STATS 217: Introduction to Stochastic Processes I

Lecture 4

Branching processes

- Consider a single bacterium in an ideal environment. We call this the generation 0 bacterium.
- This bacterium gives birth to ξ bacteria, where ξ is a non-negative integer valued random variable. We call these the generation 1 bacteria.
- Generally, let the generation k bacteria be $b_1, \ldots, b_k$. Then, b_i gives birth to ξ_i bacteria where $\xi_1, \ldots, \xi_k$ are i.i.d. copies of ξ.
- What is the probability that the bacteria population goes extinct?
- This problem was studied by Galton and Watson in relation to the propagation of last names in Victorian England.

Branching processes

Let Z_n denote the number of bacteria in generation n and let $(\xi_{i,j})$ denote i.i.d. copies of ξ. Then,

- $Z_0 = 1$,
- $Z_1 = \xi_{0,1}$,
- $Z_2 = \sum_{i=1}^{Z_1} \xi_{1,i}$, ...
- $Z_k = \sum_{i=1}^{Z_{k-1}} \xi_{k-1,i}$.

Note that if $Z_i = 0$ for some $i \geq 1$, then $Z_j = 0$ for all $j \geq i$. This corresponds to the extinction of the population.

Formally, we say that 0 is an **absorbing state** for the process $(Z_n)_{n \geq 0}$.

Branching processes

- We have a branching process $(Z_n)_{n\geq 0}$ with **offspring distribution** ξ.
- We are interested in the probability that the population survives i.e.

$$\mathbb{P}[Z_n \geq 1 \quad \forall n].$$

- Trivial case: Suppose $\mathbb{P}[\xi \geq 1] = 1$. Then, $\mathbb{P}[Z_n \geq 1 \quad \forall n] = 1$.
- Hence, we may assume that for all integers $k \geq 0$,

$$\mathbb{P}[\xi = k] =: p_k$$

with $0 < p_0 < 1$.

Expected size of generation n

Suppose that $\mu := \mathbb{E}[\xi]$. What is the expectation of Z_n?

- $\mathbb{E}[Z_0] = 1$.
- $\mathbb{E}[Z_1] = \mathbb{E}[\xi_{0,1}] = \mu$.
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$$\begin{aligned}
\mathbb{E}[Z_2] &= \mathbb{E}\left[\sum_{i=1}^{Z_1} \xi_{1,i}\right] \\
&= \sum_{z \geq 0} \mathbb{E}\left[\sum_{i=1}^{z} \xi_{i,1}\right] \mathbb{P}[Z_1 = z] \\
&= \sum_{z \geq 0} z\mu \mathbb{P}[Z_1 = z] \\
&= \mu \sum_{z \geq 0} z \mathbb{P}[Z_1 = z] \\
&= \mu \cdot \mathbb{E}[Z_1] = \mu^2.
\end{aligned}$$

Subcritical case

- Similarly, $\mathbb{E}[Z_n] = \mu\mathbb{E}[Z_{n-1}] = \mu^n$.
- This shows that if $\mu < 1$, then with probability 1, the population becomes extinct.
- Indeed, if $\mu < 1$ (this is called the **subcritical case**), then

$$\mathbb{P}[Z_n \geq 1] \leq \mathbb{E}[Z_n] = \mu^n \to 0.$$

- What about the case when $\mu \geq 1$?
- If $\mu = 1$, then $\mathbb{E}[Z_n] = 1$ and if $\mu > 1$, then $\mathbb{E}[Z_n] \to \infty$, but this doesn't say anything about the probability of survival.

First step analysis

- To analyse the case $\mu \geq 1$, we will use first step analysis.
- Let ρ denote the probability that the population eventually dies out so that

$$\rho = \mathbb{P}[Z_n = 0 \text{ for some } n \geq 1].$$

- Suppose that the bacterium b in generation 0 has k children $b_1, \ldots, b_k$. Then, the population dies out if and only if the subpopulations starting at $b_1, \ldots, b_k$ die out. Moreover, the probability of each of these subpopulations dying out is also ρ.

First step analysis

- Therefore,

$$\rho = \sum_{k=0}^{\infty} \mathbb{P}[\xi_{0,1} = k]\rho^k = \sum_{k=0}^{\infty} p_k \rho^k = \phi(\rho),$$

  where

$$\phi(z) := \sum_{k=0}^{\infty} p_k z^k$$

  is the **generating function** of $(p_k)_{k \geq 0}$.
- So, we see that the probability of extinction is a fixed point of the generating function i.e. a solution of

$$\rho = \phi(\rho).$$

First step analysis

- We saw that the probability of extinction is a solution of

$$\rho = \phi(\rho) = \sum_{k \geq 0} p_k \rho^k.$$

- Since

$$\phi(1) = \sum_{k \geq 0} p_k = 1,$$

  we see that 1 is always a solution of $\rho = \phi(\rho)$.
- However, this does not mean that the extinction probability is 1, since there may be other solutions to $\rho = \phi(\rho)$.

Properties of the generating function

Recall that $\phi(z) = \sum_{k \geq 0} p_k z^k$.

- ϕ is non-decreasing on $[0, 1]$.
- ϕ is continuous on $[0, 1]$.
- $\phi(0) = p_0 \in (0, 1)$.
- $\phi(1) = 1$.
- $\phi'(z) = \sum_{k \geq 1} k p_k z^{k-1}$.
- Hence, $\phi'(1) = \sum_{k \geq 1} k p_k = \mu$.
- $\phi''(z) = \sum_{k \geq 2} k(k-1) p_k z^{k-2} > 0$ for $z \in (0, 1]$.
- Hence, ϕ is strictly convex on $(0, 1]$.

Properties of the generating function

Let $g(\rho) = \phi(\rho) - \rho$. We are interested in the solutions of $g(\rho) = 0$ for $\rho = [0, 1]$.

- We have $g(0) = p_0 \in (0, 1)$, $g(1) = 0$.
- $g''(\rho) = \phi''(\rho) > 0$ for $\rho \in (0, 1]$.
- $g'(\rho) = \phi'(\rho) - 1$.
- So, we have two cases:
  - If $\phi'(1) \leq 1$, then $g'(1) \leq 0$ and $g'(\rho) < 0$ for all $\rho \in [0, 1)$. Hence, the only solution of $g(\rho) = 0$ is at $\rho = 1$.
  - If $\phi'(1) > 1$, then $g'(1) > 0$. So, there exists exactly one $\rho \in (0, 1)$ such that $g(\rho) = 0$.

Critical case

- We know that the extinction probability ρ is a solution of $\phi(\rho) = \rho$.
- We also saw that when $\mu = \phi'(1) = 1$, this equation has only one solution: $\rho = 1$.
- Therefore, if $\mu = 1$ (this is called the **critical case**), we see that $\rho = 1$.

Supercritical case

- It remains to deal with the case when $\mu > 1$ (this is called the **supercritical case**).
- In this case, $\phi(\rho) = \rho$ has two solutions: $\rho^* < 1$ and 1.
- We claim that the extinction probability in this case is ρ^*.
- To see this, let

$$\rho_n = \mathbb{P}[Z_n = 0].$$

- Then, by first step analysis, we have

$$\rho_n = \sum_{k \geq 0} p_k \rho_{n-1}^k = \phi(\rho_{n-1}).$$

Supercritical case

- We have $\rho_n = \phi(\rho_{n-1})$.
- Since ϕ is a non-decreasing function, $\rho_0 \leq \rho_1 \leq \rho_2 \leq \ldots$.
- Since $\rho_0 \leq \rho^*$, it follows that

$$\rho_1 = \phi(\rho_0) \leq \phi(\rho^*) = \rho^*.$$

- Iterating this shows that $\rho_n \leq \rho^*$ for all n.
- Therefore,

$$\rho = \lim_{n\to\infty} \mathbb{P}[Z_n = 0] = \lim_{n\to\infty} p_n \leq \rho^*.$$

- Finally, since $\rho = \phi(\rho)$, it must be the case that $\rho = \rho^*$.

Summary

Thus, we have established the following theorem.

- Let $(Z_n)_{n\geq 0}$ be a branching process with $Z_0 = 0$ and common offspring distribution ξ.
- Let $\mu = \mathbb{E}[\xi]$ and let $\phi(z) = \sum_{k\geq 0} \mathbb{P}[\xi = k] z^k$.
- Suppose that $0 < p_0 = \mathbb{P}[\xi = 0] < 1$.
- Let ρ be the probability of extinction.
- Then, ρ is the smallest solution of $\phi(z) = z$, $z \in [0, 1]$.
- If $\mu \leq 1$, then $\rho = 1$.
- If $\mu > 1$, then $\rho < 1$.