The Differential Equations of Fluid Flow

- Compressible / Incompressible
- Viscous / Inviscid
- Laminar / Turbulent
- 2-D / 3-D
- Hyperbolic / Parabolic / Elliptic
- One Phase / Multi-Phase
- Steady / Unsteady

→ The Differential Equations of Fluid Flow are based on conservation principles of

i) Mass

ii) Momentum

iii) Energy

→ Which Fluid Flow Equations are common in Aerospace?

- Navier-Stokes, Reynolds-Averaged Navier-Stokes, Boundary Layer Equations, Euler Equations

- Laplace Equation, Wave Equation, Heat Equation, Transonic Equations

→ Why do we need "Numerical Methods"?

- Known exact solutions are rare compared to the wealth of engineering applications.
- Example: No exact solution is known for a general steady, incompressible, and laminar flow around an airfoil. (One of the exceptions: "Blasius" solution for laminar boundary layer along Flat Plate)
- However, we can use 'numerical' discretization techniques to solve for a solution.
- One has to be careful though, as the solution space may not be unique!

Example: Non-Uniqueness of constellation of shock waves on airfoil in transonic flow.

→ What is the goal of this course?

1) Knowledge of the Fluid Flow Equations relevant to Aerospace Engineering.

2) Ability to classify the various equations and device a numerical algorithm (solution technique) to solve model problems.

3) Knowledge and programming experience of various numerical schemes and the ability to demonstrate a scheme's

$$\left.\begin{array}{l}\text{i) consistency}\\ \text{ii) accuracy}\\ \text{iii) stability}\end{array}\right\}\ \text{Essentials of } \underline{\text{any}} \text{ numerical technique}$$

4.) Provide a firm foundation for future studies in Computational Fluid Dynamics (CFD).

Navier-Stokes Equations

- may be obtained by using infinitesimal or finite control volume approaches
- Here: Differential form obtained from infinitesimal control volume moving along a streamline w/ velocity vector $\vec{V}(u,v,w)$, which equals the flow velocity at each point.
- Assumptions: Three-Dimensional & Incompressible Flow

Continuity Equation

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0 \qquad (1)$$

Substantial Derivative

$$\frac{D(\ )}{Dt} = \frac{\partial(\ )}{\partial t} + u\frac{\partial(\ )}{\partial x} + v\frac{\partial(\ )}{\partial y} + w\frac{\partial(\ )}{\partial z}$$

$$= \frac{\partial(\ )}{\partial t} + \vec{V}\cdot\nabla(\ ) \qquad (2)$$

X-Momentum Equation

$$\rho \frac{Du}{Dt} = -\frac{\partial p}{\partial x} + \left(\frac{\partial \tau_{xx}}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + \frac{\partial \tau_{xz}}{\partial z}\right) + \rho f_x \quad (3)$$

Y-Momentum Equation

$$\rho \frac{Dv}{Dt} = -\frac{\partial p}{\partial y} + \left(\frac{\partial \tau_{yx}}{\partial x} + \frac{\partial \tau_{yy}}{\partial y} + \frac{\partial \tau_{yz}}{\partial z}\right) + \rho f_y \quad (4)$$

Z-Momentum Equation

$$\rho \frac{Dw}{Dt} = -\frac{\partial p}{\partial z} + \left(\frac{\partial \tau_{zx}}{\partial x} + \frac{\partial \tau_{zy}}{\partial y} + \frac{\partial \tau_{zz}}{\partial z}\right) + \rho f_z \quad (5)$$

- Equations (3) - (5) are a differential formulation of Newton's 2nd law of motion.

  L.H.S : Mass acceleration per unit volume

  R.H.S : Sum of net forces per unit volume acting on fluid $\Longrightarrow$ Surface & Body Forces

  $\rightarrow$ Surface Forces : Molecular stresses in the fluid (3 terms)

- "Pressure" <u>normal</u> to surface
- "Viscous stresses" <u>normal</u> & <u>tangential</u> to surface

→ <u>Body Forces</u> :

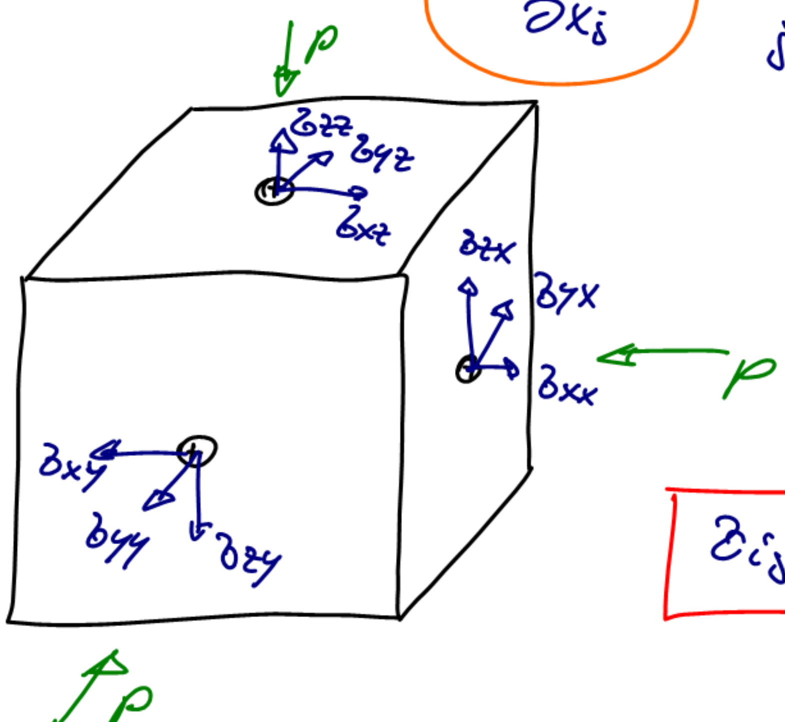
- Gravity forces, Electrical forces, etc.
  (f terms)

→ <u>Pressure Gradient</u> : (p terms)

- $\ominus$ sign, because pressure drop acts to <u>increase</u> the momentum

- The <u>viscous terms</u> in the momentum equation are often written in tensor notation as: $\dfrac{\partial \tau_{ij}}{\partial x_j}$

$i \triangleq$ Direction of stress
$j \triangleq$ 'Normal' direction of surface element



$$\boxed{\tau_{ij} = \tau_{ji}}$$

- For a constant density "Newtonian" viscous fluid, the normal viscous stress τ_{ii} and shear (tangential) stresses τ_{ij} are obtained from the viscous stress tensor given by

$$\tau_{ij} = \mu \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right) \qquad (6)$$

- Occasionally, (6) is written as:

$$\tau_{ij} = 2\mu S_{ij}$$
$$S_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right) \qquad (7)$$

S_{ij} is called the rate of strain tensor.

- In terms of (6), the Navier-Stokes equations can be simplified. For example, the x-momentum equation (3) becomes

$$\frac{Du}{Dt} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu \nabla^2 u + f_x \qquad (8)$$

Here, $\nu = \frac{\mu}{\rho}$ is the 'kinematic' viscosity and

$$\nabla^2 \equiv \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \qquad (9)$$

is the Laplacian operator. Similarly, the y- and z-momentum equations in (4) & (5) can be simplified. The resulting set of equations can be written in vector form as:

$$\boxed{\frac{D\vec{V}}{Dt} = -\frac{1}{\rho}\nabla p + \nu \nabla^2 \vec{V} + \vec{f}} \qquad (10)$$

- Equation (10) in conjunction with (1) represents a system of 4 equations with 4 unknowns u, v, w, p.

Reynolds-Averaged Navier-Stokes Equations

- The conservation equations for turbulent flow are obtained by replacing the instantaneous quantities by the sum of their mean & fluctuating parts.

$$u = \bar{u} + u' \; ; \; v = \bar{v} + v' \; ; \; w = \bar{w} + w' \; ; \; p = \bar{p} + p' \qquad (11)$$

For example, $\bar{u}$ is the ensemble average of u defined by

$$\bar{u} = \lim_{N \to \infty} \frac{1}{N} \sum_{i=1}^{N} u_i \qquad (12)$$

Note that the ensemble average of the corresponding fluctuating component u' is <u>zero</u>.

$$\overline{u'} = \lim_{N \to \infty} \frac{1}{N} \sum_{i=1}^{N} u' = 0 \qquad (13)$$

- Substituting (11) into (1) & (10) and using (13), one can find the Reynolds-Averaged Navier-Stokes Equations (RANS)

$$\frac{\partial \bar{u}}{\partial x} + \frac{\partial \bar{v}}{\partial y} + \frac{\partial \bar{w}}{\partial z} = 0 \qquad (14)$$

$$\rho \frac{D\bar{u}}{Dt} = -\frac{\partial \bar{p}}{\partial x} + \mu \nabla^2 \bar{u} + \rho f_x \qquad (15)$$

$$-\rho \frac{\partial}{\partial x}(\overline{u'^2}) - \rho \frac{\partial}{\partial y}(\overline{u'v'}) - \rho \frac{\partial}{\partial z}(\overline{u'w'})$$

$$\rho \frac{D\bar{v}}{Dt} = -\frac{\partial \bar{p}}{\partial y} + \mu \nabla^2 \bar{v} + \rho f_y \qquad (16)$$

$$-\rho \frac{\partial}{\partial x}(\overline{v'u'}) - \rho \frac{\partial}{\partial y}(\overline{v'^2}) - \rho \frac{\partial}{\partial z}(\overline{v'w'})$$

$$\rho \frac{D\bar{w}}{Dt} = -\frac{\partial \bar{p}}{\partial z} + \nu \nabla^2 \bar{w} + \rho f_z \qquad (17)$$

$$- \rho \frac{\partial}{\partial x}\left(\overline{w'u'}\right) - \rho \frac{\partial}{\partial y}\left(\overline{w'v'}\right) - \rho \frac{\partial}{\partial z}\left(\overline{w'^2}\right)$$

- It is common to drop the overbars in (14)–(17). This results in a description that looks quite similar to the 'laminar' equations (1) & (10) with the addition of the Reynolds normal and shear stress terms, i.e. the Reynolds stresses.
- The Reynolds stresses represent turbulent contributions to a e.g. $\tau_{xx}, \tau_{xy}, \tau_{xz}$ term.
- This results in the extension of a laminar stress tensor to a turbulent stress tensor

$$\boxed{\tau_{ij} = \mu\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) - \rho\, \overline{u_i' u_j'}} \qquad (18)$$

or $$\tau_{ij} = \tau_{ij}^{l} + \tau_{ij}^{t} \qquad (19)$$

where τ_{ij}^{t} denotes the Reynolds stresses. In three-dimensional flow, they become ...

$$\tau_{xx}^t = -\rho\,\overline{u'^2} \;;\; \tau_{yy}^t = -\rho\,\overline{v'^2} \;;\; \tau_{zz}^t = -\rho\,\overline{w'^2}$$

$$\tau_{xy}^t = \tau_{yx}^t = -\rho\,\overline{u'v'}$$

$$\tau_{xz}^t = \tau_{zx}^t = -\rho\,\overline{u'w'} \qquad (20)$$

$$\tau_{yz}^t = \tau_{zy}^t = -\rho\,\overline{v'w'}$$

- The Reynolds stresses introduce additional unknowns in the momentum equations. Therefore, additional assumptions are necessary regarding the relationship between the unknowns and the mean (averaged) flow variables.

- This is referred to as the "closure" problem in turbulent flows. – Turbulence models add additional transport equations for new quantities to the problem.

  Examples: $k \hat{=}$ Turbulent kinetic energy

  $\varepsilon \hat{=}$ Eddy Dissipation Rate

  $\omega \hat{=}$ Eddy frequency

- The transport equations contain models for the Reynolds stresses in (20).

Reduced Forms of the Navier-Stokes Equations

- The conservation equations can be reduced to simpler forms for specific applications.

  $\Longrightarrow$ Order-of-Magnitude Analysis

- It is common to introduce length scales ($L \sim$ chord; $\delta \sim$ Boundary Layer thickness) parallel & normal to the wall, to estimate a typical 'external' velocity to be of order u_e as well as magnitudes of inertia, pressure, viscous and body force terms in the Navier-Stokes equations.

$\Longrightarrow$ Thin-Layer Navier-Stokes Equations

- Retaining only the viscous terms derivatives in the direction normal to the body surface (y) or normal to a free thin shear layer.

Continuity:
$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0 \qquad (1)$$

x-Momentum:
$$\rho \frac{Du}{Dt} = -\frac{\partial p}{\partial x} + \mu \frac{\partial^2 u}{\partial y^2} - \rho \frac{\partial}{\partial y}\overline{u'v'} + \rho f_x \qquad (21)$$

y-Momentum:
$$\rho \frac{Dv}{Dt} = -\frac{\partial p}{\partial y} + \mu \frac{\partial^2 v}{\partial y^2} - \rho \frac{\partial}{\partial y}\overline{v'^2} + \rho f_y \qquad (22)$$

z-momentum: $\rho \frac{Dw}{Dt} = -\frac{\partial p}{\partial z} + \mu \frac{\partial^2 w}{\partial y^2} - \rho \frac{\partial}{\partial y} \overline{v'w'} + \rho f_z$ (23)

⟹ Inviscid Flow Equations (Euler Equations)

- If all viscous forces (τ-stresses) are negligible, this corresponds to inviscid flow.

Hence, (10) becomes

$$\boxed{\frac{D\vec{V}}{Dt} = -\frac{1}{\rho}\nabla p + \vec{f}} \qquad (24)$$

- Equation (24) is also known as the "Euler Equations". For steady flow w/o body forces, (24) simplifies to

$$(\vec{V}\cdot\nabla)\vec{V} = -\frac{\nabla p}{\rho} \qquad (25)$$

- Taking a dot product of (25) w/ a differential element of a streamline $d\vec{s}$, we can integrate along the streamline (ρ=const., incompressible) to find

$$p + \frac{1}{2}\rho V^2 = \text{const.} \qquad (26)$$

where $V^2 = u^2 + v^2 + w^2$. Equation (26) is the well known Bernoulli Equation.

- Additional simplifications arise for irrotational flow, i.e. flow w/ zero vorticity.

$$\vec{\omega} = \nabla \times \vec{V} = \vec{0} \qquad (27)$$

- For two-dimensional, incompressible, inviscid & irrotational flow, we have

$$\begin{aligned} &\text{Continuity:} && \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \\ &\text{Zero Vorticity:} && \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} = 0 \end{aligned} \qquad (28)$$

Equation set (28) is often referred to as the Cauchy-Riemann equations.

In such case, both a potential function $\phi(x,y)$ and stream function $\psi(x,y)$ exist with

$$\begin{aligned} u &= \frac{\partial \phi}{\partial x} = -\frac{\partial \psi}{\partial y} \\ v &= \frac{\partial \phi}{\partial y} = \frac{\partial \psi}{\partial x} \end{aligned} \qquad (29)$$

Substituting the relations for ϕ into 'continuity' and ψ into 'irrotationality' we obtain

$$\left.\begin{aligned} \frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} &= 0 \\ \frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2} &= 0 \end{aligned}\right\} \quad (30)$$

which is the well-known "Laplace Equation".

Equation (30) is "elliptic" and often used to solve the "outer" inviscid flow around airfoils & wings. For some problems, viscous effects can be introduced into the solution of (30).

"Interactive Boundary Layer Theory"

⟹ Boundary-Layer Equations

- Another simplification to the Navier-Stokes equations occurs when the ratio δ/L of the boundary-layer thickness δ to a reference length L (e.g. chord) is sufficiently small such that terms that are by a factor δ/L smaller than the leading terms can be neglected.

- For two-dimensional steady flows, it is assumed that

$$u \sim u_e \ , \ p \sim \rho u_e^2 \ , \ x \sim L \ , \ y \sim \delta \quad (31)$$

$$\overline{u'^2} \sim \overline{v'^2} \sim \overline{u'v'} \quad (32)$$

- Equations (31) & (32) lead to

$$\text{i)} \quad v \sim \frac{u_e \delta}{L}$$

$$\text{ii)} \quad \frac{\partial p}{\partial y} = \mathcal{O}(\delta) \longrightarrow \underline{\underline{0}} \ !$$

$\Longrightarrow$ Boundary Layer Approximations

- Thus, the Navier-Stokes equations reduce to

$$\begin{cases} \dfrac{\partial u}{\partial x} + \dfrac{\partial v}{\partial y} = 0 & (28a) \\ u\dfrac{\partial u}{\partial x} + v\dfrac{\partial u}{\partial y} = -\dfrac{1}{\rho}\dfrac{dp}{dx} + \nu\dfrac{\partial^2 u}{\partial y^2} - \dfrac{\partial}{\partial y}\left(\overline{u'v'}\right) & (33) \\ \dfrac{\partial p}{\partial y} = 0 & (34) \end{cases}$$

- The Reynolds stress term $-\frac{\partial}{\partial y}(\overline{u'v'})$ in (33) is closed w/ an appropriate turbulence model. It becomes, of course, <u>zero</u> in laminar flow.
- One might think that (28a) & (33) constitute 2 equations for 3 unknowns u, v, p.

However, p is <u>not</u> unknown, as the pressure is constant throughout the boundary layer ($\partial p/\partial y = 0$). Hence it is equal to the pressure in the outer inviscid flow where Bernoulli's equation applies. Thus,

$$\frac{dp}{dx} = -\rho u_e \frac{du_e}{dx} \qquad (35)$$

Now (33) becomes

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = u_e\frac{du_e}{dx} + \nu\frac{\partial^2 u}{\partial y^2} - \frac{\partial}{\partial y}(\overline{u'v'}) \qquad (36)$$

- The Boundary Layer Equations are "parabolic" PDEs and much easier and less costly to solve than the RANS equations, which are elliptic.

⇒ Some additional Equations of Fluid Flow

- Linear Convection/Diffusion Equation
- Burger's Equation
- Wave Equation
- Transonic Small Disturbance Equation (TSD)