Probabilistic Robotics Hw #1

1.) Coding Algorithm (uploaded as a seperate file and github link)

GitHub: https://github.com/jaiselsingh1/Probabilistic-Robotics/blob/main/prob_rob_hw1.py

↳ Link also included in the comments of the canvas submission

2.)

$$\Sigma_{xx} = E((x-\mu_x)(x-\mu_x)^T) \quad \text{Covariance}$$

$$K_{xy} = E((x-\mu_x)(y-\mu_y)^T) \quad \text{Cross covariance}$$

a.) Random Variables → $z = x + y$

$$\mu_z = E[z] = E[x+y] = E[x] + E[y] = \mu_x + \mu_y$$

$$z - \mu_z = (x+y) - E[z] = (x+y) - (\mu_x + \mu_y)$$

$$z - \mu_z = (x-\mu_x) + (y-\mu_y)$$

$$\Sigma_{zz} = E[(z-\mu_z)(z-\mu_z)^T] = E[[(x-\mu_x)+(y-\mu_y)][(x-\mu_x)+(y-\mu_y)]^T]$$

$$= E[[(x-\mu_x)+(y-\mu_y)][(x-\mu_x)^T + (y-\mu_y)^T]]$$

$$= E[(x-\mu_x)(x-\mu_x)^T + (x-\mu_x)(y-\mu_y)^T + (y-\mu_y)(x-\mu_x)^T + (y-\mu_y)(y-\mu_y)^T]$$

Using linearity property of expectation

$$= E[(x-\mu_x)(x-\mu_x)^T] + E[(x-\mu_x)(y-\mu_y)^T] + E[(y-\mu_y)(x-\mu_x)^T] + E[(y-\mu_y)(y-\mu_y)^T]$$

$$\Sigma_{zz} = \Sigma_{xx} + K_{xy} + K_{yx} + \Sigma_{yy}$$

b.) Cross covariance of 2 independant random variables is 0

$E[xy] = E[x]E[y]$ for independance

$$= E[(x-\mu_x)(y-\mu_y)^T] = E[(x-\mu_x)]\, E[(y-\mu_y)^T]$$

$$E[(x-\mu_x)] = E[x] - E[\mu_x] = \mu_x - \mu_x = 0$$

$$E[(y-\mu_y)^T] = (E[y] - E[\mu_y])^T = (\mu_y - \mu_y)^T = 0^T = 0$$

Hence $K_{xy} = 0 \cdot 0 = 0$

c.) The corollary of the above 2 results is that if X and y are independent random variables then the covariance of their sum (the random variable z) would simplify by a large degree to just become the sum of their individual covariances.

$$\Sigma_{zz} = \Sigma_{xx} + \Sigma_{yy}$$ The covariance matrix of the sum is the sum of the independent covariance matrices.

3.) Sensor has 10 (unreliable sensors) - independent.

$P(obj) = 0.1$

a.) how many sensors detect an object

$\downarrow$

number of successes in 10 independent trials with $p = 0.1$

Binomial Distribution $\Rightarrow$ $p_X(k) = \binom{n}{k} p^k (1-p)^{n-k}$ for $k = 0 \ldots n$

$X \sim \text{Binomial}(n=10, p=0.1)$

Probability Mass function $P(X=k) = \binom{10}{k}(0.1)^k (0.9)^{10-k}$ For $k = 0, \ldots 10$

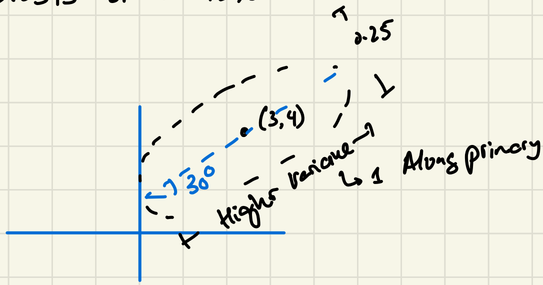
b.) All measurements are occuring in parallel hence model using the Binomial distribution

$P(X \geq 1) = 1 - P(X=0)$ Using the complement formula

$$P(X=0) = \binom{10}{0}(0.1)^0 (0.9)^{10}$$
$$= 1 \cdot 1 \cdot 0.9^{10} = 0.3487$$

$P(X \geq 1) = 1 - 0.9^{10} = 0.6513$ or 65.13%

4.)



Uncertainty Ellipse centered at (3,4)

Primary Axis $\theta = 30°$

$\mu_x = 3$ $\lambda_1 = 1$

$\mu_y = 4$ $\lambda_2 = 0.25$

$$f(x) = \frac{1}{2\pi^{\frac{k}{2}}\sqrt{|\Sigma|}} \exp\left[-\frac{1}{2}(x-\mu)^T \Sigma^{-1}(x-\mu)\right]$$

$$\Sigma = \begin{bmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{12} & \sigma_{22} \end{bmatrix}$$

$$\det(\Sigma - \lambda I) = 0$$

$$\det\left[\begin{bmatrix}\sigma_{11} & \sigma_{12}\\ \sigma_{12} & \sigma_{22}\end{bmatrix} - \begin{bmatrix}\lambda & 0\\ 0 & \lambda\end{bmatrix}\right] = 0 = \quad (\sigma_{11}-\lambda)(\sigma_{22}-\lambda) - \sigma_{12}^2 = 0$$

$$\lambda^2 - (\sigma_{11}+\sigma_{22})\lambda + (\sigma_{11}\sigma_{22} - \sigma_{12}^2) = 0$$

θ = angle between eigenvectors $\Rightarrow$

$\tan\theta = \frac{v_2}{v_1}$ Ratio between eigenvectors

$$\Sigma \vec{v} = \lambda \vec{v}$$

$$[\Sigma - \lambda I]\vec{v} = 0$$

$$\begin{bmatrix}\sigma_{11}-\lambda & \sigma_{12}\\ \sigma_{12} & \sigma_{22}-\lambda\end{bmatrix}\begin{bmatrix}v_1\\ v_2\end{bmatrix} = \begin{bmatrix}0\\0\end{bmatrix}$$

$$(\sigma_{11}-\lambda)v_1 + \sigma_{12}v_2 = 0$$
$$\sigma_{12}v_1 + (\sigma_{22}-\lambda)v_2 = 0$$

$$\sigma_{12}v_2 = (\lambda - \sigma_{11})v_1$$

$$\frac{v_2}{v_1} = \left(\frac{\lambda - \sigma_{11}}{\sigma_{12}}\right)$$

$$\tan\theta = \frac{\lambda - \sigma_{11}}{\sigma_{12}} = \frac{1}{\sqrt{3}}$$

30° corresponds to $\lambda_1 = 1$

$$\sigma_{12} = \frac{\lambda - \sigma_{11}}{\tan\theta}$$

$$\sigma_{12} = \frac{1-\sigma_{11}}{\frac{1}{\sqrt{3}}}$$

$$\lambda_1 + \lambda_2 = \sigma_{11} + \sigma_{22}$$
$$(\lambda_1)(\lambda_2) = \sigma_{11}\sigma_{22} - \sigma_{12}^2$$

① $1.25 = \sigma_{11} + \sigma_{22}$

② $\sigma_{11}\sigma_{22} - \sigma_{12}^2 = 0.25$

$$\sigma_{11} = 0.8125 = \frac{13}{16}$$

$$\sigma_{22} = 0.4375 = \frac{7}{16}$$

$$\sigma_{12} = \frac{1 - \frac{13}{16}}{\frac{1}{\sqrt{3}}} = 0.325$$

Hence covariance matrix

$$\Sigma = \begin{bmatrix}\frac{13}{16} & 0.325\\ 0.325 & \frac{7}{16}\end{bmatrix}$$

$$\mu = [3,4]^T$$

$$|\Sigma| = \left(\frac{13}{16}\right)\left(\frac{7}{16}\right) - \left(\frac{3\sqrt{3}}{16}\right)^2 = \frac{64}{256} = \frac{1}{4}$$

$$\sqrt{|\Sigma|} = \frac{1}{2}$$

$$\Sigma^{-1} = \frac{1}{|\Sigma|}\begin{bmatrix}\frac{7}{16} & -\frac{3\sqrt{3}}{16}\\ -\frac{3\sqrt{3}}{16} & \frac{13}{16}\end{bmatrix}$$

$$= \begin{bmatrix}\frac{7}{4} & -\frac{3\sqrt{3}}{4}\\ -\frac{3\sqrt{3}}{4} & \frac{13}{4}\end{bmatrix}$$

$$(\bar{x}-\bar{\mu})^T\Sigma^{-1}(\bar{x}-\bar{\mu}) = [x-3 \quad y-4]\begin{bmatrix}\frac{7}{4} & -\frac{3\sqrt{3}}{4}\\ -\frac{3\sqrt{3}}{4} & \frac{13}{4}\end{bmatrix}\begin{bmatrix}x-3\\ y-4\end{bmatrix}$$

$$f(x,y) = \frac{1}{\pi}\exp\left[-\frac{1}{2}\left(\frac{7}{4}(x-3)^2 - \frac{3\sqrt{3}}{2}(x-3)(y-4) + \frac{13}{4}(y-4)^2\right]\right.$$

5.) If boy stop → success → Avg number of trials till first success.
girl continue
↓
Geometric Distribution

$$p_X(k) = (1-p)^{k-1} p \quad \text{for } k = 1, 2 \ldots$$

Expectation is modeled as $\frac{1}{p}$ = Average number of children per family.

Assuming $P_{boy} = P_{girl} = 0.5$

Average number of children $= \frac{1}{0.5} = 2$ children.