FINANCIAL MARKETS IN DISCRETE TIME

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1. General ideas in discrete time

- **Market model:** Is a tuple $(\Omega, \mathcal{F}, P, \mathbb{T}, \mathbb{F}, S)$ where
  - $(\Omega, \mathcal{F}, P)$ is a probability space.
  - $\mathbb{T} = [0, \dots, T]$ is a time horizon.
  - $\mathbb{F} = \{\mathcal{F}_t\}_{t\in\mathbb{T}}$ is a filtration on $(\Omega, \mathcal{F}, P)$.
  - $\{S_t\}_{t\in\mathbb{T}}$ is an adapted random process $S_t = (S_t^0, S_t^1, \dots, S_t^d)$ with S_t^0 riskless (bond) and S_t^i for $i \geqslant 1$ are risky assets (stocks). The riskless bond evolves following the deterministic process
  $$S_{t+1}^0 = (1+r)S_t^0.$$
- **Strategy:** A strategy $\{\theta_t\}_{1\leqslant t\leqslant T}$ is a predictable process (with respect to the sequence of filtrations $\mathcal{F}_t$). The strategy represents the sequence of positions of an investor. At time $t-1$ we take a position and hold this position during the interval $[t-1, t]$.
- **Portfolio**: The portfolio at time $t \geqslant 1$ consists of
  - θ_t^0 bonds
  - θ_t^i units of the risky asset S_t^i.

  Its value is given by
  $$V_t(\theta) = \theta_t \cdot S_t, \qquad t \geqslant 1.$$
  The initial investment needed to construct the portolio is $V_0(\theta) = \theta_1 \cdot S_0$.
- **Put call parity:** derive from simple argument.

2. Arbitrage and first fundamental theorem

It is a **property of the market** itself. We say that there is an arbitrage oportunity in a market if there exists an admissible($V_t(\theta) \geqslant 0$) strategy such that
$$V_0(\theta) = 0 \qquad \text{and} \qquad \mathbb{E}(V_t(\theta)) > 0 \qquad a.s.$$

Remark 1. Note that given two equivalent probability measures arbitrage for one if and only if arbitrage for the other.

We say that a market is viable if no-arbitrage oportunities. A probability measure Q is an **equivalent martingale measure** (EMM) if

- $Q \sim P$.
- The discounted process $\bar{S}_t = \beta_t(S_t^1, \dots, S_t^d)$ is a martingale under Q (here $\beta_t = (1+r)^{-t}$).

Remark 2. Example of a non-viable market: binomial with $r < a < b$ (see later).

2.1. **The value process of an admissible strategy.** We define the **value process** of a strategy θ by

$$V_t(\theta) = \theta_t \cdot S_t$$

Denote by $\bar{S}_t$ the discounted process and let $\bar{V}_t(\theta) = \theta_t \cdot \bar{S}_t$ be the **discounted value process**. If θ is **self-financing**

$$\bar{V}_t(\theta) = \bar{V}_{t-1}(\theta) + \theta_t \Delta \bar{S}_t = V_0(\theta) + (\theta \cdot \bar{S})_t$$

Thus, under a EMM the discounted value process $\bar{V}_t$ of any admissible strategy is a **martingale**.

2.2. **Buyer and seller price.**

- The seller defines its price as the infimum initial investment which allows to construct a hedge
$$\pi_s(H) = \inf\{z \colon \exists \theta \in \Theta_a \text{ such that } z + G_T(\theta) = H_T\}.$$
- The buyer defines its price as the maximum such that he can construct an strategy such that together with the value of the claim allows him to pay the money borrowed to buy
$$\pi_b(H) = \sup\{z \colon \exists \theta \in \Theta_a \text{ such that } H_T + G_T(\theta) \geqslant z\}.$$

2.3. **The first fundamental theorem of asset pricing.**

Theorem 2.1 (First fundamental theorem of asset pricing)**.** *In a* ***finite*** *market model there exists an EMM if and only if the market is viable.*

We only do the proof for finite market models (Ω is finite dimensional). Can be adapted to general discrete market models. Do not know about continuous markets?

Proof. First notice that viability can be characterized in terms of $\hat{\theta} = (\theta^1, \dots, \theta^d)$ as this portfolio can always be extended to a self-financing $\theta = (\theta^0, \hat{\theta})$ with $V_0(\theta) = 0$. Now viability of the market iff

$$L = \{(\hat{\theta} \cdot \bar{S})_T \colon \hat{\theta} \text{ is predictable }\}$$

satisfies $L \cap C = \varnothing$ (where C is the "positive cone"). A EMM can now be represented by a vector $q \in L^\perp$ with $q_i > 0$ and $|q|_1 = 1$ (see Figure...). That this vector exists if $L \cap C = \varnothing$ is an easy consequence of the separation theorem for the vector subspace L and the compact convex set $K = \{\mathbb{E}_P(X) = 1\}$.

□

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Viability of the market is very relevant when pricing contingent claims whose underlying asset belongs to the market. It is automatic in

- Binomial model

- Continuous time (Girsanov)

What about other models? See the discrete-time Heston model.

2.3.1. *Viability of binomial model.* If $S_1(\omega_1) = (1+b)S_0$ and $S_1(\omega_2) = (1+a)S_0$ we obtain that

$$\Delta\bar{S}_1(\omega_1) = \frac{b-r}{1+r} \qquad \Delta\bar{S}_1(\omega_2) = \frac{a-r}{1+r}$$

We need $\Delta\bar{S}_1(\omega_1)\Delta\bar{S}_1(\omega_2) < 0$ which is equivalent to $a < r < b$.

3. Completeness and second fundamental theorem

It is again a **property of the market** itself. Let $(\Omega, \mathcal{F}, P, \mathbb{T}, \mathbb{F}, S)$ be a viable market model with a EMM Q. We say that the market is **complete** if, under martingale transform, the discounted process $\bar{S}$ serves as basis for the vector space of $(\mathbb{F}, Q)$ martingales. That is, given any martingale $M \in (\mathbb{F}, Q)$ there exists an adapted process θ such that

$$M_t = M_0 + (\theta \cdot \bar{S})_t.$$

Theorem 3.1 (Second fundamental theorem of asset pricing)**.** *A **finite** viable market is complete if and only if there exists a unique martingale measure.*

Proof. If market complete, for any $\mathcal{F}$-measurable X there exists $\theta \in \Theta_a$ such that $V_T(\theta) = \beta_T^{-1}X$. But then $\mathbb{E}_Q(X) = V_0(\theta)$ for any EMM Q and the conclusion follows since we can take $X = \mathbb{1}_A$ for any $A \in \mathcal{F}$ and we obtain that $\mathbb{E}_Q(\mathbb{1}_A) = \mathbb{E}_{Q'}(\mathbb{1}_A) =$ for any two EMMs Q, Q'.

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If market is viable but not complete we show how to construct a family of EMMs. Indeed consider

$$\mathcal{L} = \{\boldsymbol{c} + (\hat{\theta} \cdot \bar{S})_T : \hat{\theta} \text{ is predictable and } \boldsymbol{c} = (c_1, \ldots, c_n)\}$$

We see that $\mathcal{L}$ is a proper closed subspace of $\mathbb{R}^\Omega$. There exists $Z \in \mathbb{R}^\Omega$ such that $\mathbb{E}_Q(ZY) = 0$ for all $Y \in \mathcal{L}$. Take for example the measure

$$Q'(\omega) = Q(\omega)(1 + \frac{Z(\omega)}{2|Z|_\infty})$$

Easy to see that Q' is a different EMM. Convex combinations also form EMM. □

Completeness of the market is of extreme relevance to option pricing and the **construction of hedging strategies**$\longrightarrow$ Law of one price.

Remark 3. Examples of non-complete markets ("trinomial").

3.1. Incomplete market models: the arbitrage interval.

We are given a European contingent claim H. If the market is viable and complete there exists a unique price $\pi(H)$ which precludes arbitrage in the extended market model (or equivalent buyer and seller price agree). This price is given by $\mathbb{E}_Q(\beta_T H)$ where Q is the unique EMM for this market.

What about **incomplete markets**? We consider the extended market model $\widetilde{S} = (S, S^{d+1})$ where S^{d+1} satisfies

$$S_T^{d+1} = H \qquad S_t^{d+1} \geqslant 0$$

We want to determine the set of $\Pi(H)$ of values of $\beta_T S_0^{d+1}$ for which the extended market is viable.

Remark 4. This is natural. We have a market model. Depending on the dynamics of the process S^{d+1} the model will be viable or not. As $S_T^{d+1} = H(S_T)$ it turns out that viability (related to the fact that $\bar{\tilde{S}}$ is a martingale for a certain probability measure) only depends on the initial value we assign to S_0^{d+1}.

Theorem 3.2. *Let H be European claim in viable model. Let $\mathcal{P}$ be set of EMMs for the reduced model. Then,*

$$\Pi(H) = \{\mathbb{E}_Q(\beta_T H) \colon Q \in \mathcal{P},\ \mathbb{E}_Q(H) < \infty\}.$$

Proof. Notice that Q is also a EMM for the extended model. Thus, for S_t^{d+1} to be a martingale we need that (in particular) $S_0^{d+1} = \mathbb{E}_Q(\beta_T H)$. □

4. Asset pricing

The first remark is that it is in general **not possible** to assign a unique price to a given contingent claim. This possibility will depend on

- viability and completeness of the market.
- nature of the contingent claim.

Remark 5. Even in complete viable markets pricing of complex assets such as American puts might not be straightforward! For example gap between buyer-seller price due to early exercise.

4.1. **Risk-neutral pricing.** Consider a complete viable model with EMM Q. The discounted process $\bar{S}_t$ is a martingale. The **key idea behind risk-neutral** pricing is that given any self-financing $\theta \in \Theta_a$ which replicates a (European) claim then $\bar{V}_t(\theta)$ is also a martingale! In particular, $V_0(\theta)$ is uniquely determined!

4.2. **General ideas.** In order to "price" a contingent claim depending on an asset:

- **Choose a model:** for the dynamics of the underlying asset. Examples: binomial model, trinomial model, finite differences, continuous time...
- **Calibrate parameters**: For example, in the binomial model set r (interest rate) to match historic data and a, b (up-down factors) to match the volatility (see CRR model).
- **Existence of EMM:** In the binomial model is automatic. In some continuous time models (Black-Scholes) existence is automatic from Girsanov. Finite difference models coming from these continuous time models already implicit as well.

  What about other models? Are there popular models (discrete or continuous) in which one has to care about existence and uniqueness of the EMM?
- **Computation of EMM:** In the binomial model or Black-Sholes automatic. In models with non-uniqueness such as trinomial models? Choose based on minimization of the risk function, historical data, intuition, etc...
- **Pricing:** Below we show how to price
  - European claims in discrete complete viable models.
  - European claims in continuous time.
  - American claims in discrete complete viable models.

5. Pricing of European contingent claims in complete viable models

The properties of a market studied above are of relevance to price contingent claims.

5.1. **Pricing of European options.** In a complete viable model **risk-neutral pricing**: if different price we could trade in the "extended market" market+option and see that it is possible to construct an "extended strategy" in which, with zero initial investment we cannot loose money and we make profit with positive probability. That is, "there would exist an arbitrage oportunity in this extended market".

Theorem 5.1. *Under a EMM Q we have $\pi_b(H) \leqslant E_Q(\beta_T H) \leqslant \pi_s(H)$ for any European claim H. If the European claim is attainable then*

$$\pi_b(H) = E_Q(\beta_T H) = \pi_s(H)$$

5.2. **The Binomial model.** It is a complete viable market. The **Cox-Ross-Rubinstein** (CRR) formula for option pricing of a European option reads:

$$C_0 := V_0 = S_0 \Psi(A; T, q') - K(1+r)^{-T} \Psi(A; T, q)$$

where $\Psi(m; n, p) = \sum_{j=m}^{n} \binom{n}{j} p^j (1-p)^{n-j}$. We can also obtain formulas for the hedging strategy.

5.3. **From CRR to Black-Scholes model.** Consider a binomial model with

- N steps
- Interest rate ρ_N
- Up and down factors a_N and b_N.

This determines uniquely a EMM Q_N. Under this measure the return rates

$$R_n^N = \frac{S_n}{S_{n-1}} \qquad \text{are } i.i.d$$

From the CRR formula we obtain that the price of a European put option is given by

$$P_N = \mathbb{E}_{Q_N} \left(K(1+\rho_N)^{-N} - S_0 \prod R_n^N \right)^+$$

Writing

$$Z^N = \sum_n Y_n^N \qquad Y_n^N = \log \left(\frac{R_n^N}{(1+\rho_N)} \right)$$

so

$$P_N = \mathbb{E}_{Q_N} \left(K(1+\rho_N)^{-N} - S_0 e^{Z_N} \right)^+$$

Adjusting properly the parameters ρ_N, a_N and b_N one may check that, using the Lindenberg-Feller CLT for triangular arrays Z_N **converges in law** to a normal random variable $\mathcal{N}(-\frac{1}{2}\sigma^2 T, \sigma^2 T)$. This yields the Black-Scholes formula. It is interesting that in this derivation we do not obtain/need any information on the dynamics of the limiting process.

6. Pricing of American contingent claims in complete viable models

Here the buyer can **exercise at any time prior** to the expiration day. Suppose Q is the EMM of the market model and let $f(S_t)$ be the payoff of the claim at time t.

Remark 6. Think for example that $f(S_t) = (S_t - K)^+$.

The idea behind option pricing is more involved than for European claims. For American options:

- Construct the Snell envelope Z_t of the payoff process. It dominates the payoff process.

- Z_t is a supermartingale $Z_t = M_t - A_t$. It is a martingale up to the largest optimal stopping time[1].
- Construct a portfolio θ which replicates M_T[2]. Clearly M_t ($= \bar{V}_t(\theta)$) dominates the payoff process.
- Exercise is only optimal at optimal stopping times because at not optimal times $\bar{f}(S_t) < Z_t \leqslant M_t$ (if we exercise we recieve $f(S_t)$ so we could instead have not buyed the option at $t = 0$ but sold it and construct a portfolio with discounted value $\bar{V}_t(\theta)$.

For European options:

- Just construct the portfolio which replicates the random variable $f(S_T)$.

Definition 6.1. Given a process X_t we define its **Snell envelope** to be the process

$$Z_t = \max\{X_t, \mathbb{E}_Q(Z_{t+1}|\mathcal{F}_t)\} \qquad\qquad Z_T = X_T.$$

Remark 7. Think of $X_t = f_t(S_t)$. Notice that in general, the fact that S_t is a martingale does not imply that $f(S_t)$ is a martingale (in particular for $f(S_t) = (S_t - K)^+$ is not).

Lemma 6.2. *The Snell envelope is the smallest supermartingale which dominates X_t.*

Definition 6.3. A stopping time σ is **optimal** if it satisfies

$$\mathbb{E}(X_\sigma) = \sup_\tau \mathbb{E}(X_\tau).$$

Lemma 6.4. *Let Z_t be the Snell envelope of X_t. The stopping time $\tau_* = \min\{t \geqslant 0 \colon X_t = Z_t\}$ is optimal for X_t (think of X_t as the payoff $f(S_t)$) and*

$$Z_0 = \mathbb{E}(X_{\tau_*}).$$

The following is key to the analysis of option pricing for American claims.

Theorem 6.5. *The stopped process $Z_{t\wedge\tau_*}$ is a martingale (recall Z_t is a submartingale only).*

Proof. Notice that while $\tau_* \geqslant t$ we have that $Z_t = \mathbb{E}(Z_{t+1}|\mathcal{F}_t)$. □

Remark 8. Discuss also the largest optimal stopping time.

6.1. **Construction of a minimal hedge.** Let Z_t be the snell envelope of the discounted payoff $\bar{f}_t(S_t) = (1+r)^{-t} f_t(S_t)$. Since Z_t is a supermartingale it admits a **Doob decomposition** as

$$Z_t = M_t - A_t$$

with M_t a martingale and A_t a predictable increasing process. Since market is viable, there exists $\theta \in \Theta_a$ such that $\bar{V}_T(\theta) = M_T(\theta)$. We need

$$x = V_0(\theta) = \sup_\tau\{\mathbb{E}((1+r)^{-\tau} f_\tau(S_\tau))\}$$

as initial investment to construct this hedge.

[1] Notice that A_t will always be zero if $f(S_t)$ is a submartingale (for example for American calls). This explains why in American call option there is no benefit from early exercise.

[2] Notice that M_T is in general different from $Z_T = \bar{f}(S_T)$ though.

6.2. **Optimal exercise.** Exercise is only optimal at optimal times ($\tau_* \leqslant t \leqslant \nu$, with ν the largest optimal time).

Remark 9. This is already intuitive from the fact that optimal stopping times maximize the expected payoff.

Indeed suppose that we buy the option and exercise it at a non-optimal time t and get a payoff $f_t(S_t)$. Instead of buying the option, at time $t = 0$ we could have sold the option and use this money to create the portfolio θ above which, by definition, would have value at this time $V_t(\theta) > f_t(S_t)$ (notice that at any t either $f_t(S_t) = Z_t(\theta)$ or $f_t(S_t) < Z_t(\theta) \leqslant \bar{V}_t(\theta)$).

Remark 10. It seems that early exercise is not optimal because we would have $f(S_t) < Z_t = \bar{V}_t(\theta)$. But late exercise seems worse? because we would have $f(S_t) < Z_t = \bar{V}_t(\theta) - A_t(\theta) < \bar{V}_t(\theta)$.

6.3. **American call options.**

Theorem 6.6 (No benefit from early exercise)**.** *$\tau_* = T$ is an optimal exercise time.*

Proof. Notice that $f_t = (S_t - K)^+$ behaves a submartingale for Q. In particular $C_t := \mathbb{E}_Q(f_T|\mathcal{F}_t) \geqslant f_t$[3]. On the other hand
$$Z_t \geqslant \mathbb{E}_Q(Z_T|\mathcal{F}_t) = \mathbb{E}_Q(f_T|\mathcal{F}_t) \geqslant f_t.$$
Thus, C_t dominates the process f_t and it must coincide with Z_t. □

7. Pricing of contingent claims in non complete and/or viable models

Talk about risk function. Minimize risk.

Appendix A. Stopping times

Let τ be a stopping time. We define the information available at time τ (see also figure ... for a set not in $\mathcal{F}_\tau$)
$$\mathcal{F}_\tau = \{A \in \mathcal{F} \colon A \cap \{\tau \leqslant t\} \in \mathcal{F}_t\}.$$

[3]Notice $\mathbb{E}_Q(f_T|\mathcal{F}_t)$ is the value process for European call