Integration of Linearized Compressible Flow Equations

- What we know so far ...

→ Linearized "Supersonic" Potential Flow

$$\boxed{-\beta^2 \frac{\partial^2 \varphi}{\partial x^2} + \frac{\partial^2 \varphi}{\partial y^2} = 0}$$

"Wave Equation" (Hyperbolic)

$\beta = \sqrt{M_\infty^2 - 1} > 0$; $M_\infty \triangleq$ Free Stream Mach #

→ "Incompressible" Potential Flow

$$\boxed{\frac{\partial^2 \varphi}{\partial x^2} + \frac{\partial^2 \varphi}{\partial y^2} = 0}$$

"Laplace Equation" (Elliptic)

Recall that in potential flow, the Cartesian velocity perturbations u' & v' are defined as

$$\boxed{u' = \frac{\partial \varphi}{\partial x} \quad ; \quad v' = \frac{\partial \varphi}{\partial y}} \qquad (1)$$

One major advantage of potential flow is that the conservation equations for mass, momentum, and energy are reduced to 1 equation w/ 1 unknown.

- Question:

  Can we also find a potential equation for general subsonic, transonic, and supersonic flow?

$\Longrightarrow$ Transonic Small Disturbance Equation (TSD)

- Potential flow implies the assumptions of inviscid & irrotational flow.

- The general "inviscid" Equations of motion read in steady & 2-D conditions ...

$$\boxed{\frac{\partial}{\partial x}(\rho u) + \frac{\partial}{\partial y}(\rho v) = 0} \quad \text{"Continuity"} \quad (2)$$

$$\boxed{\begin{aligned} u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} &= -\frac{1}{\rho}\frac{\partial p}{\partial x} \\ u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} &= -\frac{1}{\rho}\frac{\partial p}{\partial y} \end{aligned}} \quad \text{"Momentum"} \quad (3)$$

Introducing the local "speed of sound" a via

$$\boxed{a^2 = \frac{dp}{d\rho}} \qquad (4)$$

we can rewrite equation (3) as

$$\boxed{\begin{aligned} u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} &= -\frac{a^2}{\rho}\frac{\partial \rho}{\partial x} \\ u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} &= -\frac{a^2}{\rho}\frac{\partial \rho}{\partial y} \end{aligned}} \qquad (5)$$

which was an elegant way to eliminate the pressure p. Next, we multiply the first equation in (5) by u and the second equation by V to obtain

$$\boxed{\begin{aligned} u^2\frac{\partial u}{\partial x} + uv\frac{\partial u}{\partial y} + uv\frac{\partial v}{\partial x} + v^2\frac{\partial v}{\partial y} \\ = -\frac{a^2}{\rho}\left(u\frac{\partial \rho}{\partial x} + v\frac{\partial \rho}{\partial y}\right) \end{aligned}} \qquad (6)$$

At this stage, it would be advantageous to also eliminate the density ρ. Let us expand the continuity equation (2) ...

$$\frac{\partial}{\partial x}(\rho u) + \frac{\partial}{\partial y}(\rho v) = 0$$

$$\rho \frac{\partial u}{\partial x} + u \frac{\partial \rho}{\partial x} + \rho \frac{\partial v}{\partial y} + v \frac{\partial \rho}{\partial y} = 0$$

$$\boxed{-\frac{1}{\rho}\left(u \frac{\partial \rho}{\partial x} + v \frac{\partial \rho}{\partial y}\right) = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}} \qquad (7)$$

Equation (7) in (6) yields the following

$$\boxed{\left(\frac{u^2}{a^2} - 1\right)\frac{\partial u}{\partial x} + \frac{uv}{a^2}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) + \left(\frac{v^2}{a^2} - 1\right)\frac{\partial v}{\partial y} = 0} \qquad (8)$$

As a next step, we ...

i) introduce the velocity potential according to (1)

ii) apply the condition of irrotationality through $\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} = \cancelto{0}{\left(\frac{\partial u}{\partial y} - \frac{\partial v}{\partial x}\right)} + 2\frac{\partial v}{\partial x}$

Thus, equation (8) becomes

$$\boxed{\left(\frac{u^2}{a^2}-1\right)\frac{\partial^2\varphi}{\partial x^2}+\left(\frac{2uv}{a^2}\right)\frac{\partial^2\varphi}{\partial x\partial y}+\left(\frac{v^2}{a^2}-1\right)\frac{\partial^2\varphi}{\partial y^2}=0} \quad (9)$$

An expression for the sonic speed a can be obtained from the energy equation.

$$\boxed{\frac{\gamma}{\gamma-1}\frac{p}{\rho}+\frac{1}{2}q^2=\frac{\gamma}{\gamma-1}\frac{p_\infty}{\rho_\infty}+\frac{1}{2}q_\infty^2} \quad (10)$$

which is the "Compressible Bernoulli Equation".

$$q^2=(V_\infty+u')^2+v'^2 \quad ; \quad q_\infty=V_\infty^2 \quad (11)$$

are the total velocities. In (10), γ is the isentropic exponent. The ideal gas law gives us $\boxed{\frac{p}{\rho}=RT}$ (12) and we further use the classical definition for the sonic speed $\boxed{a^2=\gamma RT}$ (13) to write equation (10) in the following way

$$\boxed{a^2 + \frac{1}{2}(\gamma-1)\,q^2 = a_\infty^2 + \frac{1}{2}(\gamma-1)\,q_\infty^2} \qquad (14)$$

- In theory, (14) can replace a^2 in (9) and all velocity perturbations can be written in terms of the potential φ. This would indeed result in a single equation with a single unknown φ, however this equation would be highly non-linear and impractical to solve.

- Instead, let us consider small flow perturbations as do occur around thin bodies at small angles-of-attack. Then we have

$$u = V_\infty + u' = V_\infty + \frac{\partial \varphi}{\partial x} \quad ; \quad v' = \frac{\partial \varphi}{\partial y}$$

We can hence write equation (9) as

$$\boxed{\left(\frac{V_\infty^2 + 2u'V_\infty + u'^2}{a^2} - 1\right)\frac{\partial^2 \varphi}{\partial x^2} + 2\left(\frac{V_\infty v' + u'v'}{a^2}\right)\frac{\partial^2 \varphi}{\partial x \partial y} + \left(\frac{v'^2}{a^2} - 1\right)\frac{\partial^2 \varphi}{\partial y^2} = 0} \qquad (15)$$

and equation (14) in the following form:

$$a^2 + \frac{1}{2}(\gamma-1)\left(\cancel{V_\infty^2} + 2V_\infty u' + u'^2\right)$$

$$= a_\infty^2 + \cancel{\frac{1}{2}(\gamma-1)V_\infty^2} \qquad ; \quad \boxed{M_\infty = \frac{V_\infty}{a_\infty}}$$

$$\left(\frac{a}{a_\infty}\right)^2 = 1 - \frac{\gamma-1}{2} M_\infty^2 \left(2\frac{u'}{V_\infty} + \frac{u'^2}{V_\infty^2} + \frac{v'^2}{V_\infty^2}\right) \qquad (16)$$

- Assuming small perturbations we can neglect the higher-order terms in (16) and obtain

$$\boxed{\left(\frac{a}{a_\infty}\right)^2 \approx 1 - (\gamma-1) M_\infty^2 \frac{u'}{V_\infty}} \qquad (17)$$

We can also neglect in (15) the terms that are of higher order in the perturbations

$$\boxed{\left(\frac{V_\infty^2 + 2u'V_\infty}{a^2} - 1\right)\frac{\partial^2\varphi}{\partial x^2} + \left(\frac{2V_\infty v'}{a^2}\right)\frac{\partial^2\varphi}{\partial x\partial y} - \frac{\partial^2\varphi}{\partial y^2}} \qquad (15a)$$

Multiply (15a) by $(a/a_\infty)^2$ and obtain

$$\boxed{\left(M_\infty^2 + 2M_\infty^2\frac{u'}{V_\infty} - \left(\frac{a}{a_\infty}\right)^2\right)\frac{\partial^2\varphi}{\partial x^2} + \left(2M_\infty^2\frac{v'}{V_\infty}\right)\frac{\partial^2\varphi}{\partial x\partial y} - \left(\frac{a}{a_\infty}\right)^2\frac{\partial^2\varphi}{\partial y^2} = 0}$$

(18)

Now use equation (17) and find that

$$\left(M_\infty^2 + 2M_\infty^2 \frac{u'}{V_\infty} - 1 + (\gamma-1) M_\infty^2 \frac{u'}{V_\infty}\right) \frac{\partial^2 \varphi}{\partial x^2}$$

$$+ \left(2M_\infty^2 \frac{v'}{V_\infty}\right) \frac{\partial^2 \varphi}{\partial x \partial y} - \left(1 - (\gamma-1) M_\infty^2 \frac{u'}{V_\infty}\right) \frac{\partial^2 \varphi}{\partial y^2} = 0 \quad (19)$$

Use definitions for u' & v'...

$$\left[(M_\infty^2 - 1) + (\gamma+1) \frac{M_\infty^2}{V_\infty} \frac{\partial \varphi}{\partial x}\right] \frac{\partial^2 \varphi}{\partial x^2}$$

$$\cancel{+ \underbrace{\left(2 \frac{M_\infty^2}{V_\infty} \frac{\partial \varphi}{\partial y}\right) \frac{\partial^2 \varphi}{\partial x \partial y}}_{\text{Small!}}} - \left(1 - \cancel{\underbrace{(\gamma-1) \frac{M_\infty^2}{V_\infty} \frac{\partial \varphi}{\partial x}}_{\text{Small compared to 1!}}}\right) \frac{\partial^2 \varphi}{\partial y^2} = 0$$

$$\boxed{\left[(M_\infty^2 - 1) + (\gamma+1) M_\infty^2 \frac{\varphi_x}{V_\infty}\right] \varphi_{xx} - \varphi_{yy} = 0} \quad (20)$$

Equation (20) is the

Transonic Small Disturbance (Potential) Equation (TSD)

- It is apparent that (20) is a non-linear PDE in the perturbation potential φ.
However, the equation becomes linear when

$$\| M_\infty^2 - 1 \| \gg (\gamma+1) M_\infty^2 \frac{\varphi_x}{V_\infty}$$

- This is the case for $M_\infty \to 0$ which is the incompressible limit and (20) reduces to

$$-\varphi_{xx} - \varphi_{yy} = 0 \quad \text{or}$$

$$\boxed{\varphi_{xx} + \varphi_{yy} = 0} \quad \text{"Laplace Equation"}$$

- But this can also happen for supersonic flow when $M_\infty \gg 1$ and (20) becomes

$$(M_\infty^2 - 1)\varphi_{xx} - \varphi_{yy} = 0 \quad \text{or}$$

$$\boxed{-(M_\infty^2 - 1)\varphi_{xx} + \varphi_{yy} = 0} \quad \text{"Wave Equation"}$$

- The non-linear term in (20) becomes important only if $M_\infty \approx 1$ ...

…, i.e. in transonic flow. While equation (20) is elliptic in subsonic flow and hyperbolic in supersonic flow, it can be of mixed type if the free stream Mach number M_∞ is close to one ($0.85 \lesssim M_\infty \lesssim 1.15$)

- The numerical solution of equation (20) in the early 1970's was a milestone in computing transonic flows (w/ shocks!) over airfoils. The original procedure was published by Murman & Cole in 1970. However, the scheme was not conservative across the shock.
- Therefore, Murman corrected the scheme in his 1973 paper.

The Four-Operator Scheme [Murman, 1973]

i) Supersonic Point ($u_{i-1,j} > u^*$; $u_{i,j} > u^*$)

$$\left((1-M_\infty^2)-(\gamma+1)M_\infty^2\,\frac{\varphi_{i,j}^{n}-\varphi_{i-2,j}^{n+1}}{2\Delta x}\right)\frac{\varphi_{i,j}^{n+1}-2\varphi_{i-1,j}^{n+1}+\varphi_{i-2,j}^{n+1}}{\Delta x^2}$$

$$+\ \frac{\varphi_{i,j+1}^{n+1}-2\varphi_{i,j}^{n+1}+\varphi_{i,j-1}^{n+1}}{\Delta y^2}=0$$

ii) Subsonic Point ($u_{i-1,j} < u^*$; $u_{i,j} < u^*$)

$$\left((1-M_\infty^2)-(\gamma+1)M_\infty^2\,\frac{\varphi_{i+1,j}^{n}-\varphi_{i-1,j}^{n+1}}{2\Delta x}\right)\frac{\varphi_{i+1,j}^{n+1}-2\tilde{\varphi}_{i,j}+\varphi_{i-1,j}^{n+1}}{\Delta x^2}$$

$$+\ \frac{\tilde{\varphi}_{i,j+1}-2\tilde{\varphi}_{i,j}+\tilde{\varphi}_{i,j-1}}{\Delta y^2}=0$$

iii) Sonic Point ($u_{i-1,j} < u^*$; $u_{i,j} > u^*$)

$$\frac{\varphi_{i,j+1}^{n+1}-2\varphi_{i,j}^{n+1}+\varphi_{i,j-1}^{n+1}}{\Delta y^2}=0$$

iv) Shock Point $\quad (u_{i-1,j} > u^* \; ; \; u_{i,j} < u^*)$

$$\left((1-M_\infty^2) - (\gamma+1) M_\infty^2 \frac{\varphi_{i,j}^n - \varphi_{i-2,j}^{n+1}}{2\Delta x}\right) \frac{\varphi_{i,j}^{n+1} - 2\varphi_{i-1,j}^{n+1} + \varphi_{i-2,j}^{n+1}}{\Delta x^2}$$

$$+ \left((1-M_\infty^2) - (\gamma+1) M_\infty^2 \frac{\varphi_{i+1,j}^n - \varphi_{i-1,j}^{n+1}}{2\Delta x}\right) \frac{\varphi_{i+1,j}^n - 2\tilde{\varphi}_{i,j} + \varphi_{i-1,j}^{n+1}}{\Delta x^2}$$

$$+ \frac{\tilde{\varphi}_{i,j+1} - 2\tilde{\varphi}_{i,j} + \tilde{\varphi}_{i,j-1}}{\Delta y^2} = 0$$

- Over-Relaxation can be used whenever there is a part of "subsonic" flow, i.e. the equation has an "elliptic" component.
- The final values are then obtained as

$$\varphi^{n+1} = \varphi^n + \omega(\tilde{\varphi} - \varphi^n)$$

- The "Four-Operator Scheme" works well with SLOR.