p]\Gamma[\frac{1}{2}(l+k+1)-p]}$$
$$\times\ {}_4F_3\left(\begin{matrix} \frac{1}{2}(m-n)+p+1, & \frac{1}{2}(m-n)-p, & -\frac{1}{2}(l-m-1), & -\frac{1}{2}(l-m) \\ \frac{1}{2}(k-l+m-n)+1, & -\frac{1}{2}(l+k-1-m+n), & m+1 & \end{matrix} ;1\right)$$

for $\Re[\frac{1}{2}(m+n)-p] > 0$, for $k-n \ge l-m$.

[159]

$$\int_{-1}^{1} P_l^m(x) P_k^m(x)(1-x^2)^{-1}dx = \frac{1}{m}\frac{(l+m)!}{(l-m)!}, \quad k \ge l \ge m > 0. \tag{7.2}$$

[159]

$$\int_{-1}^{1} P_l^m(x) P_k^m(x)(1-x^2)^{-2}dx = \frac{(l+m)!}{2(l-m)!(m-1)m(m+1)}$$
$$\times [l(l+1)+(m-1)(m+1)+\frac{1}{2}(k-l)(m+1)(k+l+1)], \quad k \ge l \ge m > 1.$$

[159]

$$\int_{-1}^{1} P_l^m(x) P_k^m(x) = \frac{2}{2l+1}\frac{(l+m)!}{(l-m)!}\delta_{kl} \tag{7.3}$$

[159]

$$\int_{-1}^{1} P_l^m(x) P_k^m(x)(1-x^2)dx =$$
$$(-1)^{(k-l)/2}\frac{(l+m)![l(l+1)+(m-1)(m+1)+\frac{1}{2}(k-l)(m+1)(k+l+1)]}{(l-m)![\frac{1}{2}(k-l)+1]!\Gamma[\frac{1}{2}(l-k)+2](k+l-1)(k+l+1)(k+l+3)}$$

for $k \ge l \ge m \ge 0$, zero for $k-l \ge 4$.

7.2. **Associated Legendre functions, powers, and trigonometric functions.**
[26]

(7.4) $$\int_0^z P_n[\cos(z-t)]P_n^{-m}(\cos t)\frac{dt}{\sin t} = \frac{P_n^{-m}(\cos z}{m}, \quad \Re m > 0.$$

(7.5)
$$\int_0^z \sin^m t P_{n-m-1}[\cos(z-t)]P_n^{-m}(\cos t)dt = \frac{\Gamma(m+1/2)}{2^{1/2}\Gamma(m+1)}\sin^{m+1/2} z P_{n-1/2}^{-m-1/2}(\cos z), \quad \Re m > -1/2.$$

(7.6)
$$\int_0^z \sin^{m+1} t P_{n-m-2}[\cos(z-t)]P_n^{-m}(\cos t)dt = \frac{\Gamma(m+3/2)}{2^{1/2}\Gamma(m+2)}\sin^{m+3/2} z P_{n-1/2}^{-m-1/2}(\cos z), \quad \Re m > -1.$$

(7.7)
$$\int_0^z \sin^{-1-k} t \sin^k(z-t) P_n^{-m}(\cos t) P_n^{-k}[\cos(z-t)] = \frac{2^k\Gamma(m-k)\Gamma(k+1/2)}{\sqrt{\pi}\Gamma(k+m+1)}\sin^k z P_n^{-m}(\cos z), \quad \Re m > \Re k > -1/$$

(7.8)
$$\int_0^z \sin^m t \sin^{-m}(z-t) P_n^{-m}(\cos t) P_{n-1}^m[\cos(z-t)] = \frac{\sin[(m+n)z]}{(m+n)\cos m\pi}, \quad -1/2 < \Re m < 1/2.$$

(7.9)
$$\int_0^z P_{n+1}^{-m}(\cos t)\sin[n(z-t)]dt = 2^{3/2}\sin^{1/2} z \sum_{r=0}^\infty \frac{(-1)^r(m+n+2)_{2r}(3/2)_r\Gamma(m+r+3/2)}{r!\Gamma(m+r+1)} \times \frac{m+2r+3/2}{(m+2r+1)(m+2r+2)} P_{n-1/2}^{-(m+3/2+2r)}(\cos z).$$

[146]

(7.10) $$\int_0^\pi d\theta \sin^{|m|+1}\theta \exp(\pm iR\cos\theta) P_n^{|m|}(\cos\theta) = 2(\pm i)^{n+|m|}\frac{(n+|m|)!}{(n-|m|)!}\frac{j_n(R)}{R^{|m|}}.$$

[146, (23)]
(7.11)
$$\int_0^\pi d\theta \sin\theta \exp(iR\cos\alpha\cos\theta) P_n^m(\cos\theta) J_m(R\sin\alpha\sin\theta) = 2i^{n-m}P_n^m(\cos\alpha)j_n(R).$$

[15, §11.4,13][123]

(7.12) $$\int_0^\infty e^{-c^2x^2}H_{2m}(ax)H_{2k}(bx)dx =$$
$$(-)^{m+k}2^{2m+2k-1}\frac{\Gamma(m+1/2)\Gamma(k+1/2)}{\pi c^{2m+2k+1}}(c^2-a^2)^m(c^2-b^2)^k\,{}_2F_1(-m,-k;\frac{1}{2};\frac{a^2b^2}{(c^2-a^2)(c^2-b^2}),$$
$c^2-a^2-b^2 > 0.$
[15, §11.4,14]

(7.13)
$$\int_0^\infty e^{-c^2x^2}H_{2m+1}(ax)H_{2k+1}(bx)dx = (-)^{m+k}2^{2m+2k+1}\frac{\Gamma(m+3/2)\Gamma(k+3/2)}{\Gamma(3/2)} \times \frac{ab(c^2-a^2)^m(c^2-b^2)^k}{c^{2m+2k+3}}\,{}_2F_1(-m,-k;\frac{3}{2};\frac{a^2b^2}{(c^2-a^2)(c^2-b^2}).$$

[15, §11.4,11]

$$\int_{-\infty}^{\infty} e^{-2x^2}\left[H_n(x)\right]^2 dx = 2^{n-1/2}\Gamma(n+1/2). \tag{7.14}$$

[15, §11.4,20]

$$\int_{-\infty}^{\infty} e^{-(a^2+b^2)x^2} H_{2m}(ax)H_{2k}(bx)dx = (-)^{m+k}2^{2(m+k)}\Gamma(m+k+\frac{1}{2})\frac{a^{2k}b^{2m}}{(a^2+b^2)^{m+k+1/2}}. \tag{7.15}$$

[15, §11.4,21]

$$\int_{-\infty}^{\infty} e^{-(a^2+b^2)x^2} H_{2m+1}(ax)H_{2k+1}(bx)dx = (-)^{m+k}2^{2(m+k+1)}\Gamma(m+k+\frac{3}{2})\frac{a^{2k+1}b^{2m+1}}{(a^2+b^2)^{m+k+3/2}}. \tag{7.16}$$

[111]

$$\int_0^{\infty} x^{\nu+1}e^{-\alpha x^2}L_m^{\nu-\sigma}(\alpha x^2)L_n^{\sigma}(\alpha x^2)J_\nu(xy)dx = \frac{(-1)^{m+n}}{2\alpha}\left(\frac{y}{2\alpha}\right)^{\nu}\exp\left(-\frac{y^2}{4\alpha}\right)L_m^{\sigma-m+n}(\frac{y^2}{4\alpha})L_n^{\nu-\sigma+m-n}(\frac{y^2}{4\alpha}), \tag{7.17}$$

for $y>0$, $\Re\alpha>0$, $Re\nu>-1$.

$$\int_0^{\infty} x^{(\alpha+\beta)/2}e^{-x}L_m^{\alpha}(x)L_n^{\beta}(x)J_{\alpha+\beta}(2\sqrt{ax})dx = (-1)^{m+n}a^{(\alpha+\beta)/2}L_m^{\beta-m+n}(a)L_n^{\alpha+m-n}(a);\quad \Re(\alpha+\beta)>-1. \tag{7.18}$$

7.3. **Hypergeometric Functions.** [73, p 238]

$$\frac{1}{\Gamma(2\lambda+2\nu)}\int_0^{\infty} dte^{-pt}t^{2\lambda+2\nu-1}\,{}_1F_2(\nu;\lambda+\nu,\lambda+\nu+\frac{1}{2};-\frac{1}{4}a^2t^2) = \frac{1}{p^{2\lambda}}\frac{1}{(p^2+a^2)^{\nu}};\quad \Re(\lambda+\nu)>0. \tag{7.19}$$

previous formula at $\lambda=0$ with [1, 9.1.69] gives [91, 6.623.1]

$$\begin{aligned}\frac{1}{\Gamma(2\nu)}\int_0^{\infty} dte^{-pt}t^{2\nu-1}\,{}_0F_1(;\nu+\frac{1}{2};-\frac{1}{4}a^2t^2) &= \frac{\Gamma(\nu+1/2)}{\Gamma(2\nu)}\left(\frac{a}{2}\right)^{1/2-\nu}\int_0^{\infty} e^{-pt}t^{\nu-1/2}J_{\nu-1/2}(at)\\ &= \frac{1}{(p^2+a^2)^{\nu}};\quad \Re(\nu)>0.\end{aligned} \tag{7.20}$$

[133]

$$\int_0^{\infty} J_\mu^2(\omega\rho)\,{}_3F_2\left(\begin{array}{c}3/2,(\sigma+\nu)/2,(\sigma-\nu)/2\\ 1-\mu,1+\mu\end{array}\mid -4\omega^2\right)\omega d\omega = -\frac{\mu\rho^{\sigma-2}K_\nu(\rho)}{2^{\sigma-1}\Gamma[(\sigma+\nu)/2]\Gamma[(\sigma-\nu)/2]} \tag{7.21}$$

for $\Re\sigma>1+|\Re\nu|$, $\mu=l+1/2$ with $l\in\mathbb{N}$, $\Re(\sigma+\nu)>0$, $\Re\rho>0$.

[133]

$$\int_0^{\infty} J_\mu^2(\omega\rho)J_{\mu+1}(\omega\rho)\,{}_3F_2\left(\begin{array}{c}3/2,(\sigma+\nu)/2,(\sigma-\nu)/2\\ 1-\mu,2+\mu\end{array}\mid -4\omega^2\right)\omega^2 d\omega = -\frac{\mu(\mu+1)\rho^{\sigma-3}K_\nu(\rho)}{2^{\sigma-1}\Gamma[(\sigma+\nu)/2]\Gamma[(\sigma-\nu)/2]} \tag{7.22}$$

for $\Re\sigma>2+|\Re\nu|$, $\mu=l+1/2$ with $l\in\mathbb{N}$, $\Re(\sigma+\nu)>0$, $\Re\rho>0$.

8. Special Functions

8.1. The exponential integral and related functions.
[30] If

$$\int_0^x \frac{\sin u}{u} du = \frac{\pi}{2} - r\cos(x-\theta); \int_0^x \frac{1-\cos u}{u} du = \gamma + \log x - r\sin(x-\theta) \tag{8.1}$$

then

$$r\cos\theta \sim \sum_{k\geq 0} \frac{(-)^k(2k)!}{x^{2k+1}}; \; r\sin\theta \sim \sum_{k\geq 1} \frac{(-)^{k+1}(2k-1)!}{x^{2k}}; \; r \sim \sum_{k\geq 1} \frac{(-)^{k+1}(2k-1)!}{kx^{2k}} \tag{8.2}$$

for $x \to \infty$.

8.2. The error function and Fresnel integrals.
[34]

$$2\sqrt{\pi}e^{b^2}\operatorname{erf}(b) = \sum_{j=0}^{\infty} \frac{j!b^{2j+1}}{(2j+1)!2^{2j+2}}. \tag{8.3}$$

8.3. The gamma function.

$$\frac{1}{a+r} = \frac{1}{a}\frac{(a)_r}{(a+1)_r}. \tag{8.4}$$

[174, 2.2.3.1]

$$(a)_{m-r} = \frac{(-1)^r(a)_m}{(1-a-m)_r}. \tag{8.5}$$

[174, (2.4.5.2.)][64, 166]

$$(a)_{2r} = (a/2)_r(\frac{a+1}{2})_r2^{2r}; \quad (a)_{qr} = (a/q)_r(\frac{a+1}{q})_r\cdots(\frac{a+q-1}{q})_rq^{qr} \tag{8.6}$$

$$\Gamma(n+1-k) = \frac{(-1)^k\Gamma(n+1)}{(-n)_k}. \tag{8.7}$$

[174, (I.4)]

$$(a+kn)_n = \frac{(a)_{(k+1)n}}{(a)_{kn}}. \tag{8.8}$$

[174, (I.6)]

$$(a-kn)_n = \frac{(-1)^n(1-a)_{kn}}{(1-a)_{(k-1)n}}. \tag{8.9}$$

[174, (I.9)][166]

$$(a)_{N-n} = \frac{(-1)^n(a)_N}{(1-a-N)_n}. \tag{8.10}$$

[174, (I.11)]

$$(a+kn)_{N-n} = \frac{(a)_N(a+N)_{(k-1)n}}{(a)_{kn}}. \tag{8.11}$$

[174, (I.13)]

$$(a-kn)_{N-n} = \frac{(-1)^n(a)_N(1-a)_{kn}}{(1-a-N)_{(k+1)n}}. \tag{8.12}$$

[70, (§1.2)]

$$\frac{\Gamma(n+\frac{1}{2}+z)\Gamma(n+\frac{1}{2}-z)}{\Gamma^2(n+\frac{1}{2})} = \frac{1}{\cos(\pi z)}\prod_{l=1}^{n}\left[1-\frac{4z^2}{(2l-1)^2}\right], \tag{8.13}$$

for $n = 1, 2, 3, \ldots$.

[194][175, A073006]

$$\Gamma(2/3) = \frac{2\pi}{\sqrt{3}}\frac{1}{\Gamma(1/3)}. \tag{8.14}$$

[194][175, A068465]

$$\Gamma(3/4) = \pi\sqrt{2}\frac{1}{\Gamma(1/4)}. \tag{8.15}$$

[194][175, A175379]

$$\Gamma(1/6) = \frac{\sqrt{3}}{\sqrt{\pi}2^{1/3}}\Gamma^2(1/3). \tag{8.16}$$

[194]

$$\Gamma(3/5) = \frac{\pi\sqrt{2}\sqrt{\phi^*}}{\sqrt{5}}\frac{1}{\Gamma(2/5)}, \tag{8.17}$$

where $\phi^* = 5-\sqrt{5}$.

[194][175, A203145]

$$\Gamma(5/6) = \frac{\pi^{3/2}2^{4/3}}{\sqrt{3}}\frac{1}{\Gamma^2(1/3)}. \tag{8.18}$$

[194]

$$\Gamma(4/5) = \frac{\pi\sqrt{2}\sqrt{\phi}}{\sqrt{5}}\frac{1}{\Gamma(1/5)}, \tag{8.19}$$

where $\phi = 5+\sqrt{5}$.

[194][175, A203143]

$$\Gamma(3/8) = \sqrt{\pi}\sqrt{\sqrt{2}-1}\frac{\Gamma(1/8)}{\Gamma(1/4)}. \tag{8.20}$$

[194][175, A203144]

$$\Gamma(5/8) = \sqrt{\pi}2^{3/4}\frac{\Gamma(1/4)}{\Gamma(1/8)}. \tag{8.21}$$

[194][175, A203146]

$$\Gamma(7/8) = \pi 2^{3/4}\sqrt{\sqrt{2}+1}\frac{1}{\Gamma(1/8)}. \tag{8.22}$$

[194]

$$\Gamma(1/10) = \frac{\sqrt{\phi}}{\sqrt{\pi}2^{7/10}}\Gamma(1/5)\Gamma(2/5), \tag{8.23}$$

where $\phi = 5+\sqrt{5}$.

[194]

$$\Gamma(3/10) = \frac{\sqrt{\pi}\phi^*}{2^{3/5}\sqrt{5}}\frac{\Gamma(1/5)}{\Gamma(2/5)}, \tag{8.24}$$

where $\phi^* = 5 - \sqrt{5}$.

[194]

$$\Gamma(7/10) = \sqrt{\pi}2^{3/5}\frac{\Gamma(2/5)}{\Gamma(1/5)}. \tag{8.25}$$

[194]

$$\Gamma(9/10) = \frac{\pi^{3/2}2^{7/10}\sqrt{\phi}}{\sqrt{5}}\frac{1}{\Gamma(2/5)\Gamma(1/5)}. \tag{8.26}$$

[194]

$$\Gamma(1/12) = \frac{3^{3/8}\sqrt{\sqrt{3}+1}}{\sqrt{\pi}2^{1/4}}\Gamma(1/3)\Gamma(1/4). \tag{8.27}$$

[194]

$$\Gamma(5/12) = \frac{\sqrt{\pi}2^{1/4}\sqrt{\sqrt{3}-1}}{3^{1/8}}\frac{\Gamma(1/4)}{\Gamma(1/3)}. \tag{8.28}$$

[164]

(8.29)

$$(N+\frac{1}{2},K)(N+\frac{1}{2},L) = \sum_{j=\max(K,L)}^{\min(N,K+L)} \binom{2j-K-L}{j-K}(j+\frac{1}{2},K+L-j)(N+\frac{1}{2},j),$$

where $(n+\frac{1}{2},j) = \frac{(n+j)!}{j!(n-j)!}$ is Hankel's symbol.

[164]

(8.30)

$$(-)^N\frac{(N+\frac{1}{2},K)(N+\frac{1}{2},L)}{\binom{K+L}{K}} = \sum_{j=\max(K,L)}^{\min(N,K+L)} (-)^j\binom{2j-K-L}{j-K}\frac{(j+1/2,K+L-j)(N+1/2,j)}{\binom{K+L}{j}},$$

where $(n+\frac{1}{2},j) = \frac{(n+j)!}{j!(n-j)!}$ is Hankel's symbol.

[104]

$$\log\frac{\Gamma(z+1/2)}{\sqrt{z}\Gamma(z)} = -\sum_{r=1}^{k}\frac{(1-2^{-2r})B_{2r}}{r(2r-1)z^{2r-1}} + O(z^{-2k+1/2}). \tag{8.31}$$

[104]

$$\log\frac{\Gamma(z+3/4)}{\sqrt{z}\Gamma(z+1/4)} = -\sum_{r=1}^{k}\frac{E_{2r}}{4r(4z)^{2r}} + O(z^{-2k+1/2}). \tag{8.32}$$

[104]

$$\frac{1}{z}\left[\frac{\Gamma(z+3/4)}{\Gamma(z+1/4)}\right]^2 = 1 + \frac{2u}{1+}\frac{9u}{1+}\frac{25u}{1+}\frac{49u}{1+}\cdots \tag{8.33}$$

where $u = 1/(64z^2)$.

8.4. **The psi function ψ.** [6]

$$\sum_{k=0}^{\infty}\frac{(-1)^k}{2k+1}[\psi(k+1)+\gamma] = G-\frac{\pi}{2}\log 2. \tag{8.34}$$

[6]

$$\sum_{k=0}^{\infty}\frac{(-1)^k}{2k+1}[\psi(k+3/2)+\gamma] = G-\frac{\pi}{4}\log 2. \tag{8.35}$$

[6]

$$\frac{(-1)^n}{q^n(n-1)!}\sum_{k=1}^{q-1}e^{2\pi ikp/q}\psi^{n-1}(k/q) = \mathrm{Li}_n(e^{2\pi ip/q})-\frac{\zeta(n)}{q^n}. \tag{8.36}$$

[6]

$$\frac{1}{4}\sum_{k=0}^{\infty}\frac{2^k k!^2}{(2k+1)!}[\psi(k+3/2)+\gamma] = G-\frac{\pi}{4}\log 2. \tag{8.37}$$

[6]

$$q\sum_{k=0}^{q-1}\mathrm{Li}_2(e^{2\pi ik/q}t) = \mathrm{Li}_2(t^q). \tag{8.38}$$

8.5. **Integral representations of the functions J_ν and N_ν.** [185, La. 4.13]

$$J_{\mu+\nu+1}(t) = \frac{t^{\nu+1}}{2^\nu\Gamma(\nu+1)}\int_0^1 J_\mu(ts)s^{\mu+1}(1-s^2)^\nu ds. \tag{8.39}$$

[71, 7.2.7][118]

(8.40)

$$\Gamma(\nu+1)\Gamma(\mu+1)J_\nu(z)J_\mu(z) = (z/2)^{\nu+\mu}\ {}_2F_3(\frac{1+\nu+\mu}{2}, 1+\frac{\nu+\mu}{2}; 1+\nu, 1+\mu, 1+\nu+\mu; -z^2).$$

[24]

$$\begin{aligned}\frac{z}{2}J_\mu(z\cos\phi\cos\Phi)J_\nu(z\sin\phi\sin\Phi) &= (\cos\phi\cos\Phi)^\mu(\sin\phi\sin\Phi)^\nu \\ &\times\sum_{n=0}^{\infty}(-1)^n(\mu+\nu+2n+1)J_{\mu+\nu+2n+1}(z)\frac{\Gamma(\mu+\nu+n+1)\Gamma(\nu+n+1)}{n!\Gamma(\mu+n+1)\Gamma^2(\nu+1)} \\ &\times F(-n,\mu+\nu+n+1;\nu+1;\sin^2\phi)F(-n,\mu+\nu+n+1;\nu+1;\sin^2\Phi),\end{aligned} \tag{8.41}$$

where ν and μ are not negative integers.

[63]

$$J_\mu(Xz)J_\nu(xz) = \frac{X^\mu x^\nu}{\pi}\int_{-\pi/2}^{\pi/2}e^{i(\mu-\nu)\theta}(\lambda_1/\lambda_2)^{\mu+\nu}J_{\mu+\nu}(z\lambda_1\lambda_2)d\theta \tag{8.42}$$

where $\lambda_1 = +\sqrt{(e^{i\theta}+e^{-i\theta})}$, $\lambda_2 = (X^2e^{i\theta}+x^2e^{-i\theta})^{1/2}$.

[71, p 99]

$$(z/2)^{\gamma-\mu-\nu} J_\mu(\alpha z)J_\nu(\beta z) = \frac{\alpha^\mu\beta^\nu}{\Gamma(\mu+1)\Gamma(\nu+1)}\sum_{m=0}^{\infty}\frac{(\gamma+2m)\Gamma(\gamma+m)}{m!} \times F_4(-m,\gamma+m;\mu+1,\nu+1;\alpha^2,\beta^2)J_{\gamma+2m}(z) \tag{8.43}$$

$$= \frac{\alpha^\mu\beta^\nu}{\Gamma(\nu+1)}\sum_{m=0}^{\infty}(\gamma+2m)J_{\gamma+2m}(z)\sum_{n=0}^{\infty}\frac{(-1)^n\Gamma(\gamma+m+n)\alpha^{2n}}{n!(m-n)!\Gamma^2(n+\mu+1)}\,{}_2F_1(-n,-n-\mu;\nu+1;\frac{\beta^2}{\alpha^2}). \tag{8.44}$$

Application of [81, (3.5)] to [1, (9.1.14)]

$$J_\rho(cz)J_\nu(z)J_\mu(z) = \frac{c^\rho(z/2)^{\nu+\mu+\rho}}{\Gamma(\rho+1)\Gamma(\nu+1)\Gamma(\mu+1)}\sum_{n\ge 0}\frac{(-c^2z^2/4)^n}{(\rho+1)_n n!} \times {}_4F_3\left(-n,-n-\rho,1+\frac{\nu+\mu}{2},\frac{1+\nu+\mu}{2};\nu+1,\mu+1,\nu+\mu+1;\frac{4}{c^2}\right). \tag{8.45}$$

[31]

$$x^{-\alpha}I_\alpha(2\sqrt{x}) = \sum_{r=0}^{\infty}\frac{x^r}{r!\Gamma(r+\alpha+1)}. \tag{8.46}$$

[31][175, A002426]

$$\exp(t)I_0(2t) = \sum_{n=0}^{\infty}\frac{t^n}{n!}c_n; \quad c_n \equiv \sum_{k=0}^{[n/2]}\frac{n!}{(k!)^2(n-2k)!} = i^n\sqrt{3^n}P_n\left(-\frac{i}{\sqrt{3}}\right). \tag{8.47}$$

[31]

$$\exp(yt)\exp(xt^2)^{-\alpha/2}I_\alpha(2t\sqrt{x}) = \sum_{n=0}^{\infty}\frac{t^n}{n!}\Pi_n^\alpha(x,y); \quad \Pi_n^\alpha(x,y) \equiv n!\sum_{k=0}^{[n/2]}\frac{x^ky^{n-2k}}{(n-2k)!k!\Gamma(k+\alpha+1)}. \tag{8.48}$$

$$\frac{\exp(t)}{t}I_1(t) = \sum_{n=0}^{\infty}\frac{t^n}{n!}\Pi_n^1(1,1). \tag{8.49}$$

[146]

$$j_s(R) = \frac{R^s}{2^{s+1}s!}\int_0^\pi d\theta\sin\theta\cos(R\cos\theta)(\sin\theta)^{2s}. \tag{8.50}$$

[175, A122848]

$$\frac{d^s}{dt^s}t^\nu Z_\nu(t) = \sum_{l=\lfloor(s+1)/2\rfloor}^{s}\alpha_{l,s}t^{\nu-(s-l)}Z_{\nu-l}(t). \tag{8.51}$$

$$\frac{d^s}{dt^s}t^\nu K_\nu(t) = (-)^s\sum_{l=\lfloor(s+1)/2\rfloor}^{s}\alpha_{l,s}t^{\nu-(s-l)}K_{\nu-l}(t) \tag{8.52}$$

with Bessel polynomial coefficients [97]

$$\alpha_{l,s} = \frac{s!}{(s-l)!(2l-s)!2^{s-l}}.$$

[164]

(8.53)

$$\sum_{k=1}^{n} \frac{2k-1}{\sqrt{\pi}} K_{k-1/2}(x) K_{k-1/2}(y) = \sum_{k=1}^{n} \frac{n(n-1+k)!}{k!(n-k)!} \left(\frac{x+y}{2xy}\right)^{k-1/2} K_{k-1/2}(x+y).$$

[164]

$$\frac{1}{\sqrt{\pi}} K_{n+1/2}(x) K_{n+1/2}(y) = \sum_{\mu=0}^{n} \frac{(n+\mu)!}{\mu!(n-\mu)!} \left(\frac{x+y}{2xy}\right)^{\mu+1/2} K_{k+1/2}(x+y). \quad (8.54)$$

[87]

$$\sum_{n=1}^{\infty} (-1)^n \frac{J_{2m}(n\pi)}{a^2-n^2} = \frac{\pi J_{2m}(a\pi)}{2a \sin a\pi}. \quad (8.55)$$

[87]

$$\sum_{n=1}^{\infty} (-1)^n \frac{n J_{2m-1}(n\pi)}{a^2-n^2} = \frac{\pi J_{2m-1}(a\pi)}{2 \sin a\pi}. \quad (8.56)$$

[168]

$$z^m [K_{\nu+m}(z) - K_{\nu-m}(z)] = \sum_{j=0}^{m-1} (-)^{m-j-1} b_m(j) z^j K_{\nu+j}(z), \quad (8.57)$$

$$z^m [I_{\nu-m}(z) - I_{\nu+m}(z)] = \sum_{j=0}^{m-1} b_m(j) z^j I_{\nu+j}(z), \quad (8.58)$$

$$z^m [J_{\nu-m}(z) - (-1)^m I_{\nu-m}(z)] = \sum_{j=0}^{m-1} b_m(j) (-z)^j J_{\nu+j}(z), \quad (8.59)$$

with $m = 1, 2, 3, \ldots.$ and

$$b_m(j) = 2^{m-j} \binom{m}{j} \nu(\nu-1)\cdots(\nu+j-m+1) = \Gamma(\nu+1) \frac{2^{m-j}\binom{m}{j}}{\Gamma(\nu+j-m+1)}.$$

[71, §7.10.1]

$$f(z) = \frac{1}{z^\nu} \sum_{n=0}^{\infty} a_n J_{\nu+n}(z); \quad a_n = (\nu+n) 2^{\nu+n} \sum_{s=0}^{\lfloor n/2 \rfloor} \frac{\Gamma(\nu+n-s)}{s! 2^{2s}} b_{n-2s}, \quad (8.60)$$

where $f(z) = \sum_{n=0}^{\infty} b_n z^n$.

[71, §7.10.1]

$$f(z) = \frac{1}{z^\nu} \sum_{n=0}^{\infty} a_n z^n J_{\nu+n}(z); \quad a_n = \sum_{s=0}^{n} \frac{\Gamma(\nu+s+1)}{(n-s)!} 2^{2s-n+\nu} b_s, \quad (8.61)$$

$$\Gamma(\nu+n+1) b_n = \sum_{s=0}^{\infty} (-1)^s 2^{-\nu-n-s} \frac{a_{n-s}}{s!}. \quad (8.62)$$

where $f(z) = \sum_{l=0}^{\infty} b_l z^{2l}$.

[25]
(8.63)
$$[z^{2n}]\,\frac{1}{z^{2\nu}}\sum_{r=0}^{\infty}A_rJ^2_{\nu+r}(z)=\frac{(-)^n\Gamma(\nu+n+1/2)}{n!\sqrt{\pi}\Gamma(2\nu+n+1)\Gamma(\nu+n+1)}\sum_{r=0}^{n}\frac{A_r(-n)_r}{(2\nu+n+1)_r}.$$

[25]
(8.64)
$$\frac{1}{z^{2\nu}}\sum_{r=0}^{\infty}\frac{(2\nu)_r(\nu+1)_r(a_3)_r(a_4)_r}{r!(\nu)_r(1+2\nu-a_3)_r(1+2\nu-a_4)_r}J^2_{\nu+r}(z)=\frac{1}{4^\nu\Gamma^2(1+\nu)}\,{}_2F_3\left(\begin{array}{c}\nu+1/2,1+2\nu-a_3-a_4\\ \nu+1,1+2\nu-a_3,1+2\nu-a_4\end{array}\mid -z^2\right)$$

[25]

$$z^\nu J_\nu(2z)=\frac{2\sqrt{\pi}}{\Gamma(\nu+1/2)}\sum_{r=0}^{\infty}(-)^r(\nu+r)\frac{\Gamma(2\nu+r)}{r!}J^2_{\nu+r}(z). \tag{8.65}$$

[25]

$$z^{-1/2}J_{2\nu+1/2}(2z)=-\frac{\Gamma(\nu+1)}{\Gamma(\nu+1/2)}\sum_{r=0}^{\infty}(\nu+r)\frac{\Gamma(2\nu+r)\Gamma(r-1/2)}{r!\Gamma(2\nu+r+3/2)}J^2_{\nu+r}(z). \tag{8.66}$$

[25]
(8.67)
$$[z^{2n}]\,\frac{1}{z^{2\nu}}\sum_{r=0}^{\infty}A_rJ^2_{\nu+2r}(z)=\frac{(-)^n\Gamma(\nu+n+1/2)}{n!\sqrt{\pi}\Gamma(2\nu+n+1)\Gamma(\nu+n+1)}\sum_{r=0}^{n}\frac{A_r(-n)_{2r}}{(2\nu+n+1)_{2r}}.$$

[25]
(8.68)
$$\frac{1}{z^{2\nu}}\sum_{r=0}^{\infty}\frac{(\nu)_r(1+\nu/2)_r(a_3)_r}{r!(\nu/2)_r(1+\nu-a_3)_r}J^2_{\nu+2r}(z)=\frac{1}{4^\nu\Gamma^2(1+\nu)}\,{}_1F_2\left(\begin{array}{c}1/2+\nu-a_3\\ \nu+1,1+2\nu-2a_3,\end{array}\mid -z^2\right).$$

[25]
(8.69)
$$\frac{J_\lambda(2z\sin\frac{1}{2}\phi)}{(2z\sin\frac{1}{2}\phi)^{\lambda-2\nu}}=\sum_{n=0}^{\infty}\frac{2^{1-\lambda+2\nu}\sqrt{\pi}\Gamma(\nu+1)}{n!\Gamma(\nu+1/2)\Gamma(\lambda+1)}\sin^{2\nu}\frac{1}{2}\phi(\nu+n)\Gamma(2\nu+n)J^2_{\nu+n}(z)\,{}_3F_2\left(\begin{array}{c}\nu+1,2\nu+n,-n\\ \nu+1/2,\lambda+\end{array}\mid \sin^2\frac{1}{2}\phi\right)$$

$$J_0(z\cos a)+J_0(z\sin a)\equiv 2\sum_{n=0}^{\infty}b_{2n}(a)J_{2n}(z); \tag{8.70}$$

where

$2n$	b_{2n}
0	1
2	1
4	$1+6\cos^4 a-6\cos^2 a=\frac{1}{4}+\frac{3}{4}\cos(4a)$
6	$1+6\cos^4 a-6\cos^2 a=\frac{1}{4}+\frac{3}{4}\cos(4a)$
8	$1+70\cos^8 a-20\cos^2 a-140\cos^6 a+90\cos^4 a=\frac{9}{64}+\frac{35}{64}\cos(8a)+\frac{5}{16}\cos(4a)$
10	$1+70\cos^8 a-20\cos^2 a-140\cos^6 a+90\cos^4 a=\frac{9}{64}+\frac{35}{64}\cos(8a)+\frac{5}{16}\cos(4a)$
12	$1-42\cos^2 a+420\cos^4 a-1680\cos^6 a+3150\cos^8 a-2772\cos^{10} a+924\cos^{12} a$ $=\frac{25}{256}+\frac{104}{512}\cos(4a)+\frac{63}{256}\cos(8a)+\frac{231}{512}\cos(12a)$

Apparently, the b_{2n} count numbers of Delannoy paths, A109983 in [175].

[122, p31]

$$e^{ik\cos\varphi} = 2^{\nu}\Gamma(\nu)\sum_{m=0}^{\infty}(\nu+m)i^m J_{\nu+m}(k\rho)(k\rho)^{-\nu}C_m^{(\nu)}(\cos\varphi), \tag{8.71}$$

$\nu \neq 0, -1, -2, -3, \ldots$.

[82]

(8.72)

$$S_{\mu,\nu}(z) \sim z^{\mu-1}\sum_{m=0}^{\infty}(-1)^m\left(\frac{1-\mu+\nu}{2}\right)_m\left(\frac{1-\mu-\nu}{2}\right)_m (z/2)^{-2m},\ |z|\to\infty, |\arg z|<\pi.$$

The series terminates and is equal to $S_{\mu,\nu}(z)$ when $\mu \pm \nu$ is a positive odd integer.

[158]

(8.73)

$$\left(\frac{\sin\beta}{\sin\alpha}\right)^{m+n} P_{m+n}^m(\cos\alpha) = \sum_{r=0}^{n}\binom{2m+n}{r}\left(\frac{\sin(\beta-\alpha)}{\sin\alpha}\right)^r P_{n+m-r}^m(\cos\beta).$$

8.6. **Orthogonal Polynomials.** [197]

(8.74)

$$L_m^{(\alpha)}L_n^{(\alpha)} = \sum_{M=|m-n|}^{m+n}\frac{L_M^{(\alpha)}(z)(-)^{m+n-M}2^{m+n-M}M!}{(M-m)!(M-n)!(m+n-M)!}\,{}_3F_2\left(\begin{matrix}\alpha+M+1, \frac{M-m-n}{2}, \frac{(M-m-n+1}{2} \\ M-m+1, M-n+1\end{matrix}\ \Big|\ 1\right).$$

[54]

$$\sum_{n\ge 0} t^n L_n^{a+bn}(x) = \frac{(1+v)^{a+1}}{1-bv}\exp(-xv), \tag{8.75}$$

where $v = t(1+v)^{b+1}$, $v(0)=0$.

[54]

$$\sum_{n\ge 0} t^n L_n^{v+bn}(x(1+an)) = \frac{(1-z)^{1-v}}{1-z(b+2-ax)+z^2(b+1)}e^{xz/(z-1)}, \tag{8.76}$$

where $t = z(1-z)^b\exp[axz/(1-z)]$ and $|t|<1$.

[54]

(8.77)

$$\sum_{n\ge 0}\frac{t^n}{v+bn+n}L_n^{v+bn}(x(1+an)) = \frac{\exp[xz/(z-1)]}{v(1-z)^v}\,{}_1F_1\left(\begin{matrix}1 \\ \frac{v+1+b}{1+b}\end{matrix}\ \Big|\ \frac{xz(1+b-av)}{(1-z)(1+b)}\right),$$

where $t = z(1-z)^b\exp[axz/(1-z)]$ and $|t|<1$.

[54]

$$\sum_{n\ge 0}\frac{t^n}{1+an}L_n^{v+avn-n}(x(1+an)) = \frac{1}{(1-z)^v}e^{xz/(z-1)}, \tag{8.78}$$

where $t = z(1-z)^{av-1}\exp[axz/(1-z)]$ and $|t|<1$.

[54]

$$\sum_{n\ge 0}\frac{t^n}{1+an}L_n^{v+bn}(x(1+an)) = \frac{\exp[xz/(z-1)]}{(1-z)^v}\,{}_2F_1\left(\begin{matrix}1, \frac{1+b-av}{a} \\ \frac{a+1}{a}\end{matrix}\ \Big|\ z\right), \tag{8.79}$$

where $t = z(1-z)^b\exp[axz/(1-z)]$ and $|t|<1$.

[54]

(8.80)

$$\sum_{n\geq 0}\frac{t^n}{1+n}L_n^{v+bn}(x(1+an)) = \frac{\exp[x(1-a-z)/(1-z)]}{(1-z)^v z(v-b)}\Big\{ {}_1F_1\left(\begin{matrix} v-b \\ v-b+1\end{matrix} \;\Big|\; \frac{x(a-1)}{1-z}\right) - (1-z)^{v-b}\,{}_1F_1\left(\begin{matrix} v-b \\ v-b+1\end{matrix} \;\Big|\; x(a-1)\right)\Big\},$$

where $t = z(1-z)^b \exp[axz/(1-z)]$ and $|t| < 1$.

[54]

$$\sum_{n\geq 0}\frac{(1+bn)^{n/2}}{n!}t^n H_n[x(1+an)/(1+bn)^{1/2}] = \frac{e^{-z^2-2xz}}{1+2bz^2+2axz}, \tag{8.81}$$

where $t = (-z)e^{bz^2+2axz}$ and $|2azx\exp[bz^2+2axz+1]| < 1$.

[54]

(8.82)

$$\sum_{n\geq 0}\frac{(1+bn)^{n/2-1}}{n!}t^n H_n[x(1+an)/(1+bn)^{1/2}] = e^{-z^2-2xz}\,{}_1F_1\left(\begin{matrix} 1 \\ 1+1/b\end{matrix} \;\Big|\; 2xz(b-a)/b\right)$$

where $t = (-z)e^{bz^2+2axz}$ and $|2azx\exp[bz^2+2axz+1]| < 1$.

[54]

$$\sum_{n\geq 0}\frac{(1+an)^{n/2-1}}{n!}t^n H_n[x(1+an)] = e^{-z^2-2xz} \tag{8.83}$$

where $t = (-z)e^{az^2+2axz}$ and $|2azx\exp[az^2+2axz+1]| < 1$.

[83]

(8.84)

$$\int_0^1 f(t)t^r dt = c_r,\ (r=0,1,\dots n) \rightsquigarrow f(t) = \sum_{i=0}^{n}(2i+1)\sum_{r=0}^{i}\left\{(-)^r\frac{(i+r)!}{(i-r)!}\frac{1}{(r!)^2}c_r\right\}P_i(1-2t).$$

9. Special Functions II.

9.1. **Hypergeometric Functions.** [29]

$$ {}_2F_1(a,b;c;z) = (a)_m(b)_m z^m\,{}_2F_1(a+m,b+m;1+m;z). \tag{9.1}$$

[29]

$$ {}_2F_1(a,b;c;z) = (1-z)^{c-a-b}\,{}_2F_1(c-a,c-b;c;z). \tag{9.2}$$

[29]

$$ {}_2F_1(a,b;c;z) = (a)_m(b)_m z^m(1-z)^{1-a-b-m}\,{}_2F_1(1-b,1-a;1+m;z). \tag{9.3}$$

[29]

$$ {}_2F_1(a,b;c;z) = (1-z)^{-a}\,{}_2F_1(a,c-b;c;z/(z-1)). \tag{9.4}$$

[145]

(9.5)

$$\frac{(z/2)^c}{\Gamma(1+c)}\,{}_2F_1(a,b;c+1;-z^2/(4ab)) = \sum_{\nu\geq 0}\frac{1}{\nu!}\,{}_3F_0(-\nu,a,b;1/(ab))(z/2)^\nu J_{c+\nu}(z).$$

[29]

$$(9.6)\quad {}_2F_1(a,b;c;z) = (1-z)^{-a}\left(\frac{z}{z-1}\right)^{1-c}\left\{\frac{\Gamma(|m|)}{\Gamma(a+\bar{m})\Gamma(c-a-\underline{m})}\left(\frac{1}{1-z}\right)^{\underline{m}}\right.$$

$$\sum_{n=0}^{|m|-1}\frac{(1-a-\bar{m})_n(1-c+a+\underline{m})_n}{(1-|m|)_n\Gamma(1+n)}\left(\frac{1}{1-z}\right)^n - \frac{(-)^m}{\Gamma(a+\underline{m})\Gamma(c-a-\bar{m})}\left(\frac{1}{1-z}\right)^{\bar{m}}$$

$$\sum_{n=0}^{\infty}\frac{(1-a-\underline{m})_n(1-c+a+\bar{m})_n}{\Gamma(1+|m|+n)\Gamma(1+n)}\left(\frac{1}{1-z}\right)^n$$

$$\left.[-\ln(1-z)-\pi\cot\pi(c-a)-\pi\cot\pi a+\psi(1-a-\underline{m}+n)+\psi(1-c+a+\bar{m}+n)-\psi(1+|m|+n)-\psi(1+n)]\right\},$$

where $\bar{m}\equiv\max(0,m)$, $\underline{m}\equiv\min(0,m)$.

[163]

(9.7)

$$ {}_1F_1(1/2;9/2;z) = -\frac{525+280z+140z^2}{128z^3}e^z + \frac{525+630z+420z^2+280z^3}{256z^{7/2}}\sqrt{\pi}\,\mathrm{erfi}(\sqrt{z}).$$

[173]

(9.8)

$$ {}_1F_1\left(\frac{1}{2}+\frac{1}{2}a-b, 1+a-b; x\right) = e^{x/2}\sum_{r=0}^{\infty}\frac{(a)_r(b)_r(-x/4)^r}{r!(a/2)_r(1+a-b)_r}\,{}_0F_1\left(;\frac{1}{2}a+r+1;(x/4)^2\right).$$

[174, 1.7.7]

$$(9.9)\qquad {}_2F_1(a,-m;c,1) = \frac{(c-a)_m}{(c)_m}.$$

[47]

$$(9.10)\qquad {}_2F_1(a,-n;c;p) = \frac{(a)_n}{(c)_n}(-p)^n\,{}_2F_1(1-c-n,-n;1-a-n;1/p).$$

[193]

$$(9.11)\qquad {}_2F_1(a+n,b;a-b;-1) = P(n)\frac{\Gamma(a-b)\Gamma(\frac{a+1}{2})}{\Gamma(a)\Gamma(\frac{a+1}{2}-b)} + Q(n)\frac{\Gamma(a-b)\Gamma(\frac{a}{2})}{\Gamma(a)\Gamma(\frac{a}{2}-b)}$$

where

$$P(n) = \frac{1}{2^{n+1}}\,{}_3F_2(-n/2,-(n+1)/2,a/2-b;1/2,a/2;1);$$

$$Q(n) = \frac{n+1}{2^{n+1}}\,{}_3F_2(-(n-1)/2,-n/2,(a+1)/2-b;3/2,(a+1)/2;1).$$

[13]

(9.12)

$$ {}_2F_1(-2n,b;-2n+2r-b;-1) = \frac{(1/2)_n(b+1-r)_n}{(b/2+1-r)_n(b/2+1/2-r)_n}\sum_{i=0}^{r-1}\frac{2^{2i}i!\binom{r+i-1}{2i}}{(b-r+1)_i}\binom{n}{i}.$$

[193]

(9.13)

$$ {}_2F_1(-a,1/2;2a+3/2+n;1/4) = \frac{2^{n+3/2}}{3^{n+1}} \frac{\Gamma(a+5/4+n/2)\Gamma(a+3/4+n/2)\Gamma(a+1/2)}{\Gamma(a+7/6+n/3)\Gamma(a+5/6+n/3)\Gamma(a+1/2+n/3)} K(n) - (-3)^{n-2} 2^{3/2} \frac{\Gamma(a+5/4+n/2)\Gamma(a+3/4+n/2)\Gamma(a+1)}{\Gamma(a+3/2)\Gamma(a+1/2+n/2)\Gamma(a+1+n/2)} L(n) $$

where $K(n)$ and $L(n)$ are defined in the reference.

[30]

(9.14)

$$ \pi {}_2F_1(1/2,1/2;1;1-x) = \log\frac{16}{x} {}_2F_1(1/2,1/2;1;x) - 4 \sum_{k\geq 1} \frac{(1/2)_k^2}{(k!)^2} \sum_{j=1}^{k} \frac{x^k}{(2j-1)(2j)}. $$

[30]

(9.15)

$$ {}_2F_1(\frac{1}{2},\frac{1}{2};1;\frac{1+x}{2}) = \frac{\sqrt{\pi}}{\Gamma^2(3/4)} {}_2F_1(1/4,1/4;1/2;x^2) + \frac{\Gamma^2(3/4)}{\pi^{3/2}} {}_2F_1(3/4,3/4;3/2;x^2). $$

[30]

(9.16)

$$ {}_2F_1(\frac{1}{2},\frac{1}{2};1;\frac{1}{2}+\frac{x}{1+x^2}) = \frac{\sqrt{\pi}}{\Gamma^2(3/4)} \sqrt{1+x^2} {}_2F_1(1/4,1/2;3/4;x^4) + \frac{\Gamma^2(3/4)}{\pi^{3/2}} x(1+x^2)^{3/2} {}_2F_1(1/2,3/4;5/4;x^4). $$

[30]

$$ {}_2F_1(n,-n;1/2;x^2) = \cos(2n\sin^{-1}x). \tag{9.17} $$

[30]

$$ 2nx {}_2F_1(\frac{1}{2}+n,\frac{1}{2}-n;3/2;x^2) = \sin(2n\sin^{-1}x). \tag{9.18} $$

[30]

$$ {}_2F_1(\frac{1}{2}+n,\frac{1}{2}-n;1/2;x^2) = (1-x^2)^{-1/2}\cos(2n\sin^{-1}x). \tag{9.19} $$

[163]

$$ {}_2F_1(-3/2,-1/2;1/2,z) = \frac{2+z}{2}\sqrt{1-z} + \frac{3\sqrt{z}}{2}\arcsin\sqrt{z}. \tag{9.20} $$

[163]

$$ \frac{z^{\mu+1}}{2^{\mu}\sqrt{\pi}\Gamma(\frac{3}{2}+\mu)} {}_2F_1(1,3/2;3/2+\mu,z^2/4) = L_\mu(z). \tag{9.21} $$

[163]

$$ {}_2F_1(-3/2,1/2;3/2,z) = \frac{5-2z}{8}\sqrt{1-z} + \frac{3}{8\sqrt{z}}\arcsin\sqrt{z}. \tag{9.22} $$

[163]

$$ {}_2F_1(-3/2,-1/2;3/2,z) = \frac{13+2z}{16}\sqrt{1-z} + \frac{3+12z}{16\sqrt{z}}\arcsin\sqrt{z}. \tag{9.23} $$

[163]

(9.24)
$$ {}_1F_2(-3/2;-1/2,1/2;z) = (1+2z)\cosh(2\sqrt{z}) + \sqrt{z}\sinh(2\sqrt{z}) - 4z^{3/2}\,\mathrm{Shi}(2\sqrt{z}). $$

[163]

$$ {}_1F_2(-3/2;-1/2,2;z) = -\frac{4+24z-28z^2}{15z\pi}K(\sqrt{z}) + \frac{4+56z+4z^2}{15z\pi}E(\sqrt{z}). \quad (9.25) $$

[195]

$$ {}_2F_1\left(1/4,-1/12;2/3;\frac{x(4+x)^3}{4(2x-1)^3}\right) = (1-2x)^{-1/4}.. \quad (9.26) $$

[195]

$$ {}_2F_1\left(5/4,-1/12;5/3;\frac{x(4+x)^3}{4(2x-1)^3}\right) = \frac{1+x}{(1+\frac{1}{4}x)^2}(1-2x)^{-1/4}. \quad (9.27) $$

[195]

$$ {}_2F_1\left(1/4,7/12;4/3;\frac{x(4+x)^3}{4(2x-1)^3}\right) = \frac{1}{1+\frac{1}{4}x}(1-2x)^{3/4}. \quad (9.28) $$

[195]

$$ {}_2F_1\left(1/4,-5/12;1/3;\frac{x(4+x)^3}{4(2x-1)^3}\right) = (1+\frac{5}{2}x)(1-2x)^{-5/4}. \quad (9.29) $$

[195]

$$ {}_2F_1\left(1/2,-1/6;2/3;\frac{x(2+x)^3}{(2x+1)^3}\right) = (1+2x)^{-1/2}. \quad (9.30) $$

[195]

$$ {}_2F_1\left(1/2,5/6;2/3;\frac{x(2+x)^3}{(2x+1)^3}\right) = \frac{1}{(1-x)^2}(1+2x)^{3/2}. \quad (9.31) $$

[195]

$$ {}_2F_1\left(1/6,5/6;4/3;\frac{x(2+x)^3}{(2x+1)^3}\right) = \frac{1}{1+\frac{1}{2}x}(1+2x)^{1/2}(1+x)^{1/3}. \quad (9.32) $$

[195]

$$ {}_2F_1\left(1/6,-1/6;1/3;\frac{x(2+x)^3}{(2x+1)^3}\right) = (1+2x)^{-1/2}(1+x)^{1/3}. \quad (9.33) $$

[195]

$$ {}_2F_1\left(7/24,-1/24;3/4;\frac{108x(x-1)^4}{(x^2+14x+1)^3}\right) = (1+14x+x^2)^{-1/8}. \quad (9.34) $$

[195]

$$ {}_2F_1\left(7/24,23/24;7/4;\frac{108x(x-1)^4}{(x^2+14x+1)^3}\right) = \frac{1+2x-\frac{1}{11}x^2}{(1-x)^2}(1+14x+x^2)^{7/8}. \quad (9.35) $$

[195]

$$ {}_2F_1\left(5/24,13/24;5/4;\frac{108x(x-1)^4}{(x^2+14x+1)^3}\right) = \frac{1}{1-x}(1+14x+x^2)^{5/8}. \quad (9.36) $$

[195]

$$_2F_1\left(5/24, -11/24; 1/4; \frac{108x(x-1)^4}{(x^2+14x+1)^3}\right) = \frac{1-22x-11x^2}{(1+14x+x^2)^{11/8}}. \tag{9.37}$$

[195]

$$_2F_1\left(19/60, -1/60; 4/5; \varphi_1(x)\right) = (1-228x+494x^2+228x^3+x^4)^{-1/20}. \tag{9.38}$$

(9.39)

$$_2F_1\left(19/60, 59/60; 4/5; \varphi_1(x)\right) = \frac{(1+66x-11x^2)(1-228x+494x^2+228x^3+x^4)^{19/20}}{(1+x^2)(1+522x-10006x^2-522x^3+x^4)}.$$

$$_2F_1\left(11/60, 31/60; 6/5; \varphi_1(x)\right) = \frac{(1-228x+494x^2+228x^3+x^4)^{11/20}}{1+11x-x^2}. \tag{9.40}$$

$$_2F_1\left(11/60, -29/60; 1/5; \varphi_1(x)\right) = \frac{1+435x-6670x^2-3335x^4-87x^5}{(1-228x+494x^2+228x^3+x^4)^{29/20}}. \tag{9.41}$$

$$_2F_1\left(13/60, -7/60; 3/5; \varphi_1(x)\right) = \frac{1-7x}{(1-228x+494x^2+228x^3+x^4)^{7/20}}. \tag{9.42}$$

(9.43)

$$_2F_1\left(13/60, 53/60; 3/5; \varphi_1(x)\right) = \frac{(1+119x+187x^2+17x^3)(1-228x+494x^2+228x^3+x^4)^{13/20}}{(1+x^2)(1+522x-10006x^2-522x^3+x^4)}.$$

(9.44)

$$_2F_1\left(17/60, 37/60; 7/5; \varphi_1(x)\right) = \frac{(1+\frac{1}{7}x)(1-228x+494x^2+228x^3+x^4)^{17/20}}{(1+11x-x^2)^2}.$$

$$_2F_1\left(17/60, -23/60; 2/5; \varphi_1(x)\right) = \frac{(1+107x-391x^2+1173x^3+46x^4)}{(1-228x+494x^2+228x^3+x^4)^{23/20}}. \tag{9.45}$$

Where

$$\varphi_1(x) = \frac{1728x(x^2-11x-1)^5}{(x^4+228x^3+494x^2-228x+1)^3}. \tag{9.46}$$

[195]

$$_2F_1\left(7/20, -1/20; 4/5; \varphi_2(x)\right) = \frac{(1+x)^{7/20}}{(1-x)^{1/20}(1-4x-x^2)^{1/4}}. \tag{9.47}$$

(9.48)

$$_2F_1\left(7/20, 19/20; 4/5; \varphi_2(x)\right) = \frac{(1+3x)(1+x)^{7/20}(1-x)^{19/20}(1-4x-x^2)^{7/4}}{(1+x^2)(1+22x-6x^2-22x^3+x^4)}.$$

Where

$$\varphi_2(x) = \frac{64x(x^2-x-1)^5}{(x^2-1)(x^2+4x-1)^5}. \tag{9.49}$$

[151, 51]

$$_2F_2(a, d; b, c; x) = e^x \sum_{n\geq 0} \frac{(c-d)_n}{(c)_n n!}(-x)^n \, _2F_2(b-a, d; b, c+n; -x). \tag{9.50}$$

[140]

$$_{r+1}F_{r+1}(a, (f_r+1); b, (f_r); y) = e^y \, _{r+1}F_{r+1}(b-a-r, (\xi_r+1); b, (\xi_r); -y), \tag{9.51}$$

where ξ_r are nonvanishing zeros of an associated parametric polynomial of Q degree r,

$$Q_r(t) = \sum_{j=0}^{r} s_{r-j} \sum_{l=0}^{j} \left\{ \begin{array}{c} j \\ l \end{array} \right\} (a)_l (t)_l (b-a-r-t)_{r-l}, \tag{9.52}$$

and the s_{r-j} are determined by

$$(f_1+x)\cdots(f+r+x) = \sum_{j=0}^{r} s_{r-j} x^j. \tag{9.53}$$

[11]

$${}_3F_2(1,1,2-t;2,3;1) = \frac{2(1-\gamma-\psi(t+1))}{1-t}. \tag{9.54}$$

[166]
(9.55)

$${}_3F_2(-n,-a,-b;c,2-n-a-b-c;1) = \frac{(c+b-1)_n(c+a)_n}{(c+a+b-1)_n(c)_n}\left[1-\frac{a}{(c+b-1)(a+c+n-1)}\right].$$

for $n = 1,2,3,\ldots$
[113][174, (2.3.1.3)]
(9.56)

$${}_3F_2(-m,b,c;e,-m+b+c-e+1;1) = \frac{(e-b)_m(e-c)_m}{(e)_m(e-b-c)_m}; \quad m = 0,1,2,\ldots, \quad e, e-b-c \neq 0,-1,-2,\ldots$$

[64]
(9.57)

$${}_3F_2(a,b/2,(b+1)/2;c/2,(c+1)/2;1) = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-b)\Gamma(c-a)}\,{}_2F_1(a,b;c-a;-1), \quad \Re c > \Re b > 0, \Re(c-a-b) > 0.$$

[64]
(9.58)

$${}_3F_2(a,b/2,(b+1)/2;c/2,(c+1)/2;1/2) = 2^a \sum_{k=0}^{\infty} \binom{-a}{k} \frac{(c-b)_k}{(c)_k}\,{}_2F_1(-k,b;c+k;-1), \quad \Re c > \Re b > 0.$$

[47]

$${}_0F_1(a;px)\,{}_0F_1(c';qx) = \sum_{n\ge 0} \frac{(px)^n}{n!(c)_n}\,{}_2F_1(1-c-n,-n;c';q/p). \tag{9.59}$$

[47]
(9.60)

$${}_1F_1(a;c;px)\,{}_1F_1(a';c';qx) = \sum_{n\ge 0} \frac{(a)_n(px)^n}{n!(c)_n}\,{}_3F_2(a',1-c-n,-n;c',1-a-n;-q/p).$$

[47]
(9.61)

$${}_2F_0(a,b;px)\,{}_2F_0(a',b';qx) = \sum_{n\ge 0} \frac{(a)_n(b)_n(px)^n}{n!}\,{}_3F_2(a',b',-n;1-a-n,1-b-n;-q/p).$$

[47]
(9.62)
$${}_2F_1(a,b;c;px)\,{}_2F_1(a',b';c';qx) = \sum_{n\geq 0} \frac{(a)_n(b)_n(px)^n}{n!(c)_n}\,{}_4F_3(a',b',1{-}c{-}n,-n;c',1{-}a{-}n,1{-}b{-}n;q/p).$$

[202]
(9.63)
$$(a{+}b{+}1){}_3F_2[-c,-a,1;b{+}1,\frac{1-a-c}{2};\frac{1}{2}] = (b{+}1){}_3F_2[-\frac{a}{2},\frac{1-a}{2},1;\frac{1-a-b}{2},\frac{1-a-c}{2};1]$$

where a, b and c are positive integers of the same parity.
[64]
(9.64)
$${}_3F_2(-n,b/q,(b{+}1)/2;c/2,(c{+}1)/2;1) = \frac{(c-b)_n}{(c)_n}\,{}_2F_1(-n,b;c{+}n;-1), \quad \Re c > \Re b > 0.$$

[113]

$$\begin{aligned}{}_3F_2(a,b,c;e,a{+}b{+}c{-}e{+}1;1)+&\frac{\Gamma(e-1)\Gamma(a-e+1)\Gamma(b-e+1)\Gamma(c-e+1)\Gamma(a+b+c-e+1}{\Gamma(1-e)\Gamma(a)\Gamma(b)\Gamma(c)\Gamma(a+b+c-2e+2)}\\ &\times_3 F_2(a-e+1,b-e+1,c-e+1;2-e,a+b+c-2e+2;1)\\ &\qquad= \frac{\Gamma(a-e+1)\Gamma(b-e+1)\Gamma(c-e+1)\Gamma(a+b+c-e+1)}{\Gamma(1-e)\Gamma(b+c-e+1)\Gamma(a+c-e+1)\Gamma(a+b-e+1)}.\end{aligned}$$

[113]
(9.65)
$${}_3F_2(a,b,c;e,a{+}b{+}c{-}e{+}1;1) = \frac{\Gamma(e)\Gamma(a+b+c-e+1)}{\Gamma(a)\Gamma(b+1)\Gamma(c+1)}\,{}_3F_2(e{-}a,b{+}c{-}e{+}1,1;b{+}1,c{+}1;1); \quad \Re a > 0.$$

[160][139]

$$ {}_3F_2(a,b,c;f,e;1) $$

(9.66)
$$ = \frac{\Gamma(f)\Gamma(e)\Gamma(f+e-a-b-c)}{\Gamma(e-b+f-c)\Gamma(f+e-a-c)}\, {}_3F_2(e+f-a-b-c, f-c, e-c; e-b+f-c, e+f-a-c; 1) $$

(9.67)
$$ = \frac{\Gamma(f)\Gamma(e)\Gamma(f+e-a-b-c)}{\Gamma(e-b+f-c)\Gamma(e-b+f-a)}\, {}_3F_2(e+f-a-b-c, f-b, e-b; e-b+f-c, e-b+f-a; 1) $$

(9.68)
$$ = \frac{\Gamma(e)\Gamma(f+e-a-b-c)}{\Gamma(e-a)\Gamma(e-b+f-c)}\, {}_3F_2(f-c, f-b, a; e-b+f-c, f; 1) $$

(9.69)
$$ = \frac{\Gamma(f)\Gamma(f+e-a-b-c)}{\Gamma(f-a)\Gamma(e-b+f-c)}\, {}_3F_2(e-c, e-b, a; e-b+f-c, e; 1) $$

(9.70)
$$ = \frac{\Gamma(e)\Gamma(f)\Gamma(f+e-a-b-c)}{\Gamma(e+f-a-c)\Gamma(e-b+f-a)}\, {}_3F_2(e+f-a-b-c, f-a, e-a; e+f-a-c, e-b+f-a; 1) $$

(9.71)
$$ = \frac{\Gamma(e)\Gamma(f+e-a-b-c)}{\Gamma(e-b)\Gamma(e+f-a-c)}\, {}_3F_2(f-c, f-a, b; e+f-a-c, f; 1) $$

(9.72)
$$ = \frac{\Gamma(f)\Gamma(f+e-a-b-c)}{\Gamma(f-b)\Gamma(e+f-a-c)}\, {}_3F_2(e-c, e-a, b; e+f-a-c, e; 1) $$

(9.73)
$$ = \frac{\Gamma(e)\Gamma(f+e-a-b-c)}{\Gamma(e-c)\Gamma(e-b+f-a)}\, {}_3F_2(f-b, f-a, c; e+f-a-b, f; 1) $$

(9.74)
$$ = \frac{\Gamma(f)\Gamma(f+e-a-b-c)}{\Gamma(f-c)\Gamma(e-b+f-a)}\, {}_3F_2(e-b, e-a, c; e+f-a-b, e; 1). $$

[157, (6)]

(9.75)
$$ {}_3F_2(-n,\alpha,\beta;\gamma,\delta;1) = \frac{\Gamma(\gamma)\Gamma(\gamma+n-\alpha)}{\Gamma(\gamma+n)\Gamma(\gamma-\alpha)}\, {}_3F_2(-n,\alpha,\delta-\beta; 1+\alpha-\gamma-n, \delta; 1). $$

[30]

(9.76)
$$ {}_3F_2(-2\alpha,-2\beta,\gamma;\gamma+1/2,2\gamma;x) = {}_2F_1^2(-\alpha,-\beta;\gamma+1/2;x). $$

[30]

(9.77)
$$ {}_3F_2(\alpha,\beta,\gamma;\delta,\epsilon;1) = \frac{\Gamma(\delta)\Gamma(\delta-\alpha-\beta)}{\Gamma(\delta-\alpha)\Gamma(\delta-\beta)} {}_3F_2(\alpha,\beta,\epsilon-\gamma;\alpha+\beta-\delta+1,\epsilon;1) $$
$$ + \frac{\Gamma(\delta)\Gamma(\epsilon)\Gamma(\alpha+\beta-\gamma)\Gamma(\delta+\epsilon-\alpha-\beta-\gamma)}{\Gamma(\alpha)\Gamma(\beta)\Gamma(\epsilon-\gamma)\Gamma(\delta+\epsilon-\alpha-\beta)} {}_3F_2(\delta-\alpha,\delta-\beta,\delta+\epsilon-\alpha-\beta-\gamma;\delta-\alpha-\beta+1,\delta+\epsilon-\alpha-\beta;1). $$

[174, (2.2.3.2)]

(9.78)
$$ {}_{A+1}F_B[(a),-m;(b);z] = \frac{((a))_m(-z)^m}{((b))_m}\, {}_{B+1}F_A[1-(b),1-m;1-(a);\frac{(-1)^{A+B}}{z}]. $$

[174, (4.3.5.1)][200][105]

(9.79)
$$ {}_4F_3(x,y,z,-n;u,v,w;1) = \frac{(v-z)_n(w-z)_n}{(v)_n(w)_n}\, {}_4F_3(u-x,u-y,z,-n;1-v+z-n,1-w+z-n,w;1), $$

if $u+v+w=1+x+y+z-n$.

[174]

(9.80)
$$[\delta(\delta+b_1-1)(\delta+b_2-1)\cdots(\delta+b_B-1)-z(\delta+a_1)(\delta+a_2)\cdots(\delta+a_A)]\,{}_AF_B((a),(b),z)=0,$$

where $\delta=z\frac{d}{dz}$.

[47]

(9.81)
$$\frac{(\delta+h)_m}{(\delta+k)_m}\,{}_rF_s\left(\begin{array}{c}a,\ldots\\c,\ldots\end{array}\mid x\right)=\frac{(h)_m}{(k)_m}\,{}_{r+2}F_{s+2}\left(\begin{array}{c}a,\ldots,h+m,k\\c,\ldots,h,k+m\end{array}\mid x\right).$$

[47]

(9.82)
$$\sum_{n\geq0}\frac{(a)_n}{n!}F\left(\begin{array}{c}-n,A,\ldots\\C\ldots\end{array}\mid p\right)x^n=(1-x)^{-a}F\left(\begin{array}{c}a,A,\ldots\\C,\ldots\end{array}\mid -\frac{px}{1-x}\right).$$

[47]

(9.83)
$$\sum_{n\geq0}\frac{(a)_n}{n!}F\left(\begin{array}{c}-n,a+n,A,\ldots\\C\ldots\end{array}\mid p\right)x^n=(1-x)^{-a}F\left(\begin{array}{c}a/2,(1+a)/2,A,\ldots\\C,\ldots\end{array}\mid -\frac{4px}{(1-x)^2}\right).$$

[47]

(9.84)
$$\sum_{n\geq0}\frac{(a)_n}{n!}F\left(\begin{array}{c}-n,A,\ldots\\1-a-n,C,\ldots\end{array}\mid p\right)x^n=(1-x)^{-a}F\left(\begin{array}{c}A,\ldots\\C,\ldots\end{array}\mid px\right).$$

[47]

(9.85)
$$[\delta(\delta+c-1)-px(\delta+a)(\delta-n)]F_n=0,$$

where $F_n\equiv\frac{(c)_n}{n!}\,{}_2F_1(a,-n;c;px)$.

[47]

(9.86)
$$nF_n-[2n+c-2-p(n+a-1)]F_{n-1}+(1-p)(n+c-2)F_{n-2}=0$$

where $F_n\equiv\frac{(c)_n}{n!}\,{}_2F_1(a,-n;c;p)$.

[47]

(9.87)
$$[\delta(\delta+c-1)(\delta+c'-1)-(\delta+a)(\delta+a')(\delta-n)]F_n=0,$$

where $F_n\equiv\frac{(c)_n}{n!}\,{}_3F_2(a,a',-n;c,c';x)$.

[47]

(9.88)
$$n(n+c'-1)F_n-[2(n-1)^2+(2c+2c'-a-a'-1)(n-1)+cc'-aa']F_{n-1}+(n+c-2)(n+c+c'-a-a'-2)F_{n-2}=0$$

where $F_n\equiv\frac{(c)_n}{n!}\,{}_3F_2(a,a',-n;c,c';1)$.

[174]

(9.89)
$$\left[\sum_{\nu=1}^{B}z^{\nu-1}(a_\nu z-b_\nu)\frac{d^\nu}{dz^\nu}+a_0+z^B(1-z)\frac{d^{B+1}}{dz^{B+1}}\right]{}_{B+1}F_B((a),(b),z)=0.$$

[163]

(9.90)
$${}_2F_2(-3/2,-1/2;-5/2,1;z)=\frac{5-4z}{5}e^{z/2}I_0(z/2)+\frac{4z}{5}e^{z/2}I_1(z/2).$$

[163]

$$ {}_2F_3(-1/2,1;1/4,1/2,3/4;z) = 1 + z^{1/4}\sqrt{2}\sqrt{\pi}e^{2\sqrt{z}}\operatorname{erf}(\sqrt{2}z^{1/4}) - z^{1/4}\sqrt{2}\sqrt{\pi}e^{-2\sqrt{z}}\operatorname{erfi}(\sqrt{2}z^{1/4}) - 2\sqrt{z}\pi\operatorname{erf}(\sqrt{2}z^{1/4})\operatorname{erfi}(\sqrt{2}z^{1/4}). $$

[163]

$$ {}_1F_2(3/2;5/2,5;z) = -\frac{432-24z+96z^2}{5z^3}I_0(2\sqrt{z}) + \frac{432+192z+48z^2}{5z^{7/2}}I_1(2\sqrt{z}) - \frac{48}{5z}\pi\left(I_0(2\sqrt{z})L_1(2\sqrt{z}) - I_1(2\sqrt{z})L_0(2\sqrt{z})\right). $$

[163]

$$ {}_3F_2(-1/2,1,2;3,4;z) = -\frac{480+3472z-2100z^2}{525z^3} + \frac{480+3712z-1024z^2+192z^3}{525z^3}\sqrt{1-z} - \frac{32}{5z^2}\log\left(\frac{1}{2}+\frac{\sqrt{1-z}}{2}\right). $$

[30]

$$ {}_2F_3(-\beta,\beta+\gamma;\gamma;\gamma/2,\frac{1+\gamma}{2};x^2/4) = {}_1F_1(-\beta;\gamma;-x){}_1F_1(-\beta;\gamma;x). \tag{9.91} $$

[30]

$$ {}_2F_3(1,n;n+1;(n+1)/2,2+n2;x^2/4) = {}_1F_1(1;n+1;-x){}_1F_1(1;n+1;x). \tag{9.92} $$

[30]

(9.93)

$$ {}_4F_1(-\alpha,-\beta,-\frac{\alpha+\beta}{2}-\frac{\alpha+\beta-1}{2};-\alpha-\beta;4x^2) = {}_2F_0(-\alpha,-\beta;x){}_2F_0(-\alpha,-\beta;-x). $$

if α or β a nonnegative integer.

[174]

(9.94)

$$ {}_{A+1}F_{B+1}\left(c,(a);d,(b);z\right) = \frac{\Gamma(d)}{\Gamma(c)\Gamma(d-c)}\int_0^1 t^{c-1}(1-t)^{d-c-1}\,{}_AF_B\left((a),(b),tz\right)dt. $$

[53, 64]

(9.95)

$$ {}_{q+1}F_q\left(\begin{array}{c} a,\frac{b}{q},\frac{b+1}{q},\cdots,\frac{b+q-1}{q} \\ \frac{c}{q},\frac{c+1}{q},\cdots,\frac{c+q-1}{q}\end{array}\mid x\right) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)}\int_0^1 t^{b-1}(1-t)^{c-b-1}(1-xt^q)^{-a}dt $$
$$ = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)}\sum_{k\ge 0}\binom{c-b-1}{k}\frac{(-)^k}{b+k}\,{}_2F_1(a,(b+k)/q;1+(b+k)/q;x) $$

[64]

(9.96)

$$ {}_{p+k}F_{q+k}\left(\begin{array}{c} a_1,\ldots,a_p,\frac{\alpha}{k},\frac{\alpha+1}{k},\ldots,\frac{\alpha+k-1}{k} \\ b_1,\ldots,b_q,\frac{\alpha+\beta}{k},\frac{\alpha+\beta+1}{k},\ldots,\frac{\alpha+\beta+k-1}{k}\end{array}\mid ct^k\right) = \frac{t^{1-\alpha-\beta}}{B(\alpha,\beta)}\int_0^t x^{\alpha-1}(t-x)^{\beta-1}\,{}_pF_q\left(\begin{array}{c} a_1,\ldots,a_p \\ b_1,\ldots,b_q\end{array}\mid cx^k\right)dx. $$

[141]
(9.97)

$$ {}_{r+2}F_{r+1}\left(\begin{matrix} a,b, & (f_r+1) \\ c, & (f_r) \end{matrix} \mid x \right) = (1-x)^{-a}\, {}_{r+2}F_{r+1}\left(\begin{matrix} a,\lambda, & (\xi_r+1) \\ c, & (\xi_r) \end{matrix} \mid \frac{x}{x-1} \right) $$

[141]
(9.98)

$$ {}_{r+2}F_{r+1}\left(\begin{matrix} a,b, & (f_r+1) \\ c, & (f_r) \end{matrix} \mid x \right) = (1-x)^{c-a-b-r}\, {}_{r+2}F_{r+1}\left(\begin{matrix} \lambda,\lambda', & (\xi_r+1) \\ c, & (\xi_r) \end{matrix} \mid x \right) $$

[4]

$$ {}_{p+1}F_p\left(\begin{matrix} a,a,\ldots,a \\ a+1,\ldots,a+1 \end{matrix} \mid 1 \right) = \frac{a^p}{(a-1)!}\sum_{k=0}^{a-1}(-)^{a-k-1}\zeta(p-k)\left[\begin{matrix} a \\ k+1 \end{matrix} \right] \tag{9.99} $$

for a a positive integer, where the [] is Stirling numbers of the first kind.
[148]
(9.100)

$$ {}_{p+1}F_p\left(a_1,\ldots,a_{p+1};b_1\ldots b_p \mid z\zeta\right) = (1-z)^{-a_1}\sum_{k\geq 0}\frac{(a_1)_k}{k!}\,{}_{p+1}F_p\left(-k,a_2,\ldots,a_{p+1};b_1,\ldots,b_p \mid \zeta\right)\left(\frac{z}{z-1}\right)^k. $$

[4]

$$ {}_{p+1}F_p\left(\begin{matrix} a,a,\ldots,a \\ a+1,\ldots,a+1 \end{matrix} \mid 1 \right) = \frac{(-)^{p-1}\pi a^p}{\sin(a\pi)(p-1)!}w(a,p-1) \tag{9.101} $$

where

$$ w(n,m) = \frac{1}{(n-1)!}\sum_{i=m+1}^{n}\left[\begin{matrix} n \\ i \end{matrix} \right](i-m)_m(-1)^{n-i}n^{i-m-1}, \tag{9.102} $$

recursively

$$ w(n,0)=1, \quad w(n,m) = \sum_{k=0}^{m-1}(1-m)_k H_{n-1}^{(k+1)}w(n,m-1-k) \tag{9.103} $$

and the Harmonic numbers defined in (0.88).
[78]

$$ \begin{aligned} {}_{q+1}F_q\left(\begin{matrix} a_1,\ldots,a_{q+1} \\ b_1,\ldots,b_q \end{matrix} \mid 1 \right) &+ {}_{q+1}F_q\left(\begin{matrix} a_1,\ldots,a_{q+1} \\ b_1,\ldots,b_q \end{matrix} \mid -1 \right) \\ &= 2\,{}_{2q+2}F_{2q+1}\left(\begin{matrix} a_1/2, a_1/2+1/2\ldots,a_{q+1}/2+1/2 \\ b_1/2, b_1/2+1/2,\ldots,b_q/2+1/2,1/2 \end{matrix} \mid 1 \right). \end{aligned} \tag{9.104} $$

[78]

$$ {}_{q+1}F_q\left(\begin{matrix} a_1,\ldots,a_{q+1} \\ b_1,\ldots,b_q \end{matrix} \mid 1 \right) - {}_{q+1}F_q\left(\begin{matrix} a_1,\ldots,a_{q+1} \\ b_1,\ldots,b_q \end{matrix} \mid -1 \right) \tag{9.105} $$

$$ = 2\frac{a_1a_2\cdots a_{q+1}}{b_1b_2\cdots b_q}\,{}_{q+1}F_q\left(\begin{matrix} a_1/2+1/2, a_1/2+1\ldots,a_{q+1}/2+1/2,a_{q+1}/2+1 \\ b_1/2+1/2,b_1/2+1,\ldots,b_q/2+1,3/2 \end{matrix} \mid 1 \right). $$

[78]

$$
{}_{q+1}F_q\left(\begin{matrix} a, a+2, a+4\ldots, a+2q \\ a+1, a+3, a+5\ldots a+2q-1\end{matrix} \mid 1\right) + {}_{q+1}F_q\left(\begin{matrix} a, a+2, a+4, \ldots, a+2q \\ a+1, a+3, a+5, a+2q-1\end{matrix} \mid -1\right) = {}_2F_1(a/2, a+q+1/2; 1/2; 1). \tag{9.106}
$$

[78]

$$
{}_{q+1}F_q\left(\begin{matrix} a, a+2, a+4\ldots, a+2q \\ a+1, a+3, a+5\ldots a+2q-1\end{matrix} \mid 1\right) - {}_{q+1}F_q\left(\begin{matrix} a, a+2, a+4, \ldots, a+2q \\ a+1, a+3, a+5, a+2q-1\end{matrix} \mid -1\right) = \frac{2a(a+2)(a+4)\cdots(a+2q)}{(a+1)(a+3)\cdots(a+2q-1)}\, {}_2F_1(a/2+1/2, a/2+q+1; 3/2; 1). \tag{9.107}
$$

[78]

$$
{}_{q+1}F_q\left(\begin{matrix} a, a+2, a+4\ldots, a+2q \\ a+1, a+3, a+5\ldots a+2q-1\end{matrix} \mid 1\right) = \frac{\sqrt{\pi}\Gamma(-a-q)}{\Gamma(1/2-a)\Gamma(-a/2-q)} + \frac{2(a/2)_q}{(a/2+1/2)_{q-1}\Gamma(1-a/2)\Gamma(1/2-a/2-q)}, \tag{9.108}
$$

if $\Re(a+q) < 0$, and a similar expression for argument -1.
[139]

$$
{}_3F_2(a, m, b; c, m-n; 1) = \Gamma(a+n-m+1) \times \left(\sum_{L=0}^{m-1} \frac{(-1)^L\Gamma(-b+c-a-n+L)\Gamma(1-b+L)}{\Gamma(m-L)\Gamma(-b-n+1+L)\Gamma(-b+c-m+1+L)\Gamma(L+1)}\right) \times \frac{\Gamma(c)\Gamma(m-n)(-1)^n}{\Gamma(c-a)\Gamma(a)}, \quad 0 < n < m. \tag{9.109}
$$

$$
{}_3F_2(a, b, -n; c, m-n; 1) = \Gamma(a+n-m+1) \times \left(\sum_{L=0}^{m-1} \frac{(-1)^L\Gamma(c-a+L)\Gamma(1+b-m+n+L)}{\Gamma(m-L)\Gamma(b-m+1+L)\Gamma(c+n-m+1+L)\Gamma(L+1)}\right) \times \frac{\Gamma(c)\Gamma(m-n)(-1)^n}{\Gamma(c-a)\Gamma(a)}, \quad 0 < n < m. \tag{9.110}
$$

$$
{}_3F_2(a, -n, b; c, a-c+b-n+m; 1) = \Gamma(c-b+n-m+1)\Gamma(-a+c-b-m+1) \times \left(\sum_{L=0}^{m-1} \frac{(-1)^L\Gamma(b+L)\Gamma(1-a+c-m+n+L)}{\Gamma(m-L)\Gamma(-a+c-m+1+L)\Gamma(c+n-m+1+L)\Gamma(L+1)}\right) \times \frac{\Gamma(c)\Gamma(m)}{\Gamma(n+1-a+c-b-m)\Gamma(b)\Gamma(c-b)}, \quad 0 < n, m. \tag{9.111}
$$

(9.112)
$$
{}_3F_2(a,1,b;n{+}1,c;1) = \frac{\Gamma(a-n)\Gamma(b-n)\Gamma(n+1)\Gamma(c)\Gamma(-b+c-a+n)\Gamma(1-b)\Gamma(1+n)\Gamma(c)}{\Gamma(b)\Gamma(c-b)\Gamma(a)\Gamma(c-a)\Gamma(a)} \times \left(\sum_{L=0}^{n-1} \frac{(-1)^L\Gamma(a-1-L)}{\Gamma(c-1-L)\Gamma(n-L)\Gamma(2+L-b)} \right), \quad 0 < n, m.
$$

and 60 others.
[64]

(9.113)
$$
{}_4F_3(a, b/3, (b+1)/3, (b+2)/3; c/3, (c+1)/3, (c+2)/3; 1) = \frac{\Gamma(c)\Gamma(c-b-a)}{\Gamma(c-a)\Gamma(c-b)} \sum_{k=0}^{\infty} \frac{(a)_k(-1)^k(b)_k}{k!(c-a)_k} \, {}_2F_1(-k, b+k; c-a+k; -1).
$$

[105]

(9.114)
$$
{}_4F_3\left(\frac{\alpha}{2}, \frac{\alpha+1}{2}, \beta+n, -n; 1+\alpha, \frac{\beta}{2}, \frac{\beta+1}{2}; 1\right) = \frac{(\beta-\alpha)_n}{(\beta)_n}.
$$

[105]

(9.115)
$$
{}_4F_3\left(\alpha, -\alpha, -\frac{m}{2}, \frac{1-m}{2}; \frac{1}{2}, \beta, 1-m-\beta; 1\right) = \frac{(\alpha+\beta)_m + (\beta-\alpha)_m}{2(\beta)_m}.
$$

[105]
(9.116)
$$
{}_4F_3\left(a, b, \frac{1}{2}-a-b-n, -n; a+b-\frac{1}{2}, 1-a-n, 1-b-n; 1\right) = \frac{(2a)_n(2b)_n(a+b)_n}{(2a+2b-1)_n(a)_n(b)_n}.
$$

[105]
(9.117)
$$
{}_4F_3\left(a-\frac{1}{2}, b-\frac{1}{2}, \frac{1}{2}-a-b-n, -n; a+b-\frac{1}{2}, \frac{1}{2}-a-n, \frac{1}{2}-b-n; 1\right) = \frac{(2a)_n(2b)_n(a+b)_n}{(2a+2b-1)_n(a+\frac{1}{2})_n(b+\frac{1}{2})_n}.
$$

[105]
(9.118)
$$
{}_4F_3\left(a+1, b, \frac{1}{2}-a-b-n, -n; a+b+\frac{1}{2}, 1-a-n, 1-b-n; 1\right) = \frac{(2a+1)_n(2b)_n(a+b)_n}{(2a+2b)_n(a)_n(b)_n}.
$$

[105]
(9.119)
$$
{}_4F_3\left(a+\frac{1}{2}, b-\frac{1}{2}, \frac{1}{2}-a-b-n, -n; a+b+\frac{1}{2}, \frac{1}{2}-a-n, \frac{1}{2}-b-n; 1\right) = \frac{(2a+1)_n(2b)_n(a+b)_n}{(2a+2b)_n(a+\frac{1}{2})_n(b+\frac{1}{2})_n}.
$$

[105]
(9.120)
$$
{}_4F_3\left(a, b, \frac{1}{2}-a-b-n, -n; a+b+\frac{1}{2}, 1-a-n, 1-b-n; 1\right) = \frac{(2a)_n(2b)_n(a+b)_n}{(2a+2b)_n(a)_n(b)_n}.
$$

[105]

(9.121)
$$ {}_4F_3(a,b,a+b-\frac{1}{2}-n,-n;a+b+\frac{1}{2},\frac{1}{2}+a-n,\frac{1}{2}+b-n;1)=\frac{(1/2)_n(a-b+\frac{1}{2})_n(b-a+\frac{1}{2})_n}{(a+b+\frac{1}{2})_n(\frac{1}{2}-a)_n(\frac{1}{2}-b)_n}. $$

[105]

(9.122)
$$ {}_4F_3(a,-a,\frac{1}{2}-n,-n;\frac{1}{2},\frac{1}{2}+b-n,\frac{1}{2}-b-n;1)=\frac{(\frac{1}{2}+a+b)_n(\frac{1}{2}-a-b)_n+(\frac{1}{2}+a-b)_n(\frac{1}{2}-a+b)_n}{2(\frac{1}{2}+b)_n(\frac{1}{2}-b)_n}. $$

[105]

(9.123)
$$ {}_4F_3(a,-a,-\frac{1}{2}-n,-n;\frac{1}{2},b-n,-b-n;1)=\frac{(a+b)_{n+1}(1-a-b)_n+(b-a)_{n+1}(1+a-b)_n}{2b(1+b)_n(1-b)_n}. $$

[105]

(9.124)
$$ {}_4F_3(\frac{1}{2}+a,\frac{1}{2}-a,-\frac{1}{2}-n,-n;\frac{1}{2},\frac{1}{2}+b-n,\frac{1}{2}-b-n;1)=\frac{(a+b)_{n+1}(1-a-b)_n+(b-a)_{n+1}(1+a-b)_n}{2b(\frac{1}{2}+b)_n(\frac{1}{2}-b)_n}. $$

[105]

$$ (9.125)\quad {}_4F_3(a,-a,-\frac{1}{2}-n,-n;\frac{1}{2},\frac{1}{2}+b-n,\frac{1}{2}-b-n;1) = \frac{(a+b)(\frac{1}{2}+a+b)_n(\frac{1}{2}-a-b)_n+(b-a)(\frac{1}{2}+b-a)_n(\frac{1}{2}+a-b)_n}{2b(\frac{1}{2}+b)_n(\frac{1}{2}-b)_n}. $$

[105]

(9.126)
$$ {}_4F_3(a,-a,\frac{1}{2}-n,-n;\frac{1}{2},1+b-n,1-b-n;1)=\frac{(a+b)_n(-a-b)_n+(a-b)_n(b-a)_n}{2(b)_n(-b)_n}. $$

[30]

(9.127)
$$ {}_4F_3(\alpha,\beta,\frac{\alpha+\beta}{2},\frac{\gamma+\delta}{2};\gamma,\delta,\alpha+\beta;x)={}_2F_1(\alpha,\beta;\gamma;\frac{1-\sqrt{1-x}}{2})\,{}_2F_1(\alpha,\beta;\delta;\frac{1-\sqrt{1-x}}{2}) $$
if $\alpha+\beta+1=\gamma+\delta$.

[45]

(9.128)
$$ \frac{1}{\Gamma(b_1)\Gamma(b_2)\Gamma(b_3)}\,{}_4F_3\left(\begin{array}{c} a_1,a_2,a_3,a_4 \\ b_1,b_2,b_3 \end{array} \mid 1\right)=\frac{\Gamma(s)}{\Gamma(a_1+s)\Gamma(a_2+s)\Gamma(a_3)\Gamma(a_4)} $$
$$ \times\sum_{k=0}^{\infty}\frac{(b_1+b_3-a_3-a_4)_k(b_2+b_3-a_3-a_4)_k(s)_k}{(a_1+s)_k(a_2+s)_k k!}\,{}_3F_2\left(\begin{array}{c} b_3-a_3,b_3-a_4,-k \\ b_1+b_3-a_3-a_4,b_2+b_3-a_3-a_4 \end{array} \mid 1\right). $$

[78]

$$ (9.129)\quad {}_4F_3(a/2,a/2+1/2,b/2,b/2+1/2;1/2+a/2-b/2,1+a/2-b/2,1/2;1) = \frac{\Gamma(1+a-b)\Gamma(1-2b)}{\Gamma(1-b)\Gamma(1+a-2b)}+\frac{\Gamma(1+a-b)\Gamma(1+a/2)}{\Gamma(1+a)\Gamma(1+a/2-b)}. $$

where $\Re b<1/2$.

[78]

(9.130)
$$\frac{2ab}{1+a-b}\,{}_4F_3(a/2+1/2,a/2+1,b/2+1/2,b/2+1;1+a/2-b/2,3/2+a/2-b/2,3/2;1)$$
$$=\frac{\Gamma(1+a-b)\Gamma(1-2b)}{\Gamma(1-b)\Gamma(1+a-2b)}+\frac{\Gamma(1+a-b)\Gamma(1+a/2)}{\Gamma(1+a)\Gamma(1+a/2-b)}.$$

where $\Re b < 1/2$.
[45]

$$\frac{1}{\prod_{j=1}^{p}\Gamma(b_j)}\,{}_{p+2}F_{p+1}\left(\begin{array}{c} a_1,a_2,\ldots,a_{p+1},-m \\ b_1,b_2,\ldots,b_p,1-s-m\end{array}\mid 1\right) \tag{9.131}$$
$$=\frac{(a_1+s)_m(a_2+s)_m\prod_{j=3}^{p+1}(a_j)_m}{(s)_m\prod_{j=1}^{p}\Gamma(b_j+m)}\sum_{k=0}^{m}\frac{(s)_k(-m)_k}{(a_1+s)_k(a_2+s)_k k!}$$
$$\times (S)_k\,{}_{p+1}F_p\left(\begin{array}{c} 1-b_1-m,1-b_2-m,\ldots,1-b_p-m,-k \\ 1-a_3-m,1-a_4-m,\ldots 1-a_{p+1}-m,1-S-k\end{array}\mid 1\right)$$

where $s=\sum_{j=1}^{p}b_j-\sum_{j=1}^{p+1}a_j$ and $S=a_1+a+2+s+m-1$ and s not a negative integer or zero.
[45]
(9.132)
$$\frac{1}{\Gamma(b_1)\Gamma(b_2)}\,{}_4F_3\left(\begin{array}{c} a_1,a_2,a_3,-m \\ b_1,b_2,1-s-m\end{array}\mid 1\right)=\frac{(a_1+s)_m(a_2+s)_m(a_3)_m}{(s)_m\Gamma(b_1+m)\Gamma(b_2+m)}\,{}_4F_3\left(\begin{array}{c} b_1-a_3,b_2-a_3,s,-m \\ a_1+s,a_2+s,1-a_3-m\end{array}\mid 1\right)$$

where $s=b_1+b_2-a_1-a_2-a_3$ not a negative integer or zero, $m=0,1,2\ldots$.
[30]
(9.133)
$${}_0F_2(m+n+1,n+1;x)\,{}_0F_2(m+1,1-n;-x)=1+\sum_{k\geq 1}\frac{\alpha_k(\frac{2m+n+k+2}{2})_k(2x)^k}{(m+n+1)_k(m+1)_k k!}$$

where

$$\alpha_k=\begin{cases}\frac{n}{(n^2-1^2)(n^2-3^2)\cdots(n^2-k^2)}, & \text{if} k \text{odd};\\ \frac{1}{(n^2-2^2)(n^2-4^2)\cdots(n^2-k^2)}, & \text{if} k \text{even}\end{cases} \tag{9.134}$$

[54]
(9.135)
$$\sum_{k\geq 0}t^k\frac{(\alpha+1+sk)_k}{k!}\,{}_rF_r\left(\begin{array}{c}\Delta(-k,r)\\ \Delta(-\alpha-sk-k,r)\end{array}\mid x(\beta+sk)\right)=(1-z)^{\alpha+1}\frac{1}{1+sz+rsy}e^{-\beta y},$$

where $t=(-z)(1-z)^{-s-1}\exp(sy)$, $x=(-y)[1-1/z]^r$, $(a)_k=\Gamma(a+k)/\Gamma(a)$ and $\Delta(-k,r)\equiv -k/r,(-k+1)/r,\ldots(-k+r-1)/r$, $|t|<1$.

[54]

(9.136)
$$\sum_{k\ge 0} t^k \frac{(\alpha+1+sk)_k}{(\alpha+l+1+sk)k!((\alpha+l+1+s''+sk)/s'')_{l'}}\, {}_rF_r\left(\begin{array}{c}\Delta(-k,r)\\ \Delta(-\alpha=sk-k,r)\end{array}\mid x(\beta+sk)\right)$$
$$=\frac{(1-z)^{\alpha+l+1}e^{-\beta y}}{l'!}\sum_{q=0}^{l'}\sum_{k=0}^{l+s''q}\frac{(-z)^k(1-z)^{s''q-k}(-l')_q(-l-s''q)_k}{k!q!(\alpha+l+1+s''q+sk)}$$
$$\times\ {}_1F_1\left(\begin{array}{c}1\\ (\alpha+l+1+s'+s''r+sk)/s'\end{array}\mid y(\beta-\alpha-l-1-s''q)\right),$$

where $t=(-z)(1-z)^{-s-1}e^{sy}$, $x=-y[1-1/z]^r$ and $|t|<1$.

[54]

$$\sum_{p\ge 0} t^p\frac{(\beta+s'p)^p}{p!}\, {}_rF_0\left(\begin{array}{c}\Delta(-p,r)\\ -\end{array}\mid \frac{xr^r(\alpha+s'p)}{(\beta+s'p)^r}\right)=\frac{e^{-\beta y-\alpha z}}{1+s'y+rs'z}, \tag{9.137}$$

where $t=-ye^{s'y+s'z}$, $x=-z/y^r$ and $|s'y\exp(s'y+s'z+1)|<1$.

[54]

(9.138)
$$\sum_{p\ge 0} t^p\frac{(\beta+s'p)^p}{p!(\alpha+s'p)}\, {}_rF_0\left(\begin{array}{c}\Delta(-p,r)\\ -\end{array}\mid \frac{xr^r(\alpha+s'p)}{(\beta+s'p)^r}\right)=\frac{e^{-\beta y-\alpha z}}{\alpha}\, {}_1F_1\left(\begin{array}{c}1\\ \alpha/s'+1\end{array}\mid y(\beta-\alpha)\right)$$

where $t=-ye^{s'y+s'z}$, $x=-z/y^r$ and $|s'y\exp(s'y+s'z+1)|<1$.

[170]

$$F_1\left(\begin{array}{c}a;b,b\\ 1+a-b\end{array}\mid e^{2\pi i/3},e^{-2\pi i/3}\right)=\frac{\Gamma(1+a-b)\Gamma(1+a/3)}{\Gamma(1+a)\Gamma(1+a/3-b)}. \tag{9.139}$$

[170]

$$F_D\left(\begin{array}{c}a;b,b,\dots b\\ c\end{array}\mid \omega_{1,n},\dots\omega_{n-1,n}\right)=\frac{\Gamma(a-b+1)\Gamma(1+a/n)}{\Gamma(1+a)\Gamma(1+a/n-b)}, \tag{9.140}$$

where $\omega_{k,n}\equiv e^{2k\pi i/n}$.

[170]

(9.141)
$$F_D\left(\begin{array}{c}2mb-a;b,b,\dots b\\ 2mb\end{array}\mid x_1,\dots x_{2m}\right)=\frac{1}{2m}\frac{\Gamma(a/(2m))\Gamma(2mb)\Gamma((2mb-a)/(2m))}{\Gamma(a)\Gamma(b)\Gamma(2mb-a)},$$

where $x_k=1+e^{(2k-1)\pi i/(2m)}$, for $k=1,\dots 2m$, and $2m$ is an even integer, $a>0$, $b>0$, $nb>a$.

[170]

$$F_D\left(\begin{array}{c}(2m-1)b-a;b,b,\dots b\\ (2m-1)b\end{array}\mid y_1,\dots y_{2m-1}\right) \tag{9.142}$$
$$=\frac{1}{2m-1}\frac{\Gamma(a/(2m-1))\Gamma((2m-1)b)\Gamma(((2m-1)b-a)/(2m-1))}{\Gamma(a)\Gamma(b)\Gamma((2m-1)b-a)},$$

where $y_k=1+e^{(2k-1)\pi i/(2m-1)}$, and $2m-1$ is an odd integer, $a>0$, $b>0$, $nb>a$.

[18, p 20]

$$\alpha F_4(\alpha+1,\beta;\gamma,\gamma';x,y) - \beta F_4(\alpha,\beta+1;\gamma,\gamma';x,y) = (\alpha-\beta)F_4(\alpha,\beta;\gamma,\gamma';x,y). \tag{9.143}$$

[18, p 21]

$$\frac{\beta}{\gamma} x F_4(\alpha+1,\beta+1;\gamma+1,\gamma';x,y) + \frac{\beta}{\gamma'} y F_4(\alpha+1,\beta+1;\gamma,\gamma'+1;x,y) = F_4(\alpha+1,\beta;\gamma,\gamma';x,y) - F_4(\alpha,\beta;\gamma,\gamma';x,y). \tag{9.144}$$

[18, p 26]

$$\begin{aligned} F_4(\alpha,\beta;\gamma,\gamma';x,y) &= \sum \frac{(\alpha)_m(\beta)_m}{(\gamma)_m m!} \frac{\Gamma(\gamma')\Gamma(\beta-\alpha)}{\Gamma(\gamma'-\alpha-m)\Gamma(\beta+m)}(-y)^{-\alpha-m} \\ &\quad \times {}_2F_1(\alpha+m,\alpha+m+1-\gamma';\alpha+1-\beta;\frac{1}{y})x^m \\ &\quad + \sum \frac{(\alpha)_m(\beta)_m}{(\gamma)_m m!} \frac{\Gamma(\gamma')\Gamma(\alpha-\beta)}{\Gamma(\gamma'-\beta-m)\Gamma(\alpha+m)}(-y)^{-\beta-m} \\ &\quad \times {}_2F_1(\beta+m,\beta+m+1-\gamma';\beta+1-\alpha;\frac{1}{y})x^m. \end{aligned} \tag{9.145}$$

[18]

$$\begin{aligned} F_4(\alpha,\beta;\gamma,\gamma';x,y) &= \frac{\Gamma(\gamma')\Gamma(\beta-\alpha)}{\Gamma(\gamma'-\alpha)\Gamma(\beta)}(-y)^{-\alpha} F_4(\alpha,\alpha+1-\gamma';\gamma,\alpha+1-\beta;\frac{x}{y},\frac{1}{y}) \\ &\quad + \frac{\Gamma(\gamma')\Gamma(\alpha-\beta)}{\Gamma(\gamma'-\beta)\Gamma(\alpha)}(-y)^{-\beta} F_4(\beta+1-\gamma',\beta;\gamma,\beta+1-\alpha;\frac{x}{y},\frac{1}{y}). \end{aligned} \tag{9.146}$$

[27, 85]

$$F_4\left[\alpha,\beta;\gamma,1+\alpha+\beta-\gamma;x(1-y),y(1-x)\right] = F(\alpha,\beta;\gamma;x)F(\alpha,\beta;1+\alpha+\beta-\gamma;y). \tag{9.147}$$

$$F_4\left[\alpha,\beta;\gamma,\beta;-\frac{x}{(1-x)(1-y)},-\frac{y}{(1-x)(1-y)}\right] = (1-x)^\alpha(1-y)^\alpha F_1[\alpha;\gamma-\beta,1+\alpha-\gamma;\gamma;x,xy]. \tag{9.148}$$

$$F_4\left[\alpha,\beta;\alpha,\beta;-\frac{x}{(1-x)(1-y)},-\frac{y}{(1-x)(1-y)}\right] = (1-xy)^{-1}(1-x)^\beta(1-y)^\alpha. \tag{9.149}$$

$$F_4\left[\alpha,\beta;\beta,\beta;-\frac{x}{(1-x)(1-y)},-\frac{y}{(1-x)(1-y)}\right] = (1-x)^\alpha(1-y)^\alpha F[\alpha,1+\alpha-\beta;\beta;xy]. \tag{9.150}$$

$$F_4\left[\alpha,\beta;1+\alpha-\beta,\beta;-\frac{x}{(1-x)(1-y)},-\frac{y}{(1-x)(1-y)}\right] = (1-y)^\alpha F[\alpha,,\beta;1+\alpha-\beta;-\frac{x(1-y)}{1-x}]. \tag{9.151}$$

[105]

$$F^{p:2;2}_{q:1;1}\left[\begin{array}{lcc} \alpha_1,\dots\alpha_p: & a_1,b_1 & a_2,b_2 \\ \gamma_1,\dots\gamma_q: & c_1 & c_2 \end{array} \mid X,X\right] ={}_{p+3} F_{q+2}(\alpha_1,\dots,\alpha_p,\beta_1,\beta_2,\beta_3;\gamma_1,\dots,\gamma_q,\delta_1,\delta_2;X) \tag{9.152}$$

a_1	b_1	c_1	a_2	b_2	c_2	β_1	β_2	β_3	δ_1	δ_2
a	b	$a+b-\frac{1}{2}$	a	b	$a+b+\frac{1}{2}$	$2a$	$2b$	$a+b$	$2a+2b-1$	$a+b+\frac{1}{2}$
$a-\frac{1}{2}$	$b-\frac{1}{2}$	$a+b-\frac{1}{2}$	$a+\frac{1}{2}$	$b+\frac{1}{2}$	$a+b+\frac{1}{2}$	$2a$	$2b$	$a+b$	$2a+2b-1$	$a+b+\frac{1}{2}$

and more of that format.

[183]

(9.153)
$$F^{1:2;2}_{1:1;1}\left[\begin{array}{ccc} \alpha: & -M,\beta: & -N,\beta': \\ \beta+\beta': & \alpha-\beta'-M+1 & \alpha-\beta-N+1 \end{array} \mid 1,1\right] = \frac{(\beta+\beta'-\alpha)_{M+N}(\beta')_M(\beta)_N}{(\beta+\beta')_{M+N}(\beta'-\alpha)_M(\beta-\alpha)_N}$$
for $M,N=0,1,2,\ldots$.

[183]

$$F^{1:2;2}_{1:1;1}\left[\begin{array}{ccc} \beta+\beta'-\alpha: & \beta,\gamma: & \beta',\gamma': \\ \beta+\beta': & \delta & \delta' \end{array} \mid 1,1\right] \tag{9.154}$$
$$= \frac{\Gamma(\delta)\Gamma(\delta')\Gamma(\alpha-\beta-\gamma+\delta)\Gamma(\alpha-\beta'-\gamma'+\delta')}{\Gamma(\delta-\gamma)\Gamma(\delta'-\gamma')\Gamma(\alpha-\beta+\delta)\Gamma(\alpha-\beta'+\delta')} F^{1:2;2}_{1:1;1}\left[\begin{array}{ccc} \alpha: & \beta',\gamma: & \beta,\gamma': \\ \beta+\beta': & \alpha-\beta+\delta & \alpha-\beta'+\delta' \end{array} \mid 1,1\right].$$

[183]

(9.155)
$$F^{1:r;u}_{1:s;v}\left[\begin{array}{ccc} \alpha: & (a_r): & (c_u): \\ \gamma: & (b_s): & (d_v); \end{array} \mid x,y\right] = \sum_{n=0}^{\infty} \frac{(\alpha)_n(\gamma-\alpha)_n\prod_{j=1}^r(a_j)_n\prod_{j=1}^u(c_j)_n}{(\gamma+n-1)_n(\gamma)_{2n}\prod_{j=1}^s(b_j)_n\prod_{l=1}^v(d_l)_n}\frac{(xy)^n}{n!}$$
$$\times\ {}_{r+1}F_{s+1}\left(\begin{array}{cc} (a_r)+n, & \alpha+n \\ (b_s)+n, & \gamma+2n \end{array} \mid x\right) {}_{u+1}F_{v+1}\left(\begin{array}{cc} (c_u)+n, & \alpha+n; \\ (d_v)+n, & \gamma+2n; \end{array} \mid y\right).$$

[85]

$${}_3F_2\left(\begin{array}{c} -n,n+a,b \\ c,d \end{array} \mid 1\right) {}_3F_2\left(\begin{array}{c} -n,n+a,e \\ c,f \end{array} \mid 1\right) \tag{9.156}$$
$$= \frac{(-)^n(a-c+1)_n}{(c)_n} F^{2:2;2}_{2:1;1}\left[\begin{array}{ccc} -n,n+a: & b,e; & d-b,f-e \\ d,f: & c; & a-c+1 \end{array} \mid 1,1\right].$$

[85]

$$F^{2:2;2}_{2:1;1}\left[\begin{array}{ccc} a,b: & -x,y+e; & -y,x+d \\ d,e: & c; & b \end{array} \mid 1,1\right] \tag{9.157}$$
$$= \frac{(d-a)_x(e-a)_y}{(d)_x(e)_y} F^{2:2;1}_{2:1;0}\left[\begin{array}{ccc} a,-x: & 1+a-c,-y; & c-b \\ c,1+a-d-x: & 1+a-e-y; & - \end{array} \mid 1,1\right]$$
if $x,y=0,1,\ldots$

[85]

(9.158)
$$F^{2:2;2}_{2:1;1}\left[\begin{array}{ccc} a,b: & -x,y+e; & -y,x+d \\ d,e: & c; & c' \end{array} \mid 1,1\right] = \sum_{r=0}^{\min(x,y)} \frac{(a)_r(b)_r(a+b-c-c'+1)_r}{r!(c)_r(c')_r}\frac{(-x)_r(-y)_r}{(d)_r(e)_r}$$
$$\times\ {}_3F_2\left(\begin{array}{c} r+a,r+b,r-x \\ r+c,r+c \end{array} \mid 1\right) {}_3F_2\left(\begin{array}{c} r+a,r+b,r-y \\ r+c',r+e \end{array} \mid 1\right)$$
if $x,y=0,1,\ldots$

9.2. **The Confluent Hypergeometric Function.**

9.3. The Meijer G-Function. [2]

(9.159) $$G^{m+1,n+1}_{p+1,q+2}\left(z \mid \begin{array}{c} a, a_p \\ a, b_q, b \end{array}\right) = (-)^{a-b} G^{m+1,n}_{p,q+1}\left(z \mid \begin{array}{c} a_p \\ b_q, b \end{array}\right)$$
$$- (-1)^{a-b} \sum_{k=1}^{a-b} \operatorname{Res}_{s=k-a}\left[\frac{\Gamma(b+s)\prod_{i=1}^{m}\Gamma(b_i+s)\prod_{i=1}^{n}\Gamma(1-a_i-s)}{\prod_{i=n+1}^{p}\Gamma(a_i+s)\prod_{i=m+1,q}\Gamma(1-b_i-s)} z^{-s}\right]$$

if $a-b>0$.

[2]

(9.160) $$G^{m+1,n+1}_{p+1,q+2}\left(z \mid \begin{array}{c} a, a_p \\ a, b_q, b \end{array}\right) = (-)^{a-b} G^{m+1,n}_{p,q+1}\left(z \mid \begin{array}{c} a_p \\ b_q, b \end{array}\right)$$

if $a-b \le 0$.

[2]

(9.161) $$G^{m,n+1}_{p+2,q+1}\left(z \mid \begin{array}{c} a, a_p, b \\ b_q, b \end{array}\right) = (-)^{a-b} G^{m,n}_{p+1,q}\left(z \mid \begin{array}{c} a_p, a \\ b_q \end{array}\right)$$

if $a-b$ is an integer.

9.4. The MacRobert E-function.

9.5. Riemann and Hurwitz zeta functions. [188, (2.4.1)]

(9.162) $$\zeta(s) = \frac{1}{\Gamma(s)}\int_0^\infty \frac{x^{s-1}}{e^x-1}dx, \quad \sigma > 1.$$

[184, 5]

(9.163) $$\zeta(3) = \frac{5}{2}\sum_{k\ge 1}\frac{(-)^{k-1}}{k^3\binom{2k}{k}} = -\frac{4\pi^2}{7}\sum_{k\ge 0}\frac{\zeta(2k)}{(2k+1)(2k+2)2^{2k}}.$$

[153]

(9.164) $$\zeta(4) = \sum_{n=1}^{\infty}\frac{1}{n!^4}\left(\frac{4n+1}{2n^2}L_n + \frac{7n^3}{4}L_{n-1}\right)$$

where $L_0 = 0$, $L_1 = 1/3$ and

(9.165) $$4(4n+3)(4n+5)L_{n+1}+2(n+1)^3(6n^3+9n^2+5n+1)L_n-n^6(n+1)^3L_{n-1}=0.$$

[142]

(9.166) $$\zeta'(2k) = (-)^{k+1}\frac{(2\pi)^{2k}}{2(2k)!}\{2k\zeta'(1-2k) - [\psi(2k)-\log(2\pi)]B_{2k}\}.$$

[142]

(9.167)
$$\zeta'(1-2k, p/q) = \frac{[\psi(2k)-\log(2\pi q)]B_{2k}(p/q)}{2k} - \frac{[\psi(2k)-\log(2\pi)]B_{2k}(p/q)}{q^{2k}2k}$$
$$+ (-)^{k+1}\frac{\pi}{(2\pi q)^{2k}}\sum_{n=1}^{q-1}\sin\left(\frac{2\pi pn}{q}\right)\psi^{(2k-1)}(n/q)$$
$$+ (-)^{k+1}\frac{2(2k-1)!}{(2\pi q)^{2k}}\sum_{n=1}^{q-1}\cos\left(\frac{2\pi pn}{q}\right)\zeta'(2k, n/q) + \frac{\zeta'(1-2k)}{q^{2k}}.$$

[142]

$$\zeta'(1-2k,1/2) = -\frac{B_{2k}\log 2}{k4^k} - \frac{(2^{2k-1}-1)\zeta'(1-2k)}{2^{2k-1}}. \tag{9.168}$$

[142]

$$\zeta'(1-2k,1/3) = -\frac{(9^k-1)B_{2k}\pi}{\sqrt{3}(3^{2k-1}-1)8k} - \frac{B_{2k}\log 3}{3^{2k-1}4k} + (-)^k\frac{\psi^{(2k-1)}(1/3)}{2\sqrt{3}(6\pi)^{2k-1}} - \frac{(3^{2k-1}-1)\zeta'(1-2k)}{2\times 3^{2k-1}}. \tag{9.169}$$

[153]

$$\sum_{k=1}^{\infty}\frac{k}{k^4-x^2k^2-y^4} = \sum_{n,m=0}^{\infty}\binom{n+m}{n}\zeta(2n+4m+3)x^{2n}y^{4m} = \frac{1}{2}\sum_{n=1}^{\infty}\frac{(-)^{n-1}r(n)}{n\binom{2n}{n}}\frac{\prod_{m=1}^{n-1}((m^2-x^2)^2+4y^4}{\prod_{m=n}^{2n}(m^4-x^2m^2-y^4)} \tag{9.170}$$

where

$$r(n) = 205n^6-160n^5+(32-62x^2)n^4+40x^2n^3+(x^4-8x^2-25y^4)n^2+10y^4n+y^4(x^2-2). \tag{9.171}$$

[5]

$$\sum_{k\ge 1}\frac{(-)^k}{k}[\zeta(nk)-1] = \log\left(\prod_{j=0}^{n-1}\Gamma[2-(-)^{(2j+1)/n}]\right). \tag{9.172}$$

[5]

$$\sum_{k=2}^{\infty}(-)^k[\zeta(k)-1]k^n = -1+\frac{1-2^{n+1}}{n+1}B_{n+1} - \sum_{k=1}^{n}(-)^k k!\zeta(k+1)S(n+1,k+1), \tag{9.173}$$

with S the Stirling numbers of the second kind.

[5]

$$\sum_{k=2}^{\infty}[\zeta(k)-1]k^n = 1+\sum_{k=1}^{n}k!\zeta(k+1)S(n+1,k+1), \tag{9.174}$$

[6]

$$-\sum_{k=1}^{\infty}\frac{k}{4^{2k}}\zeta(2k+1) = G-1. \tag{9.175}$$

[184]

$$\sum_{k\ge 0}\frac{\zeta(2k)}{(2k+1)4^{2k}} = -\frac{G}{\pi} - \frac{1}{4}\log 2. \tag{9.176}$$

[184]

$$\sum_{k\ge 0}\frac{\zeta(2k)}{2k+1}\left(\frac{3}{4}\right)^{2k} = \frac{G}{3\pi} - \frac{1}{4}\log 2. \tag{9.177}$$

[6]

$$\frac{1}{16}\sum_{k=1}^{\infty}\frac{3^k-1}{4^k}(k+1)\zeta(k+2)=G. \tag{9.178}$$

[5]

$$\sum_{k=1}^{\infty}\frac{\zeta(2k+1)-1}{k+2}=\frac{2}{3}-\frac{\gamma}{2}+\log 2+6\zeta'(-1). \tag{9.179}$$

[5]

$$\sum_{k=2}^{\infty}\frac{\zeta(k)}{(k+1)(k+2)}=\frac{1-\gamma}{6}-2\zeta'(-1). \tag{9.180}$$

[5]

$$\sum_{k=2}^{\infty}\frac{k^2}{k+1}[\zeta(k)-1]=\frac{3}{2}-\frac{\gamma}{2}+\frac{\pi^2}{6}-\frac{1}{2}\log(2\pi). \tag{9.181}$$

[5]

$$\sum_{k=2}^{\infty}[\zeta(4k)-1]=\frac{7}{8}-\frac{\pi}{4}\coth\pi \tag{9.182}$$

[142]

$$4\sum_{k=0}^{\infty}\frac{1-\zeta(2k)}{(2k+1)3^{2k}}=\log(192)-\pi\frac{2\sqrt{3}}{9}+\frac{\sqrt{3}}{3\pi}\psi^{(1)}(1/3). \tag{9.183}$$

[5]

$$\sum_{k=1}^{\infty}[\zeta(4k)-1]z^{4k}=\frac{3z^4-1}{2(z^4-1)}-\frac{\pi z}{4}[\cot(\pi z)+\coth(\pi z)],\quad |z|<2. \tag{9.184}$$

[5]

(9.185)

$$\sum_{k=1}^{\infty}[\zeta(2k)-1]\sin k=-\frac{1}{2}\cot(1/2)+\frac{\pi}{2}\frac{\sin(1/2)\sin[2\pi\cos(1/2)]-\cos(1/2)\sinh[2\pi\sin(1/2)]}{\cos[2\pi\cos(1/2)]-\cosh[2\pi\sin(1/2)]}$$

[5]

$$\sum_{k=1}^{\infty}\binom{p+k}{k}\zeta(p+k+1,a)z^k=\frac{(-)^p}{p!}[\psi^{(p)}(a)-\psi^{(p)}(a-z)]. \tag{9.186}$$

[5]

$$\sum_{k=1}^{\infty}\frac{t^k}{k^2}\zeta(2k)=\log[\pi\sqrt{t}\csc(\pi\sqrt{t}]. \tag{9.187}$$

[6]

$$\frac{1}{8}\sum_{k=2}^{\infty}\frac{k}{2^k}\zeta(k+1,3/4)=G. \tag{9.188}$$

[6]

$$-\frac{1}{8}\sum_{k=2}^{\infty}\frac{k}{2^k}\zeta(k+1,5/4)=G-1. \tag{9.189}$$

[3]

(9.190)

$$\zeta'(1,p/q)-\zeta'(1,1-p/q)=\pi\cot\frac{\pi p}{q}[\log(2\pi q)+\gamma]-2\pi\sum_{j=1}^{q-1}\log\left(\Gamma(j/q)\right)\sin\frac{2\pi jp}{q}.$$

[75]

$$\int_0^1 \sin(2\pi q)\zeta(z,q)dq=\frac{(2\pi)^z}{4\Gamma(z)}\csc\frac{z\pi}{2}. \tag{9.191}$$

[75]

$$\int_0^1 \sin(2k\pi q)\zeta(z,q)dq=\frac{(2\pi)^z k^{z-1}}{4\Gamma(z)}\csc\frac{z\pi}{2}. \tag{9.192}$$

[75]

$$\int_0^1 \cos(2k\pi q)\zeta(z,q)dq=\frac{(2\pi)^z k^{z-1}}{4\Gamma(z)}\sec\frac{z\pi}{2}. \tag{9.193}$$

[75]

$$\int_0^1 \zeta(z',q)\zeta(z,q)dq=-\zeta(z+z'-1)B(1-z,1-z')\frac{\cos\frac{\pi(z-z')}{2}}{\cos\frac{\pi(z+z')}{2}}. \tag{9.194}$$

[75]

$$\int_0^1 q^n\zeta(z,q)dq=-n!\sum_{j=1}^{n}\frac{\zeta(z-j)}{(z-j)_j(n-j+1)!}. \tag{9.195}$$

[75]

$$\int_0^1 \ln(\sin\pi q)\zeta(z,q)dq=-\frac{\Gamma(1-z)}{(2\pi)^{1-z}}\sin\frac{\pi z}{2}\zeta(2-z). \tag{9.196}$$

[95]

$$\Phi(z,s,u)=2^{-s}\left[\Phi(z^2,s,u/2)+z\Phi(z^2,s,(u+1)/2)\right]. \tag{9.197}$$

[57]

$$\zeta(r,s)\equiv\sum_{m<n}\frac{1}{n^s m^r}. \tag{9.198}$$

[57]

$$\zeta(r,s)+\zeta(s,r)=\zeta(r)\zeta(s)-\zeta(r+s). \tag{9.199}$$

[57]

$$\zeta(r,s)=-\frac{1}{2}\zeta(r+s)+\sum_{j=1,j\mathrm{odd}}\left(\binom{j-1}{s-1}+\binom{j-1}{r-1}\right)\zeta(j)\zeta(r+s-j). \tag{9.200}$$

for r even and s odd.

[57]

$$\zeta(1,s)=\frac{s}{2}\zeta(s+1)-\frac{1}{2}\sum_{j=1}^{s-2}\zeta(j+1)\zeta(s-j). \tag{9.201}$$

[57]

$$\zeta(r,s)=-\frac{1}{2}\zeta(r+s)+\sum_{k=3,k\mathrm{odd}}^{\infty}\Phi_k\sum_{j=0,j\mathrm{even}}^{k-1}\binom{k}{j}\eta(r-j)\eta(s-k+j), \tag{9.202}$$

where

$$\Phi_k\equiv-\frac{2}{\pi}\sum_{d=1,d\mathrm{odd}}^{k-2}(-1)^{(d-1)/2}\frac{\pi^d}{d!}\zeta(k-d+1), \tag{9.203}$$

and

$$\eta(s)=(1-2^{1-s})\zeta(s). \tag{9.204}$$

9.6. **Bernoulli Numbers and Polynomials.** [95]

$$B_m(x)=\sum_{n=0}^{m}\frac{1}{n+1}\sum_{k=0}^{n}(-1)^k\binom{n}{k}(x+k)^m. \tag{9.205}$$

[48]

$$\begin{aligned}\sum_{k=0}^{m}\frac{\binom{m}{k}}{\binom{n+k+s}{s}}y^{m-k}B_{n+k+s}(x)&=\sum_{k=0}^{n}\frac{\binom{n}{k}}{\binom{m+k+s}{s}}(-y)^{n-k}B_{m+k+s}(x+y)\\&+\sum_{j=0}^{s-1}\sum_{i=0}^{s-1-j}\binom{s-1-j}{i}\binom{s-1}{j}\frac{(-1)^{n+1+i}sy^{m+n+s-j}B_j(x)}{(m+n+1+i)\binom{m+n+l}{n}}\end{aligned} \tag{9.206}$$

and

(9.207)
$$\sum_{k=0}^{m}\binom{m}{k}\binom{n+k}{s}y^{m-k}B_{n+k-s}(x)=\sum_{k=0}^{n}\binom{n}{k}\binom{m+k}{s}(-y)^{n-k}B_{m+k-s}(x+y).$$

[48]

$$\sum_{j=0}^{k}\binom{k+1}{j}(k+j+1)B_{k+j}=0,\quad k\geq 1. \tag{9.208}$$

[57]

(9.209)
$$B_s(t)B_r(t)=\sum_{j>0,j\equiv r+s \bmod 2}\frac{1}{j}\left(r\binom{s}{j-r}+s\binom{r}{j-s}\right)B_{r+s-j}B_j(t)+\frac{1}{2}((-1)^{r+s}+1)\frac{(-)^rB_{r+s}}{\binom{r+s}{s}}$$

[95]

$$E_m(x)=\sum_{n=0}^{m}\frac{1}{2^n}\sum_{k=0}^{n}(-1)^k\binom{n}{k}(x+k)^m. \tag{9.210}$$

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