Convex Optimization Examples

Jaime Tenorio

April 2023

1 Convex Optimization

A *convex optimization problem* can be stated as

$$\begin{array}{ll} \text{minimize} & f_0(x) \\ \text{subject to} & f_i(x) \leq 0, \qquad i = 1, \dots, m \\ & a_i^T x = b_i, \qquad i = 1, \dots, p, \end{array} \tag{1}$$

where the function $f_0(x)$ and the inequality constraints $f_i(x)$ are convex, and the equality constraint must be affine.

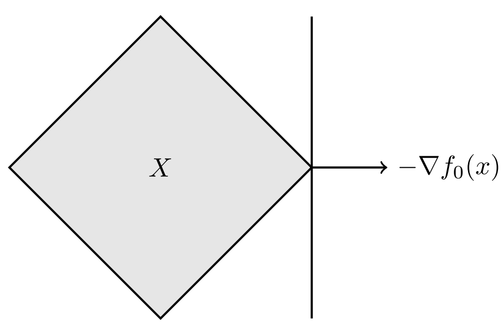
The feasible set is convex because it is the intersection of of the domain of the constraints, which are convex.

$$D = \bigcap_i^m \mathbf{dom} f_i \tag{2}$$

Suppose we have a convex optimization problem where the objective function f_0 is differentiable. Let X be the feasible set of the problem. Then $x \in X$ is optimal if and only if:

$$\nabla f_0(x)^T (y - x) \geq 0 \quad \forall y \in X \tag{3}$$

Geometrically, this means that the gradient of the objective function at x points in the direction of the feasible set, in other words, it is a supporting hyperplane to X.



2 Unconstrained Problems

Suppose we have no constraint inequalities f_i and the objective function f_0 is differentiable. Then $x \in X$ is optimal if and only if:

$$\nabla f_0(x) = 0 \tag{4}$$

Take as an example the *unconstrained quadratic optimization*:

$$\text{minimize} \quad f_0(x) = \frac{1}{2}x^T Px + q^T x + r \tag{5}$$

where $P \in \mathbf{S}_+^n$. The optimality condition gives:

$$\nabla f_0(x) = Px + q = 0 \tag{6}$$

We can visualize this in two dimensions in the following figure: