Computational Aspects of Robotics HW 1

Problem 1

$p_A = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}$ in frame $\{A\}$

$\{B\}$ rotated $90°$ about z-axis

$t = \begin{bmatrix} 1 \\ -2 \\ 0 \end{bmatrix}$ m

1.) $p_A^{cart} = \begin{bmatrix} 2 \\ 1 \\ 1 \\ 1 \end{bmatrix}$ in homogenous coordinates

2.) $^A T_B = \begin{bmatrix} R & t \\ 0\,0\,0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & -1 & 0 & 1 \\ 1 & 0 & 0 & -2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

$R(\theta, z) = \begin{bmatrix} c\theta & -s\theta & 0 \\ s\theta & c\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$p_A = {}^A T_B \, p_B$

↳ frame $\{B\}$ in $\{A\}$ coordinates

$\begin{bmatrix} R & t \\ 0 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} R^T & -R^T t \\ 0 & 1 \end{bmatrix}$

3.) p_{cart} in B coordinates?

We have $p_A^{cart} = \begin{bmatrix} 2 \\ 1 \\ 1 \\ 1 \end{bmatrix}$

hence now we need

$p_B^{cart} = {}^B T_{\cancel{A}} \, p_{\cancel{A}}^{cart}$

$^B T_A = [{}^A T_B]^{-1} \rightarrow \quad p_B^{cart} = [{}^A T_B]^{-1} \, p_A^{CART}$

$^B R_A = {}^A R_B^T = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$\underset{\text{defined}}{-{}^B R_A t} = -\begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ -2 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$

$^B T_A = \begin{bmatrix} 0 & 1 & 0 & 2 \\ -1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

$^B p_{CART} = {}^B T_A \, p_A^{CART} = \begin{bmatrix} 0 & 1 & 0 & 2 \\ -1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \\ 1 \\ 1 \end{bmatrix}$

$p_B^{CART} = \begin{bmatrix} -3 \\ 1 \\ 1 \\ 1 \end{bmatrix}$

Problem 2

1.) a.) $q = (x, y, \theta) \in \mathbb{R}^2 \times S^1$

$$Q = [0,5] \times [0,4] \times (-\pi, \pi] \subset \mathbb{R}^2 \times S^1$$

$$Q = \{ q = (x,y,\theta) : x \in [0,5], y \in [0,4], \theta \in (-\pi, \pi] \}$$

$$q \in Q \subset \mathbb{R}^2 \times S^1$$

b.)

There are 3 DOFs in this robot, 2D translational (x,y) and 1 rotational (θ)

2.) a.) $W = [0,5] \times [0,4]$

$$W = \{ (x,y) : x \in [0,5], y \in [0,4] \}$$

$$W \subset \mathbb{R}^2 \times S^1$$

b.) Footprint → Disk $r_R = 0.35m$

To include footprint, we perform a "Minkowski Sum"

$$A \oplus B : \{ a+b \mid a \in A, b \in B \}$$

hence

$$W_{clear} = W \oplus (-\text{Footprint})$$

$$W_{clear} = \{ (x,y) : x \in [0.35, 4.65], y \in [0.35, 3.65] \}$$

$$W_{clear} \subset \mathbb{R}^2 \times S^1$$

c.) The point $(x,y) = (0.30, 0.30)$ is not reachable without collision

as $(0.30, 0.30) \notin W_{clear}$ as defined in part b.

3.)

c.) $O = \{ (x,y) : \| (x,y) - (0.9, 0.3) \| \leq 0.10 \}$

Obstacle is a circular object with center at $(x,y) = (0.9, 0.3)$ and radius of 0.10m but we must also consider the robot footprint hence:

$$\sqrt{(x-0.9)^2 + (y-0.3)^2} \leq r_o + r_{footprint} = 0.10 + 0.35 = 0.45$$

$$\sqrt{(x-0.90)^2 + (y-0.3)^2} \leq 0.45$$

$$(x-0.9)^2 + (y-0.3)^2 \leq 0.2025$$

$Q_{obstacle} = \{ (x,y,\theta) : \| (x,y) - (0.9, 0.3) \| \leq 0.2025 \}$

The obstacle does not depend on θ under this approximation.

b.) $q^* = (1.20,\ 0.40,\ \theta = 0.524)$

$q^* \in Q_{obs}$?

$$(x - 0.90)^2 + (y - 0.3)^2 \leq 0.2025$$

$$(x,y) = (1.20\ \ 0.40)$$

$$(1.20 - 0.90)^2 + (0.4 - 0.3)^2 = d^2 = 0.10 \leq 0.2025$$

Hence $q^* \in Q_{obs}$ since the point $(1.20, 0.40)$ is inside the C-Space obstacle

4.)

$Q_{free} = Q \setminus Q_{obs}$

Q_{free} is considered to be "path connected" if there exists a path in the configuration space $c : [0,1] \to Q$ for any two points $q \in Q$

Problem 3

1.)

a.)

$$P_E = R(\theta_1) \begin{bmatrix} L_1 \\ 0 \end{bmatrix} + R(\theta_1 + \theta_2) \begin{bmatrix} L_2 \\ 0 \end{bmatrix} =$$

$$P_E = \begin{bmatrix} c\theta_1 & -s\theta_1 \\ s\theta_1 & c\theta_1 \end{bmatrix} \begin{bmatrix} L_1 \\ 0 \end{bmatrix} + \begin{bmatrix} c(\theta_1+\theta_2) & -s(\theta_1+\theta_2) \\ s(\theta_1+\theta_2) & c(\theta_1+\theta_2) \end{bmatrix} \begin{bmatrix} L_2 \\ 0 \end{bmatrix}$$

$$P_E = P_1 + P_2 = \begin{bmatrix} L_1 c\theta_1 \\ L_1 s\theta_1 \end{bmatrix} + \begin{bmatrix} L_2 c(\theta_1+\theta_2) \\ L_2 s(\theta_1+\theta_2) \end{bmatrix}$$

$$P_E = \begin{bmatrix} L_1 c\theta_1 + L_2 c(\theta_1+\theta_2) \\ L_1 s\theta_1 + L_2 s(\theta_1+\theta_2) \end{bmatrix}$$

Think of vector expression as Rotate by θ and translate along local x-axis

b.)

$$x(\theta_1, \theta_2) = L_1 \cos\theta_1 + L_2 \cos(\theta_1 + \theta_2)$$

$$y(\theta_1, \theta_2) = L_1 \sin(\theta_1) + L_2 \sin(\theta_1 + \theta_2)$$

This is based on $R(\alpha) \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} \cos\alpha \\ \sin\alpha \end{bmatrix}$

c.) $\phi(\theta_1, \theta_2) = \theta_1 + \theta_2$

The end effector orientation is the sum of both the joint angles as each joint angle contributes additively to the total rotation. This also makes sense since both rotations are CCW relative to the same axis, resulting in a total orientation of $\theta_1 + \theta_2$ relative to the base frame.

2.)

a.) $A_i = A_i(q_i)$ rotation of frame i in reference to i-1

$$A_i(q_i) = \begin{bmatrix} R_n^0 & o_n^0 \\ 0 & 1 \end{bmatrix}$$ origin in reference

$$H = T_n^0 = A_1(q_1) \dots A_n(q_n)$$

From Base → EE

2 Planar Rotations + Translations

To apply this to our problem

$$T_1^0 = \begin{bmatrix} c\theta_1 & -s\theta_1 & 0 \\ s\theta_1 & c\theta_1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad T_2^1 = \begin{bmatrix} 1 & 0 & L_1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad T_3^2 = \begin{bmatrix} c\theta_2 & -s\theta_2 & 0 \\ s\theta_2 & c\theta_2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad T_4^3 = \begin{bmatrix} 1 & 0 & L_2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$T_E^0 = H_0^4 = H_0^1 H_2^1 H_3^2 H_4^3 = \begin{bmatrix} c(\theta_1+\theta_2) & -s(\theta_1+\theta_2) & L_1 c\theta_1 + L_2 c(\theta_1+\theta_2) \\ s(\theta_1+\theta_2) & c(\theta_1+\theta_2) & L_1 s\theta_1 + L_2 s(\theta_1+\theta_2) \\ 0 & 0 & 1 \end{bmatrix}$$

Since $A_i(q_i) = \begin{bmatrix} R_4^0 & o_4^0 \\ 0 & 1 \end{bmatrix}$

we can see

$$o_4^0 = \begin{bmatrix} L_1 c\theta_1 + L_2 c(\theta_1+\theta_2) \\ L_1 s\theta_1 + L_2 s(\theta_1+\theta_2 \end{bmatrix} = PE$$

$L_1 = 1m$
$L_2 = 0.8m$

hence $$P_E = \begin{bmatrix} c\theta_1 + 0.8c(\theta_1+\theta_2) \\ s\theta_1 + 0.8s(\theta_1+\theta_2) \end{bmatrix}$$

3.) Numeric Evaluation

$\theta_1 = 30°$
$\theta_2 = 60°$ → $\theta_1 = \frac{\pi}{6}$, $\theta_2 = \frac{\pi}{3}$

$x = 1\cos(\frac{\pi}{6}) + 0.8\cos(\frac{\pi}{2}) = 0.866$

$y = \sin(\frac{\pi}{6}) + 0.8\sin(\frac{\pi}{2}) = 1.300$

$\phi = 1.571$ rad

$$\phi = \theta_1 + \theta_2 = \frac{\pi}{6} + \frac{\pi}{3} = \frac{\pi}{2}$$

$$\therefore T_E^0 = \begin{bmatrix} 0 & -1 & 0.866 \\ 1 & 0 & 1.30 \\ 0 & 0 & 1 \end{bmatrix}$$

4.) $\{G\}$ is translated along $\{x_E\}$ by $d_g = 0.10m$

let $\phi = \theta_1 + \theta_2$

$${}^{E}T_G = \begin{bmatrix} P_G^E & O_G^E \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & d_g \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0.1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$${}^{0}T_G = {}^{0}T_E \, {}^{E}T_G = \begin{bmatrix} c\phi & -s\phi & x \\ s\phi & c\phi & y \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & d_g \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$T_G^0 = \begin{bmatrix} c\phi & -s\phi & x + d_g c\phi \\ s\phi & c\phi & y + d_g s\phi \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} c(\theta_1+\theta_2) & -s(\theta_1+\theta_2) & L_1 c\theta_1 + (L_2+d_g)c(\theta_1+\theta_2) \\ s(\theta_1+\theta_2) & c(\theta_1+\theta_2) & L_1 s\theta_1 + (L_2+d_g)s(\theta_1+\theta_2) \\ 0 & 0 & 1 \end{bmatrix}$$

Now numerical substitution where $\theta_1 = \frac{\pi}{6}$ $\theta_2 = \frac{\pi}{3}$ $L_1 = 1.0$ $L_2 = 0.8$

we know $P_E = \begin{bmatrix} x_E \\ y_E \end{bmatrix} = \begin{bmatrix} \cos\frac{\pi}{6} + 0.8\cos\frac{\pi}{2} \\ \sin\frac{\pi}{6} + 0.8\sin\frac{\pi}{2} \end{bmatrix} = \begin{bmatrix} 0.866 \\ 1.300 \end{bmatrix}$

$$x_G = x_E + d_g \cos\phi = 0.866 + 0.1\cos\frac{\pi}{2} = 0.866$$
$$y_G = y_E + d_g \sin\phi = 1.33 + 0.1\sin\frac{\pi}{2} = 1.44$$

$$x_G = L_1 c\theta_1 + (L_2+d_g)c(\theta_1+\theta_2) = x_E + d_g c(\theta_1+\theta_2)$$
$$y_G = L_1 s\theta_1 + (L_2+d_g)s(\theta_1+\theta_2) = y_E + d_g s(\theta_1+\theta_2)$$

Problem 4 Inverse kinematics

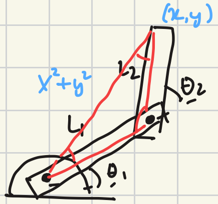
1.)

$$r = \sqrt{x^2+y^2}$$

$$|L_1 - L_2| < r \le L_1 + L_2$$

The point must lie in the annulus between the "inner" circle (when the arm is folded back) and the "outer" circle (arm fully stretched).

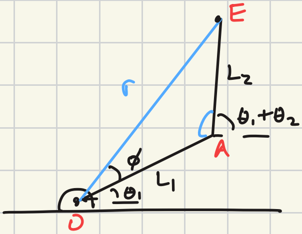
2.)



$$a^2 = b^2 + c^2 - 2bc\cos A$$

$$\theta_2 = a\cos\left(\frac{x^2+y^2-L_1^2-L_2^2}{2L_1L_2}\right)$$

$x_E = L_1\cos\theta_1 + L_2\cos(\theta_1+\theta_2)$ $\quad y_E = L_1\sin\theta_1 + L_2\sin(\theta_1+\theta_2)$



$r^2 = L_1^2 + L_2^2 - 2L_1L_2\cos(\text{int angle})$

$\pi - \theta_2$ since defined relative to θ_1

$r^2 = L_1^2 + L_2^2 - 2L_1L_2\cos(\pi - \theta_2)$

$$r^2 = L_1^2 + L_2^2 - 2l_1 l_2 \cos(\pi - \theta_2)$$

$$r^2 = l_1^2 + l_2^2 + 2l_1 l_2 \cos(\theta_2)$$

since $\cos(-\theta) = -\cos(\theta)$

Due to the even property of the cosine function we have 2 θ_2 values that can create the orientation pose.

where $r^2 = L_1^2 + L_2^2$

$$r^2 - L_1^2 - L_2^2 = 2L_1L_2\cos(\theta_2)$$

$$\theta_2 = \arccos\left[\pm\frac{r^2 - L_1^2 - L_2^2}{2L_1L_2}\right]$$

$\theta_2 =$

Positive branch = Elbow up

Negative branch = Elbow down since $-\theta_2$ means bent downward relative to L_1/θ_1

3.)

$$\begin{bmatrix} x_E \\ y_E \end{bmatrix} = \begin{bmatrix} L_1 c\theta_1 + L_2 c(\theta_1+\theta_2) \\ L_1 s\theta_1 + L_2 s(\theta_1+\theta_2) \end{bmatrix} = L_1\begin{bmatrix} c\theta_1 \\ s\theta_1 \end{bmatrix} + L_2\begin{bmatrix} c(\theta_1+\theta_2) \\ s(\theta_1+\theta_2) \end{bmatrix}$$

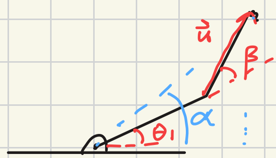
$$= R(\theta_1)\left[\begin{bmatrix} L_1 \\ 0 \end{bmatrix} + R(\theta_2)\begin{bmatrix} L_2 \\ 0 \end{bmatrix}\right]$$

$$= R(\theta_1)\left[\begin{bmatrix} L_1 \\ 0 \end{bmatrix} + \begin{bmatrix} c\theta_2 & -s\theta_2 \\ s\theta_2 & c\theta_2 \end{bmatrix}\begin{bmatrix} L_2 \\ 0 \end{bmatrix}\right]$$

$L_1 + R(\theta_2)L_2$

$$P_E = R(\theta_1)\begin{bmatrix} L_1 + L_2 c\theta_2 \\ L_2 s\theta_2 \end{bmatrix}$$

Hence think of $p = R(\theta_1)\vec{u}$ as $\vec{u} = \begin{bmatrix} L_1 + L_2 c\theta_2 \\ L_2 s\theta_2 \end{bmatrix}$



$\alpha \rightarrow$ Absolute angle for p_E is $\operatorname{atan}(y_E, x_E)$

$\beta \rightarrow$ Absolute angle for vector $\vec{u}$

$$\theta_1 = \alpha - \beta = \operatorname{atan2}(y_E, x_E) - \operatorname{atan2}(L_2\sin\theta_2,\ L_1 + L_2\cos(\theta_2))$$

There are 2 possible values for the elbow up and the elbow down configuration based on the θ_2 value which is used.

$+\theta_2 \Rightarrow +\sin\theta_2$ Elbow up

$-\theta_2 \Rightarrow -\sin\theta_2$ Elbow down

Elbow up: $(s\theta_2 \geq 0)$

$$\theta_1^{up} = \text{atan2}(y,x) - \text{atan2}(L_2 \sin\theta_2,\ L_1 + L_2\cos\theta_2)$$

Elbow down: $(s\theta_2 \leq 0)$

$$\theta_1^{down} = \text{atan2}(y,x) - \text{atan2}(-L_2\sin\theta_2,\ L_1 + L_2\cos\theta_2)$$
$$= \text{atan2}(y,x) + \text{atan2}(L_2|\sin\theta_2|,\ L_1 + L_2\cos\theta_2)$$

4.) Target point $x^* = (x^*, y^*) = (1.20, 0.40)$

$\theta_1 \in [-\pi, \pi)$

$\theta_2 \in \left[-\frac{3\pi}{4}, \frac{3\pi}{4}\right]$

$L_1 = 1.0m$

$L_2 = 0.8m$

$$\theta_2 = \pm \arccos\left[\frac{r^2 - L_1^2 - L_2^2}{2L_1L_2}\right]$$

$$\theta_2 = \pm \arccos\left[\frac{1.6 - 1^2 - 0.8^2}{2(1)(0.8)}\right] \Rightarrow \pm 1.596 \text{ rad}$$

$\theta_2^{up} = 1.596 \text{ rad}$ $\quad \theta_2^{down} = -1.596 \text{ rad}$

which are within

$\theta_2 \in \left[-\frac{3\pi}{4}, \frac{3\pi}{4}\right]$

$$P_E = \begin{bmatrix} x_E \\ y_E \end{bmatrix} = \begin{bmatrix} L_1\cos\theta_1 + L_2\cos(\theta_1+\theta_2) \\ L_1\sin\theta_1 + L_2\sin(\theta_1+\theta_2) \end{bmatrix}$$

$$r = \left[x_E^2+y_E^2\right]^{1/2} \Rightarrow \left[L_1 c\theta_1 + L_2 c(\theta_1+\theta_2)\right]^2 + \left[L_1 s\theta_1 + L_2 s(\theta_1+\theta_2)\right]^2$$

$$=(L_1 c\theta_1)^2 + 2(L_1 c\theta_1)(L_2 c(\theta_1+\theta_2)) + [L_2 c(\theta_1+\theta_2)]^2 + (L_1 s\theta_1)^2 + 2(L_1 s\theta_1)(L_2 s(\theta_1+\theta_2)) + [L_2 s(\theta_1+\theta_2)]^2$$

$$= L_1^2 c^2\theta_1 + 2L_1L_2 c\theta_1 c(\theta_1+\theta_2) + L_2^2 c^2(\theta_1+\theta_2) + L_1^2 s^2\theta_1 + 2L_1L_2 s\theta_1 s(\theta_1+\theta_2) + L_2^2 s^2(\theta_1+\theta_2)$$

$$= L_1^2 + 2L_1L_2\left(c\theta_1 c(\theta_1+\theta_2) + s\theta_1 s(\theta_1+\theta_2)\right) + L_2^2$$

$$= c\theta_1(c\theta_1 c\theta_2 - s\theta_1 s\theta_2) + s\theta_1(s\theta_1 c\theta_2 + s\theta_2 c\theta_1)$$

$$= c^2\theta_1 c\theta_2 - \cancel{c\theta_1 s\theta_1 s\theta_2} + s^2\theta_1 c\theta_2 + \cancel{s\theta_1 s\theta_2 c\theta_1}$$

$$c\theta_2(s^2\theta_1 + c^2\theta_1) = c\theta_2$$

$$x_E^2+y_E^2 = L_1^2 + 2L_1L_2 c\theta_2 + L_2^2$$

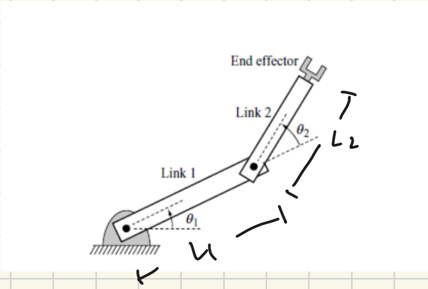
readability $= r = \sqrt{x_E^2+y_E^2} = (L_1^2 + 2L_1L_2 c\theta_2 + L_2^2)^{1/2}$

now plug back in

$L_1 = 1.0m$ $\quad \theta_1 = \frac{\pi}{6}$ $\quad \rightarrow \quad (1^2 + 2(1)(0.8)c\frac{\pi}{3} + 0.8^2)^{1/2}$

$L_2 = 0.8m$ $\quad \theta_2 = \frac{\pi}{3}$ $\quad r = 1.562m$

Problem 3



$P_E = [x,y]^T$ in $\{0\}$ frame

$\{0\} \rightarrow \{1\}$ in the base frame

$$R_0^1 = \begin{bmatrix} c\theta_1 & -s\theta_1 & 0 \\ s\theta_1 & c\theta_1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

The movement along L_1 can be characterized by a translation L_1 in the $\hat{x}_0$ dir

$$H_0^1 = \begin{bmatrix} c\theta_1 & -s\theta_1 & 0 & L_1 \\ s\theta_1 & c\theta_1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$H_0^1 = \begin{bmatrix} c\theta_1 & -s\theta_1 & L_1 \\ s\theta_1 & c\theta_1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

in 2D

$\{1\} \to \{2\}$ is similarly defined as

$$H_1^2 = \begin{bmatrix} c\theta_2 & -s\theta_2 & L_2 \\ s\theta_2 & c\theta_2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$H_1^2 = \begin{bmatrix} c\theta_2 & -s\theta_2 & 0 & L_2 \\ s\theta_2 & c\theta_2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

This is in the 1 frame

$$H_0^E = H_0^2 = H_1^2 \, H_0^1 = \begin{bmatrix} c\theta_2 & -s\theta_2 & 0 & L_2 \\ s\theta_2 & c\theta_2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} c\theta_1 & -s\theta_1 & 0 & L_1 \\ s\theta_1 & c\theta_1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \\ 1 \end{bmatrix}_{\hat{0}} = \begin{bmatrix} c\theta_2 & -s\theta_2 & L_2 \\ s\theta_2 & c\theta_2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \underbrace{\begin{bmatrix} c\theta_1 & -s\theta_1 & L_1 \\ s\theta_1 & c\theta_1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{3\times 3} \underbrace{\begin{bmatrix} x \\ y \\ 1 \end{bmatrix}}_{3\times 1}$$

$$= \begin{bmatrix} c\theta_2 & -s\theta_2 & L_2 \\ s\theta_2 & c\theta_2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x c\theta_1 - y s\theta_1 + L_1 \\ x s\theta_1 + y c\theta_1 \\ 1 \end{bmatrix}$$

$c\theta_1 c\theta_2 - s\theta_1 s\theta_2 = c(\theta_1+\theta_2)$

$s\theta_1 c\theta_2 + c\theta_1 s\theta_2 = s(\theta_1+\theta_2)$

$$= \begin{bmatrix} c\theta_2 (x c\theta_1 - y s\theta_1 + L_1) - s\theta_2 (x s\theta_1 + y c\theta_1) + L_2 \\ s\theta_2 (x c\theta_1 - y s\theta_1 + L_1) + c\theta_2 (x s\theta_1 + y c\theta_1) \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \\ 1 \end{bmatrix}_{\hat{0}} = \begin{bmatrix} x c\theta_1 c\theta_2 - y s\theta_1 c\theta_2 + L_1 c\theta_2 - x s\theta_1 s\theta_2 - y c\theta_1 s\theta_2 + L_2 \\ x c\theta_1 s\theta_2 - y s\theta_1 s\theta_2 + L_1 s\theta_2 + x s\theta_1 c\theta_2 + y c\theta_1 c\theta_2 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \\ 1 \end{bmatrix}_{\hat{B}} = \begin{bmatrix} x(c\theta_1 c\theta_2 - s\theta_1 s\theta_2) - y(s\theta_1 c\theta_2 + c\theta_1 s\theta_2) + L_1 c\theta_2 + L_2 \\ x(c\theta_1 s\theta_2 + s\theta_1 c\theta_2) + y(c\theta_1 c\theta_2 - s\theta_1 s\theta_2) + L_1 s\theta_2 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \\ 1 \end{bmatrix}_{\hat{B}} = \begin{bmatrix} x\, c(\theta_1+\theta_2) - y\, s(\theta_1+\theta_2) + L_1 c\theta_2 + L_2 \\ x\, s(\theta_1+\theta_2) + y\, c(\theta_1+\theta_2) + L_1 s\theta_2 \\ 1 \end{bmatrix}$$

$$\therefore \begin{bmatrix} x \\ y \end{bmatrix} \text{ in base frame} = \begin{bmatrix} x\, c(\theta_1+\theta_2) - y\, s(\theta_1+\theta_2) + (L_1 c\theta_2 + L_2) \\ x\, s(\theta_1+\theta_2) + y\, c(\theta_1+\theta_2) + L_1 s\theta_2 \end{bmatrix}$$

b.) scalar formulas for $x(\theta_1, \theta_2)$, $y(\theta_1, \theta_2)$

Based on the geometry of the 2 link planar arm, we know that it cannot be in an orientation that is longer/larger than the length $L_1 + L_2$

Hence $D = \sqrt{x_B^2 + y_B^2} < L_1 + L_2$