HOMEWORK 8

DUE 03/13 AT 7:00PM PST

(1) (due to Durrett) At the beginning of each day, a piece of equipment is inspected to determine its working condition, which is classified as $1 = \text{new}$, $2, 3$, or $4 = \text{broken}$. We assume that the state is a Markov chain with the following transition matrix:

$$\begin{array}{cc} & \begin{array}{cccc} \mathbf{1} & \mathbf{2} & \mathbf{3} & \mathbf{4} \end{array} \\ \begin{array}{c} \mathbf{1} \\ \mathbf{2} \\ \mathbf{3} \end{array} & \begin{bmatrix} 0.95 & 0.05 & 0 & 0 \\ 0 & 0.9 & 0.1 & 0 \\ 0 & 0 & 0.875 & 0.125 \end{bmatrix} \end{array}$$

(a) Suppose that a broken machine requires three days to fix. To incorporate this into the Markov chain, we add states 5 and 6 and suppose that $p(4,5) = 1$, $p(5,6) = 1$, and $p(6,1) = 1$. Find the fraction of time that the machine is working.

(b) Suppose now that we have the option of performing preventative maintenance when the machine is in state 3, and that this maintenance takes one day and returns the machine to state 1. This changes the transition probability to

$$\begin{array}{cc} & \begin{array}{ccc} \mathbf{1} & \mathbf{2} & \mathbf{3} \end{array} \\ \begin{array}{c} \mathbf{1} \\ \mathbf{2} \\ \mathbf{3} \end{array} & \begin{bmatrix} 0.95 & 0.05 & 0 \\ 0 & 0.9 & 0.1 \\ 1 & 0 & 0 \end{bmatrix} \end{array}$$

Find the fraction of time the machine is working under this new policy.

(2) Let P be an aperiodic, irreducible, transition matrix on a finite state space S. Let π denote the unique stationary distribution.

(a) Let $(\widehat{X}_t, \widehat{Y}_t)_{t \geq 0}$ be a coupling of two copies of the Markov chain with transition matrix P and with initial distributions $\widehat{X}_0 \sim \mu$ and $\widehat{Y}_0 \sim \nu$, where μ and ν are probability distributions on S. As before, let

$$\tau_{\text{couple}} = \min\{t \geq 0 : \widehat{X}_t = \widehat{Y}_t\}.$$

Show that

$$\text{TV}(\mu P^t, \nu P^t) \leq \mathbb{P}[\tau_{\text{couple}} > t].$$

(b) Fix $x \in S$. Let $\mu = \delta_x$ (i.e. $\mu(\{x\}) = 1$) and let $\nu = \pi$. Show that for the corresponding independent coupling $(\widehat{X}_t, \widehat{Y}_t)$, there exists some $r_0 \geq 1$ and some $\epsilon > 0$ such that

$$\mathbb{P}[\tau_{\text{couple}} > kr_0] \leq (1 - \epsilon)^k.$$

This provided another proof of the convergence theorem.

(3) A birth-and-death chain has state space $S = \{0, 1, \ldots, n\}$ and in each step, can change its position (in absolute value) by at most 1. Formally, the transition matrix P is completely specified by the collection $(p_k, q_k, r_k)_{k=0}^n$ where for all $0 \leq k \leq n$, $p_k + q_k + r_k = 1$ and

- $p_n = q_0 = 0$.
- $P_{k,k+1} = p_k$, $0 \leq k < n$.
- $P_{k,k-1} = q_k$, $0 < k \leq n$.
- $P_{k,k} = r_k$, $0 \leq k \leq n$.

We further assume that $p_k > 0$ for all $0 \leq k < n$ and $q_k > 0$ for all $0 < k \leq n$.

(a) Show that every such birth-and-death chain is irreducible and reversible and find the unique stationary distribution π.

(b) Fix $\ell \in \{1, \dots, n\}$ and let τ_ℓ denote the first time that the chain visits ℓ. Show that

$$\mathbb{E}[\tau_\ell \mid X_0 = \ell - 1] = \frac{1}{q_\ell \pi_\ell} \sum_{j=0}^{\ell-1} \pi_j.$$

(c) Consider the special case when $(q_k, r_k, p_k) = (q, r, p)$ for $1 \leq k < n$, $(q_0, r_0, p_0) = (0, r+q, p)$ and $(q_n, r_n, p_n) = (q, r+p, 0)$, where $p, q > 0$ and $p + q + r = 1$. Show that, if $p \neq q$, then

$$\mathbb{E}[\tau_\ell \mid X_0 = \ell - 1] = \frac{1}{p-q}\left(1 - \left(\frac{q}{p}\right)^\ell\right)$$

and deduce that

$$\mathbb{E}[\tau_n \mid X_0 = 0] = \frac{1}{p-q}\left(n - q\left(\frac{1-(q/p)^n}{p-q}\right)\right).$$

(4) $(*)$ A group is a set G endowed with an operation $\cdot : G \times G \to G$ and a special identity element $\mathrm{id} \in G$ such that

- $\mathrm{id} \cdot g = g = g \cdot \mathrm{id} \quad \forall g \in G$.
- $(g \cdot h) \cdot k = g \cdot (h \cdot k) \quad \forall g, h, k \in G$.
- For every $g \in G$, there exists an inverse $g^{-1} \in G$ such that $g \cdot g^{-1} = \mathrm{id} = g^{-1} \cdot g$.

Let μ be a probability distribution on the group G. Consider the (left) random walk on G whose transition matrix is given by

$$P_\mu(g, h \cdot g) = \mu(h) \quad \forall g, h \in G.$$

(a) Let $\mathrm{Unif}(G)$ denote the uniform distribution on G. Show that $\mathrm{Unif}(G)$ is a stationary distribution for the random walk on G with transition matrix P_μ.

(b) Suppose that $\mu(g) = \mu(g^{-1})$ for every $g \in G$ if and only if P_μ is reversible with respect to $\mathrm{Unif}(G)$.

(c) Consider the reversed distribution $\widehat{\mu}$ defined by $\widehat{\mu}(g) := \mu(g^{-1})$ for all $g \in G$. Let $\pi = \mathrm{Unif}(G)$. Show that

$$\mathrm{TV}(P_\mu^t(\mathrm{id}, \cdot), \pi) = \mathrm{TV}(P_{\widehat{\mu}}^t(\mathrm{id}, \cdot), \pi).$$

(d) Recall the library chain on Problem 8 of Homework 7. Suppose that $p_{i_1} = \cdots = p_{i_n}$, so that the chain is irreducible, aperiodic, and the unique stationary distribution is the uniform distribution on all permutations of $\{1, \dots, n\}$. Show that for any $\varepsilon \in (0, 1)$,

$$\tau_{\mathrm{mix}}(\varepsilon) \leq n \log n + n \log(\varepsilon^{-1}).$$