FINANCIAL MARKETS IN CONTINUOUS TIME

Contents

1. Continuous time stochastic processes

We think of (continuous time) stochastic process as **probability measures** on $[0,T]^{\mathbb{R}}$, i.e. the on the **space of functions** $f:[0,T]\to\mathbb{R}$.

1.1. Construction of Brownian motion.

Definition 1.1. We say that the random process B_t is a **Brownian motion** if:

- $B_0(\omega)=0$ a.s.
- (*Continuity*): $t\mapsto B_t(\omega)$ is continuous a.s.
- (*Stationary independent Gaussian increments*): For all $s<t$ the random variable B_t-B_s is a normal random variable with zero mean and variance $t-s$ and is independent of $\mathcal{F}_s=\sigma(B_u,u\leqslant s)$.

We now show that this process actually exists. Before doing that let us point out that we think of Brownian motion as a **probability measure** on the probability space $(C,\mathcal{C})$ where $C=\{\text{continuous }\omega:[0,\infty)\to\mathbb{R}\}$ and $\mathcal{C}$ is the σ-algebra generated by the coordinate maps $t\mapsto\omega(t)$. What we ask in the definition corresponds to this measure assigning probability zero to non-continuous functions and fixing the finite dimensional distributions.

Remark 1. Given the above probability measure on $(C,\mathcal{C})$ the random process B_t is simply given by $B_t(\omega)=\omega(t)$.

Theorem 1.2. *The Brownian motion random process exists.*

Proof. The proof follows from the **Kolmogorov extension theorem**. In order to deal with continuity this has to be done in a slightly indirect way.

- **Step 1:** Check consistency of finite-dimensional distributions (see Durrett).
- **Step 2:** Consider the (countable) product space $\Omega_q=\{\omega:\mathbb{Q}_{dyad}\to\mathbb{R}\}$ and let $\mathcal{F}_q$ denote the product (cylinder) σ-algebra. It is then possible (see Durrett) to construct a (unique) probability measure ν on Ω_q which:
  - Assigns probability one to paths $\omega:\Omega_q\to\mathbb{R}$ which are uniformly continous on $\mathbb{Q}_2\cap[0,T]$ (for any $T<\infty$) and
  - Has the given finite dimensional distributions, i.e.

$$\nu\{\omega(t_i)\in A_i\}=\int_{A_1}\mathrm{d}x_1\cdots\int_{A_n}\mathrm{d}x_n\prod_{i=1}^{n}p_{t_i-t_{i-1}}(x_{i-1},x_i)$$

where $p_t(a,b) = \frac{1}{\sqrt{2\pi t}} \exp(-(b-a)^2/2t)$.

- **Step 3:** Extend to a process defined for $t \in \mathbb{R}$. Then, let $\tilde{\Omega}_q = \{\omega \in \Omega_q : \omega \text{ is uniformly continuous}\}$. Define the measure $P = \nu \circ \psi^{-1}$ on $(C, \mathcal{C})$ using that the map

$$\psi : \tilde{\Omega}_q \to C$$

that assigns the (unique) continuous extension is measurable (see Durrett).

□

It is not difficult to prove the following.

Proposition 1.3. *The function $t \mapsto B_t(\omega)$ is not differentiable a.s.*

1.2. **The Ito integral.** The material in this section is well explained in Oksendal's book. We construct the Ito integral in three steps:

-) Definition for elementary functions: We say a function ϕ is elementary if

$$\phi(t,\omega) = \sum e_j(\omega) \mathbb{1}_{[t_j, t_{j+1}]}$$

where $e_j(\omega)$ is $\mathcal{F}_{t_j}$-measurable. For elementary functions we define

$$\int_0^T \phi(t,\omega) \mathrm{d}B_t = \sum_j e_j(\omega)(B_{t_{j+1}} - B_{t_j})(\omega).$$

-)Ito isometry: If ϕ is bounded and elementary then

$$\mathbb{E}\left(\int_0^T \phi(t,\omega)\mathrm{d}B_t\right)^2 = \mathbb{E}\int_0^T \phi(t,\omega)^2 \mathrm{d}t$$

-)Extension to $f \in \mathcal{V}$: This is done via an approximation procedure which exploits the Ito isometry. First we construct a sequence $\{\phi_n\} \subset \mathcal{V}$ such that

$$\mathbb{E}\int_0^T (f - \phi_n)^2 \mathrm{d}t \to 0.$$

We then define (note the limit exists as an element of L^2)

$$I(f)[\omega] = \int_0^T f(t,\omega)\mathrm{d}B_t(\omega) := \lim_{n\to\infty} \int_0^T \phi_n(t,\omega)\mathrm{d}B_t(\omega).$$

1.3. **Ito's formula.** We now give Ito's lemma, which can be seen as a generalization of the fundamental theorem of calculus to functions of Brownian motion.

Lemma 1.4 (Ito's lemma)**.** *Let $f(t,x) \in C^2$. Then,*

$$f(t, B_t) - f(0, B_0) = \int_0^t \left(\partial_t f(s, B_s) + \frac{1}{2}\partial^2_{x^2} f(s, B_s)\right) \mathrm{d}s + \int \partial_x f(s, B_s)\mathrm{d}B_s.$$

Remark 2. The appearence of the second term is due to the fact that B_t has quadratic variation different from zero (see below).

Proof. The complete proof can be found in Durrett. Here we just deal with the second term. Also, we only analyze f independent of t so, if $s < t$, by the mean value theorem

$$f(B_t) - f(B_s) = \partial_x f(B_s)(B_t - B_s) + \frac{1}{2}\partial^2_{x^2} f(c(B_s, B_t))(B_t - B_s)^2$$

where $c(B_s, B_t)$ is a number between B_s and B_t. We want to compute

$$\sum_i \partial^2_{x^2} f(c(B_i, B_{i-1}))(B_i - B_{i-1})^2 \tag{1.1}$$

To that end we show that $\sum_i (B_i - B_{i-1})^2 \to t$ a.s. That is, the cumulative distribution (separate monotone increasing and monotone decreasing see construction of Lebesgue-Stieltjes integral) of the measure converges to t. Hence the measure converges weakly to dt. To check (1.1) we show that

$$\mathbb{E}(Q(B_t) - t)^2 = 0$$

where the **quadratic variation** operator Q is defined by

$$Q(f) = \sup_{\mathcal{P}} \sum_{i \in \mathcal{P}} (f(t_i) - f(t_{i-1}))^2.$$

To that end we write

$$\begin{aligned}\mathbb{E}\sum_i ((B_{t_i} - B_{t_{i-1}})^2 - (t_i - t_{i-1}))^2 &= \sum_i \mathbb{E}((B_{t_i} - B_{t_{i-1}})^2 - (t_i - t_{i-1}))^2 \\ &= \sum_i \mathbb{E}(B_{t_i} - B_{t_{i-1}})^4 - 2\mathbb{E}(B_{t_i} - B_{t_{i-1}})^2(t_i - t_{i-1}) + (t_i - t_{i-1})^2\end{aligned}$$

Now use that, since $B_{t_i} - B_{t_{i-1}}$ is normally distributed with variance $t_i - t_{i-1}$ we have

$$E(B_{t_i} - B_{t_{i-1}})^2 = t_i - t_{i-1} \qquad E(B_{t_i} - B_{t_{i-1}})^4 = (t_i - t_{i-1})^2$$

and our claim follows easily. □

1.4. **The martingale representation theorem.** It can be shown by means of the Doob martingale inequality that, if $f \in \mathcal{V}$, then $\int_0^T f(t, \omega)\mathrm{d}B_t(\omega)$ is an $\mathcal{F}_t$-martingale. The converse to this statement is key to establish completeness of the Black-Scholes model.

Theorem 1.5. *Suppose that M_t is a $\mathcal{F}_t$-martingale and $M_t \in L^2$. Then, there exists a unique $g \in \mathcal{V}$ such that*

$$M_t(\omega) = \mathbb{E}M_t + \int_0^t g(s, \omega)\mathrm{d}B_s(\omega)$$

Sketch of the proof. We use that:

- The linear span of the set of functions of the form $Y = \exp(\int_0^T h\mathrm{d}B_t - \frac{1}{2}\int_0^T h\mathrm{d}t)$ with $h(t) \in L^2[0, T]$ is dense in $L^2(\mathcal{F}_T)$.
- For Y as above $Y = 1 + \int_0^T Y_t(s, \omega)h(s)\mathrm{d}B_t$ where $Y_t = \exp(\int_0^t h\mathrm{d}B_s - \frac{1}{2}\int_0^t h\mathrm{d}s)$.
- Approximate $F \in L^2(\mathcal{F}_T)$ by linear combination of Y as above and obtain a sequence $\{f_n\}$ such that

$$F_n = \mathbb{E}F_n + \int_0^T f_n \mathrm{d}B_t.$$

- Use the Ito isometry to show that f_n form a Cauchy sequence in $L^2(\mathcal{F}_T)$.

□

1.5. **Cameron-Martin-Girsanov's theorem.** The following result is of key importance to construct an equivalent martingale measure in the case where the risky assett is driven by a Brownian motion with drift.

Theorem 1.6. *Suppose that the random process B_t is a Brownian motion under the probability measure P and let Q_θ be the probability measure defined by*

$$\frac{\mathrm{d}Q_\theta}{\mathrm{d}P} = Z_\theta := \exp(\theta B_t - \theta^2 t/2).$$

Then, under Q_θ the random process B_t is a Brownian motion with drift $-\theta$.

Remark 3. Notice that Z_θ is a random variable which satisfies $\mathbb{E}_P(Z_\theta) = 1$. Thus, Q_θ is a probability measure (it is defined by $Q_\theta(A) = \mathbb{E}_P(Z_\theta \mathbb{1}_A)$).

Proof. First note that if we denote by $U = B_1$ we have that

$$\begin{aligned} Q_\theta(U \leqslant x) =& \frac{1}{\sqrt{2\pi}} \int_{-\infty}^x \exp(\theta u - \theta^2/2) \exp(-u^2/2) \mathrm{d}u \\ =& \frac{1}{\sqrt{2\pi}} \int_{-\infty}^x \exp(-(u-\theta)^2/2) \mathrm{d}u \\ =& P(U - \theta \leqslant x). \end{aligned}$$

(so the result seems plausible). To actually check the result we just need to check that the finite dimensional distributions of B_t under Q_θ and of $B_t - \theta t$ under P agree. We can do this using for example characteristic functions. □

1.6. **The classical Wiener space.** The classical Wiener space $(\Omega, \mathcal{F}, P)$ is the probability space where

- $\Omega = C = \{$**continuous** $\omega : [0, \infty) \to \mathbb{R}\}$.
- $\mathcal{F} = \mathcal{C}$ is the σ-algebra generated by the coordinate maps $t \mapsto \omega(t)$ (cylinder σ-algebra).
- $P = \nu$ is the unique measure on $(\Omega, \mathcal{F})$ for which the coordinate process is Brownian motion.

2. One-dimensional market on the classical Wiener space

We consider the classical Wiener space $(\Omega, \mathcal{F}, P)$ and consider a market consisting of

- S_t^0 is a **bond**, whose dynamics is given by the ODE
$$\mathrm{d}S_t^0 = rS_t^0 \mathrm{d}t$$
- S_t^1 is a **risky asset** whose dynamics is given by the SDE
$$\mathrm{d}S_t^1 = S_t^1(\mu \mathrm{d}t + \sigma \mathrm{d}B_t).$$

2.1. **Completeness of the market model.** Completeness of the market model follows from the martingale representation theorem.

2.2. **Equivalent martingale measure and strategies.** We first construct the **equivalent martingale measure** P_μ using Girsanov's theorem. Namely we choose

$$\frac{\mathrm{d}P_\mu}{\mathrm{d}P} = \exp(-\theta B_t - \frac{1}{2}\theta^2 t) \qquad \theta = \frac{\mu - r}{\sigma}.$$

Under the measure P_μ, the process $B_t^\mu = \frac{\mu-r}{\sigma}t + B_t$ is the standard Brownian measure.

Lemma 2.1. *Under the measure P_μ the discounted process $\tilde{S}_t^1 = e^{-rt}S_t^1$ satisfies the SDE*

$$\mathrm{d}\tilde{S}_t^1 = \tilde{S}_t^1 \sigma \mathrm{d}B_t^\mu.$$

2.2.1. *The value process.* Given a strategy (adapted process) $\phi = (H_t^0, H_t^1)$ we define the associated value process

$$V_t(\phi) = H_t^0 S_t^0 + H_t^1 S_t^1.$$

Let $\tilde{V}_t(\phi)$ denote the discounted value process with $\phi \in SF$ (i.e. self-financing). Then, from Ito's differentiation rule

$$\begin{aligned}\tilde{V}_t(\phi) =& V_0(\phi) + \int_0^t H_u^1 \; e^{-ru}\mathrm{d}(S_u^1) \\ =& V_0(\phi) + \int_0^t H_u^1 \; \sigma e^{-ru} S_u^1 \; \mathrm{d}B_u^\mu.\end{aligned}$$

Lemma 2.2. *The process $\tilde{V}_t(\phi)$ is a martingale.*

2.3. **Pricing of European claims.** We consider a European claim of the form $f(S_T^1)$.

Definition 2.3. The **investment price** $C(T, f_T)$ for a European claim f_T at time T is the smallest initial investment with which the investor can attain an amount f_T using strategies $\phi \in SF(0)$.

We now show how to price European options.

Theorem 2.4. *Consider an European option with payoff $f(S_T)$*[1]*. Then the rational price for the option is*

$$C(T, f_T) = e^{-rT}F(T, S_0^1)$$

where

$$F(T, S_0^1) = \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} f\left(S_0^1 \exp\left((r - \frac{1}{2}\sigma^2)T + \sigma\sqrt{T}y\right)\right) \; e^{-y^2/2}\mathrm{d}y.$$

Remark 4. Also possible to obtain formula for the **hedge**.

Proof. Observe that

$$C(T, f_T) = \mathbb{E}^\mu(e^{-rT}f(S_T)) = F(T, S_0^1).$$

Using martingale representation theorem one can show that there is a strategy for which the initial investment is $C(T, f_T)$. □

[1] Satisfying certain integrability condition.

2.4. **The PDE approach.** Let ϕ be the strategy for which $V_T(\phi) = f_T$ and $V_0(\phi) = C(T, f_T)$. Then, using the Markov property for the process S_t (this is also what we used in the proof of Theorem 2.4)

$$\tilde{V}_t(\phi) = \mathbb{E}^\mu(e^{-rT} f(S_T)|\mathcal{F}_t)$$

Theorem 2.5 (Black-Scholes equation)**.** *Consider a European claim whose payoff is of the form $f(S_T)$ where $f \in C^2$. Its value process $V(t,x)$ satisfies the* ***linear parabolic PDE***

$$\partial_t V + rS_t^1 \partial_x V_u + \frac{1}{2}\sigma^2 (S_t^1)^2 \partial_{x^2}^2 V - rV = 0$$

with terminal condition $V(T, S_T) = f(S_T)$.

Some remarks are in order:

- **Existence and uniqueness** of solutions to this PDE with the given terminal condition is standard.
- For $f \notin C^2$ we can approximate by C^2, **convergence**?

Proof. It follows from Ito's differentiation rule that

$$e^{-rt}V_t(S_t^1) - V_0(S_0^1) = \int_0^t e^{-ru} L(V) \mathrm{d}u + \int_0^t \sigma S_u^1 \partial_x V \mathrm{d}B_u^\mu$$

where

$$L(V) = \partial_t V + rS\partial_x V + \frac{1}{2}\sigma^2 S^2 \partial_{x^2}^2 V - rV$$

However, by construction of the value process and since S_t is a martingale, $e^{-rt}V(t, S_t)$ is a martingale. It then follows that $L(V) = 0$ (see Lemma 6.4.2, I think the proof they give is strange but easy to fix). $\square$