DISCRETIZATION OF SDES

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Definition 0.1 (Euler-Maruyama discretization)**.** Given a SDE

$$\mathrm{d}X_t = b(t, X_t)\mathrm{d}t + \sigma(t, X_t)\mathrm{d}B_t,$$

we define the discrete process

$$\bar{X}_{t^n_{k+1}} = \bar{X}_{t^n_k} + b(t^n_k, \bar{X}_{t^n_k})\ \frac{T}{n} + \sigma(t^n_k, \bar{X}_{t^n_k})\ \sqrt{\frac{T}{n}}\ Z_k$$

where $\{Z_k\}$ are i.i.d normal random variables with zero mean and variance one.

The following result is crucial for the numerical simulation of the geometric Brownian motion.

Theorem 0.2 (Euler-Maruyama)**.** *The Euler-Maruyama discretization* $\{\bar{X}_{t^n_k}\}$ *of the SDE*

$$\mathrm{d}X_t = \sigma X_t \mathrm{d}B_t$$

satisfies

$$\|\sup_k |X_{t^n_k} - \bar{X}_{t^n_k}|\|_2 \lesssim \frac{\sigma^2}{\sqrt{n}}.$$

Sketch of the proof. (For more details see Pagès) In this case the EM discretization is given by (set $T = 1$)

$$\bar{X}_{t^n_{k+1}} = \bar{X}_{t^n_k}(1 + \sigma\ \sqrt{\frac{1}{n}}\ Z_k).$$

The proof is then an easy consequence of the fact that (use Taylor with remainder of order 3, independence of Z_j, Z_k with $j \neq k$, and the fact that $\frac{1}{n}\sum Z_k^2 = 1 + O(1/\sqrt{n})$)

$$\|\exp(\sigma(Z_1 + \cdots + Z_n)/\sqrt{n} - \sigma^2/2) - \prod_{i=1}^{n}(1 + \sigma Z_k/\sqrt{n})\|_2 \lesssim \frac{\sigma^2}{\sqrt{n}}.$$

Indeed, to conclude the proof it is enough to note that

$$\exp(\sigma(Z_1 + \cdots + Z_n)/\sqrt{n} - \sigma^2/2) = \exp(\sigma \int_0^1 \mathrm{d}B_t - \frac{1}{2}\sigma^2 \int_0^1 \mathrm{d}t) = X^n_n$$

while

$$\prod_{i=1}^{n}(1 + \sigma Z_k/\sqrt{n}) = \bar{X}^n_n.$$

□