Partial Differential Equations

- **The mystery of PDEs:**

→ Flow equations solved in one region of a flow field can exhibit very different solution behavior in another flow region.

Example: Subsonic, transonic, supersonic flow around an airfoil

→ Be aware that a given PDE may have quite different mathematical behavior.

→ How can you know?

- By "classifying" the governing PDE and investigating whether or not it may change type/behavior for a given physical problem.

1.) Classification of Quasi-Linear Partial Differential Equations

- Consider a system of quasi-linear PDEs:

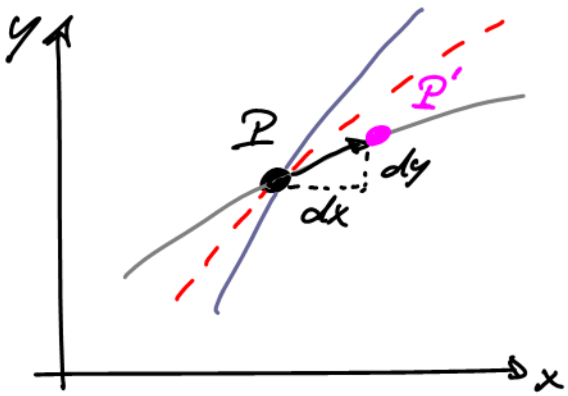
$$\boxed{\begin{aligned} a_1 \frac{\partial u}{\partial x} + b_1 \frac{\partial u}{\partial y} + c_1 \frac{\partial v}{\partial x} + d_1 \frac{\partial v}{\partial y} &= f_1 \\ a_2 \frac{\partial u}{\partial x} + b_2 \frac{\partial u}{\partial y} + c_2 \frac{\partial v}{\partial x} + d_2 \frac{\partial v}{\partial y} &= f_2 \end{aligned}} \qquad (1)$$

where f_1 & f_2 can be functions of x, y, u, and v.

- Equation (1) actually constitutes 2 equations for the 2 unknown functions $u(x,y)$ and $v(x,y)$.
- At any given point in the x-y plane, there exist unique values of u and v. Furthermore, the derivatives $\frac{\partial u}{\partial x}$, $\frac{\partial u}{\partial y}$, $\frac{\partial v}{\partial x}$, and $\frac{\partial v}{\partial y}$ have finite values at this point.
- What is so mysterious about that?

We can find directions through any given point P such that the derivatives of u and v

are indeterminant along these directions and may even allow discontinuities.



"Characteristic Line"

"Lines w/ determinant derivatives"

- The total differentials of the continuous functions u and v are:

$$\boxed{\begin{aligned} du &= \frac{\partial u}{\partial x}\,dx + \frac{\partial u}{\partial y}\,dy \\ dv &= \frac{\partial v}{\partial x}\,dx + \frac{\partial v}{\partial y}\,dy \end{aligned}} \qquad (2)$$

- Equations (1) & (2) describe 4 linear equations for the 4 unknowns

$$\frac{\partial u}{\partial x},\ \frac{\partial u}{\partial y},\ \frac{\partial v}{\partial x},\ \frac{\partial v}{\partial y}\ .$$

- Let us write (1) & (2) in matrix form

$$\underbrace{\begin{bmatrix} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ dx & dy & 0 & 0 \\ 0 & 0 & dx & dy \end{bmatrix}}_{[A]} \begin{bmatrix} \partial u/\partial x \\ \partial u/\partial y \\ \partial v/\partial x \\ \partial v/\partial y \end{bmatrix} = \begin{bmatrix} f_1 \\ f_2 \\ du \\ dv \end{bmatrix} \qquad (3)$$

where [A] is the coefficient matrix. One can solve for the unknowns using e.g. Cramer's rule.

- Example:

$$\boxed{\frac{\partial u}{\partial x} = \frac{|B|}{|A|}} \qquad (4)$$

where the matrix [B] is obtained by replacing the first column of [A] with the R.H.S. of (3). ⇒ The remaining unknowns can be found in a similar way.

- What determines the actual value of $\partial u/\partial x$?

⇒ The values for dx, dy, du, dv in the matrices [A] and [B].

$\Rightarrow$ dx & dy define the direction from P to P'. The associated velocities at those locations become $u(x_P, y_P)$, $v(x_P, y_P)$ and $u(x_{P'}, y_{P'})$, $v(x_{P'}, y_{P'})$, which determine du & dv.

- In the limit when $dx, dy, du, dv \to 0$ the same value $\partial u / \partial x$ is obtained at the given point P. However, there may be certain directions dx, dy such that the combination of dx, dy, du, dv results in $|A|$ becoming zero in (4)!

- In such a case, $\partial u / \partial x$ governed by (4) becomes indeterminant.

- The pathline along this direction through P is called a "Characteristic Line".

- The relation $|A| = 0$ is also called

the "Characteristic Equation".

$$\boxed{|A| = 0} \qquad (5)$$

- Equation (5) leads to identifiable curves $y(x)$ in the x-y plane. In particular, one can find the slopes dy/dx of the characteristic lines at a point P.

- How many "Characteristic Lines" exist @ P ?
  - → Determined by the number of solutions that (5) produces for dy/dx.
  - → This will classify the PDE.

- Let's do it ...

$$|A| = \begin{vmatrix} \overset{+}{a_1} & b_1 & c_1 & d_1 \\ \overset{-}{a_2} & b_2 & c_2 & d_2 \\ \overset{+}{dx} & \overset{-}{dy} & 0 & 0 \\ 0 & 0 & dx & dy \end{vmatrix} = 0$$

$$|A| = dx \begin{vmatrix} b_1 & c_1 & d_1 \\ b_2 & c_2 & d_2 \\ 0 & dx & dy \end{vmatrix} - dy \begin{vmatrix} a_1 & c_1 & d_1 \\ a_2 & c_2 & d_2 \\ 0 & dx & dy \end{vmatrix}$$

$$= dx\left[-dx(b_1 d_2 - d_1 b_2) + dy(b_1 c_2 - c_1 b_2)\right]$$

$$-dy\left[-dx(a_1 d_2 - d_1 a_2) + dy(a_1 c_2 - c_1 a_2)\right]$$

$$= -(dy)^2(a_1 c_2 - a_2 c_1) - (dx)^2(b_1 d_2 - b_2 d_1)$$

$$+ dx\,dy(b_1 c_2 - b_2 c_1 + a_1 d_2 - a_2 d_1)$$

$$\overset{!}{=} 0$$

$$\boxed{(a_1 c_2 - a_2 c_1)(dy)^2 - (a_1 d_2 - a_2 d_1 + b_1 c_2 - b_2 c_1)\,dx\,dy + (b_1 d_2 - b_2 d_1)(dx)^2 = 0} \quad (6)$$

- Divide (6) by $(dx)^2$ ...

$$\boxed{(a_1 c_1 - a_2 c_1)(dy)^2 - (a_1 d_2 - a_2 d_1 + b_1 c_2 - b_2 c_1)\frac{dy}{dx} + (b_1 d_2 - b_2 d_1) = 0}$$

(7)

- Equation (7) is a quadratic equation in dy/dx. Let us simplify (7) to

$$\boxed{\begin{aligned} & a\left(\frac{dy}{dx}\right)^2 + b\frac{dy}{dx} + c = 0 \\ & a = (a_1 c_2 - a_2 c_1) \\ & b = -(a_1 d_2 - a_2 d_1 + b_1 c_2 - b_2 c_1) \\ & c = (b_1 d_2 - b_2 d_1) \end{aligned}} \tag{8}$$

- Equation (8) is a nonlinear ODE for the characteristic curve $y = y(x)$. At first, we are only interested in the slope dy/dx of the characteristics through the point P.
- The solution of the quadratic equation (8) has the following solutions for dy/dx:

$$\boxed{\frac{dy}{dx} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}} \tag{9}$$

- The characteristic lines through a given point P can be of different nature depending on the discriminant D that defines the slope dy/dx.

$$\boxed{D = b^2 - 4ac} \qquad (10)$$

- The mathematical classification of PDEs is determined by the value of D in (10).

$D > 0$	Two real distinct characteristics exist through each point in the x-y plane. $\longrightarrow$ Hyperbolic
$D = 0$	One real characteristic $\longrightarrow$ Parabolic
$D < 0$	The characteristic lines are imaginary $\longrightarrow$ Elliptic

(10a)

- Note that PDEs of hyperbolic, parabolic, and elliptic type have very different behavior.

- Why are they called ...
"hyperbolic", "parabolic", and "elliptic"?

→ The general equation for conic sections is given as:

$$\boxed{a \cdot x^2 + b \cdot xy + c \cdot y^2 + d \cdot x + e \cdot y + f = 0} \quad (11)$$

→ Note the similarity between (11) and (8). Indeed the conic section is a

• Hyperbola	$b^2 - 4ac > 0$
• Parabola	$b^2 - 4ac = 0$
• Ellipse	$b^2 - 4ac < 0$

- What makes (4) or $\frac{\partial u}{\partial x} = \frac{|B|}{|A|}$ "indeterminant"?

→ "Characteristic Equation" $\boxed{|A| = 0}$ (5)

→ "Compatibility Equation" $\boxed{|B| = 0}$ (5a)

- What do equations (5) & (5a) give us?

→ "Characteristic Equation"

Relation for "characteristic" lines $y(x)$

→ "Compatibility Equation"

$$B = \begin{bmatrix} f_1 & b_1 & c_1 & d_1 \\ f_2 & b_2 & c_2 & d_2 \\ du & dy & 0 & 0 \\ dv & 0 & dx & dy \end{bmatrix}$$

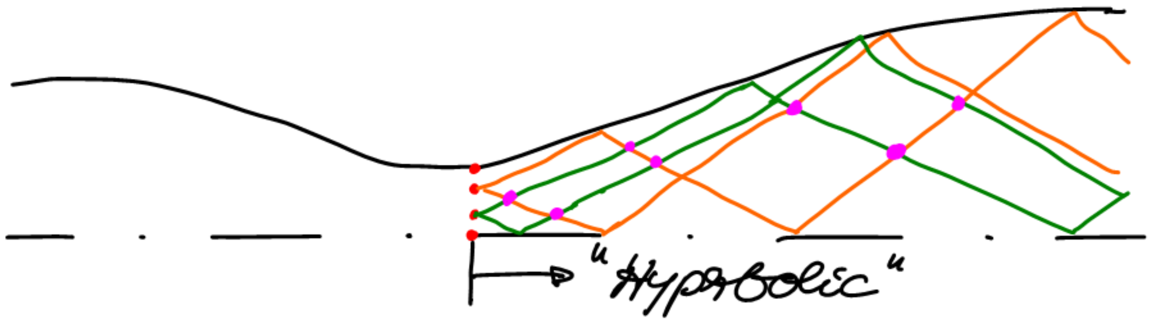
"Ordinary Differential Equation" for u & v along "characteristic lines".

Note: The ODE resulting from (5a) only holds along the respective characteristic line.

⟹ "Solution Method for Hyperbolic PDEs"

$$\boxed{\text{"Method of Characteristics"}}$$

Example: "Supersonic Flow in a Nozzle"



- Initial Conditions
- Points of Intersection between characteristic lines that originated from different •

$\Longrightarrow$ 2 ODEs for 2 unknowns u & v.

- What is the "Characteristic Speed"?

The characteristic speed denotes the slope of a characteristic. The terminology derives from the fact that the slope is a speed if one of the dependent variables is time t.

2.) A General Method of Determining the Classification of Partial Differential Equations: The Eigenvalue Method

- Consider again the system of quasi-linear PDEs in (1).

$$a_1 \frac{\partial u}{\partial x} + b_1 \frac{\partial u}{\partial y} + c_1 \frac{\partial v}{\partial x} + d_1 \frac{\partial v}{\partial y} = f_1$$
$$a_2 \frac{\partial u}{\partial x} + b_2 \frac{\partial u}{\partial y} + c_2 \frac{\partial v}{\partial x} + d_2 \frac{\partial v}{\partial y} = f_2$$

- Defining a column vector

$$W = \begin{Bmatrix} u \\ v \end{Bmatrix} \qquad (12)$$

the system of equations in (1) can be written as:

$$\begin{bmatrix} a_1 & c_1 \\ a_2 & c_2 \end{bmatrix} \frac{\partial W}{\partial x} + \begin{bmatrix} b_1 & d_1 \\ b_2 & d_2 \end{bmatrix} \frac{\partial W}{\partial y} = 0 \qquad (13)$$

or :

$$\boxed{[K]\frac{\partial \bar{W}}{\partial x} + [M]\frac{\partial \bar{W}}{\partial y} = 0} \qquad (13a)$$

- Note that $[K]$ and $[M]$ are 2×2 matrices. If $[K]^{-1}$ is the inverse of $[K]$, then

$$\frac{\partial \bar{W}}{\partial x} + [K]^{-1}[M]\frac{\partial \bar{W}}{\partial y} = 0$$

or:

$$\boxed{\frac{\partial \bar{W}}{\partial x} + [N]\frac{\partial \bar{W}}{\partial y} = 0} \qquad (14)$$

where $[N] = [K]^{-1}[M]$.

- The eigenvalues of $[N]$ determine the classification of (1).

  → If the eigenvalues are all "real", then the system is "hyperbolic".

  → If the eigenvalues are all "complex", then the system is "elliptic".

  → If the eigenvalues are mixed, i.e. "real & complex", then the situation is not as easy. One can investigate D governed by (10) to see if the system is parabolic.

→ Keep in mind though that a PDE can actually be of mixed type.

3) EXAMPLES - PDE CLASSIFICATION

EXAMPLE: "COMPRESSIBLE POTENTIAL FLOW"

$$\boxed{(1-M_\infty^2)\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0} \qquad (15)$$

- Equation (15) is a 2nd order PDE. Let us first transform (15) into a system of 2 1st order PDEs by introducing

$$\boxed{u = \frac{\partial \phi}{\partial x} \quad ; \quad v = \frac{\partial \phi}{\partial y}} \qquad (16)$$

- Thus, equation (15) can be written as:

$$\boxed{\begin{aligned} (1-M_\infty^2)\frac{\partial u'}{\partial x} + \frac{\partial v'}{\partial y} &= 0 \\ \frac{\partial u'}{\partial y} - \frac{\partial v'}{\partial x} &= 0 \end{aligned}} \qquad (17)$$

where we used that

$$u = V_\infty + u' \quad ; \quad v = v'$$

and $$\frac{\partial}{\partial y}\left(\frac{\partial \phi}{\partial x}\right) = \frac{\partial u}{\partial y} = \frac{\partial}{\partial x}\left(\frac{\partial \phi}{\partial y}\right) = \frac{\partial v}{\partial x}$$

- Note that the 2nd relation in (17) is the condition of "irrotationality" (zero vorticity!)

$$\vec{\omega} = \vec{\nabla} \times \vec{V}' = \begin{pmatrix} \partial/\partial x \\ \partial/\partial y \\ \partial/\partial z \end{pmatrix} \times \begin{pmatrix} u' \\ v' \\ w' \end{pmatrix}$$

$$= \begin{pmatrix} \frac{\partial w'}{\partial y} - \frac{\partial v'}{\partial z} \\ \frac{\partial u'}{\partial z} - \frac{\partial w'}{\partial x} \\ \frac{\partial v'}{\partial x} - \frac{\partial u'}{\partial y} \end{pmatrix} \overset{!}{=} \vec{0}$$

<u>METHOD 1:</u>

- Now let us compare (17) to (1) and identify the coefficients:

$$\boxed{\begin{array}{ll} a_1 = 1 - M_\infty^2 & a_2 = 0 \\ b_1 = 0 & b_2 = 0 \\ c_1 = 0 & c_2 = -1 \\ d_1 = 1 & d_2 = 0 \end{array}} \qquad (18)$$

- From (8) we know that

$$a = a_1 c_2 - a_2 c_1 = -(1 - M_\infty^2)$$

$$b = 0$$

$$c = -1$$

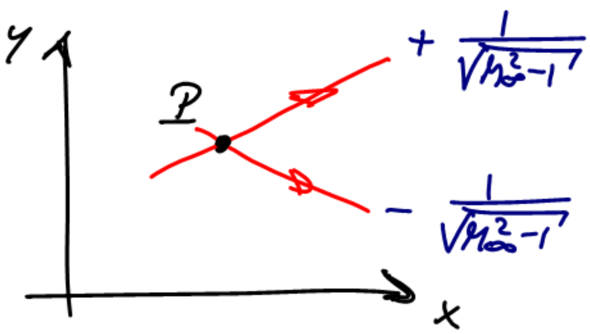
(9) $\Rightarrow$

$$\frac{dy}{dx} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \pm \frac{\sqrt{\cancel{4}(M_\infty^2 - 1)}}{\cancel{2}(M_\infty^2 - 1)}$$

$$\boxed{\frac{dy}{dx} = \pm \frac{1}{\sqrt{M_\infty^2 - 1}}} \qquad (19)$$

- $M_\infty > 1 \Rightarrow$ Two real characteristics through each point.

"Hyperbolic"



- $M_\infty < 1 \Rightarrow$ The characteristics are imaginary.

"Elliptic"

METHOD 2:

- For the eigenvalue method, we need to write (17) in matrix form as:

$$\boxed{\begin{bmatrix} 1-M_\infty^2 & 0 \\ 0 & -1 \end{bmatrix} \frac{\partial W}{\partial x} + \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \frac{\partial W}{\partial y} = 0} \qquad (20)$$

where we used (12) with $W = \begin{Bmatrix} u \\ v \end{Bmatrix}$.

- Let us recall that (20) can be also written in the form

$$[K] \frac{\partial W}{\partial x} + [M] \frac{\partial W}{\partial y} = 0 \qquad (13a)$$

- Now we can find the inverse matrix of $[K]$ through

$$[K]^{-1} = -\frac{1}{1-M_\infty^2} \begin{bmatrix} -1 & 0 \\ 0 & 1-M_\infty^2 \end{bmatrix}$$

and

$$[N] = [K]^{-1}[M] = \begin{bmatrix} \frac{1}{1-M_\infty^2} & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & \frac{1}{1-M_\infty^2} \\ -1 & 0 \end{bmatrix}$$

- Now let us find the eigenvalues of $[N]$:

$$|[N]-\lambda[I]| = 0 \Longrightarrow \begin{vmatrix} -\lambda & \frac{1}{1-M_\infty^2} \\ -1 & -\lambda \end{vmatrix} = 0$$

$$\lambda^2 + \frac{1}{1-M_\infty^2} = 0$$

$$\boxed{\lambda = \pm \frac{1}{\sqrt{M_\infty^2 - 1}}} \qquad (21)$$

- $M_\infty > 1$ $\Longrightarrow$ All eigenvalues are real.

  "Hyperbolic"

- $M_\infty < 1$ $\Longrightarrow$ All eigenvalues are complex.

  "Elliptic"

- As we inverted the matrix $[K]$, the eigenvalues λ describe the slope of the characteristic $\frac{dy}{dx}$.

- Note that we can obtain the same result by inverting the matrix $[M]$.

$$[M]^{-1} = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$$

- In this case, $[N] = [M]^{-1}[K]$

$$[N] = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} 1-M_\infty^2 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ M_\infty^2-1 & 0 \end{bmatrix}$$

- The eigenvalues of $[N]$ are

$$\left| [N] - \lambda [I] \right| = 0 \implies \begin{vmatrix} -\lambda & 1 \\ M_\infty^2-1 & -\lambda \end{vmatrix} = 0$$

$$\lambda^2 - (M_\infty^2 - 1) = 0$$

$$\boxed{\lambda = \pm\sqrt{M_\infty^2 - 1}} \qquad (21a)$$

- The same conditions apply for the equation type governed by M_∞ as stated above.

- However, as the matrix $[M]$ was inverted, the eigenvalues λ describe the slope of the characteristics $\frac{dx}{dy}$.

Example: "LINEAR CONVECTION EQUATION"

$$\boxed{\frac{\partial u}{\partial t} + c\,\frac{\partial u}{\partial x} = 0} \qquad (22)$$

"A single 1st order PDE is always hyperbolic."

METHOD 1:

- Compare to (1):

$$\boxed{\begin{array}{ll} a_1 = 1 & a_2 = 0 \\ b_1 = c & b_2 = 0 \\ c_1 = 0 & c_2 = 0 \\ d_1 = 0 & d_2 = 0 \end{array}} \qquad (23)$$

- From (8) we find that

$$\boxed{a = 0\ , \quad b = 0\ , \quad c = 0} \qquad (24)$$

- Then it becomes clear from (9) that

$$\frac{dx}{dt} = \frac{0}{0}$$ which is "indeterminant".

- Note though that the discriminant in (10) becomes

$$\boxed{D = b^2 - 4ac = 0} \qquad (25)$$

which could be deceiving in naming the equation parabolic. But as there is only one 1st order equation, the system can have only one real root and is therefore hyperbolic.

METHOD 2 :

- $W = \{u\} \implies [1]\frac{\partial w}{\partial t} + [c]\frac{\partial w}{\partial x} = 0$

- $[K] = [1] \quad ; \quad [M] = [c]$

  $[K]^{-1} = [1] \quad ; \quad [N] = [K]^{-1}[M] = [1][c]$
  $= c$

- Eigenvalues ...

$$|[N] - \lambda[I]| = 0 \implies |c - \lambda| = 0$$

$$\boxed{\lambda = c} = \frac{dx}{dt} \qquad (26)$$

- Here it is clear: All eigenvalues are real, and the system is hyperbolic.

EXAMPLE: "Burger's Equation (Inviscid)"

$$\boxed{\frac{\partial u}{\partial t} + \frac{\partial}{\partial x}\left(\frac{u^2}{2}\right) = 0} \qquad (27)$$

- Write in quasi-linear form ...

$$\boxed{\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = 0} \qquad (27a)$$

- Again, is it hyperbolic?

  Yes, showing the same analysis as for (22) it can be shown that (27a) is hyperbolic.

Example: "Laplace's Equation"

$$\boxed{\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0} \qquad (29)$$

- Note that this example is similar to (15). As it is 2^{nd} order in x and y, we need to introduce two new variables.

$$\boxed{v = \frac{\partial u}{\partial x} \quad ; \quad w = \frac{\partial u}{\partial y}} \qquad (30)$$

- Hence (29) can be written as:

$$\boxed{\begin{aligned} \frac{\partial v}{\partial x} + \frac{\partial w}{\partial y} &= 0 \\ \frac{\partial v}{\partial y} - \frac{\partial w}{\partial x} &= 0 \end{aligned}} \qquad (31)$$

METHOD 2:

- For the eigenvalue method, write (31) in matrix form: $\bar{W} = \begin{Bmatrix} v \\ w \end{Bmatrix}$

$$\boxed{\underbrace{\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}}_{[K]} \frac{\partial \bar{W}}{\partial x} + \underbrace{\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}}_{[M]} \frac{\partial \bar{W}}{\partial y} = 0} \qquad (32)$$

- Determine $[K]^{-1}$ ...

$$[K]^{-1} = -\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

- Determine $[N] = [K]^{-1}[M]$

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

- Find the eigenvalues of $[N]$.

$$\left| [N] - \lambda [I] \right| = 0 \rightarrow \left| \begin{pmatrix} -\lambda & 1 \\ -1 & -\lambda \end{pmatrix} \right| = 0$$

$$\boxed{\lambda^2 + 1 = 0} \qquad (33)$$

- Equation (33) allows only imaginary solutions for λ.

- Hence Laplace's equation is elliptic !