Integration of Convection/Diffusion Equations

- So far ...

  → 1^{st} order convection equations (Hyperbolic)
  Example: "Linear Convection Equation"

  → Diffusion equations (Parabolic)
  Example: "Heat Equation"

- In the presence of both convection & diffusion phenomena, a combined equation is in general parabolic. Example: "Boundary Layer Equations"

- A simpler model equation is the viscous Burger's equation

$$\boxed{\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = \nu\frac{\partial^2 u}{\partial x^2}} \qquad (1)$$

- Equation (1) was derived to study the internal structure of a weak normal shock in the near-sonic regime.

- The velocity $u(x,t)$ represents the perturbation from a uniform sonic stream ; $u=0$ corresponds to the sonic condition.
- As the equation is parabolic, initial- and boundary conditions must be added to the problem.
- A steady-state, i.e. $\partial u/\partial t \mapsto 0$, is reached when convection and diffusion are in balance.
- For the following boundary conditions

$$\boxed{\begin{cases} u(0,t) = u_0 \\ u(L,t) = 0 \end{cases}} \qquad (2)$$

an exact steady solution to (1) is given by

$$\boxed{u(x) = u_0 \, \bar{u} \, \frac{1 - e^{-\bar{u}\, Re_L \left(1 - \frac{x}{L}\right)}}{1 + e^{-\bar{u}\, Re_L \left(1 - \frac{x}{L}\right)}}} \qquad (3)$$

where $\bar{u}$ is defined through

$$\boxed{\frac{\bar{u}-1}{\bar{u}+1} = e^{-\bar{u}\, Re_L}} \qquad (4)$$

and

$$\boxed{Re_L = \frac{u_0 L}{\nu}} \qquad (5)$$

is sort of a Reynolds number defined by the shock width and describes the ratio of convection to diffusion.

- Two limiting cases:

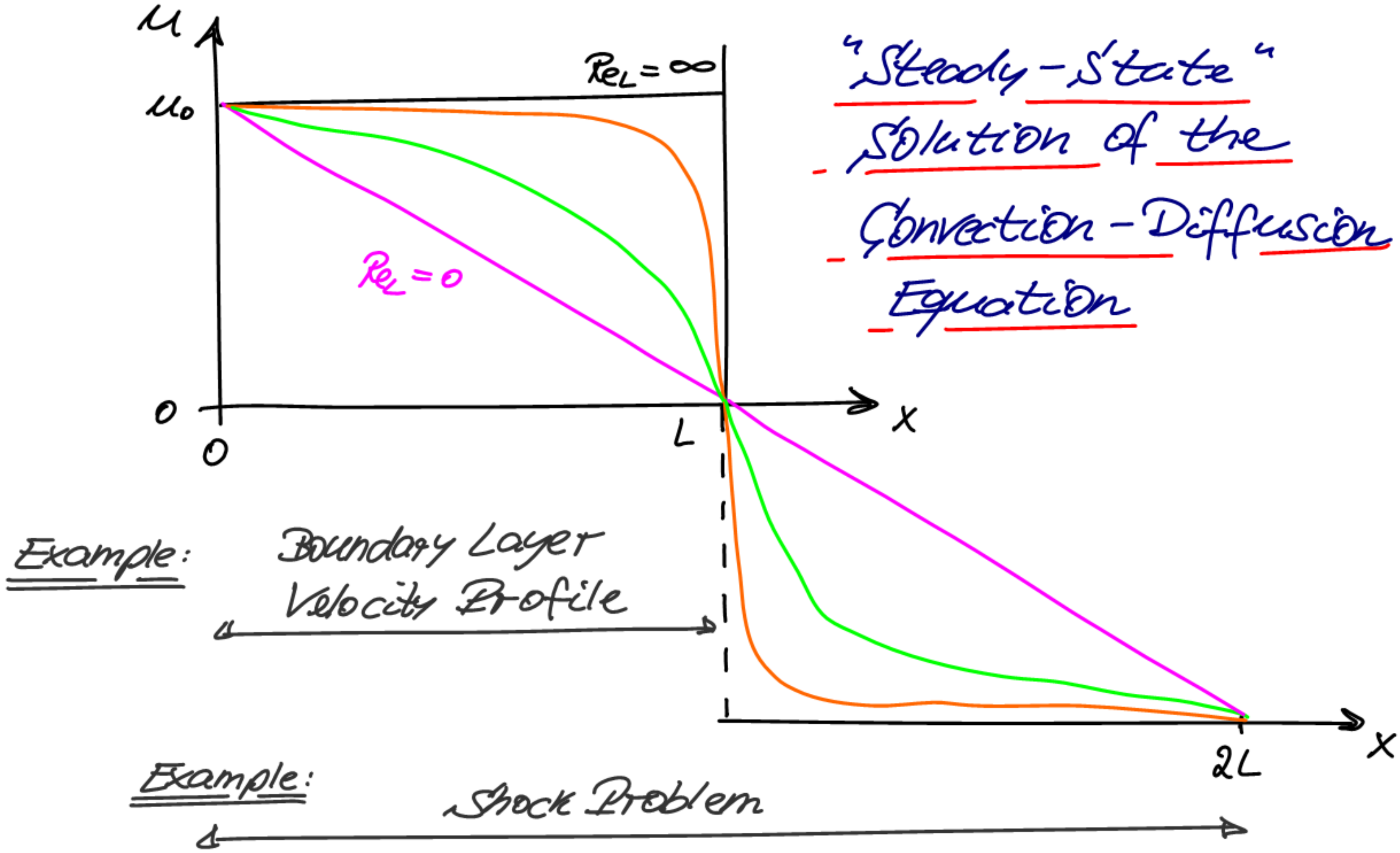
a.) $Re_L \to 0$

$$\bar{u} \approx \sqrt{\frac{2}{Re_L}} \quad ; \; u(x) = u_0\left(1-\frac{x}{L}\right)$$

b.) $Re_L \to \infty$

$$\bar{u} \to 1 \quad \Longrightarrow \text{singular inviscid limit} \qquad \left.\begin{array}{l} u(x) = u_0 \\ u(L) = 0 \end{array}\right\}$$

"Steady-State" Solution of the Convection-Diffusion Equation



- One can also device a simpler model problem, i.e. the linearized Burger's equation

$$\boxed{\frac{\partial u}{\partial t} + c\,\frac{\partial u}{\partial x} = \nu\,\frac{\partial^2 u}{\partial x^2}} \qquad (6)$$

It can be shown that in this case $\bar{u} = 1$ and that the exact steady-state solution is given by

$$\boxed{u(x) = u_0\,\frac{1 - e^{-Re_L\left(1-\frac{x}{L}\right)}}{1 - e^{-Re_L}}} \quad (7) \qquad \boxed{Re_L = \frac{cL}{\nu}} \quad (8)$$

FTCS Method

[Roach, 1972]

FTCS ≙ Forward-in-Time and Centered-in-Space

$$\boxed{\frac{u_i^{n+1} - u_i^n}{\Delta t} + c\,\frac{u_{i+1}^n - u_{i-1}^n}{2\Delta x} = \nu\,\frac{u_{i+1}^n - 2u_i^n + u_{i-1}^n}{\Delta x^2}} \qquad (9)$$

- The analysis of the truncation error TE is straight forward about (i,n)

$$\varepsilon_i^n = \cancel{\frac{\partial u_i^n}{\partial t}} + \frac{\Delta t}{2}\frac{\partial^2 u_i^n}{\partial t^2} + \mathcal{O}(\Delta t^2)$$

$$+ c\left(\cancel{\frac{\partial u_i^n}{\partial x}} + \frac{\Delta x^2}{3!}\frac{\partial^3 u_i^n}{\partial x^3} + \mathcal{O}(\Delta x^4)\right)$$

$$- \nu\left(\cancel{\frac{\partial^2 u_i^n}{\partial x^2}} + \frac{\Delta x^2}{4!}\frac{\partial^4 u_i^n}{\partial x^4} + \mathcal{O}(\Delta x^4)\right) \quad = \mathcal{O}(\Delta t, \Delta x^2)$$

$\Longrightarrow$ The FTCS scheme is consistent and 1st order accurate in time and second order accurate in space.

- **Stability:**

$$\boxed{u_i^n = g^n e^{\hat{i} i\beta}} \qquad (10)$$

Bring (9) in update form ...

$$u_i^{n+1} = u_i^n - \frac{1}{2}\underbrace{\frac{c\Delta t}{\Delta x}}_{\mathcal{C}}\left(u_{i+1}^n - u_{i-1}^n\right) + \underbrace{\frac{\nu\Delta t}{\Delta x^2}}_{\tau}\left(u_{i+1}^n - 2u_i^n + u_{i-1}^n\right)$$

$$g^{n+1} e^{\hat{i} i\beta} = g^n e^{\hat{i} i\beta} - \frac{1}{2}\mathcal{C}\left(g^n e^{\hat{i}(i+1)\beta} - g^n e^{\hat{i}(i-1)\beta}\right)$$

$$+ \tau\left(g^n e^{\hat{i}(i+1)\beta} - 2 g^n e^{\hat{i} i\beta} + g^n e^{\hat{i}(i-1)\beta}\right)$$

Divide by $u_i^n = g^n e^{\hat{i} i\beta}$

$$g = 1 - \frac{1}{2}\mathcal{C}\left(e^{\hat{i}\beta} - e^{\hat{i}(-\beta)}\right) + \tau\left(e^{\hat{i}\beta} - 2 + e^{\hat{i}(-\beta)}\right)$$

$$g = 1 - \frac{1}{2}\mathcal{C}\left(\cancel{\cos\beta} + \hat{i}\sin\beta - \cancel{\cos(-\beta)} - \hat{i}\sin(-\beta)\right)$$

$$+ \tau\left(\cos\beta + \cancel{\hat{i}\sin\beta} - 2 + \cos(-\beta) + \cancel{\hat{i}\sin(-\beta)}\right)$$

$$g = 1 - \hat{i}\,\mathcal{C}\sin\beta - 2\tau(1 - \cos\beta)$$

$$g = [1 - 2r(1-\cos\beta)] - i\, \mathfrak{c}\, \sin\beta$$

$$\|g\|^2 = [1-2r(1-\cos\beta)]^2 + \mathfrak{c}^2 \sin^2\beta$$

$$\|g\|^2 = 1 - 4r(1-\cos\beta) + 4r^2(1-\cos\beta)^2 + \mathfrak{c}^2(1-\cos^2\beta)$$

$$\|g\|^2 = 1 - 4r(1-\cos\beta) + 4r^2(1-\cos\beta)^2 + \mathfrak{c}^2(1-\cos\beta)(1+\cos\beta)$$

$$\|g\|^2 = 1 + [4r^2(1-\cos\beta) - 4r + \mathfrak{c}^2(1+\cos\beta)](1-\cos\beta)$$

$$\boxed{\|g\|^2 = 1 + [8r^2 - 4r + (\mathfrak{c}^2 - 4r^2)(1+\cos\beta)](1-\cos\beta)} \quad (11)$$

- Remember that the scheme will be stable if

$$\|g\|^2 \leq 1 \quad \forall \alpha$$

i) $\underline{\mathfrak{c}^2 - 4r^2 \leq 0}$

Worst case $\Longrightarrow$ $\cos\beta = -1$

$$8r^2 - 4r = 4r(2r-1) \leq 0$$

$$\Longrightarrow \boxed{2r \leq 1} \Longrightarrow \boxed{\mathfrak{c}^2 \leq 1} \quad (12)$$

ii) $\sigma^2 - 4\tau^2 \geq 0$

Worst case $\Rightarrow$ $\cos\beta = 1$

$$\cancel{8\tau^2} - 4\tau + 2(\sigma^2 - \cancel{4\tau^2}) \leq 0$$

$$-4\tau + 2\sigma^2 \leq 0$$

$$\boxed{\sigma^2 \leq 2\tau} \qquad (13)$$

- Combinations of the conditions (12), (13) yields the stability condition for the FTCS scheme as

$$\boxed{\sigma^2 \leq 2\tau \leq 1} \qquad (14)$$

- In the special case of the inviscid Burger's equation $\frac{\partial u}{\partial t} + c\frac{\partial u}{\partial x} = 0$, the FTCS scheme is unstable.

- With reference to "Integration of Linear Hyperbolic Equations" a stable scheme for the inviscid Burger's equation reads:

$$\boxed{\frac{u_i^{n+1} - u_i^n}{\Delta t} + c\,\underbrace{\frac{u_i^n - u_{i-1}^n}{\Delta x}}_{\text{Backward Differencing}} = 0} \quad (15)$$

"Inviscid Burger's Equation"

What's the relation between (15) and (9) ?

→ Consider the convective term

$$c\,\frac{u_i^n - u_{i-1}^n}{\Delta x} = c\,\frac{u_i^n - u_{i-1}^n}{\Delta x} + c\,\frac{u_{i+1}^n - u_{i-1}^n}{2\Delta x} - c\,\frac{u_{i+1}^n - u_{i-1}^n}{2\Delta x}$$

$$= c\,\frac{u_{i+1}^n - u_{i-1}^n}{2\Delta x} - \frac{1}{2}c\,\frac{u_{i+1}^n - 2u_i^n + u_{i-1}^n}{\Delta x}$$

$$= c\,\frac{u_{i+1}^n - u_{i-1}^n}{2\Delta x} - \frac{1}{2}(c\cdot\Delta x)\,\frac{u_{i+1}^n - 2u_i^n + u_{i-1}^n}{\Delta x^2} \quad (16)$$

→ (16) in (15) ...

Artificial Viscosity

$$\boxed{\frac{u_i^{n+1} - u_i^n}{\Delta t} + c\,\frac{u_{i+1}^n - u_{i-1}^n}{2\Delta x} = \underbrace{\frac{c\cdot\Delta x}{2}\,\frac{u_{i+1}^n - 2u_i^n + u_{i-1}^n}{\Delta x^2}}_{\text{Dissipation}}} \quad (17)$$

→ 2nd Derivative

- What makes equation (15) stable is the equivalent "Artificial Viscosity" in (17). A comparison to (9) suggests that

$$\boxed{\nu_{AV} = \frac{c \cdot \Delta x}{2}} \qquad (18)$$

- We have shown earlier in "Integration of Linear Hyperbolic Equations" that equation (15) is stable for

$$\sigma = \frac{c\,\Delta t}{\Delta x} \leq 1$$

- Linking equation (17) to the FTCS scheme in (9) and the stability criterion in (14) we find that ...

$$\tau = \frac{\nu_{AV} \cdot \Delta t}{\Delta x^2} = \frac{c \cdot \cancel{\Delta x}}{2} \frac{\Delta t}{\Delta x^{\cancel{2}}} = \frac{1}{2}\sigma$$

$$(14) \longrightarrow \quad \sigma^2 \leq 2\tau \leq 1$$

$$\sigma^2 \leq \sigma \leq 1$$ True!

- By looking at (16) we can argue that ...

$$\boxed{\text{Backward Differencing} = \text{Central Differencing} + \text{Artificial Dissipation}} \quad (16a)$$

... and this "shifting operation" gains stability.

- So it appears the following:

> Artificial viscosity is required for the numerical stability of Finite Difference approximations of "convective terms".

- But what are the problems that come with it?

   i) Loss of accuracy in truncation error.

   ii) Oscillations in solutions of steep gradients such as shocks.

   iii) Artificial dissipation of physical flow structures, e.g. vortices.

- How does "Artificial Viscosity" cause oscillations ?

From (3) and (4) in "Integration of Linear Hyperbolic Equations" we write the truncation error for (15) as

$$\boxed{\varepsilon_i^n = \underbrace{\nu_{AV}(\sigma - 1)\frac{\partial^2 u}{\partial x^2}}_{\mathcal{O}(\Delta x)} + \mathcal{O}(\Delta t^2, \Delta x^2)} \qquad (17)$$

The 2nd order term (as the leading term of the TE) allows oscillations. The stability condition (14) gives us for the FTCS scheme

$$\sigma^2 \leq 2\tau \leq 1$$

$$\sigma^2 \leq 2\tau \quad \Rightarrow \quad \frac{c^2 \Delta t^{\cancel{2}}}{\cancel{\Delta x^2}} \leq 2\,\frac{\nu_{AV}\cancel{\Delta t}}{\cancel{\Delta x^2}}$$

$$\boxed{\Delta t \leq \frac{2\nu_{AV}}{c^2} = \Delta t_1} \qquad (18)$$

$$2\tau \leq 1 \quad \Rightarrow \quad 2\,\frac{\nu_{AV}\Delta t}{\Delta x^2} \leq 1 \quad \Rightarrow \quad \boxed{\Delta t \leq \frac{\Delta x^2}{2\nu_{AV}} = \Delta t_2}$$

Keep in mind that in general the scheme (15) has $\nu_{AV} = \frac{c \cdot \Delta x}{2}$

$$\Rightarrow \Delta t_1 = \frac{\cancel{2} \cdot \cancel{c} \cdot \frac{\Delta x}{\cancel{2}}}{c^{\cancel{2}}} = \frac{\Delta x}{c}$$

$$\Delta t_2 = \frac{\Delta x^{\cancel{2}}}{\cancel{2}\, \frac{c \cdot \cancel{\Delta x}}{\cancel{2}}} = \frac{\Delta x}{c}$$

"Shock Speed"

One can show that oscillations in the solution can be suppressed when additional artificial viscosity ν_{add} is added to (17). This can become a trial-and-error task, especially when the shock speed c becomes u in a non-linear case.