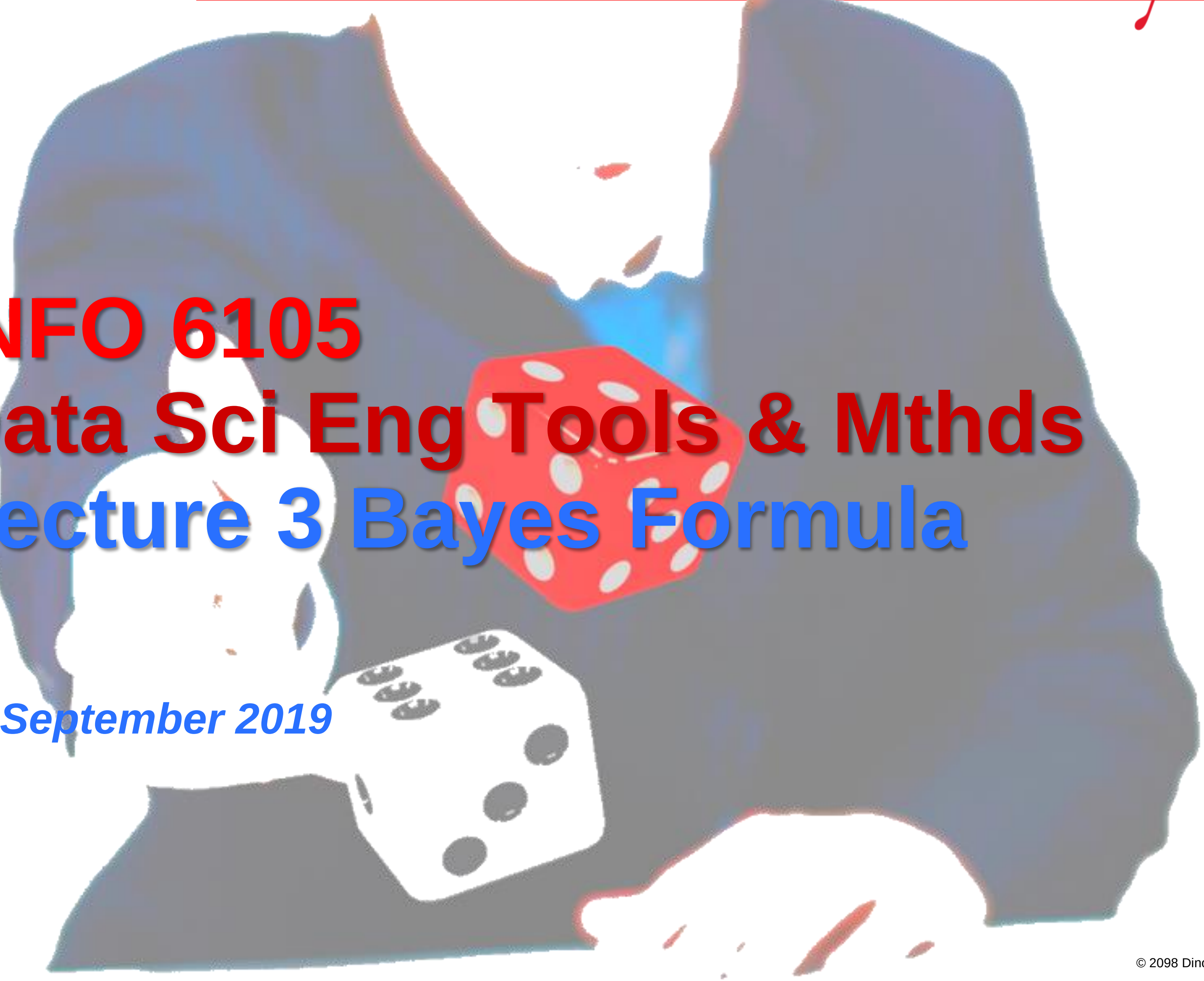
Northeastern University

INFO 6105
Data Sci Eng Tools & Mthds
Lecture 3 Bayes Formula



18 September 2019

Bayes' Formula

- We have a prior belief in event A , beliefs formed by previous information, e.g., our prior belief about bugs being in our code before performing tests
- Secondly, we observe our evidence. if our code passes X tests, we want to update our belief to incorporate this. We call this new belief the posterior probability
- Updating our belief is done via the following equation, known as Bayes' Theorem, after its discoverer *Thomas Bayes*:

$$P(A|X) = \frac{P(X|A)P(A)}{P(X)}$$

The formula is not unique to Bayesian inference: it is a mathematical fact with uses outside Bayesian inference

- Bayesian inference merely uses it to connect prior probabilities P(A) with updated posterior probabilities P(A|X)

A useful Formulation

- **When multiple events A_i form an exhaustive set with another event B**
- B can be written as $B = \sum_{i=1}^{n} B \cap A_i$
- So, probability of B can be written as $P(B) = \sum_{i=1}^{n} P(B \cap A_i)$
- Since $P(B \cap A_i) = P(B|A_i) \times P(A_i)$
- The marginal probability of all events A_i together is the sum of all possible terms above, for all *i*
- Replacing P(B) in the equation of conditional probability:

$$P(A_i|B) = (P(B|A_i) \times P(A_i))/(\sum_{i=1}^{n}(P(B|A_i) \times P(A_i)))$$

Bayes' Rule

- As Bayesians, we start with a belief, called a **prior**
- Then we obtain some data and use it to update our **belief**
- The outcome is called a **posterior**
- Should we obtain even more data, the ***old*** **posterior** becomes a ***new*** **prior** and the cycle repeats
- **All components are probability distributions**
- This process employs the **Bayes rule**:
  - **P(A | B) = P(B | A) * P(A) / P(B)**
- In Bayesian machine learning we use the Bayes rule to infer model parameters (theta) from data (D):
  - **P(theta | D) = P(D | theta) * P(theta) / P(data)**
  - ***This is how we can ask machines what parameters they use to make decisions***

Does this make sense?

- Let’s look at our intuition
  - (in black)

Intuition: What does “Bayesian inference” mean?

Inference = *Educated guessing*

Thomas Bayes = A nonconformist Presbyterian minister in London back when the United States were still The Colonies.

He also wrote an essay on probability. His friend Richard Price edited and published it after he died.

Bayesian inference = Guessing *in the style of Bayes*

Dilemma at the movies…

You see a person that dropped their ticket in the hallway. From the back, you see that person has *long har*

Do you call out

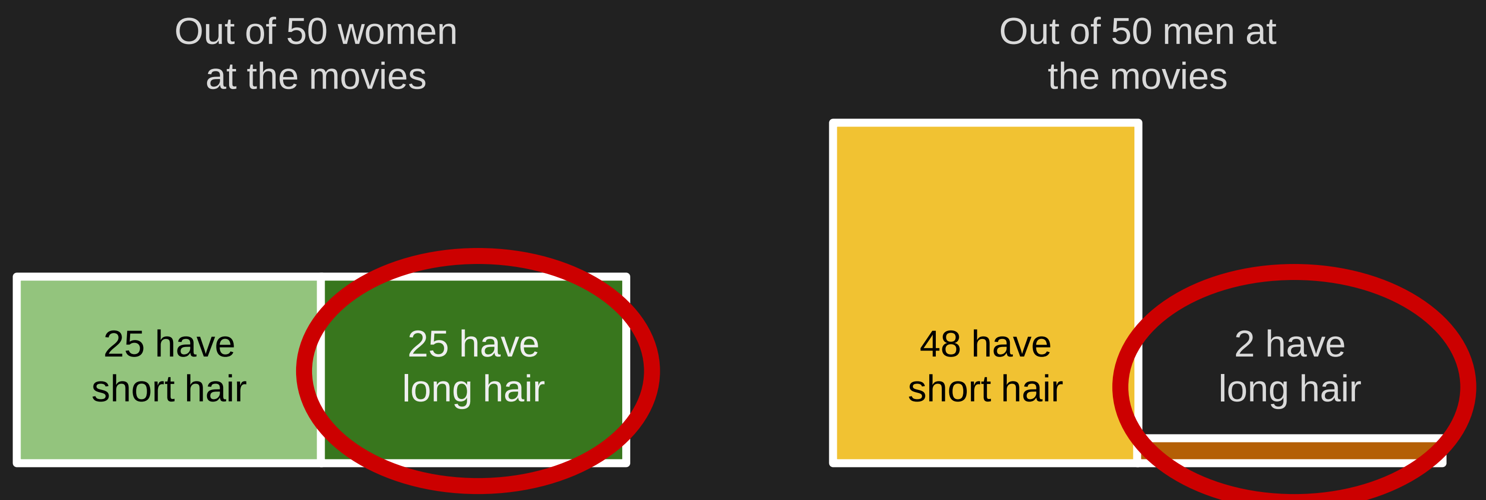
“Excuse me, ma’am!”

or

“Excuse me, sir!”

You have to make a ***guess***!

In the theater, there are 50 women & 50 men…



About 12 times more women have long hair than men.

Conditional probabilities

P(long hair | woman)

If I know that a person is a woman, what is the probability that person has long hair?

P(long hair | woman)

= # favorable outcomes / # all outcomes

= # women with long hair / # women

= 25 / 50 = **.5**

Out of 100 people
at the movies

50 are women

Conditional probabilities

If I know that a person is a man, what is the probability that person has long hair?

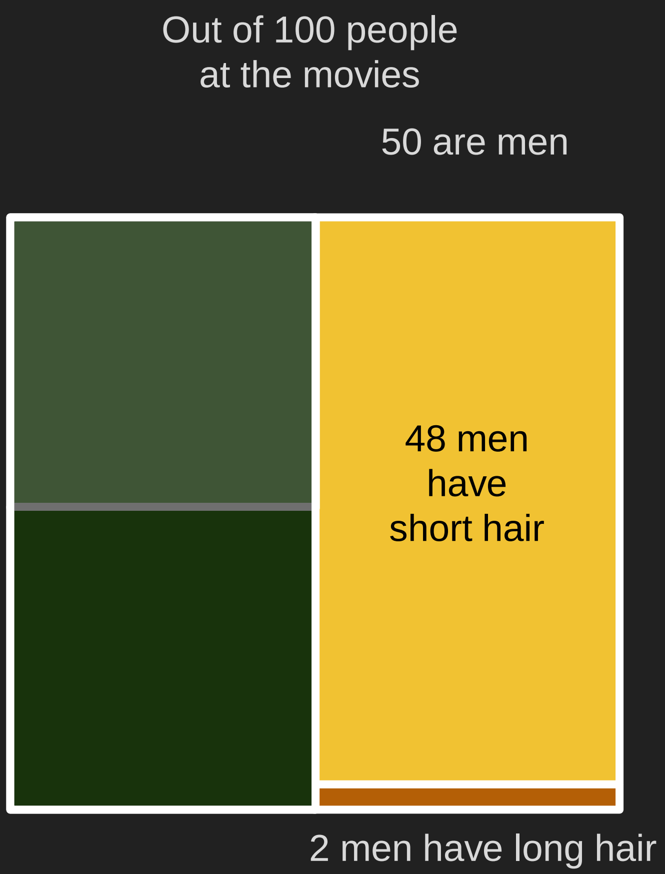
P(long hair | man)

= # favorable outcomes / # all outcomes

= # men with long hair / # men

= 2 / 50 = **.04**

Whether in line or not!



Joint probabilities (∩)

P(A and B) is the probability that both A and B are the case.

Also written P(A, B) or P(A ∩ B)

P(A + B) = P(A) * P(B)

P(A and B) is the same as P(B and A)

The probability that I am having a jelly donut with my milk is the same as the probability that I am having milk with my jelly donut.

P(donut and milk) = P(milk and donut)

Joint (∩) probabilities

What is the probability that a person is both a woman and has ***long*** hair?

P(woman with long hair)

= P(woman) * P(long hair | woman)

= .5 * .5 = **.25**

Out of probability of 1

P(woman) = .5 P(man) = .5

Joint (∩) probabilities

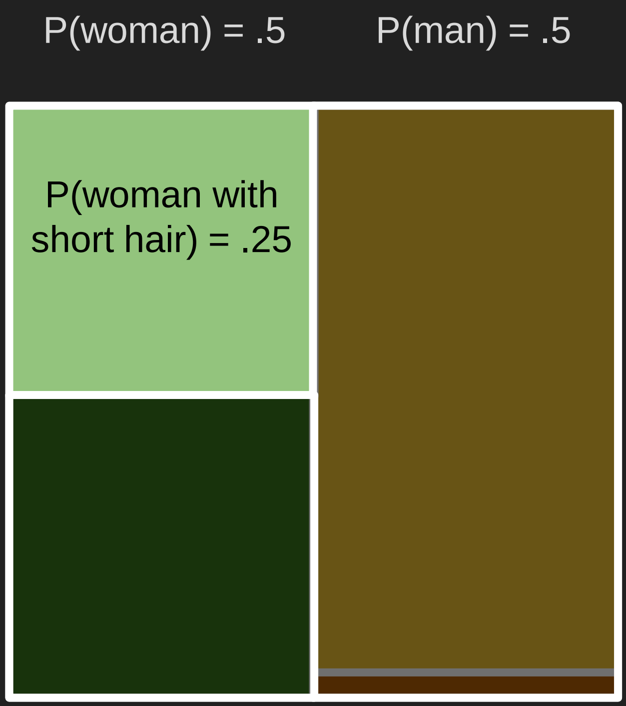
What is the probability that a person is both a woman and has ***short*** hair?

P(woman with short hair)

= P(woman) * P(short hair | woman)

= .5 * .5 = **.25**

Out of probability of 1

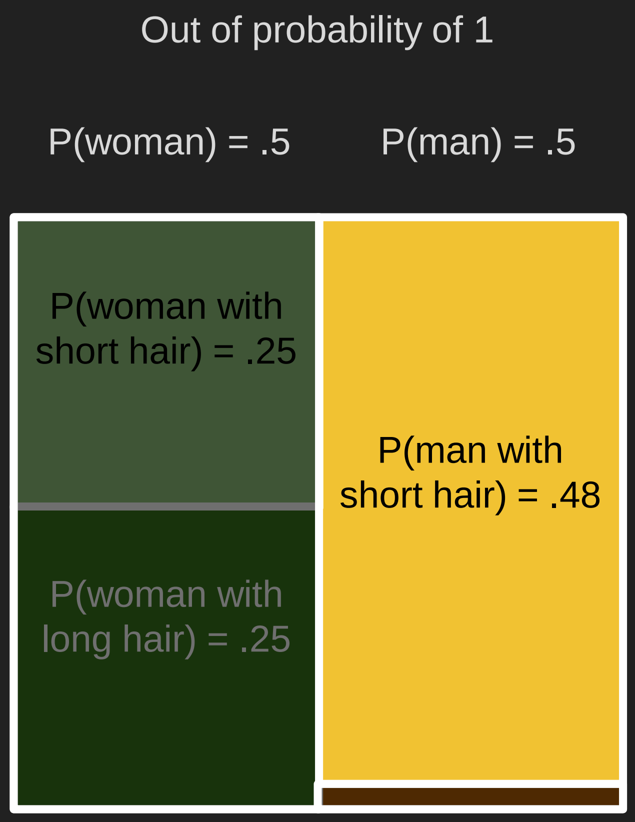


Joint (∩) probabilities

P(man with short hair)

= P(man) * P(short hair | man)

= .5 * .96 = **.48**



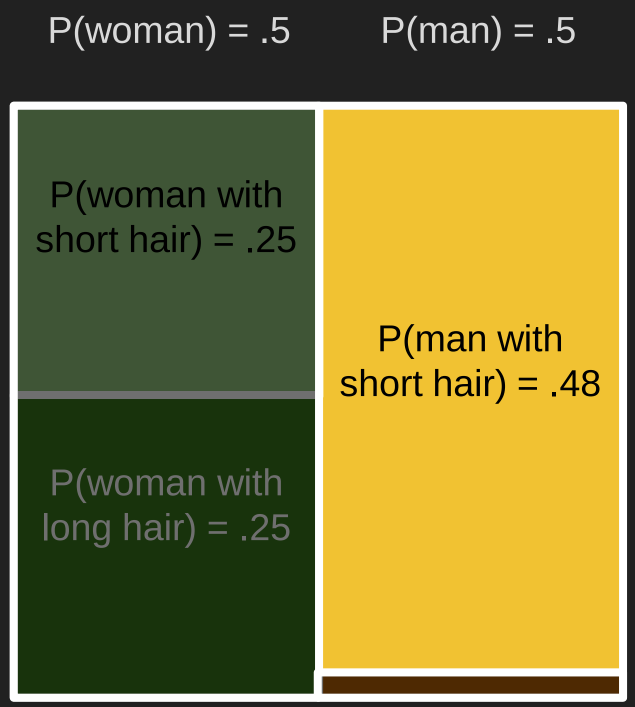
Joint (∩) probabilities

P(man with long hair)

= P(man) * P(long hair | man)

= .5 * .04 = **.02**

Out of probability of 1



Marginal probabilities (∪)

P(A or B) is the probability that either A or B is the case

Also written P(A | B) or P(A ∪ B)

$P(A \mid B) = P(A) + P(B) - P(a \cap b)$

Because otherwise you count this twice!

Marginal probabilities

P(long hair) = P(woman with long hair)

+ P(man with long hair)

= .25 + .02 = **.27**

Marginal probabilities

P(short hair) = P(woman with short hair)

+ P(man with short hair)

= .25 + .48 = **.73**

What we really care about

We know the person **has long hair**. Are they a **man** or a **woman**?

P(man | long hair) = ??

We don't know this answer yet!

Thomas Bayes noticed something cool

P(man with long hair) = P(long hair) * P(man | long hair) *(assume long hair event #1)*

P(long hair and man) = P(man) * P(long hair | man) *(assume man event #1)*

Because P(man and long hair) = P(long hair and man) :

P(long hair) * P(man | long hair) = P(man) * P(long hair | man)

Thomas Bayes noticed something cool

P(man with long hair) = P(long hair) * P(man | long hair)

P(long hair and man) = P(man) * P(long hair | man)

Because P(man and long hair) = P(long hair and man)

P(long hair) * P(man | long hair) = P(man) * P(long hair | man)

P(man | long hair) = P(man) * P(long hair | man) / P(long hair)

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P(man with long hair) = P(long hair) * P(man | long hair)

P(long hair and man) = P(man) * P(long hair | man)

Because P(man and long hair) = P(long hair and man)

P(long hair) * P(man | long hair) = P(man) * P(long hair | man)

P(man | long hair) = P(man) * P(long hair | man) / P(long hair)

P(A | B) = P(B | A) * P(A) / P(B)

Bayes' Theorem

$$P(A \mid B) = \frac{P(B \mid A)\ P(A)}{P(B)}$$

Back to the movie theater, this time with Bayes

$$P(\text{man} \mid \text{long hair}) = \frac{P(\text{man}) * P(\text{long hair} \mid \text{man})}{P(\text{long hair})}$$

$$= \frac{P(\text{man}) * P(\text{long hair} \mid \text{man})}{P(\text{woman with long hair}) + P(\text{man with long hair})}$$

slide #8

slide #11

$$= \frac{.5 * .04}{.25 + .02} = .02 / .27 = \boxed{.07}$$

slide #13

slide #16

..so now you ***know***!

P(woman) = .5 P(man) = .5

P(long hair | man) = .04
P(long hair | woman) = .5

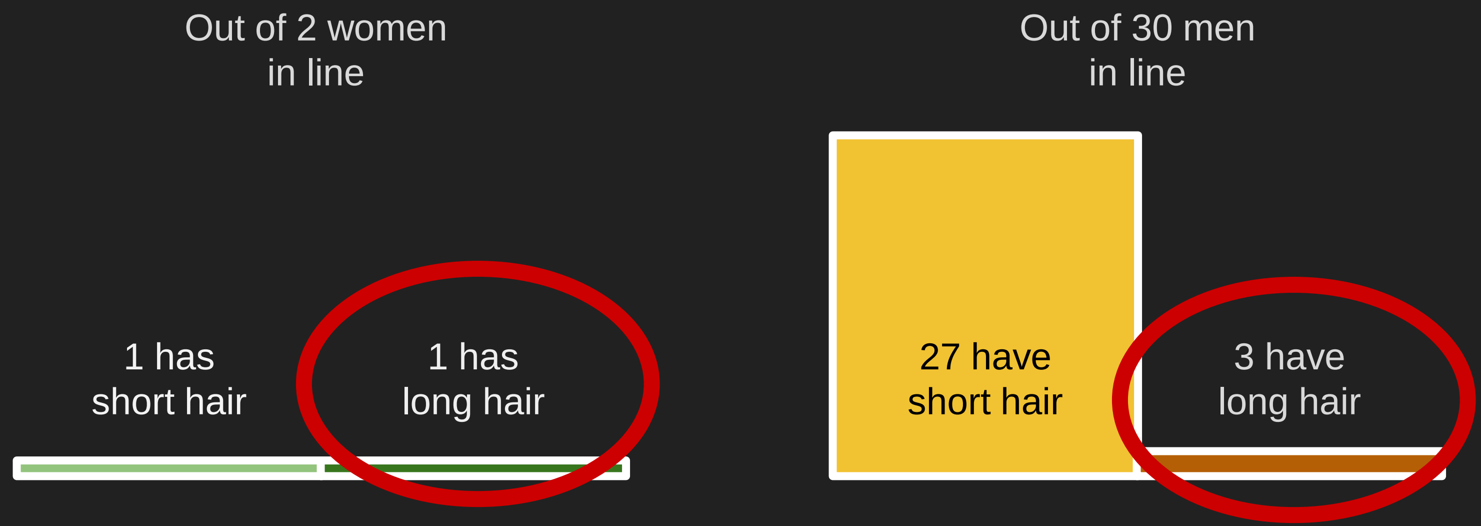
Conclusion

- You update your belief based on ***evidence***

What happens if you receive *different* evidence?

- Suppose they have beer and wine at the movies
- ***Everyone knows beer makes you pee..***
- So there will be many more ***men*** in line at the bathroom than ***women***

Meanwhile, at the bathroom line…



In the line, more men have long hair than women. The evidence has ***changed because there are more men in line,*** but *let’s assume* P(long hair | man), P(woman with long hair), P(man with long hair) are *population statistics* and *don’t change*

Back to the bathroom line, this time with Bayes

$$P(\text{man} \mid \text{long hair}) = \frac{P(\text{man}) * P(\text{long hair} \mid \text{man})}{P(\text{long hair})}$$

$$= \frac{P(\text{man}) * P(\text{long hair} \mid \text{man})}{P(\text{woman with long hair}) + P(\text{man with long hair})}$$

New

$$= \frac{.9375 * .04}{.01 + .04} = \boxed{.14}$$

..the probability has ***doubled*** given the *new* evidence

Update your posterior probability with Bayes' formula

- Probabilities change with new data!

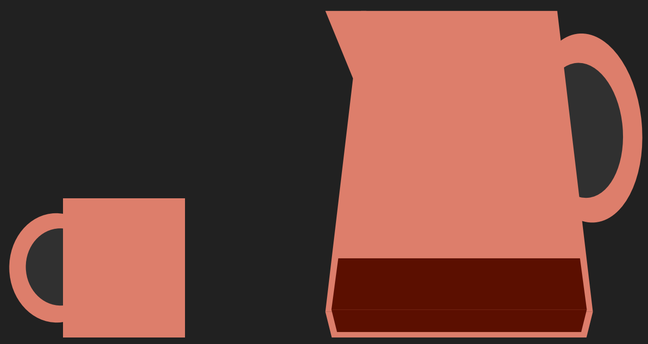
Believe the impossible, at least a little bit

Leave room for believing the unlikely. Leave a non-zero probability unless you are absolutely certain.

> “When you have excluded the impossible, whatever remains, however improbable, must be the truth”
>
> - Sherlock Holmes (Sir Arthur Conan Doyle)

Probability distributions

Probability is like a pot with just one cup of coffee left in it.

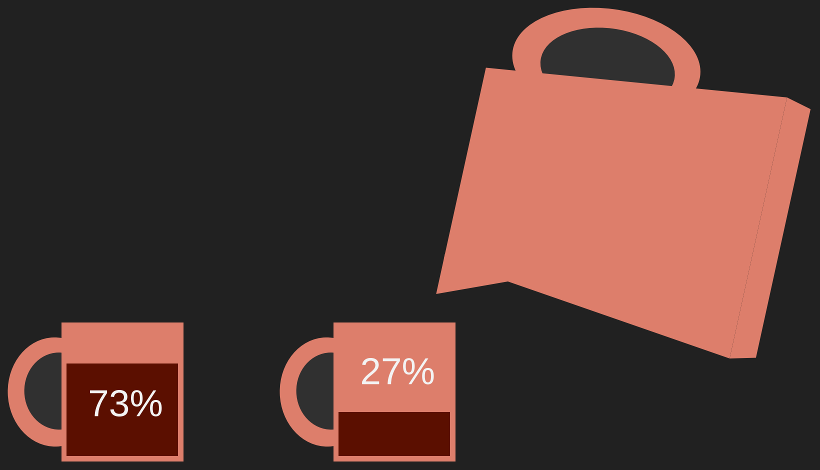


Probability distributions

If you only have one cup, you can fill it completely.

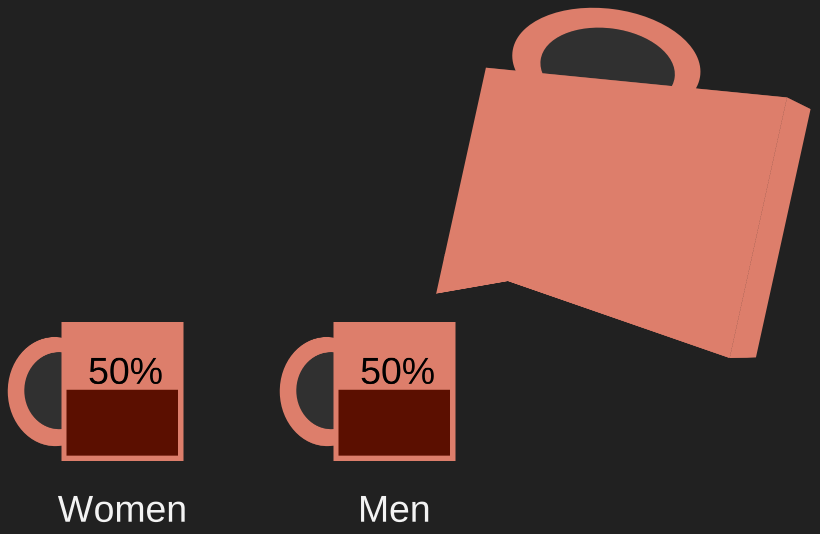
Probability distributions

If you have two cups, you have to decide how to share (distribute) it.



Probability distributions

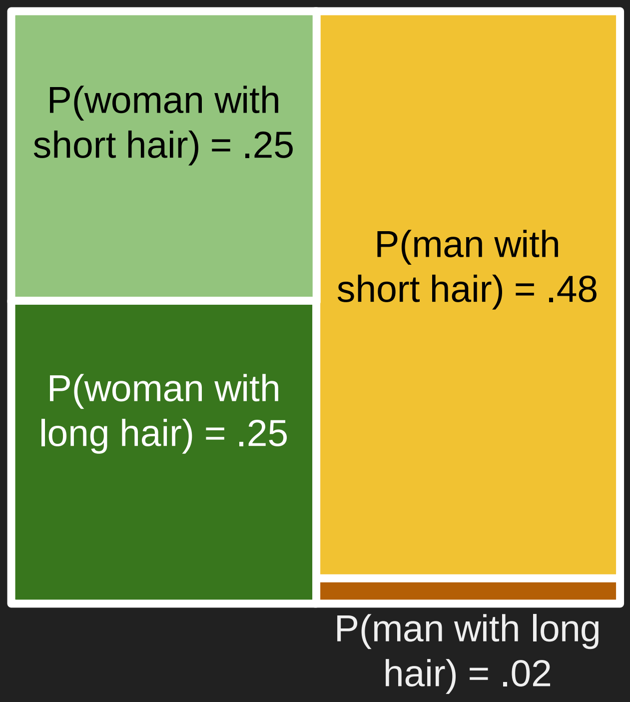
Our people are distributed between two groups, women and men.



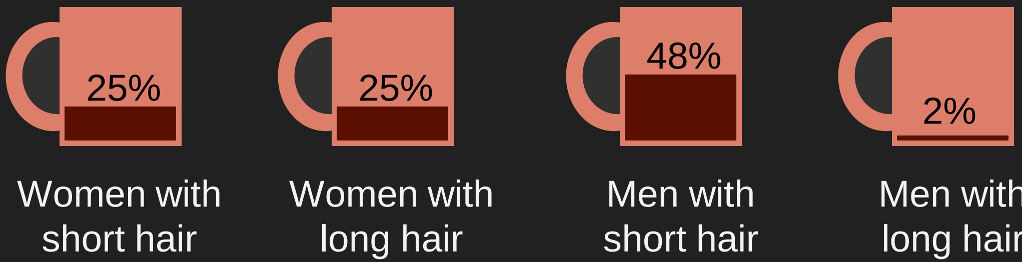
Probability distributions

We can distribute them more.

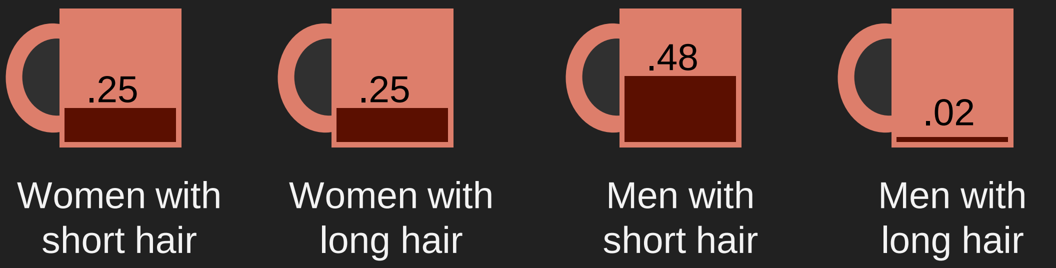
Probability distributions



Probability distributions

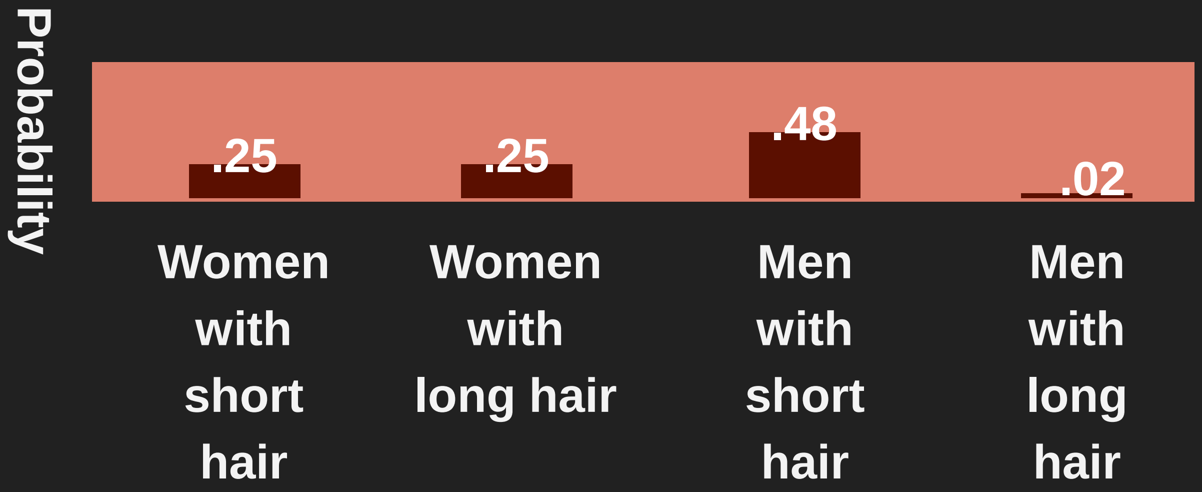


Probability distributions



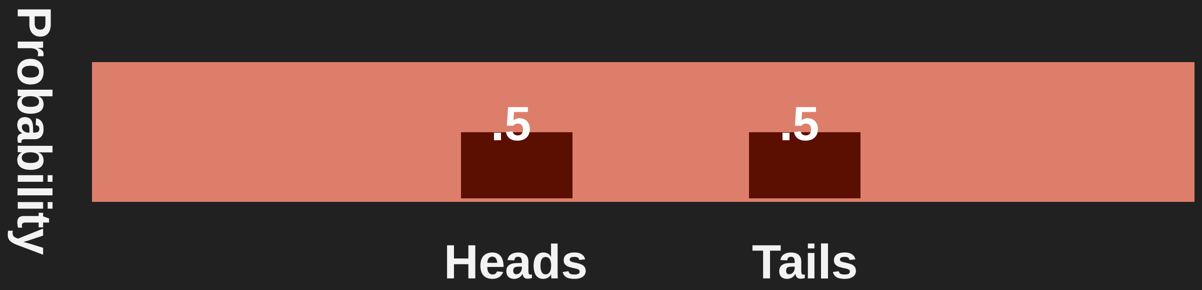
Probability distributions

It’s helpful to think of probabilities as beliefs



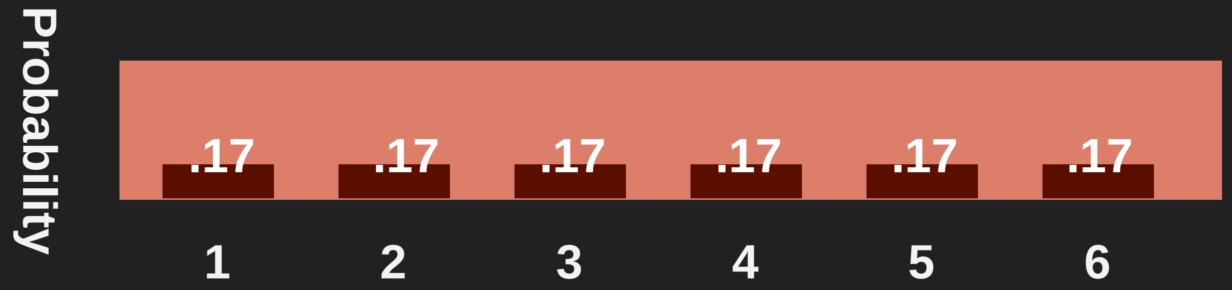
Probability distributions

Flipping a fair coin



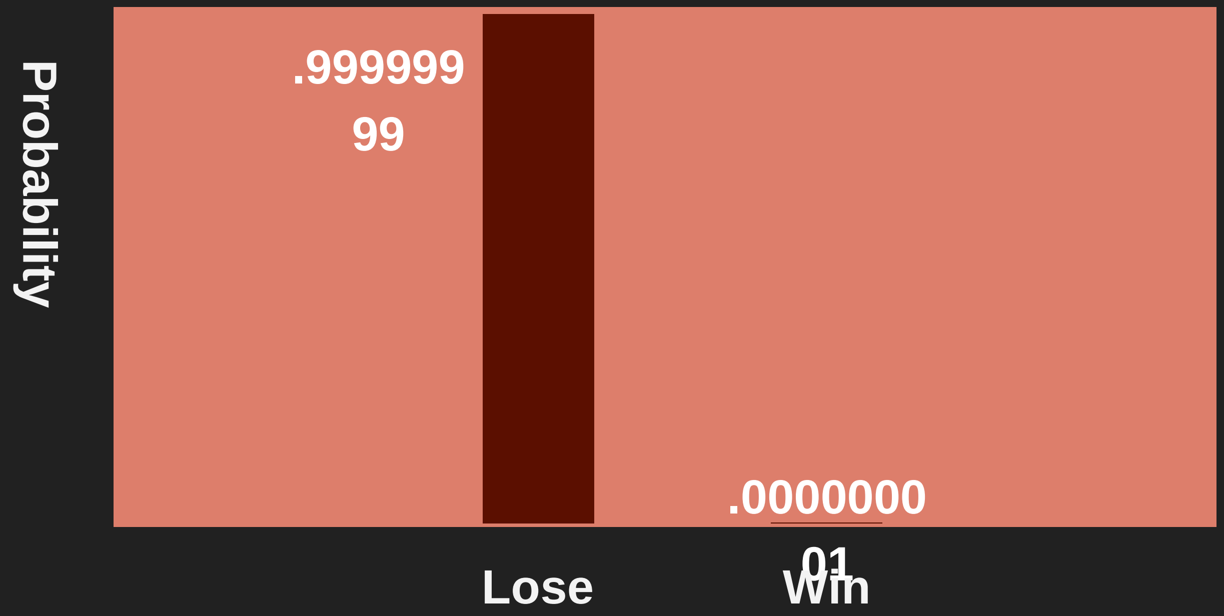
Probability distributions

Rolling a fair die



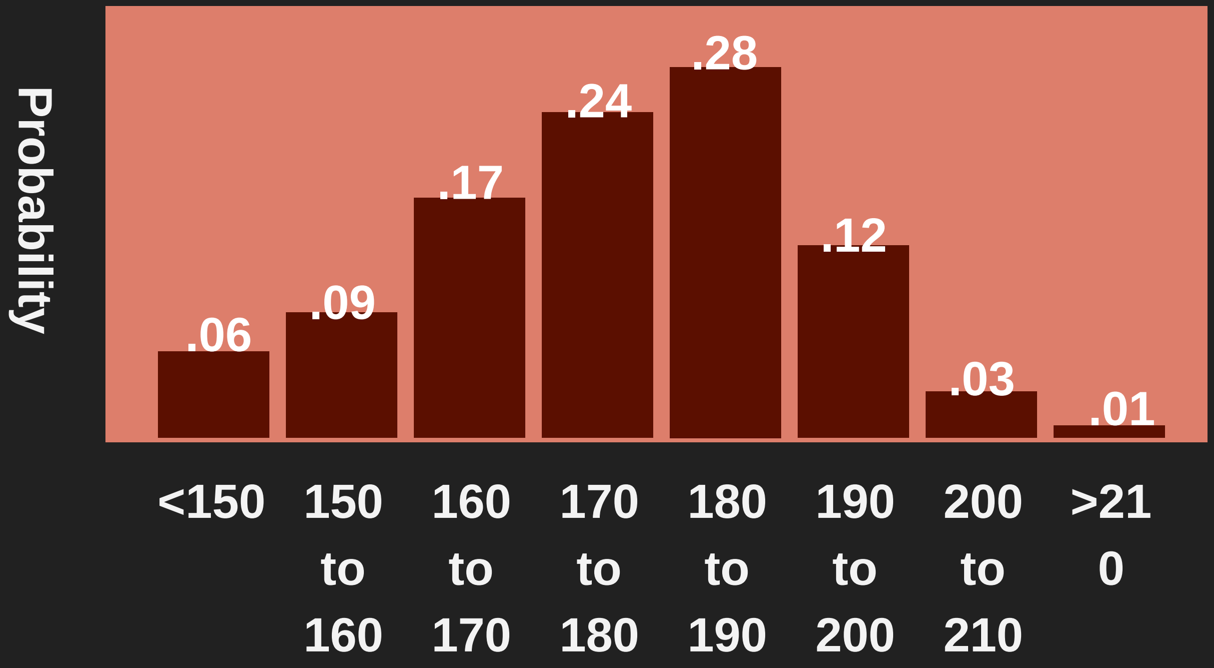
Probability distributions

Playing for the Powerball jackpot



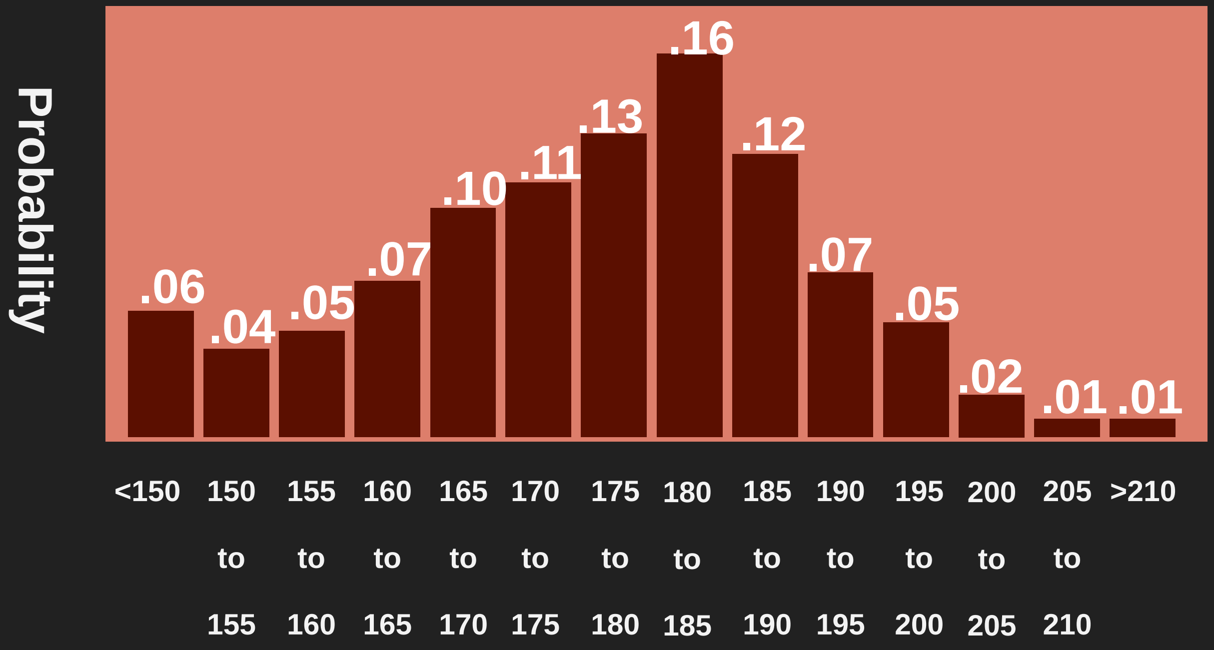
Probability distributions

Height of adults in cm



Probability distributions

Height of adults in cm



Probability distributions

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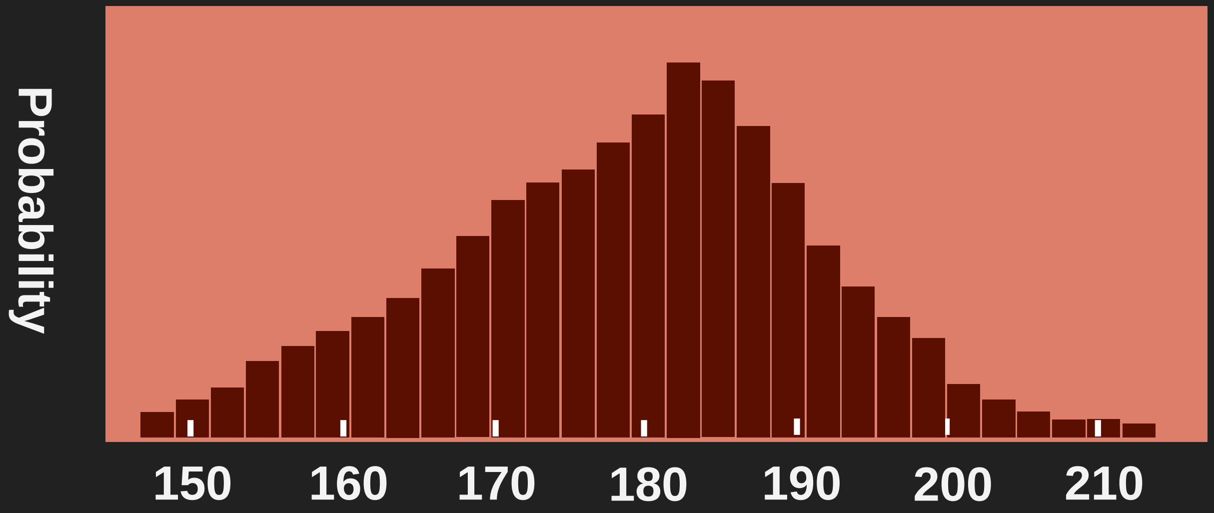
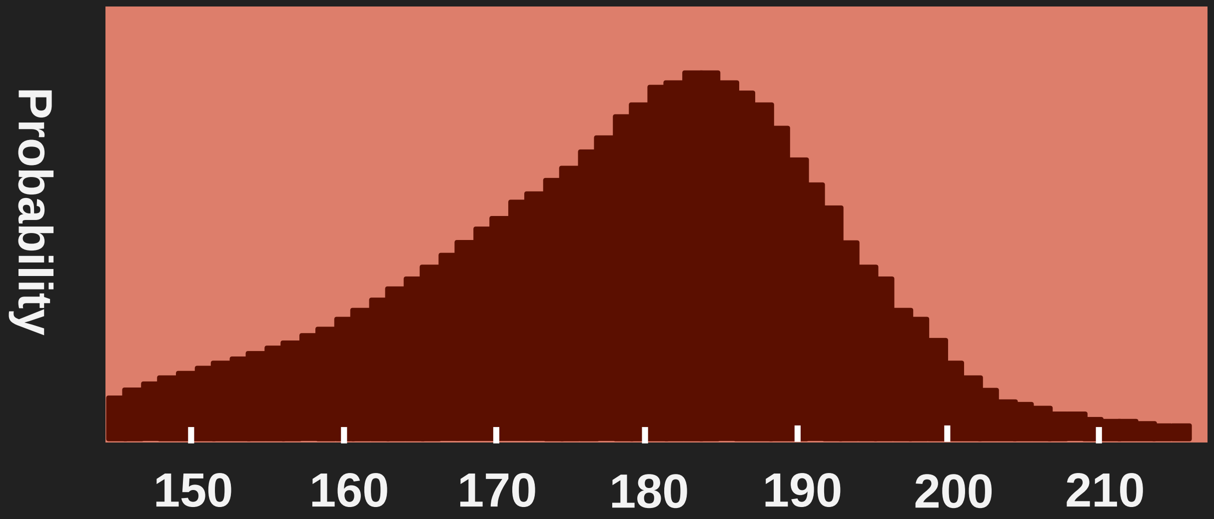


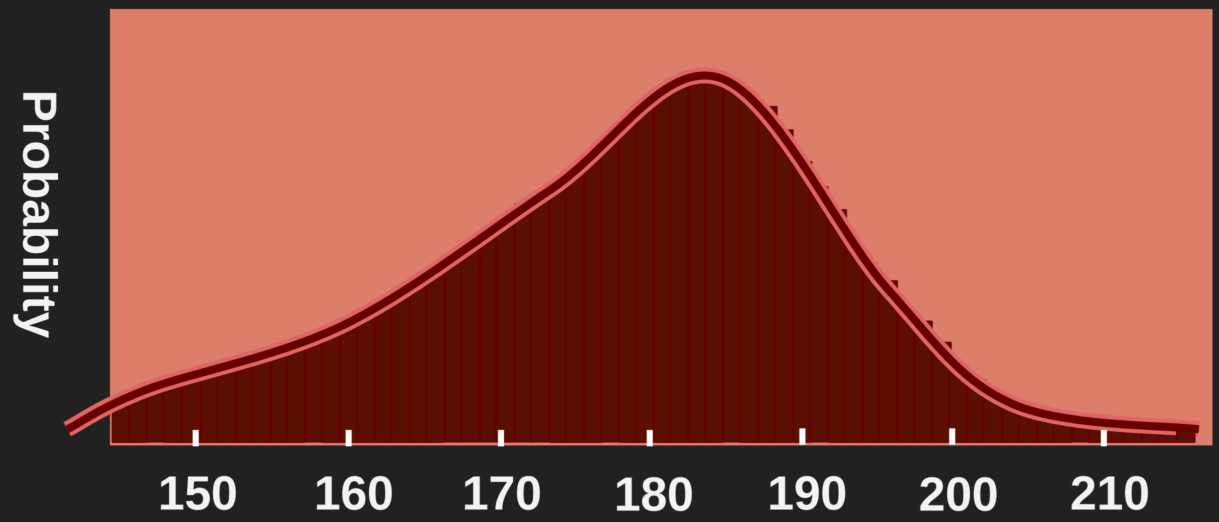
Probability distributions

Height of adults in cm



Probability distributions

Height of adults in cm



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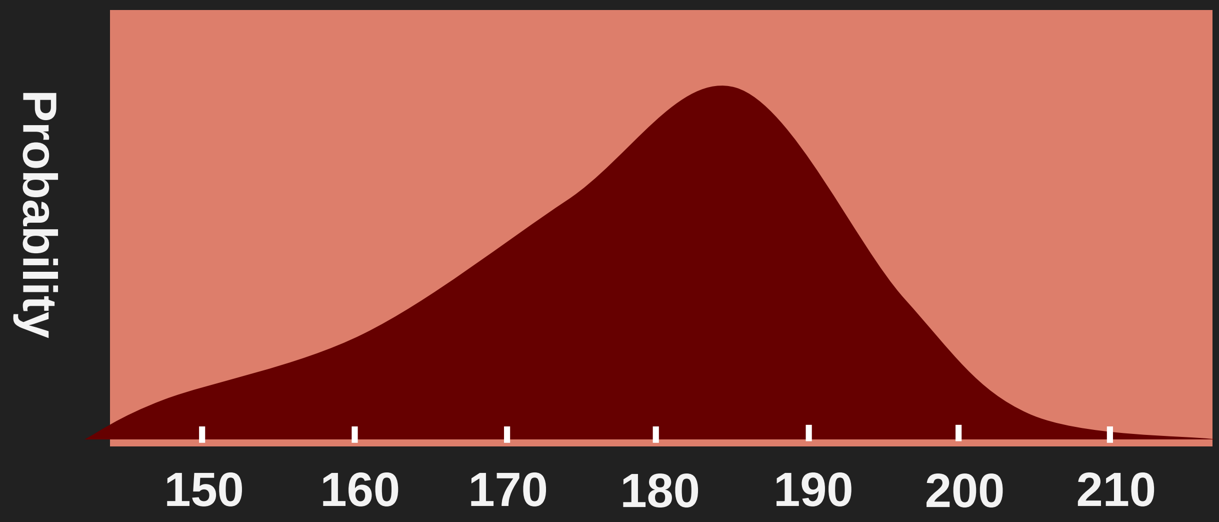

Probability distributions

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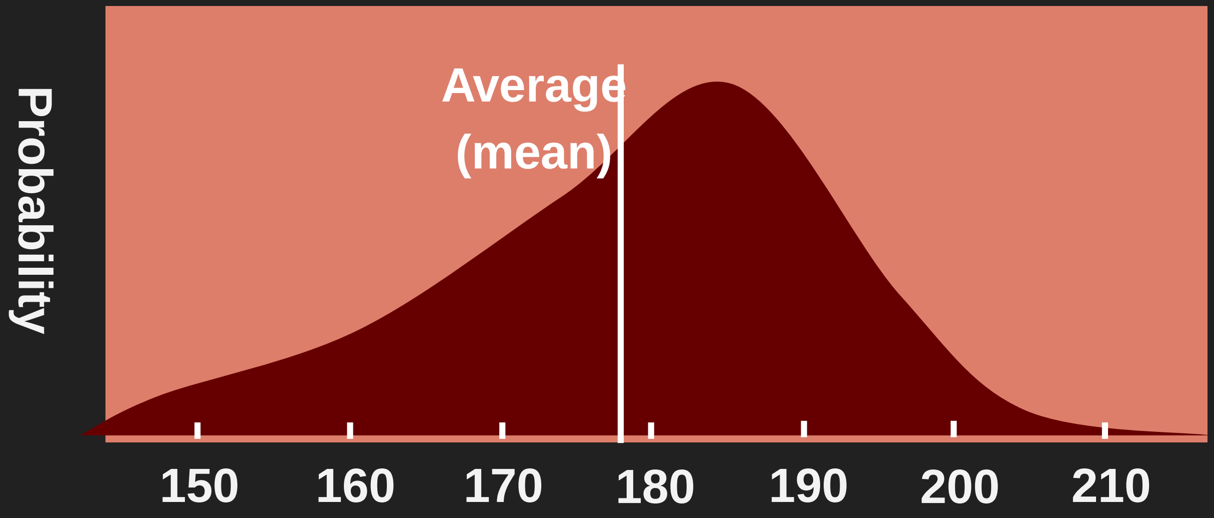

Probability distributions

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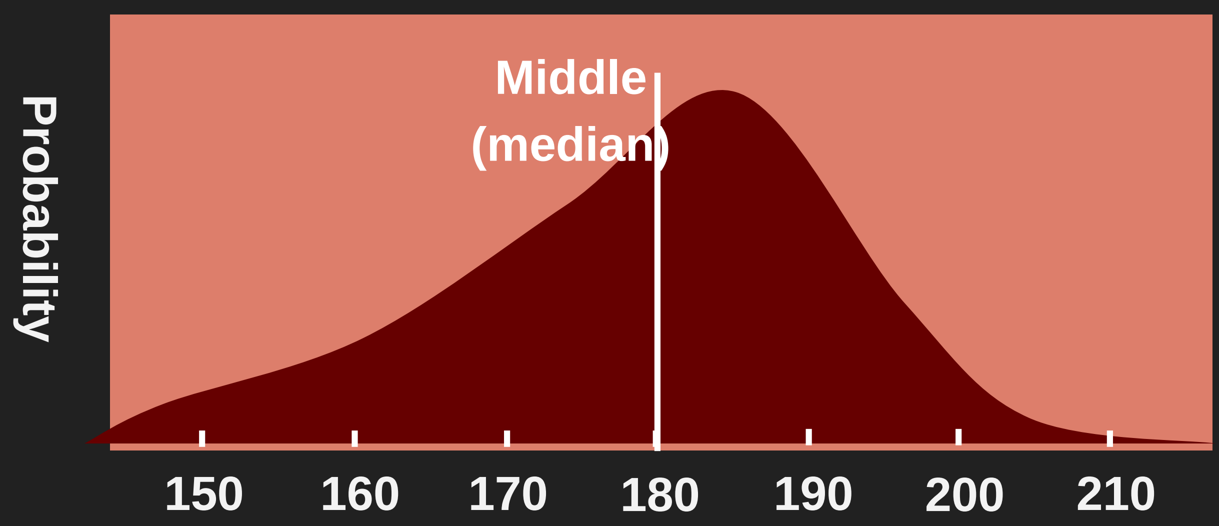
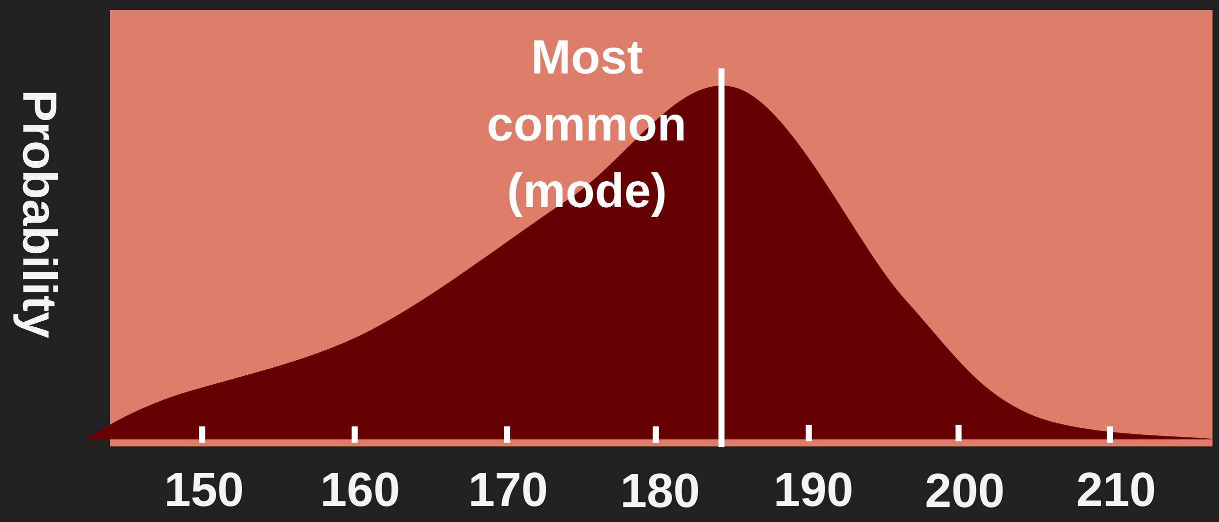


Probability distributions

Height of adults in cm



Bayes' Theorem

$$P(w \mid m) = \frac{P(m \mid w)\ P(w)}{P(m)}$$

Bayes' Theorem

prior

$$P(w \mid m) = \frac{P(m \mid w)\ \boxed{P(w)}}{P(m)}$$

Bayes' Theorem

likelihood

$$P(w \mid m) = \frac{\boxed{P(m \mid w)}\ P(w)}{P(m)}$$

Bayes' Theorem

posterior

$$\boxed{P(w \mid m)} = \frac{P(m \mid w)\ P(w)}{P(m)}$$

Bayes' Theorem

$$P(w \mid m) = \frac{P(m \mid w)\; P(w)}{\boxed{P(m)}}$$

marginal likelihood

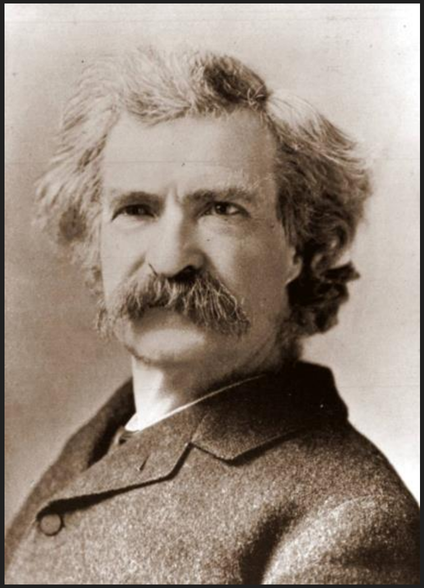
Why Bayesian inference makes us nervous

We’re not always aware of what we believe.

Putting what we believe into a distribution correctly is tricky.

We want to be able to be surprised by our data.
Inaccurate beliefs can make it hard or impossible to learn.

> “It ain't what you don't know that gets you into trouble.
> It's what you know for sure that just ain't so.”
>
> - Mark Twain



Believe the impossible, at least a little bit

"Alice laughed: "There's no use trying," she said; "one can't believe impossible things."

"I daresay you haven't had much practice," said the Queen. "When I was younger, I always did it for half an hour a day. Why, sometimes I've believed as many as six impossible things before breakfast."

- Lewis Carroll (Alice’s Adventures in Wonderland)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