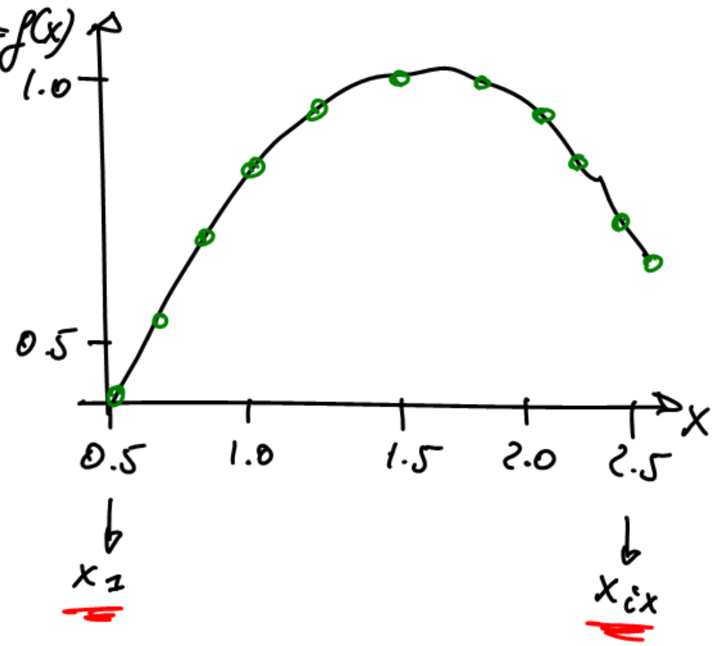
Basics of the Finite-Difference Method

1.) Representing Functions by Discrete Values

- Let $u = f(x)$ be a continuous & differentiable function to a sufficient order in the interval $[a, b]$. A discretization of $[a,b]$ with constant step h is introduced:



$$\left.\begin{aligned} & x_1 = a \;;\; x_{ix} = b \\ & x_i = a + (i-1)\cdot h \\ & h = \frac{b-a}{ix - 1} \end{aligned}\right\} \quad (1)$$

- In (1), ix is the total number of points of the discretization, h is the discretization step.

$$\Longrightarrow \boxed{\text{Let } u_i = f(x_i)}$$

- Note: Going from the continuous $\longrightarrow$ discrete carries w/ it a loss of information. If discrete values $f(x_i)$ are given, it is <u>not</u> possible to find a unique $u = f(x)$ such that $u_i = f(x_i)$ w/o further information about f (e.g. polynomial, Fourier Series).

- In the Finite-Difference (FD) method, there is <u>no</u> hypothesis concerning the variation of $f(x)$ between the discrete points x_i.
- This is in contrast to Finite-Volume (FV) and Finite-Element (FE) methods, where some assumption is made concerning the variation of the function between the points x_i.

2.) Representing Derivatives by Discrete Values

- Let $f(x)$ be a continuous function differentiable as many times as needed in $[a, b]$. In such a case, a Taylor expansion exists about <u>any</u> point x_i

$$
\begin{aligned}
u_{i+1} &= f(x_{i+1}) \\
&= f(x_i + h) \\
&= f(x_i) + h\, f'(x_i) + \frac{h^2}{2!} f''(x_i) + \frac{h^3}{3!} f'''(x_i) \\
&\quad + O(h^4)
\end{aligned}
\tag{2}
$$

- In (2), $O(h^4)$ is the remainder that represents the unwritten terms that are of 4th order and higher. Recall that $u_i = f(x_i)$ and we find that

$$
\boxed{\frac{u_{i+1} - u_i}{h} = f'(x_i) + \frac{h}{2!} f''(x_i) + \frac{h^2}{3!} f'''(x_i) + O(h^3)}
\tag{3}
$$

- The L.H.S. of (3) is a FD approximation of the 1st derivative $f'(x_i)$. The error is governed by the leading term ...

$$
\frac{h}{2!} f''(x_i) = O(h) \quad , \text{ as } h \to 0
$$

- The "numerical" scheme in (3) is 1st order accurate. It is also called a forward Finite-Difference (FD) scheme.

- Substitute h by $-h$ in (2) to obtain

$$\begin{aligned} u_{i-1} &= f(x_{i-1}) \\ &= f(x_i - h) \\ &= f(x_i) - h f'(x_i) + \frac{h^2}{2!} f''(x_i) - \frac{h^3}{3!} f'''(x_i) + O(h^4) \end{aligned} \qquad (4)$$

- Using again that per definition $u_i = f(x_i)$ we find that

$$\boxed{\frac{u_i - u_{i-1}}{h} = f'(x_i) - \frac{h}{2!} f''(x_i) + \frac{h^2}{3!} f'''(x_i) - O(h^3)} \quad (5)$$

- The L.H.S. of (5) is another FD approximation of $f'(x_i)$. The "numerical" scheme is again 1st order accurate and is called "backward Finite-Difference (FD) scheme".

- If we take the average of (3) & (5), i.e. $\frac{(3)+(5)}{2}$,

$$\boxed{\frac{u_{i+1} - u_{i-1}}{2h} = f'(x_i) + \frac{h^2}{3!} f'''(x_i) + O(h^4)} \quad (6)$$

- Equation (6) is a "Centred" Finite-Difference FD scheme. It is 2nd order accurate; the remainder is 4th order.
- In the construction process of FD schemes, one can combine various Taylor expansions in a suitable way such that certain derivatives vanish.
- It is wrong to assume though that non-centred schemes are necessarily 1st order accurate.
- Using instead the combination $\frac{(3)-(5)}{h}$

$$\boxed{\frac{u_{i+1}-2u_i+u_{i-1}}{h^2} = f''(x_i) + \frac{h^2}{4!} f^{(iv)}(x_i) + \mathcal{O}(h^4)} \quad (7)$$

which is a "Centred" 2nd order accurate scheme for the 2nd derivative $f''(x_i)$. The scheme has u_i as a dominant coefficient, which is relevant for a numerical solution method.

$\longrightarrow$ Laplace Equation

- A centered scheme for the 1^{st} derivative in (6) seems attractive due to its 2^{nd} order accuracy. However, it does not contain the actual value for u_i!

$\Longrightarrow$ Consequently, the solution may become "decoupled"; i.e. ② independent solutions may exist. — Why?

$$@\ x_{i-1} \quad : \quad \frac{u_i - u_{i-2}}{2h} = f'(x_{i-1}) + \mathcal{O}(h^2)$$

$$@\ x_i \quad : \quad \frac{u_{i+1} - u_{i-1}}{2h} = f'(x_i) + \mathcal{O}(h^2)$$

$$@\ x_{i+1} \quad : \quad \frac{u_{i+2} - u_i}{2h} = f'(x_{i+1}) + \mathcal{O}(h^2)$$

$$@\ x_{i+2} \quad : \quad \frac{u_{i+3} - u_{i+1}}{2h} = f'(x_{i+2}) + \mathcal{O}(h^2)$$

$\Longrightarrow$ The scheme may allow 2 independent solutions ○ and ○ that are "decoupled" from one another.

Note that this would not be possible with schemes (3) & (5) for the 1st derivative and scheme (7) for the 2nd derivative, as the corresponding u_i @ x_i is included in the scheme w/ at least one of its neighbors.

3.) Complements on Taylor Expansion

- The Taylor expansion in two and more variables can be obtained by simply repeating the operations variable by variable.

$$f(x_{i+1}, y_{j+1}) = f_{i,j} + \Delta x \frac{\partial f_{i,j}}{\partial x} + \Delta y \frac{\partial f_{i,j}}{\partial y}$$

$$+ \frac{\Delta x^2}{2!} \frac{\partial^2 f_{i,j}}{\partial x^2} + \Delta x \Delta y \frac{\partial^2 f_{i,j}}{\partial x \partial y} + \frac{\Delta y^2}{2!} \frac{\partial^2 f_{i,j}}{\partial y^2}$$

$$+ \frac{\Delta x^3}{3!} \frac{\partial^3 f_{i,j}}{\partial x^3} + \frac{\Delta x^2 \Delta y}{2!} \frac{\partial^3 f_{i,j}}{\partial x^2 \partial y} + \frac{\Delta x \Delta y^2}{2!} \frac{\partial^3 f_{i,j}}{\partial x \partial y^2}$$

$$+ \frac{\Delta y^3}{3!} \frac{\partial^3 f_{i,j}}{\partial y^3} + \mathcal{O}(\Delta x + \Delta y)^4 \qquad (8)$$

- Equation (8) can be applied in a similar way to more than 2 independent variables.

4.) Consistency & Accuracy

- Let us introduce a truncation error (TE). The concept of a TE is important in qualifying a FD scheme.

- Example: "Heat Equation"

$$\boxed{\frac{\partial u}{\partial t} = \alpha \frac{\partial^2 u}{\partial x^2} \quad ; \alpha > 0} \qquad (9)$$

- We can rearrange terms and write a FD equation as:

$$\boxed{\frac{u_i^{n+1} - u_i^n}{\Delta t} - \alpha \frac{u_{i+1}^n - 2u_i^n + u_{i-1}^n}{\Delta x^2} = 0} \qquad (10)$$

In (10), subscripts indicate the position in space, and superscripts in time, i.e...

$$\boxed{x_{i+1} = x_i + \Delta x \quad ; \quad t^{n+1} = t^n + \Delta t} \qquad (11)$$

- Definition: The truncation error (TE) ε_i^n is obtained by putting the exact solution $\hat{u}(x,t)$ in the FDE in place of the approximate solution.

- In general, the exact solution does not satisfy the FDE exactly, i.e. ...

$$\varepsilon_i^n = \frac{\hat{u}_i^{n+1} - \hat{u}_i^n}{\Delta t} - \alpha \frac{\hat{u}_{i+1}^n - 2\hat{u}_i^n + \hat{u}_{i-1}^n}{\Delta x^2} \neq 0$$

- We can gather information about the TE of a numerical scheme by expanding formally the <u>exact</u> solution in Taylor series about the point (i,n). This is possible as long as we assume that the exact solution is continuous and differentiable to sufficient order. Here we can adopt (3) & (7) w/ the appropriate change of variables to obtain

$$\boxed{\varepsilon_i^n = \left(\frac{\partial \hat{u}_i^n}{\partial t} - \alpha \frac{\partial^2 \hat{u}_i^n}{\partial x^2}\right) + \frac{\Delta t}{2}\frac{\partial^2 \hat{u}_i^n}{\partial t} - \alpha \frac{\Delta x^2}{24}\frac{\partial^4 \hat{u}_i^n}{\partial x^4} + \mathcal{O}(\Delta t^2, \Delta x^4)}$$

Since the exact solution satisfies the underlying Partial Differential Equation (PDE), the terms in the first bracket cancel, and the TE reduces to

$$\boxed{\varepsilon_i^n = \frac{\Delta t}{2}\frac{\partial^2 \hat{u}_i^n}{\partial t^2} - \alpha \frac{\Delta x^2}{24}\frac{\partial^4 \hat{u}_i^n}{\partial x^4} + \mathcal{O}(\Delta t^2, \Delta x^4)} \quad (12)$$

- In (12), the leading terms in the TE are of order $\mathcal{O}(\Delta t, \Delta x^2)$. They dominate the error ε_i^n for small Δt and Δx.

- Definition: A Finite-Difference scheme is said to be consistent if $\varepsilon_i^n \to 0$ as $\Delta t, \Delta x \to 0$ independently. If the TE is of the form $\varepsilon_i^n = \mathcal{O}(\Delta t^p, \Delta x^q)$ w/ $p, q > 0$, the scheme is said to be of order p in t and of q in x.

- Remark: The truncation error (ε) is obtained by introducing the exact solution in the FDE. Any point can be chosen for the Taylor expansion. The TE must go to zero for a consistent scheme, regardless how the discretization step goes to zero. Hence the term independently.

Not all schemes pass the test of consistency.

- Example: ... or rather counterexample

"Du Fort Frankel" scheme for Heat Equation

$$\boxed{\frac{u_i^{n+1} - u_i^{n-1}}{2\Delta t} = \alpha \frac{u_{i+1}^n - u_i^{n+1} - u_i^{n-1} + u_{i-1}^n}{\Delta x^2}} \quad (13)$$

- The truncation error (TE) becomes

$$\boxed{\varepsilon_i^n = \frac{u_i^{n+1} - u_i^{n-1}}{2\Delta t} - \alpha \frac{u_{i+1}^n - u_i^{n+1} - u_i^{n-1} + u_{i-1}^n}{\Delta x^2}} \quad (14)$$

- How to evaluate TE?

  i.) Replace terms in FDE w/ known Taylor expansions

  ii.) If a term in the FDE is unknown, then construct term w/ individual Taylor expansions.

- From (6) we know that ...

$$\frac{u_i^{n+1} - u_i^{n-1}}{2\Delta t} = \frac{\partial u_i^n}{\partial t} + O(\Delta t^2) \quad (15)$$

$$\frac{u_{i+1}^n - u_i^{n+1} - u_i^{n-1} + u_{i-1}^n}{\Delta x^2} = \overbrace{\frac{u_{i+1}^n + u_{i-1}^n}{\Delta x^2}}^{A} - \overbrace{\frac{u_i^{n+1} + u_i^{n-1}}{\Delta x^2}}^{B}$$

Sort by "opposing" index, i.e. n, whenever possible.

• Now compare (A) to (7) ... and subtract/add $2u_i^n$ ...

$$(A)-(B) \longmapsto \underbrace{\frac{u_{i+1}^n - 2u_i^n + u_{i-1}^n}{\Delta x^2}}_{C} - \underbrace{\frac{u_i^{n+1} - 2u_i^n + u_i^{n-1}}{\Delta x^2}}_{D}$$

• Recognize that (C) relates to $\frac{\partial^2 u_i^n}{\partial x^2}$ through (7). We can adjust (D) such that it approximates $\frac{\partial^2 u_i^n}{\partial t^2}$ as:

$$\frac{u_i^{n+1} - 2u_i^n + u_i^{n-1}}{\Delta x^2} = \underbrace{\frac{\Delta t^2}{\Delta x^2}\,\frac{u_i^{n+1} - 2u_i^n + u_i^{n-1}}{\Delta t^2}}_{E}$$

• As a next step, we write down the known Taylor expansions for (C) & (E)

$$C \quad \frac{u_{i+1}^n - 2u_i^n + u_{i-1}^n}{\Delta x^2} = \frac{\partial^2 u_i^n}{\partial x^2} + \mathcal{O}(\Delta x^2) \qquad (16)$$

$$E \quad \frac{\Delta t^2}{\Delta x^2}\,\frac{u_i^{n+1} - 2u_i^n + u_i^{n-1}}{\Delta t^2} = \frac{\Delta t^2}{\Delta x^2}\left(\frac{\partial^2 u_i^n}{\partial t^2} + \mathcal{O}(\Delta t^2)\right)$$

$$= \frac{\Delta t^2}{\Delta x^2}\,\frac{\partial^2 u_i^n}{\partial t^2} + \mathcal{O}\left(\frac{\Delta t^4}{\Delta x^2}\right) \qquad (17)$$

• Find ε_i^n by putting (15)–(17) into (14) to obtain

$$\varepsilon_i^n = \cancel{\frac{\partial u_i^n}{\partial t}} + O(\Delta t^2) - \alpha \cancel{\frac{\partial^2 u_i^n}{\partial x^2}} + O(\Delta x^2) + \alpha \frac{\Delta t^2}{\Delta x^2} \frac{\partial^2 u_i^n}{\partial t^2} + O\left(\frac{\Delta t^4}{\Delta x^2}\right) \qquad (18)$$

- We assume that $u_i^n = \tilde{u}_i^n$ is the exact solution. Then the terms underlined by ~~~ cancel out, as the exact solution satisfies the equation. Hence (18) becomes

$$\varepsilon_i^n = \alpha \frac{\Delta t^2}{\Delta x^2} \frac{\partial^2 u_i^n}{\partial t^2} + O(\Delta t^2) + O(\Delta x^2) + O\left(\frac{\Delta t^4}{\Delta x^2}\right) \qquad (19)$$

- If $\Delta t \to 0$ faster than Δx, then $\varepsilon_i^n \to 0$ for $\Delta t, \Delta x \to 0$. The scheme is consistent.
- If $\Delta t \to 0$ slower than Δx, then the is not consistent.

$\Longrightarrow$ The "Du Fort Frankel" for the Heat Equation is not consistent.

Why??? Because $\varepsilon_i^n \not\to 0$ for $\Delta x, \Delta t \to 0$ independently.

5.) Stability

- **Lax's Equivalence Theorem:**

Given a properly posed initial value problem and a finite-difference approximation to it that satisfies the consistency condition, stability is the necessary & sufficient condition for convergence. (Note that the theorem has not been proven for general nonlinear equations.)

- **Convergence:**

In a certain norm, the difference between the exact and the discrete solutions $\|\bar{u}_i^n - u_i^n\|$ will tend to zero as the small parameters tend to zero themselves.

- **What about Stability?**

Stability is concerned w/ round-off errors occurring in the solution of the FDE on a computer w/ finite accuracy.

- Consistency & Accuracy $\longrightarrow$ Taylor Expansion

  Stability $\longrightarrow$ "Von Neumann Method"

- "Von Neumann Method"

→ Local analysis of wave mode amplification of the round-off errors.

→ Can only be applied to a linear homogeneous equation or system, as it is based on superposition of wave modes.

→ Nonlinear equations must first be linearized by "freezing" the coefficients of the partial derivatives.

→ Linearity of the equation insures that, if the scheme is stable for each mode, it will be stable for any superposition of modes.

→ Time & space are treated differently.

→ The complex Fourier Mode:

$$\boxed{u_i^n = g^n \cdot e^{\underline{i}\, i\beta} = g^n \left(\cos(i\beta) + \underline{i}\,\sin(i\beta)\right)} \quad (20)$$

- In (20),

$g^n \hat{=}$ Complex amplitude raised to the n^{th} power (n corresponds to time)

$\beta \hat{=}$ wave number corresponding to the Fourier mode (i corresponds to space)

- In order for a scheme to be stable, i.e. for round-off errors to be damped, ...

$$\boxed{\| g \| \leq 1 \qquad \forall \beta} \qquad (21)$$

- Example: FDE for "Heat Equation" in (10)

$$\frac{u_i^{n+1} - u_i^n}{\Delta t} = \alpha \frac{u_{i+1}^n - 2u_i^n + u_{i-1}^n}{\Delta x^2}$$

Write in "update form" (i.e. form used to calculate new values of the unknowns.

$$\boxed{u_i^{n+1} = u_i^n + \alpha \frac{\Delta t}{\Delta x^2}\left(u_{i+1}^n - 2u_i^n + u_{i-1}^n\right)} \qquad (10*)$$

• Substituting the complex Fourier mode (20) into (10*) yields

$$g^{n+1} e^{\underline{i}\, i\beta} = g^{n} e^{\underline{i}\, i\beta} + \alpha \frac{\Delta t}{\Delta x^2} g^{n}\left(e^{\underline{i}\,(i+1)\beta} - 2 e^{\underline{i}\, i\beta} + e^{\underline{i}\,(i-1)\beta}\right)$$

• Defining $r = \alpha \frac{\Delta t}{\Delta x^2}$ and dividing by $g^n e^{\underline{i}\, i\beta}$ we obtain

$$g = 1 + r\left(e^{\underline{i}\beta} - 2 + e^{-\underline{i}\beta}\right)$$

$$g = 1 + r\left(\cos\beta + \cancel{\underline{i}\sin\beta} - 2 + \cos(-\beta) + \cancel{\underline{i}\sin(-\beta)}\right)$$

$$g = 1 + r\left[2\cos\beta - 2\right]$$

$$\boxed{g = 1 - 2r\left[1 - \cos\beta\right]} \qquad (22)$$

• Now, (22) has to comply to (21) for stability.

$$-1 \leq 1 - \underbrace{2r\left[1 - \cos\beta\right]}_{>0} \leq 1 \qquad (\checkmark)$$

$$-1 \leq 1 - 4r \implies \boxed{r \leq \frac{1}{2}} \qquad (23)$$

- In terms of the time step (23) means

$$\boxed{\Delta t \leq \frac{\Delta x^2}{2\alpha}} \qquad (23*)$$

- Equation (23*) is a quite severe restriction on the time step since, if the space step is divided by 2 the max. allowable time step for stability will be divided by 4.

<u>Example:</u> "Linear Convection Equation"

$$\boxed{\frac{\partial u}{\partial t} + c\,\frac{\partial u}{\partial x} = 0} \qquad (24)$$

- In 1954, <u>Lax</u> proposed the following scheme:

$$\boxed{\frac{u_i^{n+1} - \dfrac{u_{i-1}^n + u_{i+1}^n}{2}}{\Delta t} + c\,\frac{u_{i+1}^n - u_{i-1}^n}{2\Delta x} = 0} \qquad (25)$$

- Find <u>consistency</u> & <u>stability</u> of Lax's scheme.

... by adding & subtracting u_i^n to the numerator of the time derivative term.

$$\boxed{\underbrace{\frac{u_i^{n+1} - u_i^n}{\Delta t}}_{\to (3)} - \underbrace{\frac{u_{i+1}^n - 2u_i^n + u_{i-1}^n}{2\Delta t}}_{\to \frac{\Delta x^2}{2\Delta t}\,(7)} + C\,\underbrace{\frac{u_{i+1}^n - u_{i-1}^n}{2\Delta x}}_{\to C\cdot(6)} = 0} \qquad (26)$$

$\Longrightarrow$ Truncation Error (TE)

$$\boxed{\begin{aligned} \varepsilon_i^n = {} & \frac{\partial u_i^n}{\partial t} + \frac{\Delta t}{2}\frac{\partial^2 u_i^n}{\partial t^2} + \mathcal{O}(\Delta t^2) \\ & - \frac{\Delta x^2}{2\Delta t}\frac{\partial^2 u_i^n}{\partial x^2} + \mathcal{O}\left(\frac{\Delta x^4}{\Delta t}\right) \\ & + C\frac{\partial u_i^n}{\partial x} + \mathcal{O}(\Delta x^2) \end{aligned}} \qquad (27)$$

- ~~ cancels, as it is the exact solution. Hence (27) reduces to

$$\boxed{\varepsilon_i^n = \frac{\Delta t}{2}\frac{\partial^2 u_i^n}{\partial t^2} - \frac{\Delta x^2}{2\Delta t}\frac{\partial^2 u_i^n}{\partial x^2} + \mathcal{O}\left(\Delta t, \Delta x^2, \frac{\Delta x^4}{\Delta t}\right)} \qquad (28)$$

- Consistency requires that ...

... $\Delta t = \mathcal{O}(\Delta x^{2-r})$ w/ $0 < r < 2$. Then (28) reduces to

$$\boxed{\varepsilon_i^n = \frac{\Delta x^{2-r}}{2}\frac{\partial^2 u_i^n}{\partial t^2} - \frac{\Delta x^r}{2}\frac{\partial^2 u_i^n}{\partial x^2} + \mathcal{O}(\Delta x^{2-r})} \quad (29)$$

$\Rightarrow$ Lax's scheme is conditionally consistent.

The scheme requires large time steps for consistency. (unlike the Du Fort Frankel scheme)

- Stability will put a restriction on how large of a time step one can choose.

$\rightarrow$ Write (25) in update form as:

$$\boxed{u_i^{n+1} = \frac{1}{2}\left(u_{i-1}^n + u_{i+1}^n\right) - \frac{\tau}{2}\left(u_{i+1}^n - u_{i-1}^n\right)} \quad (30)$$

- In (30), $\tau = c \cdot \frac{\Delta t}{\Delta x}$. Performing the "von Neumann" method with $u_i^n = g^n \cdot e^{i\, i\beta}$ in (30) we obtain

$$\boxed{g^{n+1} e^{i\, i\beta} = \frac{1}{2} g^n \left(e^{i(i-1)\beta} + e^{i(i+1)\beta}\right) - \frac{\tau}{2} g^n \left(e^{i(i+1)\beta} - e^{i(i-1)\beta}\right)}$$

- Now divide by $u_i^n = g^n e^{i i \beta}$ and get

$$g = \frac{1}{2}\left(e^{-i\beta} + e^{i\beta}\right) - \frac{\sigma}{2}\left(e^{i\beta} - e^{-i\beta}\right)$$

$$g = \frac{1}{2}\left(\cos(-\beta) + \cancel{\underline{i}\sin(-\beta)} + \cos(\beta) + \cancel{\underline{i}\sin(\beta)}\right)$$
$$- \frac{\sigma}{2}\left(\cancel{\cos(\beta)} + \underline{i}\sin\beta - \cancel{\cos(-\beta)} - \underline{i}\sin(-\beta)\right)$$

$$\boxed{g = \cos(\beta) - \underline{i}\,\sigma\,\sin(\beta)} \qquad (31)$$

- Taking the square of the modulus

$$\|g\|^2 = \cos^2(\beta) + \sigma^2\sin^2\beta$$
$$= \underbrace{\cos^2(\beta) + \sin^2(\beta)}_{=1} + (\sigma^2 - 1)\sin^2\beta$$

$$\boxed{\|g\|^2 = 1 - (1 - \sigma^2)\sin^2\beta} \qquad (32)$$

- For stability $\quad -1 \le g \le 1$

$$-1 \le 1 - (1 - \sigma^2)\sin^2\beta \le 1$$

$$-1 \le 1-(1-\sigma^2) \qquad \sigma^2 \le 1$$

$$-1 \le \sigma^2 \qquad \text{or } \sigma \le 1$$

Always.

- Thus, the limitation on the time step is

$$\boxed{\Delta t \le \frac{\Delta x}{c}} \quad ; \quad c > 0 \qquad (33)$$

- Equation (33) is also known as the "Courant-Friedrichs-Lewy" condition (CFL) (1928). CFL Number: σ

6.) Tips & Tricks in evaluating Truncation Error (TE)

- The truncation error is independent of the point chosen to evaluate it. This allows for some shortcuts in determining the TE.
- Instead of evaluating the TE directly by using Taylor expansions for each point about a single point, say (i,n), it may be easier to proceed in the following way.

Recipe to evaluate TE:

i.) Expand each FD grouping representing <u>one</u> partial derivative w.r.t. the most convenient point for that term.

ii.) Expand the resulting Taylor expansion w.r.t. the point chosen for the full FDE. This can be accomplished by a shifting operation (again Taylor expansion) for <u>each term</u>. The expansion has to be carried far enough to retain the order of the remainder.

<u>Example:</u> Consider the FDE for the "Heat Equation" in (10)

$$\boxed{\frac{u_i^{n+1} - u_i^n}{\Delta t} - \alpha \frac{u_{i+1}^n - 2u_i^n + u_{i-1}^n}{\Delta x^2}}$$

The TE evaluated about (i,n) was given by (12)

$$\boxed{\varepsilon_i^n = \frac{\Delta t}{2} \frac{\partial^2 u_i^n}{\partial t^2} - \alpha \frac{\Delta x^2}{24} \frac{\partial^4 u_i^n}{\partial x^4} + \mathcal{O}(\Delta t^2, \Delta x^4)}$$

• Now we choose to find the TE by expanding about point $(i, n+1)$; ε_i^{n+1} – How do we do that?

$\Longrightarrow$ Use "Shifting Operations" for FD groupings

• "Backward Differencing" for the 1st term

$$u_i^n = u_i^{n+1} - \Delta t \frac{\partial u_i^{n+1}}{\partial t} + \frac{\Delta t^2}{2} \frac{\partial^2 u_i^{n+1}}{\partial t^2} + O(\Delta t^3)$$

$$\boxed{\frac{u_i^{n+1} - u_i^n}{\Delta t} = \frac{\partial u_i^{n+1}}{\partial t} - \frac{\Delta t}{2} \frac{\partial^2 u_i^{n+1}}{\partial t^2} + O(\Delta t^2)} \quad (34)$$

• The 2nd term is best evaluated about (i, n)

$\rightarrow$ (7)

$$\boxed{\frac{u_{i+1}^n - 2u_i^n + u_{i-1}^n}{\Delta x^2} = \frac{\partial^2 u_i^n}{\partial x^2} + \frac{\Delta x^2}{24} \frac{\partial^4 u_i^n}{\partial x^4} + O(\Delta x^4)} \quad (7)$$

Shift individually

$(i, n) \rightarrow (i, n+1)$

$\Longrightarrow$ $\boxed{t^n = t^{n+1} - \Delta t}$

- $$\frac{\partial^2 u_i^n}{\partial x^2} = \frac{\partial^2 u_i^{n+1}}{\partial x^2} - \Delta t \frac{\partial^3 u_i^{n+1}}{\partial t \, \partial x^2} + O(\Delta t^2) \tag{35}$$

- $$\frac{\Delta x^2}{24} \frac{\partial^4 u_i^n}{\partial x^4} = \frac{\Delta x^2}{24}\left(\frac{\partial^4 u_i^{n+1}}{\partial x^4} - \Delta t \frac{\partial^5 u_i^{n+1}}{\partial t \, \partial x^4} + O(\Delta t^2) \right) \tag{36}$$

- (35), (36) in (7) and together w/ (34) we can expand the TE about $(i, n+1)$ as:

$$\begin{aligned} \varepsilon_i^{n+1} = {} & \cancel{\frac{\partial u_i^{n+1}}{\partial t}} - \frac{\Delta t}{2} \frac{\partial^2 u_i^{n+1}}{\partial t^2} + O(\Delta t^2) \\ & - \alpha \left(\cancel{\frac{\partial^2 u_i^{n+1}}{\partial x^2}} - \Delta t \frac{\partial^3 u_i^{n+1}}{\partial t \, \partial x^2} + O(\Delta t^2) \right. \\ & \left. + \frac{\Delta x^2}{24} \frac{\partial^4 u_i^{n+1}}{\partial x^4} + O(\Delta x^2 \Delta t) \right) \end{aligned}$$

Exact Solution. Rearranging the terms gives

$$\boxed{\begin{aligned} \varepsilon_i^{n+1} &= -\frac{\Delta t}{2} \frac{\partial^2 u_i^{n+1}}{\partial t^2} + \alpha \Delta t \frac{\partial^3 u_i^{n+1}}{\partial t \, \partial x^2} \\ &\quad - \alpha \frac{\Delta x^2}{24} \frac{\partial^4 u_i^{n+1}}{\partial x^4} + O(\Delta t^2, \Delta t \Delta x^2) \\ &= O(\Delta t, \Delta x^2) \end{aligned}} \tag{37}$$

$\Rightarrow$ The FDE in (10) is 1st order accurate in t and 2nd order accurate in x.