COMS W4733: Computational Aspects of Robotics
Homework 1

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Problem 1: Homogeneous Transformations

1. Convert to homogeneous coordinates (1 point)

Convert $p_A^{\text{cart}} = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}$ to homogeneous:

$$p_A = \begin{bmatrix} p_A^{\text{cart}} \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 1 \\ 1 \end{bmatrix}.$$

2. Construct ${}^{A}T_B$ (4 points)

Rotation 90° about $+z$:

$$R = R_z(90^\circ) = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \qquad t = \begin{bmatrix} 1 \\ -2 \\ 0 \end{bmatrix}.$$

Therefore

$${}^{A}T_B = \begin{bmatrix} R & t \\ 0\ 0\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & -1 & 0 & 1 \\ 1 & 0 & 0 & -2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

3. Transform the point (3 points)

Use $({}^{A}T_B)^{-1} = \begin{bmatrix} R^\top & -R^\top t \\ 0 & 1 \end{bmatrix}$:

$${}^{B}T_A = \begin{bmatrix} 0 & 1 & 0 & 2 \\ -1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \qquad p_B = {}^{B}T_A\, p_A = \begin{bmatrix} 3 \\ -1 \\ 1 \\ 1 \end{bmatrix}.$$

So $p_B^{\text{cart}} = (3, -1, 1)^\top$.

4. Interpret the result (2 points)

The 90° rotation swaps $(x, y) \mapsto (-y, x)$; the translation adds $(+1, -2, 0)$, yielding $(3, -1, 1)$ as expected.

Problem 2: Configuration Space and Workspace (Mobile Robot)

1. C-space and DOF (2 points)

(a) $q = (x, y, \theta) \in \mathbb{R}^2 \times S^1$ with

$$\mathcal{Q} = [0, 5] \times [0, 4] \times (-\pi, \pi].$$

(b) DOF $= 3$ (two translational, one rotational).

2. Position workspace of P (5 points)

(a) Ignoring footprint:

$$\mathcal{W} = [0, 5] \times [0, 4].$$

(b) With disc footprint $r_R = 0.35$:

$$\mathcal{W}_{\text{clear}} = \{(x, y) : 0.35 \le x \le 4.65,\ 0.35 \le y \le 3.65\}.$$

(c) $(0.30, 0.30) \notin \mathcal{W}_{\text{clear}}$ (collision).

3. Workspace → C-space obstacles (2 points)

Obstacle $\mathcal{O} = \{(x, y) : \|(x, y) - (0.9, 0.3)\| \le 0.10\}$. Grow by r_R:

$$\|(x, y) - (0.9, 0.3)\| \le 0.10 + 0.35 = 0.45 \iff (x - 0.9)^2 + (y - 0.3)^2 \le 0.45^2 = 0.2025.$$

Thus $\mathcal{Q}_{\text{obs}} = \{(x, y, \theta) : (x - 0.9)^2 + (y - 0.3)^2 \le 0.2025\}$. For $q^\star = (1.20, 0.40, 0.524)$: $(1.20 - 0.9)^2 + (0.40 - 0.3)^2 = 0.10 < 0.2025 \Rightarrow q^\star \in \mathcal{Q}_{\text{obs}}$.

4. Connectivity (1 point)

$\mathcal{Q}_{\text{free}}$ is *path-connected* if any two configurations in it are connected by a continuous collision-free path.

Problem 3: Forward Kinematics (2R Planar Arm)

1. Geometric FK for position & orientation (4 points)

(a) Vector form:

$$p_E = \underbrace{R(\theta_1)\begin{bmatrix} L_1 \\ 0 \end{bmatrix}}_{\text{Link 1}} + \underbrace{R(\theta_1 + \theta_2)\begin{bmatrix} L_2 \\ 0 \end{bmatrix}}_{\text{Link 2}}, \quad R(\alpha) = \begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix}.$$

(b) Scalars:

$$x = L_1 \cos\theta_1 + L_2 \cos(\theta_1 + \theta_2), \quad y = L_1 \sin\theta_1 + L_2 \sin(\theta_1 + \theta_2).$$

(c) Orientation: $\phi = \theta_1 + \theta_2$ (since θ_2 is relative).

2. Pose in SE(2) (3 points)

$$
{}^{0}T_E = \begin{bmatrix} \cos\phi & -\sin\phi & x \\ \sin\phi & \cos\phi & y \\ 0 & 0 & 1 \end{bmatrix} = \underbrace{R_z(\theta_1)T_x(L_1)}_{{}^{0}T_1}\,\underbrace{R_z(\theta_2)T_x(L_2)}_{{}^{1}T_E}.
$$

3. Numeric evaluation (2 points)

For $\theta_1 = 30^\circ = \pi/6$ and $\theta_2 = 60^\circ = \pi/3$ with $L_1 = 1.0$, $L_2 = 0.8$:

$$
\phi = 1.571, \quad x = 0.866, \quad y = 1.300, \qquad {}^{0}T_E = \begin{bmatrix} 0 & -1 & 0.866 \\ 1 & 0 & 1.300 \\ 0 & 0 & 1 \end{bmatrix}.
$$

4. Tool offset (gripper) (1 point)

With $d_g = 0.10$ along x_E:

$$
{}^{E}T_G = T_x(d_g), \quad {}^{0}T_G = {}^{0}T_E\,{}^{E}T_G = \begin{bmatrix} \cos\phi & -\sin\phi & x + d_g\cos\phi \\ \sin\phi & \cos\phi & y + d_g\sin\phi \\ 0 & 0 & 1 \end{bmatrix}.
$$

Numerically $(x_G, y_G) = (0.866, 1.400)$.

Problem 4: Inverse Kinematics (2R Planar Arm)

1. Reachability condition (2 points)

Let $r = \sqrt{x^2 + y^2}$. The point (x, y) is reachable iff

$$
\boxed{|L_1 - L_2| \le r \le L_1 + L_2},
$$

i.e., the target lies in the annulus between the inner (arm folded) and outer (arm stretched) circles.

2. Elbow angle θ_2 (3 points)

Law of cosines on triangle (L_1, L_2, r) with elbow interior angle $\pi - \theta_2$:

$$
\cos\theta_2 = \frac{r^2 - L_1^2 - L_2^2}{2L_1L_2} =: c_2, \qquad s_2 = \pm\sqrt{1 - c_2^2}, \qquad \boxed{\theta_2 = \operatorname{atan2}(s_2, c_2)}.
$$

Two branches: *elbow-up* ($s_2 > 0$) and *elbow-down* ($s_2 < 0$).

3. Shoulder angle θ_1 (3 points)

Let $\alpha = \operatorname{atan2}(y, x)$ and $\beta = \operatorname{atan2}\big(L_2s_2,\ L_1 + L_2c_2\big)$. Then

$$
\boxed{\theta_1 = \alpha - \beta = \operatorname{atan2}(y, x) - \operatorname{atan2}\big(L_2s_2,\ L_1 + L_2c_2\big)}.
$$

This yields one θ_1 for each choice of $\operatorname{sign}(s_2)$.

4. Numeric test & joint limits (2 points)

Target $(x^\star, y^\star) = (1.200, 0.400)$, $L_1 = 1.0$, $L_2 = 0.8$. Limits: $\theta_1 \in [-\pi, \pi)$, $\theta_2 \in [-3\pi/4, 3\pi/4] = [-2.356, 2.356]$.

Step 1: r, c_2, s_2 (3 d.p.).

$$r^2 = 1.200^2 + 0.400^2 = 1.600, \qquad r = \sqrt{1.600} = \boxed{1.265}.$$

$$c_2 = \frac{r^2 - L_1^2 - L_2^2}{2L_1L_2} = \frac{1.600 - 1.000 - 0.640}{2(1.0)(0.8)} = \frac{-0.040}{1.600} = \boxed{-0.025},$$

$$s_2 = \pm\sqrt{1 - c_2^2} = \pm\sqrt{1 - 0.025^2} = \boxed{\pm 0.999}.$$

Step 2: α and β (3 d.p.).

$$\alpha = \operatorname{atan2}(0.400, 1.200) = \operatorname{atan}(1/3) = \boxed{0.322}\ \text{rad},$$

$$L_1 + L_2c_2 = 1.0 + 0.8(-0.025) = 0.980, \quad L_2s_2 = 0.8 \times 0.999 = 0.800.$$

$$\beta_\pm = \operatorname{atan2}(\pm 0.800,\ 0.980) = \boxed{\pm 0.684}\ \text{rad}.$$

Step 3: Joint angles (3 d.p.).

$$\theta_2 = \operatorname{atan2}(s_2, c_2) = \begin{cases} \boxed{+1.596} & (s_2 > 0,\ \text{elbow-up}), \\ \boxed{-1.596} & (s_2 < 0,\ \text{elbow-down}), \end{cases} \qquad \theta_1 = \alpha - \beta_\pm = \begin{cases} \boxed{-0.363} & (\text{elbow-up}), \\ \boxed{+1.006} & (\text{elbow-down}). \end{cases}$$

Step 4: Joint-limit check.

$$-2.356 \le \pm 1.596 \le 2.356, \qquad -\pi < -0.363,\ 1.006 < \pi.$$

$$\boxed{(-0.363,\ +1.596)}\ \text{and}\ \boxed{(1.006,\ -1.596)}\ \text{both satisfy the limits.}$$

Step 5: Forward check (using the rounded angles).

$$\hat{x} = L_1\cos\theta_1 + L_2\cos(\theta_1 + \theta_2), \quad \hat{y} = L_1\sin\theta_1 + L_2\sin(\theta_1 + \theta_2).$$

Elbow-up $(\theta_1, \theta_2) = (-0.363,\ 1.596)$:

$$\cos\theta_1 \approx 0.935,\ \sin\theta_1 \approx -0.355,\ \cos(\theta_1 + \theta_2) = \cos(1.233) \approx 0.332,\ \sin(1.233) \approx 0.944.$$

$\hat{x} = 1(0.935) + 0.8(0.332) = \underline{1.201} \approx 1.200$, $\quad \hat{y} = 1(-0.355) + 0.8(0.944) = \underline{0.400}$ ($\le 10^{-3}$ from $y^\star$).

Elbow-down $(\theta_1, \theta_2) = (1.006,\ -1.596)$:

$$\cos\theta_1 \approx 0.535,\ \sin\theta_1 \approx 0.845,\ \cos(\theta_1 + \theta_2) = \cos(-0.590) \approx 0.831,\ \sin(-0.590) \approx -0.556.$$

$$\hat{x} = 1(0.535) + 0.8(0.831) = \underline{1.200}, \quad \hat{y} = 1(0.845) + 0.8(-0.556) = \underline{0.400}.$$

In both branches, $\|(\hat{x}, \hat{y}) - (x^\star, y^\star)\| \le 10^{-3}$.