COMS W4733: Computational Aspects of Robotics
Homework 1

Jaisel Singh
js6897

September 20, 2025

Problem 1: Homogeneous Transformations

1. Convert to homogeneous coordinates (1 point)

Convert $p_A^{\text{cart}} = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}$ to homogeneous:

$$p_A = \begin{bmatrix} p_A^{\text{cart}} \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 1 \\ 1 \end{bmatrix}.$$

2. Construct ${}^{A}T_B$ (4 points)

Rotation 90° about $+z$:

$$R = R_z(90^\circ) = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \qquad t = \begin{bmatrix} 1 \\ -2 \\ 0 \end{bmatrix}.$$

Therefore

$${}^{A}T_B = \begin{bmatrix} R & t \\ 0\ 0\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & -1 & 0 & 1 \\ 1 & 0 & 0 & -2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

3. Transform the point (3 points)

Use $({}^{A}T_B)^{-1} = \begin{bmatrix} R^\top & -R^\top t \\ 0 & 1 \end{bmatrix}$:

$${}^{B}T_A = \begin{bmatrix} 0 & 1 & 0 & 2 \\ -1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \qquad p_B = {}^{B}T_A\, p_A = \begin{bmatrix} 3 \\ -1 \\ 1 \\ 1 \end{bmatrix}.$$

So $p_B^{\text{cart}} = (3, -1, 1)^\top$.

4. Interpret the result (2 points)

The 90° rotation swaps $(x, y) \mapsto (-y, x)$; the translation adds $(+1, -2, 0)$, yielding $(3, -1, 1)$ as expected. ${}^{B}T_{A}$ is the inverse transformation, it first translates the point by $-t$ and then performs a 90° *clockwise* rotation about the z-axis. This explicitly works out to be: $p_B = R^\top(p_A - t)$.

Problem 2: Configuration Space and Workspace (Mobile Robot)

1. C-space and DOF (2 points)

(a) $q = (x, y, \theta) \in \mathbb{R}^2 \times S^1$ with

$$\mathcal{Q} = [0, 5] \times [0, 4] \times (-\pi, \pi].$$

(b) DOF = 3 (two translational, one rotational).

2. Position workspace of P (5 points)

(a) Ignoring footprint:

$$\mathcal{W} = [0, 5] \times [0, 4].$$

(b) With disc footprint $r_R = 0.35$:

$$\mathcal{W}_{\text{clear}} = \{(x, y) : 0.35 < x < 4.65,\ 0.35 < y < 3.65\}.$$

(c) The point $(0.30, 0.30) \notin \mathcal{W}_{\text{clear}}$ (collision).

3. Workspace → C-space Obstacles (2 points)

(a) The C-space obstacle set is obtained by growing the obstacle radius by r_R:

$$Q_{\text{obs}} = \{(x, y) \mid \|(x, y) - (0.9, 0.3)\| \le 0.10 + r_R\} = \{(x, y) \mid \|(x, y) - (0.9, 0.3)\| \le 0.45\}.$$

This set does not depend on θ because the robot is a disc (isotropic).

(b) For $q^\star = (1.20,\ 0.40,\ \theta = 0.524)$, compute the distance:

$$d = \|(1.20,\ 0.40) - (0.9,\ 0.3)\| = \sqrt{0.3^2 + 0.1^2} = 0.316 < 0.45.$$

Hence, $q^\star \in Q_{\text{obs}}$ (the configuration is in collision).

4. Connectivity (1 point)

$\mathcal{Q}_{\text{free}}$ is *path-connected* if any two configurations in it are connected by a continuous collision-free path. This means that every pair of collision-free configurations can be connected by a continuous collision free path. This property is essential for motion planning because it gurantees that a planner can find a path between any two configurations in $\mathcal{Q}_{\text{free}}$ if one exists.

Problem 3: Forward Kinematics (2R Planar Arm)

1. Geometric FK for position & orientation (4 points)

(a) Vector form:

$$p_E = \underbrace{R(\theta_1)\begin{bmatrix} L_1 \\ 0 \end{bmatrix}}_{\text{Link 1}} + \underbrace{R(\theta_1+\theta_2)\begin{bmatrix} L_2 \\ 0 \end{bmatrix}}_{\text{Link 2}}, \quad R(\alpha) = \begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix}.$$

(b) Scalars:

$$x = L_1\cos\theta_1 + L_2\cos(\theta_1+\theta_2), \quad y = L_1\sin\theta_1 + L_2\sin(\theta_1+\theta_2).$$

(c) Orientation:

$$\phi = \theta_1 + \theta_2$$

(since θ_2 is relative to link 1).

2. Pose in SE(2) (3 points)

(a) General pose of E in frame 0:

$$^0T_E = \begin{bmatrix} \cos\phi & -\sin\phi & x \\ \sin\phi & \cos\phi & y \\ 0 & 0 & 1 \end{bmatrix}, \qquad \phi = \theta_1 + \theta_2.$$

(b) Compose link transforms explicitly:

$$^0T_E = {}^0T_1\, {}^1T_E,$$

with

$$^0T_1 = \begin{bmatrix} \cos\theta_1 & -\sin\theta_1 & L_1\cos\theta_1 \\ \sin\theta_1 & \cos\theta_1 & L_1\sin\theta_1 \\ 0 & 0 & 1 \end{bmatrix}, \quad {}^1T_E = \begin{bmatrix} \cos\theta_2 & -\sin\theta_2 & L_2\cos\theta_2 \\ \sin\theta_2 & \cos\theta_2 & L_2\sin\theta_2 \\ 0 & 0 & 1 \end{bmatrix}.$$

Multiplying,

$$^0T_E = \begin{bmatrix} \cos(\theta_1+\theta_2) & -\sin(\theta_1+\theta_2) & L_2\cos(\theta_1+\theta_2) + L_1\cos\theta_1 \\ \sin(\theta_1+\theta_2) & \cos(\theta_1+\theta_2) & L_2\sin(\theta_1+\theta_2) + L_1\sin\theta_1 \\ 0 & 0 & 1 \end{bmatrix}.$$

(c) Extract the end-effector point explicitly:

$$p_E^0 = {}^0T_E\, p_E^E, \qquad p_E^E = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \Rightarrow p_E^0 = \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}.$$

3. Numeric evaluation (2 points)

For $\theta_1 = 30° = \pi/6$, $\theta_2 = 60° = \pi/3$, $L_1 = 1.0$, $L_2 = 0.8$:

$$\phi = \theta_1 + \theta_2 = \tfrac{\pi}{2}, \qquad x = 0.8\cos\tfrac{\pi}{2} + 1\cos\tfrac{\pi}{6} = 0.866, \qquad y = 0.8\sin\tfrac{\pi}{2} + 1\sin\tfrac{\pi}{6} = 1.300.$$

$$^{0}T_E = \begin{bmatrix} 0 & -1 & 0.866 \\ 1 & 0 & 1.300 \\ 0 & 0 & 1 \end{bmatrix}.$$

4. Tool offset (gripper) (1 point)

Offset $d_g = 0.10$ along x_E:

$$^{E}T_G = \begin{bmatrix} 1 & 0 & d_g \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \qquad {}^{0}T_G = {}^{0}T_E\, {}^{E}T_G = \begin{bmatrix} \cos\phi & -\sin\phi & x + d_g\cos\phi \\ \sin\phi & \cos\phi & y + d_g\sin\phi \\ 0 & 0 & 1 \end{bmatrix}.$$

Numerically,

$$(x_G, y_G) = (0.866, 1.400).$$

Problem 4: Inverse Kinematics (2R Planar Arm)

1. Reachability condition (2 points)

Let $r = \sqrt{x^2 + y^2}$. The point (x, y) is reachable if:

$$\boxed{|L_1 - L_2| \le r \le L_1 + L_2},$$

i.e., the target lies in the annulus between the inner (arm folded) and outer (arm stretched) circles.

2. Elbow angle θ_2 (3 points)

Law of cosines on triangle (L_1, L_2, r) with elbow interior angle $\pi - \theta_2$:

$$\cos\theta_2 = \frac{r^2 - L_1^2 - L_2^2}{2L_1L_2} =: c_2, \qquad s_2 = \pm\sqrt{1 - c_2^2}, \qquad \boxed{\theta_2 = \operatorname{atan2}(s_2, c_2)}.$$

Two branches: *elbow-up* ($s_2 > 0$) and *elbow-down* ($s_2 < 0$).

3. Shoulder angle θ_1 (3 points)

Let $\alpha = \operatorname{atan2}(y, x)$ and $\beta = \operatorname{atan2}\big(L_2 s_2,\ L_1 + L_2 c_2\big)$. Then

$$\boxed{\theta_1 = \alpha - \beta = \operatorname{atan2}(y, x) - \operatorname{atan2}\big(L_2 s_2,\ L_1 + L_2 c_2\big)}.$$

This yields one θ_1 for each choice of $\operatorname{sign}(s_2)$.

4. Numeric test & joint limits (2 points)

Target $(x^\star, y^\star) = (1.200, 0.400)$, $L_1 = 1.0$, $L_2 = 0.8$. Limits: $\theta_1 \in [-\pi, \pi)$, $\theta_2 \in [-3\pi/4, 3\pi/4] = [-2.356, 2.356]$.

Step 1: r, c_2, s_2 (3 d.p.).

$$r^2 = 1.200^2 + 0.400^2 = 1.600, \qquad r = \sqrt{1.600} = \boxed{1.265}.$$

$$c_2 = \frac{r^2 - L_1^2 - L_2^2}{2L_1L_2} = \frac{1.600 - 1.000 - 0.640}{2(1.0)(0.8)} = \frac{-0.040}{1.600} = \boxed{-0.025},$$

$$s_2 = \pm\sqrt{1 - c_2^2} = \pm\sqrt{1 - 0.025^2} = \boxed{\pm 0.999}.$$

Step 2: α and β (3 d.p.).

$$\alpha = \operatorname{atan2}(0.400, 1.200) = \operatorname{atan}(1/3) = \boxed{0.322} \text{ rad},$$

$$L_1 + L_2c_2 = 1.0 + 0.8(-0.025) = 0.980, \quad L_2s_2 = 0.8 \times 0.999 = 0.800.$$

$$\beta_\pm = \operatorname{atan2}(\pm 0.800,\ 0.980) = \boxed{\pm 0.684} \text{ rad}.$$

Step 3: Joint angles (3 d.p.).

$$\theta_2 = \operatorname{atan2}(s_2, c_2) = \begin{cases} \boxed{+1.596} & (s_2 > 0,\ \text{elbow-up}), \\ \boxed{-1.596} & (s_2 < 0,\ \text{elbow-down}), \end{cases} \qquad \theta_1 = \alpha - \beta_\pm = \begin{cases} \boxed{-0.363} & (\text{elbow-up}), \\ \boxed{+1.006} & (\text{elbow-down}). \end{cases}$$

Step 4: Joint-limit check.

$$-2.356 \le \pm 1.596 \le 2.356, \qquad -\pi < -0.363,\ 1.006 < \pi.$$

$$\boxed{(-0.363,\ +1.596)} \text{ and } \boxed{(1.006,\ -1.596)} \text{ both satisfy the limits.}$$

Step 5: Forward check (using the rounded angles).

$$\hat{x} = L_1\cos\theta_1 + L_2\cos(\theta_1 + \theta_2), \quad \hat{y} = L_1\sin\theta_1 + L_2\sin(\theta_1 + \theta_2).$$

Elbow-up $(\theta_1, \theta_2) = (-0.363,\ 1.596)$:

$$\cos\theta_1 \approx 0.935,\ \sin\theta_1 \approx -0.355,\ \cos(\theta_1 + \theta_2) = \cos(1.233) \approx 0.332,\ \sin(1.233) \approx 0.944.$$

$$\hat{x} = 1(0.935) + 0.8(0.332) = \underline{1.201} \approx 1.200, \quad \hat{y} = 1(-0.355) + 0.8(0.944) = \underline{0.400}\ (\le 10^{-3} \text{ from } y^\star).$$

Elbow-down $(\theta_1, \theta_2) = (1.006,\ -1.596)$:

$$\cos\theta_1 \approx 0.535,\ \sin\theta_1 \approx 0.845,\ \cos(\theta_1 + \theta_2) = \cos(-0.590) \approx 0.831,\ \sin(-0.590) \approx -0.556.$$

$$\hat{x} = 1(0.535) + 0.8(0.831) = \underline{1.200}, \quad \hat{y} = 1(0.845) + 0.8(-0.556) = \underline{0.400}.$$

In both branches, $\|(\hat{x}, \hat{y}) - (x^\star, y^\star)\| \le 10^{-3}$.

4.4 Numerical IK Computation

The following Python script implements the inverse kinematics computation for Problem 4:

Listing 1: Python Implementation of 2R Planar Arm Inverse Kinematics

```python
import math

L1, L2 = 1.0, 0.8
x, y = 1.20, 0.40

r = math.hypot(x, y)
c = (r*r - L1*L1 - L2*L2) / (2*L1*L2)
s = math.sqrt(1 - c*c)

theta2_down = math.atan2(+s, c)
theta2_up   = math.atan2(-s, c)

a_down = L1 + L2*math.cos(theta2_down)
b_down = L2*math.sin(theta2_down)
theta1_down = math.atan2(y, x) - math.atan2(b_down, a_down)

a_up = L1 + L2*math.cos(theta2_up)
b_up = L2*math.sin(theta2_up)
theta1_up = math.atan2(y, x) - math.atan2(b_up, a_up)

print(f"Elbow-down: theta1={theta1_down:.3f}, theta2={theta2_down:.3f}")
print(f"Elbow-up:   theta1={theta1_up:.3f}, theta2={theta2_up:.3f}")

xd = L1*math.cos(theta1_down) + L2*math.cos(theta1_down +
     theta2_down)
yd = L1*math.sin(theta1_down) + L2*math.sin(theta1_down +
     theta2_down)
xu = L1*math.cos(theta1_up)   + L2*math.cos(theta1_up   + theta2_up)
yu = L1*math.sin(theta1_up)   + L2*math.sin(theta1_up   + theta2_up)

print(f"Forward check: down = ({xd:.3f}, {yd:.3f})")
print(f"Forward check: up   = ({xu:.3f}, {yu:.3f})")
```

Output:

```
Elbow-down: theta1=-0.363, theta2=1.596
Elbow-up:   theta1=1.006, theta2=-1.596
Forward check: down = (1.200, 0.400)
Forward check: up   = (1.200, 0.400)
```

Citations

I used ChatGPT in order to assist me in the structure of my LaTeX write up and formatting.

Handwritten Work

The following pages contain my handwritten derivations for selected problems.

Computational Aspects of Robotics HW 1

Problem 1

$P_A = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}$ in frame {A}

{B} rotated 90° about z-axis

$t = \begin{bmatrix} 1 \\ -2 \\ 0 \end{bmatrix} m$

1.) $P_A^{cert} = \begin{bmatrix} 2 \\ 1 \\ 1 \\ 1 \end{bmatrix}$ in homogenous coordinate

2.) ${}^{A}T_{B} = \begin{bmatrix} R & t \\ 0\,0\,0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & -1 & 0 & 1 \\ 1 & 0 & 0 & -2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

$R(\theta, z) = \begin{bmatrix} c\theta & -s\theta & 0 \\ s\theta & c\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$P_A = {}^{A}T_B \, P_B$

↳ frame {B} in {A} coordinate

$\begin{bmatrix} R & t \\ 0 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} R^T & -R^T t \\ 0 & 1 \end{bmatrix}$

3.) P_{cert} in B coordinate?

We have $P_A^{cert} = \begin{bmatrix} 2 \\ 1 \\ 1 \\ 1 \end{bmatrix}$

hence now we need

$P_B^{cert} = {}^{B}T_{\cancel{A}} \, P_{\cancel{A}}^{cert}$

${}^{B}T_A = [{}^{A}T_B]^{-1} \rightarrow P_B^{cert} = [{}^{A}T_B]^{-1} P_A^{CART}$

${}^{B}R_A = {}^{A}R_B^{T} = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$-{}^{B}R_A t = -\begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ -2 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$

↓ defined

${}^{B}T_A = \begin{bmatrix} 0 & 1 & 0 & 2 \\ -1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

${}^{B}P_{CART} = {}^{B}T_A \, P_A^{CART} = \begin{bmatrix} 0 & 1 & 0 & 2 \\ -1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \\ 1 \\ 1 \end{bmatrix}$

$P_B^{CART} = \begin{bmatrix} -3 \\ 1 \\ 1 \\ 1 \end{bmatrix}$

Problem 2

1.) a.) $q = (x, y, \theta) \in \mathbb{R}^2 \times S^1$

$$Q = [0,5] \times [0,4] \times (-\pi, \pi] \subset \mathbb{R}^2 \times S^1$$

$$Q = \{q = (x,y,\theta) : x \in [0,5],\ y \in [0,4],\ \theta \in (-\pi, \pi]\}$$

$$q \in Q \subset \mathbb{R}^2 \times S^1$$

b.)

There are 3 DOFs in this robot, 2D translational (x,y) and 1 rotational (θ)

2.) a.) $W = [0,5] \times [0,4]$

$$W = \{(x,y) : x \in [0,5],\ y \in [0,4]\}$$

$$W \subset \mathbb{R}^2 \times S^1$$

b.) Footprint → Disk $r_R = 0.35m$

To include footprint, we perform a "Minkowski Sum"

$$A \oplus B : \{a+b \mid a \in A, b \in B\}$$

hence

$$W_{clear} = W \ominus (-\text{Footprint})$$

$$W_{clear} = \{(x,y) : x \in [0.35, 4.65],\ y \in [0.35, 3.65]\}$$

$$W_{clear} \subset \mathbb{R}^2 \times S^1$$

c.) The point $(x,y) = (0.30, 0.30)$ is not reachable without collision as $(0.30, 0.30) \notin W_{clear}$ as defined in part b.

3.)

a.) $O = \{(x,y) : \|(x,y) - (0.9, 0.3)\| \le 0.10\}$

Obstacle is a circular object with center at $(x,y) = (0.9, 0.3)$ and radius of 0.10m but we must also consider the robot footprint hence:

$$\sqrt{(x-0.9)^2 + (y-0.3)^2} \le r_o + r_{footprint} = 0.10 + 0.35 = 0.45$$

$$\sqrt{(x-0.90)^2 + (y-0.3)^2} \le 0.45$$

$$(x-0.9)^2 + (y-0.3)^2 \le 0.2025$$

$Q_{obstacle} = \{ (x,y,\theta) : \| (x,y) - (0.9, 0.3) \| \leq 0.2025 \}$

The obstacle does not depend on θ under this approximation.

b.) $q^* = (1.20, 0.40, \theta = 0.524)$

$q^* \in Q_{obs}$?

$$(x - 0.90)^2 + (y - 0.3)^2 \leq 0.2025$$

$(x,y) = (1.20 \quad 0.40)$

$$(1.20 - 0.90)^2 + (0.4 - 0.3)^2 = d^2 = 0.10 \quad \leq 0.2025$$

Hence $q^* \in Q_{obs}$ since the point $(1.20, 0.40)$ is inside the C-Space obstacle

4.)

$Q_{free} = Q \setminus Q_{obs}$

Q_{free} is considered to be "path connected" if there exists a path in the configuration space $c : [0,1] \to Q$ for any two points $q \in Q$

Problem 3

1.)

a.)

$$P_E = R(\theta_1) \begin{bmatrix} L_1 \\ 0 \end{bmatrix} + R(\theta_1 + \theta_2) \begin{bmatrix} L_2 \\ 0 \end{bmatrix} =$$

$$P_E = \begin{bmatrix} c\theta_1 & -s\theta_1 \\ s\theta_1 & c\theta_1 \end{bmatrix} \begin{bmatrix} L_1 \\ 0 \end{bmatrix} + \begin{bmatrix} c(\theta_1+\theta_2) & -s(\theta_1+\theta_2) \\ s(\theta_1+\theta_2) & c(\theta_1+\theta_2) \end{bmatrix} \begin{bmatrix} L_2 \\ 0 \end{bmatrix}$$

$$P_E = P_1 + P_2 = \begin{bmatrix} L_1 c\theta_1 \\ L_1 s\theta_1 \end{bmatrix} + \begin{bmatrix} L_2 c(\theta_1+\theta_2) \\ L_2 s(\theta_1+\theta_2) \end{bmatrix}$$

$$P_E = \begin{bmatrix} L_1 c\theta_1 + L_2 c(\theta_1+\theta_2) \\ L_1 s\theta_1 + L_2 s(\theta_1+\theta_2) \end{bmatrix}$$

Think of vector expression as Rotate by θ and translate along local x-axis

b.)

$$x(\theta_1, \theta_2) = L_1 \cos\theta_1 + L_2 \cos(\theta_1+\theta_2)$$
$$y(\theta_1, \theta_2) = L_1 \sin(\theta_1) + L_2 \sin(\theta_1+\theta_2)$$

This is based on $R(\alpha) \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} \cos\alpha \\ \sin\alpha \end{bmatrix}$

c.) $\phi(\theta_1, \theta_2) = \theta_1 + \theta_2$

The end effector orientation is the sum of both the joint angles as each joint angle contributes additively to the total rotation. This also makes sense since both rotations are CCW relative to the same axis, resulting in a total orientation of $\theta_1 + \theta_2$ relative to the base frame.

2.)

a.) $A_i = A_i(q_i)$ → rotation of frame i in reference to i-1

$$A_i(q_i) = \begin{bmatrix} R_n^0 & o_n^0 \\ 0 & 1 \end{bmatrix}$$ → origin in reference

$H = T_n^0 = A_1(q_1) \dots A_n(q_n)$

From Base → EE

2 Planar Rotations + Translations

To apply this to our problem

$$T_1^0 = \begin{bmatrix} c\theta_1 & -s\theta_1 & 0 \\ s\theta_1 & c\theta_1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad T_2^1 = \begin{bmatrix} 1 & 0 & L_1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad T_3^2 = \begin{bmatrix} c\theta_2 & -s\theta_2 & 0 \\ s\theta_2 & c\theta_2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad T_4^3 = \begin{bmatrix} 1 & 0 & L_2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$T_E^0 = H_0^4 = H_0^1 H_2^1 H_3^2 H_4^3 = \begin{bmatrix} c(\theta_1+\theta_2) & -s(\theta_1+\theta_2) & L_1 c\theta_1 + L_2 c(\theta_1+\theta_2) \\ s(\theta_1+\theta_2) & c(\theta_1+\theta_2) & L_1 s\theta_1 + L_2 s(\theta_1+\theta_2) \\ 0 & 0 & 1 \end{bmatrix}$$

Since $A_i(q_i) = \begin{bmatrix} R_4^0 & o_4^0 \\ 0 & 1 \end{bmatrix}$

we can see

$$o_4^0 = \begin{bmatrix} L_1 c\theta_1 + L_2 c(\theta_1+\theta_2) \\ L_1 s\theta_1 + L_2 s(\theta_1+\theta_2) \end{bmatrix} = PE$$

$L_1 = 1m$

$L_2 = 0.8m$

hence $PE = \begin{bmatrix} c\theta_1 + 0.8\, c(\theta_1+\theta_2) \\ s\theta_1 + 0.8\, s(\theta_1+\theta_2) \end{bmatrix}$

3.) Numeric Evaluation

$\theta_1 = 30°$ → $\theta_1 = \frac{\pi}{6}$

$\theta_2 = 60°$ → $\theta_2 = \frac{\pi}{3}$

$x = 1\cos(\frac{\pi}{6}) + 0.8\cos(\frac{\pi}{2}) = 0.866$

$y = \sin(\frac{\pi}{6}) + 0.8\sin(\frac{\pi}{2}) = 1.300$

$\phi = 1.571$ rad

$\phi = \theta_1 + \theta_2 = \frac{\pi}{6} + \frac{\pi}{3} = \frac{\pi}{2}$

$$\therefore T_E^0 = \begin{bmatrix} 0 & -1 & 0.866 \\ 1 & 0 & 1.30 \\ 0 & 0 & 1 \end{bmatrix}$$

4.) $\{G\}$ is translated along $\{x_E\}$ by $d_g = 0.10$m

let $\phi = \theta_1 + \theta_2$

$$^{E}T_G = \begin{bmatrix} R_G^E & o_G^E \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & d_g \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0.1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$^{0}T_G = {}^{0}T_E \, {}^{E}T_G = \begin{bmatrix} c\phi & -s\phi & x \\ s\phi & c\phi & y \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & d_g \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$T_G^0 = \begin{bmatrix} c\phi & -s\phi & x + d_g c\phi \\ s\phi & c\phi & y + d_g s\phi \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} c(\theta_1+\theta_2) & -s(\theta_1+\theta_2) & L_1 c\theta_1 + (L_2+d_g)c(\theta_1+\theta_2) \\ s(\theta_1+\theta_2) & c(\theta_1+\theta_2) & L_1 s\theta_1 + (L_2+d_g)s(\theta_1+\theta_2) \\ 0 & 0 & 1 \end{bmatrix}$$

Now numerical substitution where $\theta_1 = \frac{\pi}{6}$ $\theta_2 = \frac{\pi}{3}$ $L_1 = 1.0$ $L_2 = 0.8$

we know $P_E = \begin{bmatrix} x_E \\ y_E \end{bmatrix} = \begin{bmatrix} \cos\frac{\pi}{6} + 0.8\cos\frac{\pi}{2} \\ \sin\frac{\pi}{6} + 0.8\sin\frac{\pi}{2} \end{bmatrix} = \begin{bmatrix} 0.866 \\ 1.300 \end{bmatrix}$

$$x_G = x_E + d_g\cos\phi = 0.866 + 0.1\cos\frac{\pi}{2} = 0.866$$

$$y_G = y_E + d_g\sin\phi = 1.33 + 0.1\sin\frac{\pi}{2} = 1.44$$

$$x_G = L_1 c\theta_1 + (L_2+d_g)c(\theta_1+\theta_2) = x_E + d_g c(\theta_1+\theta_2)$$

$$y_G = L_1 s\theta_1 + (L_2+d_g)s(\theta_1+\theta_2) = y_E + d_g s(\theta_1+\theta_2)$$

Problem 4 Inverse kinematics

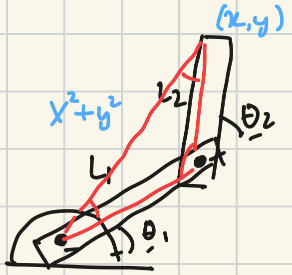
1.)

$$r = \sqrt{x^2+y^2}$$

$$|L_1 - L_2| < r \leq L_1 + L_2$$

The point must lie in the annulus between the "inner" circle (when the arm is folded back) and the "outer" circle (arm fully stretched).

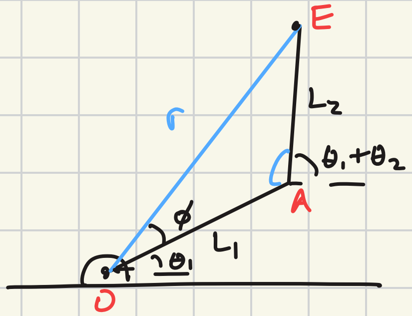
2.)



$$a^2 = b^2 + c^2 - 2bc\cos A$$

$$\theta_2 = a\cos\left(\frac{x^2+y^2-L_1^2-L_2^2}{2L_1L_2}\right)$$

$x_E = L_1\cos\theta_1 + L_2\cos(\theta_1+\theta_2)$ $y_E = L_1\sin\theta_1 + L_2\sin(\theta_1+\theta_2)$



$r^2 = L_1^2 + L_2^2 - 2L_1L_2\cos(\text{int angle})$

$r^2 = L_1^2 + L_2^2 - 2L_1L_2\cos(\pi-\theta_2)$

$\pi-\theta_2$ since defined relative to θ_1

$$r^2 = L_1^2 + L_2^2 - 2l_1 l_2\cos(\pi-\theta_2)$$

$$r^2 = l_1^2 + l_2^2 + 2l_1 l_2\cos(\theta_2)$$

since $\cos(-\theta) = -\cos(\theta)$

Due to the even property of the cosine function we have 2 θ_2 values that can create the orientation pose.

where $r^2 = L_1^2 + L_2^2$

$$r^2 - L_1^2 - L_2^2 = 2L_1L_2\cos(\theta_2)$$

$$\theta_2 = \arccos\left[\pm\frac{r^2-L_1^2-L_2^2}{2L_1L_2}\right]$$

$\theta_2 =$

Positive branch = Elbow up

Negative branch = Elbow down since $-\theta_2$ means bent downward relative to L_1/θ_1

3.)

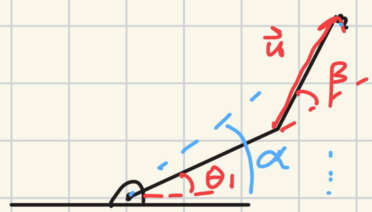
$$\begin{bmatrix} x_E \\ y_E \end{bmatrix} = \begin{bmatrix} L_1 c\theta_1 + L_2 c(\theta_1+\theta_2) \\ L_1 s\theta_1 + L_2 s(\theta_1+\theta_2) \end{bmatrix} = L_1\begin{bmatrix} c\theta_1 \\ s\theta_1 \end{bmatrix} + L_2\begin{bmatrix} c(\theta_1+\theta_2) \\ s(\theta_1+\theta_2) \end{bmatrix}$$

$$= R(\theta_1)\left[\begin{bmatrix} L_1 \\ 0 \end{bmatrix} + R(\theta_2)\begin{bmatrix} L_2 \\ 0 \end{bmatrix}\right]$$

$$= R(\theta_1)\left[\overbrace{\begin{bmatrix} L_1 \\ 0 \end{bmatrix} + \begin{bmatrix} c\theta_2 & -s\theta_2 \\ s\theta_2 & c\theta_2 \end{bmatrix}\begin{bmatrix} L_2 \\ 0 \end{bmatrix}}^{L_1 + R(\theta_2)L_2}\right]$$

$$P_E = R(\theta_1)\begin{bmatrix} L_1 + L_2 c\theta_2 \\ L_2 s\theta_2 \end{bmatrix}$$

Hence think of $p = R(\theta_1)\vec{u}$ as $\vec{u} = \begin{bmatrix} L_1 + L_2 c\theta_2 \\ L_2 s\theta_2 \end{bmatrix}$



$\alpha \to$ Absolute angle for p_E is $\operatorname{atan}(y_E, x_E)$

$\beta \to$ Absolute angle for vector $\vec{u}$

$$\theta_1 = \alpha - \beta = \operatorname{atan2}(y_E, x_E) - \operatorname{atan2}(L_1\sin\theta_2,\ L_1 + L_2\cos(\theta_2))$$

There are 2 possible values for the elbow up and the elbow down configuration based on the θ_2 value which is used.

$+\theta_2 \Rightarrow +\sin\theta_2$ Elbow up

$-\theta_2 \Rightarrow -\sin\theta_2$ Elbow down

Elbow up: $(s\theta_2 \geq 0)$

$$\theta_1^{up} = \text{atan2}(y,x) - \text{atan2}(L_2 \sin\theta_2,\ L_1 + L_2\cos\theta_2)$$

Elbow down: $(s\theta_2 \leq 0)$

$$\theta_1^{down} = \text{atan2}(y,x) - \text{atan2}(-L_2\sin\theta_2,\ L_1 + L_2\cos\theta_2)$$
$$= \text{atan2}(y,x) + \text{atan2}(L_2|\sin\theta_2|,\ L_1 + L_2\cos\theta_2)$$

4.) Target point $x^* = (x^*, y^*) = (1.20, 0.40)$

$\theta_1 \in [-\pi, \pi)$

$\theta_2 \in \left[-\frac{3\pi}{4}, \frac{3\pi}{4}\right]$

$L_1 = 1.0m$
$L_2 = 0.8m$

$$\theta_2 = \pm \arccos\left[\frac{r^2 - L_1^2 - L_2^2}{2L_1L_2}\right]$$

$$\theta_2 = \pm \arccos\left[\frac{1.6 - 1^2 - 0.8^2}{2(1)(0.8)}\right] \Rightarrow \pm 1.596 \text{ rad}$$

$\theta_2^{up} = 1.596$ rad $\quad \theta_2^{down} = -1.596$ rad
which are within
$\theta_2 \in \left[-\frac{3\pi}{4}, \frac{3\pi}{4}\right]$

$$p_E = \begin{bmatrix} x_E \\ y_E \end{bmatrix} = \begin{bmatrix} L_1 \cos\theta_1 + L_2 \cos(\theta_1+\theta_2) \\ L_1 \sin\theta_1 + L_2 \sin(\theta_1+\theta_2) \end{bmatrix}$$

$$r = \left[x_E^2 + y_E^2 \right]^{1/2} \Rightarrow \left[L_1 c\theta_1 + L_2 c(\theta_1+\theta_2) \right]^2 + \left[L_1 s\theta_1 + L_2 s(\theta_1+\theta_2) \right]^2$$

$$= (L_1 c\theta_1)^2 + 2(L_1 c\theta_1)(L_2 c(\theta_1+\theta_2)) + [L_2 c(\theta_1+\theta_2)]^2 + (L_1 s\theta_1)^2 + 2(L_1 s\theta_1)(L_2 s(\theta_1+\theta_2)) + [L_2 s(\theta_1+\theta_2)]^2$$

$$= L_1^2 c^2\theta_1 + 2L_1L_2 c\theta_1 c(\theta_1+\theta_2) + L_2^2 c^2(\theta_1+\theta_2) + L_1^2 s^2\theta_1 + 2L_1L_2 s\theta_1 s(\theta_1+\theta_2) + L_2^2 s^2(\theta_1+\theta_2)$$

$$= L_1^2 + 2L_1L_2 \left(c\theta_1 c(\theta_1+\theta_2) + s\theta_1 s(\theta_1+\theta_2) \right) + L_2^2$$

$$= c\theta_1 (c\theta_1 c\theta_2 - s\theta_1 s\theta_2) + s\theta_1 (s\theta_1 c\theta_2 + s\theta_2 c\theta_1)$$

$$= c^2\theta_1 c\theta_2 - \cancel{c\theta_1 s\theta_1 s\theta_2} + s^2\theta_1 c\theta_2 + \cancel{s\theta_1 s\theta_2 c\theta_1}$$

$$c\theta_2 (s^2\theta_1 + c^2\theta_1) = c\theta_2$$

$$x_E^2 + y_E^2 = L_1^2 + 2L_1L_2 c\theta_2 + L_2^2$$

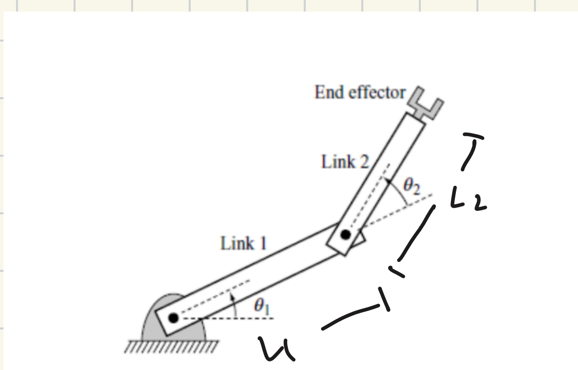
readability $= r = \sqrt{x_E^2 + y_E^2} = \left(L_1^2 + 2L_1L_2 c\theta_2 + L_2^2 \right)^{1/2}$

now plug back in

$L_1 = 1.0\text{m}$ $\quad \theta_1 = \frac{\pi}{6}$ $\quad \rightarrow \quad \left(1^2 + 2(1)(0.8) c\frac{\pi}{3} + 0.8^2 \right)^{1/2}$

$L_2 = 0.8\text{m}$ $\quad \theta_2 = \frac{\pi}{3}$ $\quad r = 1.562\text{m}$

Problem 3



$p_E = [x, y]^T$ in $\{0\}$ frame

$\{0\} \rightarrow \{1\}$ in the base frame

$$R_0^1 = \begin{bmatrix} c\theta_1 & -s\theta_1 & 0 \\ s\theta_1 & c\theta_1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

The movement along L_1 can be characterized by a translation L_1 in the $\hat{x}_0$ dir

$$H_0^1 = \begin{bmatrix} c\theta_1 & -s\theta_1 & 0 & L_1 \\ s\theta_1 & c\theta_1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$H_0^1 =$ in 2D $\begin{bmatrix} c\theta_1 & -s\theta_1 & L_1 \\ s\theta_1 & c\theta_1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$\{1\} \to \{2\}$ is similarly defined as

This is in the 1 frame

$$H_1^2 = \begin{bmatrix} c\theta_2 & -s\theta_2 & L_2 \\ s\theta_2 & c\theta_2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$H_1^2 = \begin{bmatrix} c\theta_2 & -s\theta_2 & 0 & L_2 \\ s\theta_2 & c\theta_2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$H_0^E = H_0^2 = H_1^2 \, H_0^1 = \begin{bmatrix} c\theta_2 & -s\theta_2 & 0 & L_2 \\ s\theta_2 & c\theta_2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} c\theta_1 & -s\theta_1 & 0 & L_1 \\ s\theta_1 & c\theta_1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \\ 1 \end{bmatrix}_{\hat{0}} = \begin{bmatrix} c\theta_2 & -s\theta_2 & L_2 \\ s\theta_2 & c\theta_2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \underbrace{\begin{bmatrix} c\theta_1 & -s\theta_1 & L_1 \\ s\theta_1 & c\theta_1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{3\times 3} \underbrace{\begin{bmatrix} x \\ y \\ 1 \end{bmatrix}}_{3\times 1}$$

$$= \begin{bmatrix} c\theta_2 & -s\theta_2 & L_2 \\ s\theta_2 & c\theta_2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x c\theta_1 - y s\theta_1 + L_1 \\ x s\theta_1 + y c\theta_1 \\ 1 \end{bmatrix}$$

$c\theta_1 c\theta_2 - s\theta_1 s\theta_2 = c(\theta_1+\theta_2)$

$s\theta_1 c\theta_2 + c\theta_1 s\theta_2 = s(\theta_1+\theta_2)$

$$= \begin{bmatrix} c\theta_2 (x c\theta_1 - y s\theta_1 + L_1) - s\theta_2 (x s\theta_1 + y c\theta_1) + L_2 \\ s\theta_2 (x c\theta_1 - y s\theta_1 + L_1) + c\theta_2 (x s\theta_1 + y c\theta_1) \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \\ 1 \end{bmatrix}_{\hat{0}} = \begin{bmatrix} x c\theta_1 c\theta_2 - y s\theta_1 c\theta_2 + L_1 c\theta_2 - x s\theta_1 s\theta_2 - y c\theta_1 s\theta_2 + L_2 \\ x c\theta_1 s\theta_2 - y s\theta_1 s\theta_2 + L_1 s\theta_2 + x s\theta_1 c\theta_2 + y c\theta_1 c\theta_2 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \\ 1 \end{bmatrix}_{\hat{B}} = \begin{bmatrix} x(c\theta_1 c\theta_2 - s\theta_1 s\theta_2) - y(s\theta_1 c\theta_2 + c\theta_1 s\theta_2) + L_1 c\theta_2 + L_2 \\ x(c\theta_1 s\theta_2 + s\theta_1 c\theta_2) + y(c\theta_1 c\theta_2 - s\theta_1 s\theta_2) + L_1 s\theta_2 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \\ 1 \end{bmatrix}_{\hat{B}} = \begin{bmatrix} x\, c(\theta_1+\theta_2) - y\, s(\theta_1+\theta_2) + L_1 c\theta_2 + L_2 \\ x\, s(\theta_1+\theta_2) + y\, c(\theta_1+\theta_2) + L_1 s\theta_2 \\ 1 \end{bmatrix}$$

$\therefore$ $\begin{bmatrix} x \\ y \end{bmatrix}$ in base frame $= \begin{bmatrix} x\, c(\theta_1+\theta_2) - y\, s(\theta_1+\theta_2) + (L_1 c\theta_2 + L_2) \\ x\, s(\theta_1+\theta_2) + y\, c(\theta_1+\theta_2) + L_1 s\theta_2 \end{bmatrix}$

b.) scalar formulas for $x(\theta_1,\theta_2)$, $y(\theta_1,\theta_2)$

Based on the geometry of the 2 link planar arm, we know that it cannot be in an orientation that is longer/larger than the length $L_1 + L_2$

Hence $D = \sqrt{x_B^2 + y_B^2} < L_1 + L_2$