STATS 217: Introduction to Stochastic Processes I

Lecture 7

Recap

The **inhomogeneous PPP** may be described infinitesimally by

- $N(t)$ is the number of points in $[0, t]$.
- $\mathbb{P}[\text{there is a point in } [t, t + dt]] = \lambda(t)dt$ i.e.,
  - $\mathbb{P}[N(t+\epsilon) - N(t) = 0] = 1 - \lambda\epsilon + o(\epsilon)$.
  - $\mathbb{P}[N(t+\epsilon) - N(t) = 1] = \lambda\epsilon + o(\epsilon)$.
  - $\mathbb{P}[N(t+\epsilon) - N(t) > 1] = o(\epsilon)$,

  where $o(\epsilon)/\epsilon \to 0$ as $\epsilon \to 0$.
- The number of points in disjoint intervals are independent.

We saw that for any $0 \leq s \leq t$, $N(t) - N(s) \sim \text{Pois}\left(\int_s^t \lambda(u)du\right)$.

Construction: $N(t) = N^{\text{hom}}(\Lambda(0, t))$ is an inhomogeneous PPP, where $N^{\text{hom}}(\cdot)$ is a (homogeneous) PPP with rate 1.

Homogeneous case: Taking $\lambda(t) \equiv \lambda$ gives a (homogeneous) PPP of rate λ, in which case the waiting (interarrival) times are i.i.d. $\text{Exp}(\lambda)$ (not true in the general inhomogeneous case).

Superposition of Poisson processes

Let $N_1(t), \ldots, N_k(t)$ be independent Poisson processes with rates $\lambda_1, \ldots, \lambda_k$.

- Then, $N_1(t) + \cdots + N_k(t)$ is a Poisson process with rate $\lambda_1 + \cdots + \lambda_k$.
- Why? Recall properties (P1), (P2), (P3) from last lecture.
- (P1) and (P3) are immediate.
- As for (P2), for any $s \leq t$

$$
\begin{aligned}
& N_1(t) + \cdots + N_k(t) - (N_1(s) + \cdots + N_k(s)) \\
& = (N_1(t) - N_1(s)) + \cdots + (N_k(t) - N_k(s)) \\
& \sim \mathrm{Pois}(\lambda_1(t-s)) + \cdots + \mathrm{Pois}(\lambda_k(t-s)) \\
& \sim \mathrm{Pois}((\lambda_1 + \cdots + \lambda_k)(t-s)).
\end{aligned}
$$

Poisson thinning

- Let $\{N(s)\}_{s \geq 0}$ denote a Poisson process with rate λ.
- Let $\alpha_1, \alpha_2, \ldots$ denote the (random) "arrival times".
- Let $Y_1, Y_2, \ldots$ denote a sequence of iid random variables.
- For each $j \in \text{supp}(Y_1)$, let $p_j = \mathbb{P}[Y_1 = j]$
- For each $j \in \text{supp}(Y_1)$, define

$$N_j(s) := |\{i \in \{1, 2, \ldots, \lfloor N(s) \rfloor\} : Y_i = j\}|.$$

- Then,
  - $\{N_j(s)\}_{s \geq 0}$ is a Poisson process with rate $p_j \lambda$ and
  - $\{N_1(s)\}_{s \geq 0}, \{N_2(s)\}_{s \geq 0}, \ldots$ are independent.

Poisson thinning

- Why? Can check by direct calculation, but for intuition, consider the case when $Y \sim \text{Ber}(p)$.
- From the infinitesimal description, it is clear that $N_0(t)$ is a PPP with rate $(1-p)\lambda$ and $N_1(t)$ is a PPP with rate $p\lambda$.
- For independence, note that

$$
\begin{aligned}
&\mathbb{P}[N_0 \text{ has a point in}[t, t+dt] \mid N_1 \text{ has a point in}[t, t+dt]] \\
&= \frac{\mathbb{P}[N_0 \text{ and } N_1 \text{ have points in } [t, t+dt]]}{p\lambda dt} \\
&\approx (p\lambda dt)^{-1} \cdot \left(e^{-\lambda dt} \cdot \frac{(\lambda dt)^2}{2} \cdot (p(1-p) + (1-p)p) + o((\lambda dt)^2) \right) \\
&\approx (1-p)\lambda dt \\
&= \mathbb{P}[N_0 \text{ has a point in } [t, t+dt]].
\end{aligned}
$$

Poisson thinning

The analogous result also holds in the inhomogeneous case using the same argument.

- Let $\{N(s)\}_{s\geq 0}$ denote an inhomogeneous PPP with rate $\lambda(s)$.
- Let $\{Y(s)\}_{s\geq 0}$ denote a collection of independent random variables, each with support $\{1,\ldots,k\}$.
- Let $\alpha_1,\alpha_2,\ldots$ denote the random arrival times.
- For $j=1,\ldots,k$, define

$$N_j(s) := |\{i \in \{1,2,\ldots,\lfloor N(s)\rfloor\} : Y(\alpha_i) = j\}|.$$

- Then,
  - $N_j(s)$ is an inhomogeneous PPP with rate $\lambda(s)\mathbb{P}[Y(s)=j]$.
  - $\{N_1(s)\}_{s\geq 0},\ldots,\{N_k(s)\}_{s\geq 0}$ are independent processes

Example

Example 2.5 from Durrett. Given a Poisson process of red arrivals with rate λ and an independent Poisson process of green arrivals with rate μ, what is the probability that we will get 6 red arrivals before a total of 4 green ones?

- Equivalently, at least 6 red arrivals in the first 9.
- By thinning,

$$\sum_{k=6}^{9} \left(\frac{\lambda}{\lambda+\mu}\right)^{k} \left(\frac{\mu}{\lambda+\mu}\right)^{9-k}.$$

Example

Example 2.4 from Durrett. Consider a model of telephone traffic in which the system starts empty at time 0. Suppose that the starting times of the calls is a Poisson process with rate λ and that the probability a call started at time s has ended by time t is $G(t-s)$, where G is some CDF with $G(0) = 0$ and mean μ. What is the distribution of the number of calls still in progress at time t?

- Call starting at time $\alpha \in [0, t]$ is kept with probability $(1 - G(t - \alpha))$.
- Therefore, by thinning, number of calls in progress at time t is Poisson with mean
$$\int_{s=0}^{t} \lambda(1 - G(t-s))ds = \lambda \int_{r=0}^{t} (1 - G(r))dr.$$
- Let $t \to \infty$ to see that in the long run, the number of calls in the system is Poisson with mean
$$\lambda \int_{r=0}^{\infty} \mathbb{P}(G \geq r)dr = \lambda\mu.$$

Compound Poisson processes

Each of the 'thinned' Poisson processes is a special case of a **compound Poisson process**.

- Let $\{N(s)\}_{s\geq 0}$ denote a Poisson process with rate λ.
- Let $\alpha_1, \alpha_2, \ldots$ denote the (random) "arrival times".
- Let $Y_1, Y_2, \ldots$ denote a sequence of iid random variables. Let $Y_0 = 0$.
- Let
$$S(t) = Y_0 + Y_1 + \cdots + Y_{N(t)}.$$
- Then, $\mathbb{E}[S(t)] = \mathbb{E}[Y_1] \cdot \mathbb{E}[N(t)]$ by the same argument as for branching processes.
- Also, by the same argument as on this week's homework,
$$\mathrm{Var}[S(t)] = \mathbb{E}[N(t)] \cdot \mathrm{Var}(Y_1) + \mathrm{Var}[N(t)] \cdot \mathbb{E}[Y_1]^2.$$

Poisson conditioning

- Let $\{N(s)\}_{s \geq 0}$ be a Poisson process with rate λ.
- Let $\alpha_1, \alpha_2, \ldots,$ denote the (random) "arrival times".
- Conditioned on $N(t) = n$, what is the distribution of $\alpha_1, \ldots, \alpha_n$?
- It turns out that

  $$\{\alpha_1, \ldots, \alpha_n\} \sim \{u_1, \ldots, u_n\},$$

  where $u_1, \ldots, u_n$ are iid uniformly distributed in $[0, t]$.
- Why? Again, this is intuitive from the infinitesimal description of the process.

Poisson conditioning

Formally,

$$
\begin{aligned}
&\mathbb{P}[\text{arrival times } \alpha_1, \ldots, \alpha_n \mid N(t) = n] \\
&= \mathbb{P}[N(t) = n]^{-1} \mathbb{P}[W_1 = \alpha_1, \ldots, W_n = \alpha_n - \alpha_{n-1}, W_{n+1} > t - \alpha_n] \\
&= \mathbb{P}[N(t) = n]^{-1} \lambda e^{-\lambda \alpha_1} \cdot \ldots \lambda e^{-\lambda(\alpha_n - \alpha_{n-1})} \cdot e^{-\lambda(t - \alpha_n)} \\
&= \mathbb{P}[N(t) = n]^{-1} \cdot \lambda^n e^{-\lambda t},
\end{aligned}
$$

which does not depend on $\alpha_1, \ldots, \alpha_n$.

Example: simulating a PPP

Here is a practical application of Poisson conditioning.

- How might one generate (on a computer) a PPP with rate λ in the time interval $[0, t]$?
- Poisson conditioning shows that we can do this in two easy steps.
  - First, generate $N(t) \sim \text{Pois}(\lambda t)$.
  - Next, generate $\alpha_1, \ldots, \alpha_{N(t)}$ iid uniformly in $[0, t]$.