Filtrado en el dominio de la frecuencia (Parte 1)

Agenda

- Contexto.
- Conceptos preliminares.
- El muestreo y la transformada de Fourier de funciones muestreadas.
- Transformada discreta de Fourier (DFT) de una variable.

Repaso de números complejos

$$C = R + jI$$

$$C^* = R - jI$$

$$|C| = \sqrt{R^2 + I^2}$$

$$\tan\theta = \frac{I}{R}$$

$$\theta = \tan^{-1}\left(\frac{I}{R}\right)$$

$$C = |C|e^{j\theta}$$

$$e^{j\theta} = \cos\theta + j\sin\theta$$

$$e = 2.71828...$$

$$C = |C|\left(\cos\theta + j\sin\theta\right)$$

Series de Fourier

- Jean Baptiste Joseph Fourier.
- En 1822 se publica el libro "*La Théorie Analitique de la Chaleur*" (La teoría analítica del calor).
- Cualquier función periódica se puede expresar como la suma de senos y-o cosenos de diferentes frecuencias multiplicados por coeficientes diferentes (serie de Fourier).

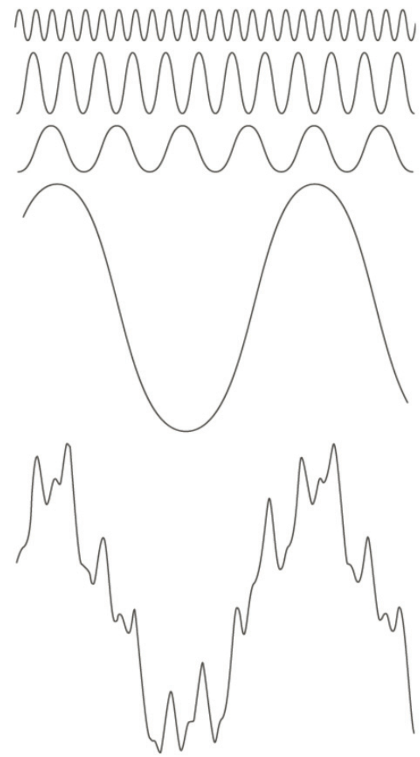


FIGURE 4.1 The function at the bottom is the sum of the four functions above it. Fourier's idea in 1807 that periodic functions could be represented as a weighted sum of sines and cosines was met with skepticism.

$$f(t) = \sum_{n=-\infty}^{\infty} c_n e^{j\left(\frac{2\pi n}{T}\right)t}$$

$$c_n = \frac{1}{T}\int_{-T/2}^{T/2} f(t) e^{-j\left(\frac{2\pi n}{T}\right)t} dt \quad para \quad n = 0, \pm 1, \pm 2, \dots$$

Impulso unitario

$$\delta(t) = \begin{cases} \infty & si \quad t = 0 \\ 0 & si \quad t \neq 0 \end{cases}$$

$$\int_{-\infty}^{\infty} \delta(t)\,dt = 1$$

$$\int_{-\infty}^{\infty} f(t)\delta(t)\,dt = f(0)$$

$$\int_{-\infty}^{\infty} f(t)\delta(t-t_0)\,dt = f(t_0)$$

Impulso unitario discreto

$$\delta(x) = \begin{cases} 1 & si \quad x = 0 \\ 0 & si \quad x \neq 0 \end{cases}$$

$$\sum_{x=-\infty}^{x=\infty} \delta(x) = 1$$

$$\sum_{x=-\infty}^{x=\infty} f(x)\delta(x) = f(0)$$

$$\sum_{x=-\infty}^{x=\infty} f(x)\delta(x - x_0) = f(x_0)$$

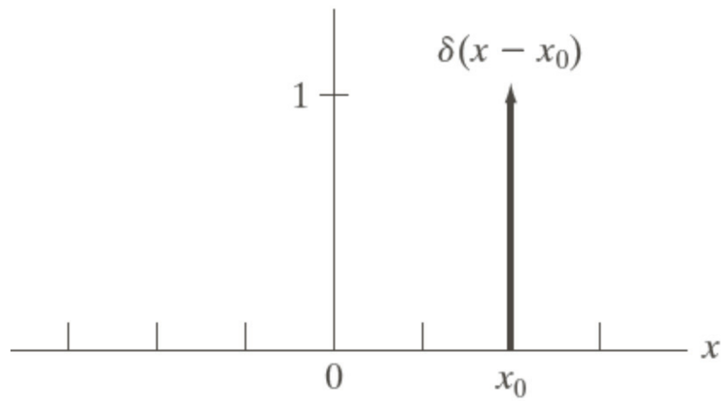


FIGURE 4.2 A unit discrete impulse located at $x = x_0$. Variable x is discrete, and δ is 0 everywhere except at $x = x_0$.

Tren de impulsos

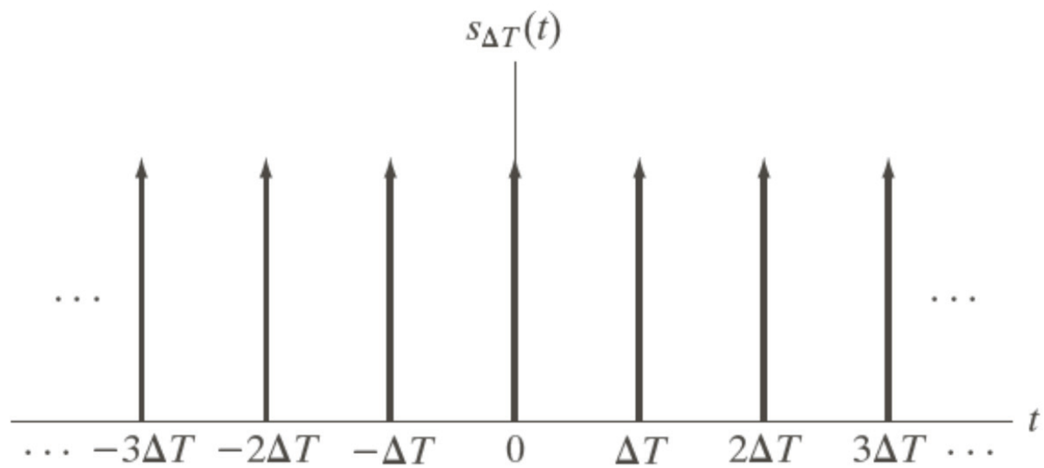


FIGURE 4.3 An impulse train.

$$s_{\Delta T}(t) = \sum_{n=-\infty}^{n=\infty} \delta(t - n\Delta T)$$

Transformada de Fourier de una variable continua

$$\mathcal{F}\left\{f(t)\right\} = F(u) = \int_{-\infty}^{\infty} f(t)\, e^{-j2\pi ut}\, dt$$

$$\mathcal{F}^{-1}\left\{F(u)\right\} = f(t) = \int_{-\infty}^{\infty} F(u)\, e^{j2\pi ut}\, du$$

- Algunas transformadas de utilidad:

$$\mathcal{F}\left[\delta(t)\right] = 1$$

$$\mathcal{F}\left[\delta(t - t_0)\right] = e^{-j2\pi ut_0}$$

$$\mathcal{F}\left[s_{\Delta T}(t)\right] = \mathcal{F}\left[\sum_{n=-\infty}^{n=\infty} \delta(t - n\Delta T)\right] = \frac{1}{\Delta T} \sum_{n=-\infty}^{n=\infty} \delta\left(u - \frac{n}{\Delta T}\right)$$

Transformada de Fourier de una ventana rectangular

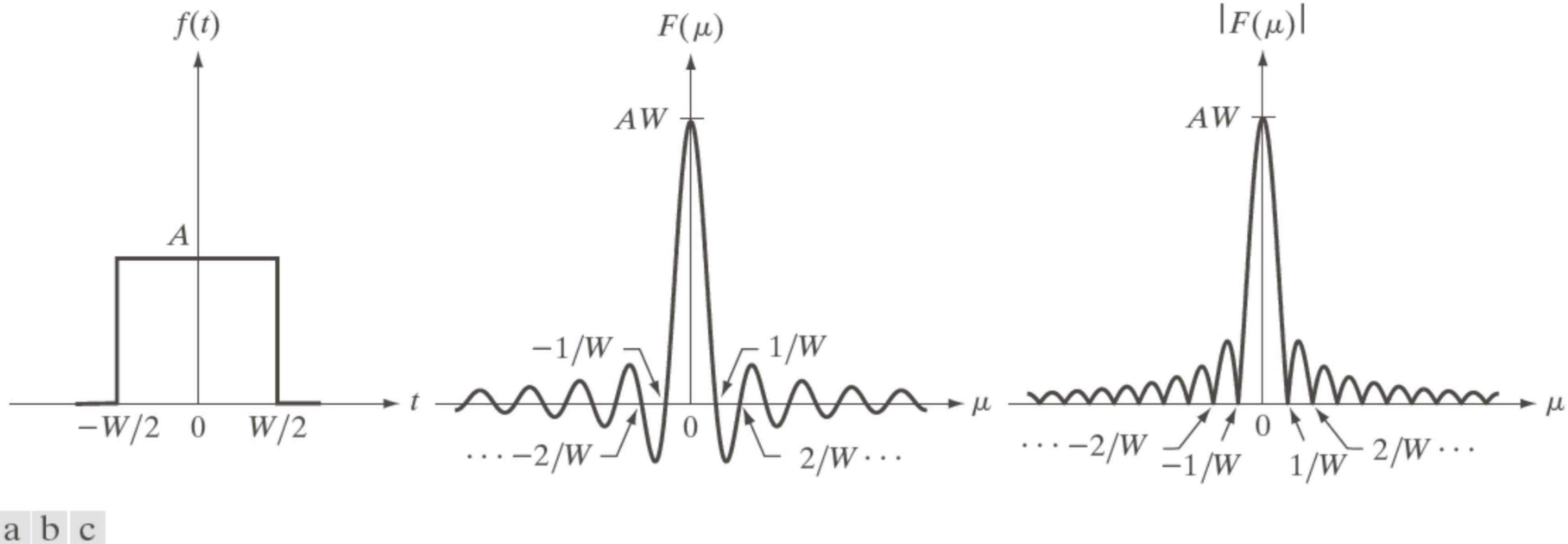


FIGURE 4.4 (a) A simple function; (b) its Fourier transform; and (c) the spectrum. All functions extend to infinity in both directions.

$$f(t) = \begin{cases} A & si \quad |t| \leq \dfrac{W}{2} \\ 0 & si \quad |t| > \dfrac{W}{2} \end{cases}$$

$$F(u) = AW \frac{\sin(\pi u W)}{\pi u W} = AWsinc(uW)$$

$$sinc(m) = \frac{\sin(\pi m)}{(\pi m)}$$

Convolución y pares de transformación

Operador de convolución

$$f(t) * h(t) = \int_{-\infty}^{\infty} f(\tau) h(t-\tau) d\tau$$

$$\mathcal{F}\left[f(t) * h(t)\right] = F(u) H(u)$$

$$\mathcal{F}\left[f(t) h(t)\right] = F(u) * H(u)$$

$$f(t) * h(t) \leftrightarrow F(u) H(u)$$

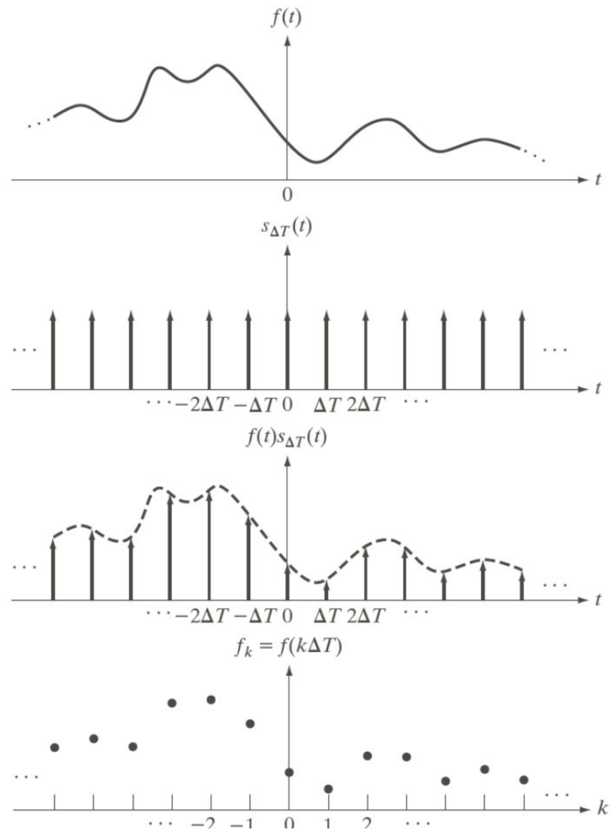
$$f(t) h(t) \leftrightarrow F(u) * H(u)$$

Funciones muestreadas

$f(t)$

$s_{\Delta T}(t)$

$\tilde{f}(t)$



a
b
c
d

FIGURE 4.5
(a) A continuous function. (b) Train of impulses used to model the sampling process. (c) Sampled function formed as the product of (a) and (b). (d) Sample values obtained by integration and using the sifting property of the impulse. (The dashed line in (c) is shown for reference. It is not part of the data.)

$$\tilde{f}(t) = f(t)\,s_{\Delta T}(t) = \sum_{n=-\infty}^{n=\infty} f(t)\,\delta(t - n\Delta T)$$

$$f_k = f(k\Delta T) = \int_{-\infty}^{\infty} f(t)\,\delta(t - k\Delta T)\,dt$$

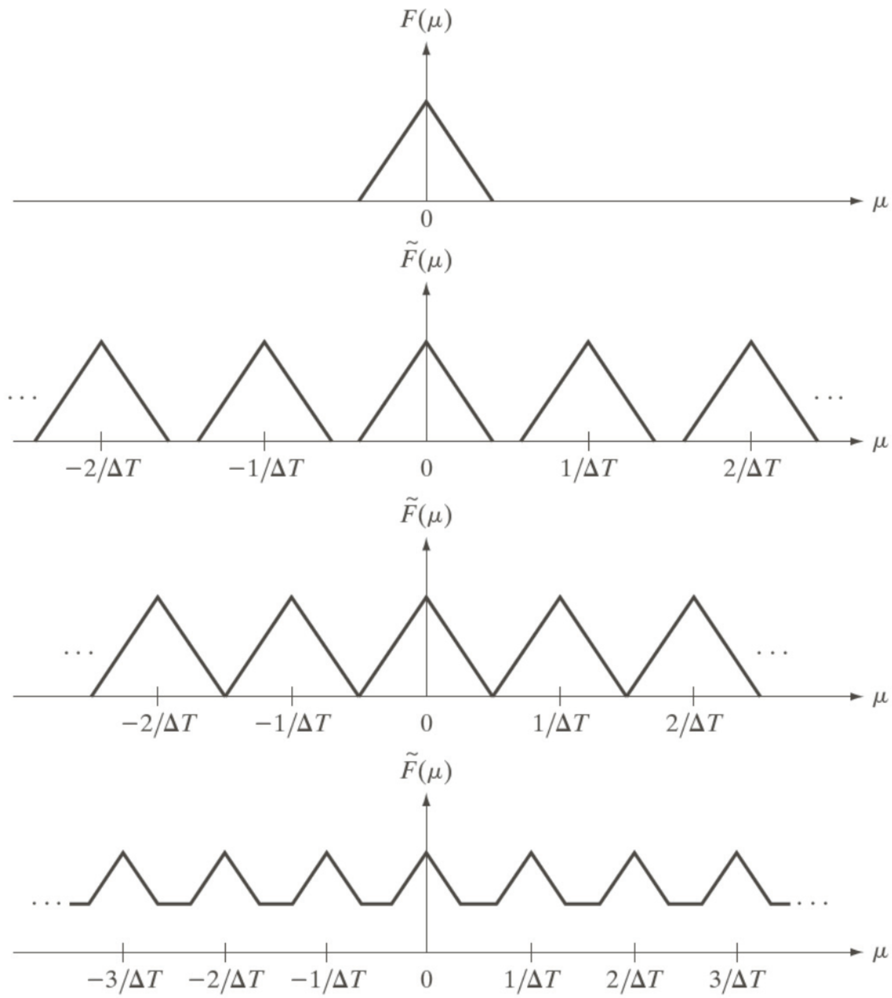
Transformada de Fourier de funciones muestreadas

$$\tilde{f}(t) = f(t)s_{\Delta T}(t) = \sum_{n=-\infty}^{n=\infty} f(t)\delta(t - n\Delta T)$$

$$\tilde{F}(u) = F(u) * S_{\Delta T}(u)$$

$$\boxed{\tilde{F}(u) = \frac{1}{\Delta T}\sum_{n=-\infty}^{n=\infty} F\left(u - \frac{n}{\Delta T}\right)}$$

Sobre muestreo, muestreo crítico y aliasing en señales de banda limitada

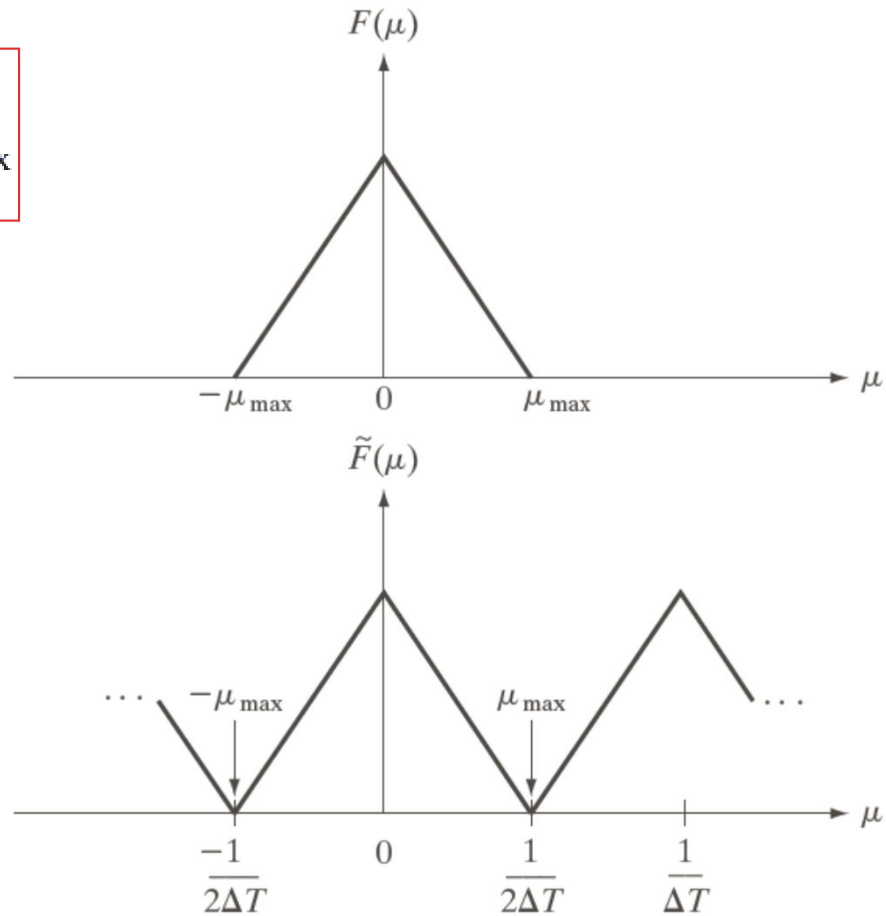


a
b
c
d

FIGURE 4.6
(a) Fourier transform of a band-limited function. (b)–(d) Transforms of the corresponding sampled function under the conditions of over-sampling, critically-sampling, and under-sampling, respectively.

Teorema del muestreo (Nyquist)

$$\boxed{f_s = \frac{1}{\Delta T} > 2u_{\max}}$$



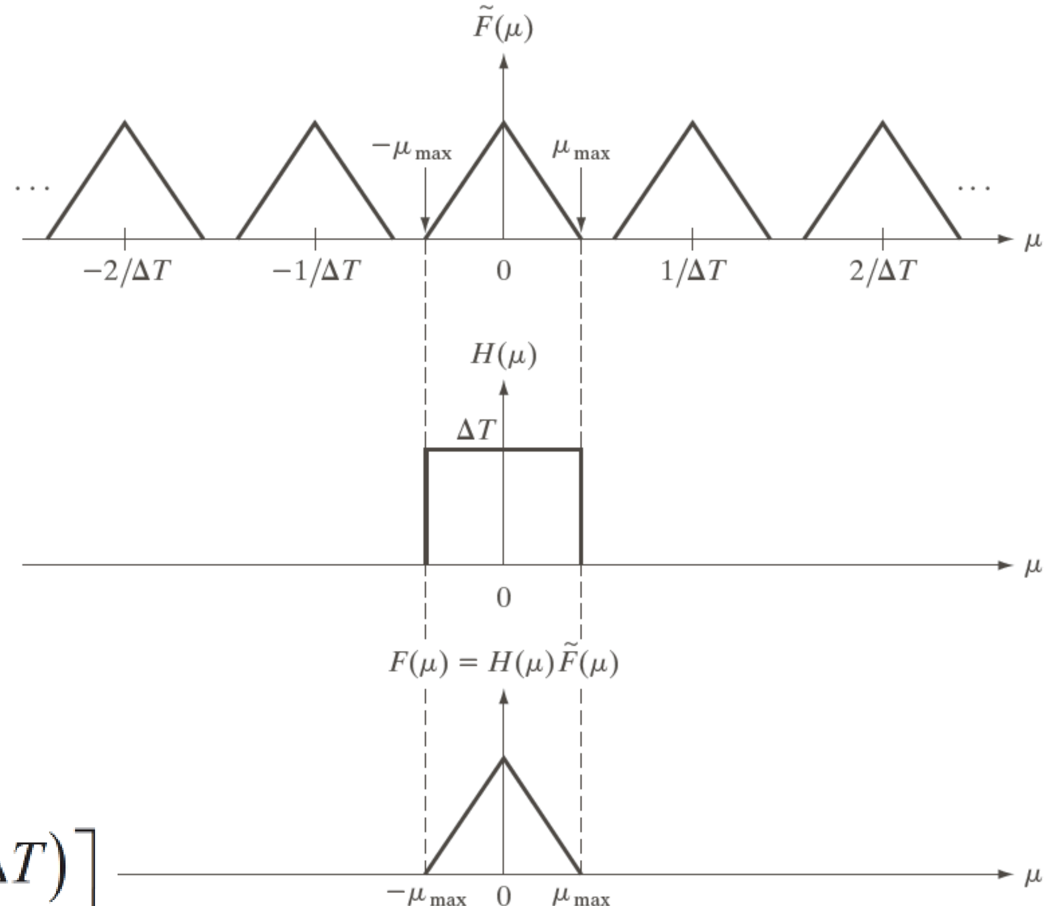
a
b

FIGURE 4.7
(a) Transform of a band-limited function.
(b) Transform resulting from critically sampling the same function.

Reconstrucción de una señal continua a partir del espectro de la señal muestreada correspondiente

a
b
c

FIGURE 4.8 Extracting one period of the transform of a band-limited function using an ideal lowpass filter.



$$H(u) = \begin{cases} \Delta T & si \quad |u| \le u_{max} \\ 0 & si \quad |u| > u_{max} \end{cases}$$

$$F(u) = \tilde{F}(u)H(u)$$

$$f(t) = \int_{-\infty}^{\infty} F(u)e^{j2\pi ut}du$$

$$f(t) = \sum_{n=-\infty}^{n=\infty} f(n\Delta T)\,sinc\left[\frac{(t-n\Delta T)}{\Delta T}\right]$$

Aliasing

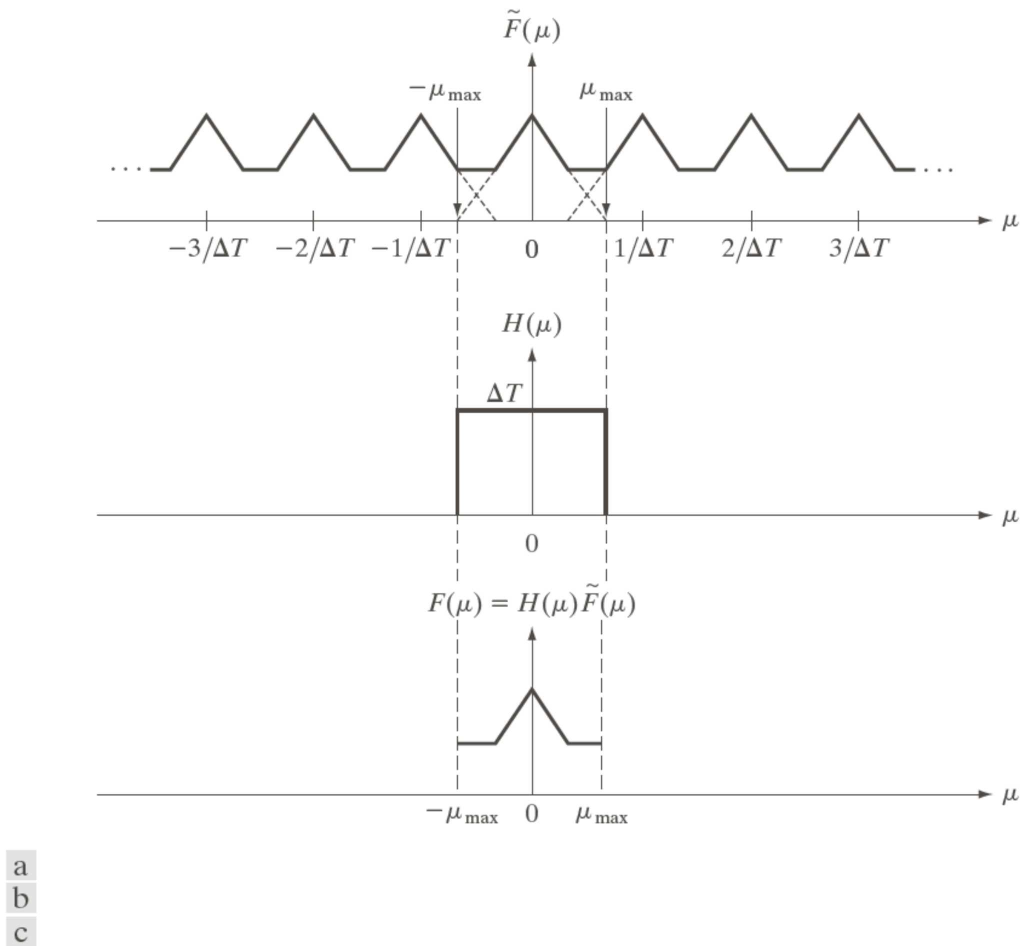


FIGURE 4.9 (a) Fourier transform of an under-sampled, band-limited function. (Interference from adjacent periods is shown dashed in this figure). (b) The same ideal lowpass filter used in Fig. 4.8(b). (c) The product of (a) and (b). The interference from adjacent periods results in aliasing that prevents perfect recovery of $F(\mu)$ and, therefore, of the original, band-limited continuous function. Compare with Fig. 4.8.

Efectos del aliasing

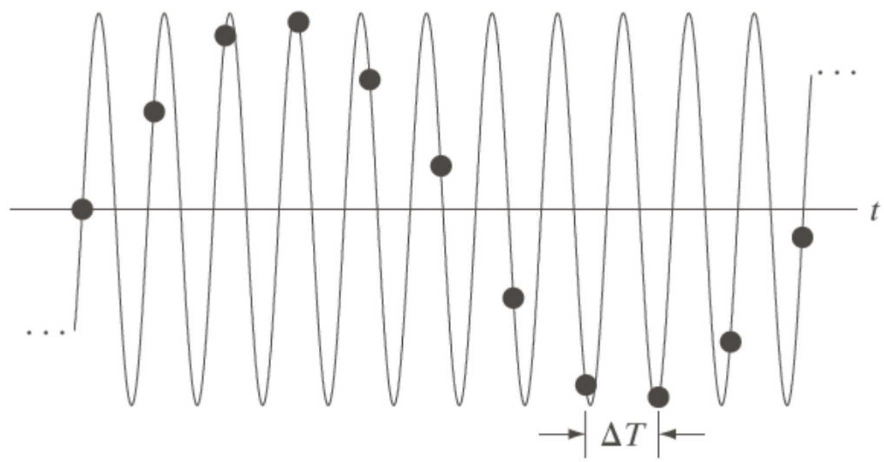


FIGURE 4.10 Illustration of aliasing. The under-sampled function (black dots) looks like a sine wave having a frequency much lower than the frequency of the continuous signal. The period of the sine wave is 2 s, so the zero crossings of the horizontal axis occur every second. ΔT is the separation between samples.

Transformada discreta de Fourier (DFT) de una variable

Definición

$$\mathcal{F}\left[\tilde{f}(t)\right] = \tilde{F}(u) = \sum_{n=-\infty}^{n=\infty} f_n e^{-j2\pi u n\Delta T}$$

- Suponga que se desean obtener *M* muestras igualmente espaciadas de $\tilde{F}(u)$ tomadas en el periodo de $u = 0 \quad a \quad u = 1/\Delta T$. Estas muestras se van a denotar como F_m.

$$u = \left(\frac{1}{M\Delta T}\right)m \qquad para \quad m = 0,1,2,\ldots,M-1$$

$$\boxed{F_m = \sum_{n=0}^{M-1} f_n e^{-j2\pi \frac{m}{M} n} \qquad para \quad m = 0,1,2,\ldots,M-1}$$

$$\boxed{f_n = \frac{1}{M}\sum_{m=0}^{M-1} F_m e^{j2\pi \frac{m}{M} n} \qquad para \quad n = 0,1,2,\ldots,M-1}$$

- Para simplificar la notación:

$$F(u) = \sum_{x=0}^{M-1} f(x) e^{-j2\pi \frac{u}{M} x} \quad para \quad u = 0,1,2,\ldots, M-1$$

$$f(x) = \frac{1}{M} \sum_{u=0}^{M-1} F(u) e^{j2\pi \frac{u}{M} x} \quad para \quad x = 0,1,2,\ldots, M-1$$

- Tanto la transformada directa como la inversa son periódicas:

$$F(u) = F(u + kM)$$
$$f(x) = f(x + kM)$$

Donde *k* es un entero.

- Convolución circular:

$$f(x) \star h(x) = \sum_{m=0}^{M-1} f(m) h(x - m)$$

Preguntas??