Solution of Incompressible Navier-Stokes Equations

- So far...
  - We have learned basic concepts in Finite-Difference Methods to solve simple partial differential equations.
- Now...
  - Introduction to solution methods for incompressible viscous flow, i.e. the (incompressible) Navier Equations

$$\boxed{\begin{aligned} &\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \quad (\vec{\nabla}.\vec{u} = 0) && \text{"mass"} \\ &\rho \frac{du}{dt} = -\frac{\partial p}{\partial x} + \left(\frac{\partial \tau_{xx}}{\partial x} + \frac{\partial \tau_{yx}}{\partial y}\right) && \text{"x-momentum"} \\ &\rho \frac{dv}{dt} = -\frac{\partial p}{\partial y} + \left(\frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \tau_{yy}}{\partial y}\right) && \text{"y-momentum"} \end{aligned}} \quad (1)$$

- In equation (1), $\frac{d(\ldots)}{dt} = \frac{\partial(\ldots)}{\partial t} + u\frac{\partial(\ldots)}{\partial x} + v\frac{\partial(\ldots)}{\partial y}$ is the total (or substantial) derivative and τ_{ij} are shear stresses acting on a fluid particle

$$\boxed{\begin{aligned} &\tau_{xx} = 2\mu\frac{\partial u}{\partial x} - \frac{2}{3}\mu\vec{\nabla}.\vec{u} \\ &\tau_{yy} = 2\mu\frac{\partial v}{\partial y} - \frac{2}{3}\mu\vec{\nabla}.\vec{u} \end{aligned} \qquad \tau_{xy} = \mu\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)} \quad (2)$$

- Using the mass principle (or incompressibility condition) with $\vec{\nabla}.\vec{u}=0$, we find the following:

$$\frac{\partial \tau_{xx}}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} = 2\mu \frac{\partial^2 u}{\partial x^2} + \mu \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)$$

$$= 2\mu \frac{\partial^2 u}{\partial x^2} + \mu \frac{\partial^2 u}{\partial y^2} + \mu \frac{\partial}{\partial x}\underbrace{\left(\frac{\partial v}{\partial y} + \frac{\partial u}{\partial x}\right)}_{=\vec{\nabla}.\vec{u}=0} - \mu \frac{\partial^2 u}{\partial x^2}$$

$$= \mu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) = \mu \nabla^2 u$$

$$\frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \tau_{yy}}{\partial y} = \mu \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) + 2\mu \frac{\partial^2 v}{\partial y^2}$$

$$= \mu \frac{\partial^2 v}{\partial x^2} + \mu \frac{\partial}{\partial y}\underbrace{\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right)}_{=\vec{\nabla}.\vec{u}=0} - \mu \frac{\partial^2 v}{\partial y^2} + 2\mu \frac{\partial^2 v}{\partial y^2}$$

$$= \mu\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right) = \mu \nabla^2 v$$

- Remember that shear stresses τ_{ij} are a result of Stokes' Postulate. We can hence write Equation (1) in vector form as: $\vec{u} = \begin{pmatrix} u \\ v \end{pmatrix}$

$$\boxed{\begin{aligned} \vec{\nabla}.\vec{u} &= 0 \\ \rho \frac{d\vec{u}}{dt} &= -\vec{\nabla}p + \mu \vec{\nabla}^2 \vec{u} \end{aligned}} \qquad (3)$$

- Equation (3) reads in component form:

$$
\boxed{
\begin{aligned}
&\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \\
&\rho\left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y}\right) = -\frac{\partial p}{\partial x} + \mu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) \\
&\rho\left(\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y}\right) = -\frac{\partial p}{\partial y} + \mu\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right)
\end{aligned}
}
\qquad (4)
$$

- Equations (3) and (4) describe the incompressible Navier-Stokes Equations. In 2-D, they describe a system of 3 equations for 3 unknowns u, v, p.

- The "mass" principle is a 1st order linear PDE of 'hyperbolic' type, while the "momentum" equations are 2nd order non-linear PDEs of mixed 'parabolic/elliptic' type.

- The question arises as to how to solve (4) using Finite-Difference Methods.

  → Primitive Variable Methods
    - Artificial Compressibility Method (ACM) [Chorin, 1967]
    - Pressure Correction Methods e.g. [Patankar & Spalding, 1972]

→ Vortex Methods

- based on Vorticity Transport Equation

- Today, the most common methods are Pressure Correction Methods. They are widely used for incompressible flows to keep the pressure field from oscillating. These pressure oscillations may arise due to difficulties in preserving the mass principle (or incompressibility condition) as the sound speed becomes much higher than convection velocity components.

- Note that the mass principle ($\vec{\nabla}.\vec{u}=0$) was applied to Equation (2) in order to obtain Equation (4). This may be problematic as $\vec{\nabla}.\vec{u}\neq 0$ during an iterative solution process (as described above). It is therefore helpful revisiting Equation (2) in conjunction with Navier's Equations (1):

$$\left.\begin{aligned}\tau_{xx} &= 2\mu\frac{\partial u}{\partial x}-\frac{2}{3}\mu\vec{\nabla}.\vec{u}\\ \tau_{yy} &= 2\mu\frac{\partial v}{\partial y}-\frac{2}{3}\mu\vec{\nabla}.\vec{u}\\ \tau_{xy} &= \tau_{yx}=\mu\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right)\end{aligned}\right\} \xrightarrow{(2)} \xrightarrow{(1)} \left\{\begin{aligned}\frac{\partial \tau_{xx}}{\partial x}+\frac{\partial \tau_{yx}}{\partial y} &= ?\\ \frac{\partial \tau_{xy}}{\partial x}+\frac{\partial \tau_{yy}}{\partial y} &= ?\end{aligned}\right.$$

• Let us write $\frac{\partial \tau_{xx}}{\partial x} + \frac{\partial \tau_{yx}}{\partial y}$ and $\frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \tau_{yy}}{\partial y}$ in a more familiar form, though accounting for $\vec{\nabla}.\vec{u} \neq 0$ during an iterative solution procedure.

$$\frac{\partial \tau_{xx}}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} = \frac{\partial}{\partial x}\left(2\mu \frac{\partial u}{\partial x} - \frac{2}{3}\mu\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right)\right)$$

$$+ \frac{\partial}{\partial y}\left(\mu\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)\right)$$

$$= 2\mu \frac{\partial^2 u}{\partial x^2} - \frac{2}{3}\mu \frac{\partial^2 u}{\partial x^2} - \frac{2}{3}\mu \frac{\partial}{\partial x}\left(\frac{\partial v}{\partial y}\right)$$

$$+ \mu \frac{\partial^2 u}{\partial y^2} + \mu \frac{\partial}{\partial x}\left(\frac{\partial v}{\partial y}\right)$$

$$= \frac{4}{3}\mu \frac{\partial^2 u}{\partial x^2} + \mu \frac{\partial^2 u}{\partial y^2} + \frac{1}{3}\mu \frac{\partial}{\partial x}\left(\frac{\partial v}{\partial y}\right)$$

$$- \frac{1}{3}\mu \frac{\partial^2 u}{\partial x^2} + \frac{1}{3}\mu \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right)$$

$$= \mu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) + \frac{1}{3}\mu \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right)$$

$$= \mu \vec{\nabla}^2 u + \frac{1}{3}\mu \frac{\partial}{\partial x}(\vec{\nabla}.\vec{u})$$

$$\frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \tau_{yy}}{\partial y} = \frac{\partial}{\partial x}\left(\mu\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)\right)$$

$$+ \frac{\partial}{\partial y}\left(2\mu \frac{\partial v}{\partial y} - \frac{2}{3}\mu\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right)\right)$$

$$= \mu \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x}\right) + \mu \frac{\partial^2 v}{\partial x^2} + 2\mu \frac{\partial^2 v}{\partial y^2}$$

$$- \frac{2}{3}\mu \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x}\right) - \frac{2}{3}\mu \frac{\partial^2 v}{\partial y^2}$$

$$= \mu \frac{\partial^2 v}{\partial x^2} + \frac{4}{3}\mu \frac{\partial^2 v}{\partial y^2} + \frac{1}{3}\mu \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x}\right)$$

$$- \frac{1}{3}\mu \frac{\partial^2 v}{\partial y^2} + \frac{1}{3}\mu \frac{\partial}{\partial y}\left(\frac{\partial v}{\partial y}\right)$$

$$= \mu\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right) + \frac{1}{3}\mu\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right)$$

$$= \mu\vec{\nabla}^2 v + \frac{1}{3}\mu\frac{\partial}{\partial y}(\vec{\nabla}\cdot\vec{u})$$

- We can now write the (incompressible) Navier-Stokes Equations in equivalent form as:

$$\boxed{\begin{aligned} &\vec{\nabla}\cdot\vec{u} = 0 \\ &\rho\frac{du}{dt} = -\frac{\partial p}{\partial x} + \mu\vec{\nabla}^2 u + \frac{1}{3}\mu\frac{\partial}{\partial x}(\vec{\nabla}\cdot\vec{u}) \\ &\rho\frac{dv}{dt} = -\frac{\partial p}{\partial y} + \mu\vec{\nabla}^2 v + \frac{1}{3}\mu\frac{\partial}{\partial y}(\vec{\nabla}\cdot\vec{u}) \end{aligned}} \qquad (5)$$

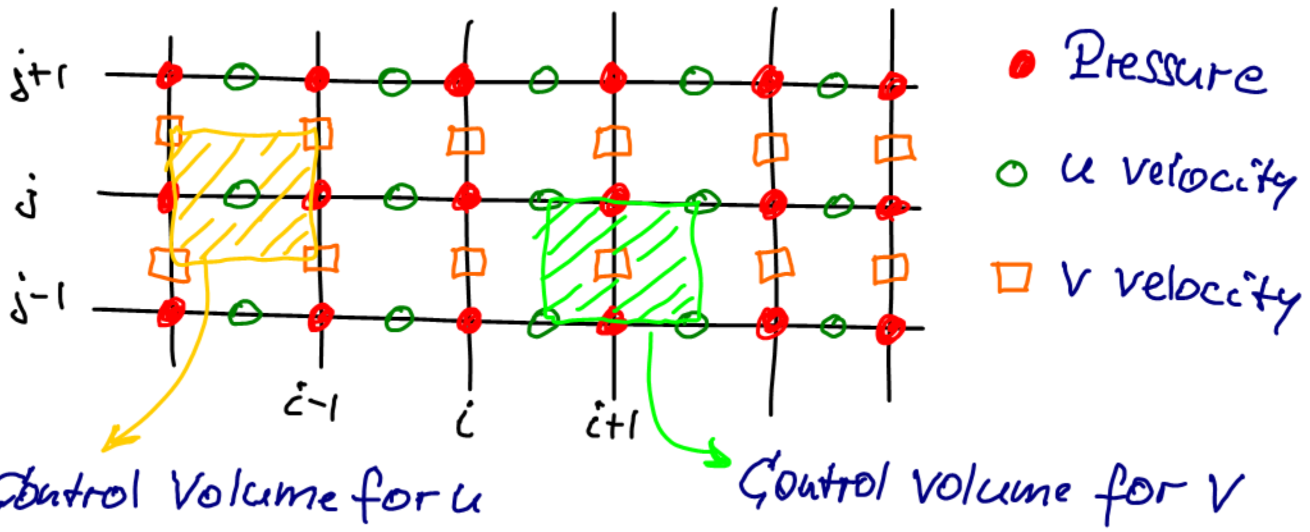
- Equation (5) is a more suitable form for Pressure Correction Methods as $\vec{\nabla}\cdot\vec{u} \neq 0$ during the numerical solution procedure. In the following, we are going look at one example of pressure-based solvers, the SIMPLE algorithm.

Semi-Implicit Method For Pressure-Linked Equations

SIMPLE [Patankar & Spalding, 1972]

- It is known that Finite-Difference Equations written in control volume grids lead to non-physical oscillations of velocity components. As a consequence, mass is not conserved ($\vec{\nabla}\cdot\vec{u} \neq 0$), thus causing the pressure to undergo similar oscillations.

- These difficulties can be remedied by using "staggered grids".



- In the SIMPLE algorithm, the velocity nodes are staggered with respect to the pressure nodes. In this method, the predictor-corrector procedure with successive pressure correction steps is used:

$$\boxed{p = \bar{p} + p' \;;\; u = \bar{u} + u' \;;\; v = \bar{v} + v'} \quad (6)$$

where the "overbar" $(\bar{\;})$ denotes an estimated quantity and the "prime" $(')$ is the correction.

- The pressure corrections are related to the velocity corrections by approximate momentum equations,

$$\boxed{\begin{aligned} \rho \frac{\partial u'}{\partial t} &= \frac{\partial p'}{\partial x} \\ \rho \frac{\partial v'}{\partial t} &= \frac{\partial p'}{\partial y} \end{aligned}} \quad (7) \qquad \underline{\underline{\text{OR}}} \qquad \boxed{\begin{aligned} u' &= -\frac{\Delta t}{\rho}\frac{\partial p'}{\partial x} \\ v' &= -\frac{\Delta t}{\rho}\frac{\partial p'}{\partial y} \end{aligned}} \quad (8)$$

- Next, we substitute Equations (6) and (7) into the mass principle in (5) to obtain

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = \frac{\partial \bar{u}}{\partial x} + \frac{\partial \bar{v}}{\partial y} - \frac{\Delta t}{\rho}\left(\frac{\partial^2 p'}{\partial x^2} + \frac{\partial^2 p'}{\partial y^2}\right)$$

$$\boxed{\frac{\partial^2 p'}{\partial x^2} + \frac{\partial^2 p'}{\partial y^2} = -\frac{\rho}{\Delta t}\left(\frac{\partial \bar{u}}{\partial x} + \frac{\partial \bar{v}}{\partial y}\right)} \quad (9)$$

$$\underline{\underline{\text{or}}}$$

$$\boxed{\vec{\nabla}^2 p' = -\frac{\rho}{\Delta t}\, \vec{\nabla} . \vec{\bar{u}}} \quad (9a)$$

- Equation (9) is the so-called pressure-correction Poisson equation of an "elliptic form". It is apparent that $p' \to 0$ and $\vec{\nabla} . \vec{u} \to 0$ as the estimated solution (overbar) gets closer to the actual solution.

- In other words, Equation (9) provides a correction for the pressure such that the approximate solution $\tilde{u}, \tilde{v}$ satisfies the mass principle.

- Though the Navier-Stokes equations (5) are of mixed type, the pressure correction equation (9) is always "elliptic". Hence it can be solved with an appropriate iterative procedure, e.g. "Jacobi", "Gauss-Seidel", "SOR", "SLOR", etc.

- An iterative procedure to solve the Navier-Stokes equations (5) is as follows:
  [Raithby & Schneider, 1979]

  a.) Guess the pressure $\bar{p}$ at each grid point.

  b.) Solve the momentum equations to find $\tilde{u}$ and $\tilde{v}$ at the staggered grid ($i+1/2$, $i-1/2$, $j+1/2$, $j-1/2$)
  → see "Solution of Conv./Diff. Equation" with known pressure term.

  c.) Solve the pressure-correction equation (9) for p' on the actual grid (i,j) using a method suitable for "elliptic" PDEs.

d.) Correct the pressure and velocity

$$\boxed{\begin{aligned} p &= \bar{p} + p' \\ u &= \bar{u} - \frac{\Delta t}{2\rho\Delta x}\left(p'_{i+1,j} - p'_{i-1,j}\right) - \frac{\Delta t}{\rho}\left(A^{(1)}_{i+\frac{1}{2},j} - A^{(1)}_{i-\frac{1}{2},j}\right) \\ v &= \bar{v} - \frac{\Delta t}{2\rho\Delta y}\left(p'_{i,j+1} - p'_{i,j-1}\right) - \frac{\Delta t}{\rho}\left(A^{(2)}_{i,j+\frac{1}{2}} - A^{(2)}_{i,j-\frac{1}{2}}\right) \end{aligned}} \quad (10)$$

where

$$\boxed{\begin{aligned} A^{(1)} &= \rho\left(u'\frac{\partial u'}{\partial x} + v'\frac{\partial u'}{\partial y}\right) - \mu\left(\frac{\partial^2 u'}{\partial x^2} + \frac{\partial^2 u'}{\partial y^2} + \frac{1}{3}\left(\frac{\partial u'}{\partial x} + \frac{\partial v'}{\partial y}\right)\right) \\ A^{(2)} &= \rho\left(u'\frac{\partial v'}{\partial x} + v'\frac{\partial v'}{\partial y}\right) - \mu\left(\frac{\partial^2 v'}{\partial x^2} + \frac{\partial^2 v'}{\partial y^2} + \frac{1}{3}\left(\frac{\partial u'}{\partial x} + \frac{\partial v'}{\partial y}\right)\right) \end{aligned}} \quad (11)$$

e.) Replace the previous estimated values for $\bar{p}, \bar{u}, \bar{v}$ with the new ones p, u, v and return to b.)

f.) Repeat steps b.) through e.) until convergence.

- Note that the convergence of the above process may not be satisfactory because of the tendency for overestimation of p'. This problem can be solved by using an under-relaxation parameter α,

$$\boxed{p = \bar{p} + \alpha p'} \qquad (12)$$

- A good choice for α is $\alpha \approx 0.8$.