Convex Optimization Examples

Jaime Tenorio

April 2023

1 Convex Optimization

A *convex optimization problem* can be stated as

$$\begin{aligned} \text{minimize} \quad & f_0(x) \\ \text{subject to} \quad & f_i(x) \leq 0, \qquad i = 1, \ldots, m \\ & a_i^T x = b_i, \qquad i = 1, \ldots, p, \end{aligned} \tag{1}$$

where the function $f_0(x)$ and the inequality constraints $f_i(x)$ are convex, and the equality constraint must be affine.

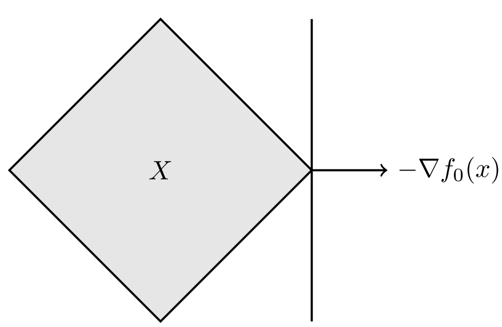
The feasible set is convex because it is the intersection of of the domain of the constraints, which are convex.

$$D = \bigcap_{i}^{m} \mathbf{dom} f_i \tag{2}$$

Suppose we have a convex optimization problem where the objective function f_0 is differentiable. Let X be the feasible set of the problem. Then $x \in X$ is optimal if and only if:

$$\nabla f_0(x)^T (y - x) \geq 0 \quad \forall y \in X \tag{3}$$

Geometrically, this means that the gradient of the objective function at x points in the direction of the feasible set, in other words, it is a supporting hyperplane to X.



2 Unconstrained Problems

Suppose we have no constraint inequalities f_i and the objective function f_0 is differentiable. Then $x \in X$ is optimal if and only if:

$$\nabla f_0(x) = 0 \tag{4}$$

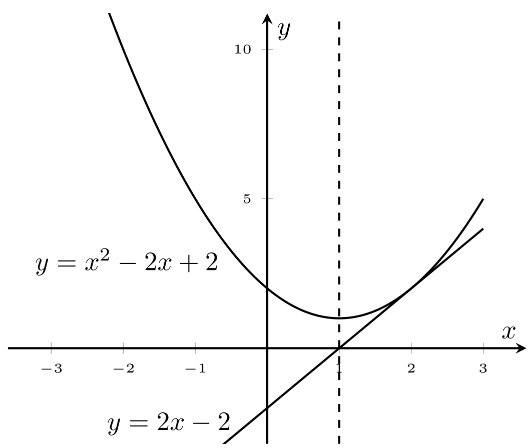
Take as an example the *unconstrained quadratic optimization*:

$$\text{minimize} \quad f_0(x) = \frac{1}{2}x^T P x + q^T x + r \tag{5}$$

where $P \in \mathbf{S}^n_+$. The optimality condition gives:

$$\nabla f_0(x) = Px + q = 0 \tag{6}$$

We can visualize this in two dimensions in the following figure:



3 Equality Constraints

If we have only equality constraints

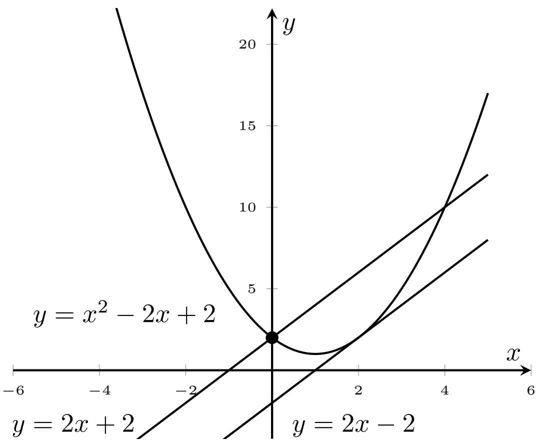
$$\begin{aligned} \text{minimize} \quad & f_0(x) \\ \text{subject to} \quad & Ax = b \end{aligned} \tag{7}$$

the feasible set is affine. The optimality condition for $x \in X$ is

$$\exists \nu \in \mathbb{R}^p \quad s.t. \quad \nabla f_0(x) + A^T \nu = 0 \quad \text{and} \quad Ax = b \tag{8}$$

which is the Lagrange multiplier optimality condition, also known as the Karush-Kuhn-Tucker (KKT) condition.

We can visualize this in two dimensions in the following figure:



4 Nonnegative Orthant

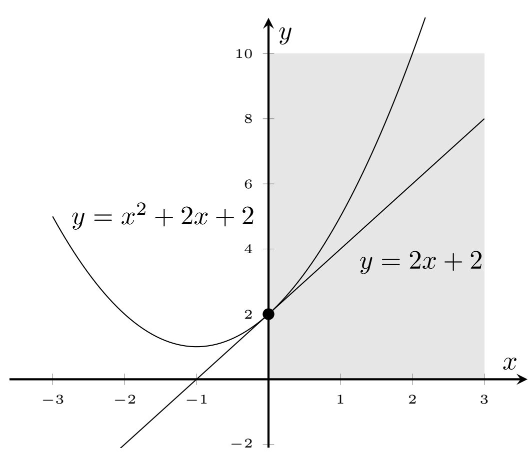
If we consider a problem like

$$\begin{array}{rl} \text{minimize} & f_0(x) \\ \text{subject to} & x \succeq 0 \end{array} \tag{9}$$

where the only constraint is the nonnegativity of x. The feasible set is the nonnegative orthant. The optimality condition for $x \succeq 0, x \in X$ is

$$\begin{cases} \nabla f_0(x)_i \geq 0 & x_i = 0 \\ \nabla f_0(x)_i = 0 & x_i > 0 \end{cases} \tag{10}$$

We can visualize this in two dimensions in the following figure:



5 Equivalent Problems

We can obtain **equivalent** convex problems using transformations that preserve convexity.

Original		Equivalent	
minimize	$f_0(x)$	minimize (over z)	$f_0(Fz + x_0)$
subject to	$f_i(x) \leq 0$	subject to	$f_i(Fz + x_0)$
	$Ax = b$	$Ax + b \iff x = Fz + x_0$	
minimize	$f_0(A_0 x + b_0)$	minimize (over x, y_i)	$f_0(y_0)$
subject to	$f_i(A_i x + b_i) \leq 0$	subject to	$f_i(y_i) \leq 0$
			$y_i = A_i x + b_i$
minimize	$f_0(x)$	minimize (over x, s)	$f_0(x)$
subject to	$a_i^T x \leq b_i$	subject to	$a_i^T x + s_i = b_i$
			$s_i \geq 0$
minimize	$f_0(x)$	minimize (over x, t)	t
subject to	$f_i(x) \leq 0$	subject to	$f_0(x) - t \leq 0$
	$a_i^T x = b_i$		$f_i(x) \leq 0$
			$Ax = b$
minimize	$f_0(x_1, x_2)$	minimize	$\tilde{f}_0(x_1)$
subject to	$f_i(x_1) \leq 0$	subject to	$f_i(x_1) \leq 0$
		$\tilde{f}_0(x_1) = \inf_{x_2} f_0(x_1, x_2)$	

In the table we can see equivalent problems for the following transformations:

1. Eliminating equality constraints
2. Introducing equality constraints
3. Introducing slack variables for linear inequality constraints
4. Epigraph form of the standard convex problem
5. Minimizing over some variables

6 Neural Network Training

Training a neural network can be posed as a convex optimization problem.

$$\text{minimize} \quad L(y; \hat{y}) \tag{11}$$

where $L(y; \hat{y})$ is the loss function, y is the ground truth, and $\hat{y}$ is the prediction. The prediction is given by the neural network, which is a function of the weights W and the input x. If we consider a network of one layer, the prediction is given by

$$\hat{y} = \sigma(Wx) \tag{12}$$

where σ is the activation function. For this problem to be convex we need the loss function to be convex. We can consider the following loss functions:

- **Mean squared error (MSE)**: $L(y;\hat{y}) = \frac{1}{2}(y-\hat{y})^2$
- **Cross-entropy loss**: $L(y;\hat{y}) = -y\log(\hat{y}) - (1-y)\log(1-\hat{y})$
- **Hinge loss**: $L(y;\hat{y}) = \max(0, 1-y\hat{y})$

For example, YOLO is a convolutional neural network for object detection. It is trained using the following loss function

$$
\begin{aligned}
L = &= \lambda_{\text{coord}} \sum_{i=0}^{S^2} \sum_{i=0}^{B} \mathbf{1}_{ij}^{\text{obj}} \left[(x_i - \hat{x}_i)^2 + (y_i - \hat{y}_i)^2 \right] \\
&+ \lambda_{\text{coord}} \sum_{i=0}^{S^2} \sum_{i=0}^{B} \mathbf{1}_{ij}^{\text{obj}} \left[(\sqrt{w_i} - \sqrt{\hat{w}_i})^2 + (\sqrt{h_i} - \sqrt{\hat{h}_i})^2 \right] \\
&+ \sum_{i=0}^{S^2} \sum_{i=0}^{B} \mathbf{1}_{ij}^{\text{obj}} \left(C_i - \hat{C}_i \right)^2 \\
&+ \lambda_{\text{noobj}} \sum_{i=0}^{S^2} \sum_{i=0}^{B} \mathbf{1}_{ij}^{\text{noobj}} \left(C_i - \hat{C}_i \right)^2 \\
&+ \sum_{i=0}^{S^2} \mathbf{1}_{i}^{\text{obj}} \sum_{c \in \text{classes}} (p_i(c) - \hat{p}_i(c))^2
\end{aligned}
\tag{13}
$$

where $\hat{x}, \hat{y}, \hat{w}, \hat{h}, \hat{c}$ are the output from the convolutional neural network. This is a modified mean squared error loss function, which is convex.

Usually, the optimization of a neural network is not a convex problem.