Aerospace 423 HW 6

Laplace Equation $\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} = 0$ $0 \le x \le L$

$0 \le y \le W$

Boundary Conditions $\rightarrow$
$\phi(x,0) = 0$
$\phi(x,W) = 1$
$\phi(0,y) = 0$
$\phi(L,y) = 0$

$$\phi(x,y) = \frac{2}{\pi}\sum_{n=1}^{\infty} \frac{(-1)^{n+1}+1}{n} \sin\left(\frac{n\pi x}{L}\right)\frac{\sinh\left(\frac{n\pi y}{L}\right)}{\sinh\left(\frac{n\pi W}{L}\right)}$$

$L = W = 1$ n from 1:50

$0 \le x \le 1$
$0 \le y \le 1$

$ix = jx = 50$ ①
$ix = jx = 100$ ② Based on other parts

residual = 1E05

$$\Delta x = \frac{x_1 - x_0}{ix} = \frac{1-0}{ix} = \frac{1}{ix}$$

$$\Delta y = \frac{1}{jx}$$

Laplace Equation

Jacobi Method

$$\frac{u^n_{i+1} - 2u^{n+1}_{ij} + u^n_{i-1\,j}}{\Delta x^2} + \frac{u^n_{i\,j+1} - 2u^{n+1}_{ij} + u^n_{i\,j-1}}{\Delta y^2} = 0$$

For our case

$i = 2 \rightarrow ix+1$ $j = 2 \rightarrow jx+1$

$$u^{n+1}_{ij} = h^2\left[\frac{1}{\Delta x^2}\left(u^n_{i+1,j} + u^n_{i-1\,j}\right) + \frac{1}{\Delta y^2}\left(u^n_{i\,j+1} + u^n_{i\,j-1}\right)\right]$$

where $\frac{1}{h^2} = 2\left(\frac{1}{\Delta x^2} + \frac{1}{\Delta y^2}\right)$

$\hookrightarrow h^2 = \frac{\left(\frac{1}{\Delta x^2} + \frac{1}{\Delta y^2}\right)}{2}$

Gauss Seidel Method

Also a new "time" step

$$\frac{u^n_{i+1\,j} - 2u^{n+1}_{ij} + u^{n+1}_{i-1\,j}}{\Delta x^2} + \frac{u^n_{i\,j+1} - 2u^{n+1}_{ij} + u^{n+1}_{i\,j-1}}{\Delta y^2} = 0$$

$f_{ij} = 0$

Provisional new value

$$u^{n+1}_{ij} = u^n_{ij} + \omega\left(\hat{u}_{ij} - u^n_{ij}\right)$$

real new value

relaxation factor = 1 for Gauss Seidel.

Substituting in yields.

$$\frac{u_{i+1}^{n} - 2\left[u_{ij}^{n} + \frac{1}{\omega}(u_{ij}^{n+1} - u_{ij}^{n})\right] + u_{i-1j}^{n+1}}{\Delta x^2} + \frac{u_{ij+1}^{n} - 2\left[u_{ij}^{n} + \frac{1}{\omega}(u_{ij}^{n+1} - u_{ij}^{n})\right] + u_{ij-1}^{n+1}}{\Delta y^2} = 0$$

Update Form

$$\frac{1}{\omega}\frac{1}{h^2}u_{ij}^{n+1} = \frac{1}{\Delta x^2}(u_{i+1j}^{n} + u_{i-1j}^{n+1}) + \frac{1}{\Delta y^2}(u_{ij+1}^{n} + u_{ij-1}^{n+1}) - \frac{1}{h^2}\left(1 - \frac{1}{\omega}\right)u_{ij}^{n}$$

↳ 0 for Gauss Seidel.

$\omega = 1$ For Gauss Seidel

$$\frac{1}{h^2}u_{ij}^{n+1} = \frac{1}{\Delta x^2}(u_{i+1j}^{n} + u_{i-1j}^{n+1}) + \frac{1}{\Delta y^2}(u_{ij+1}^{n} + u_{ij-1}^{n+1})$$

Our relaxation Method $(\omega = 1.8)$

<u>SLOR</u>

$$\frac{u_{i+1j}^{n} - 2\tilde{u}_{ij} + u_{i-1j}^{n+1}}{\Delta x^2} + \frac{\tilde{u}_{ij+1} - 2\tilde{u}_{ij} + \tilde{u}_{ij-1}}{\Delta y^2} = 0$$

SOR

$$\frac{u_{i+1j}^{n} - 2\tilde{u}_{ij} + u_{i-1j}^{n+1}}{\Delta x^2} + \frac{u_{i,j+1}^{n} - 2\tilde{u}_{ij} + u_{ij-1}^{n+1}}{\Delta y^2} = 0$$

$\hat{u}_{ij}$ is a provisional update

$$u_{ij}^{n+1} = u_{ij}^{n} + \omega(\tilde{u}_{ij} - u_{ij}^{n})$$

$$\hat{u}_{ij} = u_{ij}^{n} + \frac{1}{\omega}(u_{ij}^{n+1} - u_{ij}^{n})$$

$\hat{u}_{ij}$ solved first

$$\begin{bmatrix} i = \text{const} \\ j = 2, \ldots, jx-1 \end{bmatrix}$$

$$\begin{bmatrix} q_1 & r_1 & & & & \\ p_2 & q_2 & r_2 & & & \\ & & p_j & q_j & r_j & \\ & & & p_{jx-1} & q_{jx-1} & r_{jx-1} \\ & & & & p_{jx} & q_{jx} \end{bmatrix} \vec{x} = \vec{s}$$

$$\frac{u_{i+1,j}^{n} - 2\hat{u}_{ij} + u_{i-1j}^{n+1}}{\Delta x^2} + \frac{\tilde{u}_{ij+1} - 2\tilde{u}_{ij} + \tilde{u}_{ij-1}}{\Delta y^2} = 0$$

$$\frac{1}{\Delta x^2}\left(u_{i+1j}^{n} - 2\hat{u}_{ij} + u_{i-1j}^{n+1}\right) + \frac{1}{\Delta y^2}\left(\tilde{u}_{ij+1} - 2\hat{u}_{ij} + \tilde{u}_{ij-1}\right) = 0$$

$$\frac{u_{i+1j}^{n} + u_{i-1j}^{n+1}}{\Delta x^2} - \frac{2\tilde{u}_{ij}}{\Delta x^2} + \frac{\tilde{u}_{ij+1}}{\Delta y^2} - \frac{2\hat{u}_{ij}}{\Delta y^2} + \frac{\hat{u}_{ij-1}}{\Delta y^2} = 0$$

$$\frac{\hat{u}_{i,j-1}}{\Delta y^2} - 2\left(\frac{1}{\Delta x^2}+\frac{1}{\Delta y^2}\right)\hat{u}_{ij} + \frac{\tilde{u}_{i,j+1}}{\Delta y^2} = -\left[\frac{u^{n}_{i+1,j} + u^{n+1}_{i-1,j}}{\Delta x^2}\right]$$

$$= \frac{-1}{\Delta x^2}\left(u^{n}_{i+1,j} + u^{n+1}_{i-1,j}\right)$$

$$\begin{bmatrix} q_1 & r_1 & & & & & \\ p_2 & q_2 & r_2 & & & & \\ & \frac{1}{\Delta y^2} & -2\left(\frac{1}{\Delta x^2}+\frac{1}{\Delta y^2}\right) & \frac{1}{\Delta y^2} & & & \\ & p_j & q_j & r_j & & & \\ & & & p_{jx-1} & q_{jx-1} & r_{jx-1} \\ & & & & p_{jx} & q_{jx} \end{bmatrix} \begin{bmatrix} \vdots \\ \hat{u}_{ij} \\ \vdots \end{bmatrix} = \tilde{S}_{jx}$$

$$\begin{bmatrix} b_1 & c_1 & & & & \\ a_2 & b_2 & c_2 & & & \\ & a_3 & b_3 & c_3 & & \\ & & a_4 & b_4 & c_4 & \\ & & & a_5 & \ddots & c_{jx-1} \\ & & & & a_{jx} & b_{jx} \end{bmatrix}$$

$$a_i = \frac{1}{\Delta y^2}$$

$$c_i = \frac{1}{\Delta y^2}$$

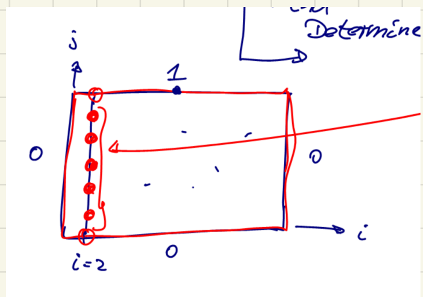
$$b_i = -2\left(\frac{1}{\Delta x^2}+\frac{1}{\Delta y^2}\right)$$

$$d_i = -\frac{1}{\Delta x^2}\left(u^{n}_{i+1,j} + u^{n+1}_{i-1,j}\right)$$

HW # 6:

SLOR

while solution not converged
  for $i = 2 : ix-1$
    Determine $p_j\ q_j\ r_j\ ;\ S_j$
    Call Thomas algorithm
    "update" i-line
  end
  Determine Residual



update after Thomas Also

$$u^{n+1}_{ij} = u^{n}_{ij} + \omega\left(\tilde{u}^{n}_{ij} - u^{n}_{ij}\right)$$

For constant i

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