

HOMEWORK 2

DUE 01/30 AT 7:00PM PST

- (1) Let $(Z_n)_{n \geq 0}$ be a branching process with $Z_0 = 1$ and offspring distribution ξ with $\mathbb{E}[\xi] = \mu > 0$ and $\text{Var}(\xi) = \sigma^2$. Show that

$$\text{Var}(Z_n) = \begin{cases} \sigma^2 n & \mu = 1 \\ \sigma^2 \mu^{n-1} \left(\frac{1-\mu^n}{1-\mu} \right) & \mu \neq 1. \end{cases}$$

Hint: Just as in our calculation of $\mathbb{E}[Z_n]$, try to relate $\text{Var}(Z_n)$ to $\text{Var}(Z_{n-1})$.

- (2) Let $(Z_n)_{n \geq 0}$ be a branching process with $Z_0 = 1$ and offspring distribution ξ . Find the probability of extinction in each of the following situations.

- (a) There exists some $p \in (0, 1)$ such that

$$\mathbb{P}[\xi = 0] = p, \quad \mathbb{P}[\xi = 2] = (1 - p).$$

- (b) There exists some $p \in (0, 1)$ such that $\xi \sim \text{Geom}(p)$, i.e.,

$$\mathbb{P}[\xi = k] = p(1 - p)^k, \quad k = 0, 1, 2, \dots$$

- (c) $\xi \sim \text{Pois}(1.1)$. Also compute the probability of extinction in 0 generations, 1 generation, 2 generations, and 3 generations.

- (3) (due to Pinsky and Karlin) At time 0, a blood culture starts with one red cell. At the end of 1 minute, the red cell dies and is replaced by one of the following combinations with the probabilities as indicated:

$$\begin{cases} \text{Two red cells} & \text{with probability } \frac{1}{4}, \\ \text{One red cell, one white cell} & \text{with probability } \frac{2}{3}, \\ \text{Two white cells} & \text{with probability } \frac{1}{12}. \end{cases}$$

Each red cell lives for 1 minute and gives birth to offspring in the same way as the parent cell. Each white cell lives for 1 minute and dies without reproducing. Assume that individual cells behave independently.

- (a) At time $n + 1/2$ after the culture begins, what is the probability that no white cells have yet appeared?
- (b) What is the probability that the entire culture dies out entirely?
- (4) (due to Grimmett and Stirzaker) Let $(Z_n)_{n \geq 0}$ denote a branching process with $Z_0 = 1$ and offspring distribution ξ . Suppose that $\mathbb{E}[\xi] = \mu > 1$ and let $u \in (0, 1)$ denote the extinction probability. Let ϕ denote the generating function of the offspring distribution and let ϕ_n denote the generating function of the distribution of Z_n . Show that

$$\mathbb{E}[s^{Z_n} \mid \text{extinction}] = \frac{1}{u} \phi_n(su).$$

- (5) (*) Consider the branching process with alternating distributions, which is defined as follows. Consider two offspring distributions ξ and η with generating functions ϕ_1 and ϕ_2 respectively. We start with a single individual i.e. $Z_0 = 1$. The individuals in even generations reproduce independently according to

the distribution η whereas those in odd generations reproduce independently according to the distribution ξ i.e.

$$Z_{2n+1} = \sum_{i=1}^{Z_{2n}} \eta_{2n,i}$$

$$Z_{2n} = \sum_{i=0}^{Z_{2n-1}} \xi_{2n-1,i}.$$

- (a) Let ρ_{21} denote the probability of extinction. Show that ρ_{21} is the smallest solution in $(0, 1]$ of the equation

$$x = \phi_2(\phi_1(x)).$$

- (b) (**) (due to Munford) Let ρ_{12} denote the probability of extinction of the branching process in which the even generations reproduce according to ξ and the odd generations reproduce according to η . Also, let ρ_1 (respectively ρ_2) denote the probability of extinction of a branching process with offspring distribution ξ (respectively η). Show that

$$\rho_1 \leq \rho_2 \implies \rho_1 \leq \rho_{12} \leq \rho_{21} \leq \rho_2.$$

Hint: Use the characterization of $\rho_1, \rho_2, \rho_{12}, \rho_{21}$ to show that $\rho_{12} = \phi_1(\rho_{21})$ and $\rho_{21} = \phi_2(\rho_{12})$. Also, by the characterization of ρ_1, ρ_2 , note that $\rho_1 \leq \rho_2$ is equivalent to $\phi_1(\rho_2) \leq \rho_2$.

- (6) Consider a simple random walk $(S_n)_{n \geq 0}$ on \mathbb{Z} starting from $S_0 = 0$ for which the transitions are as follows:

$$\begin{cases} S_n = S_{n-1} - 1 & \text{with probability } q \\ S_n = S_{n-1} + 1 & \text{with probability } p = 1 - q, \end{cases}$$

where $p \in (0, 1)$.

- (a) Let τ_1 denote the first time that the random walk hits 1 and let ϕ denote the generating function of the distribution of τ_1 . Show that

$$\phi(s) = \frac{1 - \sqrt{1 - 4pqs^2}}{2qs}, \quad |s| \leq \frac{1}{2\sqrt{pq}}.$$

Hint: Use first step analysis to argue that

$$\tau_1 \sim \begin{cases} 1 & \text{with probability } p \\ 1 + \tau'_1 + \tau''_1 & \text{with probability } q, \end{cases}$$

where τ'_1, τ''_1 are i.i.d. random variables with the same distribution as τ .

- (b) Let p_1 denote the probability that the walk ever reaches 1. Show that

$$p_1 = \min \left\{ \frac{p}{q}, 1 \right\}.$$

Hint: What is $\phi(1)$?

- (c) Show that

$$\mathbb{E}[\tau_1] = \begin{cases} \infty & p \leq 1/2 \\ \frac{1}{p-q} & p > 1/2. \end{cases}$$

Hint: Use part (a).

- (7) (*) Consider a random walk $(S_n)_{n \geq 0}$ on \mathbb{Z} starting from $S_0 = 0$ for which the transitions are as follows:

$$\begin{cases} S_n = S_{n-1} - 2 & \text{with probability } 1/2 \\ S_n = S_{n-1} + 1 & \text{with probability } 1/2. \end{cases}$$

Let p_1 denote the probability that the walk ever reaches 1. Show that

$$p_1 = \frac{\sqrt{5} - 1}{2}.$$

- (8) (*) Let $(Z_n)_{n \geq 0}$ denote a branching process with $Z_0 = 1$ and offspring distribution ξ . Let

$$Y_n = Z_0 + Z_1 + \cdots + Z_n$$

denote the total number of individuals up to and including generation n (this is called the **total progeny**). Let ψ_n denote the generating function of Y_n and let ϕ denote the generating function of ξ . Let ρ^* denote the extinction probability of the branching process.

- (a) Show that

$$\psi_n(s) = s \cdot \phi(\psi_{n-1}(s)).$$

- (b) Show that for all $0 < s < 1$, $\psi_n(s)$ is a decreasing sequence in n .

- (c) Let $\psi(s) = \lim_{n \rightarrow \infty} \psi_n(s)$. Show that for all $0 < s < 1$,

$$\psi(s) = s \cdot \phi(\psi(s)).$$

- (d) For this, and subsequent parts, you may use that

$$\psi(s) = \sum_{k=0}^{\infty} q_k s^k$$

for some $(q_k)_{k \geq 0}$ with $q_k \geq 0$ for all k and $\sum_{k=0}^{\infty} q_k \leq 1$.
Show that for $0 < s < 1$, the equation

$$x = s \cdot \phi(x)$$

has a unique solution in $(0, 1]$ and in fact, this solution lies in $(0, \rho^*]$.

- (e) Show that $\psi(1) = \rho^*$.