STATS 217: Introduction to Stochastic Processes I

Lecture 1

Course information

- **Instructor**: Vishesh Jain
- **TAs**: Sohom Bhattacharya, Michael Feldman, Disha Ghandwani.
- **Final grade based entirely on 9 problem sets**. See "Grading" section of course website for policies and further details.
- **Course website**: `jainvishesh.github.io/STATS217_Winter2021.html`.
- There are also associated **Canvas** and **Gradescope** sites that you should be enrolled in.

Symmetric simple random walk

- $X_1, X_2, \ldots$ is a sequence of independent and identically distributed (i.i.d.) **Rademacher random variables** i.e.,

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- So, after n rounds of betting, the wealth of the gambler is

$$S_n := S_0 + X_1 + \cdots + X_n.$$

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- **Question 5:** How do these answers change if $\mathbb{P}[X_i = 1] = 0.49$?

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- **Question 3:** $A = 200, B = 100$, find $\mathbb{E}[\tau \mid S_0 = 0]$.
- **Question 4:** "$A = \infty$", $B = 100$, find (i) $\mathbb{P}[\tau < \infty \mid S_0 = 0]$ and (ii) $\mathbb{E}[\tau \mid S_0 = 0]$.

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- Since $f(A) = 1$, we must have

$$x = \frac{1}{A+B}.$$

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We have proved that

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- **Answer 2:** $A = 200, B = 100,\ f(0) = 1/3.$

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- Another interpretation of this scenario is the following: suppose Alice and Bob bet on the outcomes of fair coin tosses. If the outcome is heads, then Bob pays \$1 to Alice, otherwise Alice pays \$1 to Bob. If Alice starts with $\$A$ and Bob starts with $\$B$ then the probability that Alice wins everything ('Alice ruins Bob') is

$$\frac{A}{A+B}.$$

Application: symmetric simple random walk on the circle

Consider the symmetric simple random walk on the circle with $n + 1$ points, starting from the point marked 0.

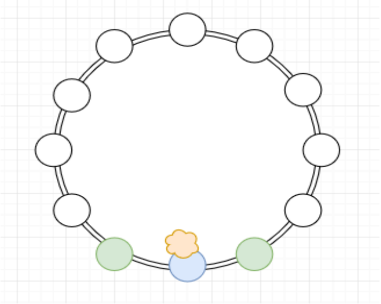

Image courtesy of user 'mark' on math.stackexchange.com

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- Let $(\Delta h)(k) := h(k+1) - h(k)$.
- Then, for all $-B < k < A$

$$\begin{aligned}(\Delta(\Delta g))(k-1) &= (\Delta g)(k) - (\Delta g)(k-1) \\ &= g(k+1) - g(k) - g(k) + g(k-1) \\ &= g(k+1) - (g(k+1) + g(k-1) + 2) + g(k-1) \\ &= -2.\end{aligned}$$

- "Second derivative of g is -2" so $g(k) = -k^2 + Dk + C$.
- Using boundary conditions,

$$g(k) = -(k-A)(k+B).$$

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- Then, for all $\ell \geq 1$, $\tau_2(\ell) \leq \tau_1$ so that

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- Then, for all $\ell \geq 1$, $\tau_2(\ell) \leq \tau_1$ so that

$$100\ell = \mathbb{E}[\tau_2(\ell) \mid S_0 = 0] \leq \mathbb{E}[\tau_1 \mid S_0 = 0],$$

and now take $\ell \to \infty$.

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- So, a symmetric simple random walk starting at 0 visits -100 with probability 1. Again, there is nothing special about -100 here.