HOMEWORK 2

DUE 01/30 AT 7:00PM PST

(1) Let $(Z_n)_{n\geq 0}$ be a branching process with $Z_0 = 1$ and offspring distribution ξ with $\mathbb{E}[\xi] = \mu$ and $\mathrm{Var}(\xi) = \sigma^2$. What is $\mathrm{Var}(Z_n)$?

(2) Let $(Z_n)_{n\geq 0}$ be a branching process with $Z_0 = 1$ and offspring distribution ξ. Find the probability of extinction in each of the following situations.
   (a) There exists some $p \in (0,1)$ such that
   $$\mathbb{P}[\xi = 0] = p, \quad \mathbb{P}[\xi = 2] = (1-p).$$
   (b) There exists some $p \in (0,1)$ such that $\xi \sim \mathrm{Geom}(p)$, i.e.,
   $$\mathbb{P}[\xi = k] = p(1-p)^k, \quad k = 0, 1, 2, \ldots$$
   (c) $\xi \sim \mathrm{Pois}(1.1)$. Also compute the probability of extinction in 0 generations, 1 generation, 2 generations, and 3 generations.

(3) (due to Pinsky and Karlin) At time 0, a blood culture starts with one red cell. At the end of 1 minute, the red cell dies and is replaced by one of the following combinations with the probabilities as indicated:
$$\begin{cases} \text{Two red cells} & \text{with probability } \frac{1}{4}, \\ \text{One red cell, one white cell} & \text{with probability } \frac{2}{3}, \\ \text{Two white cells} & \text{with probability } \frac{1}{12}. \end{cases}$$
Each red cell lives for 1 minute and gives birth to offspring in the same way as the parent cell. Each white cell lives for 1 minute and dies without reproducing. Assume that individual cells behave independently.
   (a) At time $n + 1/2$ after the culture begins, what is the probability that no white cells have yet appeared?
   (b) What is the probability that the entire culture dies out entirely?

(4) (due to Grimmett and Stirzaker) Let $(Z_n)_{n\geq 0}$ denote a branching process with $Z_0 = 1$ and offspring distribution ξ. Suppose that $\mathbb{E}[\xi] = \mu > 1$ and let $u \in (0,1)$ denote the extinction probability. Let ϕ denote the generating function of the offspring distribution and let ϕ_n denote the generating function of the distribution of Z_n. Show that
$$\mathbb{E}[s^{Z_n} \mid \text{extinction}] = \frac{1}{u}\phi_n(su).$$

(5) Consider the branching process with alternating distributions, which is defined as follows. Consider two offspring distributions ξ and η with generating functions ϕ_1 and ϕ_2 respectively. We start with a single individual i.e. $Z_0 = 1$. The individuals in even generations reproduce independently according to the distribution η whereas those in odd generations reproduce independently according to the distribution ξ i.e.
$$Z_{2n+1} = \sum_{i=1}^{Z_{2n}} \eta_{2n,i}$$

$$Z_{2n} = \sum_{i=0}^{Z_{2n-1}} \xi_{2n-1,i}.$$

(a) (*) Let ρ_{21} denote the probability of extinction. Show that ρ_{21} is the smallest solution in $(0, 1]$ of the equation

$$x = \phi_2(\phi_1(x)).$$

(b) (**) (due to Munford) Let ρ_{12} denote the probability of extinction of the branching process in which the even generations reproduce according to ξ and the odd generations reproduce according to η. Also, let ρ_1 (respectively ρ_2) denote the probability of extinction of a branching process with offspring distribution ξ (respectively η). Show that

$$\rho_1 \leq \rho_2 \implies \rho_1 \leq \rho_{12} \leq \rho_{21} \leq \rho_2.$$

Hint: Use the characterization of $\rho_1, \rho_2, \rho_{12}, \rho_{21}$ to show that $\rho_{12} = \phi_1(\rho_{21})$ and $\rho_{21} = \phi_2(\rho_{12})$. Also, by the characterization of ρ_1, note that $\rho_1 \leq \rho_2$ is equivalent to $\phi_1(\rho_2) \leq \rho_2$.

(6) Consider a simple random walk $(S_n)_{n\geq 0}$ on $\mathbb{Z}$ starting from $S_0 = 0$ for which the transitions are as follows:

$$\begin{cases} S_n = S_{n-1} - 1 & \text{with probability } q \\ S_n = S_{n-1} + 1 & \text{with probability } p = 1 - q, \end{cases}$$

where $p \in (0, 1)$.

(a) Let τ_1 denote the first time that the random walk hits 1 and let ϕ denote the generating function of the distribution of τ_1. Show that

$$\phi(s) = \frac{1 - \sqrt{1 - 4pqs^2}}{2qs}, \quad |s| \leq \frac{1}{2\sqrt{pq}}.$$

Hint: Use first step analysis to argue that

$$\tau_1 \sim \begin{cases} 1 & \text{with probability } p \\ 1 + \tau_1' + \tau_1'' & \text{with probability } q, \end{cases}$$

where τ_1', τ_1'' are i.i.d. random variables with the same distribution as τ.

(b) Let p_1 denote the probability that the walk ever reaches 1. Show that

$$p_1 = \min\left\{\frac{p}{q}, 1\right\}.$$

(c) Show that

$$\mathbb{E}[\tau_1] = \begin{cases} \infty & p \leq 1/2 \\ \frac{1}{p-q} & p > 1/2. \end{cases}$$

(7) (*) Consider a random walk $(S_n)_{n\geq 0}$ on $\mathbb{Z}$ starting from $S_0 = 0$ for which the transitions are as follows:

$$\begin{cases} S_n = S_{n-1} - 2 & \text{with probability } 1/2 \\ S_n = S_{n-1} + 1 & \text{with probability } 1/2. \end{cases}$$

Let p_1 denote the probability that the walk ever reaches 1. Show that

$$p_1 = \frac{\sqrt{5} - 1}{2}.$$

(8) Let $(Z_n)_{n\geq 0}$ denote a branching process with $Z_0 = 1$ and offspring distribution ξ. Let

$$Y_n = Z_0 + Z_1 + \cdots + Z_n$$

denote the total number of individuals up to and including generation n (this is called the **total progeny**). Let ψ_n denote the generating function of Y_n and let ϕ denote the generating function of ξ. Let ρ^* denote the extinction probability of the branching process.

(a) Show that

$$\psi_n(s) = s \cdot \phi(\psi_{n-1}(s)).$$

(b) Show that for all $0 < s < 1$, $\psi_n(s)$ is a decreasing sequence in n.

(c) Let $\psi(s) = \lim_{n\to\infty} \psi_n(s)$. Show that for all $0 < s < 1$,

$$\psi(s) = s \cdot \phi(\psi(s)).$$

(d) $(*)$ For this, and subsequent parts, you may use that

$$\psi(s) = \sum_{k=0}^{\infty} q_k s^k$$

for some $(q_k)_{k\geq 0}$ with $q_k \geq 0$ for all k and $\sum_{k=0}^{\infty} q_k \leq 1$.
Show that for $0 < s < 1$, the equation

$$x = s \cdot \phi(x)$$

has a unique solution in $(0, 1]$ and in fact, this solution lies in $(0, \rho^*]$.

(e) $(*)$ Show that $\psi(1) = \rho^*$.