HOMEWORK 9

DUE 03/19 **(FRIDAY)** AT 7:00PM PST

Let $(X_n)_{n\geq 0}$ denote a DTMC on Ω with transition matrix P, where Ω is either finite or countably infinite. As before P is said to be irreducible if for any pair of states $x, y \in \Omega$, there exists some $t \geq 0$ such that $P^t_{x,y} > 0$.

For a state $x \in \Omega$, let

$$\tau_x^+ := \min\{n \geq 1 : X_n = x\},$$

and for a pair of states $x, y \in \Omega$, let

$$N(x,y) = \sum_{n=0}^{\infty} P^n_{x,y} = \mathbb{E}\left[\sum_{n=0}^{\infty} 1_{\{X_n = y\}} \mid X_0 = x\right]$$

We say that x is recurrent if $\mathbb{P}[\tau_x^+ < \infty \mid X_0 = x] = 1$ and that x is transient otherwise.

The same argument as for the finite state space case shows that the following statements are equivalent for irreducible P:

- x is recurrent for some $x \in \Omega$.
- $N(x,x) = \infty$ for some $x \in \Omega$.
- $N(x,y) = \infty$ for all $x, y \in \Omega$.
- $\mathbb{P}[\tau_y^+ < \infty \mid X_0 = x] = 1$ for all $x, y \in \Omega$.

A recurrent state $x \in \Omega$ is called positive recurrent if $\mathbb{E}[\tau_x^+ \mid X_0 = x] < \infty$, and null recurrent otherwise. In class, we showed that if Ω is finite, then every recurrent state is positive recurrent. However, as the example of the simple symmetric random walk on $\mathbb{Z}$ shows, a recurrent state need not be positive recurrent.

(1) For irreducible P, show that the following are equivalent:
- $\mathbb{E}[\tau_x^+ \mid X_0 = x] < \infty$ for some $x \in \Omega$.
- $\mathbb{E}[\tau_y^+ \mid X_0 = x] < \infty$ for all $x, y \in \Omega$.

In particular, it makes sense to classify an irreducible chain as transient, positive recurrent, or null recurrent.

(2) $(*)$ For irreducible and recurrent P, suppose that there exists a stationary distribution i.e., a probability distribution π on Ω satisfying $\pi = \pi P$. Show that:
- $\pi_x > 0$ for all $x \in \Omega$.
- $\pi_x \cdot \mathbb{E}[\tau_x^+ \mid X_0 = x] = 1$ for all $x \in \Omega$.

(3) For irreducible P, show that the following are equivalent:
- P is positive recurrent.
- There exists a probability distribution π on Ω such that $\pi = \pi P$.

Moreover, show that such a probability distribution is unique.

Hint: Use Problem 2

(4) $(*)$ Let P be irreducible, aperiodic, and positive recurrent. By the previous problem, P has a unique stationary distribution π. Show that, for all $x \in \Omega$,

$$\lim_{t\to\infty} \mathrm{TV}(P^t(x,\cdot), \pi) = 0.$$

Hint: You may use a similar argument as Problem 2 of Homework 8. To show that the chains couple with probability 1, it may help to show that the product chain on $\Omega \times \Omega$ with transition matrix $Q((a,b),(x,y)) = P(a,x)P(b,y)$ is also irreducible and positive recurrent.

(5) (Polya's theorem) Let $\mathbb{Z}^d$ denote the d-dimensional integer lattice. Thus, elements of $\mathbb{Z}^d$ are of the form $(z_1, \ldots, z_d)$ for $z_i \in \mathbb{Z}$. The simple symmetric random walk on $\mathbb{Z}^d$ has the following transitions: if the current state is $(z_1, \ldots, z_d)$, then there are $2d$ possibilities for the next state, given by

$$(z_1 + 1, z_2, \ldots, z_d), (z_1 - 1, z_2, \ldots, z_d), \ldots, (z_1, z_2, \ldots, z_d + 1), (z_1, z_2, \ldots, z_d - 1),$$

and each of these $2d$ transitions are equally likely.

Show that the simple random walk on $\mathbb{Z}^d$ is recurrent for $d \leq 2$ and transient for $d \geq 3$. Therefore, "a drunk man will find his way home, but a drunk bird may get lost forever" (quote attributed to S. Kauktani).

Hint: For $i = 1, \ldots, d$, let $N_i(n)$ denote the number of steps taken in the i^{th} direction by time n. Show that there exists some constant $c_d > 0$ such that $\mathbb{P}[N_i(n) \in [n/2d, 2n/d]$ for all $i \in [d]] \geq 1 - \exp(-c_d n)$.