COMS W4733: Computational Aspects of Robotics
Homework 1

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Problem 1: Homogeneous Transformations

1. Convert to homogeneous coordinates (1 point)

Convert $p_A^{\text{cart}} = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}$ to homogeneous:

$$p_A = \begin{bmatrix} p_A^{\text{cart}} \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 1 \\ 1 \end{bmatrix}.$$

2. Construct ${}^{A}T_B$ (4 points)

Rotation 90° about $+z$:

$$R = R_z(90^\circ) = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \qquad t = \begin{bmatrix} 1 \\ -2 \\ 0 \end{bmatrix}.$$

Therefore

$${}^{A}T_B = \begin{bmatrix} R & t \\ 0\ 0\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & -1 & 0 & 1 \\ 1 & 0 & 0 & -2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

3. Transform the point (3 points)

Use $({}^{A}T_B)^{-1} = \begin{bmatrix} R^\top & -R^\top t \\ 0 & 1 \end{bmatrix}$:

$${}^{B}T_A = \begin{bmatrix} 0 & 1 & 0 & 2 \\ -1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \qquad p_B = {}^{B}T_A\, p_A = \begin{bmatrix} 3 \\ -1 \\ 1 \\ 1 \end{bmatrix}.$$

So $p_B^{\text{cart}} = (3, -1, 1)^\top$.

4. Interpret the result (2 points)

The $90°$ rotation swaps $(x, y) \mapsto (-y, x)$; the translation adds $(+1, -2, 0)$, yielding $(3, -1, 1)$ as expected. ${}^{B}T_{A}$ is the inverse transformation, it first translates the point by $-t$ and then performs a $90°$ *clockwise* rotation about the z-axis. This explicitly works out to be: $p_B = R^{\top}(p_A - t)$.

Problem 2: Configuration Space and Workspace (Mobile Robot)

1. C-space and DOF (2 points)

(a) $q = (x, y, \theta) \in \mathbb{R}^2 \times S^1$ with

$$\mathcal{Q} = [0, 5] \times [0, 4] \times (-\pi, \pi].$$

(b) DOF = 3 (two translational, one rotational).

2. Position workspace of P (5 points)

(a) Ignoring footprint:

$$\mathcal{W} = [0, 5] \times [0, 4].$$

(b) With disc footprint $r_R = 0.35$:

$$\mathcal{W}_{\text{clear}} = \{(x, y) : 0.35 < x < 4.65,\ 0.35 < y < 3.65\}.$$

(c) The point $(0.30, 0.30) \notin \mathcal{W}_{\text{clear}}$ (collision).

3. Workspace → C-space Obstacles (2 points)

(a) The C-space obstacle set is obtained by growing the obstacle radius by r_R:

$$Q_{\text{obs}} = \{(x, y) \mid \|(x, y) - (0.9, 0.3)\| \le 0.10 + r_R\} = \{(x, y) \mid \|(x, y) - (0.9, 0.3)\| \le 0.45\}.$$

This set does not depend on θ because the robot is a disc (isotropic).

(b) For $q^{\star} = (1.20,\ 0.40,\ \theta = 0.524)$, compute the distance:

$$d = \|(1.20,\ 0.40) - (0.9,\ 0.3)\| = \sqrt{0.3^2 + 0.1^2} = 0.316 < 0.45.$$

Hence, $q^{\star} \in Q_{\text{obs}}$ (the configuration is in collision).

4. Connectivity (1 point)

$\mathcal{Q}_{\text{free}}$ is *path-connected* if any two configurations in it are connected by a continuous collision-free path. This means that every pair of collision-free configurations can be connected by a continuous collision free path. This property is essential for motion planning because it gurantees that a planner can find a path between any two configurations in $\mathcal{Q}_{\text{free}}$ if one exists.

Problem 3: Forward Kinematics (2R Planar Arm)

1. Geometric FK for position & orientation (4 points)

(a) Vector form:

$$p_E = \underbrace{R(\theta_1)\begin{bmatrix} L_1 \\ 0 \end{bmatrix}}_{\text{Link 1}} + \underbrace{R(\theta_1+\theta_2)\begin{bmatrix} L_2 \\ 0 \end{bmatrix}}_{\text{Link 2}}, \quad R(\alpha) = \begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix}.$$

(b) Scalars:

$$x = L_1\cos\theta_1 + L_2\cos(\theta_1+\theta_2), \quad y = L_1\sin\theta_1 + L_2\sin(\theta_1+\theta_2).$$

(c) Orientation:

$$\phi = \theta_1 + \theta_2$$

(since θ_2 is relative to link 1).

2. Pose in SE(2) (3 points)

(a) General pose of E in frame 0:

$${}^0T_E = \begin{bmatrix} \cos\phi & -\sin\phi & x \\ \sin\phi & \cos\phi & y \\ 0 & 0 & 1 \end{bmatrix}, \qquad \phi = \theta_1+\theta_2.$$

(b) Compose link transforms explicitly:

$${}^0T_E = {}^0T_1\,{}^1T_E,$$

with

$${}^0T_1 = \begin{bmatrix} \cos\theta_1 & -\sin\theta_1 & L_1\cos\theta_1 \\ \sin\theta_1 & \cos\theta_1 & L_1\sin\theta_1 \\ 0 & 0 & 1 \end{bmatrix}, \quad {}^1T_E = \begin{bmatrix} \cos\theta_2 & -\sin\theta_2 & L_2\cos\theta_2 \\ \sin\theta_2 & \cos\theta_2 & L_2\sin\theta_2 \\ 0 & 0 & 1 \end{bmatrix}.$$

Multiplying,

$${}^0T_E = \begin{bmatrix} \cos(\theta_1+\theta_2) & -\sin(\theta_1+\theta_2) & L_2\cos(\theta_1+\theta_2) + L_1\cos\theta_1 \\ \sin(\theta_1+\theta_2) & \cos(\theta_1+\theta_2) & L_2\sin(\theta_1+\theta_2) + L_1\sin\theta_1 \\ 0 & 0 & 1 \end{bmatrix}.$$

(c) Extract the end-effector point explicitly:

$$p_E^0 = {}^0T_E\,p_E^E, \qquad p_E^E = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \Rightarrow p_E^0 = \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}.$$

3. Numeric evaluation (2 points)

For $\theta_1 = 30^\circ = \pi/6$, $\theta_2 = 60^\circ = \pi/3$, $L_1 = 1.0$, $L_2 = 0.8$:

$$\phi = \theta_1 + \theta_2 = \tfrac{\pi}{2}, \qquad x = 0.8\cos\tfrac{\pi}{2} + 1\cos\tfrac{\pi}{6} = 0.866, \qquad y = 0.8\sin\tfrac{\pi}{2} + 1\sin\tfrac{\pi}{6} = 1.300.$$

$$^0T_E = \begin{bmatrix} 0 & -1 & 0.866 \\ 1 & 0 & 1.300 \\ 0 & 0 & 1 \end{bmatrix}.$$

4. Tool offset (gripper) (1 point)

Offset $d_g = 0.10$ along x_E:

$$^ET_G = \begin{bmatrix} 1 & 0 & d_g \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \qquad ^0T_G = {}^0T_E\,{}^ET_G = \begin{bmatrix} \cos\phi & -\sin\phi & x + d_g\cos\phi \\ \sin\phi & \cos\phi & y + d_g\sin\phi \\ 0 & 0 & 1 \end{bmatrix}.$$

Numerically,

$$(x_G, y_G) = (0.866, 1.400).$$

Problem 4: Inverse Kinematics (2R Planar Arm)

1. Reachability condition (2 points)

Let $r = \sqrt{x^2 + y^2}$. The point (x, y) is reachable if:

$$\boxed{|L_1 - L_2| \le r \le L_1 + L_2},$$

i.e., the target lies in the annulus between the inner (arm folded) and outer (arm stretched) circles.

2. Elbow angle θ_2 (3 points)

Law of cosines on triangle (L_1, L_2, r) with elbow interior angle $\pi - \theta_2$:

$$\cos\theta_2 = \frac{r^2 - L_1^2 - L_2^2}{2L_1L_2} =: c_2, \qquad s_2 = \pm\sqrt{1 - c_2^2}, \qquad \boxed{\theta_2 = \operatorname{atan2}(s_2, c_2)}.$$

Two branches: *elbow-up* ($s_2 > 0$) and *elbow-down* ($s_2 < 0$).

3. Shoulder angle θ_1 (3 points)

Let $\alpha = \operatorname{atan2}(y, x)$ and $\beta = \operatorname{atan2}\big(L_2s_2,\ L_1 + L_2c_2\big)$. Then

$$\boxed{\theta_1 = \alpha - \beta = \operatorname{atan2}(y, x) - \operatorname{atan2}\big(L_2s_2,\ L_1 + L_2c_2\big)}.$$

This yields one θ_1 for each choice of $\operatorname{sign}(s_2)$.

4. Numeric test & joint limits (2 points)

Target $(x^\star, y^\star) = (1.200, 0.400)$, $L_1 = 1.0$, $L_2 = 0.8$. Limits: $\theta_1 \in [-\pi, \pi)$, $\theta_2 \in [-3\pi/4, 3\pi/4] = [-2.356, 2.356]$.

Step 1: r, c_2, s_2 (3 d.p.).

$$r^2 = 1.200^2 + 0.400^2 = 1.600, \qquad r = \sqrt{1.600} = \boxed{1.265}.$$

$$c_2 = \frac{r^2 - L_1^2 - L_2^2}{2L_1L_2} = \frac{1.600 - 1.000 - 0.640}{2(1.0)(0.8)} = \frac{-0.040}{1.600} = \boxed{-0.025},$$

$$s_2 = \pm\sqrt{1 - c_2^2} = \pm\sqrt{1 - 0.025^2} = \boxed{\pm 0.999}.$$

Step 2: α and β (3 d.p.).

$$\alpha = \operatorname{atan2}(0.400, 1.200) = \operatorname{atan}(1/3) = \boxed{0.322} \text{ rad},$$

$$L_1 + L_2c_2 = 1.0 + 0.8(-0.025) = 0.980, \quad L_2s_2 = 0.8 \times 0.999 = 0.800.$$

$$\beta_\pm = \operatorname{atan2}(\pm 0.800,\ 0.980) = \boxed{\pm 0.684} \text{ rad}.$$

Step 3: Joint angles (3 d.p.).

$$\theta_2 = \operatorname{atan2}(s_2, c_2) = \begin{cases} \boxed{+1.596} & (s_2 > 0,\ \text{elbow-up}), \\ \boxed{-1.596} & (s_2 < 0,\ \text{elbow-down}), \end{cases} \qquad \theta_1 = \alpha - \beta_\pm = \begin{cases} \boxed{-0.363} & (\text{elbow-up}), \\ \boxed{+1.006} & (\text{elbow-down}). \end{cases}$$

Step 4: Joint-limit check.

$$-2.356 \le \pm 1.596 \le 2.356, \qquad -\pi < -0.363,\ 1.006 < \pi.$$

$$\boxed{(-0.363,\ +1.596)} \text{ and } \boxed{(1.006,\ -1.596)} \text{ both satisfy the limits.}$$

Step 5: Forward check (using the rounded angles).

$$\hat{x} = L_1 \cos\theta_1 + L_2 \cos(\theta_1 + \theta_2), \quad \hat{y} = L_1 \sin\theta_1 + L_2 \sin(\theta_1 + \theta_2).$$

Elbow-up $(\theta_1, \theta_2) = (-0.363,\ 1.596)$:

$$\cos\theta_1 \approx 0.935,\ \sin\theta_1 \approx -0.355,\ \cos(\theta_1 + \theta_2) = \cos(1.233) \approx 0.332,\ \sin(1.233) \approx 0.944.$$

$\hat{x} = 1(0.935) + 0.8(0.332) = \underline{1.201} \approx 1.200$, $\quad \hat{y} = 1(-0.355) + 0.8(0.944) = \underline{0.400}$ ($\le 10^{-3}$ from $y^\star$).

Elbow-down $(\theta_1, \theta_2) = (1.006,\ -1.596)$:

$$\cos\theta_1 \approx 0.535,\ \sin\theta_1 \approx 0.845,\ \cos(\theta_1 + \theta_2) = \cos(-0.590) \approx 0.831,\ \sin(-0.590) \approx -0.556.$$

$$\hat{x} = 1(0.535) + 0.8(0.831) = \underline{1.200}, \quad \hat{y} = 1(0.845) + 0.8(-0.556) = \underline{0.400}.$$

In both branches, $\|(\hat{x}, \hat{y}) - (x^\star, y^\star)\| \le 10^{-3}$.

4.4 Numerical IK Computation

The following Python script implements the inverse kinematics computation for Problem 4:

Listing 1: Python Implementation of 2R Planar Arm Inverse Kinematics

```python
import math

L1, L2 = 1.0, 0.8
x, y = 1.20, 0.40

r = math.hypot(x, y)
c = (r*r - L1*L1 - L2*L2) / (2*L1*L2)
s = math.sqrt(1 - c*c)

theta2_down = math.atan2(+s, c)
theta2_up   = math.atan2(-s, c)

a_down = L1 + L2*math.cos(theta2_down)
b_down = L2*math.sin(theta2_down)
theta1_down = math.atan2(y, x) - math.atan2(b_down, a_down)

a_up = L1 + L2*math.cos(theta2_up)
b_up = L2*math.sin(theta2_up)
theta1_up = math.atan2(y, x) - math.atan2(b_up, a_up)

print(f"Elbow-down: theta1={theta1_down:.3f}, theta2={theta2_down:.3f}")
print(f"Elbow-up:   theta1={theta1_up:.3f}, theta2={theta2_up:.3f}")

xd = L1*math.cos(theta1_down) + L2*math.cos(theta1_down +
     theta2_down)
yd = L1*math.sin(theta1_down) + L2*math.sin(theta1_down +
     theta2_down)
xu = L1*math.cos(theta1_up)   + L2*math.cos(theta1_up   + theta2_up)
yu = L1*math.sin(theta1_up)   + L2*math.sin(theta1_up   + theta2_up)

print(f"Forward check: down = ({xd:.3f}, {yd:.3f})")
print(f"Forward check: up   = ({xu:.3f}, {yu:.3f})")
```

Output:

```
Elbow-down: theta1=-0.363, theta2=1.596
Elbow-up:   theta1=1.006, theta2=-1.596
Forward check: down = (1.200, 0.400)
Forward check: up   = (1.200, 0.400)
```