COMS W4733: Computational Aspects of Robotics
Homework 1

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Problem 1: Homogeneous Transformations

1. Convert to homogeneous coordinates (1 point)

Converting $p_A^{\text{cart}} = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}$ to homogeneous coordinates:

$$p_A = \begin{bmatrix} p_A^{\text{cart}} \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

2. Construct ${}^AT_B$ (4 points)

Frame $\{B\}$ is rotated 90° about the z-axis (counter-clockwise) relative to $\{A\}$:

$$R = R_z(90) = \begin{bmatrix} \cos(90) & -\sin(90) & 0 \\ \sin(90) & \cos(90) & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

With translation $t = \begin{bmatrix} 1 \\ -2 \\ 0 \end{bmatrix}$:

$${}^AT_B = \begin{bmatrix} R & t & & \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & -1 & 0 & 1 \\ 1 & 0 & 0 & -2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

3. Transform the point (3 points)

To find the coordinates of p in frame $\{B\}$:

$${}^BT_A = ({}^AT_B)^{-1}$$

For a homogeneous transformation matrix:

$$^{B}T_{A} = \begin{bmatrix} R^T & -R^T t & & \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Computing: $^{B}R_{A} = R^T = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$-R^T t = -\begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ -2 \\ 0 \end{bmatrix} = -\begin{bmatrix} -2 \\ -1 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$

Therefore: $^{B}T_{A} = \begin{bmatrix} 0 & 1 & 0 & 2 \\ -1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

Computing $p_B =^B T_A \cdot p_A$: $p_B = \begin{bmatrix} 0 & 1 & 0 & 2 \\ -1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \cdot 2 + 1 \cdot 1 + 0 \cdot 1 + 2 \cdot 1 \\ -1 \cdot 2 + 0 \cdot 1 + 0 \cdot 1 + 1 \cdot 1 \\ 0 \cdot 2 + 0 \cdot 1 + 1 \cdot 1 + 0 \cdot 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ -1 \\ 1 \\ 1 \end{bmatrix}$

Thus: $p_B^{\text{cart}} = \begin{bmatrix} 3 \\ -1 \\ 1 \end{bmatrix}$

4. Interpret the result (2 points)

The 90° rotation about the z-axis swapped and negated the x and y components. The translation then shifted the point in the expected direction. The result is reasonable as the point moved from $(2, 1, 1)$ in frame $\{A\}$ to $(3, -3, 1)$ in frame $\{B\}$.

Problem 2: Configuration Space and Workspace (Mobile Robot)

1. C-space and DOF (2 points)

(a) Configuration space

$$\mathcal{Q} = \{(x, y, \theta) : x \in [0, 5], y \in [0, 4], \theta \in (-\pi, \pi]\} \subset \mathbb{R}^2 \times S^1$$

(b) Degrees of freedom

The robot has 3 DOFs: 2 translational (x, y) and 1 rotational (θ).

2. Position workspace of P (5 points)

(a) Ignoring the footprint

$$\mathcal{W} = \{(x, y) : x \in [0, 5], y \in [0, 4]\} \subset \mathbb{R}^2$$

(b) With the footprint

To avoid collision, point P must stay at least $r_R = 0.35$ m away from walls:

$$\mathcal{W}_{\text{clear}} = \{(x, y) : x \in [0.35, 4.65], y \in [0.35, 3.65]\} \subset \mathbb{R}^2$$

(c) Quick check

The point $(x, y) = (0.30, 0.30)$ is **not** reachable without collision since $(0.30, 0.30) \notin \mathcal{W}_{\text{clear}}$.

3. Workspace → C-space obstacles (2 points)

(a) C-space obstacle

Given obstacle $\mathcal{O} = \{(x, y) : \|(x, y) - (0.9, 0.3)\| \leq 0.10\}$

Using Minkowski sum with robot footprint:

$$\mathcal{Q}_{\text{obs}} = \{(x, y, \theta) : \|(x, y) - (0.9, 0.3)\| \leq 0.45\}$$

The obstacle does not depend on θ under the disc approximation.

(b) Configuration check

For $q^* = (1.20, 0.40, \theta = 0.524)$:

Distance check:

$$d = \|(1.20, 0.40) - (0.9, 0.3)\| = \sqrt{(0.30)^2 + (0.10)^2} = \sqrt{0.10} \approx 0.316$$

Since $d = 0.316 < 0.45$, we have $q^* \in \mathcal{Q}_{\text{obs}}$.

4. Connectivity (1 point)

$\mathcal{Q}_{\text{free}}$ is path-connected if there exists a continuous path in configuration space between any two points in $\mathcal{Q}_{\text{free}}$. This is essential for motion planning as it ensures a robot can move between any two valid configurations.

Problem 3: Forward Kinematics (2R Planar Arm)

1. Geometric FK for position & orientation (4 points)

(a) Vector expression

$$p_E = R(\theta_1) \begin{bmatrix} L_1 \\ 0 \end{bmatrix} + R(\theta_1 + \theta_2) \begin{bmatrix} L_2 \\ 0 \end{bmatrix}$$

Expanding:

$$p_E = \begin{bmatrix} L_1 \cos\theta_1 \\ L_1 \sin\theta_1 \end{bmatrix} + \begin{bmatrix} L_2 \cos(\theta_1 + \theta_2) \\ L_2 \sin(\theta_1 + \theta_2) \end{bmatrix}$$

(b) Scalar formulas

$$x(\theta_1, \theta_2) = L_1 \cos\theta_1 + L_2 \cos(\theta_1 + \theta_2)$$
$$y(\theta_1, \theta_2) = L_1 \sin\theta_1 + L_2 \sin(\theta_1 + \theta_2)$$

(c) End-effector orientation

$$\phi(\theta_1, \theta_2) = \theta_1 + \theta_2$$

The end-effector orientation is the sum of both joint angles as each contributes additively to the total rotation.

2. Pose in SE(2) (3 points)

(a) Homogeneous transform

$$^{0}T_E = \begin{bmatrix} \cos(\theta_1+\theta_2) & -\sin(\theta_1+\theta_2) & L_1\cos\theta_1 + L_2\cos(\theta_1+\theta_2) \\ \sin(\theta_1+\theta_2) & \cos(\theta_1+\theta_2) & L_1\sin\theta_1 + L_2\sin(\theta_1+\theta_2) \\ 0 & 0 & 1 \end{bmatrix}$$

(b) Product of elementary transforms

$$^{0}T_E = ^{0}T_1 \cdot ^{1}T_E$$

where:

$$^{0}T_1 = \begin{bmatrix} \cos\theta_1 & -\sin\theta_1 & L_1\cos\theta_1 \\ \sin\theta_1 & \cos\theta_1 & L_1\sin\theta_1 \\ 0 & 0 & 1 \end{bmatrix}$$

$$^{1}T_E = \begin{bmatrix} \cos\theta_2 & -\sin\theta_2 & L_2\cos\theta_2 \\ \sin\theta_2 & \cos\theta_2 & L_2\sin\theta_2 \\ 0 & 0 & 1 \end{bmatrix}$$

3. Numeric evaluation (2 points)

For $\theta_1 = 30 = \pi/6$ rad and $\theta_2 = 60 = \pi/3$ rad:

$$\phi = \theta_1 + \theta_2 = \pi/6 + \pi/3 = \pi/2 = 1.571 \text{ rad} \quad (1)$$

$$x = 1.0\cos(\pi/6) + 0.8\cos(\pi/2) = 0.866 + 0 = 0.866 \text{ m} \quad (2)$$

$$y = 1.0\sin(\pi/6) + 0.8\sin(\pi/2) = 0.5 + 0.8 = 1.300 \text{ m} \quad (3)$$

$$^{0}T_E = \begin{bmatrix} 0 & -1 & 0.866 \\ 1 & 0 & 1.300 \\ 0 & 0 & 1 \end{bmatrix}$$

4. Tool offset (gripper) (1 point)

Gripper transform:

$$^{E}T_G = \begin{bmatrix} 1 & 0 & 0.1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

For the gripper position:

$$x_G = x_E + d_g\cos\phi = 0.866 + 0.1\cos(\pi/2) = 0.866 \text{ m} \quad (4)$$

$$y_G = y_E + d_g\sin\phi = 1.300 + 0.1\sin(\pi/2) = 1.400 \text{ m} \quad (5)$$

Problem 4: Inverse Kinematics (2R Planar Arm)

1. Reachability condition (2 points)

For a point (x, y) to be reachable:

$$|L_1 - L_2| \leq r \leq L_1 + L_2$$

where $r = \sqrt{x^2 + y^2}$. This represents the annulus between the inner circle (arm folded) and outer circle (arm extended).

2. Elbow angle θ_2 (3 points)

Using the law of cosines:

$$\cos\theta_2 = \frac{x^2 + y^2 - L_1^2 - L_2^2}{2L_1L_2}$$

Therefore:

$$\theta_2 = \pm \arccos\left(\frac{x^2 + y^2 - L_1^2 - L_2^2}{2L_1L_2}\right)$$

Two branches: - Elbow-up: $\theta_2 > 0$ (positive branch) - Elbow-down: $\theta_2 < 0$ (negative branch)

3. Shoulder angle θ_1 (3 points)

$$\theta_1 = \text{atan2}(y, x) - \text{atan2}(L_2 \sin\theta_2, L_1 + L_2 \cos\theta_2)$$

This gives two θ_1 values corresponding to the two θ_2 branches.

$\pi/4, 3\pi/4]$: - Elbow-up: $\theta_1 = -0.414$, $\theta_2 = 0.927$ (exceeds $3\pi/4 \approx 2.356$) - Elbow-down: $\theta_1 = 1.058$, $\theta_2 = -0.927$

Only the elbow-down configuration satisfies the joint limits.

Forward check verification confirms the solution is correct.