STATS 217: Introduction to Stochastic Processes I

Lecture 4

Branching processes

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- This bacterium gives birth to ξ bacteria, where ξ is a non-negative integer valued random variable. We call these the generation 1 bacteria.

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- What is the probability that the bacteria population goes extinct?
- This problem was studied by Galton and Watson in relation to the propagation of last names in Victorian England.

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Note that if $Z_i = 0$ for some $i \geq 1$, then $Z_j = 0$ for all $j \geq i$. This corresponds to the extinction of the population.

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Formally, we say that 0 is an **absorbing state** for the process $(Z_n)_{n\geq 0}$.

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- Trivial case: Suppose $\mathbb{P}[\xi \geq 1] = 1$. Then, $\mathbb{P}[Z_n \geq 1 \quad \forall n] = 1$.
- Hence, we may assume that for all integers $k \geq 0$,

$$\mathbb{P}[\xi = k] =: p_k$$

with $0 < p_0 < 1$.

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\begin{aligned}
\mathbb{E}[Z_2] &= \mathbb{E}\left[\sum_{i=1}^{Z_1} \xi_{1,i}\right] \\
&= \sum_{z \geq 0} \mathbb{E}\left[\sum_{i=1}^{z} \xi_{i,1}\right] \mathbb{P}[Z_1 = z] \\
&= \sum_{z \geq 0} z\mu \mathbb{P}[Z_1 = z] \\
&= \mu \sum_{z \geq 0} z \mathbb{P}[Z_1 = z] \\
&= \mu \cdot \mathbb{E}[Z_1] = \mu^2.
\end{aligned}
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- What about the case when $\mu \geq 1$?
- If $\mu = 1$, then $\mathbb{E}[Z_n] = 1$ and if $\mu > 1$, then $\mathbb{E}[Z_n] \to \infty$, but this doesn't say anything about the probability of survival.

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- Suppose that the bacterium b in generation 0 has k children $b_1, \ldots, b_k$. Then, the population dies out if and only if the subpopulations starting at $b_1, \ldots, b_k$ die out. Moreover, the probability of each of these subpopulations dying out is also ρ.

First step analysis

- Therefore,

$$\rho = \sum_{k=0}^{\infty} \mathbb{P}[\xi_{0,1} = k]\rho^k = \sum_{k=0}^{\infty} p_k \rho^k = \phi(\rho),$$

  where

$$\phi(z) := \sum_{k=0}^{\infty} p_k z^k$$

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- So, we see that the probability of extinction is a fixed point of the generating function i.e. a solution of

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  we see that 1 is always a solution of $\rho = \phi(\rho)$.
- However, this does not mean that the extinction probability is 1, since there may be other solutions to $\rho = \phi(\rho)$.

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- $\phi''(z) = \sum_{k\geq 2} k(k-1)p_k z^{k-2} > 0$ for $z \in (0, 1]$.
- Hence, ϕ is strictly convex on $(0, 1]$.

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- So, we have two cases:
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  - If $\phi'(1) \leq 1$, then $g'(1) \leq 0$ and $g'(\rho) < 0$ for all $\rho \in [0, 1)$. Hence, the only solution of $g(\rho) = 0$ is at $\rho = 1$.
  - If $\phi'(1) > 1$, then $g'(1) > 0$. So, there exists exactly one $\rho \in (0, 1)$ such that $g(\rho) = 0$.

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- Therefore, if $\mu = 1$ (this is called the **critical case**), we see that $\rho = 1$.

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- Then, by first step analysis, we have

$$\rho_n = \sum_{k \geq 0} p_k \rho_{n-1}^k = \phi(\rho_{n-1}).$$

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$$\rho = \lim_{n\to\infty} \mathbb{P}[Z_n = 0] = \lim_{n\to\infty} p_n \le \rho^*.$$

- Finally, since $\rho = \phi(\rho)$, it must be the case that $\rho = \rho^*$.

Summary

Thus, we have established the following theorem.

- Let $(Z_n)_{n\geq 0}$ be a branching process with $Z_0 = 1$ and common offspring distribution ξ.
- Let $\mu = \mathbb{E}[\xi]$ and let $\phi(z) = \sum_{k\geq 0} \mathbb{P}[\xi = k] z^k$.
- Suppose that $0 < p_0 = \mathbb{P}[\xi = 0] < 1$.
- Let ρ be the probability of extinction.
- Then, ρ is the smallest solution of $\phi(z) = z$, $z \in [0, 1]$.
- If $\mu \leq 1$, then $\rho = 1$.
- If $\mu > 1$, then $\rho < 1$.