HOMEWORK 5

DUE 02/20 AT 7:00PM PST

(1) (due to Pinsky and Karlin) Consider the Markov chain whose transition probability matrix is given by

$$P = \begin{array}{c} \\ A \\ B \\ C \\ D \end{array} \begin{bmatrix} A & B & C & D \\ 1 & 0 & 0 & 0 \\ 0.1 & 0.2 & 0.5 & 0.2 \\ 0.1 & 0.2 & 0.6 & 0.1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

(a) Starting in state $X_0 = B$, determine the mean time to absorption (i.e. reaching either state A or state D).

(b) Starting in state $X_0 = B$, determine the mean time that the process spends in state B prior to absorption and the mean time that the process spends in state C prior to absorption. Verify that the sum of these is the mean time to absorption.

(2) (The coupon collector problem) Each box of a brand of cereals contains a coupon. There are N different types of coupons, and the coupon in each box is equally likely to be of any of the N types. You keep buying cereal boxes until you have collected all N different types of coupons. Let T_N denote the number of boxes you have bought. Show that

$$\mathbb{E}[T_N] = N\left(1 + \frac{1}{2} + \cdots + \frac{1}{N}\right) \approx N \log N, \text{ and}$$
$$\mathrm{Var}[T_N] < N^2\left(1 + \frac{1}{2^2} + \cdots + \frac{1}{N^2}\right) < \frac{\pi^2}{6}N^2.$$

(3) (due to Durrett) The simplex method minimizes linear functions by moving between extreme points of a polyhedral region so that each transition decreases the objective function. Suppose there are n extreme points and they are numbered, from 1 to n, in increasing order of their values. Consider the Markov chain for which $P_{1,1} = 1$ and $P_{i,j} = 1/(i-1)$ for $j < i$. In words, when we leave j, we are equally likely to go to any of the extreme points with a better value.

(a) Let T_1 denote the time when the chain is absorbed in state 1. Use first step analysis to show that

$$\mathbb{E}[T_1 \mid X_0 = i] = 1 + 1/2 + \cdots + 1/(i-1).$$

(b) Let $I_j = 1$ if the chain visits j on the way from n to 1. Show that for $j < n$,

$$\mathbb{P}[I_j = 1 \mid I_{j+1}, \ldots, I_n] = 1/j.$$

Use this to get another proof of part (a) and show that $I_1, \ldots, I_{n-1}$ are independent.

(4) (due to Pinsky and Karlin) (a) A Markov chain $X_0, X_1, \ldots$ has the transition probability matrix

$$P = \begin{array}{c} \\ A \\ B \\ C \end{array} \begin{bmatrix} A & B & C \\ 0.3 & 0.2 & 0.5 \\ 0.5 & 0.1 & 0.4 \\ 0 & 0 & 1 \end{bmatrix}$$

and is known to start in state $X_0 = A$. Eventually, the process will end up in state C. What is the probability that when the process moves into state C, it does so from state B?

(b) For the same Markov chain as in (a), let $T = \min\{n \geq 0 : X_n = 2\}$. What is the probability that T is an odd number?

(5) (a) Consider a Markov chain with finite state space S and transition matrix P. Let T denote the set of all transient states. For a recurrent state y, let C_y denote the set of all states which communicate with y. Let $f_{x\to y}$ denote the probability that starting from state x, the process ever visits state y. Show that for any $x \in T$ and any recurrent state y,

$$f_{x\to y} = \sum_{z\in T} P_{x,z} f_{z\to y} + \sum_{z\in C_y} P_{x,z}$$

(b) Consider a Markov chain with finite state space S. Show that if j is accessible from k, then j can be reached from k with positive probability in at most $|S|$ steps.

(6) Roll a 6-sided unbiased die repeatedly. What is the expected number of rolls until you see a 6? What is the expected number of rolls until you see the pattern 66? What is the expected number of rolls until you see the pattern 61?

(7) ($*$) (a) A fair coin is tossed repeatedly. Show that the expected waiting time for the pattern HHH is 14; for HTH, it is 10; for HHT, it is 8; for HTT, it is 8.

($*$) (b) Consider a game where Player 1 picks a three coin pattern (for example HHH) following which player 2 picks another three coin pattern (say THH). A fair coin is flipped repeatedly until one of the two patterns appears. Given the previous part, it may perhaps come as a surprise that player 2 has an advantage in this game, in the sense that no matter what player 1 picks, player 2 can win with probability $\geq 2/3$. Show this by verifying the table below.

case	Player 1	Player 2	Prob. 2 wins
1	HHH	THH	7/8
2	HHT	THH	3/4
3	HTH	HHT	2/3
4	HTT	HHT	2/3

(8) ($*$) (due to Pinsky and Karlin) A well-disciplined man, who smokes exactly one half of a cigar each day, buys a box containing N cigars. He cuts a cigar in half, smokes half, and returns the other half to the box. In general, on a day in which his cigar box contains w whole cigars and h half cigars, he will pick one of the $w + h$ smokes at random, each whole and half cigar being equally likely, and if it is a half cigar, he smokes it. If it is a whole cigar, he cuts it in half, smokes once piece, and returns the other to the box. Let T be the day on which the last whole cigar is selected from the box? Show that

$$\mathbb{E}[T] = 2N - \sum_{k=1}^{N} \frac{1}{k}.$$

Hint: Let X_n be the number of whole cigars in the box after the n^{th} smoke. Study $v_n(w) = \mathbb{E}[T \mid X_n = w]$ using first step analysis.

(9) ($*$) Let $(X_n)_{n\geq 0}$ be a DTMC on a finite state space S with transition matrix P. A function $h\colon S \to \mathbb{R}$ is said to be harmonic at $x \in S$ if

$$h(x) = \sum_{y\in S} P_{x,y} h(y) = \mathbb{E}[h(X_1) \mid X_0 = x].$$

(a) Show that if P is irreducible and h is harmonic at every point $x \in S$, then h is constant (i.e. it takes the same value at every point).

(b) Show that if P is irreducible, then the column rank of $P - I$ is $|S| - 1$. Use this to argue that the stationary distribution of P must be unique.

(c) Let $B \subseteq S$ be non-empty and suppose P is irreducible. Let $h : S \to \mathbb{R}$ be harmonic at all states $x \notin B$. Show that

$$\max_{y\in B} h(y) = \max_{x\in S} h(x).$$