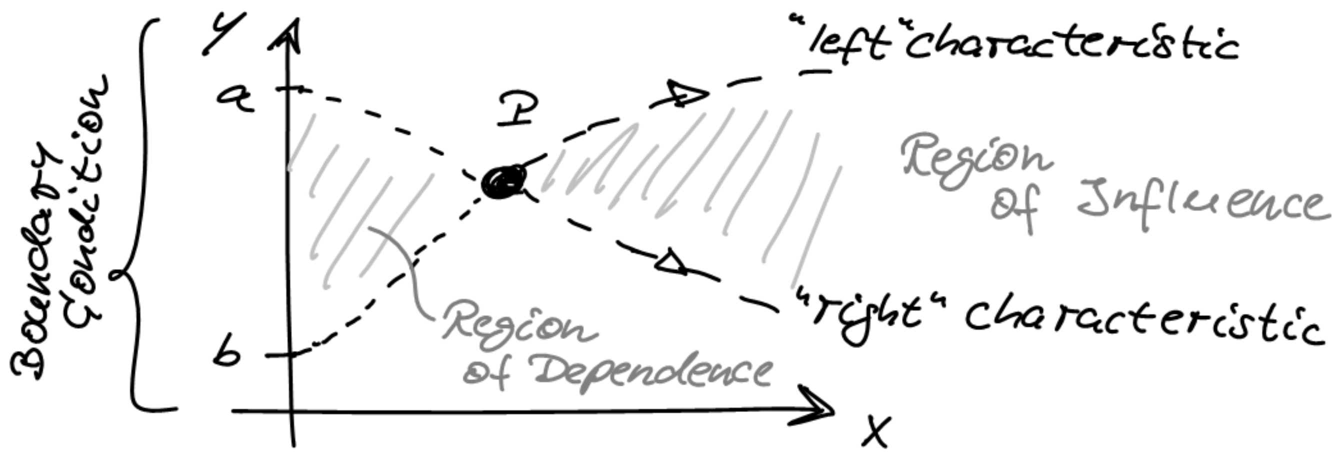
Integration of Linear Hyperbolic Equations

- Hyperbolic PDEs $\longrightarrow$ 2 real characteristics



- Solution @ P only influenced from the boundary that lies between a & b.

- Solution can be marched in x-direction starting @ x=0 (Boundary Condition).
  $\Longrightarrow$ x is time-like.

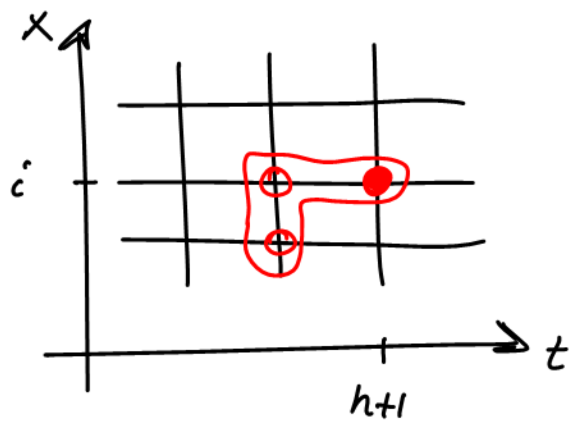
Example: Linear Convection Equation

$$\boxed{\frac{\partial u}{\partial t} + c\,\frac{\partial u}{\partial x} = 0} \quad , \quad c > 0 \qquad (1)$$

i) Numerical Scheme

$$\boxed{\frac{u_i^{n+1} - u_i^n}{\Delta t} + c\,\frac{u_i^n - u_{i-1}^n}{\Delta x} = 0} \qquad (2)$$

- The "Computational Molecule"



- Value @ new time step

$\Longrightarrow$ Scheme is explicit!

- Truncation Error TE : ε_i^n

$$u_i^{n+1} = u_i^n + \Delta t\,\frac{\partial u_i^n}{\partial t} + \frac{\Delta t^2}{2}\,\frac{\partial^2 u_i^n}{\partial t^2} + \mathcal{O}(\Delta t^3)$$

$$u_{i-1}^n = u_i^n - \Delta x\,\frac{\partial u_i^n}{\partial x} + \frac{\Delta x^2}{2}\,\frac{\partial^2 u_i^n}{\partial x^2} + \mathcal{O}(\Delta x^3)$$

$$\Longrightarrow \quad \frac{u_i^{n+1} - u_i^n}{\Delta t} = \frac{\partial u_i^n}{\partial t} + \frac{\Delta t}{2}\,\frac{\partial^2 u_i^n}{\partial t^2} + \mathcal{O}(\Delta t^2)$$

$$\frac{u_i^n - u_{i-1}^n}{\Delta x} = \frac{\partial u_i^n}{\partial x} - \frac{\Delta x}{2}\,\frac{\partial^2 u_i^n}{\partial x^2} + \mathcal{O}(\Delta x^2)$$

- Combine ..

$$\varepsilon_i^n = \cancel{\frac{\partial u_i^n}{\partial t}} + \frac{\Delta t}{2} \frac{\partial^2 u_i^n}{\partial t^2} + \mathcal{O}(\Delta t^2)$$

$$+ c \cdot \cancel{\frac{\partial u_i^n}{\partial x}} - c \frac{\Delta x}{2} \frac{\partial^2 u_i^n}{\partial x^2} + \mathcal{O}(\Delta x^2) \qquad (3)$$

$$\varepsilon_i^n = \mathcal{O}(\Delta t, \Delta x)$$

$\Longrightarrow$ Scheme is consistent and 1st order accurate in t and x.

- Try something else ...

(1) $\mapsto$ $\dfrac{\partial u}{\partial t} = -c \dfrac{\partial u}{\partial x} \quad \Big| \dfrac{\partial}{\partial t}(\dots)$

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial}{\partial t}\left(-c \frac{\partial u}{\partial x}\right) = -c \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial t}\right)$$

$$= c^2 \frac{\partial^2 u}{\partial x^2} \qquad (4)$$

- (4) in (3) yields

$$\varepsilon_i^n = \frac{\Delta t}{2} c^2 \frac{\partial^2 u}{\partial x^2} - c \frac{\Delta x}{2} \frac{\partial^2 u}{\partial x^2} + \mathcal{O}(\Delta t^2, \Delta x^2)$$

$$\boxed{\nu = c \frac{\Delta t}{\Delta x}} \qquad (5)$$

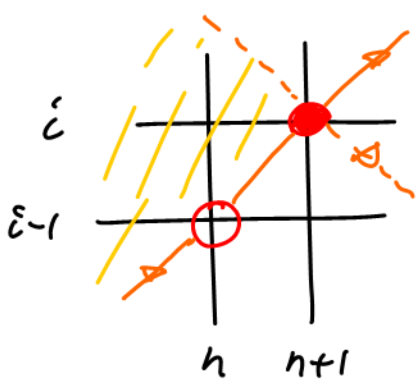
$$\varepsilon_i^n = \gamma\, c \frac{\Delta x}{2} \frac{\partial^2 u}{\partial x^2} - c \frac{\Delta x}{2} \frac{\partial^2 u}{\partial x^2} + O(\Delta t^2, \Delta x^2)$$

$$= c \frac{\Delta x}{2} (\gamma - 1) \frac{\partial^2 u}{\partial x^2} + O(\Delta t^2, \Delta x^2)$$

- Note that the scheme becomes of higher accuracy if $\gamma = 1$. – In fact, it can be shown with higher derivatives of (4) that it is exact.

- How is that possible?

$$\gamma = 1 \quad \Longrightarrow \quad \Delta t = \frac{\Delta x}{c}$$



$$\overset{(2)}{\Longrightarrow} \cancel{c}\, \frac{u_i^{n+1} - \cancel{u_i^n}}{\cancel{\Delta x}} + \cancel{c}\, \frac{\cancel{u_i^n} - u_{i-1}^n}{\cancel{\Delta x}} = 0$$

$$u_i^{n+1} = u_{i-1}^n \qquad ???$$

- Both points lie on the same characteristic.

  $\Longrightarrow$ A constant value of u is propagated along that characteristic.

  $\Longrightarrow$ Truncation error goes to zero.

- **Is that desirable ?**

Probably not, as the numerical solution is determined solely by the "left" characteristic of $(i-1, n)$ and not by the "right" characteristic of $(i+1, n)$.

- **Stability**

(2) in update form (explicit scheme)

$$\boxed{u_i^{n+1} = u_i^n - \gamma\left(u_i^n - u_{i-1}^n\right)} \qquad (2a)$$

Introduce the complex Fourier mode

$$u_i^n = g^n e^{\underline{i}\, i \beta}$$

$$g = 1 - \gamma\left(1 - e^{\underline{i}(-\beta)}\right)$$

$$= 1 - \gamma + \gamma\left(\cos(-\beta) + \underline{i} \sin(-\beta)\right)$$

$$= 1 - \gamma + \gamma \cos\beta + \underline{i}\, \gamma \sin\beta$$

$$= 1 - \gamma(1 - \cos\beta) + \underline{i}\, \gamma \sin\beta$$

$$\|g\|^2 \leq 1$$

$$[1-\gamma(1-\cos\beta)]^2+\gamma^2\sin^2\beta \leq 1$$

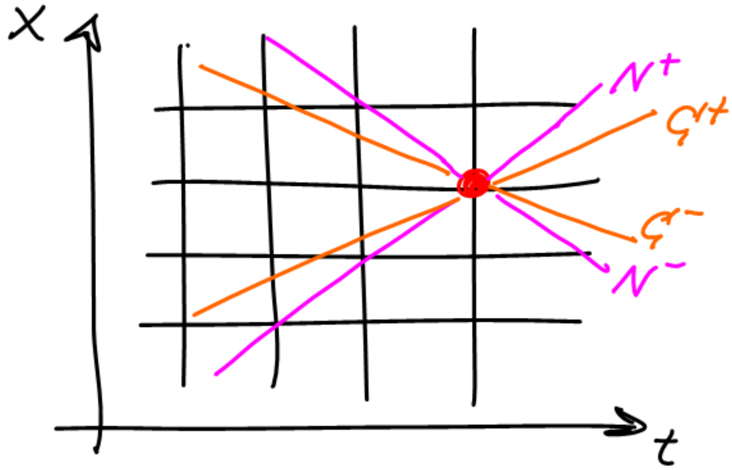
$$1-2\gamma(1-\cos\beta)+\gamma^2(1-2\cos\beta+\cos^2\beta)+\gamma^2\sin^2\beta \leq 1$$

$$\cancel{1}-2\gamma(1-\cos\beta)+\gamma^2+\gamma^2-2\gamma^2\cos\beta \leq \cancel{1}$$

$$-2\gamma(1-\cos\beta)+2\gamma^2(1-\cos\beta) \leq 0$$

$$\underbrace{-2\gamma(1-\cos\beta)}_{\leq 0}\underbrace{(1-\gamma)}_{! \geq 0} \leq 0 \qquad \boxed{\gamma \leq 1}$$

- *What does this mean for characteristics?*



$N \hat{=}$ Numerical characteristic

$\gamma \leq 1 \Longrightarrow \left|\left(\frac{dx}{dt}\right)_N\right| > c$

$C \hat{=}$ Mathematical characteristics

$\Longrightarrow$ For stability, the numerical characteristics have to include the mathematical ones.

$\boxed{\text{Numerics has to include "Region of Dependence".}}$

Example: Wave Equation

$$\boxed{\frac{\partial^2 u}{\partial t^2} - c^2 \frac{\partial^2 u}{\partial x^2} = 0} \qquad (6)$$

- Equation (6) is a 2^{nd} order PDE in x & t; it models the vibrations of a string instrument.
- Introduce $v = \partial u/\partial t$ and $w = \partial u/\partial x$ to transform (6) into a system of 1^{st} order PDEs.

$$\boxed{\begin{aligned} \frac{\partial v}{\partial t} - c^2 \frac{\partial w}{\partial x} &= 0 \\ \frac{\partial w}{\partial t} - \frac{\partial v}{\partial x} &= 0 \end{aligned}} \qquad (6a)$$

- Write in matrix form: $\bar{W} = \begin{Bmatrix} v \\ w \end{Bmatrix}$

$$\underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}}_{[K]} \frac{\partial \bar{W}}{\partial t} + \underbrace{\begin{bmatrix} 0 & -c^2 \\ -1 & 0 \end{bmatrix}}_{[\eta]} \frac{\partial \bar{W}}{\partial x} = 0$$

- Use "Eigenvalue Method" for classification

$$[K]^{-1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \Rightarrow [N] = [K]^{-1}[M]$$

$$[N] = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & -c^2 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & -c^2 \\ -1 & 0 \end{pmatrix}$$

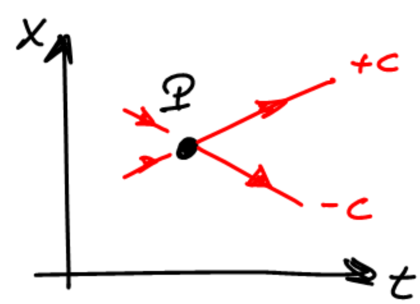
- Now find eigenvalues of $[N]$...

$$\left|[N] - \lambda[I]\right| = \left|\begin{pmatrix} -\lambda & -c^2 \\ -1 & -\lambda \end{pmatrix}\right| \stackrel{!}{=} 0$$

$$\lambda^2 - c^2 = 0 \Rightarrow \underline{\lambda = \pm c}$$

- All eigenvalues are <u>real</u> $\Rightarrow$ <u>"Hyperbolic"</u>

- The <u>slope</u> of the characteristics is $\underline{\left(\frac{dx}{dt}\right) = \pm c}$ (7)



- Integrating (7) we can solve for the shape of the characteristics $C^{\pm}$ as:

$$\boxed{x \mp ct = \text{const.}} \qquad (8)$$

The characteristics are straight lines in the x-t plane.

- What determines the const. value along $C^{\pm}$?

  → The initial/boundary conditions for u & v.

- How ? → Let's look at the exact solution

→ Combine (6a) such that

$$\left(\frac{\partial v}{\partial t} - c^2 \frac{\partial w}{\partial x}\right) - c\left(\frac{\partial w}{\partial t} - \frac{\partial v}{\partial x}\right) = 0$$

$$\boxed{\frac{\partial}{\partial t}(v - c\cdot w) + c\frac{\partial}{\partial x}(v - c\cdot w) = 0} \qquad (9)$$

→ Combine (6a) such that

$$\left(\frac{\partial v}{\partial t} - c^2 \frac{\partial w}{\partial x}\right) + c\left(\frac{\partial w}{\partial t} - \frac{\partial v}{\partial x}\right) = 0$$

$$\boxed{\frac{\partial}{\partial t}(v + c\cdot w) - c\frac{\partial}{\partial x}(v + c\cdot w) = 0} \qquad (10)$$

→ One can notice that for (9), (10) the substantial derivative appears as

$$\boxed{\frac{D}{Dt}(\dots) = \frac{\partial}{\partial t}(\dots)\frac{dt}{dt} + \frac{\partial}{\partial x}(\dots)\frac{dx}{dt}} \qquad (11)$$

We know that ...

$$\left(\frac{dx}{dt}\right)^{C^+} = +c \longrightarrow x - ct = \text{const.}$$

$$\left(\frac{dx}{dt}\right)^{C^-} = -c \longrightarrow x + ct = \text{const.}$$

→ Thus, we can conclude that (9), (10) are valid along the characteristics such that

$$\begin{cases} \frac{\partial}{\partial t}(v - c\cdot w) + c\cdot\frac{\partial}{\partial x}(v - c\cdot w) = 0 \quad , \text{ on } C^+ & (12a) \\ \frac{\partial}{\partial t}(v + c\cdot w) - c\cdot\frac{\partial}{\partial x}(v + c\cdot w) = 0 \quad , \text{ on } C^- & (12b) \end{cases}$$

→ It becomes clear that (12a), (12b) are satisfied if

$$\underline{v - c\cdot w = \text{const. on } C^+} \; ; \; \underline{v + c\cdot w = \text{const. on } C^-}$$

→ This can be true if

$$\begin{cases} v - c\cdot w = f(x - c\cdot t) & \text{on } C^+ \quad (13a) \\ v + c\cdot w = g(x + c\cdot t) & \text{on } C^- \quad (13b) \end{cases}$$

where f and g are arbitrary functions of single arguments $x - c\cdot t$ and $x + c\cdot t$ that are constant along the respective characteristic C^+ or C^-.

→ Substituting $v = \frac{\partial u}{\partial t}$ and $w = \frac{\partial u}{\partial x}$ into (13a), (13b) and using the same strategy we can integrate to obtain the exact solution for $u(x,t)$ as:

$$\boxed{u(x,t) = F(x - c\cdot t) + G(x + c\cdot t)} \quad (14)$$

→ Equation (14) is d'Alembert's solution to the wave equation (6).

Example: Linearized Supersonic Potential Flow

$$\boxed{-\beta^2 \frac{\partial^2 \varphi}{\partial x^2} + \frac{\partial^2 \varphi}{\partial y^2} = 0} \qquad (15)$$

where $\beta = \sqrt{M_0^2 - 1} > 0$ as $M_0 \geq 1$.

- φ is the "perturbation potential" and the actual (physical) velocity components are:

$$\boxed{\begin{cases} u = u_0 \left(1 + \frac{\partial \varphi}{\partial x}\right) \\ v = u_0 \frac{\partial \varphi}{\partial y} \end{cases}} \qquad (16)$$

- Let us compare (15) to the wave equation (6)

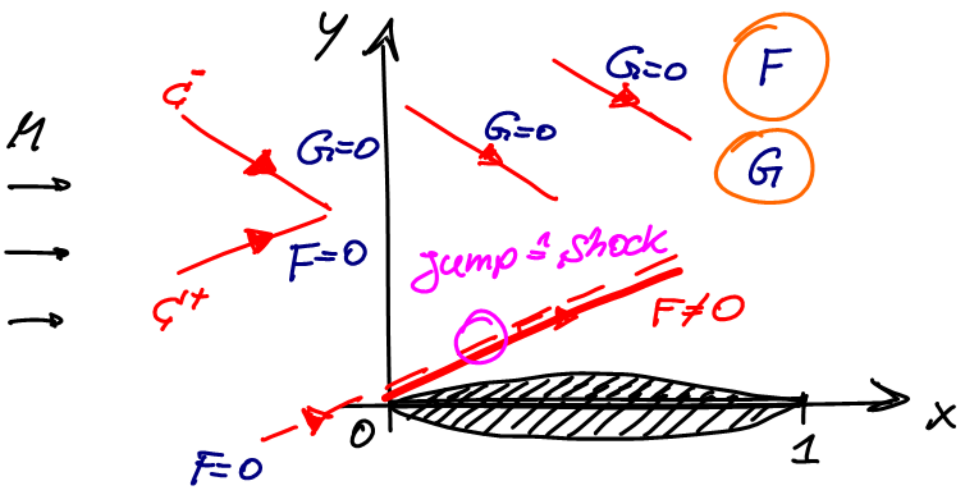
$$\frac{\partial^2 u}{\partial t^2} - c^2 \frac{\partial^2 u}{\partial x^2} = 0$$

we realize that (15) is a hyperbolic equation with characteristics $C'^+ : \left(\frac{dx}{dy}\right) = +\beta$

$$C'^- : \left(\frac{dx}{dy}\right) = -\beta$$

- Hence the general solution for the perturbation potential $\varphi(x,y)$ is of the form

$$\boxed{\varphi(x,y) = F(x-\beta y) + G(x+\beta y)} \qquad (17)$$



C^+: $x-\beta y = \text{const.}$

C^-: $x+\beta y = \text{const.}$

<u>Parabolic Profile:</u>

$f^{\pm}(x) = \pm 2 e_m x(1-x)$

- <u>x is time-like, y is space-like.</u>

- <u>Initial condition:</u> $\varphi(x,y) = 0$ for $x<0$

(17) $\rightarrow$ $\quad 0 = F(x-\beta\cdot y) + G(x+\beta y)$

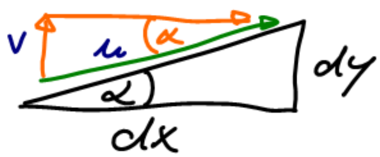
$$\forall\, x<0;\, y$$

$\Longrightarrow$ can only be generally true for

$$\underline{F = G = 0} \qquad \forall\, x<0;\, y$$

- On the upper half: $G = 0$

→ However, $F=0$ will be "blocked" by the profile for $x > 0$. A new boundary condition is introduced by the profile that satisfies flow tangency. (zero flow ⊥ surface)

→ 

$$\tan\alpha = \frac{\Delta y}{\Delta x} = f'(x)$$
$$\approx$$
$$\sin\alpha = v/u \approx v/u_0$$

$$\boxed{\frac{\partial \varphi}{\partial y} \overset{!}{=} f'(x) \;;\; 0 \le x \le 1 \;;\; y = 0} \quad (18)$$

- As $G=0$, equation (17) reduces to

$$\boxed{\varphi(x,y) = F(x - \beta y) \quad ; \; 0 \le x \le 1 \; ; \; y = 0} \quad (17a)$$

- Taking $\frac{\partial}{\partial y}(\cdots)$ of (17a) we obtain

$$\boxed{\frac{\partial \varphi}{\partial y}(x,0) = -\beta F'(x) = \Theta} \quad (17b)$$

which is an ODE for F.

- Integration of (17b) yields

$$F(x) = -\frac{\Theta}{\beta}\cdot x + C_1 \qquad (17c)$$

where we can determine the integration constant by noting that (still) $\varphi(0,0)=0$.

Hence, $F(0) \overset{!}{=} 0 \implies \underline{C_1 = 0}$

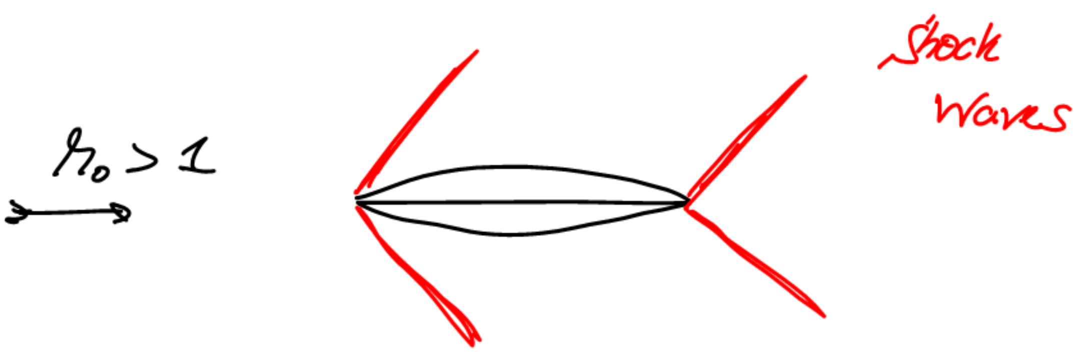
$$F(x) = -\frac{\Theta}{\beta}\cdot x \quad \Big\} \text{ Functional form of } F$$

or $F(\xi) = -\frac{\Theta}{\beta}\cdot \xi \quad ; \quad \xi \geq 0$

$$\boxed{\varphi(x,y) = -\frac{\Theta}{\beta}(x-\beta y)} \qquad (19)$$

__Example:__ Wave Drag of Parabolic Profile @ zero incidence

$f^{\pm} = \pm 2 e_m x(1-x) \quad ; \quad e_m \hat{=} \text{ thickness}$



- Consider upper half only.

Exact Solution: (18) $\mapsto$

$$\frac{\partial\varphi}{\partial y}(x,0) = f'^{+}(y) \Rightarrow \boxed{\begin{aligned} f'^{+}(x) &= 2e_m\,[1-x+x(-1)] \\ &= 2e_m\,(1-2x) \end{aligned}} \quad (20)$$

(17b) $\mapsto$
$$2e_m\,(1-2x) = -\beta\,F'(x)$$

$$F'(x) = -\frac{2e_m}{\beta}(1-2x)$$

$$F(x) = -\frac{2e_m}{\beta}(x-x^2) + C_1$$

$$= -\frac{2e_m}{\beta}x(1-x) + C_1$$

$$\varphi(0,0) = 0 \Rightarrow F(0) = 0 \Rightarrow C_1 = 0$$

$$\boxed{\varphi(x,y) = F(x-\beta y) = -\frac{2e_m}{\beta}(x-\beta y)[1-(x-\beta y)]} \quad (21)$$

- The wave drag for a symmetric profile is obtained from

$$\boxed{C_d = -4\int_0^1 \frac{\partial\varphi(x,0)}{\partial x} f'(x)\,dx} \quad (22)$$

• From (21) we find that

$$\frac{\partial\varphi}{\partial x} = -\frac{2e_m}{\beta}\left\{1\left[1-(x-\beta y)\right]+(x-\beta y)(-1)\right\}$$

$$\boxed{\frac{\partial\varphi(x,0)}{\partial x} = -\frac{2e_m}{\beta}\left\{1-x-x\right\} = -\frac{2e_m}{\beta}(1-2x)} \qquad (23)$$

• (20), (23) in (22)

$$C_d = -4\int_0^1 -\frac{2e_m}{\beta}(1-2x)\cdot 2e_m(1-2x)\,dx$$

$$= \frac{16e_m^2}{\beta}\int_0^1 (1-2x)^2 dx = \frac{16e_m^2}{\beta}\int_0^1 1-4x+4x^2\,dx$$

$$= \frac{16e_m^2}{\beta}\left[x-2x^2+\frac{4}{3}x^3\right]\Big|_0^1 = \frac{16e_m^2}{\beta}\Big[\underbrace{1-2+\frac{4}{3}}_{=\frac{1}{3}}\Big]$$

$$\Longrightarrow \boxed{C_d = \frac{16}{3}\frac{e_m^2}{\beta}} \qquad (24)$$

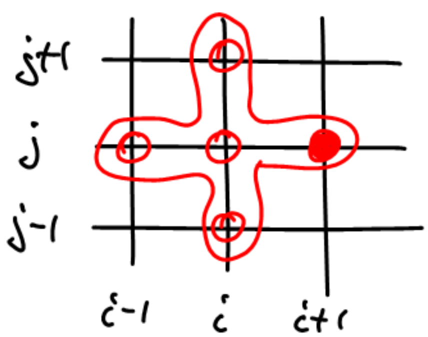
Numerical Solution of Equation (15)

Discretization: (Explicit Scheme)

$$\boxed{-\beta^2 \frac{\varphi_{i+1,j} - 2\varphi_{i,j} + \varphi_{i-1,j}}{\Delta x^2} + \frac{\varphi_{i,j+1} - 2\varphi_{i,j} + \varphi_{i,j-1}}{\Delta y^2} = 0} \qquad (25)$$

→ This is a central difference scheme. It is 2nd order accurate in x & y. It is stable for $\gamma = \frac{\Delta x}{\beta \Delta y} \leq 1$.

→ The "Computational Molecule" is:

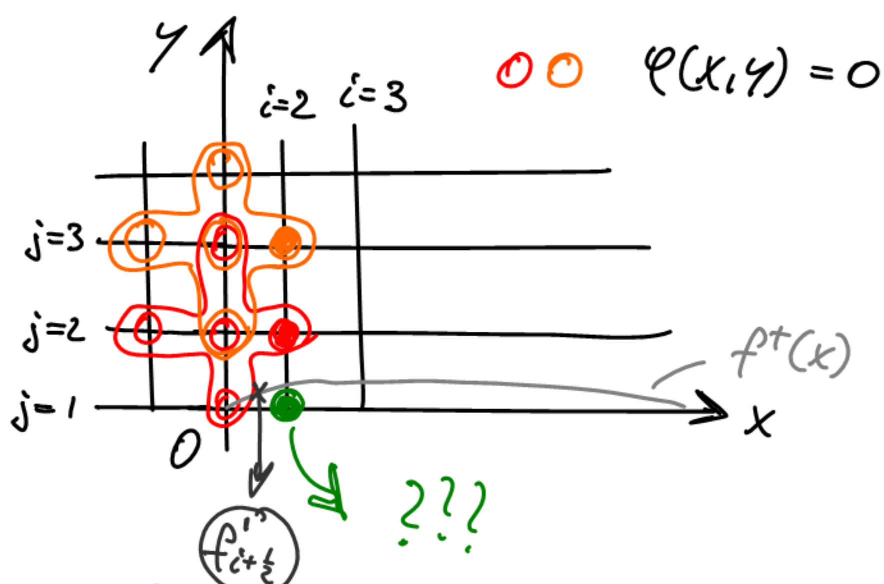


The solution is marched in x. But what happens on the profile wall?

- Boundary/Tangency Condition (18)

$$\frac{\varphi_{i+1,2} - \varphi_{i+1,1}}{\Delta y} = f'_{i+\frac{1}{2}}$$

$\Longrightarrow$ ● = ...



How to integrate for the wave drag?

→ Write (22) in discrete form:

$$\boxed{C_d = -4 \sum_{i=1}^{ix-1} \left(\varphi_{i+1,1} - \varphi_{i,1} \right) f'_{i+\frac{1}{2}}} \qquad (26)$$

How to write a computer code?

Initialization

$$dy = \frac{y_{max}}{jx-1}$$

$$phi(i,j) = 0$$

$ix = \ldots$ read in

$$dx = \frac{1}{ix-1}$$

$$\mu_0 = \sqrt{1 + \left(\frac{dx}{dy}\right)^2} \qquad (\tau = 1)$$

$$C_d = 0$$

Solve the problem

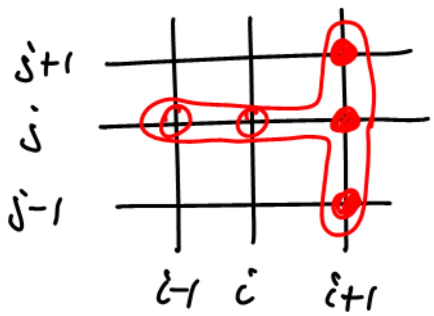
```
do 1    i = 2, ix
   do 2    j = 2, jx-1
      phi(i,j) = ....          (update form)
2  continue
      j = 1
      phi(i,j) = ....          (boundary condition)
      Cd = Cd + ...
1  continue
```

Write & plot results.

Discretization: (Implicit Scheme)

$$\boxed{-\beta^2 \frac{\varphi_{i+1,j} - 2\varphi_{i,j} + \varphi_{i-1,j}}{\Delta x^2} + \frac{\varphi_{i+1,j+1} - 2\varphi_{i+1,j} + \varphi_{i+1,j-1}}{\Delta y^2} = 0} \tag{27}$$

→ The "Computational Molecule" is :



The solution is marched in x.
3 new values are computed at once!

- Truncation Error TE : $\varepsilon_{i+1,j}$

→ $$\frac{\varphi_{i+1,j+1} - 2\varphi_{i+1,j} + \varphi_{i+1,j-1}}{\Delta y^2} = \frac{\partial^2 \varphi_{i+1,j}}{\partial y^2} + \frac{\Delta y^2}{4!}\frac{\partial^4 \varphi_{i+1,j}}{\partial y^4} + \mathcal{O}(\Delta y^4)$$

→ $$\varphi_{i,j} = \varphi_{i+1,j} - \Delta x \frac{\partial \varphi_{i+1,j}}{\partial x} + \frac{\Delta x^2}{2}\frac{\partial^2 \varphi_{i+1,j}}{\partial x^2} - \frac{\Delta x^3}{6}\frac{\partial^3 \varphi_{i+1,j}}{\partial x^3} + \mathcal{O}(\Delta x^4)$$

$$\varphi_{i-1,j} = \varphi_{i+1,j} - (2\Delta x)\frac{\partial \varphi_{i+1,j}}{\partial x} + (2\Delta x^2)\frac{\partial^2 \varphi_{i+1,j}}{\partial x^2} - \left(\frac{4}{3}\Delta x^3\right)\frac{\partial^3 \varphi_{i+1,j}}{\partial x^3} + \mathcal{O}(\Delta x^4)$$

⇒ Now construct $\frac{\partial^2 \varphi_{i+1,j}}{\partial x^2}$ . . .

$$\frac{\varphi_{i+1,j} - 2\varphi_{i,j} + \varphi_{i-1,j}}{\Delta x^2}$$

$$= \frac{1}{\Delta x^2}\Big[\cancel{\varphi_{i+1,j}} - 2\Big(\cancel{\varphi_{i+1,j}} - \cancel{\Delta x \frac{\partial \varphi_{i+1,j}}{\partial x}} + \frac{\Delta x^2}{2} \frac{\partial^2 \varphi_{i+1,j}}{\partial x^2}$$

$$- \frac{\Delta x^3}{6} \frac{\partial^3 \varphi_{i+1,j}}{\partial x^3} + O(\Delta x^4)\Big) + \cancel{\varphi_{i+1,j}} - \cancel{(2\Delta x)\frac{\partial \varphi_{i+1,j}}{\partial x}}$$

$$+ (2\Delta x^2) \frac{\partial^2 \varphi_{i+1,j}}{\partial x^2} - \left(\frac{4}{3}\Delta x^3\right) \frac{\partial^3 \varphi_{i+1,j}}{\partial x^3} + O(\Delta x^4) \Big]$$

$$= \frac{1}{\Delta x^2}\Big[\Delta x^2 \frac{\partial^2 \varphi_{i+1,j}}{\partial x^2} - (3\Delta x^3) \frac{\partial^3 \varphi_{i+1,j}}{\partial x^3} + O(\Delta x^4)\Big]$$

$$= \frac{\partial^2 \varphi_{i+1,j}}{\partial x^2} - (3\Delta x) \frac{\partial^3 \varphi_{i+1,j}}{\partial x^3} + O(\Delta x^2)$$

→ Now we can write the combined TE $\varepsilon_{i+1,j}$ as:

$$\varepsilon_{i+1,j} = -\beta^2\Big(\cancel{\frac{\partial^2 \varphi_{i+1,j}}{\partial x^2}} - (3\Delta x) \frac{\partial^3 \varphi_{i+1,j}}{\partial x^3} + O(\Delta x^2)\Big)$$

$$+ \cancel{\frac{\partial^2 \varphi_{i+1,j}}{\partial y^2}} + \frac{\Delta y^2}{4!} \frac{\partial^4 \varphi_{i+1,j}}{\partial y^4} + O(\Delta y^4)$$

Exact Solution

$$\varepsilon_{i+1,j} = 3\beta^2 \Delta x \frac{\partial^3 \varphi_{i+1,j}}{\partial x^3} + \mathcal{O}(\Delta x^2)$$

$$+ \frac{\Delta y^2}{4!} \frac{\partial^4 \varphi_{i+1,j}}{\partial y^4} + \mathcal{O}(\Delta y^2)$$

$$= \mathcal{O}(\Delta x, \Delta y^2)$$

$\Longrightarrow$ The implicit scheme in (27) is 1st order accurate in x and 2nd order accurate in y.

- Stability:

Complex Fourier mode: $\boxed{\varphi_{i,j} = g^i e^{\imath j \beta}}$ (28)

(i) is time-like.

(j) is space-like.

$\rightarrow$ Write (27) as:

$$\boxed{\begin{aligned} &\varphi_{i+1,j+1} - 2\varphi_{i+1,j} + \varphi_{i+1,j-1} \\ &= \left(\frac{\beta \Delta y}{\Delta x}\right)^2 \left(\varphi_{i+1,j} - 2\varphi_{i,j} + \varphi_{i-1,j}\right) \end{aligned}} \quad (29)$$

• (28) in (29) yields $\qquad \sigma = \frac{\beta \Delta y}{\Delta x}$

$$g^{i+1} e^{i(j+1)\beta} - 2 g^{i+1} e^{i j \beta} + g^{i+1} e^{i(j-1)\beta}$$

$$= \sigma^2 \left(g^{i+1} e^{i j \beta} - 2 g^{i} e^{i j \beta} + \underline{g^{i-1} e^{i j \beta}} \right)$$

$\longrightarrow$ Divide by $g^{i-1} e^{i j \beta}$

$$g^2 \left(e^{i\beta} - 2 + e^{i(-\beta)} \right) = \sigma^2 \left(g^2 - 2g + 1 \right)$$

$$g^2 \left(\cos\beta + \cancel{i \sin\beta} - 2 + \cos(-\beta) + \cancel{i \sin(-\beta)} \right)$$

$$= \sigma^2 \left(g^2 - 2g + 1 \right)$$

$$2g^2 (\cos\beta - 1) = \sigma^2 \left(g^2 - 2g + 1 \right)$$

$$g^2 \left(\sigma^2 + 2(1 - \cos\beta) \right) - 2\sigma^2 g + \sigma^2 = 0$$

$$\left[1 + \frac{2(1-\cos\beta)}{\sigma^2} \right] g^2 - 2g + 1 = 0$$

$$\overbrace{\;}^{a} g^2 \overbrace{- 2 \frac{\sigma^2}{\sigma^2 + 2(1-\cos\beta)}}^{b} g + \overbrace{\frac{\sigma^2}{\sigma^2 + 2(1-\cos\beta)}}^{c} = 0$$

$$\boxed{g = \frac{-b \pm \sqrt{b^2-4ac}}{2a}} \quad (30) \qquad ; \qquad \boxed{\begin{array}{l} a = 1 \\ b < 0 \\ c = -\tfrac{1}{2}\cdot b \geq 0 \end{array}} \quad (31)$$

→ Let us investigate the Discriminant

$$b^2 - 4ac = b^2 + 2b = b(b+2)$$

$$\underline{b(b+2) \geq 0}$$

$$(b \leq 0 \wedge b+2 \leq 0) \vee \cancel{(b \geq 0 \wedge b+2 \geq 0)}$$

NO.

$$b \leq -2 \Rightarrow \frac{\sigma^2}{\sigma^2 + 2(1-\cos\beta)} \geq 1 \Rightarrow \underbrace{2(1-\cos\beta)}_{\geq 0} \leq 0$$

⟹ Not possible!

$$\underline{b(b+2) \leq 0}$$

$$(b \leq 0 \wedge b+2 \geq 0) \vee \cancel{(b \geq 0 \wedge b+2 \leq 0)}$$

No.

$$b \leq 0 \wedge b \geq -2$$

$$b \leq 0 \wedge \frac{\sigma^2}{\sigma^2 + 2(1-\cos\beta)} \leq 1$$

$$b \leq 0 \quad \wedge \quad 2(1-\cos\beta) \geq 0$$

That's ok !

$\Longrightarrow$ g is complex.

$$g = -\frac{b}{2} \pm \frac{1}{2} i \sqrt{-b(b+2)}$$

$$\|g\|^2 = \frac{b^2}{4} - \frac{b^2}{4} - \frac{b}{2} \leq 1$$

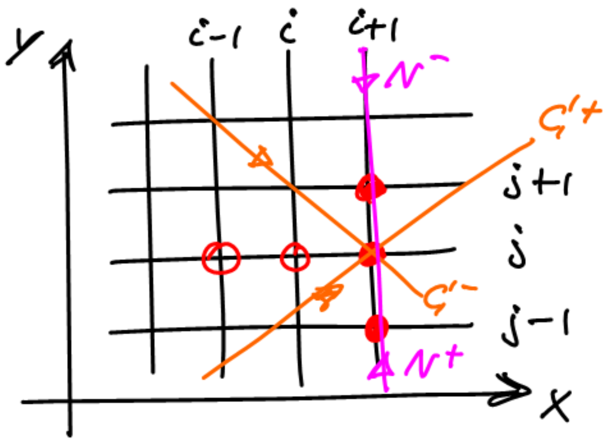
$$-\frac{b}{2} \leq 1 \quad \Longrightarrow \quad \frac{\sigma^2}{\sigma^2 + 2(1-\cos\beta)} \leq 1$$

$$2(1-\cos\beta) \geq 0$$

Yes, always true !

$\Longrightarrow$ The implicit scheme is unconditionally stable !

- What does this mean for the characteristics?



$C \hat{=}$ Mathematical characteristics

Slope: $\pm\beta$

$N \hat{=}$ Numerical Characteristics

Infinite slope $\sigma = \frac{\beta \Delta y}{\Delta x}$

$\Longrightarrow$ For an implicit scheme, the numerical characteristics always include the mathematical ones. – Hence, the implicit scheme is unconditionally stable.

- Any disadvantage of "implicit" schemes?

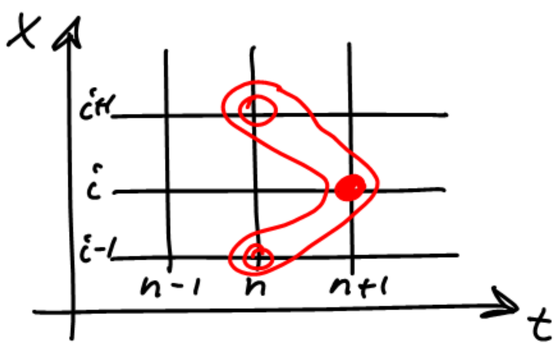
Accuracy in the marching direction.

Example: "Numerical Schemes for the Linear Convection Equation"

$$\boxed{\frac{\partial u}{\partial t} + c \cdot \frac{\partial u}{\partial x} = 0} \quad ; \quad c > 0 \qquad (1)$$

The Lax Scheme

$$\boxed{\frac{u_i^{n+1} - \frac{u_{i-1}^n + u_{i+1}^n}{2}}{\Delta t} + c\,\frac{u_{i+1}^n - u_{i-1}^n}{2\Delta x} = 0} \qquad (33)$$



"Computational Molecule"

→ see lecture on "Basics of Finite Difference Method"

- The truncation error TE reads:

$$\varepsilon_i^n = \frac{\Delta t}{2}\frac{\partial^2 u_i^n}{\partial t^2} - \frac{\Delta x^2}{2\Delta t}\frac{\partial^2 u_i^n}{\partial x^2} + \mathcal{O}\left(\Delta x^2, \frac{\Delta x^4}{\Delta t}\right)$$

$$= \mathcal{O}\left(\Delta t, \frac{\Delta x^2}{\Delta t}, \Delta x^2\right)$$

Therefore, Lax's scheme is conditionally consistent.

- It is stable under the classical CFL condition with $\Delta t \le \frac{\Delta x}{c}$.

The Lax-Wendroff Scheme

- This scheme is obtained from a systematic procedure. Let us consider the following Taylor expansion of u_i^{n+1} :

$$\boxed{u_i^{n+1} = u_i^n + \Delta t \frac{\partial u_i^n}{\partial t} + \frac{\Delta t^2}{2} \frac{\partial^2 u_i^n}{\partial t^2} + O(\Delta t^3)} \quad (34)$$

- From (1) we know that

$$\frac{\partial u_i^n}{\partial t} = -c \frac{\partial u_i^n}{\partial x}$$

- From (4) we know that

$$\frac{\partial^2 u_i^n}{\partial t^2} = c^2 \frac{\partial^2 u_i^n}{\partial x^2}$$

- Use in (34) to obtain

$$u_i^{n+1} = u_i^n - c \cdot \Delta t \frac{\partial u_i^n}{\partial x} + c^2 \frac{\Delta t^2}{2} \frac{\partial^2 u_i^n}{\partial x^2} + O(\Delta t^3)$$

- Rearrange to find

$$\underbrace{\frac{u_i^{n+1} - u_i^n}{\Delta t}}_{C} + c \cdot \underbrace{\frac{\partial u_i^n}{\partial x}}_{A} - c^2 \frac{\Delta t}{2} \underbrace{\frac{\partial^2 u_i^n}{\partial x^2}}_{B} = 0 \quad (35)$$

- Use 2nd order Finite Difference Approximations for A and B.

$$A \quad \frac{u_{i+1}^n - u_{i-1}^n}{2\Delta x} = \frac{\partial u_i^n}{\partial x} + \frac{\Delta x^2}{3!} \frac{\partial^3 u_i^n}{\partial x^3} + O(\Delta x^4) \quad (36)$$

$$B \quad \frac{u_{i+1}^n - 2u_i^n + u_{i-1}^n}{\Delta x^2} = \frac{\partial^2 u_i^n}{\partial x^2} + \frac{\Delta x^2}{4!} \frac{\partial^4 u_i^n}{\partial x^4} + O(\Delta x^4) \quad (37)$$

- Now write (35) as:

$$\boxed{\frac{u_i^{n+1} - u_i^n}{\Delta t} + c \frac{u_{i+1}^n - u_{i-1}^n}{2\Delta x} - c^2 \frac{\Delta t}{2} \frac{u_{i+1}^n - 2u_i^n + u_{i-1}^n}{\Delta x^2} = 0} \quad (38)$$

- Is (38) a valid Finite Difference Approximation of (1)? — How do you know?

→ Recall that C

$$\frac{u_i^{n+1} - u_i^n}{\Delta t} = \frac{\partial u_i^n}{\partial t} + \frac{\Delta t}{2} \boxed{\frac{\partial^2 u_i^n}{\partial t^2}} + \frac{\Delta t^2}{3!} \frac{\partial^3 u_i^n}{\partial t^3} + O(\Delta t^3)$$

$$\frac{u_i^{n+1} - u_i^n}{\Delta t} = \frac{\partial u_i^n}{\partial t} + c^2 \frac{\Delta t}{2} \frac{\partial^2 u_i^n}{\partial x^2} + \mathcal{O}(\Delta t^2) \qquad (39)$$

- Now determine the truncation error ε_i^n

$$\varepsilon_i^n = \cancel{\frac{\partial u_i^n}{\partial t}} + \cancel{c^2 \frac{\Delta t}{2} \frac{\partial^2 u_i^n}{\partial x^2}} + \frac{\Delta t^2}{3!} \frac{\partial^3 u_i^n}{\partial t^3} + \mathcal{O}(\Delta t^3)$$
$$+ c\left(\cancel{\frac{\partial u_i^n}{\partial x}} + \frac{\Delta x^2}{3!} \frac{\partial^3 u_i^n}{\partial x^3} + \mathcal{O}(\Delta x^4)\right)$$
$$- c^2 \frac{\Delta t}{2} \left(\cancel{\frac{\partial^2 u_i^n}{\partial x^2}} + \frac{\Delta x^2}{4!} \frac{\partial^4 u_i^n}{\partial x^4} + \mathcal{O}(\Delta x^4)\right)$$

$$\varepsilon_i^n = \mathcal{O}(\Delta t^2, \Delta x^2)$$

$\Longrightarrow$ The Lax-Wendroff scheme is consistent and 2nd order accurate in time & space.

The 2nd order term in the FDE (38) acts as "artificial viscosity".

- How about stability?

Complex Fourier mode: $u_i^n = g^n e^{i i \beta}$

→ We define $\boxed{\sigma = \frac{c\Delta t}{\Delta x}}$ and write (38)

in update form as:

$$\boxed{u_i^{n+1} = u_i^n - \frac{\sigma}{2}\left(u_{i+1}^n - u_{i-1}^n\right) + \frac{\sigma^2}{2}\left(u_{i+1}^n - 2u_i^n + u_{i-1}^n\right)}$$

(38a)

$$g^{n+1} e^{\hat{\imath} i\beta} = g^n e^{\hat{\imath} i\beta} - \frac{\sigma}{2}\left(g^n e^{\hat{\imath}(i+1)\beta} - g^n e^{\hat{\imath}(i-1)\beta}\right)$$

$$+ \frac{\sigma^2}{2}\left(g^n e^{\hat{\imath}(i+1)\beta} - 2g^n e^{\hat{\imath} i\beta} + g^n e^{\hat{\imath}(i-1)\beta}\right)$$

... divide by $g^n e^{\hat{\imath} i\beta}$

$$g = 1 - \frac{\sigma}{2}\left(e^{\hat{\imath}\beta} - e^{\hat{\imath}(-\beta)}\right) + \frac{\sigma^2}{2}\left(e^{\hat{\imath}\beta} - 2 + e^{\hat{\imath}(-\beta)}\right)$$

$$g = 1 - \frac{\sigma}{2}\left(\cancel{\cos\beta} + \underline{\hat{\imath}}\sin\beta - \cancel{\cos(-\beta)} - \underline{\hat{\imath}}\sin(-\beta)\right)$$

$$+ \frac{\sigma^2}{2}\left(\cos\beta + \cancel{\hat{\imath}\sin\beta} - 2 + \cos(-\beta) + \cancel{\hat{\imath}\sin(-\beta)}\right)$$

$$g = \left(1 + \sigma^2(\cos\beta - 1)\right) - \underline{\hat{\imath}}\,\sigma\sin\beta$$

$$\boxed{g = \left(1 - \sigma^2(1 - \cos\beta)\right) - \underline{\hat{\imath}}\,\sigma\sin\beta}$$

$$\|g\|^2 = \left(1 - \gamma^2(1-\cos\beta)\right)^2 + \gamma^2 \sin^2\beta$$

$$= \cancel{1} - 2\gamma^2(1-\cos\beta) + \gamma^4(1-\cos\beta)^2 + \gamma^2\sin^2\beta$$

$$\leq \cancel{1}$$

$$-2\gamma^2(1-\cos\beta) + \gamma^4(1-\cos\beta)^2 + \gamma^2(1-\cos^2\beta) \leq 0$$

$$-2\cancel{\gamma^2}\cancel{(1-\cos\beta)} + \gamma^{\cancel{4}\,2}(1-\cos\beta)^{\cancel{2}} + \cancel{\gamma^2}\cancel{(1-\cos\beta)}(1+\cos\beta) \leq 0$$

$$-2 + \gamma^2(1-\cos\beta) + (1+\cos\beta) \leq 0$$

$$\gamma^2\cancel{(1-\cos\beta)} - \overset{1}{\cancel{(1-\cos\beta)}} \leq 0$$

$$\gamma^2 - 1 \leq 0$$

$$\gamma^2 \leq 1 \quad \Longrightarrow \quad \boxed{|\gamma| \leq 1}$$

$\Longrightarrow$ The Lax-Wendroff scheme is stable under the classical CFL condition.

Its advantage is 2^{nd} order accuracy in time (marching direction) w/o compromising on stability.

The MacCormack Scheme

- Classic CFD scheme for hyperbolic and parabolic PDEs. (MacCormack, 1969)
- It is a two-step predictor-corrector scheme.

"Predictor" $$\overline{u_i^{n+1}} = u_i^n - c\,\Delta t\,\frac{u_{i+1}^n - u_i^n}{\Delta x} \qquad (39)$$

"Corrector" $$u_i^{n+1} = \frac{1}{2}\left(u_i^n + \overline{u_i^{n+1}} - c\Delta t\,\frac{\overline{u_i^{n+1}} - \overline{u_{i-1}^{n+1}}}{\Delta x}\right) \qquad (40)$$

- In case of the "Linear Convection Equation" (1), it can be shown that the scheme is identical to the "Lax-Wendroff" scheme.

- This technique can be applied to other equations than the Linear Convection Equation. It was the "standard" CFD technique until the mid 80's.