INDIAN INSTITUTE OF TECHNOLOGY MANDI

Black Scholes Equation

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A project done on understanding
the financial behaviour of Black-Scholes Model

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1 Introduction

1.1 What Are options ?

An option is a binding contract giving the buyer the right, but not the obligation, to buy or sell an underlying asset at a specific price on or before a certain date. Offers many advantages that trading stocks and ETFs(exchange transfer funds) alone cannot.

There are two types of Options:

1. Call option - Buying an option that allows us to buy shares at a later time is called a "call option".
2. Put option - Buying an option that allows us to sell shares at a later time is called a "put option'.

1.2 Why to Buy Options ?

There are two main reason why the OPTIONS are introduced in stock market.

1. For small investors which do not have enough money to compete with big investor in large multinational company like apple, reliance etc. they can compete using the options.
2. The other reason to hedge the risk for example I bought the share of 2L and I also bought the put option, now if the stock price moves up, I only lose the premium paid. If the stock price moves down, I get profit by the put option making total loss close to zero.

1.3 Advantages from buying Options -

1. Options are derivatives contracts that give the buyer the right, but not the obligation, to either buy or sell a fixed amount of an underlying asset at a fixed price on or before the contract expires.
2. Used as a hedging device, options contracts can provide investors with risk-reduction strategies.
3. For speculators, options can offer lower-cost ways to go long or short the market with limited downside risk.
4. Options also give traders and investors more flexible and complex strategies such as spread and combinations that can be potentially profitable under any market scenario.

2 What is Black Scholes Model

The Black-Scholes mathematical model explains that the price of heavily traded assets follow a geometric Brownian motion with constant drift and volatility (Black and Scholes, 1973). This model of option pricing is based on the fundamental that in the future, the price of the option contract either increases or decreases based on the spot price of the underlying asset.
Study reveals that the volatility play an important role in establishing the relationship between theoretical and actual price, the more is the volatility the higher is the gap between actual and theoretical price

2.1 Use of Black Scholes Model

- Mathematical tool, simulates the dynamics of financial market instruments such as options, futures, forwards and swaps.
- It is used to hedge the options in an investment portfolio by buying and selling the underlying asset (such as a stock) in just the right way and as a consequence, eliminate risk.
- Black-Scholes equation gives a theoretical estimate of the correct price of European stock options.

2.2 Assumptions

The model is based on certain assumptions, including:

- The options are European and can only be exercised at expiration
- No dividends payments on stock during the life of the option
- Efficient markets (i.e., market movements cannot be predicted)
- No commissions
- There are no risk less arbitrage opportunities.
- The risk-free rate and volatility of the underlying are known and constant the prices of the stocks follow a lognormal distribution; that is, returns on the underlying are normally distributed.

3 Derivation of Black Scholes equation

Derivation of Black Scholes Equation mainly consists of three parts -

3.1 Geometric Brownian Motion

It is a Markov process which means the stock price follows a random walk and is consistent with (at the very least) the weak form of the efficient market hypothesis (EMH)—past price information is already incorporated, and the next price movement is "conditionally independent" of past price movements.

The first term is a "drift" and the second term is a "shock." For each time period, the model assumes the price will "drift" up by the expected return. But the drift will be shocked (added or subtracted) by a random shock. The random shock will be the standard deviation σ multiplied by a random number ϵ.

$$\Delta S = S(\mu dt + \sigma \epsilon dz) \tag{1}$$

where - $dz = \epsilon\sqrt{dt}$

Where, $\Delta S-$change in stock price

$S-$ Stock price

$\mu-$ the expected return

$\sigma-$ the standard deviation of returns

$\epsilon-$ the random variable

$\Delta t-$ the elapsed time period

3.2 ITO's Lemma

ITO's Lemma is a key component in the ITO Calculus, used to determine the derivative of a time-dependent function of a stochastic process. It relates the change in the price of the derivative security to the change in the price of the underlying asset.

$$dV = \frac{\delta V}{\delta S}dS + \frac{\delta V}{\delta t}dt + \frac{1}{2}\frac{\delta^2 V}{\delta^2 S}dS^2 + \frac{1}{2}\frac{\delta^2 V}{\delta^2 t}dt^2 + \frac{\delta^2 V}{\delta t \delta S}dtdS \tag{2}$$

The equation here represents the change in value of an option as a function of stock price and time.

3.3 Dekta hedge Portfolio

Delta-Hedging – it is a defensive tactic used to reduce the directional exposure of a stock or option position. It is accomplished by adding a position that brings the delta closer to zero.

$$\pi = V - \Delta S$$

or we can also write it as

$$d\pi = dV - \Delta dS \tag{3}$$

Delta - Delta (Δ) represents the rate of change between the option's price and a $1 change in the underlying asset's price.

Now squaring equation (1):

$$dS^2 = \mu^2 S^2 dt^2 + \sigma^2\ S^2 \epsilon^2 dt + 2S^2 \sigma\mu\epsilon dt^{3/2}$$

Now as $dt^2 \approx 0$ and $dt^{3/2} \approx 0$
So,

$$dS^2 = \sigma^2 S^2 dt \tag{4}$$

Now substituting equation (1) in (2):

$$dV = \frac{\delta V}{\delta S} dS + \frac{\delta V}{\delta t} dt + \frac{1}{2}\frac{\delta^2 V}{\delta^2 S} dS^2 + \frac{1}{2}\frac{\delta^2 V}{\delta^2 t} dt^2 + \frac{\delta^2 V}{\delta t \delta S} dtS(\mu dt + \sigma\epsilon\sqrt{dt})$$

Now again neglecting the terms having dt^2 or $dt^{3/2}$

$$dV = \frac{\delta V}{\delta S} dS + \frac{\delta V}{\delta t} dt + \frac{1}{2}\frac{\delta^2 V}{\delta^2 S} dS^2$$

Now subsituting dS^2 from equation 4

$$dV = \frac{\delta V}{\delta S} dS + \frac{\delta V}{\delta t} dt + \frac{1}{2}\frac{\delta^2 V}{\delta^2 S} \sigma^2 S^2 dt \tag{5}$$

Now subsituting value dV from equation 5 to 3

$$d\pi = \frac{\delta V}{\delta S} dS + \frac{\delta V}{\delta t} dt + \frac{1}{2}\frac{\delta^2 V}{\delta^2 S} \sigma^2 S^2 dt - \Delta dS$$

$$d\pi = \frac{\delta V}{\delta S} dS + \frac{\delta V}{\delta t} dt + \frac{1}{2}\frac{\delta^2 V}{\delta^2 S} \sigma^2 S^2 dt - \frac{\delta V}{\delta S} dS$$

$$d\pi = \frac{\delta V}{\delta t} dt + \frac{1}{2}\frac{\delta^2 V}{\delta^2 S} \sigma^2 S^2 dt \tag{6}$$

We know $d\pi = r\pi dt = r(v - \Delta S)dt$ So equation 6 becomes -

$$r(v - \Delta S)dt = \frac{\delta V}{\delta t} dt + \frac{1}{2}\frac{\delta^2 V}{\delta^2 S} \sigma^2 S^2 dt$$

Further simplifying the equation we get, which is our main Black-Scholes equation -

$$\frac{\delta V}{\delta t} + \frac{1}{2}\frac{\delta^2 V}{\delta^2 S} \sigma^2 S^2 + rS\frac{\delta V}{\delta S} - rV = 0$$

4 Solution Of Black Scholes Equation

First, start with the Black-Scholes equation:

$$\frac{\delta V}{\delta t} + \frac{1}{2}\sigma^2 S^2 \frac{\delta^2 V}{\delta S^2} + rS\frac{\delta V}{\delta S} - rV = 0 \tag{1}$$

Then set $t = T - \frac{\tau}{\frac{1}{2}\sigma^2}$ and solve for τ :

$$\frac{\tau}{\frac{1}{2}\sigma^2} = T - t$$

$$\tau = (T-t)\frac{1}{2}\sigma^2$$

Next set $S = Ke^x$ and solve for x :

$$e^x = \frac{S}{K}$$

$$x = \ln\frac{S}{K}$$

With both of these equations, set:

$$V(S,t) = Kv(x,t) \tag{2}$$

The next step is to take the first and second derivatives of V with respect to stock price and the first derivative with respect to time:

$$\frac{\delta V}{\delta t} = K\frac{\delta v}{\delta \tau} * \frac{\delta \tau}{\delta t} = K\frac{\delta v}{\delta \tau}[(T-t)\frac{1}{2}\sigma^2\frac{\delta}{\delta t}] = K\frac{\delta v}{\delta t} * \frac{-\sigma^2}{2}$$

$$\frac{\delta V}{\delta S} = K\frac{\delta v}{\delta x} * \frac{\delta x}{\delta S} = K\frac{\delta v}{\delta x}[\ln\frac{S}{K}\frac{\delta}{\delta S}] = K\frac{\delta v}{\delta x} * \frac{1}{S}$$

Using $\frac{\delta x}{\delta S} = \frac{1}{\frac{S}{K}} * \frac{1}{K} = \frac{1}{S}$:

$$\frac{\delta^2 V}{\delta S^2} = \frac{\delta}{\delta S}(K\frac{\delta v}{\delta x} * \frac{1}{S})$$

$$= K\frac{\delta v}{\delta x} * \frac{-1}{S^2} + K\frac{\delta}{\delta S}(\frac{\delta v}{\delta x}\frac{1}{S})$$

$$= K\frac{\delta v}{\delta x} * \frac{-1}{S^2} + K\frac{\delta}{\delta S}(\frac{\delta v}{\delta x})\frac{\delta x}{\delta S} * \frac{1}{S}$$

$$= K\frac{\delta v}{\delta x} * \frac{-1}{S^2} + K\frac{\delta^2 v}{\delta x^2} * \frac{1}{S^2}$$

with these equations , the terminal equation is set to :

$$V(S,T) = max(S-K,0) = max(ke^x - K, 0)$$

$$V(S,T) = Kv(x,0) and v(x,0) = max(e^x - 1, 0)$$

Take the derivatives and plug them back into equation (1):

$$(K\frac{\delta v}{\delta \tau} * \frac{-\sigma^2}{2}) + \frac{\sigma^2}{2}S^2(K\frac{\delta v}{\delta x} * \frac{-1}{S^2} + K\frac{\delta^2 v}{\delta x^2} * \frac{1}{S^2}) + rS(K\frac{\delta v}{\delta x} * \frac{1}{S}) - rKv = 0$$

Simplify the equation by factoring out the K values, canceling out S and S^2 :

$$(\frac{\delta v}{\delta t} * \frac{-\sigma^2}{2}) + \frac{\sigma^2}{2} S^2 (\frac{\delta v}{\delta x} * \frac{-1}{S^2} + \frac{\delta^2 v}{\delta x^2} * \frac{1}{S^2}) + rS(\frac{\delta v}{\delta x} * \frac{1}{S}) - rv = 0$$

$$(\frac{\delta v}{\delta \tau} * \frac{-\sigma^2}{2}) + \frac{\sigma^2}{2}(\frac{\delta^2 v}{\delta x^2} - \frac{\delta v}{\delta x}) + r(\frac{\delta v}{\delta x}) - rv = 0$$

Solve for $\frac{\delta v}{\delta \tau}$:

$$\frac{\delta v}{\delta \tau} * \frac{\sigma^2}{2} = \frac{\sigma^2}{2}(\frac{\delta^2 v}{\delta x^2} - \frac{\delta v}{\delta x}) + r\frac{\delta v}{\delta x} - rv$$

Factor out $\frac{\sigma^2}{2}$, let $k = \frac{r}{\frac{\sigma^2}{2}}$ to substitute, and combine like terms:

$$\frac{\delta v}{\delta \tau} = (\frac{\delta^2}{\delta x^2} - \frac{\delta v}{\delta x}) + \frac{r}{\frac{\sigma^2}{2}} * \frac{\delta v}{\delta x} - \frac{r}{\frac{\sigma^2}{2}} v$$

$$\frac{\delta v}{\delta \tau} = (\frac{\delta^2}{\delta x^2} - \frac{\delta v}{\delta x}) + k\frac{\delta v}{\delta x} - kv$$

$$\frac{\delta v}{\delta \tau} = (\frac{\delta^2}{\delta x^2} + (k-1)\frac{\delta v}{\delta x}) - kv \tag{3}$$

This leaves one parameter, k, that has no dimension. From this, re-scale the v equations so that :

$$v = \exp \alpha x + \beta \tau u(x, \tau) \tag{4}$$

Derive according to x and τ :

$$\frac{\delta v}{\delta \tau} = \beta \exp \alpha x + \beta \tau u + \exp \alpha x + \beta \tau \frac{\delta u}{\delta \tau}$$

$$\frac{\delta v}{\delta x} = \alpha \exp \alpha x + \beta \tau u + \exp \alpha x + \beta \tau \frac{\delta u}{\delta x}$$

$$\frac{\delta^2 v}{\delta x^2} = \alpha^2 \exp \alpha x + \beta \tau u + 2\alpha \exp \alpha x + \beta \tau \frac{\delta u}{\delta x} + \exp \alpha x + \beta \tau \frac{\delta^2 u}{\delta x^2}$$

Plug these derivatives into equation (3):

$$\beta \exp \alpha x + \beta \tau u + \exp \alpha x + \beta \tau \frac{\delta u}{\delta \tau} = \alpha^2 \exp \alpha x + \beta \tau u + 2\alpha \exp \alpha x + \beta \tau \frac{\delta u}{\delta x} + \exp \alpha x + \beta \tau \frac{\delta^2 u}{\delta x^2} + (k-1)(\alpha \exp \alpha x + \beta$$

Divide by $\exp \alpha x + \beta \tau$ and combine like terms:

$$\beta u + \frac{\delta u}{\delta \tau} = \alpha^2 u + 2\alpha \frac{\delta u}{\delta x} + \frac{\delta^2 u}{\delta x^2} + (k-1)(\alpha u + \frac{\delta u}{\delta x}) - ku$$

$$\beta u + \frac{\delta u}{\delta \tau} = \alpha^2 u + 2\alpha \frac{\delta u}{\delta x} + \frac{\delta^2 u}{\delta x^2} + k\alpha u + k\frac{\delta u}{\delta x} - \alpha u - \frac{\delta u}{\delta x}) - ku$$

$$\frac{\delta u}{\delta \tau} = \alpha^2 u + 2\alpha \frac{\delta u}{\delta x} + \frac{\delta^2 u}{\delta x^2} + k\alpha u + k\frac{\delta u}{\delta x} - \alpha u - \frac{\delta u}{\delta x}) - ku - \beta u \tag{5}$$

$$\frac{\delta u}{\delta\tau}=\frac{\delta^2 u}{\delta x^2}+\frac{\delta u}{\delta x}(k-1+2\alpha)+u(\alpha^2+k\alpha-\alpha-k-\beta)$$

The coefficient should be equal to zero, meaning that $u=0$ and $\frac{\delta u}{\delta x}=0$. choose $\alpha=\frac{-(k-1)}{2}$ and $\beta=\alpha^2+(k-1)a-k=\frac{-(k+1)^2}{4}$ then plug into equation (5). this will lead to the basis of the heat equation:

$$\frac{\delta u}{\delta\tau}=\frac{delta^2 u}{\delta x^2}+\frac{\delta u}{\delta x}[k-1+2(-\frac{k-1}{2})]+u[(-\frac{k-1}{2})^2+k(-\frac{k-1}{2})-(-\frac{k-1}{2})-k-(\frac{-(k+1)^2}{4})]$$

$$\frac{\delta u}{\delta\tau}=\frac{\delta^2 u}{\delta x^2}+\frac{\delta u}{\delta x}[k-1-(k-1)]+u[(\frac{k^2-2k+1}{4})-(\frac{k^2-k}{2})+(\frac{k-1}{2})-k+(\frac{k^2+2k+1}{4})]$$

$$\frac{\delta u}{\delta\tau}=\frac{\delta^2 u}{\delta x^2}+\frac{\delta u}{\delta x}[0]+u[(\frac{k^2-2k+1}{4})-(\frac{2k^2-2k}{4})+(\frac{2k-2}{4})-\frac{4k}{4}+(\frac{k^2+2k+1}{4})]$$

$$\frac{\delta u}{\delta\tau}=\frac{\delta^2 u}{\delta x^2}+u[(\frac{k^2-2k+1-2k^2+2k+2k-2-4k+k^2+2k+1}{4})]$$

$$\frac{\delta u}{\delta\tau}=\frac{\delta^u}{\delta x^2}+u[0]$$

$$\frac{\delta u}{\delta\tau}=\frac{\delta^2}{\delta x^2}$$

$$u_\tau=u_{xx}$$

The initial condition is transformed into:

$$u(x,0)=max(\exp(\frac{(k+1)}{2})x-\exp(\frac{(k-1)}{2})x,0) \tag{6}$$

This leads to the Heat equation solution, which we will transform to use for the Black-Scholes equation:

$$u(x,t)=\frac{1}{2\sqrt{\pi\tau}}\int_{-\infty}^{\infty}u_o(s)\exp\frac{-(x-s)^2}{4\tau}ds$$

Make a change of variable so that $s=z\sqrt{2\tau}+x$. The goal is to get the exponent into the form of $\frac{-y^2}{2}$, which is why $z=\frac{x-s}{\sqrt{2\tau}}$, to get the equation of the standard normal deviation. This will then be used later in this derivation to find the final solution. The derivatives of these equations will then be $ds=dx$ and $dx=\sqrt{2\tau}dz$

$$u(x,t)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}u_o(z\sqrt{2\tau}+x)\exp\frac{-(z)^2}{2}dz \tag{7}$$

From this transformation, there is a change in equation (6) for the x value:

$$u_0=\exp\frac{k+1}{2}(z\sqrt{2\tau}+x)-\exp\frac{k-1}{2}(z\sqrt{2\tau}+x) \tag{8}$$

It must happen that $u_0>0$ because the time value cannot be less than 0. So, $x>\frac{x}{\sqrt{2\tau}}$ which transforms the base of the domain of equation (7):

$$u(x,\tau)=\frac{1}{\sqrt{2\pi}}\int_{-\frac{-x}{\sqrt{2\pi}}}^{\infty}\exp\frac{k+1}{2}(x+z\sqrt{2\tau})\exp\frac{-z^2}{2}-\exp\frac{k-1}{2}(x+z\sqrt{2\tau})\exp\frac{-z^2}{2}dz$$

$$u(x,\tau) = \frac{1}{\sqrt{2\pi}} \int_{-\frac{-x}{\sqrt{2\pi}}}^{\infty} e^{\frac{k+1}{2}(x+z\sqrt{2\tau})} e^{\frac{-z^2}{2}} dz - \frac{1}{\sqrt{2\pi}} \int_{-\frac{-x}{\sqrt{2\pi}}}^{\infty} e^{\frac{k-1}{2}(x+z\sqrt{2\tau})} e^{\frac{-z^2}{2}} dz \tag{9}$$

After the split of the integral, take the first integral and complete the square of the exponent:

$$\frac{k+1}{2}(x+z\sqrt{2\tau}) - \frac{z^2}{2} = \frac{-1}{2}[z^2 - z\sqrt{2\tau}(k+1)] + \frac{x(k+1)}{2}$$

$$= \frac{-1}{2}[z^2 - z\sqrt{2\tau}(k+1) + \frac{\tau}{2}(k+1)^2] + \frac{x(k+1)}{2} - [-\frac{\tau(k+1)^2}{4}]$$

$$= -\frac{1}{2}[z - \sqrt{\frac{\tau}{2}}(k+1)]^2 + \frac{x(k+1)}{2} + \frac{\tau(k+1)^2}{4}$$

Plug this value back into the first integral of equation (9). This value is the exponent of e. The last two parts of the exponent do not have z values, so they go in front of the integral:

$$\frac{e^{\frac{x(k+1)}{2} + \frac{\tau(k+1)^2}{4}}}{\sqrt{2\pi}} \int_{\frac{-x}{\sqrt{2\pi}}}^{\infty} e^{\frac{-1}{2}[z - \sqrt{\frac{\tau}{2}}(k+1)]^2} dz$$

Set $y = z - \sqrt{\frac{\tau}{2}}(k+1)$, $dy = dz$, and $z = \frac{-x}{\sqrt{2\tau}}$ which in turn changes the domain once again:

$$\frac{e^{\frac{x(k+1)}{2} + \frac{\tau(k+1)^2}{4}}}{\sqrt{2\pi}} \int_{\frac{-x}{\sqrt{2\pi}} - \sqrt{\frac{\tau}{2}}(k+1)}^{\infty} e^{\frac{-y^2}{2}} dy$$

Equation (9) then becomes:

$$u(x,\tau) = \frac{e^{\frac{x(k+1)}{2} + \frac{\tau(k+1)^2}{4}}}{\sqrt{2\pi}} \int_{-\infty}^{\frac{x}{\sqrt{2\pi}} + \sqrt{\frac{\tau}{2}}(k+1)} e^{\frac{-y^2}{2}} dy - \frac{e^{\frac{x(k-1)}{2} + \frac{\tau(k-1)^2}{4}}}{\sqrt{2\pi}} \int_{-\frac{x}{\sqrt{2\pi}} - \sqrt{\frac{\tau}{2}}(k+1)}^{\infty} e^{\frac{-y^2}{2}} dy \tag{10}$$

The area under normal curve formula form $-\infty \to d$

$$N(d) = \frac{1}{\sqrt{2\tau}} \int_{-\infty}^{d} e^{\frac{-y^2}{2}} dy$$

where,

$$d = \frac{x}{\sqrt{2\tau}} + \sqrt{\frac{\tau}{2}}(k+1)$$

$-\infty \to d$ is the same as $-d \to \infty$. The values of d_2 is the same as d_1 with the expression that $(k+1)$ is $(k-1)$, so:

$$d_1 = \frac{x}{\sqrt{2\tau}} + \sqrt{\frac{\tau}{2}}(k+1)$$

$$d_2 = \frac{x}{\sqrt{2\tau}} + \sqrt{\frac{\tau}{2}}(k-1)$$

Plug N into the equation (10):

$$u(x,\tau) = e^{\frac{x(k+1)}{2}+\frac{\tau(k+1)^2}{4}}N(d_1) - e^{\frac{x(k-1)}{2}+\frac{\tau(k-1)^2}{4}}N(d_2) \tag{11}$$

Now plug α, β, and equation (11) into equation (4):

$$v(x,\tau) = e^{\frac{-x(k-1)}{2}-\frac{\tau(k+1)^2}{4}}u(x,\tau)$$

$$v(x,\tau) = e^{\frac{-x(k-1)}{2}-\frac{\tau(k+1)^2}{4}} * [e^{\frac{x(k+1)}{2}+\frac{\tau(k+1)^2}{4}}N(d_1) - e^{\frac{x(k-1)}{2}+\frac{\tau(k-1)^2}{4}}N(d_2)]$$

The exponents will then cancel out to get a simple equation of:

$$v(x,\tau) = e^{x}N(d_1) - e^{-k\tau}N(d_2)$$

There are two values from earlier that need to be plugged back in: $x = \ln{(S/K)}$ and $\tau = \frac{1}{2}\sigma^2(T-t)$ to get:

$$v(x,\tau) = e^{\ln{(S/K)}}N(d_1) - e^{-\frac{k}{2}\sigma^2(T-t)}N(d_2)$$

$$v(x,\tau) = \frac{S}{K}N(d_1) - e^{-\frac{k}{2}\sigma^2(T-t)}N(d_2) \tag{12}$$

Plug these values into the d-values as well:

$$d_1 = \frac{\ln{(\frac{S}{K})}}{\sqrt{2(\frac{1}{2}\sigma^2(T-t))}} + \sqrt{\frac{\frac{1}{2}\sigma^2(T-t)}{2}}(k+1)$$

$$= \frac{\ln{(\frac{S}{K})}}{\sigma\sqrt{T-t}} + \frac{\sigma}{2}\sqrt{T-t}(k+1)$$

$$= \frac{\ln{(\frac{S}{K})}}{\sigma\sqrt{T-t}} + \frac{\frac{\sigma^2}{2}\sqrt{T-t}(k+1)}{\sigma\sqrt{T-t}}$$

$$= \frac{\ln{(\frac{S}{K})}(\frac{\sigma^2}{2}k + \frac{\sigma^2}{2})(T-t)}{\sigma\sqrt{T-t}}$$

The risk-free interest rate is equal to $r = \frac{k}{2}\sigma^2$ So then equation (12) and the d-value become:

$$v(x,\tau) = \frac{S}{K}N(d_1) - e^{-r(T-t)}N(d_2) \tag{13}$$

$$d_1 = \frac{\ln{(\frac{S}{K})} + (r + \frac{\sigma^2}{2})(T-t)}{\sigma\sqrt{T-t}}$$

Equation (13) is then plugged back into equation (2):

$$V(S,t) = K\frac{S}{K}N(d_1) - Ke^{-r(T-t)}N(d_2)$$
$$= SN(d_1) - Ke^{-r(T-t)}N(d_2)$$

We then have the solution to the Black-Scholes Equation:

$$V(S,t) = SN(d_1) - Ke^{-r(T-t)}N(d_2) \tag{14}$$

Where,

$$d_1 = \frac{\ln{(\frac{S}{K})} + (r + \frac{\sigma^2}{2})(T-t)}{\sigma\sqrt{T-t}} \tag{15}$$

$$d_2 = \frac{\ln{(\frac{S}{K})} + (r - \frac{\sigma^2}{2})(T-t)}{\sigma\sqrt{T-t}} \tag{16}$$

5 Use of Black Scholes Equation

Let's take an example of Tesla stock (TSLA). Now the current stock price is \$245. The strike price set to \$294 for 101 days. the voltality of the stock was 38% and the interest rate was 2.12%.

So using above given values we can predict the fair value of option.
First we will find d_1

$$d_1 = \frac{\ln\left(\frac{S}{K}\right) + \left(r + \frac{\sigma^2}{2}\right)(T-t)}{\sigma\sqrt{T-t}}$$

$$d_1 = \frac{\ln\left(\frac{245}{294}\right) + \left(0.0212 + \frac{0.38^2}{2}\right)\left(\frac{101}{365}\right)}{0.38\frac{101}{365}}$$

$$d_1 = \frac{-0.18232 + 0.02584}{0.19989}$$

$$d_1 = -0.78280$$

$$N(d_1) = N(-0.78) = 0.2177$$

We know:

$$d_2 = d_1 - \sigma\sqrt{T-t}$$

$$d_2 = -0.78280 - 0.38\sqrt{\frac{101}{365}}$$

$$d_2 = -0.98269$$

$$N(d_2) = N(-0.98) = 0.1635$$

$$V(S,t) = SN(d_1) - Ke^{-r(T-t)}N(d_2)$$

$$V(S,t) = 245 * (0.2177) - 294 * e^{-0.0212\left(\frac{101}{365}\right)} * (0.1635)$$

$$V(S,t) = 53.34 - (294)(099415)(0.1635)$$

$$V(S,t) = 53.34 - 47.79$$

$$V(S,t) = \$5.55$$

So , the fair price of the option is \$ 5.55 .

6 Plotting Black Scholes Equation in Python

6.1 Finding value of option price

In the below code we calculate the fair value of option by putting the values of parameter in the Black Scholes equation.

```
Run Cell | Run Below | Debug cell | Go to [1]
#%%
import numpy as np
from scipy.stats import norm

#define variables
r= 0.047
s = 2080
k=2160
t=31/365
sigma=0.73

def blackscholes(r,s,k,t,sigma):
        d1=((np.log(s/k) + (r+sigma**2/2)*t))/(sigma*np.sqrt(t))
        d2= d1 - sigma*np.sqrt(t)
        price= s*norm.cdf(d1,0,1) - k*np.exp(-r*t)*norm.cdf(d2,0,1)
        return price


print("option price is ",round(blackscholes(r, s, k, t, sigma),2))
```

The output of the above code is :

```
[3] import numpy as np...

option price is  145.77
```

6.2 Finding the value of option price for RELIANCE stock

In the below code I calculated the option values of reliance stock from date 2 Feb to 13 Feb. Assuming the value of risk free interest = 0.01% and volatility = 0.30.

```
Run Cell | Run Below | Debug cell | Go to [1]
#%%
import pandas as pd
import pandas_datareader as dr
import numpy as np
from scipy.stats import norm
%matplotlib inline
df = dr.data.get_data_yahoo('RELIANCE.NS',start='2021-02-02',end='2021-02-12')
#df
#print(df.columns)
df1=df['Open']
#print(df1)
s=[]
for i in df1:
    s.append(i)
#print(s)
t=[]
from datetime import datetime
#datetime(year, month, day)
a = datetime(2021, 2 , 13)
for i in range(len(df1.index)):
    d= a- df1.index[i]
    j = (d.days)/365
    t.append(j)
#print(t)

#define variables
r= 0.01
k=2052
sigma=0.30
option=[]

def blackscholes(r,s,k,t,sigma):
        d1=(np.log(s/k) + (r+sigma**2/2)*t)/(sigma*np.sqrt(t))
        d2= d1 - sigma*np.sqrt(t)
        price= s*norm.cdf(d1,0,1) - k*np.exp(-r*t)*norm.cdf(d2,0,1)
        option.append(price)
        return price

for i in range(len(df1.index)):
    print(i+1," When the Stock price ", s[i],"On date",df1.index[i] )
    print("The calculated option price is ",round(blackscholes(r, s[i], k, t[i], sigma),2))
```

The output of the code we get is as follows:

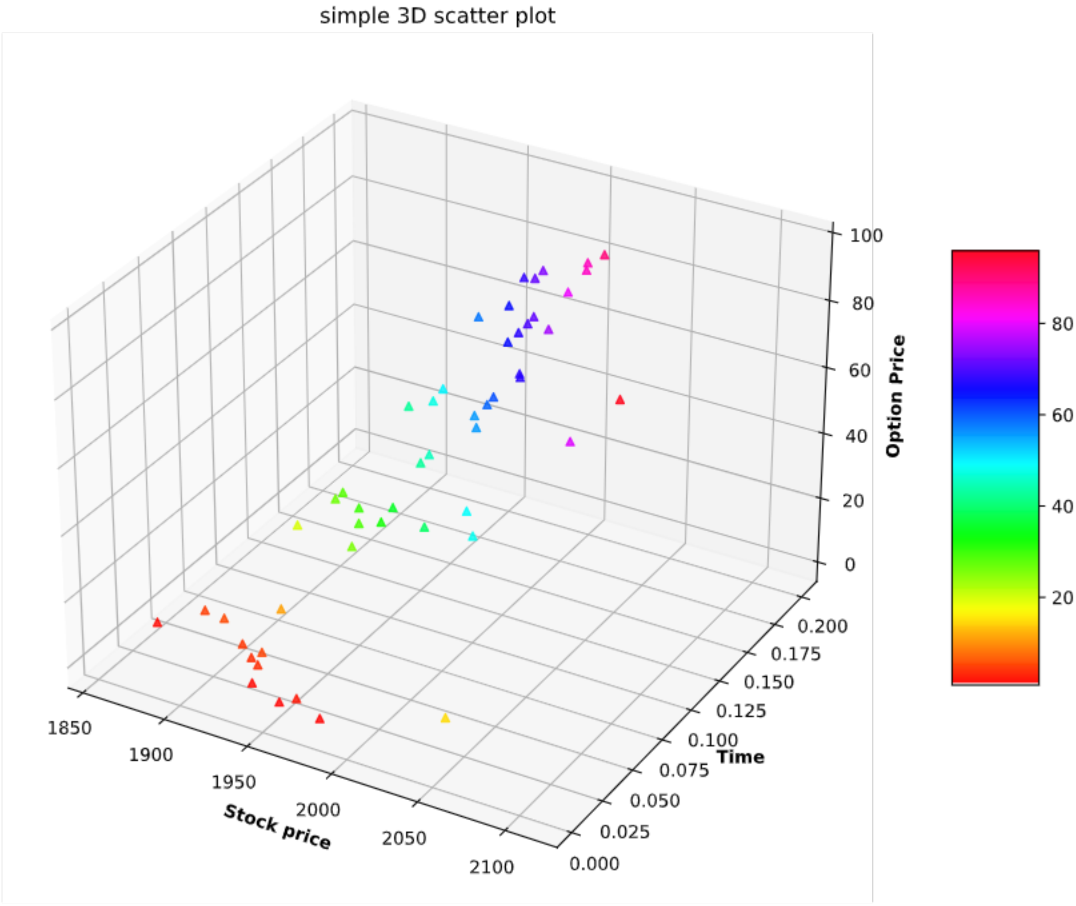
```
[1] import pandas as pd...

1  When the Stock price  1914.25 On date 2021-02-02 00:00:00
The calculated option price is  4.42
2  When the Stock price  1928.300048828125 On date 2021-02-03 00:00:00
The calculated option price is  5.03
3  When the Stock price  1924.0 On date 2021-02-04 00:00:00
The calculated option price is  3.73
4  When the Stock price  1930.050048828125 On date 2021-02-05 00:00:00
The calculated option price is  3.43
5  When the Stock price  1933.050048828125 On date 2021-02-08 00:00:00
The calculated option price is  1.29
6  When the Stock price  1961.949951171875 On date 2021-02-09 00:00:00
The calculated option price is  2.18
7  When the Stock price  1953.75 On date 2021-02-10 00:00:00
The calculated option price is  0.78
8  When the Stock price  1980.0 On date 2021-02-11 00:00:00
The calculated option price is  1.03
9  When the Stock price  2055.5 On date 2021-02-12 00:00:00
The calculated option price is  14.72
```

6.3 3d scatter plot between option price , stock price and time to maturity

Plotting scatter plot for calculated option price for the Reliance stock for 2 months that is from 2 December 2020 to 2 February 2021.

```
#Import libraries
from mpl_toolkits import mplot3d
import numpy as np
import matplotlib.pyplot as plt

x=s
y=t
z= option

# Creating figure
fig = plt.figure(figsize = (16, 9))
ax = plt.axes(projection ="3d")

# Add x, y gridlines
ax.grid(b = True, color ='grey',
        linestyle ='-.', linewidth = 0.3,
        alpha = 0.2)


# Creating plot
sctt = ax.scatter3D(x, y, z,c=z,
                    alpha = 0.8,
                    cmap = 'hsv',
                    marker ='^')

plt.title("simple 3D scatter plot")
ax.set_xlabel('Stock price', fontweight ='bold')
ax.set_ylabel('Time of maturity', fontweight ='bold')
ax.set_zlabel('Option Price', fontweight ='bold')
fig.colorbar(sctt, ax = ax, shrink = 0.5, aspect = 5)

# show plot
plt.show()

```

Output:

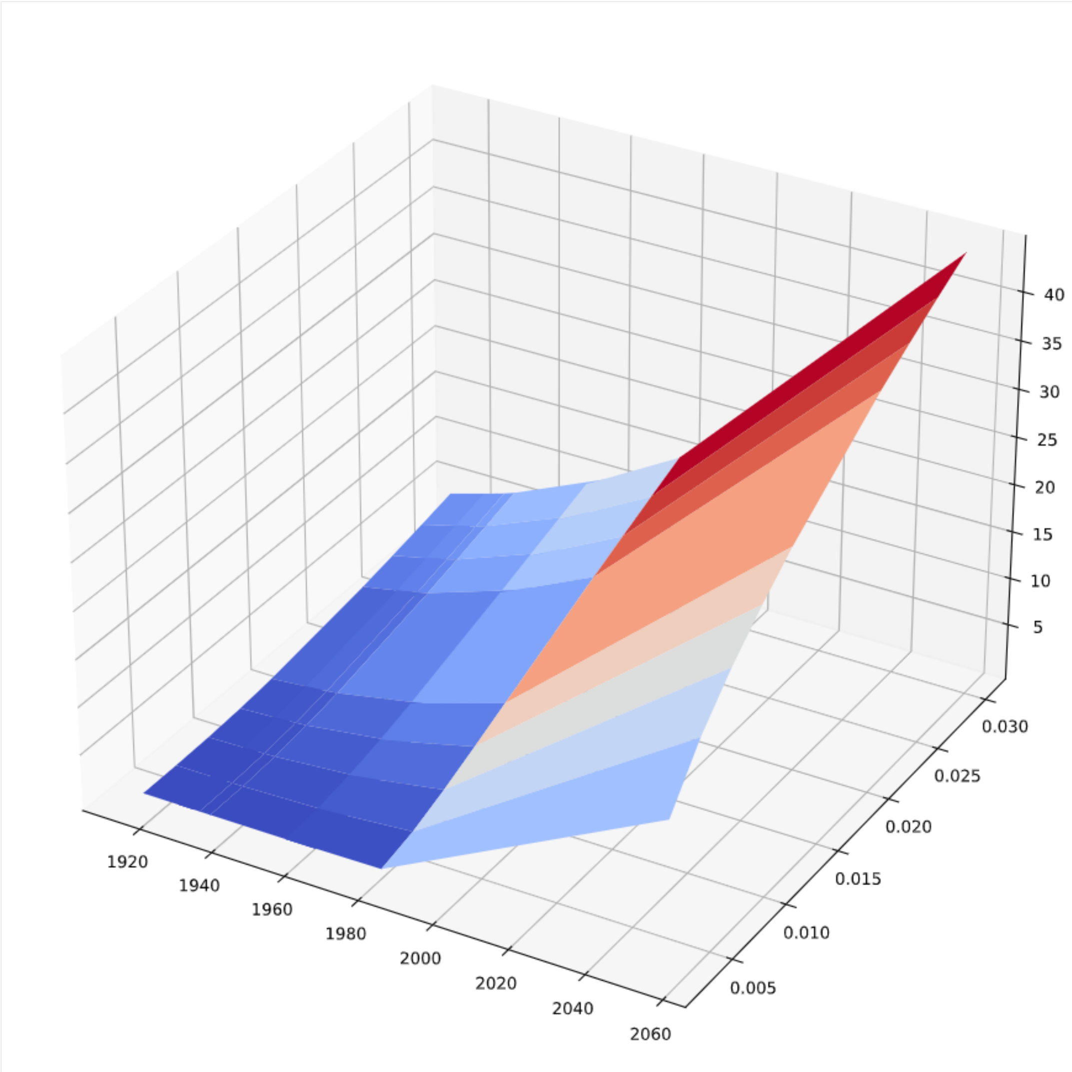

simple 3D scatter plot
Stock price
1850
1900
1950
2000
2050
2100
Time
0.000
0.025
0.050
0.075
0.100
0.125
0.150
0.175
0.200
Option Price
0
20
40
60
80
100
20
40
60
80

6.4 Surface plot

In this I draw the surface plot for option price with different combination of stock price and time of maturity.

```python
for i in range(len(df1.index)):
    print(i+1," Stock price ", s[i],"On date",df1.index[i] )
    print("option price is ",round(blackscholes(r, s[i], k, t[i], sigma),2))

import matplotlib.pyplot as plt
import numpy as np
from mpl_toolkits.mplot3d import Axes3D
from matplotlib import cm

fig=plt.figure(figsize=(12,12))
ax=fig.add_subplot(111)
x,y= np.meshgrid(s,t)
z= x*norm.cdf((np.log(x/k) + (r+sigma**2/2)*y)/(sigma*np.sqrt(y)),0,1) - k*np.exp(-r*y)*norm.cdf(((np.log(x/k) + (r+sigma**2/2)*y)/(sigma*np.sqrt(y)))-sigma*np.sqrt(y),0,1)
ax= fig.gca(projection= '3d')
ax.plot_surface(x,y,z,cmap=cm.coolwarm)
plt.show
```

Output:



7 Other numerical methods to solve Black-scholes Equation

There are various other method to solve the black scholes equation, some of the famous numerical methods are -

1. Finite difference method
2. Markov chain method
3. Monte carlo method
4. fractional generalized homotopy analysis method

8 Limitations of Analytical Solutions available

We solve the Black-Scholes equation using heat equation. In heat equation the transformation of variables are scaled exponentially and cluttered. The uniform error produced by the heat equation could be exponentially multiplied and becomes very unevenly distributed after being transformed back into financial terms, compromising accuracy in the results. For long time to maturity model the heat equation gives silly answers as it is assuming the risk free interest and voltality be constant.
There are also many examples, such as the case of multi-factor models, where variable transformation either cannot be performed or would result in a scheme not that much simpler than the original equation. Thus, we cannot expect to count on the conversion to solve the Black-Scholes model for all occasions.
To overcome such problems various different numerical method were proposed to solve black scholes equation for example finite difference method, which still produces error but giving somewhat more accurate results. Overall Black-Scholes is just a mechanical system which is doesn't include social and other factors which play major role in determining the price.